ON A POLYNOMIAL OF BYRNES

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1. *Introduction*

In [2] Littlewood asked whether there exist positive constants A_1 and A_2 such that, whenever n is sufficiently large, we can find $a_1, a_2, \ldots, a_n \in \mathbf{C}$ with $|a_j| = 1$ such that

$$A_1 n^{1/2} \leqslant \left| \sum_{k=1}^{n} a_k \exp ikt \right| \leqslant A_2 n^{1/2}$$

for all $t \in \mathbf{R}$.

In [1] Byrnes uses a very ingenious argument to produce a polynomial in $z^{1/2}$ which almost solves the problem.

THEOREM 1 (Byrnes). *Let n be the square of an even integer. Then we can find $M = O(n^{3/4})$ and complex numbers $b_1, b_2, \ldots, b_{2M}, b'_{M+1}, b'_{M+2}, \ldots, b'_{n-M}, b_{n-M+2}, \ldots, b_{n+M}$ of modulus one, such that*

$$\left| 1/2 \sum_{j=1}^{2M} b_j \exp \pi ijt + \sum_{k=M+1}^{n-M} b'_k \exp 2\pi ikt + 1/2 \sum_{j=2n-M+2}^{2n+M} b_j \exp \pi ijt \right|$$

$$= n^{1/2} + O(n^{1/4}) \qquad \textit{for all } t \in \mathbf{R}.$$

Proof. Consider the function $f+g$ described in [1] omitting the single term corresponding to b_{2n-M+1}. (The omission of this term and the choice of n as the square of an even integer are intended to simplify the argument of this paper.)

The object of this note is to show how a simple modification of Byrnes' polynomial actually solves the problem.

2. *A Preliminary Lemma*

Our argument is based on the following simple lemma.

LEMMA 2. *Given $c_{r+1}, c_{r+2}, \ldots, c_{r+N} \in \mathbf{C}$ with $|c_{r+1}|, |c_{r+2}|, \ldots, |c_{r+N}| \leqslant 1$. we can find $c'_{r+1}, c'_{r+2}, \ldots, c'_{r+N} \in \mathbf{C}$ with $|c'_{r+1}| = |c'_{r+2}| = \ldots = |c'_{r+n}| = 1$ and*

$$\left| \sum_{k=r+1}^{r+N} c_k \exp ikt - \sum_{k=r+1}^{r+N} c'_k \exp ikt \right| = O((N \log N)^{1/2})$$

for all $t \in \mathbf{R}$.

We prove Lemma 2 by appealing to a result from probability theory.

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LEMMA 3. *Let $X_1, X_2, \ldots, X_N$ be independent (but not necessarily identically distributed) random variables with $|X_j| \leqslant 1$ for each $1 \leqslant j \leqslant n$ and write $S_N = X_1 + X_2 + \ldots + X_N$. Then*

$$\Pr(|S_N - ES_N| \geqslant \lambda) \leqslant 4\exp(-\lambda^2/100N)$$

for all $\lambda \geqslant 0$.

Remark. The result is extremely plausible, since, if the X_j were identically distributed, we know that S_N would tend very fast to a normal distribution. Unfortunately this result, although part of folklore, does not appear in most standard texts, but it can be found on page 387 of [3].

Proof of Lemma 2. Without loss of generality, we may suppose $r = 0$. *Set*

$$f(t) = \sum_{k=1}^{N} c_k \exp ikt \qquad [t \in \mathbf{R}]$$

and choose $0 \leqslant \lambda_k \leqslant 1$ and $\theta_k \in \mathbf{R}$ such that $c_k = \lambda_k \exp i\theta_k$. Define $Y_1, Y_2, \ldots, Y_N$ to be independent random variables with

$$\Pr(Y_k = \exp i\theta_k) = (1+\lambda_k)/2$$

$$\Pr(Y_k = -\exp i\theta_k) = (1-\lambda_k)/2$$

and consider the random trigonometric polynomial

$$F(t) = \sum_{k=1}^{N} Y_k \exp ikt \qquad [t \in \mathbf{R}].$$

For fixed $t \in \mathbf{R}$ we see that

$$EF(t) = \sum_{k=1}^{N} (EY_k)\exp ikt$$

$$= \sum_{k=1}^{N} c_k \exp ikt = f(t).$$

Thus, setting $X_k = Y_k \exp ikt$, $S_N = F(t)$, we have, by Lemma 3,

$$\Pr\left(|F(t) - f(t)| \geqslant 20(N\log N)^{1/2}\right) \leqslant 4N^{-4}.$$

In particular, therefore

$$\Pr\left(|F(2\pi u/N^3) - f(2\pi u/N^3)| < 20(N\log N)^{1/2} \quad \text{for all } 1 \leqslant u \leqslant N^3\right)$$

$$> 1 - \sum_{u=1}^{N^3} \Pr\left(|F(2\pi u/N^3) - f(2\pi u/N^3)| > 20(N\log N)^{1/2}\right)$$

$$> 1 - N^3(4N^{-4}) = 1 - 4N^{-1} > 1/2 \quad \text{for all } N \geqslant 8.$$

Since there must be a specific instance of an event with non zero probability, it follows that there exist $c'_1, c'_2, \ldots, c'_N$ with $c'_k = \pm \exp i\theta_k$ such that

$$\left| \sum_{k=1}^{N} c_k \exp(2\pi iku/N^3) - \sum_{k=1}^{N} c'_k(2\pi iku/N^3) \right| \leqslant 20(N \log N)^{1/2}$$

for all $1 \leqslant u \leqslant N^3$. But, if $|a_1|, |a_2|, \ldots, |a_N| \leqslant 1$, then

$$\left| \sum_{k=1}^{N} a_k \exp(2\pi iku/N^3) - \sum_{k=1}^{N} a_k \exp(2\pi ikt) \right|$$

$$\leqslant \sum_{k=1}^{N} |a_k|\, |\exp(2\pi iku/N^3) - \exp(2\pi ikt)| \leqslant \sum_{k=1}^{N} \pi/N^3 \leqslant \pi/N^2$$

for all t with $|t - 2\pi iku/N^3| \leqslant \pi/N^3$. Thus

$$\left| \sum_{k=1}^{N} c_k \exp 2\pi ikt - \sum_{k=1}^{N} c'_k \exp 2\pi ikt \right| \leqslant 20(N \log N)^{1/2} + \pi N^{-3}$$

for all $t \in \mathbf{R}$ and all $N \geqslant 8$ and the lemma is proved.

3. *The Modification*

With the preliminaries out of the way, we may now argue directly towards our goal. Note first that as a trivial consequence of Theorem 1

$$\left| 1/2 \sum_{j=1}^{2M} b_j \exp 4\pi ijt + \sum_{k=M+1}^{n-M} b'_k \exp 8\pi ikt + 1/2 \sum_{j=2n-M+2}^{2n+M} b_j \exp 4\pi ijt \right|$$

$$= n^{1/2} + O(n^{1/4}) \quad \text{for all } t \in \mathbf{R}.$$

Set

$$f(z) = 1/2 \sum_{j=1}^{2M} b_j z^{2j} + \sum_{k=M+1}^{n-M} b'_k z^{4k} + 1/2 \sum_{j=2n-M+2}^{2n+M} b_j z^{2j}$$

and consider $g(z) = (1 + z - z^2 + z^3) f(z)$. Since

$$|1 + \exp i\theta - \exp 2i\theta + \exp 3i\theta|^2 = 4|\cos(3\theta/2) + i \sin(\theta/2)|^2$$

$$= 4(\cos^2(3\theta/2) + \sin^2(\theta/2)) \neq 0\,,$$

we see that, setting $E(z) = |1 + z - z^2 + z^3|$, we have

$$|g(z)| = E(z) n^{1/2} + O(n^{1/4})\,,$$

where $A_3 \leqslant E(z) \leqslant A_4$ for all $|z| = 1$ and some positive constants A_3 and A_4. Writing $g(z) = \sum_{k=1}^{4n+2M+3} c_k z^k$, we obtain the following corollary of Theorem 1:

LEMMA 4. *Under the hypotheses of Theorem* 1 *we can find* $M = O(n^{3/4})$ *and complex numbers* $c_1, c_2, \ldots, c_{4n+2M+4}$ *such that*

(i) $|c_1|, |c_2|, \ldots, c_{2M+3}| \leqslant 1$

(ii) $|c_{2M+4}| = |c_{2M+5}| = \ldots = |c_{4n-2M+3}| = 1$

(iii) $|c_{4n-2M+4}|, |c_{4n-2M+5}|, \ldots, |c_{4n+2M+3}| \leqslant 1$

(iv) $\left|\sum_{k=1}^{4n+2M+3} c_k \exp ikt\right| = E(t)\, n^{1/2} + O(n^{1/4})$.

Taking $c'_r = c_r$ for $2M+4 \leqslant r \leqslant 4n-2M+3$ and choosing the other c'_r, in accordance with Lemma 2, so that

$$\left|\sum_{k=1}^{4M+3} c_k \exp ikt - \sum_{k=1}^{4M+3} c'_k \exp ikt\right| = O((M \log M)^{1/2})$$

$$\left|\sum_{k=4n-2M+3}^{4n+2M+3} c_k \exp ikt - \sum_{k=4n-2M+3}^{4n+2M+3} c'_k \exp ikt\right| = O((m \log M)^{1/2})$$

we obtain the following modification of Lemma 4.

LEMMA 5. *Under the hypotheses of Theorem* 1 *we can find* $M = O(n^{3/4})$ *and complex numbers* $c'_1, c'_2, \ldots, c'_{4n+2M+4}$ *all of modulus one such that*

$$\left|\sum_{k=1}^{4n+2M+4} c_k \exp ikt\right| = E(t) n^{1/2} + O(n^{3/8} (\log n)^{1/2})$$

where $A_3 \leqslant E(t) \leqslant A_4$ *for all* $t \in \mathbf{R}$.

Lemma 5 is essentially the result we want. Choose $0 \leqslant A_1 < A_3/2$ and $A_2 > A_3/2$. The following theorem answers the question set out in the introduction.

THEOREM 6. *Provided that N is sufficiently large, we can find* $a_1, a\ , \ldots, a_N \in \mathbf{C}$ *with* $|a_j| = 1$ *such that*

$$A_1 N^{1/2} \leqslant \left|\sum_{k=1}^{N} a_k \exp ikt\right| \leqslant A_2 N^{1/2}$$

for all $t \in \mathbf{R}$.

Proof. We can easily find an n which is the square of an even integer together with an associated $M = M(n) = O(n^{3/4})$ such that

(i) n, M satisfy the conditions of Theorem 1

(ii) $N \geqslant 4n + 2M(n) + 3$

(iii) $N - (4n + 2M(n) + 3) = 0(N^{3/4})$.

By Lemma 5 we can find $a_1, a_2, \ldots, a_{4n+M+3}$ with $|a_1| = |a_2| = \ldots = |a_{4n+2M+3}| = 1$ and

$$\left|\sum_{k=1}^{4n+2M+3} a_k \exp ikt\right| = E(t)n^{1/2} + O(n^{3/8}(\log n)^{1/2}).$$

On the other hand applying Lemma 2 with $c_r = 0, a_r = c'_r[4n+2M+4 \leqslant r \leqslant N]$ we can find $a_{4n+2M+4}, a_{4n+2M+5}, \ldots, a_N$ with

$$|a_{4n+2M+4}| = |a_{4n+2M+5}| = \ldots = |a_N| = 1$$

and

$$\left|\sum_{k=4n+2M+4}^{N} a_k \exp ikt\right| = O(N^{3/8}(\log N)^{1/2}).$$

Thus, summing, we have

$$\left|\sum_{k=1}^{N} a_k \exp ikt\right| = E(t)n^{1/2} + O(N^{3/8}(\log N)^{1/2})$$

$$= 1/2\, E(t)N^{1/2} + O(N^{3/8}(\log N)^{1/2}).$$

We conclude with two remarks which may be obvious to the reader. The first is that the method of proof applied to

$$h(z) = 1/2 \sum_{j=1}^{2M} b_j z^j + \sum_{k=M+1}^{n-M} b'_k z^{2k} + 1/2 \sum_{j=2n-M+2}^{2n+M} b_j z^j$$

rather than to $f(z)$ yields the following interesting result.

THEOREM 7. *For each $N \geqslant 1$ we can find integers $1 \leqslant k(1) < k(2) < \ldots < k(N)$ and complex numbers $a_1, a_2, \ldots, a_N \in \mathbf{C}$ with $|a_j| = 1$ such that*

$$\left|\sum_{j=1}^{N} a_j \exp ik(j)t\right| = N^{1/2} + O(N^{3/8}(\log N)^{1/2})$$

for all $t \in \mathbf{R}$.

To state our last result compactly, we write $\mathscr{G}_n$ for the set of trigonometric polynomials of the form $\sum_{j=1}^{n} a_j \exp ijt$ with $|a_j| = 1$ $[1 \leqslant j \leqslant n]$.

LEMMA 8. *Suppose we can find $m(r) \to \infty$ and $g_{m(r)} \in \mathscr{G}_{m(r)}$ with*

$$\sup_{t\in\mathbf{R}} \frac{|g_{m(r)}(t)|}{m(r)^{1/2}} \longrightarrow 1 \quad \textit{and} \quad \inf_{t\in\mathbf{R}} \frac{|g_{m(r)}(t)|}{m^{1/2}} \longrightarrow 1 \ \textit{as}\ r \longrightarrow \infty.$$

Then we can find $g_m \in \mathscr{G}_m$ with

$$\sup_{t\in\mathbf{R}} \frac{|g_m(t)|}{m^{1/2}} \longrightarrow 1 \quad \textit{and} \quad \inf_{t\in\mathbf{R}} \frac{|g_m(t)|}{m^{1/2}} \longrightarrow 1 \quad \textit{as} \quad m \longrightarrow \infty.$$

Proof. By considering $g_{m(r)}(t)+\exp i(m(r)+1)t$, if necessary, we may suppose that the $m(r)$ are even. Fixing r for the time being, we may run through the argument of this section with

$$f(z) = 1/2 \sum_{j=1}^{2M} b_j z^{m(r)j/2} + \sum_{k=M+1}^{n-M} b'_k z^{m(r)j} + 1/2 \sum_{j=2n-M+2}^{2n+M} b_j z^{m(r)j/2}$$

and $g(z) = g_{m(r)}(z)f(z)$. We then have

$$A_3 = \inf_{t \in \mathbf{R}} |g_{m(r)}(t)|, \quad A_4 = \sup_{t \in \mathbf{R}} |g_{m(r)}(t)|$$

and $A_1 = A_3 m(r)^{-1/2}$, $A_2 = A_4 m(r)^{-1/2}$ so that A_1 and A_2 can be made as close to 1 as we want by a suitable choice of r.

Added in Proof. Whilst this paper was with the printers I have seen a preprint of an elegant paper of A. M. Odlyzko (Bell Labs., New Jersey 07974) entitled "Minima of cosines and maxima of polynomials on the unit circle" which also uses the idea of Lemma 2.

References

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