To Prove and Conjecture: Paul Erdős and His Mathematics
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To Prove and Conjecture: Paul Erdős and His Mathematics

Béla Bollobás

"Every human activity, good or bad, *except mathematics*, must come to an end"—this was a favorite saying of Paul Erdős, mathematician extraordinaire, who passed away on September 20th 1996, aged 83, after a life devoted to mathematics. He was one of the giants of 20th century mathematics: he proved fundamental results in number theory, probability theory, approximation theory, geometry, interpolation theory, real and complex analysis, set theory and combinatorics. He practically created probabilistic number theory, partition calculus for large cardinals, extremal graph theory, and the theory of random graphs, and no one did more to develop and advocate the use of probabilistic methods throughout mathematics. He had a wonderful talent for interacting with other mathematicians, and so many of his best results were obtained in collaboration. Altogether, he wrote close to 1500 papers, about five times as many as other prolific mathematicians, and he had about 500 collaborators. The rate of his output peaked in later years: like Leonhard Euler (1707–1783), who produced more than half of his works after 1765 in spite of being blind, Erdős wrote more than half of his papers in the last 20 years of his life.

He was a personal friend of more mathematicians than anybody ever; he was eager to help whomever he came in touch with, and a large number of successful mathematicians today owe their careers to him. His love of mathematics, permeating his existence, was infectious, and fired the imagination of many a young colleague. He was eager to share his ideas and to pass on his love of mathematics: there is no doubt that he attracted me to combinatorics when, as a young schoolboy, I first met him. Having known him for almost forty years, it is difficult to imagine that there is mathematics without him.

Although he was interested in medicine, history and politics, and would put incisive questions to casual acquaintances on these subjects, he had a total devotion to mathematics. He never had a 'proper' teaching job, but traveled around the world, collaborating with mathematicians of all fields. His travels started long before traveling became commonplace for mathematicians, and he had many collaborators, years before that became as common a practice as it is today. It was claimed that on a longer train journey he would write a joint paper with the conductor, and this was not much of an exaggeration. He never had a check book or credit card, never learned to drive, and was happy to travel for years on end with two half-empty suitcases. "La propriété, c'est le vol", wrote Pierre Joseph Proudhon—"Property is a nuisance", echoed Paul Erdős, and his life was a testimony to this statement. For a while he did travel with a small transistor radio, but later he abandoned even that.

With his motto, 'another roof, another proof', he would arrive on the doorstep of a mathematical friend, bringing news of discoveries and problems: "declaring 'his brain open'" he would plunge into discussions about the work of his hosts, and after a few days of furious work on their problems, he would take off for another place, often leaving his exhausted hosts to work out the details and write up the papers.

In addition to producing an immense body of results, Erdős contributed to mathematics in three important ways: he championed *elementary methods*, he introduced *probabilistic methods* and turned them into powerful tools in many branches of mathematics, and, perhaps above all, he gave mathematics hundreds of exciting *problems*.

Let us start with *problems*, as the name of Erdős is practically synonymous with problems. If one classifies mathematicians into "problem solvers" and "theory builders" then Erdős was the purest of problem solvers. Not that he had anything against building big machines, but, from his early youth on, his taste ran to problems; and this tendency was reinforced by his mentor, Louis Mordell, and close friends Harold Davenport, Paul Turán, Richard Rado, Alfréd Rényi, and others. Unlike Einstein, who chose physics instead of mathematics for fear of working on the "wrong" questions, Erdős was happy to be seduced by any beautiful problem that came his way. However, he not only solved problems, but also created many: as a *problem poser*, the world has not seen anybody remotely like him. He had a wonderful talent for keeping an open mind, for asking searching questions, the answers to which uncovered hidden connections. As Ernst Straus put it, Erdős was the prince of problem solvers and the undisputed monarch of problem posers. Through his problems Erdős has greatly influenced many branches of mathematics, especially combinatorics, and will continue to do so for many years to come. To indicate his estimate of the difficulty of his problems, from the 1950s onwards he attached to them monetary rewards. These rewards tended to increase in time, as it emerged that the problems were rather hard, after all. Ernst Specker was the first to claim a reward, and Endre Szemerédi collected the largest sum, $1000. It is hardly worth saying that nobody ever tackled an Erdős problem in order to earn money.

When, 65 years ago, Erdős started his career, *elementary methods* were out of fashion. Today this is even more so: the mathematical world is ruled by big theories straddling several branches of mathematics, and these big theories have had amazing successes, most notably the proof of Fermat's Last Theorem by Andrew Wiles. But Erdős believed that no matter how important sophisticated theories are, they cannot constitute all of mathematics. There are remarkably many natural questions in mathematics which one is unable to attack with sophisticated machines: rather than declaring that these problems are thereby uninteresting, we should be happy to answer them by whatever means. Erdős proved over and over again that elementary methods are frequently effective in attacking these 'untractable' problems, and that they often provide enlightening proofs of main-line results. Needless to say, an elementary proof need not be simple: often the opposite is the case.

The third major contribution of Erdős to mathematics was that he was the first to fully appreciate and exploit the power of *random methods* in order to attack a host of problems which have nothing to do with randomness and probability theory. In its simplest form, the probabilistic method is nothing but double counting, but at a higher level it is a delicate, complicated, and powerful tool. Just as every analyst automatically interchanges the order of integration, and every combinatorialist worth his salt double counts at the drop of a hat, Erdős was constantly on the lookout for opportunities to apply the probabilistic method. By now we all sense that frequently the best way of showing the existence of an object with seemingly contradictory properties is to work in an appropriate probability space, but this instinct has been instilled in us largely by Erdős, the apostle of random methods, who was the first to recognize their power and apply them repeatedly, many years before they became accepted. For many years, the probabilistic method was known as the *Erdős method.*

Paul Erdős was born into an intellectual Hungarian-Jewish family on March 26th 1913, in Budapest, amidst tragic circumstances: when his mother returned home from the hospital with the little Paul, she found that her two daughters had died of scarlet fever. Both his parents were teachers of mathematics and physics: his father was born Louis (Lajos) Engländer, but changed his name to the Hungarian Erdős ("of the forest", a fairly common name in Hungary), and his mother was born Anna Wilhelm. The following year, the first great Austro-Hungarian offensive of the First World War quickly turned into a disaster, and when the South flank was driven back through Lemberg (26–30 August), the present day Lvov in the Ukraine, many Hungarians were taken prisoner by the Russians. Among them was Lajos Erdős, who returned home from Siberia only six years later. In the absence of his father, the young Erdős was brought up by his mother and a German Fräulein. As a result of her terrible loss, Mrs. Erdős felt excessively protective towards her son throughout her life, and there was always an exceptionally strong bond between the two.

In 1919, at the end of the war, the Hungarian government could not accept the harsh demands of the victorious Entente, and in the ensuing turmoil a *Dictatorship of the Proletariat* was proclaimed. Practically the entire economy and cultural life were placed under state supervision, and everything was run by *Revolutionary Soviets*. Although many changes were introduced within a few weeks, the results did not come close to what had been planned. Béla Kun's Red Army could not withstand the relentless attack of the Romanians: the Dictatorship collapsed after four and a half months, and a counterrevolutionary terror followed. Mrs. Erdős, who had been a member of the soviet running her school, was dismissed from her job, and could never teach again. It is not surprising that this traumatic experience shaped Erdős's political outlook: throughout his life he remained sympathetic towards the left in every shape or form.

Erdős was a child prodigy: at the age of three he could multiply three digit numbers, and at the age of four he discovered negative numbers on his own. At the fashionable spa his mother took him to, he would ask the guests how old they were and tell them how many seconds they had lived.

His mother was so worried that he would catch diseases in a school that, for much of his school years, he was educated at home, mostly by his father, who taught him not only mathematics and physics, but English as well. As his father never really spoke English, having learned it from books, the young Erdős acquired a somewhat idiosyncratic pronunciation. Besides German and English, he learned French, Latin and Ancient Greek; later in life he picked up a smattering of Hebrew. Erdős spent two brief periods in school: first in the Tavaszmező Gymnasium and then in the St. Stephen Gymnasium. Many years later, when he met my father, they were astonished to discover that they had common classmates: my father joined the class just after Erdős had left it.

In addition to his parents, an important role was played by the *Journal of Mathematics and Physics for High Schools* (*Középiskolai Matematikai és Fizikai Lapok*) in nurturing and developing his interest in mathematics. The journal, founded by the young teacher Dániel Arany in 1893 as a mathematics journal (*Journal of Mathematics for High Schools*), appears ten times a year and specializes in publishing problems of various levels of difficulty, somewhat like the MONTHLY.

In a subsequent issue model solutions are published and the photographs of the best problem solvers are printed in the final issue of the year. The readers are encouraged to generalize and strengthen the results, so the journal provides an exciting introduction to mathematical research. Later a physics section was added, and this was also acknowledged in the title in 1925. Erdős and many of his later friends were avid readers of the Journal and cut their mathematical teeth on its problems. A photograph of Erdős was published in each of his high school years, and a model solution printed under the joint names of Paul Erdős and Paul Turán was his "first joint paper" with Turán, whom he met only some years later and who became one of Erdős's closest collaborators and best friends.

In 1930, at the age of 17, Paul Erdős entered the *Pázmány Péter Tudományegyetem*, the science university of Budapest, founded by Peter Pázmány, Primate of Hungary, in 1635, and soon became the focal point of a small group of extremely talented Jewish mathematicians, all studying mathematics and physics. The group included Paul Turán, Dezső Lázár, George Szekeres, Esther Klein, László Alpár, Martha Svéd, and others: they discussed mathematics not only at the university, but also in the afternoons and evenings. One of their favorite meeting places was the Statue of Anonymus, the chronicler of Béla III (1173–1196).

It was during this period that Erdős started to develop his own special language: he called a child an *epsilon*, a woman a *boss*, a man a *slave*; *Sam* (or better still, *sam*) was the U.S. and *Joe* (or *joe*) was the Soviet Union, and so on. In his words, a slave could be *captured* and later *liberated*, one could drink a little *poison* and listen to *noise*, and a mathematician could *preach*, usually to the converted.

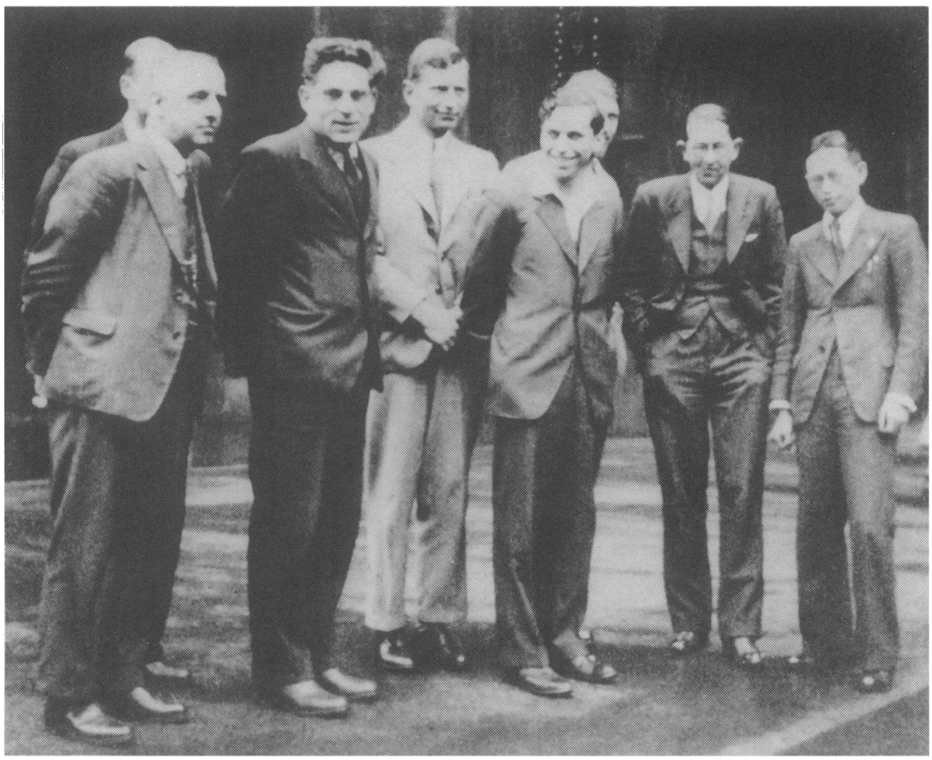

Figure 1. In Manchester, with Mordell, Heilbronn, Davenport and Ko.

At the university, Erdős learned most from Lipót (Leopold) Fejér, the great analyst, and Dénes (Dennis) König, the author of the first book on graph theory. Erdős immersed himself in number theory, which remained his love all his life, and as a second year undergraduate obtained a new proof of Chebyshev's theorem, which later prompted Nathan Fine to write the rhyme:

Chebyshev said, and I say it again,
There's always a prime between n and $2n$.

In 1934, Erdős finished university and, on the basis of his thesis on primes in arithmetic progressions, was awarded a doctorate as well. Much of the work for his thesis was done in his second year. Like most well-off Hungarians of talent, he intended to continue his studies in Germany but, as "Hitler got there first", he decided to go to England, to join Louis Mordell's exceptional group of number theorists in Manchester. On his way to Manchester, he stopped in Cambridge, where he met Harold Davenport and Richard Rado, who later became two of his closest friends.

In Manchester Erdős held various fellowships, which did not require him to teach, and he was free to do research under Mordell's guidance. In 1937, Davenport left Cambridge to join Mordell's group, and soon Erdős and Davenport became the best of friends.

I have a special reason to be grateful for the Erdős-Davenport friendship: many years later, Erdős helped me to get to Trinity College, Cambridge, precisely because Davenport was a Fellow of that college.

Figure 2. In Mordell's garden in Manchester.

In 1938 Erdős left Manchester for the Institute for Advanced Study in Princeton. In the stimulating atmosphere of the place, his talent blossomed as never before; even almost 60 years later he thought that 1938/39 was his *annus mirabilis*. He wrote outstanding papers with Mark Kac and Aurel Wintner, which practically created *probabilistic number theory*, he wrote a major paper with Paul Turán on *approximation theory*, and he solved an important problem of Witold Hurewicz in *dimension theory*. Despite the hints and verbal assurances which followed this tremendous output, his Fellowship at the Institute was not continued. The memory of this pained him even to the end, although he always added that the Institute did not have much money.

Figure 3. Sailing for America.

Figure 4. Princeton, 1939.

Without a Fellowship to support him, he was forced to embark on his travels: he visited Philadelphia, Purdue, Notre Dame, Stanford, Syracuse, Johns Hopkins, Ann Arbor, and elsewhere, and like a Wandering Scholar of the Middle Ages, he never stopped again. The war years were very hard on him, as it was difficult to hear from his parents and the news was distressing, but he kept producing mathematics at a prodigious rate. In addition to the many important papers he produced by himself, his genius for collaboration blossomed: he wrote outstanding papers with Mark Kac, Kai Lai Chung, Ivan Niven, Arye Dvoretzky, Shizuo Kakutani, Arthur Stone, Leon Alaoglu, Alfred Tarski, Irving Kaplansky, Gábor Szegő, William Feller, Fritz Herzog, George Piranian, and others.

In the fall of 1948 Erdős returned to Europe for the first time in a decade: after two months in Holland, where he worked with Nicolaas de Bruijn and Jurgen Koksma, he arrived in Budapest on 2$^{\text{nd}}$ December, to be reunited with his beloved mother. The great joy of seeing his mother was tinged with sadness, as his father had died of a stroke in 1942, and most of his relatives had been murdered in the Holocaust. In the Yalta Agreement, Hungary was given to Stalin, and the changes were already felt in the country: following a brief period of freedom and democracy, Hungary was sinking once more into a dictatorship. After a stay of about two months, Erdős left Hungary for England, before returning to the U.S. two months later.

For Erdős, 1949 was a momentous and bittersweet year. Atle Selberg found an ingenious elementary proof for his asymptotic formula concerning the distribution of primes, and after a substantial contribution by Erdős, Selberg completed an elementary proof of the Prime Number Theorem (PNT). Later both Erdős and Selberg found simpler elementary proofs. This elementary proof of the PNT was the great mathematical event of 1949, since the search for an elementary proof had been on for over 50 years, ever since 1896, when Hadamard and de la Vallée Poussin proved the PNT. There was much hope that the ideas involved in the proof would penetrate and revolutionize number theory. Subsequent events showed that this hope was unfounded. It was a great pity for Erdős that the planned back-to-back publications did not materialize, and Selberg **[92]** and Erdős **[34]** published their contributions in different journals. No doubt Erdős could have handled the delicate situation with more tact, but unquestionably he had no base motives. At the International Congress of Mathematicians at Harvard in 1950, Atle Selberg was awarded a Fields Medal, and much of the citation was about the elementary proof of the PNT.

Following the Congress, Erdős left the U.S. for Europe. After a year in Aberdeen, he spent the academic year 1951/52 at University College, London; while there, he renewed his friendship and collaboration with Harold Davenport and Richard Rado. He then returned to the U.S. and for the next two years he was mostly at Notre Dame.

The year 1954 brought a great trauma for Erdős: he left the U.S. for the International Congress of Mathematicians in Amsterdam without having obtained a reentry permit. In later life, he frequently claimed that "sam tried to starve him to death" by not allowing him to return to the American universities where he was supported. However, his claim was probably strongly colored by his desire to show that America was almost as bad as the Soviet Union. He could have gone to Amsterdam and returned to the U.S. without any difficulty, had he not insisted that he would do so with a *Hungarian passport*. The relations between the U.S. and the Hungary of Rákosi (the Hungarian Stalin) were very bad at the time and Erdős was absolutely inflexible. He left, saying that neither sam nor joe could restrict his right to travel.

In his distress, having been left with neither a country nor any means of supporting himself, he turned to Israel for support. He was received with open arms: the Hebrew University in Jerusalem offered him a job, and the state of Israel offered him a passport. He accepted the employment, but when the officials asked him whether he wanted to become an Israeli citizen, he politely refused, saying that he did not believe in citizenship. Nevertheless, from then on he visited Israel almost every year.

In the communist Hungary of the 1950s, ordinary Hungarians were not allowed to visit a Western country, not even for a short period. Although Westerners were permitted to visit Hungary, they were viewed with hostility. Thus it was a tremendous achievement when in 1955 George Alexits, a good friend of Erdős, managed to persuade the officials to permit Erdős to enter the country *and* to leave again. From then on, Erdős returned to Hungary once or twice a year, partly in order to spend more and more time with his mother, and also to collaborate with Hungarian mathematicians, especially Paul Turán and Alfréd Rényi. For many a Hungarian mathematician, Erdős was the window to the West.

As a young pupil, I first heard him lecture during one of his earlier brief visits when, in 1957, he addressed the Junior Mathematical Society in Budapest. But it was during his next visit to Hungary, in 1958, that I first came to meet him: he summoned me to the elegant hotel where he was staying with his mother, and told me numerous beautiful results and exciting problems in combinatorics, geometry, and number theory. It was impossible not to fall under his spell. His mother, whom by then most people called *Annus Néni* (Aunt Anna), was constantly by his side: they looked after each other with great affection, and she was a hostess to all his mathematical visitors. In 1961 they got to know my parents, and from then on they were frequent visitors to our house, especially for Sunday lunches. My father, who was a physician, looked after them both.

His ground-breaking work on random graphs with Alfréd Rényi started in the late fifties: in a series of brilliant papers they laid the foundation of the theory of random graphs. Their main discovery was that, for many a monotone increasing property, there is a *sharp threshold*: graphs of order n with slightly fewer edges than a certain function $f(n)$ are very unlikely to have the property, while graphs with slightly more than $f(n)$ edges are almost certain to have the property. They also discovered the *phase transition* in the component structure: for $c < 1$, in most graphs with n vertices and $\lfloor cn \rfloor$ edges every component has $O(\log n)$ vertices, while for $c > 1$ most graphs with n vertices and $\lfloor cn \rfloor$ edges have a *giant component* with at least $\gamma(c)n$ vertices, where $\gamma(c) > 0$.

The late 1950s also saw the start of much research in *transfinite combinatorics*: in 1958 he published the first of over fifty joint papers with András Hajnal, and in 1965, with Rado and Hajnal, in a difficult paper of over 100 pages they named the *giant triple paper*, he founded *partition calculus*, a detailed study of the relative sizes of large cardinals and ordinals. Erdős was rather disappointed when later "independence reared its ugly head", and many of the natural questions turned out to be undecidable.

In addition to his work with Rényi on random graphs, the highlight of the 1960s was his collaboration with András Sárközi and Endre Szemerédi on divisibility properties of sequences. This continued the work he started with Davenport in the 1930s. Another major topic of the day was *statistical group theory*: in a series of substantial papers, Erdős and Turán laid the foundations of this subject.

Erdős's brief stays in Hungary drove home the painful truth that for three decades he had seen rather little of his mother, so they started to travel together. Their first trip was to Israel in November 1964 and the second to England in 1965. They traveled all over Europe, the United States and Canada; in 1968 they even went to Australia. Erdős was truly happy then: he was with his beloved mother, his mathematics was in full bloom, and he was respected by all. If only time could have stood still and these halcyon days could have lasted longer! It was a terrible tragedy for Erdős when, in January 1971, Mrs. Erdős died during a trip to Calgary. He never recovered from the blow, and from then on was perhaps never really happy. It did not help that he lost many of his closest friends: Harold Davenport in 1969, Alfréd Rényi in 1970, and, above all, Paul Turán in 1976. One of his favorite sayings, "Pusztulunk, veszünk!" (We are perishing, vanishing!—in poor translation) seemed less and less of a joke.

Since the fifties Erdős never had anything resembling a permanent home, and even a stay of a few months was a rarity. In the mid 70s he made an exception

Figure 5. With Annus Néni.

when he came to visit me for a semester in Cambridge, where I had moved in 1969. Nevertheless, he considered Budapest his permanent base, even though he spent only a small fraction of the year there. There was only one substantial break in his visits to Hungary: when his friends from Israel were not allowed to enter Hungary for the international conference for his 60^{th} birthday, he got so upset that for three years he did not return to Hungary. While in Budapest, he liked to stay in the Guest House of the Academy, two doors away from my mother's house, and always set up his headquarters in the office of the Director of the Mathematical Institute. Most days he had a steady flow of collaborators, so that he was likely to be working at the same time on several problems, with several sets of collaborators. He was especially happy working with Vera Sós, András Hajnal, András Sárközy, Endre Szemerédi, Miklós Simonovits, and András Gyárfás.

For the last twenty years of Erdős's life, Ron Graham of AT & T Bell Laboratories provided a fixed point in his life by looking after his finances, forwarding his mail, taking care of manuscripts, and so on. When he was in the New York area, he frequently stayed with Ron Graham and Fan Chung, who arranged one of their rooms for him, but he was also keen to work with János Pach, Joel Spencer, Melvyn Nathanson, and Peter Winkler as well, to mention only some of his collaborators. He had a permanent position in Memphis, where he did much work with Ralph Faudree, Cecil Rousseau, and Dick Schelp. There were numerous other permanent ports of call: Calgary with Eric Milner, Richard Guy, and Norbert Sauer, Kalamazoo with Yousef Alavi, Atlanta with Dick Duke and Vojtěch Rödl, Lyon with Jean-Louis Nicolas, Bielefeld with Walter Deuber, Poznań with Michal

Figure 6. With Ron Graham and Peter Frankl at his 60th birthday conference at Keszthely.

Karoński, Tomasz Łuczak, and Andrzej Ruciński, Prague with Jarik Nešetřil, Athens, Georgia, with Carl Pomerance and Andrew Granville, and so on. He frequently came to see me in Cambridge and Baton Rouge; when he was in England, he always looked up the Rado family in Reading, Yael Dowker in London, and Mrs. Davenport in Cambridge.

Given his lifestyle, it is not surprising that he was a *conference-junkie*: the only reason why he would miss a conference he was interested in was that he went to another meeting. Most of us are totally exhausted if we have to attend back-to-back conferences, whereas he thrived on three or more meetings without any break. He was happy that every fifth year from his 65^{th} birthday, I organized a meeting in his honor in Cambridge, England, making sure that his birthday took place during the conference. As he liked to say, over the years he progressed from *prodigy* to *dotigy*; despite his good humor it was sad to witness this transformation. Although from his late teens he had been saying that he was old, he felt that at 80 the joke was wearing thin. Throughout 1993, the year of his 80^{th} birthday, there was a slew of international conferences dedicated to him, with the largest by far in Hungary, where most of his friends were in attendance. A beautiful film made by George Csicsery records conversations and scenes from Erdős's life in this period. Erdős did not care for honors, but was happy to receive honorary degrees from a good many places, including the ancient universities of Cambridge and Prague, and he was glad to get the Wolf Prize.

Occasionally Erdős talked of *The Book*: a transfinite book whose pages contain all the theorems and their best possible proofs. Unfortunately, went on Erdős, The Book is held by God who, being malicious towards us, only very rarely allows us to

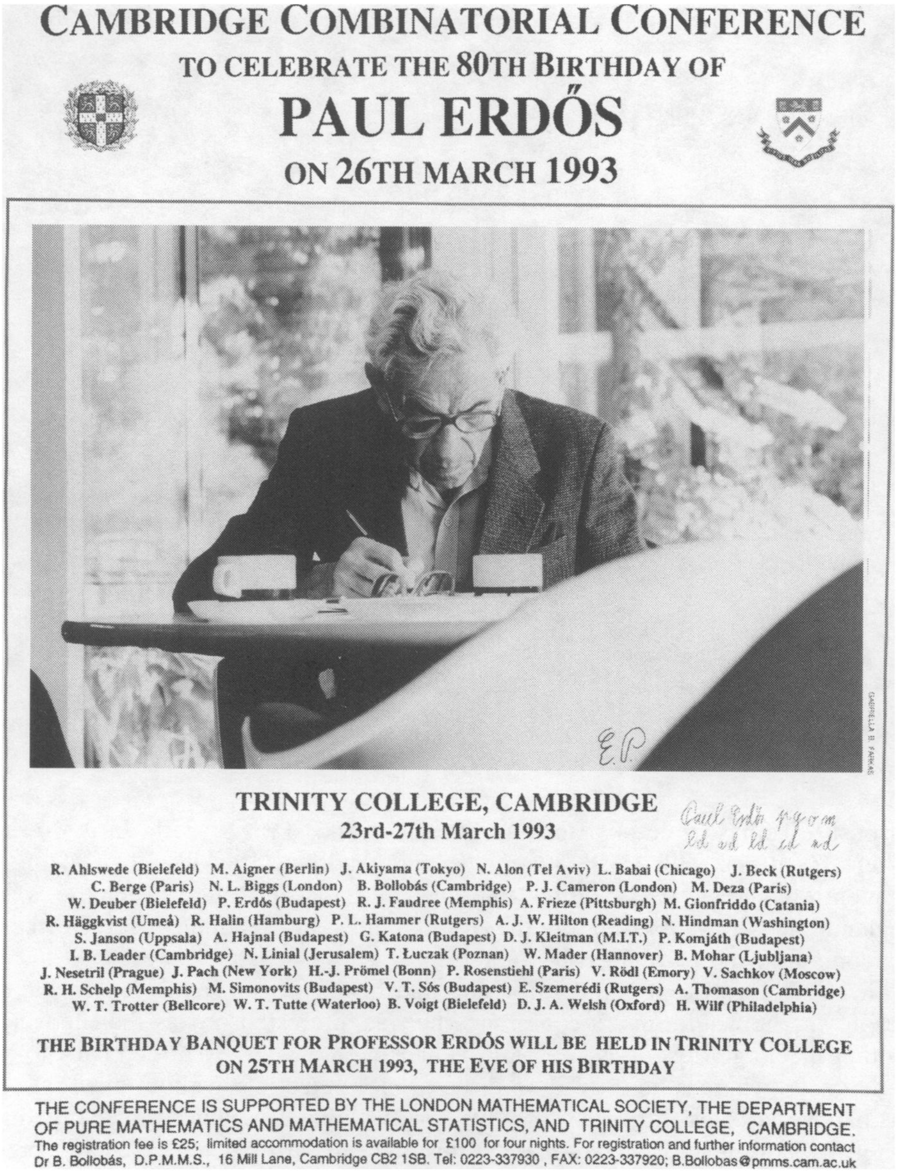

Figure 7. The poster for his 80th birthday conference in Cambridge.

catch a glimpse of a page. But when that happens, then we see mathematics in all its beauty.

In writing and talking about Erdős, The Book is frequently overemphasized: he himself always insisted that this is only a joke, which should not be taken seriously, lest it damage the mathematics we like. By its very nature, a book proof tends to be short and snappy: it is as if lightning allowed us to see some detail clearly. When it comes to substantial results, the best we can feel is that the *global idea* of the proof is from The Book.

Nevertheless, I shall present some half-a-dozen proofs Erdős himself considered to be straight from The Book; all these proofs have close ties to Erdős. Because a book proof easily becomes part of the mathematical consciousness, so that eventually every good schoolboy knows it, after many years a book proof looks much less striking than when it was first discovered. This certainly applies to the examples below.

Let us start with the following result of H.A. Schwarz. Given an acute angled triangle, the feet of the perpendiculars form the unique inscribed triangle with minimal perimeter length.

As an undergraduate in Berlin at the end of the last century, Leopold Fejér found the following proof. Let ABC be the triangle with acute angles, and let $A'B'C'$ be a triangle of minimal perimeter length, with A' on BC, B' on CA and C' on AB. Let C'' be the reflection of A' in AB, and B'' its reflection in AC. Then $A'B' = B'B''$ and $A'C' = C'C''$, so the perimeter length of $A'B'C'$ is precisely the length of the broken line $B''B'C'C''$. As we may choose for B' the intersection B^* of the segments $B''C''$ and AC, and for C' the intersection C^* of the segments $B''C''$ and AB, we find that $B' = B^*$ and $C' = C^*$. Hence the perimeter length of $A'B'C'$ is precisely the length of the segment $B''C''$ (see Figure 8).

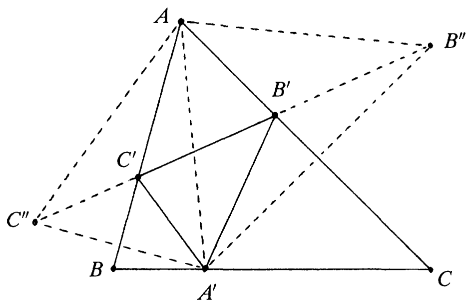

Figure 8. The perimeter length of $A'B'C'$ is $B''C''$.

Now the angle $B''AC''$ is twice the angle CAB, and $AB'' = AC'' = AA'$. Hence $B''C''$ is minimal precisely when A' is the point on BC nearest to A, that is the foot of the perpendicular from A to BC. By symmetry, B' and C' are also the feet of the corresponding perpendiculars, and we are done.

As our second example, consider the following conjecture Erdős made as an undergraduate in 1933: among n distinct points in the plane, not all on a line, there are two, whose line contains no other point.

He expected this to be easy to prove and was rather surprised that he did not succeed. However, his close friend, Tibor Grünwald (later Gallai) found the following beautiful proof. Consider all lines that contain at least two of our points, and take one, say l, whose distance from another of our points is minimal. Then l contains precisely two of our points.

The brief formulation above was intended to emphasize that the proof is from The Book: having taken a pair (l, P) at minimal distance, we are done. Nevertheless, it is not unreasonable to demand a much more detailed proof, so we give one here. No doubt, other readers will feel that this second presentation over-explains the proof and spoils its beauty.

Let $P_1, P_2, \ldots, P_n$ be the points, and $l_1, l_2, \ldots, l_m$ the lines, each containing at least two of the points. Let (P_i, l_j) be such that P_i is not on l_j, and the distance $d(P_i, l_j)$ is minimal (see Figure 9).

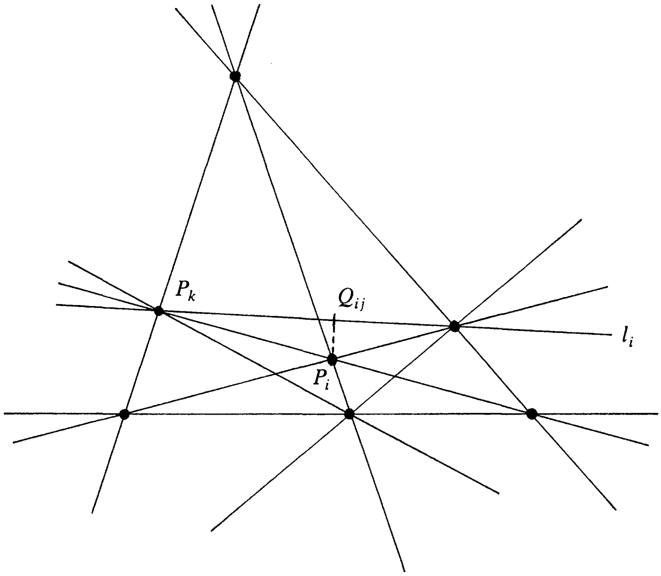

Figure 9. A pair (P_i, l_j) at minimal distance.

Let Q_{ij} be the foot of the perpendicular from P_i to l_j, and let P_k be one of our points on l_j. We claim that the closed straight line segment $[Q_{ij}, P_k]$ contains no other point P_l. Indeed, if $P_l \neq P_k$ were on $[Q_{ij}, P_k]$ then the distance of P_l from the line P_iP_k would be less than $d(P_i, l_j)$, contradicting our choice.

By our claim, each of the two half-lines of l_j determined by Q_{ij} contains at most one of our points, so l_j contains only two of our points.

Years later, Erdős submitted the problem to the MONTHLY, where in 1943 it was published as Problem 4065. Amazingly, the published proof was not that of Gallai! L.M. Kelly noticed that the problem was first posed by Sylvester in the *Educational Times* in 1893, as Mathematical Question 11851, so by now the result tends to be known as the Gallai-Sylvester theorem. It is just a trivial step to strengthen it to the assertion that n distinct points in the plane, not all on a line, determine at least n *ordinary* lines, that is, lines through precisely two of the points. For many other extensions, see Motzkin **[80]** and Pach and Agarwal [**81**, Ch. 11]. As it turned out, Motzkin also made similar conjectures in 1933.

The third example is a little result of another member of the Anonymus circle, Dezső Lázár **[75]**. To every real number x, assign a finite set of real numbers, $\Phi(x)$, not containing x. Call a set $S \subset \mathbb{R}$ *independent* if $x \in \Phi(y)$ does not hold for $x, y \in S$, that is if $S \cap \Phi(S) = \emptyset$. Then there is an independent set $S \in \mathbb{R}$ whose cardinality is that of the continuum.

The problem arose from Turán's work on interpolation theory, and Géza Grünwald had proved the weaker assertion that there is an infinite S. Erdős communicated the proof to John von Neumann, back in Budapest for the summer, who liked it so much that he invited Lázár to submit a brief note to *Compositio Mathematica*.

Here is then Lázár's proof. For every $x \in \mathbb{R}$ there is an interval $I(x) = [r_1(x), r_2(x)]$ with rational endpoints such that $x \in I(x)$ and $I(x) \cap \Phi(x) = \emptyset$. As there are only countably many intervals with rational endpoints, for some interval J the cardinality of the set $S = \{x \in \mathbb{R} : I(x) = J\}$ is that of the continuum. Since $S \subset J$ and $\Phi(S) \cap J = \emptyset$, the set S is independent.

For the fourth example, we jump to 1945, when Anning and Erdős [**3**] proved that for every n one can find n points on a circle such that their distances are all integral, but it is impossible to find infinitely many points in the plane not all on a line such that all distances are integers.

To prove the first assertion, let $p_1, p_2, \ldots$ be the primes of the form $4k + 1$, so that there are $1 \leq a_i < b_i \leq p$ such that $p_i^2 = a_i^2 + b_i^2$. Let (x_i, y_i) be a point on the circle $x^2 + y^2 = 1/4$, which is at distance a_i/p_i from $(1/2, 0)$ and so b_i/p_i from $(-1/2, 0)$. Then the sequence of points $(-1/2, 0)$, $(1/2, 0)$, (x_1, y_1), $(x_2, y_2), \ldots$ is such that the distance between any two is rational. Indeed, this is an immediate consequence of Ptolemy's theorem since for any four concyclic points $(-1/2, 0)$, $(1/2, 0)$, (x_i, y_i), (x_j, y_j), five of the distances are rational by our choice of the points, so the sixth distance, that between (x_i, y_i) and (x_j, y_j), is also rational. By enlarging the radius of the circle we can obtain n points on a circle with integral distances.

Although this argument is certainly pretty, it is not nearly as striking as the proof Erdős gave of the second part a little later [**30**]. As some readers may not have seen this proof, to give them a chance to think about the problem a little and thereby better appreciate the proof Erdős gave, we postpone its presentation to the end of this list.

Now let us turn to a beautiful early application of the probabilistic method. Starting with one of his earliest papers [**54**], written with George Szekeres in 1935, Erdős was a great devotee of Ramsey's theorem, which he generalized, applied, and popularized on numerous occasions throughout his life; it also led him to the partition calculus. A special case of Ramsey's theorem claims that, for every natural number s, if n is large enough then every red-blue coloring of the edges of K_n, a complete graph on n vertices, contains a monochromatic K_s, that is a complete graph K_s, with all edges red or all edges blue. The smallest value of n that will do is denoted by $R(s)$, so that $R(s) - 1$ is the maximal value of n for which some red-blue coloring of K_n has no monochromatic K_s. In particular, $R(1) = 1$, $R(2) = 2$ and $R(3) = 6$. The upper bound Ramsey had given for $R(s)$ was greatly improved by Erdős and Szekeres [**54**] in 1935, when they showed that

$$R(s) \leq \binom{2s-2}{s-1} \leq 2^{2s-2}.$$

Although over sixty years have passed since then, this very simple bound has hardly been improved: the best result to date is due to Thomason [**97**], who proved that there is an absolute constant A such that

$$R(s) \leq e^{A\sqrt{\log s}} \frac{1}{s} 2^{2s}.$$

But what about a lower bound? Polynomial lower bounds are easy to come by, but as one of the first unexpected applications of his probabilistic method, Erdős [**33**] obtained an exponential lower bound. In [**36**] and [**37**] he proved more sophisticated results in the same vein, about off-diagonal Ramsey numbers, and about graphs of large chromatic number and large girth. Instead of *constructing* an appropriate graph, he simply considered a random red-blue coloring of (the edges of) K_n. The expected number of monochromatic K_s subgraphs is $\binom{n}{s}2^{-\binom{s}{2}+1}$, since we have $\binom{n}{s}$ choices for the vertex set and the probability that all $\binom{s}{2}$ edges of a K_s subgraph get the same color is $2^{-\binom{s}{2}+1}$: having colored one edge, the remaining $\binom{s}{2} - 1$ edges must get the same color. Now if

$$\binom{n}{s}2^{-\binom{s}{2}+1} < 1$$

then *some* coloring of K_n has *no* monochromatic K_s, so $R(s) > n$. In particular, $R(s) > cs2^{s/2}$ for some $c > 0$.

The next result, proved by Erdős, Ko, and Rado [**46**] in 1961, is of fundamental importance in extremal combinatorics, and was the starting point of much research. Let $2 \leq r < n/2$, and let $\mathscr{A}$ be a collection of r-subsets of $[n] = \{1, 2, \dots, n\}$. Suppose that $\mathscr{A}$ is an *intersecting* family, that is $A \cap B \neq \emptyset$ for all $A, B \in \mathscr{A}$. Then

$$|\mathscr{A}| \leq \binom{n-1}{r-1}.$$

The family $\mathscr{A} = \{A \subset [n]\colon 1 \in A, |A| = r\}$ shows that the inequality cannot be improved; in fact, this is the unique extremal family.

The Erdős-Ko-Rado theorem has many beautiful proofs: here we present one from The Book, given by Katona [**70**]. Arrange the elements of $[n]$ in a *cyclic order*. How many of the sets $A \in \mathscr{A}$ can form *intervals*, i.e., sets consisting of consecutive elements? At most r, since if $(a_1, a_2, \dots, ,a_r)$ is one of these intervals then for every i, $1 \leq i \leq r - 1$, at most one interval from $\mathscr{A}$ contains precisely one of a_i and a_{i+1}.

Now the probability that a set $A \in \mathscr{A}$ is an interval in a *random* cyclic order is clearly $r!(n-r)!/(n-1)!$, so

$$|\mathscr{A}| \leq r\frac{(n-1)!}{r!(n-r)!} = \binom{n-1}{r-1},$$

as claimed.

In 1946, Erdős [**31**] conjectured in the MONTHLY that the number of unit distances among n points in the plane is at most $n^{1+o(1)}$ and proved that this number is at most $cn^{3/2}$. After various improvements by Józsa and Szemerédi [**68**], and Beck and Spencer [**6**], the best bound to date, $cn^{4/3}$, was proved by Spencer, Szemerédi, and Trotter [**93**]. Recently, Székely [**95**] found a beautiful proof: it was probably the last proof Erdős judged to have come from The Book.

The proof is based on a result of Leighton [**76**] and Ajtai, Chvátal, Newborn, and Szemerédi [**1**], stating that every planar graph with n vertices and $m \geq 4n$ edges has crossing number at least $m^3/100n^2$, i.e., that every drawing of the graph in the plane contains at least this many crossing pairs of edges. Here the constant 100 is unimportant and is easily improved, but m^3/n^2 cannot be improved.

Given n points in the plane with m unit distances, let us draw a multigraph in the plane with this set of vertices as follows. Draw a unit circle about every point that has at least three points at unit distance from it. Join the consecutive points on the unit circles by the circular arcs. In this way some pairs of points are joined by two circular arcs: discarding one of them we obtain a graph drawn in the plane with at least $m - n$ edges. The number of crossings of this graph is at most

$n(n-1)$ since any two circles intersect in at most two points. Therefore either $m - n \leq 4n$ and so $m \leq 5n$, or else $(m-n)^3/100n^2 \leq n^2$, and so $m \leq 6n^{4/3}$.

Now, as our final example, we present the Book proof from **[30]** we have postponed. This proof is so short that in his review of the paper, Irving Kaplansky **[69]** simply *reproduced the entire paper*! We can do no better. Here is Kaplansky's review:

> In the note under the same title [Erdős is referring to his paper with Anning **[3]**] it was shown that there does not exist in the plane an infinite set of noncollinear points with all mutual distances integral.
>
> It is possible to give a shorter proof of the following generalization: if A, B, C are three points not in line and $k = [\max(AB, BC)]$, then there are at most $4(k+1)^2$ points P such that $PA - PB$ and $PB - PC$ are integral. For $|PA - PB|$ is at most AB and therefore assumes one of the values $0, 1, \ldots, k$, that is, P lies on one of $k+1$ hyperbolas. Similarly P lies on one of the $k+1$ hyperbolas determined by B and C. These (distinct) hyperbolas intersect in at most $4(k+1)^2$ points. An analogous theorem clearly holds for higher dimensions.

This Book proof fared no better than Gallai's: In a collection of papers of Erdős **[38]**, the original paper of Anning and Erdős **[3]** is reproduced, but not the brief paper just quoted.

No matter how beautiful these proofs are, the reputation of Erdős rests on his more substantial results. It would be wonderful to concentrate on a dozen or so papers, but that would be most unfair to a mathematician with about 1500 papers. He himself always found it impossible to select his "Top Ten": whenever he tried to, he ended up with at least forty.

His first paper **[19]**, written as a first-year undergraduate, was on Bertrand's postulate: *for every* $n \geq 1$, *there is a prime* p *satisfying* $n < p \leq 2n$. This was first proved by Chebyshev, and Ramanujan **[84]** gave a considerably simpler proof of it in 1919. Later the great German mathematician Edmund Landau [**74**, pp. 66–68] gave a particularly simple proof of the assertion that for some $q > 1$ there is always a prime between n and qn. Landau did not give an estimate for q, but it was clear that his proof did not permit q to be taken to be 2. By sharpening Landau's argument, and concentrating on the prime divisors of $\binom{2n}{n}$, Erdős gave a simple and elementary proof of Bertrand's postulate.

Erdős further developed these ideas in **[24]** in order to apply them to primes in arithmetic progressions. In this paper the nineteen year old Erdős gave elementary proofs of extensions of some very recent results of Breusch **[12]**. Independently of Erdős, similar results were obtained by Ricci **[87]**, **[88]**.

Writing $\sigma(n)$ for the sum of the positive divisors of a natural number n, we call n *perfect* if $\sigma(n) = 2n$, *abundant* if $\sigma(n) \geq 2n$, and *deficient* if $\sigma(n) < 2n$. Several of the early papers of Erdős concerned abundant numbers, especially their density $\lim_{x \to \infty} \tilde{A}(x)/x$, where $\tilde{A}(x)$ is the number of abundant numbers not exceeding x. Schur conjectured that the abundant numbers have positive density. Using Fourier analysis, this was proved by Behrend, Chowla, and Davenport; Erdős **[20]** gave an elegant elementary proof.

A number is *primitive abundant* if it is abundant but every divisor of it is deficient. Denoting by $A(x)$ the number of primitive abundant numbers not exceeding x, Erdős [21] proved that for some positive constants one has

$$xe^{-c_1(\log x \log\log x)^{1/2}} < A(x) < xe^{-c_2(\log x \log\log x)^{1/2}}. \qquad (1)$$

From (1) it follows that the sum of the reciprocals of primitive abundant numbers is convergent, and that implies that $\lim_{x\to\infty}\tilde{A}(x)/x$ exists. Fifty years later, Ivič [66] simplified the proof of (1) and obtained better constants; recently Avidon [4] proved that here any $c_1 > \sqrt{2}$ and $c_2 < 1$ will do, provided x is large enough. However, it is still open, whether

$$A(x) = xe^{-(c+o(1))(\log x \log\log x)^{1/2}}$$

for some c, $1 \le c \le \sqrt{2}$. Erdős [40] also wondered whether one could prove that $\tilde{A}(2x)/\tilde{A}(x) \to 2$ as $x \to \infty$.

The study of abundant numbers led Erdős to a variety of problems concerning real-valued additive arithmetical functions, that is, to functions $f: \mathbb{N} \to \mathbb{R}$ such that $f(ab) = f(a) + f(b)$ whenever a and b are relatively prime. Hardy and Ramanujan [63] proved that if $g(n) \to \infty$ then

$$|\nu(n) - \log\log n| < g(n)(\log\log n)^{1/2} \qquad (2)$$

holds for almost every n, where $\nu(n)$ denotes the number of distinct prime factors of n. In other words, the density of n satisfying (2) is 1. Erdős [25] extended this result by showing that the median is about $\log\log n$: the number of integers $m \le n$ for which $\nu(m) > \log\log n$ is $\frac{1}{2}n + o(n)$.

In 1934, Turán [98] gave a brilliant elementary proof of this theorem. To be precise, Turán proved considerably more, namely that:

$$\sum_{m=1}^{n} \{\nu(m) - \log\log n\}^2 = n\log\log n + o(n\log\log n). \qquad (3)$$

What (3) says is that if we turn $[n] = \{1, 2, \ldots, n\}$ into a probability space by giving each $m \in [n]$ probability $1/n$, then ν becomes a *random variable* with mean and variance $(1 + o(1))\log\log n$. In particular, (2) is an immediate consequence of (3).

Surprisingly, G.H. Hardy, one of the greatest mathematicians of the day, did not appreciate the proof, which he acknowledged as a curiosity. The young Erdős, on the other hand, was very receptive, and was ready to develop the ideas further. In particular, by making use of Turán's ideas, he proved [22], [26] that if $f(m)$ is any non-negative arithmetical function then, for every constant c,

$$\frac{1}{n}|\{m \le n: f(m) \le c\}|$$

tends to a limit as $n \to \infty$. As a slight extension of his theorem, Turán also proved [22] that if f is a non-negative additive arithmetical function satisfying

$$\sum_{p\le n} \frac{f(p)}{p} = \psi(n) \to \infty,$$

where the summation is over primes p not exceeding n, then, for every $\varepsilon > 0$, there are only $o(n)$ integers $m \le n$ for which

$$|f(m) - \psi(n)| > \varepsilon\psi(n).$$

The case when $f(p^\alpha) = 1$ for every prime power is precisely the Hardy-Ramanujan theorem.

In 1939 Erdős joined forces with Kac to write a ground-breaking paper (see [44] for an announcement and [45] for a detailed exposition) on additive arithmetical functions, strongly extending the results above. With this paper, Erdős, the number theorist, and Kac, the probabilist, founded probabilistic number theory, although the subject did not really take off until several years later. Among others, they proved that if a bounded real-valued additive arithmetical function $f(m)$ satisfies $\Sigma_p f(p)^2/p = \infty$ then, for every fixed $x \in \mathbb{R}$,

$$\lim_{m \to \infty} A_x(m)/m = \Phi(x),$$

where $A_x(n)$ is the number of positive integers $m \leq n$ satisfying

$$f(m) < \sum_{p \leq n} f(p)/p + x\left(\sum_{p \leq n} f(p)^2/p\right)^{1/2}.$$

Here, as usual,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$$

is the *standard normal distribution.* In other words, under very mild conditions, the arithmetical function $f(m)$ satisfies the Gaussian law of error!

Given a real-valued function $f(m)$ and a real number c, denote by $N(f; x, n)$ the number of integers $m \leq n$ at which $f(m) < x$. If there is a monotone increasing function σ: $\mathbb{R} \to [0, 1]$ with $\lim_{x \to -\infty} \sigma(x) = 0$ and $\lim_{x \to \infty} \sigma(x) = 1$, such that at every point of continuity x of σ we have

$$\lim_{n \to \infty} N(f; x, n)/n = \sigma(x),$$

then f is said to have an *asymptotic distribution function*, σ. Culminating in **[27]**, Erdős proved that an additive arithmetical function $f(m)$ has an asymptotic distribution function, provided

$$\sum_p \frac{f'(p)}{p} \quad \text{and} \quad \sum_p \frac{(f'(p))^2}{p} \tag{4}$$

both converge, where $f'(p) = f(p)$ for $|f(p)| \leq 1$ and $f'(p) = 1$ otherwise. Also, if $\Sigma_{f(p) \neq 0} 1/p$ diverges then the distribution function is continuous. Following his work with Kac, Erdős proved with Wintner **[57]** that the convergence of the series in (4) is also necessary for the existence of a distribution function.

Some years later, Erdős **[32]** returned to the problem, and considered the case when the second series in (4) is divergent. He proved the stunning result that if $f(p) \to 0$ as $p \to \infty$ and $\Sigma_p (f'(p))^2/p = \infty$ then the distribution function of $F(m) = f(m) - \lfloor f(m) \rfloor$ is x. (To be precise, the distribution function $\sigma(x)$ is 0 for $x \leq 0$, x for $0 \leq x \leq 1$, and 1 for $x \geq 1$.) For many related results, see the two-volume treatise on probabilistic number theory by Elliot **[16]**, **[17]**.

The last paper Erdős wrote in Budapest before leaving for Manchester, concerned another favorite topic of his, the difference between consecutive primes. Writing p_n for the n^{th} prime, he proved **[23]** that for some $c > 0$ and infinitely many n one has

$$p_{n+1} - p_n > \frac{c \log n \log\log n}{(\log\log\log n)^2}.$$

As Erdős used to say, a little later Rankin **[85]** smuggled in a factor $\log\log\log\log n$, so that

$$p_{n+1} - p_n > \frac{c \log n \log\log n \log\log\log\log n}{(\log\log\log n)^2} \tag{5}$$

for some $c > 0$ and infinitely many n. The original value was improved by Schönhage **[91]** and later by Rankin **[86]**. Recently, Maier and Pomerance **[79]** proved (5) with a slightly larger constant: this is the best result to date. Although this improvement given by Maier and Pomerance seems to be slight, they introduced new tools, which may lead to substantial improvements.

Again as he used to say, "somewhat rashly" Erdős **[39]** offered \$10,000 for a proof that (5) holds for every c. Most unusually for him, in **[41]** he *reduced* the offer to \$5000, and instead offered \$10,000 for a proof that

$$p_{n+1} - p_n > (\log n)^{1+\varepsilon} \tag{6}$$

holds for some $\varepsilon > 0$ and infinitely many values of n. As he remarked, a proof of (6) would have actually cost him \$15,000! How sad that by the time **[41]** appeared he was not with us, so these offers became void!

Some time in the early 1800s, it was conjectured that the product of two or more consecutive integers is never a square, cube or any higher perfect number. To be precise, the equation

$$(n+1)(n+2)\cdots(n+k) = x^l \tag{7}$$

has no solution in integers with $k, l \geq 2$ and $n \geq 0$. In 1939, Erdős **[28]**, **[29]** and Rigge **[89]** proved the conjecture for $l = 2$, and they showed also that for any $l \geq 2$ there are at most finitely many solutions to (7). Later Erdős **[35]** found a different proof of this result. By making use of these ideas in **[35]**, Erdős and Selfridge **[51]** proved the full conjecture. Although the work was done in the mid-sixties, the paper was published only in 1975.

In 1940 Turán **[99]** proved a beautiful result concerning graphs, vaguely related to Ramsey's theorem: every red-blue coloring of a complete graph with many red edges contains a large red complete subgraph. To be precise, *for* $3 \leq r \leq n$, *every graph of order n that has more edges than any* $(r-1)$*-partite graph of order n contains a complete graph of order r*. With this result as the starting point, Erdős and his collaborators founded a large and lively branch of combinatorics, *extremal graph theory*. In order to formulate the basic problem of extremal graph theory, we write $|G|$ for the *order* (number of vertices) and $e(G)$ for the *size* (number of edges) of a graph G. Given graphs G and H, $H \subset G$ means that H is a *subgraph* of G. Let F be a fixed graph, usually called the *forbidden graph*. Set

$$ex(n; F) = \max\{e(G) \colon |G| = n \text{ and } F \not\subset G\}$$

and

$$EX(n; F) = \{G \colon |G| = n,\ e(G) = ex(n; F), \text{ and } F \not\subset G\}.$$

We call $ex(n; F)$ the *extremal function*, and $EX(n; F)$ the *set of extremal graphs* for the *forbidden graph* F. Then the first problem of extremal graph theory is to determine, or at least estimate, $ex(n; F)$ for a given graph F and, if possible, to determine $EX(n; F)$. A natural extension of this problem is to exclude several forbidden graphs, i.e., to determine the functions $ex(n; F_1, \ldots, F_k)$ and $EX(n; F_1, \ldots, F_k)$ for a finite family $F_1, \ldots, F_k$ of forbidden graphs.

Writing $T_k(n)$ for the unique k-partite graph of order n and maximal size (so that $T_k(n)$ is the *k-partite Turán graph* of order n), Turán proved, in fact, that $EX(n; K_r) = \{T_{r-1}(n)\}$, i.e., $T_{r-1}(n)$ is the *unique* extremal graph, and so $ex(n; K^r) = t_{r-1}(n)$, where $t_{r-1}(n) = e(T_{r-1}(n))$ is the size of $T_{r-1}(n)$.

As Erdős frequently said, he came very close to founding extremal graph theory *before* Turán proved his theorem: in 1938, in connection with sequences of integers no one of which divided the product of two others, he proved that for a *quadrilateral* C^4 we have $ex(n; C^4) = O(n^{3/2})$. However, at the time he failed to see the significance of problems of this type: he was "blind".

The importance of Turán's theorem is greatly enhanced by the fact that it is not very far from the *fundamental theorem of extremal graph theory*, proved by Erdős and Stone [53]. By Turán's theorem, the maximal size of a K_r-free graph of order n is about $(r-2)\binom{n}{2}/(r-1)$; in fact, trivially,

$$\frac{r-2}{r-1}\binom{n}{2} \le t_r(n) \le \frac{r-2}{r-1}\frac{n^2}{2}.$$

Writing $K_r(t)$ for the *complete r-partite graph with t vertices in each class*, so that $K_r(t) = T_r(rt)$, Erdős and Stone proved in 1946 that if $r \ge 2$, $t \ge 1$ and $\epsilon > 0$ are fixed and n is sufficiently large then *every graph of order n and size at least* $((r-2)/(r-1)+\epsilon)\binom{n}{2}$ *contains a* $K_r(t)$. In other words, *even* ϵn^2 *more edges* than can be found in a Turán graph *guarantee not only a* K_r *but a "thick"* K_r, one in which every vertex has been replaced by a group of t vertices.

The significance of the paper is that not only does it give much information about the *size* of extremal graphs, but it is also the starting point for the study of the *structure* of extremal graphs. If F is a non-empty r-chromatic graph, i.e., $\chi(F) = r \ge 2$, then, precisely by the definition of the chromatic number, F is not a subgraph of $T_{r-1}(n)$, so $ex(n; F) \ge t_{r-1}(n) \ge (r-2)\binom{n}{2}/(r-1)$. On the other hand, $F \subset K_r(t)$ if t is large enough (say, $t \ge |F|$), so if $\epsilon > 0$ and n is large enough then

$$ex(n; F) < \left(\frac{r-2}{r-1}+\epsilon\right)\binom{n}{2}.$$

In particular, if $\chi(F) = r \ge 2$ then

$$\lim_{n\to\infty} ex(n; F)\Big/\binom{n}{2} = \frac{r-2}{r-1},$$

that is, the asymptotic density of the extremal graphs with forbidden subgraph F is $(r-2)/(r-1)$. Needless to say, the same argument can be applied to the problem of forbidding any finite family of graphs: given graphs $F_1, F_2, \dots, F_k$, with $\min \chi(F_i) = r \ge 2$, we have

$$\lim_{n\to\infty} ex(n; F_1, \dots, F_k)\Big/\binom{n}{2} = \frac{r-2}{r-1}.$$

In an important series of papers starting in 1966, Erdős and Simonovits went considerably further than noticing this instant consequence of the Erdős-Stone theorem. Among other results, Erdős and Simonovits proved [52] that if $G \in EX(n; F)$, with $\chi(F) = r \ge 2$, then G can be obtained from $T_{r-1}(n)$ by adding and deleting $o(n^2)$ edges. Later this was refined to several results concerning the *structure* of extremal graphs.

Returning to the Erdős-Stone theorem itself, let us remark that Erdős and Stone also gave a bound for the *speed* of growth of t for which $K_r(t)$ is guaranteed to be a subgraph of every graph with n vertices and at least $((r-2)/(r-1)+\epsilon)\binom{n}{2}$ edges. Thirty years later, Erdős and I [**9**] gave a substantially better bound for the speed, which was essentially best possible. Further refinements were given in [**10**], [**14**] and [**11**].

Starting in the 1940s, Erdős frequently applied random graphs to extremal problems, but it was only in the late 1950s that he embarked, with Rényi, on a *systematic* study of random graphs. Here let us mention only one of their results, the one concerning the emergence of the 'giant component.' Let us write $G_{n,M}$ for a *random graph* on n distinguishable vertices, with M edges. Thus every graph with M edges on these n vertices has the same probability, $1/\binom{N}{M}$, where $N=\binom{n}{2}$. Erdős and Rényi [**50**] proved that if $M(n)=\lfloor cn \rfloor$ for some constant $c>0$ then, with probability tending to 1, the largest component of $G_{n,M}$ is of order $\log n$ if $c<\frac{1}{2}$, it jumps to order $n^{2/3}$ if $c=\frac{1}{2}$, and it jumps again, this time right up to order n if $c>\frac{1}{2}$. Quite understandably, Erdős and Rényi considered this phenomenon to be one of the most striking features of random graphs.

By now, all this is well known, especially since we know of similar phenomena in percolation theory, but in 1960 this was a striking discovery indeed. In fact, for over two decades not much was added to our knowledge of this *phase transition*. The investigations were reopened in 1984 in [**7**] with the main aim of deciding what happens around $M=\lfloor n/2 \rfloor$; in particular, what *scaling*, what *magnification* we should use to see the giant component growing continuously. Among other properties of the phase transition, it was shown in [**7**] that if $M=n/2+s$ and $s=o(n)$ but slightly larger than $n^{2/3}$ then, with probability tending to 1, there is a unique largest component, with about $4s$ vertices, and the second largest component has no more than $(\log n)n^2/s^2$ vertices. Thus, in a rather large range, on average every new edge adds four new vertices to the giant component!

With this result, the floodgates opened, and many more precise studies of the behavior of the components near the point of phase transition were published, notably by Kolchin [**71**], Stepanov [**94**], Flajolet, Knuth and Pittel [**58**], Łuczak [**77**], and others. The crowning achievement is a monumental paper by Janson, Knuth, Łuczak, and Pittel [**67**], which gives very detailed information about the random graph $G_{n,M}$ near its phase transition. Łuczak, Pittel, and Wierman [**78**] studied the structure of the random graph precisely at the critical point.

Rather than giving more results in number theory, combinatorics, probability theory, or any of the other areas closely associated with Erdős, let us present some results in fields tackled by Erdős only occasionally.

In the 1930s much research was done on the dimension of product spaces [**83**], [**64**], [**65**]. In those days mostly one kind of dimension was studied, that introduced by Menger and Urysohn; now it is called the *small inductive dimension*, and is denoted by *ind*. It is defined as follows: $ind X=-1$ if and only if $X=\emptyset$. For $n=0,1,\dots,$ we define $ind X \le n$ inductively: if for every $x\in X$ and every neighborhood U of x, there is a neighborhood V of x that is contained in U and whose boundary has dimension at most $n-1$ then $ind X\le n$. Also, $ind X=n$ if $ind X\le n$ and $ind X\le n-1$ does not hold. Finally, $ind X=\infty$ if $ind X\le n$ fails for every natural number n; see [**18**] for an introduction to dimension theory. It is rather trivial that

$$ind(X \times Y) \leq indX + indY, \tag{8}$$

so the natural question arises whether we have equality in (8). By constructing two compact spaces of dimension 2 whose product has dimension 3, Pontrjagin **[83]** proved that we may have strict inequality in (8). Nevertheless, the problem remained open whether in (8) we have to have equality for spaces of dimension 1. In 1939, while writing his fundamental book on dimension theory with Wallman **[65]**, Hurewicz encouraged Erdős to work on the problem, who promptly solved it, proving that there is a separable complete metric space X with $ind(X \times X) = indX = 1$. In fact, Erdős proved that for X we can take a very simple subset of the Hilbert space l_2, namely the closure of

$$\{x = (x_i)_1^\infty \in l_2 \colon x_i = 1/n_i,\ n_i \in \mathbb{Z},\ i = 1,2,\dots\}.$$

Similarly, the set of rational points of l_2 also has dimension 1. Clearly, not only do we have $ind(X \times X) = indX$, but even $X \times X \cong X$, so it is rather surprising that $indX = 1$.

It is easily seen that an algebraically closed field of characteristic 0 is determined (up to isomorphism) by its cardinality. For real-closed fields this is not the case: an additional invariant is the order type of the field. In 1955, Erdős, Gillman, and Henriksen **[43]** gave a characterization theorem for real-closed fields. As a consequence of this result they proved, among others, that, assuming the continuum hypothesis, all hyper-real fields of cardinality of the continuum are isomorphic. (A *hyper-real* field is a real-closed field of the form $C(X)/M$, where $C(X)$ is the ring of all continuous functions on a completely regular space X, and M is a maximal ideal.) The authors made use of the following beautiful result on almost disjoint sets Erdős that proved in 1934, but published only in **[43]**: For every infinite set X is of cardinality m, there exists a set of more than m subsets of X, each of cardinality m, such that the intersection of any two has cardinality less than m.

Throughout his career, Erdős was ready to apply Ramsey's theorem and its extensions. In 1976, he found an exciting application in the theory of Banach spaces. An infinite matrix of reals $A = (a_{ij})_{i,j \in \mathbb{N}}$ is a *regular method of summability* if for every sequence $(e_i)_{i \in \mathbb{N}}$ of elements of a Banach space X, converging in norm to an element e, the sequence $e_i' = \sum_{j=1}^\infty a_{ij} e_j$ also converges to e. Call a sequence $(x_i)_{i \in \mathbb{N}}$ *A-summable* if $x_i' = \sum_{j=1}^\infty a_{ij} x_j$ converges in norm. Erdős and Magidor **[47]** proved that, given a regular method of summability A, and a bounded sequence $\mathbf{x} = (x_i)_{i \in \mathbb{N}}$ in a Banach space, there is an infinite subsequence $\mathbf{x}' = (x_{i_k})_{k \in \mathbb{N}}$ such that either every infinite subsequence of $\mathbf{x}'$ is A-summable or no infinite subsequence of $\mathbf{x}'$ is A-summable. The proof is based on the Galvin-Prikry **[60]** partition theorem from set theory. The Erdős-Magidor theorem was the starting point of much research; see, e.g., **[82]**.

Every mathematical paper about Erdős should present some of his unsolved problems. We mentioned one or two; now let us give four more. The first is relatively unknown, but the second two are among his best known problems; the last carried the largest award Erdős ever offered. Erdős himself wrote many papers about his unsolved problems, including **[39]**, **[41]**, **[42]**. Recently Chung **[13]** published an excellent collection of unsolved problems of Erdős in graph theory.

At the 'Quo Vadis, Graph Theory?' conference in Fairbanks, Alaska, in 1990, while admiring the playful bears, we came up with the following problem. Let us run a graph process 'in reverse' as follows. Start with a complete graph K_n on n vertices, select one of the $\binom{n}{3}$ triangles at random, and delete its edges. Next, select one of the remaining $\binom{n}{3} - 3n + 8$ triangles at random, and delete its edges. Continue in this way: if at time t we have a graph G_t, and G_t has a triangle then select one of these at random, and delete its edges to obtain G_{t+1}. If G_t has no triangles, stop the process with G_t, and write S_n for the number of edges in G_t. (Note that $S_n = \binom{n}{3} - 3t$.) Thus the random variable S_n is the number of edges in a random triangle-free graph obtained in this peculiar way. What can we say about the random variable S_n? With Erdős we conjectured that the expectation of S_n has order $n^{3/2}$, and for some $c > 0$ the probability that $S_n > cn^{3/2}$ tends to 0 as $n \to \infty$. Grable [**61**] has proved the beautiful result that for every $\varepsilon > 0$ one has $\lim_{n\to\infty}\mathbb{P}(S_n > n^{7/4+\varepsilon}) = 0$, but it does not seem to be easy to improve the exponent $7/4$ to $3/2$.

Needless to say, the problem has many variants. Instead of triangles, we may delete a random K_4, K_5, or any graph H. Also, we need not start with a complete graph but just a graph containing many subgraphs isomorphic to H. In yet another variation, we do not delete all edges of a random copy of H but only a random edge.

In their study of intersecting set systems, Erdős and Rado [**48**], [**49**] called a family of sets $\mathscr{A} = \{A_1, A_2, \dots, A_k\}$ a *strong* Δ*-system* if all the intersections $A_i \cap A_j$, $1 \le i < j \le k$, are identical. Denoting by $f(n,k)$ the smallest integer m for which every family of n-sets $\{A_1, A_2, \dots, A_m\}$ contains k sets forming a strong Δ-system, Erdős and Rado proved that

$$2^n < f(n,3) < 2^n n!. \tag{9}$$

Furthermore, they conjectured that

$$f(n,3) < c_3^n \tag{10}$$

for some constant c_3. As, almost certainly, a similar assertion holds for $f(n,k)$, inequality (10) does not seem to be too much to ask. Nevertheless, the conjecture has turned out to be amazingly resistant, so eventually Erdős offered \$1000 for a proof or disproof of (10).

For many years, there was virtually no progress with (10), but shortly before Erdős died, Kostochka [**73**], [**72**] won a consolation prize of \$100 for the first substantial improvement of (9), and recently Axenovich, Fon-Der-Flass, and Kostochka [**2**] proved the further improvement that

$$f(n,3) < (n!)^{1/2+\varepsilon}$$

holds for every $\varepsilon > 0$, provided n is sufficiently large.

The problems Erdős probably liked best concern arithmetic progressions. In 1936, Erdős and Turán [**55**] conjectured the far-reaching extension of van der Waerden's theorem on arithmetic progressions that if $a_1 < a_2 < \cdots$ is a sequence of natural numbers such that, for some $c > 0$, we have $a_n < cn$ for infinitely many values of n, then the sequence contains arbitrarily long arithmetic progressions. In 1952, Roth [**90**] proved the existence of three-term progressions, and in 1974 Szemerédi [**96**] proved the full conjecture, amply deserving the \$1000 reward from Erdős, the largest he ever paid. In 1977, Fürstenberg [**59**] gave another proof of Szemerédi's theorem, based on ergodic theory; the proof and its ramifications revolutionized ergodic theory.

Erdős conjectured that the sequence of primes contains arbitrarily long arithmetic progressions and, even more, that every sequence $1 \leq a_1 < a_2 < \cdots$ with $\sum_{n=1}^{\infty} 1/a_n = \infty$ contains arbitrarily long arithmetic progressions. He offered \$3000 for this conjecture.

The final problem is again about the differences between consecutive primes. Writing $d_n = p_{n+1} - p_n$, Erdős and Turán **[56]** conjectured that $d_n < d_{n+1} < d_{n+2}$ holds for infinitely many values of n, but could not come close to proving it. Forty years later, Erdős **[39]** offered \$100 for a proof and \$25,000 for a *disproof*. Needless to say, he had no doubt that the conjecture was true, and it would be hard to find anybody who would disagree.

At this point we bring to an end this brief and ruefully inadequate sketch of the life and the achievements of Paul Erdős. We have not come close to doing justice

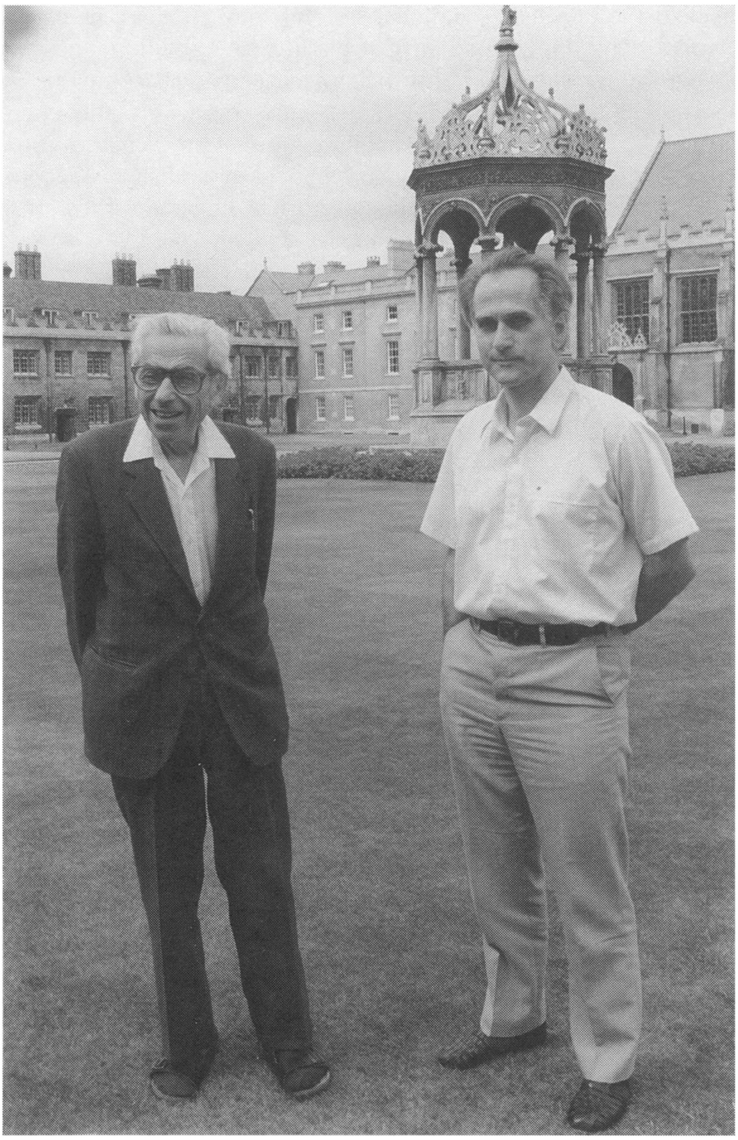

Figure 10. With B.B. in the Great Court of Trinity College, Cambridge.

to his enormous contribution to the mathematics of this century nor, in so few pages, could we.

In spite of his tremendous achievements, Paul Erdős remained outside the mathematical establishment. His eagerness to help everybody, especially those whom he thought had been badly treated, worked, to some extent, against him, as he gave the impression of not being discriminating enough. However, in private he was a remarkably shrewd and perceptive judge of mathematicians and their abilities. He was passionately, almost pathologically, keen to be free, to do as he liked, when he liked.

Did he die too early? I certainly think so, although at the very end there were signs of deterioration: occasionally even his once remarkable memory skipped a beat. It is a minor consolation that he died as he wished to, with his mathematical boots on. He always lived up to his own high moral principles, and expected others to do the same. He had a passionate desire to be free in every way and so he strongly disliked the oppressive political systems; nevertheless, he was remarkably free of personal hate. Sadly, for much of his life, he knew loneliness and sorrow, and he needed the constant stimulus of new mathematical companions and ideas to keep his unhappiness at bay. But for us, mathematicians, this companionship was a treasured gift. He was a never-failing font of magical inspiration; by his incisive mind and generous heart he enriched us all. His message to posterity, paraphrasing lines of the Hungarian poet Endre Ady, clearly showed his love for mathematics: "*Let him be blessed, who takes my place*!" Needless to say, his place will never be taken: we shall never see his like again.

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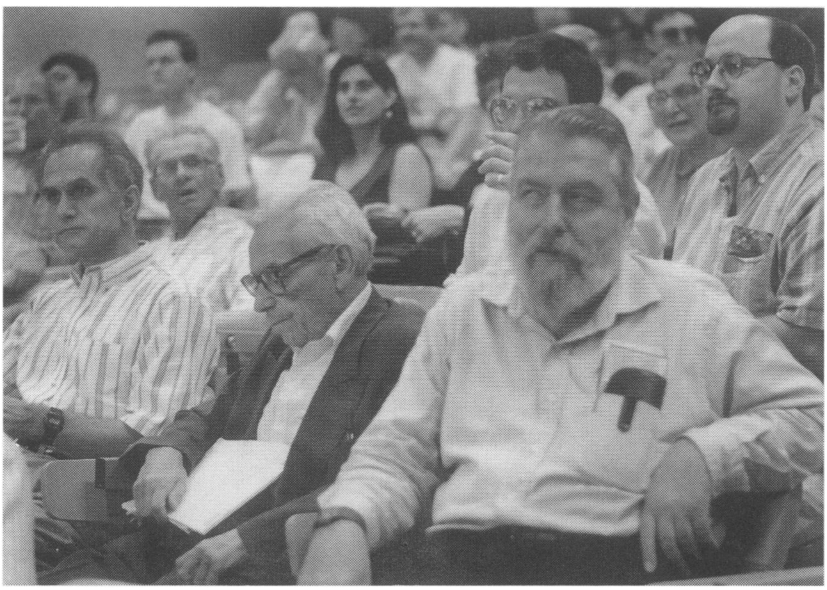

Béla Bollobás, Paul Erdős, and John Selfridge at the University of Pennsylvania, June 12–15, 1996. Photo by Stan Wagon.