University of Strathclyde

Department Of Mathematics and Statistics

MM955: Financial Econometrics Project

Name of Student: Eugene Osae

Student Number: 202222159

TABLE OF CONTENTS

INTRODUCTION AND BACKGROUND

The Standard and Poor's 500 (S&P 500) is a stock market index used to track the stock performance of the largest 500 companies listed on the stock market in the U.S. It is one of the popularly followed equity indices and it includes approximately 80% of the total market capitalization of the U.S. public companies made up of an accumulated market capitalization of US$52.2 trillion as of 31st December 2024. The S&P 500 index is a free-float market-value-weighted index made up of 34.6% market capitalization of the nine largest companies, namely Apple, Microsoft, Nvidia, Amazon, Meta, Alphabet, Berkshire Hathaway, Broadcom and Tesla. The components that increased their dividends in 25 consecutive years are called the S&P 500 Dividend Aristocrats. The index is used as a factor for the computation of the Conference Board Leading Economic Index that is used to forecast the direction of the economy. The index is represented on various stock trading platforms with ticker symbols such as ^GSPC, INX and $SPX, depending on the stock market or website. The S&P Dow Jones Indices is responsible for the maintenance of the S&P 500 index, made up of a joint venture majority-owned by S&P Global, and its components are selected by a committee. The S&P 500 index is a very important equity market, and its importance has been recognized in most academic literature since mid-1980, as demonstrated by Professor Andrei Shleifer that stocks tend to jump after being added to the S&P 500. Several other financial articles have explored this issue, for more than thirty years, the so-called "Index Inclusion" effect remains an active area of research in finance. In recent times, another financial paper has demonstrated that the way the S&P 500 index is displayed has systematic effects on financial markets, underscoring the influence of the index.

Investors mostly use the index for passive investing, which aims to maximize returns in the large part by minimizing the cost of buying and selling securities. The S&P 500 has dominated the U.S. index fund market, directing over $7 trillion of investors' money in 2022. Investment managers also use the index for mutual fund benchmarking that investors use as a reference point to measure performance. In December 2021, an amount of $15.6 trillion was globally benchmarked to the S&P 500 index. During the end of the sample period, over 40% of all dollars invested in US-focused equity mutual funds used the S&P 500 as their main benchmark, making the Index the most popular benchmarked. Investment professionals also use the index for empirical analysis, to systematically investigate and evaluate data and evidence to draw meaningful conclusions and make informed decisions. The S&P 500 simply captures the performance of one portfolio of large cap domestic equities, which involves the exercise of a substantial amount of discretion by a particular group of financial market professionals and the quantification of the extent of the discretion involved in the construction of the Index and shows that it is substantial. The S&P 500 is used by corporations and financial market participants to assess the performance of individual firms or their managers. This requires public companies to include a performance graph in their annual report that compares their total returns to those of the equity market index, which comprises companies whose equity securities are traded on the same exchange or are of similar market stature, capitalization.

Using historical closing prices of the S&P 500 Index, the project aims to transform the data using the log returns to examine the stationarity of the time series data, examine the distribution of returns, estimate the value at risk at different confidence intervals. Choose an appropriate time series model, fit the model and build a model to describe the volatility of log-returns. Financial time series models are implemented to model the volatility of log-returns and check for the adequacy of the model.

METHODOLGY

Financial Time Series is a sequence of data points that track a financial variable such as stock prices over a specified period recorded over regular intervals. Analysts use financial time series data to predict future price movements, assess investment risk and evaluate company performance to make informed decisions.

The dataset used in this project is made up of 1,010 historical closing prices from the beginning of January 1993 to the end of December 1996 of the S&P 500 Index downloaded from Yahoo Finance. In most financial studies, log returns of assets are used instead of prices (Campbell, Lo, and MacKinley, 1997), give two reasons for using asset returns instead of prices. For the average investor, the return of an asset is a complete and scale-free summary of the investment opportunity, and return series are easier to handle than price series because they have more statistical properties than price series.

Let P_t be the price of an asset at time t and P_{t-1} be the price of an asset at time t-1. The simple returns at time t are given as $r_t = \ln(P_t) - \ln(P_{t-1})$ and used to compute the log-returns of the prices and to transform the dataset to achieve normality. Time plot of the log-returns is used to visualize the change over time, to identify trends, patterns, and seasonality within the dataset, by plotting log-returns against time values, which helps to understand variable fluctuations and to forecast future behaviour based on historical patterns. The distribution of the log-returns is assessed with a plot of the histogram and density plots to provide a smooth visual representation of the frequency of data points within different ranges of the variable to identify the overall shape, central tendency, spread, skewness and potential outliers of the log-returns data. A descriptive statistic of the log returns is estimated to obtain the sample mean, variance, skewness and kurtosis. The skewness and kurtosis are used to test for the normality of the data. The Jarque-Bera test is also an applied diagnostic tool for the normality of the returns data. The Jarque-Bera statistic is defined as $JB = \frac{n}{6}\widehat{S_X^2} + \frac{n}{24}(\widehat{K_X} - 3)^2$ where $\hat{S}_X$is the skewness and $\widehat{K_X}$ is the kurtosis.

The value at risk (VaR) is used as a risk measurement metric in financial risk management to measure the risk of loss on a specific portfolio of financial assets. In each portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability of mark-to-market loss on the portfolio over the given time exceeds this value is the given probability level with the assumption of normal markets and no trading in the portfolio. The value at risk (VaR) is used as a risk measurement metric in financial risk management to measure the risk of loss on a specific portfolio of financial assets. In each portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability of mark-to-market loss on the portfolio over the given time exceeds this value is the given probability level with the assumption of normal markets and no trading in the portfolio. Under the normality assumption, if a return is normally distributed, $R \sim N(\mu, \sigma^2)$ and $\varphi^{-1}(1-\alpha)$ is the $(1-\alpha)$ quantile of a $N(0,1)$ distribution, then $VaR_\alpha = -(\mu + \varphi^{-1}(1-\alpha)\sigma)$ where $\mu \, and \, \sigma^2$ are the mean and variance respectively of the normal distribution at a confidence level α.

For a simple log-return r_t to be strictly stationary, the joint distribution of $(r_{t1}, \ldots\ldots, r_{tk})$ is identical to $(r_{t1+t}, \ldots\ldots, r_{tk+t})\ \forall t$, where k is an arbitrary positive integer and $t_1, \ldots\ldots, t_k)$ is a collection of k positive integers. A time series r_t is weakly stationary if both the mean of r_t and the covariance between r_t and $r_{t-\iota}$ are time invariant, ι is an arbitrary integer. Ie. $E(r_t) = \mu$ and $Cov(r_t, r_{t-\iota}) = \gamma_\iota$ which only depend on ι. In practice, if we have observed log-return T data points $\{r_t | t = 1, \ldots\ldots, T\}$Weak stationarity implies that the time plot of data will show that T values fluctuate with constant variation around a fixed level. The Ljung and Box test is used to also test for the stationarity and autocorrelation of the log-returns data. The statistic of the Ljung-Box test is given by $Q(m) = T(T+2)\sum_{\iota=1}^{m}\frac{\widehat{\rho_\iota^2}}{T-\iota}$, the decision rule is to reject H_0 if $Q(m) > \chi_\alpha^2$, where χ_α^2 denotes the $100(1-\alpha)th$ percentile of a chi-square distribution with m degrees of freedom.

A general Autoregressive Moving Average [ARMA(p, q)] model is given as $r_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i \, r_{t-i} + a_t - \sum_{i=1}^{q} \theta_i \, a_{t-i}$ where $\{a_t\}$ is a white noise series and p and q are non-negative integers. The AR and MA models are a special case of the ARMA(p, q) model. The model can be written as $(1 - \varphi_1 B - \cdots . -\varphi_p B^p) r_t = \varphi_1 + (1 - \theta_1 B - \cdots . -\theta_q B^q) a_t$ where $(1 - \varphi_1 B - \cdots . -\varphi_p B^p)$ is the autoregressive (AR) polynomial of the model and $(1 - \theta_1 B - \cdots . -\theta_q B^q)$ is the moving average (MA) polynomial.

For a weakly stationary return series r_t, the linear relationship between r_t and its historical values r_{t-i} is the autocorrelation. The correlation coefficient between r_t and r_{t-i} is called the lag- ι autocorrelation of r_t and it's defined by ρ_ι under the weak stationarity assumption is a function of ι only. The autocorrelation function (ACF) is given by

$$\rho_\ell = \frac{\operatorname{Cov}(r_t, r_{t-\ell})}{\sqrt{\operatorname{Var}(r_t)\operatorname{Var}(r_{t-\ell})}} = \frac{\operatorname{Cov}(r_t, r_{t-\ell})}{\operatorname{Var}(r_t)} = \frac{\gamma_\ell}{\gamma_0}$$

Where the property $Var(r_t) = Var(r_{t-\iota})$ for a weakly stationary series is used. The ACF is useful in determining the order of the AR and MA models.

The Partial Autocorrelation Function (PACF) of a stationary time series is a function of its ACF and it's a useful tool to determine the order p, of an AR(p) model. The PACF of an AR model is given as $r_t = \varphi_{0,1} + \varphi_{1,1} r_{t-1} + - - + \varphi_{p,p} r_{t-p} + e_{pt}$ where $\varphi_{0,j}, \varphi_{i,j}$ and e_{jt} are respectively the constant term, the coefficients of r_{t-i} and the error term of an AR(j) model. The Akaike Information Criterion (AIC) is useful in the determination of the order p and q of the AR and MA models. The AIC is defined as AIC $= \frac{-2}{T} \ln(likelihood) + \frac{2}{T} * (number\ of\ parameters)$ where the likelihood function is evaluated at the maximum-likelihood estimates and T is the sample size. For the Gaussian $AR(\iota)$ model, AIC is reduced to $AR(\iota) = \ln(\widetilde{\sigma_\iota^2}) + \frac{\iota \ln(T)}{T}$ where $\widetilde{\sigma_\iota^2}$ is the maximum-likelihood estimate of σ_ι^2, the variance of a_t, and T is the sample size.

For a $GARCH(m, s)$ model, let the log-return series r_t, be $a_t = r_t - \mu_t$ be the innovations at time t, then a_t follows a $GARCH(m, s)$ model if $a_t = \sigma_t r \in_t$, $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i \, a_{t-i}^2 + \sum_{j=1}^{s} \theta \beta_j \, \sigma_{t-j}^2$ where $\{\in_t\}$ is a sequence of identically distributed random variables with mean 0 and variance 1.0, $\alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0$ and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$.

These time series models are diagnosed using the quantile-quantile (qq) plot to visualise the residuals and check whether the residuals of the fitted models follow the normality assumption. The Ljung-Box test is also used to diagnose the adequacy of the models. All graphs and computations were performed using the R statistical package.

RESULTS

A plot of the daily closing price of the S&P 500 index, the daily stock prices were very low from January 1993 to January 1995. Within this period, the highest price was approximately \$475. Prices started increasing from \$475 to \$660 at the end of January 1995 to December 1995, where it dropped and increased to \$750 in January 1996, indicating that the price series are non-stationary. This is shown in the time plot below.

Figure 3.1: Daily stock prices of S&P 500 Index

The time plot shows the daily log-returns against the time index of the S&P 500 Index from January 1993 to January 1996. From the time plot, the log-returns fluctuate around a constant mean of -2% to 2% with relatively constant variation over the time. The log-returns series does not increase or decrease over time and the volatility spread is consistent. This is shown in the plot below.

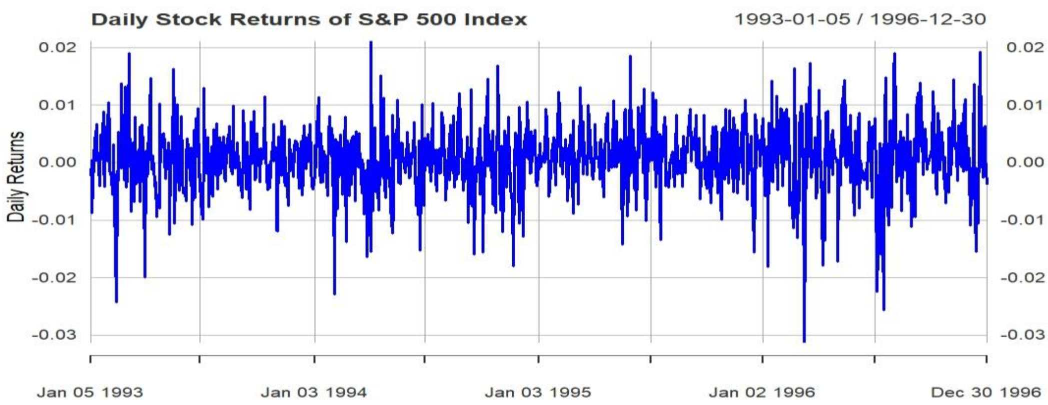


Figure 4.1: Daily stock returns of S&P 500 Index

The descriptive statistics of the daily returns of the S&P 500 Index, the average daily return is 0.0544%, there is a constant variation of 0.0037% from the average returns with a standard deviation of 0.6058%. The log-returns series is negatively skewed at -0.3970 skewness and excess kurtosis of 2.11. The Jarque-Bera test of normality distributed as a chi-squared random variable with degrees of freedom 2, p-value of 2.2-e16 at 95% confidence level to test for the normality of the log-return series.

The distribution of log-return series is shown in Figure 4.3, the histogram and the normal probability density function are evaluated using the sample and the standard deviation of log-return. The empirical density function has a higher peak around its mean and fatter tails than the normal distribution. The distribution of the log-return series is shown in Figure 4.3 below.

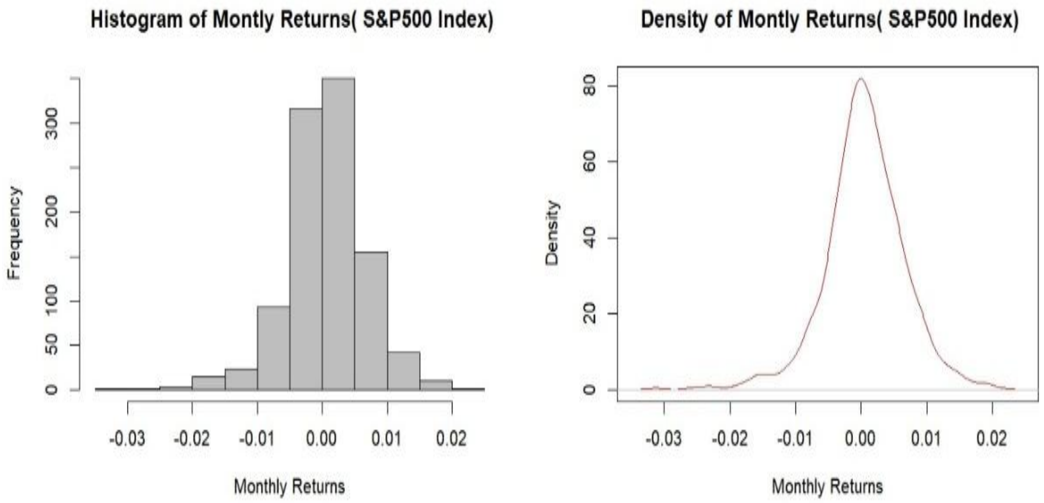


Figure 4.2: Distribution of Log-return series

The value at risk estimated at 99% confidence level is -1.65%, 90% confidence level is -1.283% and at 95% confidence level is -0.91%.

The Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) are used to determine the order of the Autoregression (AR) and Moving Average models. Further that Akaike Information Criterion (AIC) is used to determine the order of the AR (3) indicating that the AR model is significant at lag 3. From the ACF plot, the MA model is significant at lags 1 and 3. Further, the AICs were used for MA (1) (-7442.31), MA (2) (-7441.60) and MA (3) (-7443.28) indicating that MA (2) is significant with the lowest AIC. The AR (3) model and MA (2) model formed that an Autoregressive Moving Average [ARMA (3,2)] is then fitted and given by the function below.

$$r_t - 0.547r_{t-1} + 0.309r_{t-2} - 0.103r_{t-3} = 0.0005 + a_t - 0.489a_{t-1} - 0.317a_{t-2}$$

The plot of the ACF and PACF is shown below in Figure 5.2.

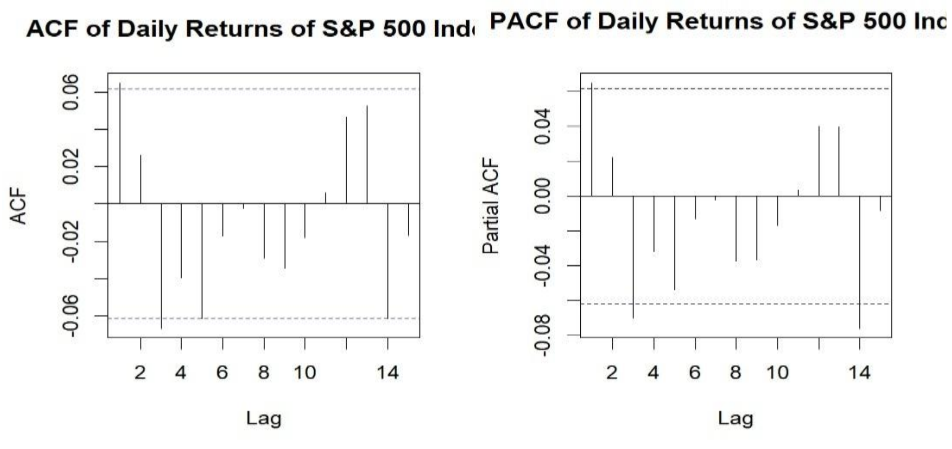


Figure 5.1: Plot of ACF and PACF of daily log-return series

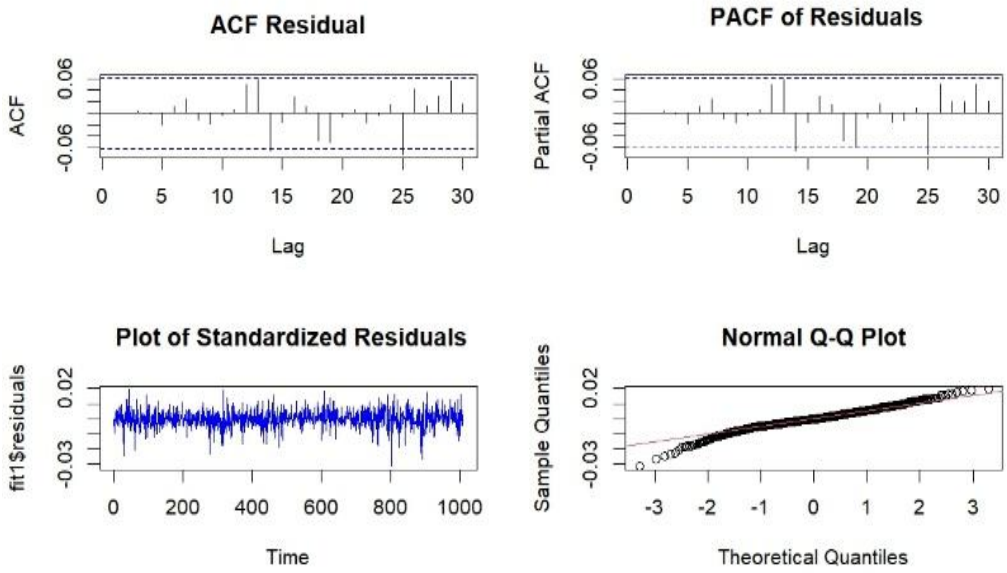


Figure 5.2: Plot of Standardized residuals, ACF, PACF and Q-Q plot

The ARMA (3,2) model is diagnosed to check the accuracy of the model using the standardised residuals to obtain the ACF, PACF and the Q-Q plots. This is shown in Figure 5.3 above. The Ljung-Box test, with degrees of freedom 10 and a p-value of 0.9984 at lag 10, is conducted to check for autocorrelation of the standard residuals, indicating that the residuals of the ARMA (3,2) model are independent.

The GARCH (1,1) model is fitted to obtain the function for the conditional variance as shown below.

$\sigma_t^2 = 0.000001128 + 0.04074a_{t-1}^2$ $+0.9311a_{t-1}^2$

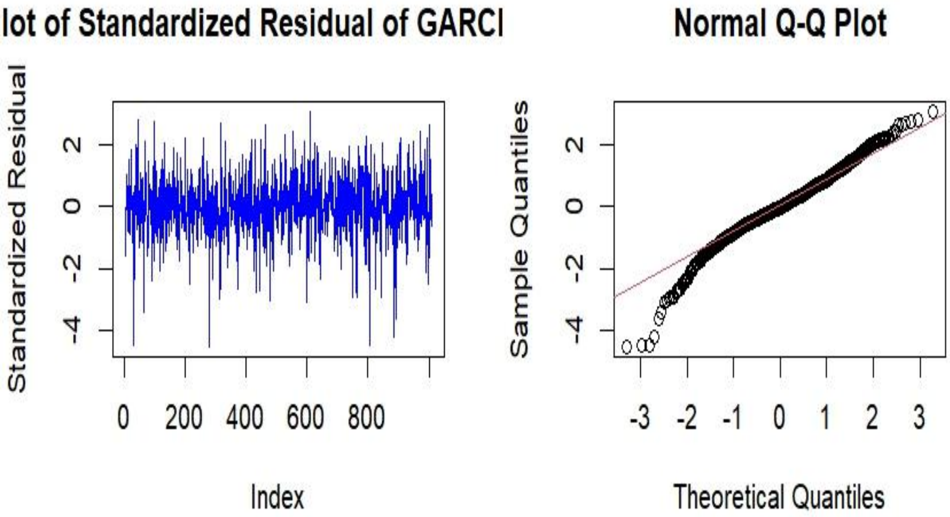
The summary of the GARCH (1,1) model is shown in Figure 5.4 below.

```
Std. Errors:
 based on Hessian

Error Analysis:
        Estimate  Std. Error  t value Pr(>|t|)
mu     6.817e-04   1.662e-04    4.103 4.08e-05 ***
omega  1.128e-06   6.749e-07    1.672  0.09460 .
alpha1 4.074e-02   1.537e-02    2.650  0.00804 **
beta1  9.311e-01   2.744e-02   33.926  < 2e-16 ***
shape  5.168e+00   8.748e-01    5.908 3.47e-09 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Figure 5.3: Summary of the GARCH (1,1) model

From the summary of the GARCH (1,1) model, all the parameters except omega are significant since they are less than the significance level. The Ljung-Box test of the residuals with degrees of freedom 20 and p-value 0.009452 is conducted to check for autocorrelation of the standard residuals. The GARCH (1,1) is further diagnosed to check the accuracy of the model using the standardised residual plot and q-q plot of the residuals. This is shown in figure 6.1 below.



The conditional variance plot is used to represent the estimated volatility of the GARCH (1,1) model changes over time. The plot showed that the highest volatility is at 0.00008 and the lowest at 0.00002. The average variation occurred at 0.00005, exhibiting periods of higher and lower volatility clustering. This can be shown in Figure 6.2 below.

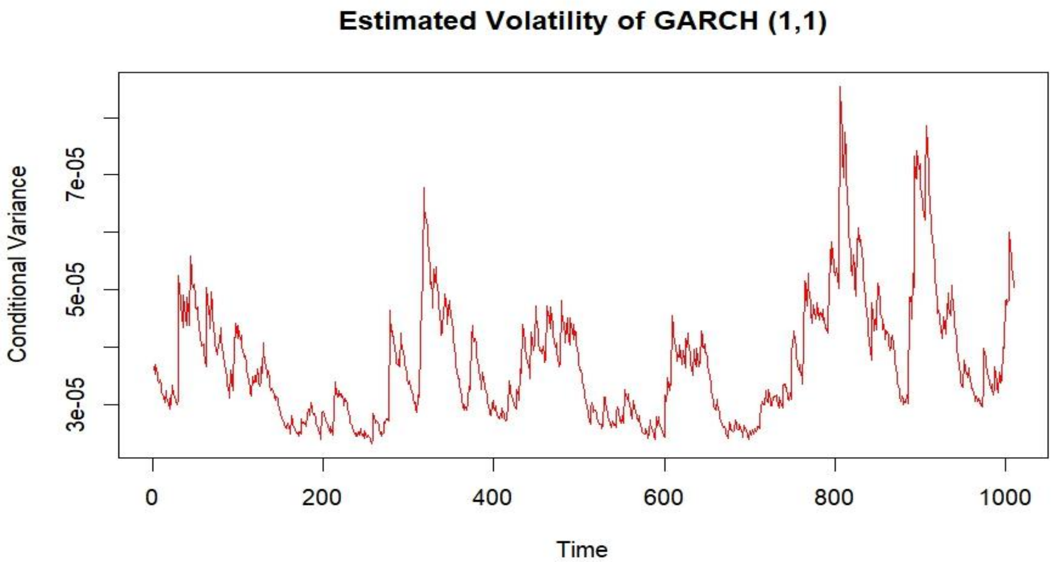


Figure 6.2: Estimated Volatility of the GARCH (1,1) Model.

DISCUSSIONS AND CONCLUSION

Since the log-return fluctuates around a constant mean and variance over time with a consistent volatility spread, the log-return series is stationary. The average log-return of 0.0544% measures the central location of the distribution. The standard deviation of 0.6058% indicates a constant variability in the log-return. The log-return exhibited a negative skewness of 0.3970, suggesting that the distribution of log-returns is skewed to the left. The excess kurtosis of 2.11 indicates that the distribution is heavy-tailed and that a random sample from this distribution tends to contain more extreme values. According to the Jarque-Bera test for normality, the normality of the log-return series is rejected since the p-value is below the significance level. The histogram and probability density function provide evidence of a lack of normality in the log-return series, rendering the normality assumption questionable. The value at risk for various confidence levels suggests that an investor can be 95% confident that the loss on an investment in the S&P 500 Index will not exceed 1.65%. Furthermore, an investor can be 90% confident that the loss will not surpass 1.283% and can be 95% confident that the loss will not be greater than 0.91% over one day. The ACF and PACF indicated significance at lags 14 and 25. The standard residual plot also demonstrated that there is ARCH effect, and the residuals of the log-return series are stationary, with constant mean and variance and a consistent volatility spread. The q-q plot revealed heavy tails at both the lower and upper limits, indicating that the residuals possess a larger proportion of extreme values at both ends of the distribution compared to a normal distribution. Since the residuals of the ARMA (3,2) model are independent, there is no significant autocorrelation. The Ljung-Box test indicates that there is potential with autocorrelation, so the null hypothesis is rejected. The q-q plot revealed heavy tails at both the lower and upper limits, indicating that the residuals possess a larger proportion of extreme values at both ends of the distribution compared to a normal distribution. The standard residual plot also demonstrated that there is ARCH effect, and the residuals of the log-return series are stationary, with constant mean and variance and a consistent volatility spread. This indicates that the model should be modified by considering another distribution assumption on innovations, higher models, such as the Integrated GARCH, Exponential GARCH, could be explored to obtain a more accurate model for prediction

References

Book by Jianqing Fan and Qiwei Yao, Cambridge University Press, Elements of Financial Econometrics (Asset Returns) Chapter 1, Pages 1-3

Book by Ruey S. Tsay, University of Chicago Booth School of Business, Analysis of Financial Time Series (Third Edition), Characteristics of Financial Time Series, Chapter 1, pages 1-3

Book by Ruey S. Tsay, University of Chicago Booth School of Business, Analysis of Financial Time Series (Third Edition), Stationarity, Chapter 2, pages 30-34

Book by Ruey S. Tsay, University of Chicago Booth School of Business, Analysis of Financial Time Series (Third Edition), Distributional Properties of Returns, Chapter 1, pages 9-10

University of Strathclyde. (2025). 'Week 2: Week2_lecture2_ slid', Value at Risk, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-3

University of Strathclyde. (2025). 'Week 2: Week2_lecture2_ slid', Calculation of Value at Risk, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-2

University of Strathclyde. (2025). 'Week 3: Week3_lecture3_ slid', Checking for Stationarity of Time Series, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-3

University of Strathclyde. (2025). 'Week 3: Week3_lecture 3 ', tutorial solutions, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-4

University of Strathclyde. (2025). 'Week 4: Week4_lecture4_ slid', Moving Average, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-11

University of Strathclyde. (2025). 'Week 3: Week3_lecture3_ slid', Forecasting AR (1), MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 2-4

University of Strathclyde. (2025). 'Week 1: Week1_lecture1_ slid', Sample Moments and Normality Test, MM955: Financial Econometrics. University of Strathclyde. 20 January. (10 March 2025) by Dr. Jiazhu Pan, pages 1-3

Article by Wikipedia.org, S&P 500 Index, 3 March 2025, paragraph 1-3: S&P 500 - Wikipedia

Article The (Mis)uses of the S&P 500 by Adriana Z. Robertson, University of Chicago Law School (Business law), paragraph 1-9

Article, Investopedia, Passive Investing: Definition, Pros and Cons, vs. Active Investing by James Chan, Michael J. Boyle and Vikki Velasquez, paragraph 1, on 27 January 2025: Passive Investing: Definition, Pros and Cons, vs. Active Investing

Article, Investopedia, Benchmarks: Definition, Types, and How to Use Them in Investing, by James Chan, Jefreda R. Brown, paragraph 1, on 7 April 2025: Benchmarks: Definition, Types, and How to Use Them in Investing

Blog by WallStreatMojo, Empirical Analysis: Empirical Analysis Meaning by Shreeya Jain, Dheeraj Vaidya on 15 Dec 2023: https://www.wallstreetmojo.com/empirical-analysis

Article, Investopedia, What Is a Time Series and How Is It Used to Analyse Data by Adam Hayes, Gordon Scott and Amanda Jackson on 12 June 2022: What Is a Time Series and How Is It Used to Analyze Data?

Article, CFA Institute.org, Time-Series Analysis, Introduction by CFA Institute, CFA® Program Level II, Quantitative Methods, 2025 Curriculum: Time-Series Analysis | CFA Institute

Training program, timescale.com, What Is a Time-Series Plot, and How Can You Create One by Anber Arif on 31 May 2023: timescale.com/blog/what-is-a-time-series-plot-and-how-can-you-create-one

Training program, Atlansian, A complete guide to histograms, what is a histogram by Mike Yi, 2024: Histograms Unveiled: Analysing Numeric Distributions