Question 1

Pareto Distribution

CDF :

$$F_X(x) = \begin{cases} 1-\left(\frac{\sigma}{x}\right)^{\lambda} & \text{if } x \geq \sigma \\ 0 & \text{if } x < \sigma \end{cases}$$

where $\sigma, \lambda > 0$

σ is the scale parameter and λ is the shape parameter

PDF :

We know,

$$F_X(x) = 1-\left(\frac{\sigma}{x}\right)^{\lambda} \text{ if } x \geq \sigma$$

$$= 1 - \frac{\sigma^{\lambda}}{x^{\lambda}}$$

$$= 1 - \sigma^{\lambda}(x^{-\lambda})$$

$$= -\left[\sigma^{\lambda}(-\lambda)\, x^{-\lambda-1}\right]$$

$$= \sigma^{\lambda}\lambda x^{-(\lambda+1)}$$

$$= \frac{\lambda\sigma^{\lambda}}{x^{\lambda+1}}$$

$$\therefore f_X(x) = \begin{cases} \lambda\sigma^{\lambda}x^{-(\lambda+1)} & \text{if } x \geq \sigma \\ 0 & \text{if } x < \sigma \end{cases}$$

Derivation of mean :

$$E[X] = \int_{\sigma}^{\infty} x f(x)\, dx$$

$$= \int_{\sigma}^{\infty} x \lambda \sigma^{\lambda} x^{-(\lambda+1)}\, dx$$

$$= \lambda \sigma^{\lambda} \int_{\sigma}^{\infty} x^{-\lambda}\, dx$$

$$= \lambda \sigma^{\lambda} \left. \frac{x^{-\lambda+1}}{-\lambda+1} \right|_{\sigma}^{\infty}$$

$$= \frac{\lambda \sigma^{\lambda}}{-\lambda+1} \lim_{b \to \infty} \left. \frac{1}{x^{\lambda-1}} \right|_{\sigma}^{b}$$

$$= \frac{\lambda \sigma^{\lambda}}{-\lambda+1} \lim_{b \to \infty} \left[\frac{1}{b^{\lambda-1}} - \frac{1}{\sigma^{\lambda-1}} \right]$$

$$= \frac{\lambda \sigma^{\lambda}}{-\lambda+1} \left[0 - \frac{1}{\sigma^{\lambda-1}} \right]$$

$$= \frac{-\lambda}{-\lambda+1} \sigma^{\lambda-1+1}$$

$$= \frac{\lambda \sigma}{\lambda-1}$$

$\therefore$ Mean $= \begin{cases} \dfrac{\sigma \lambda}{\lambda - 1} &, \lambda > 1 \\ \infty &, \lambda \leq 1 \end{cases}$

Coefficient of skewness

$$= \frac{2(\lambda+1)}{\lambda - 3} \sqrt{\frac{\lambda - 2}{\lambda}} \quad , \lambda > 3$$

Coefficient of kurtosis

$$= \frac{3(\lambda-2)(3\lambda^2+\lambda+2)}{\lambda(\lambda-3)(\lambda-4)} \quad , \lambda > 4$$

Ex. Kurtosis

$$= \text{Coefficient of kurtosis} - 3$$

$$= \frac{(3\lambda-6)(3\lambda^2+\lambda+2)}{\lambda(\lambda-3)(\lambda-4)} - 3$$

$$= \frac{(9\lambda^3+3\lambda^2+6\lambda-18\lambda^2-6\lambda-12) - 3(\lambda^2-3\lambda)(\lambda-4)}{\lambda(\lambda-3)(\lambda-4)}$$

$$= \frac{9\lambda^3+3\lambda^2+\cancel{6\lambda}-18\lambda^2-\cancel{6\lambda}-12 - 3(\lambda^3-4\lambda^2-3\lambda^2+12\lambda)}{\lambda(\lambda-3)(\lambda-4)}$$

$$= \frac{6\lambda^3+6\lambda^2-36\lambda-12}{\lambda(\lambda-3)(\lambda-4)}$$

$$= \frac{6(\lambda^3+\lambda^2-6\lambda-2)}{\lambda(\lambda-3)(\lambda-4)} \quad , \lambda > 4$$

Derivation of variance

$$E(X^2) = \int_{\sigma}^{\infty} x^2 \frac{\lambda \sigma^{\lambda}}{x^{\lambda+1}} dx$$

$$= \lambda \sigma^{\lambda} \lim_{b \to \infty} \int_{\sigma}^{b} x^{1-\lambda} dx$$

$$= \frac{\lambda \sigma^{\lambda}}{2-\lambda} \lim_{b \to \infty} x^{-(\lambda-2)} \Big|_{\sigma}^{b}$$

$$= \frac{\lambda \sigma^{\lambda}}{2-\lambda} \lim_{b \to \infty} \left[\frac{1}{b^{\lambda-2}} - \sigma^{2-\lambda} \right]$$

$$= \frac{\lambda \sigma^{\lambda} \sigma^{2-\lambda}}{\lambda - 2}$$

$$= \frac{\lambda \sigma^2}{\lambda - 2} ; \quad \lambda > 2$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \frac{\lambda \sigma^2}{\lambda - 2} - \frac{\lambda^2 \sigma^2}{(\lambda-1)^2}$$

$$= \frac{\lambda \sigma^2 (\lambda-1)^2 - \lambda^2 \sigma^2 (\lambda-2)}{(\lambda-2)(\lambda-1)^2}$$

$$= \frac{\lambda\sigma^2(\lambda^2+1-2\lambda) - \lambda^2\sigma^2(\lambda-2)}{(\lambda-2)(\lambda-1)^2}$$

$$= \frac{\cancel{\lambda^3\sigma^2} + \lambda\sigma^2 - \cancel{2\lambda^2\sigma^2} - \cancel{\lambda^3\sigma^2} + \cancel{2\lambda^2\sigma^2}}{(\lambda-1)^2(\lambda-2)}$$

$$\therefore Var = \begin{cases} \dfrac{\lambda\sigma^2}{(\lambda-1)^2(\lambda-2)} & , \lambda > 2 \\ \infty & , \lambda \le 2 \end{cases}$$

Derivation of the 95^{th} percentile and media

$$F_X(x) = 1 - \left(\frac{\sigma}{x}\right)^\lambda \quad ; \quad x \geq \sigma$$

$$y = 1 - \left(\frac{\sigma}{x}\right)^\lambda$$

$$\Rightarrow \left(\frac{\sigma}{x}\right)^\lambda = 1 - y$$

$$\Rightarrow \frac{\sigma}{x} = (1-y)^{1/\lambda}$$

$$\Rightarrow x = \frac{\sigma}{(1-y)^{1/\lambda}}$$

When $F_X(x) = y = 0.5$

$$x_{0.5} = \frac{\sigma}{(0.5)^{1/\alpha}}$$

$$= \frac{\sigma}{\frac{1}{2^{1/\alpha}}}$$

$$= 2^{1/\alpha}\sigma$$

$\therefore$ Median $= \sigma 2^{1/\alpha}, \alpha > 0$

$$x_{0.95} = \frac{\sigma}{(1-0.95)^{1/\alpha}}$$

$$= \frac{\sigma}{(0.05)^{1/\alpha}}$$

$$= \frac{\sigma}{\frac{1}{20^{1/\alpha}}}$$

95th percentile $= 20^{1/\alpha}\sigma \; , \alpha > 0$

$IQR = Q_3 - Q_1$

$$= \frac{\sigma}{(1-0.75)^{1/\lambda}} - \frac{\sigma}{(1-0.25)^{1/\lambda}}$$

$$= \frac{\sigma}{(0.25)^{1/\lambda}} - \frac{\sigma}{0.75^{1/\lambda}}$$

$$= \frac{\sigma}{\frac{1}{4^{1/\lambda}}} - \frac{\sigma}{\frac{3}{4^{1/\lambda}}}$$

$$= \sigma\left[4^{1/\lambda} - \left(\frac{4}{3}\right)^{1/\lambda}\right]; \lambda > 0$$

Proportion of values > mean + s.d.

$$P[X > x] = 1 - F(x)$$

$$= 1 - 1 + \left(\frac{\sigma}{x}\right)^{\lambda}$$

$$= \left(\frac{\sigma}{x}\right)^{\lambda}$$

where,

$$x = \frac{\lambda\sigma}{\lambda - 1} + \frac{\sigma\sqrt{\lambda}}{\lambda - 1\sqrt{\lambda - 2}}, \lambda > 2$$

Question 1

X_1, X_2, X_3

$$f_{X_{(1)}}(x) = \frac{3!}{0!1!2!} f(x)\left[1 - F(x)\right]^2$$

$$= 3 \times \frac{\lambda \sigma^{\lambda}}{x^{\lambda+1}} \left(\frac{\sigma}{x}\right)^{2\lambda}$$

$$= \frac{3\lambda \sigma^{3\lambda}}{x^{3\lambda+1}}$$

$$F_{X_{(1)}}(x) = \int_{\sigma}^{x} \frac{3\lambda \sigma^{3\lambda}}{t^{3\lambda+1}}\, dt$$

$$= 3\lambda\sigma^{3\lambda} \int_{\sigma}^{x} t^{-(3\lambda+1)}\, dt$$

$$= \frac{\cancel{3\lambda} \sigma^{3\lambda}}{-\cancel{3\lambda}} t^{-3\lambda} \Big|_{\sigma}^{x}$$

$$= -\sigma^{3\lambda}\left[x^{-3\lambda} - \sigma^{-3\lambda}\right]$$

$$= \sigma^{3\lambda}\left[\sigma^{-3\lambda} - x^{-3\lambda}\right]$$

$$P\left[X_{(1)} > \sigma 2^{1/\lambda}\right]$$

$$= 1 - F_{X_{(1)}}\left[\sigma 2^{1/\lambda}\right]$$

$$= 1 - \sigma^{3\lambda}\left[\sigma^{-3\lambda} - \left[\sigma 2^{1/\lambda}\right]^{-3\lambda}\right]$$

$$= 1 - \sigma^{3\lambda}\left[\sigma^{-3\lambda} - \frac{\sigma^{-3\lambda}}{8}\right]$$

$$= 1 - \sigma^{3\lambda}\sigma^{-3\lambda}\left[1 - \frac{1}{8}\right]$$

$$= 1 - \frac{7}{8}$$

$$= \frac{1}{8}$$

6.

Derivation of $\hat{\lambda}_{MLE}$ and $\hat{\sigma}_{MLE}$:

$$f(x) = \frac{\lambda \sigma^{\lambda}}{x^{\lambda+1}}$$

$x \geq \sigma > 0$, $\lambda > 0$

$x_1, \dots, x_n \sim \text{Pareto}(\lambda, \sigma)$

$$\text{lik}(\lambda, \sigma) = \frac{\lambda^n \sigma^{n\lambda}}{(\prod X_i)^{\lambda+1}} \prod I_{[\sigma, \infty)}(X_i)$$

$$= \frac{\lambda^n \sigma^{n\lambda}}{(\prod X_i)^{\lambda+1}} I_{[\sigma, \infty)}(\min(X_i))$$

$$I_{[\sigma,\infty)}(x) = \begin{cases} 1 & \text{if } x \in [\sigma, \infty) \\ 0 & \text{otherwise} \end{cases}$$

when $\sigma > \min(X_i) \Rightarrow \text{lik}(\lambda, \sigma) = 0$

when $\sigma \leq \min(X_i) \Rightarrow \text{lik}(\lambda, \sigma)$ is an increasing function in σ

$\therefore \hat{\sigma}_{MLE} = \min(X_i)$

$$l(\lambda) = n \log \lambda + n\lambda \log(\hat{\sigma}) - (\lambda+1) \sum \log X_i$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} + n \log(\hat{\sigma}) - \sum \log(X_i) = 0$$

$$\Rightarrow \frac{n}{\lambda} = \sum \log(X_i) - n \log(\hat{\sigma})$$

$$\frac{\lambda}{n} = \frac{1}{\sum \log(x_i) - n\log(\hat{\sigma})}$$

$$\Rightarrow \hat{\lambda}_{MLE} = \frac{n}{\sum \log(x_i) - n\log(\hat{\sigma}_{MLE})}$$

$$= \frac{n}{\sum\left(\log(x_i) - \log(\hat{\sigma}_{MLE})\right)}$$

$$= \frac{n}{\sum \log\left(\frac{x_i}{\hat{\sigma}_{MLE}}\right)}$$

$$= \frac{n}{\sum \log\left(\frac{x_i}{\min(x_i)}\right)}$$

$$f(x) = \frac{\alpha \sigma^{\alpha}}{x^{\alpha+1}} \quad ; x \geq \sigma > 0 , \alpha > 0$$

$$\log f(x) = \log \alpha + \alpha \log \sigma - (\alpha+1) \log(x)$$

$$\frac{\partial}{\partial \alpha} \log f(x) = \frac{1}{\alpha} + \log \sigma - \log(x)$$

$$\frac{\partial^2}{\partial \alpha^2} \log f(x) = -\frac{1}{\alpha^2}$$

$$\frac{\partial}{\partial \sigma} \log f(x) = \frac{\alpha}{\sigma}$$

$$\frac{\partial^2}{\partial \sigma^2} \log f(x) = \frac{-\alpha}{\sigma^2}$$

$$\frac{\partial}{\partial \alpha \partial \sigma} \log f(x) = \frac{1}{\sigma}$$

$$\frac{\partial}{\partial \sigma \partial \alpha} \log f(x) = \frac{1}{\sigma}$$

$$-E \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{1}{\sigma} \\ \frac{1}{\sigma} & \frac{-\alpha}{\sigma^2} \end{bmatrix}$$

$$I(\theta) = \begin{bmatrix} \frac{1}{\lambda^2} & -\frac{1}{\sigma} \\ -\frac{1}{\sigma} & \frac{\lambda}{\sigma^2} \end{bmatrix}$$

$$I^{-1}(\theta) = \begin{bmatrix} \frac{\lambda^2 \sigma^2}{\sigma(1-\lambda)} & \frac{\lambda \sigma^2}{\sigma(1-\lambda)} \\ \frac{\lambda \sigma^2}{\sigma(1-\lambda)} & \frac{\sigma^2}{\lambda(1-\lambda)} \end{bmatrix}$$

asymptotic variances will be $\frac{\lambda^2 \sigma^2}{\sigma(1-\lambda)}$ and $\frac{\sigma^2}{\lambda(1-\lambda)}$ if they are unbiased.

However, the Fisher information matrix cannot be used to compute the asymptotic variances here because the regularity conditions are not satisfied as the support has a parameter.