Linear Programming and Networks: Homework-3
<u>Due: July 24th (Wed), 2019</u>

1. Consider the following LP:

$$\begin{aligned} \text{Maximize} \quad & z = 5x_1 + 3x_2 \\ \text{subject to:} \quad & 4x_1 + 2x_2 \le 12 \\ & 4x_1 + x_2 \le 10 \\ & x_1 + x_2 \le 4 \\ & (x_1,\ x_2) \ge 0. \end{aligned}$$

   (a) Draw the feasible region, clearly indicating the constraints, slack variables associated with each constraint, gradient and contours of the objective function.
   (b) Identify the optimal solution graphically, clearly showing that the gradient of the objective function can be expressed as a non-negative combination of the gradients of the binding constraints.
   (c) Using KKT conditions, determine the values of *all* dual optimal solutions.

2. Consider the following LP:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + 4x_2 + 2x_3 \\ \text{subject to:} \quad & x_1 + x_2 + x_3 \le 20 \\ & x_1 + 2x_2 + x_3 = 30 \\ & (x_1,\ x_2,\ x_3) \ge 0. \end{aligned}$$

   (a) Write the dual to the given LP.
   (b) Draw the feasible region to the dual, clearly indicating the constraints, slack variables associated with each constraint, gradient and contours of the objective function.
   (c) Solve the dual problem graphically.
   (d) Solve the dual problem using the *two-phase* method. Clearly show all the steps in both Phase-I and Phase-II of the simplex method and moreover, show that the values of the *primal variables* can be obtained from the optimal dual tableau.

3. Consider the following linear program:

$$\begin{aligned} \text{Maximize} \quad & z = 3x_1 + x_2 + 4x_3 \\ \text{subject to:} \quad & 6x_1 + 3x_2 + 5x_3 \le 25 \\ & 3x_1 + 4x_2 + 5x_3 \le 20 \\ & (x_1,\ x_2,\ x_3) \ge 0. \end{aligned}$$

(a) Solve the above linear program using the simplex method.
(b) If the objective function changes to $z = 3x_1 + 3x_2 + 4x_3$. Is the solution obtained in part (a) still optimal? If not, find the new optimum.
(c) If the coefficient of x_2 in the first constraint changes to 2, does the optimal solution from part (a) change? If yes, find the new optimum.
(d) Now suppose that the only change in the original problem is that a new variable has been introduced, say x_4, with parameters $c_4 = 2$, $a_{14} = 3$, and $a_{24} = 2$. Use sensitivity analyses to determine the new optimum.

4. Consider the linear program given in Problem #4. For each of the following *independent* cases, use sensitivity analysis to find the new optimum.

(a) The RHS of constraint 1 changes to $b_1 = 15$.
(b) The RHS of constraint 2 changes to $b_2 = 5$.
(c) The coefficient of x_2 in the objective function changes to $c_2 = 4$.
(d) The coefficient of x_3 in the objective function changes to $c_3 = 3$.
(e) The coefficient of x_2 in constraint 2 changes to $a_{22} = 1$.
(f) The coefficient of x_1 in constraint 1 changes to $a_{11} = 10$.

5. Consider the following linear program:

$$\begin{array}{ll}
\text{Maximize} & z = -5x_1 + 5x_2 + 13x_3 \\
\text{subject to:} & -x_1 + x_2 + 3x_3 \le 20 \\
 & 12x_1 + 4x_2 + 10x_3 \le 90 \\
 & (x_1,\ x_2,\ x_3) \ge 0.
\end{array}$$

(a) Solve the problem using the simplex method.

For parts (b)-(i), perform sensitivity analysis and reoptimize, treating each part independently.

(b) The RHS of constraint 1 changes to $b_1 = 30$.
(c) The RHS of constraint 2 changes to $b_2 = 70$.
(d) Change the RHS to: $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix}$.
(e) Change the coefficient of x_3 in the objective function to $c_3 = 8$.

(f) Change the coefficients of x_1 to: $\begin{bmatrix} c_1 \\ a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$.

(g) Introduce a new variable x_4 with coefficients: $\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix}$.

(h) Introduce a new constraint $2x_1 + 3x_2 + 5x_3 \le 50$.

(i) Suppose that the RHS changes to $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20+2\theta \\ 90-\theta \end{bmatrix}$, where θ can take on any positive, zero, or negative values. Find the range (lower and upper bounds) of θ for which the solution in part (a) remains optimal. Plot the optimal objective function value as a function of θ between the bounds.

6. Consider the following linear program:

$$\begin{aligned} \text{Maximize} \quad & z = 2x_1 + 7x_2 - 3x_3 \\ \text{subject to:} \quad & x_1 + 3x_2 + 4x_3 \le 30 \\ & x_1 + 4x_2 - x_3 \le 10 \\ & (x_1,\ x_2,\ x_3) \ge 0. \end{aligned}$$

(a) Solve the problem using the simplex method.

For parts (b)-(f), perform sensitivity analysis and reoptimize, treating each part independently.

(b) Change the RHS to: $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$.

(c) Change the coefficients of x_3 to: $\begin{bmatrix} c_3 \\ a_{13} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$.

(d) Introduce a new variable x_4 with coefficients: $\begin{bmatrix} c_4 \\ a_{14} \\ a_{24} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.

(e) Introduce a new constraint $3x_1 + 2x_2 + 3x_3 \le 25$.

(f) Suppose that the RHS changes to $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 30+3\theta \\ 10-\theta \end{bmatrix}$, where θ can take on any positive, zero, or negative values. Find the range (lower and upper bounds) of θ for which the solution in part (a) remains optimal. Plot the optimal objective function value as a function of θ between the bounds.

Part – II

<u>Due: Aug 7th (Wed), 2019</u>

Implement the simplex algorithm (as sophisticated a version as possible) in MATLAB (or a language of your choice). You can work in pairs for this problem. Your implementation should cover the following:

(a) Solve the problem to optimality, if an optimum exists
(b) Be able to identify different types of linear programs (unbounded, infeasible, alternate optima, etc.)
(c) Identify the dual optimal solution from the primal tableau.
(d) Clearly specify the input format for your code; try and make it as intuitive as possible
(e) Be able to output information regarding each extreme point the algorithm traverses to get to the optimum (basis matrix, degeneracy, etc.)
(f) Test your algorithm on various instances available in the literature.
(g) Implement sensitivity analyses for changes in cost coefficients and RHS only. The code should determine the ranges of the cost coefficients and RHS values.

I will ask you to do a demo of your code in class on select problems. We can decide when to do this at a later date.