Statistics Term Paper
Pareto Distribution

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Question-1

The Pareto distribution has been named after the Italian economist, Vilfredo Pareto (1848-1923), who formulated the Pareto's Law to model the distribution of income over a population. The law can be stated as follows:
$N = Ax^{-a}$
where N is the number of persons having income greater than or equal to x, and A and a are parameters.

He observed that eighty percent of the land in Italy was owned by about twenty percent of the population.This is a also known as the Pareto Principle or the 80/20 rule which states that for many events, roughly eighty percent of the effects come from twenty percent of the causes. For instance, in business management, it is said that eighty percent of sales come from twenty percent of the clients.

Some other important related concepts are:

1) **Lorenz Curve**- Developed by Max O. Lorenz, the Lorenz Curve is a graphical representation of the distribution of income or wealth, with the percentage of households plotted on the x- axis and the percentage of income on the y- axis.

2) **Gini Coefficient**- It is the most commonly used measure of income inequality where a value of 0 represents perfect income equality while a value of 1 represents maximum income inequality.

In all the equations below, σ represents the scale parameter and λ represents the shape parameter respectively.
(The derivations have been sent in a separate file)

The PDF of the pareto distribution will be given by:

$$f_X(x) = \begin{cases} \lambda\sigma^{\lambda}x^{-(\lambda+1)}, & x \geq \sigma \\ 0, & x < \sigma \end{cases}$$

The CDF of the pareto distribution will be given by:

$$F_X(x) = \begin{cases} 1 - \left(\dfrac{\sigma}{x}\right)^{\lambda}, & x \geq \sigma \\ 0, & x < \sigma \end{cases}$$

The mean of the pareto distribution will be given by:

$$E[X] = \begin{cases} \frac{\sigma\lambda}{\lambda - 1}, & \lambda > 1 \\ \infty, & \lambda \leq 1 \end{cases}$$

The median of the pareto distribution will be given by:

$$Median = \sigma 2^{\frac{1}{\lambda}}, \quad \lambda > 0$$

95th percentile

$$x_{0.95} = \sigma 20^{\frac{1}{\lambda}}$$

The inter-quartile range will be given by:

$$IQR = \sigma 4^{\frac{1}{\lambda}} - \sigma \left(\frac{4}{3}\right)^{\frac{1}{\lambda}}$$

The variance of the pareto distribution will be given by:

$$Variance = \begin{cases} \frac{\lambda\sigma^2}{(\lambda - 1)^2 (\lambda - 2)}, & \lambda > 2 \\ \infty, & \lambda \leq 2 \end{cases}$$

Coefficient of skewness

$$\mu_3' = \frac{2(\lambda + 1)}{\lambda - 3}\sqrt{\frac{\lambda - 2}{\lambda}}, \quad \lambda > 3$$

Coefficient of kurtosis

$$\mu_4' = \frac{3(\lambda - 2)(3\lambda^2 + \lambda + 2)}{\lambda(\lambda - 3)(\lambda - 4)}, \quad \lambda > 4$$

Ex. kurtosis

$$\mu_4'' = \frac{6(\lambda^3 + \lambda^2 - 6\lambda - 2)}{\lambda(\lambda - 3)(\lambda - 4)}, \quad \lambda > 4$$

Therefore, moments of the Pareto distribution only exists under certain conditions of λ.

Finding the proportion of values greater than mean plus standard deviation:

$$\begin{aligned} P[X > x] &= 1 - F(x) \\ &= 1 - 1 + \left(\frac{\sigma}{x}\right)^{\lambda} \\ &= \left(\frac{\sigma}{x}\right)^{\lambda}, \quad where \quad x = \frac{\lambda\sigma}{\lambda - 1} + \frac{\sigma\sqrt{\lambda}}{(\lambda - 1)\sqrt{\lambda - 2}} \end{aligned}$$

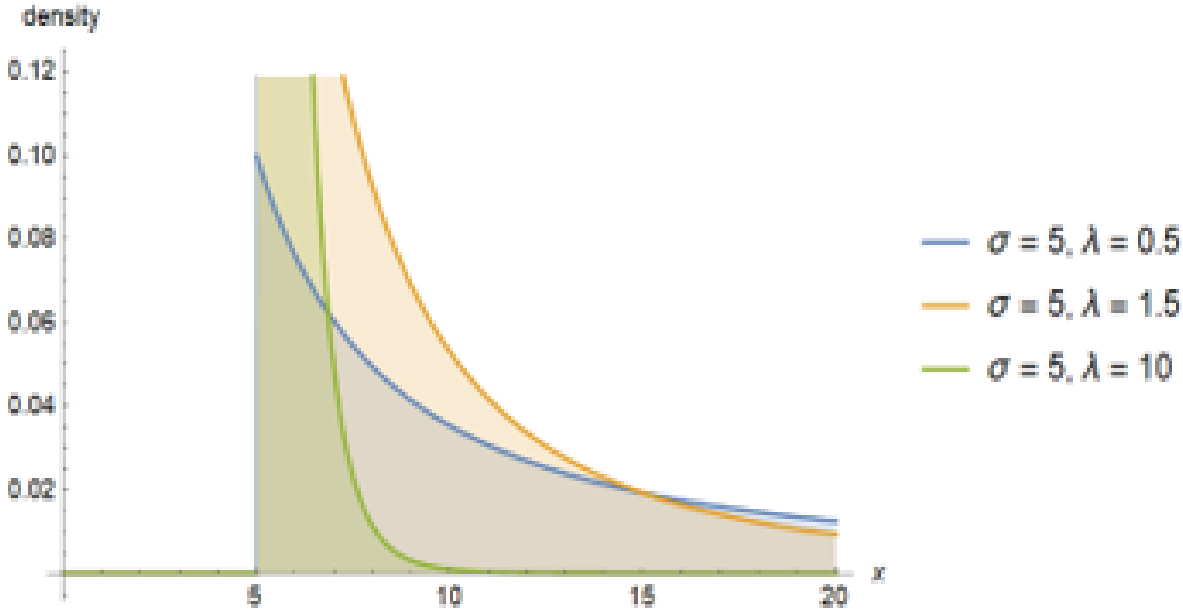

Figure 1: PDF of Pareto Distribution

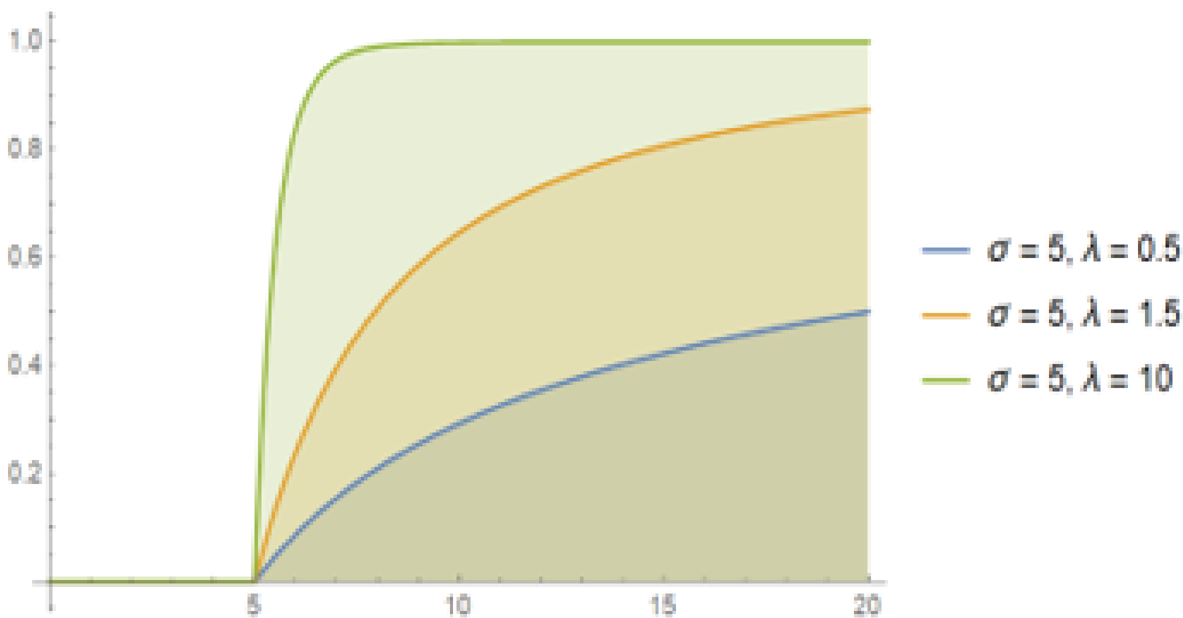

Figure 2: CDF of Pareto Distribution

Therefore, the pareto distribution is a right- tailed distribution with a positive support, which can extend from σ to ∞.

The variation of mean with the parameters is given in Figure 3.
The variation of standard deviation with the parameters is given in Figure 4.

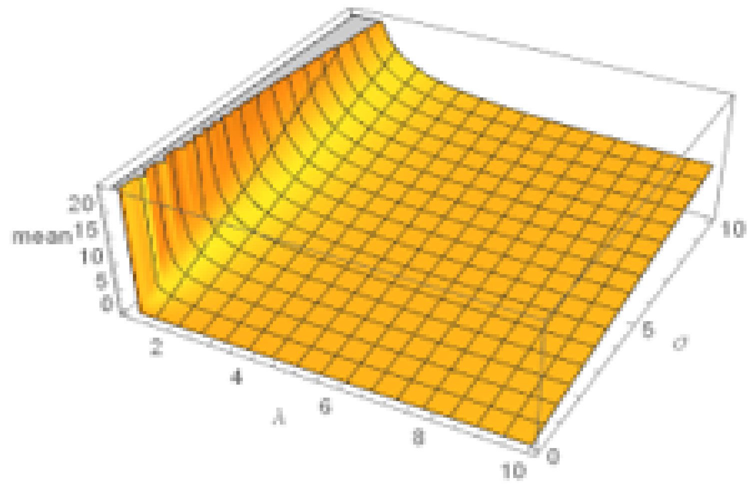

Figure 3: Variation of mean with parameters

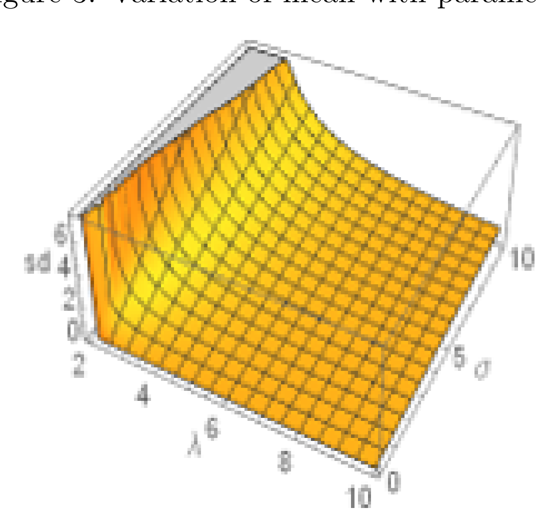

Figure 4: Variation of standard deviation with parameters

The variation of median with the parameters is given in Figure 5.

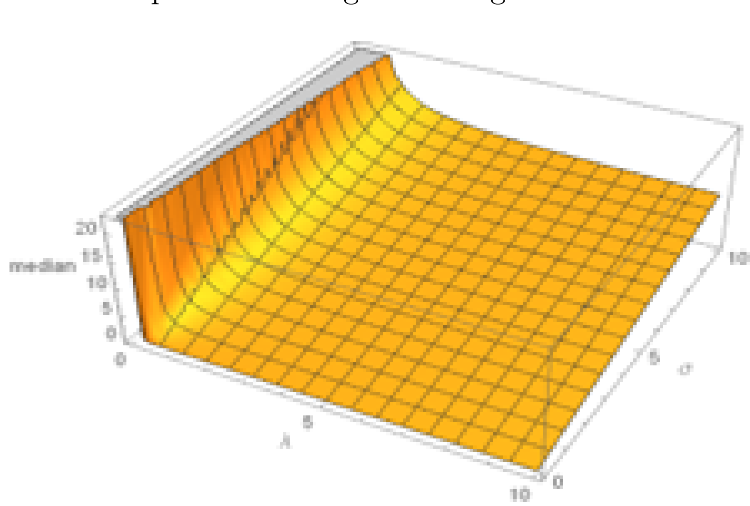

Figure 5: Variation of median with parameters

The variation of inter-quartile range with the parameters is given in Figure 6.

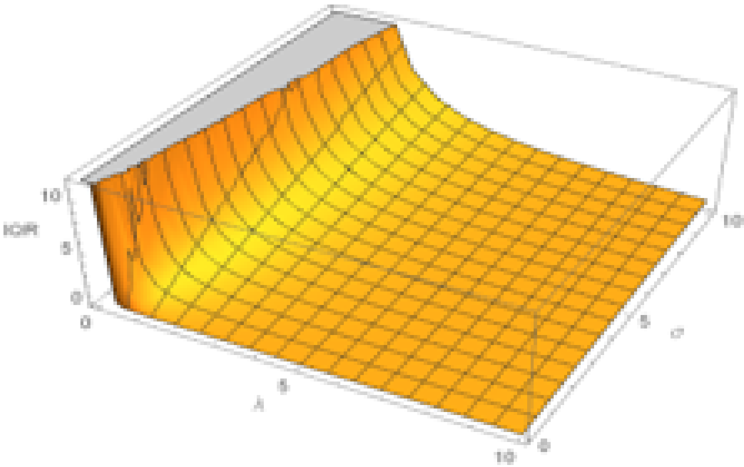

Figure 6: Variation of IQR with parameters

The variation of proportion of values greater than mean plus standard deviation with the parameters is given in Figure 8.

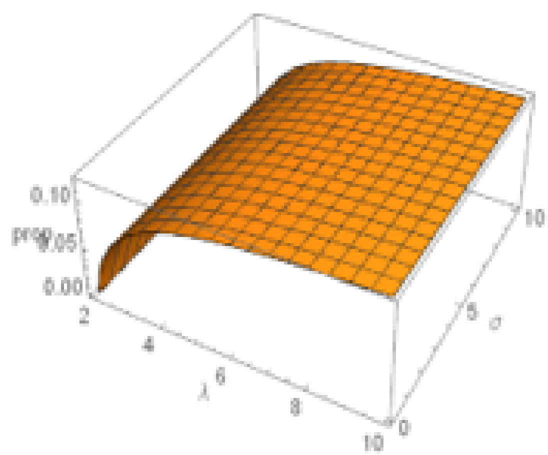

Figure 7: Variation of proportion with parameters

The variation of 95th percentile with the parameters is given in Figure 8.
The probability of detection of outliers by a box- plot by simulation for $\sigma = 5$ and $\lambda = 10$ is 0.059
The probability of a Pareto random variable lying between the boxplot fences is given by:

$$
\begin{aligned}
&P\left[X \leq Q_1 - 1 \cdot 5IQR\right] + P\left[X \geq Q_3 + 1 \cdot 5IQR\right] \\
&= F\left[Q_1 - 1 \cdot 5IQR\right] + 1 - F\left[Q_3 + 1.5IQR\right] \\
&= F\left[\left(\frac{4}{3}\right)^{\frac{1}{\lambda}} - \frac{3}{2}\left(4^{\frac{1}{\lambda}} - \left(\frac{4}{3}\right)^{\frac{1}{\lambda}}\right)\right] + 1 - F\left[4^{\frac{1}{\lambda}} + \frac{3}{2}\left(4^{\frac{1}{\lambda}} - \left(\frac{4}{3}\right)^{\frac{1}{\lambda}}\right)\right] \\
&= F\left[\frac{5}{2}\left(\frac{4}{3}\right)^{\dfrac{1}{\lambda}} - \frac{3}{2}\left(4\right)^{\dfrac{1}{\lambda}}\right] + 1 - F\left[\frac{5}{2}\left(4\right)^{\dfrac{1}{\lambda}} + \frac{3}{2}\left(\frac{4}{3}\right)^{\dfrac{1}{\lambda}}\right]
\end{aligned}
$$

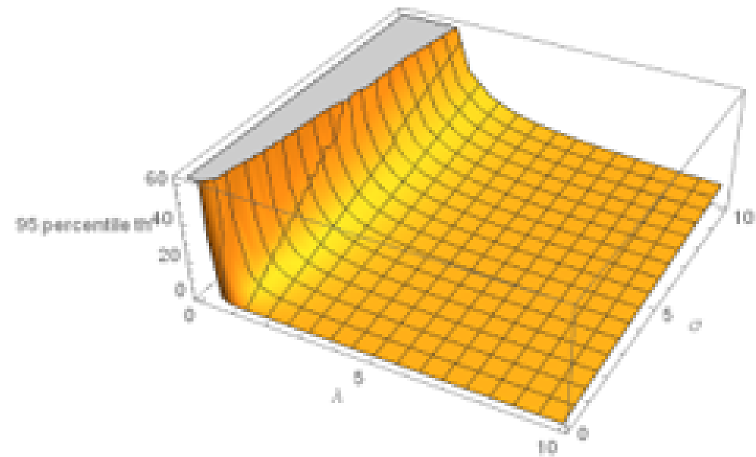

Figure 8: Variation of 95th percentile with parameters

The Pareto distribution is sensitive to outliers since the right tail has a significant probability mass.

The arithmetic mean of the Pareto distribution only exists when $\lambda > 1$

The geometric mean (γ), harmonic mean (H) and median (M), however, always exist and can be stated as follows:

$$\gamma = \sigma e^{\frac{1}{\lambda}}$$
$$H = \sigma \left(1 + \frac{1}{\lambda}\right)$$
$$M = \sigma 2^{\frac{1}{\lambda}}$$

Their maximum likelihood estimates, respectively, are as follows:

G, the Sample Geometric Mean
$$h = \sigma \left(1 + \log \frac{G}{\sigma}\right)$$
$$m = \sigma 2^{log\left(\frac{G}{\sigma}\right)}$$

Since h and m are functionally dependent on G, and G is the best estimate of its population counterpart, therefore, we can use the sample geometric mean as a measure of central tendency. The geometric mean also functions as a measure of location for the Pareto distribution.

The variance of the Pareto distribution only exists when $\lambda > 2$
Therefore, we can instead consider the standard deviation of the log of x:

$$sd\left(\log x\right) = \log \frac{\gamma}{\sigma}$$

Therfore, $\frac{\gamma}{\sigma}$ (also known as the Geometric Standard Deviation) can be used as a measure of dispersion for the Pareto distribution, and which represents the distance of the start of the curve from the origin.

The probability of the smallest element of a sample of size 3 exceeding the population median can be stated as follows:

(The derivation has been sent in a separate file)

$$
\begin{aligned}
&f_{X_{(1)}}(x) = \frac{3\lambda\sigma^{3\lambda}}{x^{3\lambda+1}} \\
&F_{X_{(1)}}(x) = \sigma^{3\lambda}\left[\sigma^{-3\lambda} - x^{-3\lambda}\right] \\
&P\left[X_{(1)} > \sigma 2^{\frac{1}{\lambda}}\right] \\
&= 1 - F_{X_{(1)}}\left[\sigma 2^{\frac{1}{\lambda}}\right] \\
&= \frac{1}{8}
\end{aligned}
$$

Similary, the probability of the smallest of a sample of size 5 exceeding the population median is 1/32. Therefore, the probability keeps declining as the sample size increases, which is intuitive.

Question-2

The necessary code has been generated in R.

Question-3

The data has been collected from:`http://data.un.org/Data.aspx?d=POP&f=tableCode%3A240%20` which provides population data for 303 US cities for the year 2016. It was collected by the United Nations Statistics Division by dispatching a set of questionnaires to national statistical offices around the United States and then published in the Demographic Yearbook collection.

We can use the Pareto distribution as a reasonable model for this data since:
1) We expect cities to have a minimum population size.
2) A few cities will be highly populated which means that the right tail will have a significant probability mass.
3) Cities which have a lower population may become even less populated through emigration while immigration may lead to highly populated cities becoming even more populated.

Therefore, only a right- tailed distribution whose support starts from a positive constant value will be a good fit for the data.

Estimates of the parameters for which Pareto is a good fit for the data is:

$$\hat{\sigma} = 1.007 * 10^5, \hat{\lambda} = 1.41$$

It passes the Kolmogorov-Smirnov test with a p-value 0.7979

Question-4

The pareto distribution has been contaminated with an exponential distribution, and an appropriate R code has been prepared for generating data from this, which can be seen in the attached R file.

A sample kernel density plot with a fifty percent contaminated distribution having pareto parameters $\sigma = 1$ and $\lambda = 2$ and, exponential parameter $\lambda = 1$ is given in Figure 9.

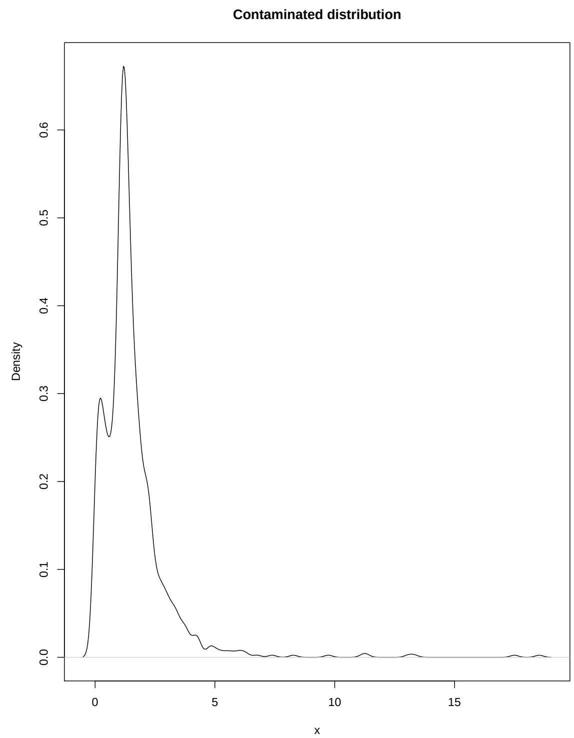

Figure 9: Contaminated distribution

Question-5

We draw 5000 samples from a Pareto distribution with parameters, $\sigma = 5$ and $\lambda = 10$, and use the Kolmogorov-Smirnov test to find the sample size at which the distribution of sample mean, sample variance, sample 95th percentile and sample IQR attain asymptotic normality. The findings are given in Table 1:

Pop. Char.	Estimator	Sample Size	p-value of KS test	Remark
Mean	Sample mean	300-400	8.91E-02	For n = 400
Variance	Sample variance	14000-15000	8.65E-02	For n = 15000
95th percentile	Sample 95th percentile	800-1000	9.77E-02	For n = 1000
IQR	Sample IQR	300-400	4.53E-02	For n = 300

Table 1: Sampling distributions of reasonable estimators of population characteristics. Sample size denotes the sample size at which asymptotic normality kicks in.

Therefore, all the distributions of the estimators (especially sample variance) attain asymptotic normality at very high sample sizes.

The same results can be shown with the following histograms and QQ Plots given in Figure 10 - Figure 13

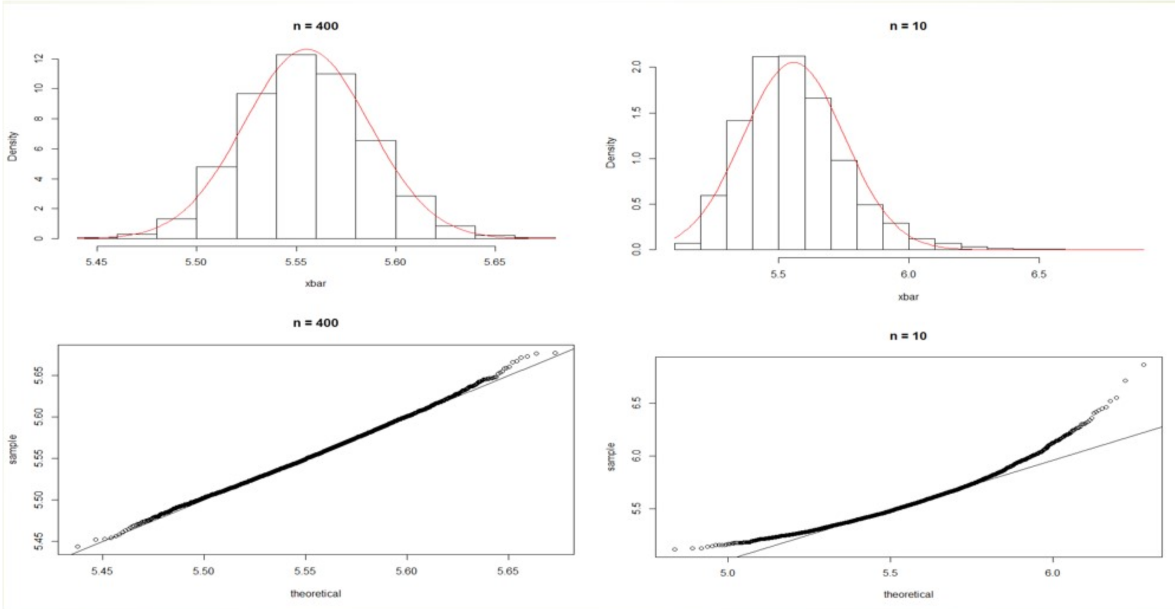

Figure 10: Sampling Distribution of Sample Mean

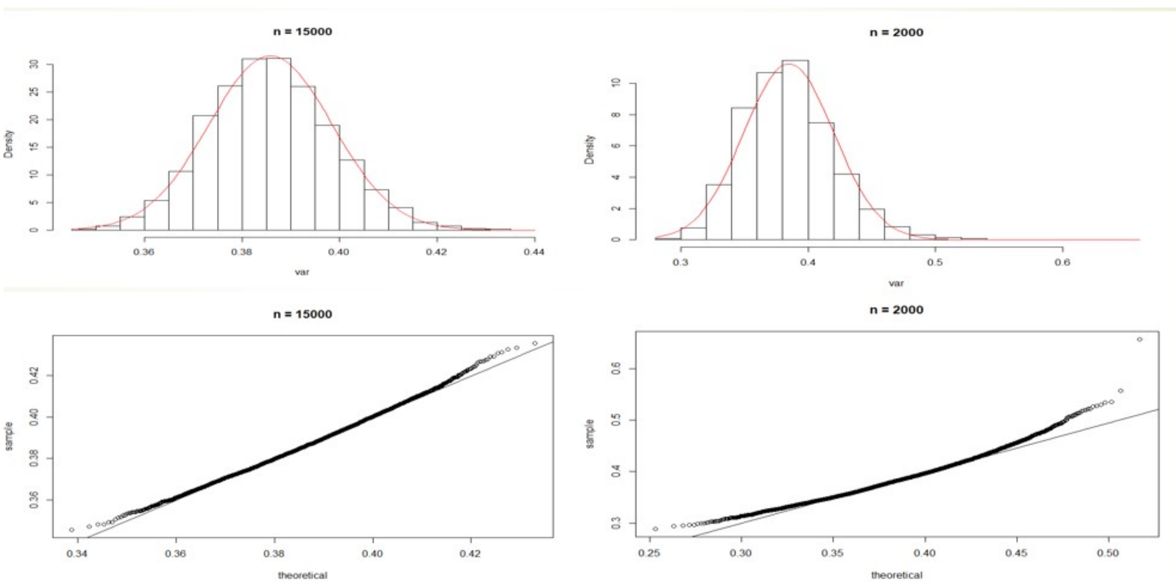

Figure 11: Sampling Distribution of Sample Variance

Therefore, the theoretical distribution of the 95th percentile fits the sample distribution the 95th percentile much better than the normal density curve (Figure 12)

We have also tried fitting a gamma distribution to the simulated data, which gives us the following estimates of the shape and scale parameters for the gamma distribution (Table 2)

Therefore, as the sample size increases, the estimated gamma shape also increases, which follows our previous conclusion that the distribution of sample variance tends towards a normal distribution at very high sample sizes.

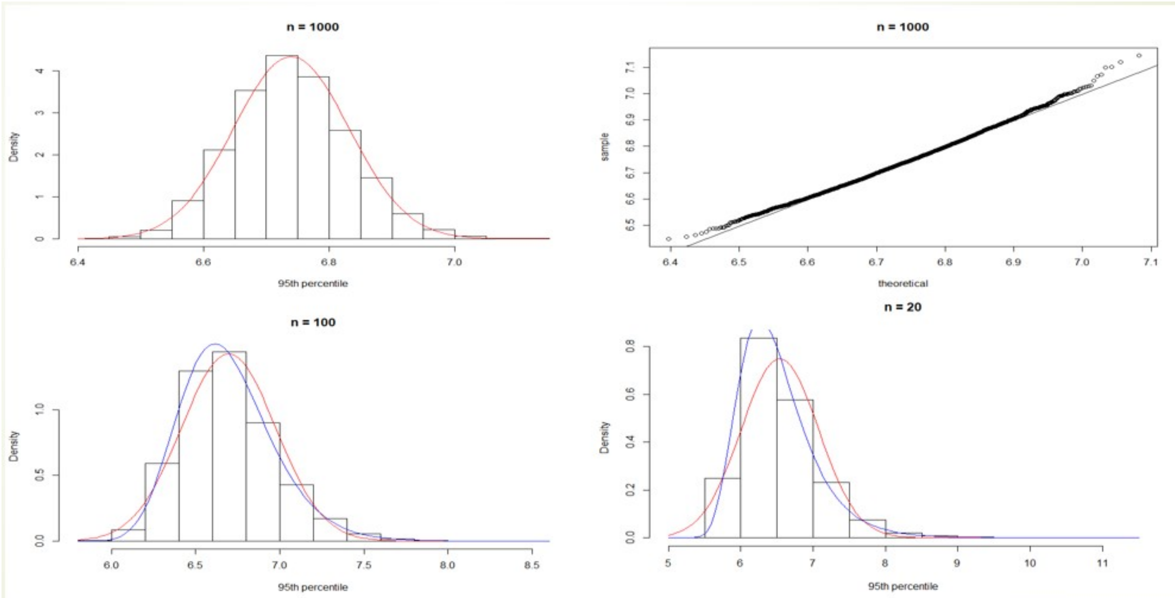

Figure 12: Sampling Distribution of Sample 95th Percentile

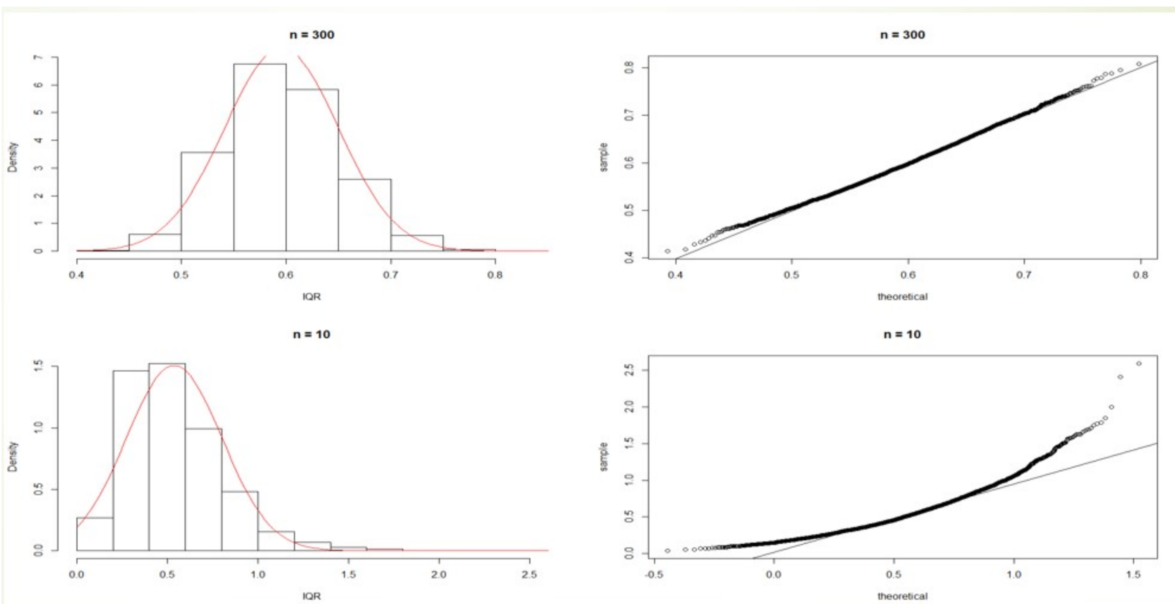

Figure 13: Sampling Distribution of Sample IQR

Sample size	Estimated Gamma shape	Estimated Gamma scale
5	7.47E-01	1.92E+00
10	1.32E+00	3.51E+00
15	1.61E+00	4.11E+00
15000	9.29E+02	2.41E+03

Table 2: Known Structure for Sampling Distribution of Sample Variance

Question-6

The MLE estimates of the population parameters for the Pareto distribution are given below: (The derivations have been sent in a separate file)

$$\hat{\lambda_{MLE}} = \frac{n}{\sum \log\left(\frac{X_i}{\min(X_i)}\right)}$$

$$\hat{\sigma_{MLE}} = min(X_i)$$

The Fisher Information Matrix will be given as follows:

$$I^{-1}(\lambda, \sigma) = \begin{bmatrix} \frac{\lambda^2\sigma^2}{\sigma(1-\lambda)} & \frac{\lambda\sigma^2}{\sigma(1-\lambda)} \\ \frac{\lambda\sigma^2}{\sigma(1-\lambda)} & \frac{\sigma^2}{\lambda(1-\lambda)} \end{bmatrix}$$

However, the Fisher information matrix cannot be used to compute the asymptotic variances because the regularity conditions are not satisfied as the support has a parameter. But if σ is assumed to be known, the Fisher information for λ can be used.

The asymptotic distributions of the MLE estimators of the population parameters are given in Table 3:

Parameter	Sample Size	MSE	Variance	Bias	Theor. Asymp. Var.	Remarks
σ	-	1.28E-07	6.43E-08	2.52E-04	-	For(n = 2000)
λ	200-300	3.52E-01	3.47E-01	6.96E-02	3.33E-01	For(n = 300)

Table 3: Asymptotic Distributions of MLE Estimators of Population Parameters. Sample size denotes the sample size at which asymptotic normality kicks in.

For σ, the estimator is $min(X_i)$, so asymptotic normality cannot be easily established, because it goes to an extreme value distribution.

For λ, asymptotically the simulated variance and calculated variance are almost the same.

The associated histograms are shown below:

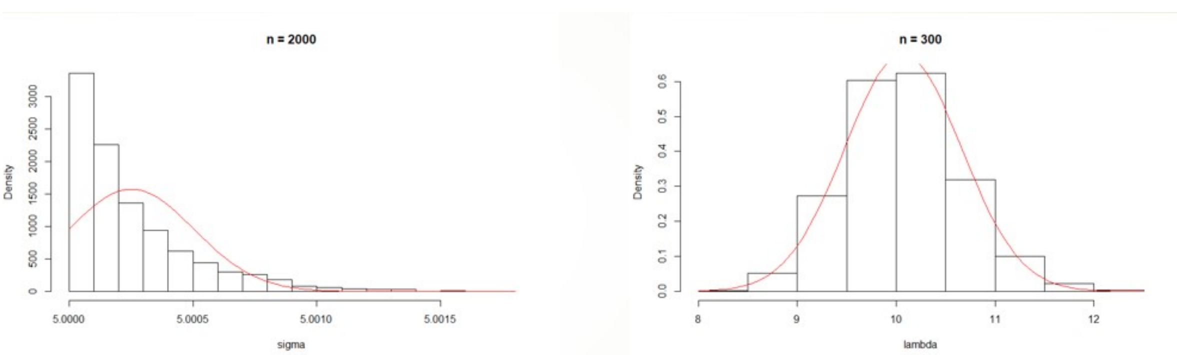

Figure 14: Asymptotic Distributions of MLE Estimators of Population Parameters

The asymptotic distributions of MLE estimators of the population characteristics are given in Table 4 :

Pop. Char.	Sample Size	MSE	Variance	Bias	TAR	95% conf. int.	Remark
Mean	200-300	1.29E-03	1.29E-03	-1.31E-04	1.27E-03	(5.49, 5.63)	For n = 300
95th perc.	200-300	1.38E-02	1.38E-02	-3.91E-03	1.36E-02	(6.51, 6.97)	For n = 300
Median	200	6.90E-04	6.89E-04	7.70E-04	6.90E-04	(5.31, 5.41)	For n = 200
IQR	200-300	1.42E-03	1.42E-03	-1.91E-03	1.40E-03	(0.522, 0.669)	For n = 300
Std. Dev.	300-400	1.99E-03	1.99E-03	-1.91E-03	1.96E-03	(0.532, 0.707)	For n = 300

Table 4: Asymptotic Distributions of MLE Estimators of Population Characteristics. NOTE: TAR denotes Theoretical Asymptotic Variance.

For all the population characteristics, asymptotically the simulated variance and calculated variance are almost the same at the given sample sizes:

Question-7

We compare MLE estimators of population mean, median, standard deviation and IQR with the corresponding MOM and non- parametric estimators by calculating the mean squared error, variance and bias. The results are given in Table 5

We see that the MLE estimators do better than the corresponding MOM and non- parametric estimators for every population characteristic, as evidenced by the lower mean squared errors.

Question-8

The 95% asymptotic confidence intervals for the population characteristics have been calculated and have been shown in the Table 4

Question-10

We here compare the variances of the MLE (parametric estimator) with the sample median (non- parametric estimator) estimates of the population median. Asymptotically both estimators go to normal. The calculations are given in Table 6 We can conclude that the MLE does much better than the non- parametric estimators due to lower variance.

We now consider a contaminated distribution with parameters $\sigma = 5$, $\lambda = 10$, for the Pareto distribution, and rate parameter = 5 for the exponential distrbution.

The calculations at different levels of contamination and sample sizes are given in Table 7.

In this case, we observe that the non- parametric estimator does much better with lower mean squared errors.

		MLE			MOM			Non-parametric		
Pop. char.	Sample Size	MSE	Variance	Bias	MSE	Variance	Bias	MSE	Variance	Bias
Mean	300	1.29E-03	1.29E-03	-1.31E-04	1.30E-03	1.30E-03	-1.84E-04	-	-	-
Median	200	6.90E-04	6.89E-04	7.70E-04	9.74E-04	9.67E-04	2.55E-03	1.41E-03	1.41E-03	1.23E-03
Standard deviation	300	1.99E-03	1.99E-03	-1.91E-03	5.03E-03	5.00E-03	-5.69E-03	-	-	-
IQR	300	1.42E-03	1.42E-03	-1.91E-03	3.46E-03	3.43E-03	-5.65E-03	2.97E-03	2.97E-03	-1.97E-03

Table 5: Comparison of MLE with MOM and Non- Parametric Estimators

Sample size	Theor. Asym. Var. of MLE	Variance of MLE(Simulated)	Theor. Asym. Var. of NP	Var. of NP(Simulated)
200	6.90E-04	6.89E-04	1.44E-03	1.41E-03
300	4.60E-04	4.68E-04	9.57E-04	9.94E-04
500	2.76E-04	2.63E-04	5.74E-04	5.60E-04
1000	1.38E-04	1.43E-04	2.87E-04	2.84E-04
2000	6.90E-05	6.95E-05	1.44E-04	1.46E-04

Table 6: Comparison of Variances for MLE and Non- Parametric Estimators of Population Median for an Uncontaminated Distribution

		MLE			NP		
Deg of contmn.	Sample size	MSE	Variance	Bias	MSE	Variance	Bias
0.25	10	1.79E+01	2.22E+00	-3.96E+00	3.61E-01	3.30E-01	-1.77E-01
	500	2.00E+01	9.86E-01	-4.35E+00	1.09E-01	9.54E-02	-1.16E-01
	1000	2.65E+01	5.36E-03	-5.15E+00	2.32E-02	5.18E-04	-1.51E-01
0.5	10	2.70E+01	3.25E-03	-5.19E+00	2.32E-02	2.72E-04	-1.51E-01
	500	2.51E+01	1.04E-01	-5.00E+00	1.15E+01	4.48E+00	-2.65E+00
	1000	2.54E+01	5.37E-02	-5.03E+00	1.00E+01	5.39E+00	-2.15E+00
0.75	10	2.78E+01	9.02E-04	-5.27E+00	1.06E+01	4.46E+00	-2.47E+00
	500	2.80E+01	5.61E-04	-5.29E+00	1.03E+01	4.32E+00	-2.44E+00
	1000	2.68E+01	1.90E-02	-5.18E+00	2.27E+01	1.63E+00	-4.59E+00

Table 7: Comparison of MLE and NP estimators for a Contaminated Distribution

Appendix

Samp. Size	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	7.73E-02	0.00E+00	9.34E-01	1.94E-42
10	4.85E-02	1.20E-10	9.63E-01	5.93E-34
15	5.23E-02	2.60E-12	9.65E-01	2.44E-33
20	4.34E-02	1.35E-08	9.79E-01	2.23E-26
30	4.10E-02	9.97E-08	9.84E-01	1.25E-23
40	3.42E-02	1.62E-05	9.91E-01	8.69E-18
50	2.67E-02	1.61E-03	9.93E-01	6.23E-15
100	2.98E-02	2.77E-04	9.94E-01	2.74E-13
200	2.13E-02	2.11E-02	9.98E-01	7.60E-07
400	1.76E-02	8.91E-02	9.98E-01	7.75E-05
500	1.25E-02	4.16E-01	9.99E-01	8.80E-02
1000	9.92E-03	7.09E-01	9.99E-01	8.13E-02
2000	8.97E-03	8.16E-01	9.99E-01	1.58E-01

Table 8: Sampling distribution of sample mean

Samp. Size.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	2.97E-01	0.00E+00	4.45E-01	1.94E-82
10	2.05E-01	0.00E+00	6.21E-01	2.69E-74
15	1.96E-01	0.00E+00	6.24E-01	3.93E-74
20	1.70E-01	0.00E+00	6.51E-01	1.40E-72
30	1.50E-01	0.00E+00	7.07E-01	5.39E-69
40	1.23E-01	0.00E+00	8.10E-01	1.21E-60
50	1.19E-01	0.00E+00	8.04E-01	3.43E-61
100	1.07E-01	0.00E+00	8.13E-01	2.82E-60
200	8.07E-02	0.00E+00	8.88E-01	3.65E-51
500	5.59E-02	5.66E-14	9.39E-01	3.46E-41
1000	4.63E-02	9.58E-10	9.70E-01	5.29E-31
2000	3.47E-02	1.16E-05	9.81E-01	1.48E-25
15000	1.77E-02	8.65E-02	9.98E-01	7.23E-07

Table 9: Sampling distribution of sample variance

Samp. Size.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	1.06E-01	0.00E+00	8.67E-01	4.19E-54
10	7.65E-02	0.00E+00	9.23E-01	6.02E-45
15	7.86E-02	0.00E+00	9.26E-01	2.81E-44
20	7.52E-02	0.00E+00	9.30E-01	1.93E-43
30	5.37E-02	6.25E-13	9.53E-01	2.21E-37
40	5.73E-02	1.11E-14	9.63E-01	4.72E-34
50	4.57E-02	1.68E-09	9.72E-01	3.47E-30
100	4.54E-02	2.33E-09	9.79E-01	7.45E-27
200	2.56E-02	2.81E-03	9.92E-01	1.83E-16
500	2.27E-02	1.18E-02	9.96E-01	6.56E-11
1000	1.74E-02	9.77E-02	9.97E-01	1.97E-08
2000	1.74E-02	9.70E-02	9.99E-01	1.58E-04

Table 10: Sampling distribution of sample 95^{th} percentile

Samp. Size	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	0.118988	0	0.866937	3.87E-54
10	0.068574	0	0.935438	3.92E-42
15	0.06337	0	0.953722	3.96E-37
20	0.062915	0	0.959896	4.45E-35
30	0.045448	2.14E-09	0.976039	3.33E-28
40	0.044667	4.33E-09	0.979834	4.44E-26
50	0.036318	3.74E-06	0.985678	4.01E-22
100	0.030082	0.000235	0.991565	1.10E-16
200	0.025036	0.003793	0.996411	1.20E-09
300	0.01946	0.045325	0.997665	6.49E-07
500	0.012133	0.453353	0.998416	6.60E-05
1000	0.012658	0.399593	0.999446	0.143347
2000	0.010374	0.654929	0.999326	0.057166

Table 11: Sampling distribution of Sample IQR

Samp. Size	Estimated Gamma Shape	Estimated Gamma Rate
5	7.47E-01	1.92E+00
10	1.32E+00	3.51E+00
15	1.61E+00	4.11E+00
20	2.05E+00	5.36E+00
30	2.73E+00	6.95E+00
40	3.64E+00	9.52E+00
50	4.18E+00	1.08E+01
100	7.19E+00	1.85E+01
200	1.39E+01	3.63E+01
500	3.15E+01	8.17E+01
1000	6.34E+01	1.64E+02
2000	1.18E+02	3.07E+02
15000	9.29E+02	2.41E+03

Table 12: Gamma distribution fits for sampling distribution of sample variance.

Samp. Size	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	1.64E-01	0.00E+00	7.91E-01	2.12E-62
10	1.60E-01	0.00E+00	8.13E-01	2.65E-60
15	1.59E-01	0.00E+00	8.13E-01	2.40E-60
20	1.56E-01	0.00E+00	8.20E-01	1.18E-59
30	1.55E-01	0.00E+00	8.21E-01	1.68E-59
40	1.55E-01	0.00E+00	8.30E-01	1.66E-58
50	1.58E-01	0.00E+00	8.13E-01	2.69E-60
100	1.55E-01	0.00E+00	8.21E-01	1.76E-59
200	1.57E-01	0.00E+00	8.20E-01	1.27E-59
300	1.61E-01	0.00E+00	8.14E-01	2.98E-60
500	1.56E-01	0.00E+00	8.13E-01	2.58E-60
1000	1.63E-01	0.00E+00	8.08E-01	7.22E-61
2000	1.60E-01	0.00E+00	8.22E-01	2.06E-59

Table 13: Sampling distribution of MLE estimator of σ

Samp. Size	MSE	Var.	Bias
5	2.16E-02	1.10E-02	1.03E-01
10	5.06E-03	2.55E-03	5.01E-02
15	2.17E-03	1.09E-03	3.29E-02
20	1.21E-03	6.01E-04	2.48E-02
30	5.46E-04	2.69E-04	1.67E-02
40	3.07E-04	1.51E-04	1.25E-02
50	1.93E-04	9.61E-05	9.85E-03
100	4.98E-05	2.45E-05	5.03E-03
200	1.19E-05	5.93E-06	2.45E-03
300	5.40E-06	2.73E-06	1.64E-03
500	2.06E-06	1.02E-06	1.02E-03
1000	5.16E-07	2.63E-07	5.03E-04
2000	1.28E-07	6.43E-08	2.52E-04

Table 14: Mean squared error of MLE estimator on σ

Samp. Size.	KS Stat.	KS p-value	SW. Stat.	SW. p-value
5	1.50E-01	0.00E+00	7.10E-01	8.61E-69
10	9.81E-02	0.00E+00	8.78E-01	1.28E-52
15	7.29E-02	0.00E+00	9.32E-01	5.71E-43
20	6.75E-02	0.00E+00	9.51E-01	4.74E-38
30	5.08E-02	1.22E-11	9.73E-01	1.67E-29
40	5.21E-02	3.14E-12	9.71E-01	1.14E-30
50	4.63E-02	1.01E-09	9.78E-01	2.64E-27
100	3.20E-02	6.93E-05	9.91E-01	5.24E-17
200	2.28E-02	1.12E-02	9.95E-01	2.13E-12
300	1.72E-02	1.05E-01	9.96E-01	1.08E-09
500	1.84E-02	6.75E-02	9.97E-01	3.15E-08
1000	1.88E-02	5.75E-02	9.99E-01	1.55E-04
2000	9.63E-03	7.42E-01	9.99E-01	7.97E-02

Table 15: Sampling distribution of MLE estimator of λ

Samp. Size.	MSE	Var.	Bias	Theor. Asymp. Var.
5	1.72E+02	1.29E+02	6.57E+00	2.00E+01
10	2.81E+01	2.23E+01	2.42E+00	1.00E+01
15	1.40E+01	1.15E+01	1.58E+00	6.67E+00
20	8.85E+00	7.51E+00	1.16E+00	5.00E+00
30	4.57E+00	4.13E+00	6.69E-01	3.33E+00
40	3.31E+00	3.05E+00	5.06E-01	2.50E+00
50	2.48E+00	2.32E+00	4.01E-01	2.00E+00
100	1.12E+00	1.08E+00	2.06E-01	1.00E+00
200	5.27E-01	5.16E-01	1.03E-01	5.00E-01
300	3.52E-01	3.47E-01	6.96E-02	3.33E-01
500	1.96E-01	1.94E-01	4.40E-02	2.00E-01
1000	1.04E-01	1.04E-01	1.56E-02	1.00E-01
2000	5.06E-02	5.05E-02	1.09E-02	5.00E-02

Table 16: Mean squared error of MLE estimator of λ

Samp. Size.	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	7.68E-02	7.68E-02	2.94E-03	7.62E-02	7.70E-02	0.00E+00	9.36E-01	4.45E-42
10	3.80E-02	3.80E-02	3.17E-03	3.81E-02	4.95E-02	4.59E-11	9.64E-01	1.17E-33
15	2.65E-02	2.65E-02	-1.12E-03	2.54E-02	4.99E-02	3.05E-11	9.67E-01	2.31E-32
20	1.94E-02	1.94E-02	-1.56E-03	1.91E-02	4.43E-02	6.11E-09	9.80E-01	7.30E-26
30	1.29E-02	1.29E-02	2.50E-03	1.27E-02	4.03E-02	1.76E-07	9.85E-01	1.48E-22
40	9.67E-03	9.67E-03	1.49E-03	9.53E-03	3.37E-02	2.27E-05	9.91E-01	1.16E-17
50	7.60E-03	7.60E-03	8.60E-04	7.62E-03	2.51E-02	3.60E-03	9.94E-01	3.53E-14
100	3.85E-03	3.85E-03	4.97E-05	3.81E-03	2.74E-02	1.10E-03	9.95E-01	4.19E-12
200	1.89E-03	1.89E-03	-1.28E-04	1.91E-03	2.17E-02	1.81E-02	9.98E-01	2.77E-06
300	1.29E-03	1.29E-03	-1.31E-04	1.27E-03	1.64E-02	1.34E-01	9.99E-01	2.97E-04
500	7.25E-04	7.25E-04	-2.65E-04	7.62E-04	9.05E-03	8.08E-01	9.99E-01	4.61E-02
1000	3.94E-04	3.94E-04	3.10E-04	3.81E-04	1.09E-02	5.96E-01	9.99E-01	1.84E-01
2000	1.92E-04	1.92E-04	-4.46E-05	1.91E-04	8.19E-03	8.91E-01	9.99E-01	1.61E-01

Table 17: Study of MLE of population mean

Samp. Size.	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	7.51E-01	7.07E-01	-2.10E-01	8.17E-01	9.13E-02	0.00E+00	9.12E-01	4.84E-47
10	3.87E-01	3.78E-01	-9.67E-02	4.08E-01	5.23E-02	2.66E-12	9.55E-01	1.31E-36
15	2.79E-01	2.73E-01	-7.35E-02	2.72E-01	5.54E-02	9.27E-14	9.60E-01	6.20E-35
20	2.06E-01	2.02E-01	-5.77E-02	2.04E-01	4.76E-02	2.88E-10	9.76E-01	2.14E-28
30	1.36E-01	1.35E-01	-2.73E-02	1.36E-01	4.24E-02	3.00E-08	9.82E-01	2.06E-24
40	1.02E-01	1.02E-01	-2.17E-02	1.02E-01	3.97E-02	2.86E-07	9.89E-01	1.71E-19
50	8.07E-02	8.04E-02	-1.82E-02	8.17E-02	2.61E-02	2.21E-03	9.92E-01	8.75E-16
100	4.11E-02	4.10E-02	-1.06E-02	4.08E-02	2.93E-02	3.81E-04	9.94E-01	2.92E-13
200	2.03E-02	2.02E-02	-5.64E-03	2.04E-02	2.30E-02	1.01E-02	9.98E-01	4.52E-07
300	1.38E-02	1.38E-02	-3.91E-03	1.36E-02	1.73E-02	1.01E-01	9.98E-01	4.38E-05
500	7.76E-03	7.75E-03	-3.06E-03	8.17E-03	1.01E-02	6.85E-01	9.99E-01	3.12E-02
1000	4.23E-03	4.23E-03	-5.53E-05	4.08E-03	1.03E-02	6.60E-01	9.99E-01	1.31E-01
2000	2.06E-03	2.06E-03	-6.84E-04	2.04E-03	8.74E-03	8.39E-01	9.99E-01	1.23E-01

Table 18: Study of MLE of population 95th percentile

Samp. Size.	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	3.64E-02	3.50E-02	3.77E-02	2.76E-02	6.72E-02	0.00E+00	9.47E-01	3.69E-39
10	1.58E-02	1.55E-02	1.94E-02	1.38E-02	4.56E-02	1.91E-09	9.72E-01	4.14E-30
15	1.04E-02	1.03E-02	1.06E-02	9.20E-03	4.24E-02	3.05E-08	9.76E-01	3.76E-28
20	7.46E-03	7.40E-03	7.59E-03	6.90E-03	3.70E-02	2.35E-06	9.86E-01	9.79E-22
30	4.90E-03	4.85E-03	7.28E-03	4.60E-03	3.65E-02	3.27E-06	9.89E-01	2.05E-19
40	3.63E-03	3.60E-03	5.22E-03	3.45E-03	2.76E-02	9.93E-04	9.93E-01	2.54E-14
50	2.83E-03	2.82E-03	3.93E-03	2.76E-03	2.17E-02	1.82E-02	9.96E-01	2.88E-11
100	1.41E-03	1.41E-03	1.77E-03	1.38E-03	2.36E-02	7.70E-03	9.96E-01	1.08E-09
200	6.90E-04	6.89E-04	7.70E-04	6.90E-04	1.94E-02	4.62E-02	9.98E-01	1.06E-04
300	4.68E-04	4.68E-04	4.84E-04	4.60E-04	1.38E-02	2.96E-01	9.99E-01	6.56E-03
500	2.63E-04	2.63E-04	1.98E-04	2.76E-04	8.56E-03	8.57E-01	9.99E-01	9.44E-02
1000	1.43E-04	1.43E-04	3.60E-04	1.38E-04	1.12E-02	5.61E-01	1.00E+00	3.36E-01
2000	6.95E-05	6.95E-05	6.05E-05	6.90E-05	8.18E-03	8.91E-01	1.00E+00	2.63E-01

Table 19: Study of MLE of population median

Samp. Size.	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	8.18E-02	7.00E-02	-1.08E-01	8.40E-02	8.21E-02	0.00E+00	9.28E-01	8.16E-44
10	4.06E-02	3.80E-02	-5.10E-02	4.20E-02	4.45E-02	4.92E-09	9.67E-01	2.57E-32
15	2.90E-02	2.77E-02	-3.69E-02	2.80E-02	4.89E-02	8.34E-11	9.70E-01	5.23E-31
20	2.14E-02	2.06E-02	-2.85E-02	2.10E-02	4.06E-02	1.40E-07	9.82E-01	5.91E-25
30	1.40E-02	1.38E-02	-1.55E-02	1.40E-02	3.72E-02	2.01E-06	9.87E-01	2.14E-21
40	1.06E-02	1.04E-02	-1.20E-02	1.05E-02	3.33E-02	3.07E-05	9.92E-01	1.89E-16
50	8.33E-03	8.24E-03	-9.79E-03	8.40E-03	2.21E-02	1.49E-02	9.94E-01	6.74E-13
100	4.24E-03	4.21E-03	-5.40E-03	4.20E-03	2.80E-02	7.77E-04	9.96E-01	4.77E-11
200	2.09E-03	2.08E-03	-2.80E-03	2.10E-03	2.17E-02	1.83E-02	9.98E-01	1.08E-05
300	1.42E-03	1.42E-03	-1.91E-03	1.40E-03	1.63E-02	1.39E-01	9.99E-01	5.64E-04
500	7.99E-04	7.98E-04	-1.38E-03	8.40E-04	9.01E-03	8.11E-01	9.99E-01	1.03E-01
1000	4.35E-04	4.35E-04	-2.21E-04	4.20E-04	9.91E-03	7.10E-01	1.00E+00	2.61E-01
2000	2.11E-04	2.11E-04	-3.20E-04	2.10E-04	8.03E-03	9.04E-01	9.99E-01	2.10E-01

Table 20: Study of MLE of population IQR

Samp. Size.	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-value	SW. Stat.	SW. p-value
5	1.17E-01	1.05E-01	-1.08E-01	1.18E-01	1.06E-01	0.00E+00	8.73E-01	2.59E-53
10	5.80E-02	5.55E-02	-5.01E-02	5.89E-02	6.30E-02	0.00E+00	9.35E-01	3.23E-42
15	4.16E-02	4.03E-02	-3.63E-02	3.93E-02	6.28E-02	0.00E+00	9.45E-01	1.00E-39
20	3.03E-02	2.95E-02	-2.83E-02	2.95E-02	5.32E-02	9.76E-13	9.67E-01	1.43E-32
30	1.98E-02	1.96E-02	-1.48E-02	1.96E-02	4.82E-02	1.58E-10	9.76E-01	5.73E-28
40	1.49E-02	1.48E-02	-1.15E-02	1.47E-02	4.32E-02	1.54E-08	9.84E-01	4.90E-23
50	1.17E-02	1.16E-02	-9.49E-03	1.18E-02	3.03E-02	2.05E-04	9.89E-01	4.90E-19
100	5.96E-03	5.94E-03	-5.33E-03	5.89E-03	3.39E-02	2.08E-05	9.92E-01	9.79E-16
200	2.93E-03	2.92E-03	-2.79E-03	2.95E-03	2.45E-02	4.90E-03	9.97E-01	1.02E-08
300	1.99E-03	1.99E-03	-1.91E-03	1.96E-03	1.98E-02	4.03E-02	9.98E-01	1.17E-06
500	1.12E-03	1.12E-03	-1.44E-03	1.18E-03	1.15E-02	5.19E-01	9.99E-01	7.24E-03
1000	6.10E-04	6.10E-04	-1.52E-04	5.89E-04	1.04E-02	6.54E-01	9.99E-01	4.87E-02
2000	2.97E-04	2.97E-04	-3.26E-04	2.95E-04	9.46E-03	7.62E-01	9.99E-01	5.91E-02

Table 21: Study of MLE of population standard deviation

Samp. Size	MSE	Var.	Bias	Theor. Asym. Var.	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	0.076633	0.076644	0.001851	0.07716	0.077336	0	0.934158	1.94E-42
10	0.037693	0.037697	0.001897	0.03858	0.048513	1.2E-10	0.962992	5.93E-34
15	0.026776	0.026779	-0.00134	0.02572	0.052316	2.6E-12	0.964605	2.44E-33
20	0.019507	0.019507	-0.00192	0.01929	0.043378	1.35E-08	0.97933	2.23E-26
30	0.013088	0.013084	0.002441	0.01286	0.041005	9.97E-08	0.983629	1.25E-23
40	0.009682	0.009683	0.000996	0.009645	0.034239	1.62E-05	0.990545	8.69E-18
50	0.00769	0.007691	0.000634	0.007716	0.026694	0.001608	0.993026	6.23E-15
100	0.003902	0.003903	0.000102	0.003858	0.029809	0.000277	0.994233	2.74E-13
200	0.001913	0.001913	-0.00026	0.001929	0.021332	0.021119	0.997693	7.6E-07
300	0.001301	0.001301	-0.00018	0.001286	0.015893	0.159878	0.998485	0.000105
500	0.000735	0.000735	-0.00027	0.000772	0.011736	0.49646	0.999295	0.044597
1000	0.0004	0.0004	0.000282	0.000386	0.010535	0.63571	0.999408	0.107461
2000	0.000194	0.000194	-9.5E-05	0.000193	0.008079	0.899894	0.999479	0.182555

Table 22: Study of MOM estimator of population mean

Samp. Size	MSE	Var.	Bias	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	4.42E-02	4.13E-02	5.42E-02	7.61E-02	0.00E+00	9.39E-01	2.30E-41
10	2.00E-02	1.89E-02	3.33E-02	4.99E-02	2.97E-11	9.67E-01	1.85E-32
15	1.30E-02	1.26E-02	2.08E-02	4.17E-02	5.56E-08	9.73E-01	1.07E-29
20	9.53E-03	9.26E-03	1.65E-02	4.43E-02	5.98E-09	9.81E-01	4.03E-25
30	6.30E-03	6.11E-03	1.39E-02	3.56E-02	6.15E-06	9.89E-01	1.71E-19
40	4.81E-03	4.68E-03	1.16E-02	2.82E-02	7.07E-04	9.93E-01	1.31E-14
50	3.71E-03	3.63E-03	8.92E-03	2.35E-02	8.02E-03	9.95E-01	2.15E-11
100	1.92E-03	1.90E-03	4.02E-03	1.74E-02	9.81E-02	9.97E-01	1.37E-08
200	9.74E-04	9.67E-04	2.55E-03	1.20E-02	4.68E-01	9.99E-01	1.84E-01
300	6.73E-04	6.70E-04	1.65E-03	1.20E-02	4.66E-01	9.97E-01	3.19E-09
500	3.81E-04	3.81E-04	8.26E-04	9.88E-03	7.13E-01	9.98E-01	1.57E-05
1000	2.01E-04	2.00E-04	7.72E-04	1.08E-02	6.09E-01	9.99E-01	6.32E-04
2000	1.02E-04	1.02E-04	3.98E-04	6.28E-03	9.89E-01	9.99E-01	1.26E-02

Table 23: Study of MOM estimator of population median

Samp. Size	MSE	Var.	Bias	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	1.28E-01	1.01E-01	-1.63E-01	1.20E-01	0.00E+00	8.28E-01	9.41E-59
10	7.39E-02	6.44E-02	-9.77E-02	8.90E-02	0.00E+00	9.00E-01	3.30E-49
15	6.35E-02	5.88E-02	-6.84E-02	9.54E-02	0.00E+00	8.86E-01	1.69E-51
20	4.92E-02	4.59E-02	-5.70E-02	8.30E-02	0.00E+00	9.05E-01	2.06E-48
30	3.76E-02	3.63E-02	-3.52E-02	8.29E-02	0.00E+00	9.09E-01	1.02E-47
40	2.74E-02	2.63E-02	-3.29E-02	7.13E-02	0.00E+00	9.42E-01	1.39E-40
50	2.39E-02	2.33E-02	-2.56E-02	6.80E-02	0.00E+00	9.40E-01	6.44E-41
100	1.40E-02	1.39E-02	-1.21E-02	6.69E-02	0.00E+00	9.35E-01	2.48E-42
200	7.11E-03	7.03E-03	-8.70E-03	5.07E-02	1.44E-11	9.55E-01	1.07E-36
300	5.03E-03	5.00E-03	-5.69E-03	4.57E-02	1.64E-09	9.59E-01	2.12E-35
500	2.97E-03	2.95E-03	-3.39E-03	4.07E-02	1.29E-07	9.77E-01	7.14E-28
1000	1.58E-03	1.58E-03	-1.52E-03	3.56E-02	6.45E-06	9.85E-01	9.90E-23
2000	8.05E-04	8.03E-04	-1.53E-03	2.55E-02	3.03E-03	9.88E-01	7.29E-20

Table 24: Study of MOM estimator of population standard deviation

Samp. Size	MSE	Var.	Bias	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	9.14E-02	6.67E-02	-1.57E-01	9.12E-02	0.00E+00	9.03E-01	1.11E-48
10	5.25E-02	4.37E-02	-9.40E-02	6.86E-02	0.00E+00	9.46E-01	2.55E-39
15	4.29E-02	3.83E-02	-6.78E-02	7.13E-02	0.00E+00	9.36E-01	5.46E-42
20	3.34E-02	3.03E-02	-5.59E-02	6.75E-02	0.00E+00	9.50E-01	3.18E-38
30	2.50E-02	2.38E-02	-3.58E-02	6.59E-02	0.00E+00	9.48E-01	9.66E-39
40	1.89E-02	1.78E-02	-3.22E-02	5.77E-02	7.11E-15	9.66E-01	9.42E-33
50	1.63E-02	1.57E-02	-2.55E-02	5.35E-02	7.27E-13	9.66E-01	7.91E-33
100	9.43E-03	9.27E-03	-1.25E-02	5.53E-02	1.03E-13	9.59E-01	2.55E-35
200	4.88E-03	4.81E-03	-8.52E-03	4.22E-02	3.68E-08	9.69E-01	2.10E-31
300	3.46E-03	3.43E-03	-5.65E-03	3.81E-02	9.81E-07	9.72E-01	4.13E-30
500	2.06E-03	2.05E-03	-3.36E-03	3.56E-02	6.21E-06	9.83E-01	5.08E-24
1000	1.10E-03	1.10E-03	-1.54E-03	3.21E-02	6.79E-05	9.88E-01	6.45E-20
2000	5.63E-04	5.61E-04	-1.42E-03	2.29E-02	1.06E-02	9.91E-01	1.08E-17

Table 25: Study of MOM estimator of population IQR

Samp. Size	MSE	Var.	Bias	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	7.15E-02	6.80E-02	5.91E-02	9.11E-02	0.00E+00	9.00E-01	3.69E-49
10	2.99E-02	2.86E-02	3.63E-02	6.22E-02	0.00E+00	9.56E-01	1.70E-36
15	2.05E-02	2.02E-02	1.62E-02	5.40E-02	4.17E-13	9.64E-01	1.34E-33
20	1.47E-02	1.45E-02	1.34E-02	5.47E-02	2.04E-13	9.73E-01	1.15E-29
30	9.56E-03	9.45E-03	1.08E-02	4.04E-02	1.58E-07	9.83E-01	3.61E-24
40	7.38E-03	7.30E-03	9.03E-03	3.91E-02	4.67E-07	9.86E-01	7.38E-22
50	5.74E-03	5.69E-03	6.79E-03	3.17E-02	8.65E-05	9.89E-01	9.34E-20
100	2.83E-03	2.82E-03	2.87E-03	2.97E-02	2.94E-04	9.93E-01	1.26E-14
200	1.41E-03	1.41E-03	1.23E-03	2.04E-02	3.15E-02	9.97E-01	1.71E-07
300	9.94E-04	9.94E-04	8.22E-04	1.57E-02	1.70E-01	9.97E-01	1.03E-08
500	5.61E-04	5.60E-04	6.71E-04	1.83E-02	7.03E-02	9.99E-01	7.58E-03
1000	2.84E-04	2.84E-04	5.82E-04	1.49E-02	2.18E-01	9.99E-01	2.93E-04
2000	1.46E-04	1.46E-04	2.27E-04	1.24E-02	4.25E-01	9.99E-01	6.63E-03

Table 26: Study of nonparametric estimator of population median

Samp. Size	MSE	Var.	Bias	KS. Stat.	KS. p-val.	SW. Stat.	SW. p-val.
5	1.43E-01	1.25E-01	-1.34E-01	1.19E-01	0.00E+00	8.67E-01	3.87E-54
10	7.35E-02	6.99E-02	-5.97E-02	6.86E-02	0.00E+00	9.35E-01	3.92E-42
15	5.17E-02	5.00E-02	-4.02E-02	6.34E-02	0.00E+00	9.54E-01	3.96E-37
20	4.04E-02	3.93E-02	-3.38E-02	6.29E-02	0.00E+00	9.60E-01	4.45E-35
30	2.82E-02	2.80E-02	-1.60E-02	4.54E-02	2.14E-09	9.76E-01	3.33E-28
40	2.14E-02	2.12E-02	-1.55E-02	4.47E-02	4.33E-09	9.80E-01	4.44E-26
50	1.67E-02	1.66E-02	-1.04E-02	3.63E-02	3.74E-06	9.86E-01	4.01E-22
100	8.41E-03	8.36E-03	-6.75E-03	3.01E-02	2.35E-04	9.92E-01	1.10E-16
200	4.35E-03	4.34E-03	-3.92E-03	2.50E-02	3.79E-03	9.96E-01	1.20E-09
300	2.97E-03	2.97E-03	-1.97E-03	1.95E-02	4.53E-02	9.98E-01	6.49E-07
500	1.74E-03	1.73E-03	-1.83E-03	1.21E-02	4.53E-01	9.98E-01	6.60E-05
1000	8.80E-04	8.80E-04	-4.08E-04	1.27E-02	4.00E-01	9.99E-01	1.43E-01
2000	4.32E-04	4.33E-04	-2.07E-04	1.04E-02	6.55E-01	9.99E-01	5.72E-02

Table 27: Study of nonparametric estimator of population IQR