HIGH GIRTH UNIT DISTANCE GRAPHS

BY PAUL O'DONNELL

A dissertation submitted to the

Graduate School—New Brunswick

Rutgers, The State University of New Jersey

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Graduate Program in Mathematics

Written under the direction of

János Komlós

and approved by

New Brunswick, New Jersey

October, 1999

UMI Number: 9947882

ABSTRACT OF THE DISSERTATION

High Girth Unit Distance Graphs

by Paul O'Donnell

Dissertation Director: János Komlós

In chapter one, high girth unit distance graphs are investigated. We answer a question of Erdős by proving the existence of arbitrary girth, 4-chromatic unit distance graphs in the plane. Included as special cases of this result are girth 9 and girth 12, 4-chromatic unit distance graphs in the plane with an interesting combinatorial structure, and girth 4, 4-chromatic unit distance graphs with relatively small numbers of vertices.

In chapter two, list colorings of complete bipartite graphs are investigated. In particular we complete the classification of the complete bipartite graphs with choice number three by showing that the choice number of $K_{6,10}$ is three. We use the techniques developed to give a shortened proof of Tesman and Shende's result that $K_{5,12}$ has choice number three. We also answer a question of Johnson about finding a nice upper bound on the smallest value for which the restricted choice number of a graph equals the choice number of that graph.

In chapter three, a problem in extremal geometric graph theory is investigated. We improve a result of Alon and Erdős by showing that a geometric graph with about $3.6n$ edges contains three disjoint, non-crossing edges.

Acknowledgements

Thanks mom and dad. You have helped me in so many ways.

Thanks Gerard. You were my first math teacher.

Thanks János. You were just the kind of advisor I wanted.

Thanks Alex. It all came to me as I drifted off to sleep on your couch.

Thanks Professors Beck, Kahn, Komlós, Roberts, and Saks for teaching me Combinatorics

Thanks Linda, Al, and Barry. You inspired me as an undergrad.

Thanks Kate. You were there from the beginning.

Thanks Rachel, for being there for all the tough years. I owe much of this to you.

Thanks to all my Drew friends, especially Dan, Kevin, Mary, Drew, Rachel, and Kate for unending love and support.

Thanks to all my ultimate and dance friends who tried to keep me sane during the insanity.

Thanks to all my math friends who helped me enjoy the insanity–especially Tony, Cyril, and TiJean: I wouldn't have made it through quals without you.

Thanks Rob, my ultimate officemate. What can I say? 11?

And thanks Carmelita, for much love and friendship, and for helping me finish.

Dedication

For Tony Brewster.

I miss you. man.

Table of Contents

Chapter 1

High Girth Unit Distance Graphs

1.1 Introduction

Erdős [7] posed the following question: For what $k \geq 3$ are there girth k, 4-chromatic, unit distance graphs in the plane? In this article and one to follow, we'll show that the answer is "for all $k \geq 3$." First we give a brief overview of the two areas which motivated the question.

In 1950, Ed Nelson posed the question: How many colors are necessary to color all the points in the Euclidean plane, $\mathbb{E}^2$, so that no two points distance one apart get the same color? The *chromatic number of the plane*, $\chi(\mathbb{E}^2)$, is the smallest number of colors which suffices. An excellent history of this problem can be found in Soifer [29]. The known bounds of $4 \leq \chi(\mathbb{E}^2) \leq 7$ have not been improved since soon after the problem was stated. In particular, no *unit distance graph in the plane* requiring more than 4 colors has been discovered.

Meanwhile in 1959, Erdős [6] proved the existence of arbitrary girth, arbitrary chromatic number graphs. In other words, given natural numbers k and l, there exist graphs with chromatic number l and no cycles with fewer than k vertices. In 1966 Erdős and Hajnal [8] generalized this to hypergraphs. In 1968 Lovász [20] came up with a general construction. Earlier in the 1950's, Mycielski [21] came up with a construction for girth 4, l-chromatic graphs, while Tutte [5] and independently Kelly and Kelly [18] came up with a construction for girth 5, l-chromatic and girth 6, l-chromatic graphs.

In the 1970's, Erdős combined the flavor of these two problems into the question mentioned above. Wormald [33] showed that the Tutte construction gives a girth 5, 4-chromatic unit distance graph in the plane. We show the existence of arbitrary girth, 4-chromatic unit distance graphs in the plane and the steps that led to that discovery.

(1) girth 5: a new embedding of the Tutte graph.

(2) girth 6: a proof of the embedding of the Tutte graph.

(3) girths 9 and 12: existence and basic structure (not based on hypergraphs).

(4) arbitrary girth: existence and basic structure (based on hypergraphs).

We also exhibit constructions of small 4-chromatic, girth 4 unit distance graphs in the plane.

There are two parts to the problem of constructing high girth, 4-chromatic unit distance graphs in the plane. We must show that the graph is 4-chromatic and has no short cycles. Then we must show that the graph is indeed a unit distance graph, by getting all adjacent vertices to be distance one apart and then adjusting the positions if any two vertices coincide. Thus we have a combinatorial part and a geometric part.

The essence of the combinatorial part is that an odd cycle is 3-chromatic. An inspiration for the high girth 4-chromatic graphs is the construction by Tutte of a girth 5, 4-chromatic graph. It goes as follows: Each 5-subset of 13 independent vertices is connected by a matching to a different 5-cycle. Clearly this graph has no 3-cycles or 4-cycles. Moreover, the graph is 4-chromatic. Four colors suffice because one color can be used for the independent vertices, leaving three colors for the "attached" odd cycles. Three colors do not suffice because in any 3-coloring, five of the independent vertices receive the same color leaving only two colors for the attached 5-cycle. Tutte's girth 6, 4-chromatic graph is obtained by attaching 7-cycles to all the 7-subsets of 19 independent vertices.

To get higher girth, k-cycles can't be attached to *all* k-subsets of an independent set. A delicate balance must be struck. Enough cycles must be attached to create a 4-chromatic graph, but they must be attached judiciously to avoid the creation of short cycles. Our idea is to number the independent vertices from 1 to n and attach odd cycles to certain arithmetic progressions. We show the following: Choosing arithmetic progressions from a restricted set of common differences gives a girth 9 graph. Van der Waerden's Theorem [32] gives chromatic number 4 for this graph. Attaching large odd cycles to an even more restricted set of arithmetic progressions (with common difference an mth power) gives a girth 12 graph using Falting's Theorem [10]. The Polynomial Szemerédi Theorem [2] gives chromatic number 4 for this graph.

Can higher girth 4-chromatic graphs be obtained by merely attaching odd cycles? Yes, but some analysis is required to determine which independent sets get odd cycles attached. There has to be enough intersection between the sets with the attached cycles to get chromatic number 4, but not so much that short circuits occur. In other words 4-chromatic high girth hypergraphs are needed. We use an idea similar to the hypergraph amalgamation of Nešetřil and Rödl [23]. The hypergraph vertices are the vertices of the graph. The hyperedges are the independent subsets to which the odd cycles are attached. Attaching the cycles gives the arbitrary girth, 4-chromatic graphs.

What about the geometric part of the problem? Given k points in the plane there are enough degrees of freedom with regard to rigidity to attach a k-cycle. However, the points must not be too close together or too far apart. A 4-coloring of the hypergraph is used to partition the independent vertices so they may be placed in the plane appropriately, then an attempt is made to attach a k-cycle. All the edges are unit length except one which is too short. Another attempt is made. All the edges are unit length except one which is too long. Continuity in the attachment process guarantees that a unit length attachment exists. Moreover, it guarantees that any coincident vertices can be separated. This gives the unit distance embeddings of the arbitrary girth 4-chromatic graphs.

Why attach cycles? Well, on the combinatorial side, graphs other than cycles can be attached. In fact, all these 4-chromatic graph constructions generalize to arbitrary chromatic number constructions by attaching more sophisticated graphs. On the geometric side, however, few tools are available for proving a graph can be attached to a set of points. A cycle has just enough degrees of freedom for the attachment procedure.

Why start with an independent set? For large graphs, the general embedding procedures use this independence. Since there are no edges between pairs of these vertices, the only constraints on their locations in the plane come from the cycles that are attached to them. For smaller graphs, though, we don't always start with an independent set. The small size of the graph permits development of more specific embedding procedures

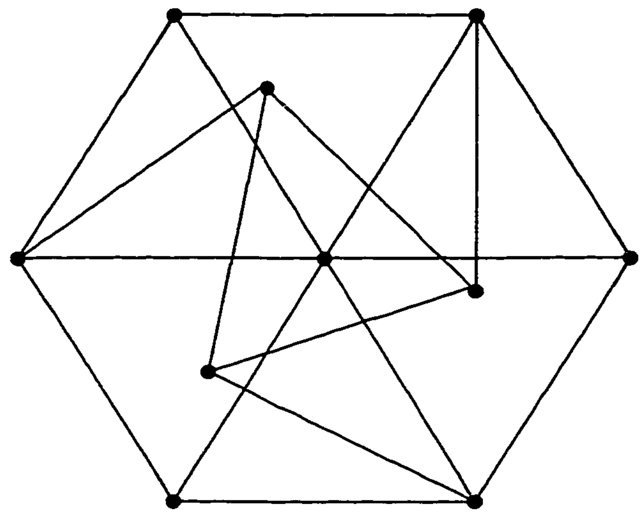

Figure 1.1: THE GOLOMB GRAPH

on a case by case basis. In either situation, attaching odd cycles is powerful enough to get the graph theoretic results and flexible enough to get the geometric results.

A construction by S. Golomb is additional inspiration for some of the work that follows. Start with the 3-chromatic unit distance graph consisting of a regular hexagon, its center point, and all possible unit length edges. Every 3-coloring of this graph has 2 colors alternating around the vertices of the hexagon. Attach a triangle to every other vertex of the hexagon to create a 4-chromatic unit distance graph (see Figure 1.1). Once again the fact that an odd cycle needs three colors is exploited.

The Galomb graph may not be particularly useful in constructing 4-chromatic unit distance graphs without short cycles, but some ideas from it are used. We start with a girth 4, 3-chromatic graph and attach large odd cycles to enough subsets of the vertices so that any 3-coloring of the underlying graph has a monochromatic set with an attached odd cycle. The resultant graph is 4 chromatic and still has girth 4 or girth 5. Starting with the Mycielski construction and a generalization, we obtain girth 4, 4-chromatic unit distance graphs on 56 and 40 vertices.

The ultimate goal is still to improve the bounds in the chromatic number of the plane problem. Although the existence of arbitrary girth 4-chromatic unit distance graphs certainly does not raise the lower bound from 4 to 5, the richness of the class of 4-chromatic unit distance graphs does make the continued search for a 5-chromatic unit distance graph seem a little more reasonable.

1.2 Definitions

Basic graph theoretical and combinatorial definitions and notation can be found in [4] and [3]. A *k-uniform hypergraph* $\mathcal{H}$ is a family of k-subsets of an n-set. The *vertices* are the elements of the underlying n-set. The *edges* (or *hyperedges*) are the k-subsets. A *cycle* of length $k \geq 2$ in a hypergraph is a sequence of distinct vertices and edges of $\mathcal{H}$,

$$v_1, E_1, v_2, E_2, \ldots, v_k, E_k,$$

such that $v_{i+1} \in E_i \cap E_{i+1}$, for $1 \leq i \leq k$ (addition modulo k). The *girth* of a hypergraph is the length of its shortest cycle. The *chromatic number* of a hypergraph is the minimum number of colors needed to color the vertices so that no edge contains vertices which are all the same color.

A k-vertex graph G with vertices $\{u_1, u_2, \ldots, u_k\}$ is *attached* to a set of vertices $\{u_1^*, u_2^*, \ldots, u_k^*\}$ if the vertices of G are connected via a matching to $\{u_1^*, u_2^*, \ldots, u_k^*\}$. The *shadow* of G, denoted G^*, is the set to which G is attached. Typically the graph G is an odd cycle. The odd cycles are attached to k-subsets of a large independent set of size n. The n independent vertices are called *foundation vertices.*

If vertices of G are placed at points in the plane so that adjacent vertices are exactly distance one apart we say this is a *unit distance embedding* of G. Thus in the plane, if the odd cycle $u_1, u_2, \ldots, u_k$ is attached to $\{u_1^*, u_2^*, \ldots, u_k^*\}$, then the vertices $u_1, u_2, \ldots, u_k, u_1^*, u_2^*, \ldots, u_k^*$ are fixed points in the plane such that for some permutation σ, u_i is distance one from $u_{\sigma(i)}^*$ and from u_{i-1} and u_{i+1} (addition modulo k) for $1 \leq i \leq k$. Since the vertices can be relabeled, we assume that u_i is adjacent to u_i^* in the attachment. Typically we do not want vertices at the same point in the plane. If vertices of G are placed at distinct points in the plane so that adjacent vertices are exactly distance one apart we say this is a *proper unit distance embedding* of G. A graph with a proper unit distance embedding is called a *unit distance graph in the* *plane*. Since higher dimensional analogues are not explored here, *unit distance graph* will mean unit distance graph in the plane. In geometric contexts, the terms point and vertex may be used interchangably, while the term edge will mean a unit length edge.

1.3 Embedding Lemmas

1.3.1 Preliminaries

Given fixed points in the plane, $u_1^*, u_2^*, \ldots, u_k^*$ (i.e. (x_1, y_1), (x_2, y_2),...,(x_k, y_k)), and a point u_1 on the unit circle centered at u_1^*, let u_i be a point distance one from both u_{i-1} and u_i^*, for $2 \leq i \leq k$. (In the following examples the distance between u_{i-1} and u_i^* is less than 2, so there are two points satisfying the distance restrictions. Let u_i be the one closer to the corresponding point in the appendix.) So for an appropriately chosen arc along the unit circle centered at u_1^*, u_i is a continuous function of u_1. If there exist two points u_1^{short} and u_1^{long} such that the distance between u_1^{short} and the corresponding u_k^{short} is less than 1, while the distance between u_1^{long} and the corresponding u_k^{long} is greater than 1, then there must be a point u_1^{unit} such that the distance between u_1^{unit} and the corresponding u_k^{unit} is exactly one. In other words, the set of points $\{u_1^*, u_2^*, \ldots, u_k^*\}$ has a k-cycle attached, namely, $u_1^{unit}, u_2^{unit}, \ldots, u_k^{unit}$.

The foundation points are distibuted among 4 regions. They are placed in δ-balls centered at the following 4 points:

$$
\begin{aligned}
C_1 &= (0, 0) \\
C_2 &= (0, 0.9) \\
C_3 &= (0.9, 0.9) \\
C_4 &= (0.9, 0).
\end{aligned}
$$

Since δ is very close to zero, it is impossible to attach a cycle to k points if they are all inside the same δ-ball. The partitioning of the foundation points is designed to prevent such an occurrence. Can a k-cycle be attached if the points are distributed among at least two of the δ-balls? Yes. First, k-cycles are attached to k foundation points placed exactly at some or all of C_1, C_2, C_3, or C_4. Next, the points are moved slightly so the k-cycles are attached to k distinct points, each placed in the appropriate δ-ball surrounding C_1, C_2, C_3, or C_4. This prevents foundation vertices from coinciding. Then some of the vertices are moved slightly to eliminate all coincidences.

1.3.2 Attaching a k-cycle to k Points in 3 Regions

Only δ-balls around points C_1, C_2, and C_3 are dealt with for the basic argument. To distinguish between the preliminary and final situations, foundation vertices coincident with C_1, C_2, C_3 are denoted $v_1^*, \ldots, v_k^*$ and their attached paths or cycles are denoted $v_1, \ldots, v_k$ while foundation vertices inside the δ-balls around C_1, C_2, C_3 are denoted $u_1^*, \ldots, u_k^*$ and their attached paths or cycles are denoted $u_1, \ldots, u_k$.

We start by attaching a triangle.

Lemma 1 *A 3-cycle can be attached to the points C_1, C_2 and C_3.*

Proof: Using the points listed in the appendix (rounded to five decimal places), two 3-vertex paths are attached to C_1, C_2 and C_3. In the first path, $T_1^{short}, T_2^{short}, T_3^{short}$, the distance from T_1^{short} to T_3^{short} is less than 1. In particular it is less than 0.99. In the second path $T_1^{long}, T_2^{long}, T_3^{long}$, the distance from T_1^{long} to T_3^{long} is greater than 1. In particular it is greater than 1.01. Since one path is obtained from the other by continuously sliding the starting vertex, there must be a path for which the distance between the first and last vertices is exactly one. This is an attached 3-cycle. □

To generalize this to k-cycles, other special points are needed. The three *triangle points*, denoted T_1, T_2, and T_3, are the points of the 3-cycle attached to C_1, C_2, and C_3. Three *spoke points*, denoted S_1, S_2, and S_3, are points such that S_i is unit distance from T_i and C_i, for $1 \leq i \leq 3$. We define "triangle" points T_i^{short} and T_i^{long}, and "spoke points" S_i^{short}, and S_i^{long} analogously for $1 \leq i \leq 3$. At first cycles or paths are attached which coincide with these triangle and spoke points. The shadows of these cycles coincide with the center points C_1, C_2, and C_3. Later we use continuity arguments to show the existence of cycles very close to these.

Lemma 2 *Let $k \geq 3$ be an odd number. For all natural numbers a_1, a_2, a_3 such that*

$a_1+a_2+a_3=k$, a k-cycle consisting only of edges from T_i to T_{i+1} (addition modulo 3), and T_i to S_i, for $1 \le i \le 3$, can be attached to the union of a_i points at C_i, $1 \le i \le 3$.

Proof: We have k vertices $v_1^*, \ldots, v_k^*$ distributed among just three points in the plane, namely C_1, C_2, and C_3. The desired attached cycle consists of an additional k vertices $v_1, \ldots, v_k$ distributed among just six points in the plane, namely T_i and S_i, for $1 \le i \le 3$. We are, in some sense, wrapping the k-cycle around the 6 edges formed by the triangle and spoke points. Since T_i and S_i are distance one from C_i, $1 \le i \le 3$, then v_j is distance one from v_j^*, $1 \le j \le k$.

The case $k=3$ is covered by lemma 1. The proof proceeds by induction. There are two possibilities to consider:

Case 1: At least one a_i is ≥ 3.

Say $a_1 \ge 3$, the other cases are similar. Let

$$v_1, v_2, \ldots, v_j, v_{j+1}, \ldots, v_{k-2}$$

be a $(k-2)$-cycle attached to a_1-2 points at C_1, a_2 points at C_2, and a_3 points at C_3 using edges of the specified type such that $v_j = T_1$. Then for $v_j' = S_1$ and $v_j'' = T_1$,

$$v_1, v_2, \ldots, v_j, v_j', v_j'', v_{j+1}, \ldots, v_{k-2}$$

is an attached k-cycle satisfying the conditions.

Case 2: At least two of the a_i are ≥ 2.

Say $a_1 \ge 2$ and $a_2 \ge 2$, the other cases are similar. Let

$$v_1, v_2, \ldots, v_j, v_{j+1}, \ldots, v_{k-2}$$

be a $k-2$ cycle attached to a_1-1 points at C_1, a_2-1 points at C_2, and a_3 points at C_3 using edges of the specified type, with $v_j = T_1$. Then for $v_j' = T_2$ and $v_j'' = T_1$,

$$v_1, v_2, \ldots, v_j, v_j', v_j'', v_{j+1}, \ldots, v_{k-2}$$

is an attached k-cycle satisfying the conditions. □

Corollary 3 *Let $k \ge 3$ be an odd number. For all natural numbers a_1, a_2, a_3 such that $a_1+a_2+a_3=k$, a k-vertex path starting at T_1^{short} and ending at T_3^{short} consisting* *only of edges from* T_i^{short} *to* T_{i+1}^{short} *and* T_i^{short} *to* S_i^{short}, *for* $1 \leq i \leq 3$, *can be attached to the union of* a_i *points at* C_i, *for* $1 \leq i \leq 3$.

Proof: The proof is similar to that of lemma 2. However, instead of a cycle, it is a path starting at T_1^{short} and ending at T_3^{short} which is attached. Since T_3^{short} and T_1^{short} are not distance one apart, we introduce an additional point T_4^{short} which is distance one from T_3^{short} and C_1. The inducton step does not change the beginning or ending vertex of the path. They are determined only by the base case: the path $T_1^{short}, T_2^{short}, T_3^{short}$. If in lemma 2, edges from T_3 to T_1 and T_1 to T_3 would be inserted into the cycle, then edges from T_3^{short} to T_4^{short} and T_4^{short} T_3^{short} are inserted into the path. □

Corollary 4 *Let* $k \geq 3$ *be an odd number. For all natural numbers* a_1, a_2, a_3 *such that* $a_1 + a_2 + a_3 = k$, *a* k*-vertex path starting at* T_1^{long} *and ending at* T_3^{long} *consisting only of edges from* T_i^{long} *to* T_{i+1}^{long} *and* T_i^{long} *to* S_i^{long}, *for* $1 \leq i \leq 3$, *can be attached to the union of* a_i *points at* C_i, *for* $1 \leq i \leq 3$.

Proof: Analogous to corollary 3. □

So if k points are all located at three center points, a "short" k-vertex path, a "long" k-vertex path, and hence a k cycle can be attached. If the k-points are located arbitrarily close to the center points, it seems reasonable that there exist an attached "short" k-vertex path, "long" k-vertex path, and k cycle arbitrarily close to the originals. Recall that v_i is distance one from both v_i^* and v_{i-1}. The following lemma states if two points are arbitrarily close to v_i^* and v_{i-1}, then there is a point distance one from both which is arbitrarily close to v_i. Continuity gives the proof.

Lemma 5 *For* $i \geq 2$, *if the distance between two points* u_i^* *and* u_{i-1} *is less than two, then for all* $\epsilon_i > 0$, *there exist* $\delta_i > 0$ *and* $\epsilon_{i-1} > 0$ *such that*

$$|u_i^* - v_i^*| < \delta_i \quad \textit{and} \quad |u_{i-1} - v_{i-1}| < \epsilon_{i-1} \Rightarrow |u_i - v_i| < \epsilon_i. \quad \Box$$

Using this lemma $k-1$ times gives the conclusion that if each u_i^* is close to v_i^* and u_1 is close to v_1, then each u_i is close to v_i.

Lemma 6 *Given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|u_i^* - v_i^*| < \delta \text{ and } |u_1 - v_1| < \delta \Rightarrow |u_i - v_i| < \epsilon \text{ , for } 1 \leq i \leq k.$$

Proof: Given ϵ_k, there exist δ_k and ϵ_{k-1} such that,

$$|u_k^* - v_k^*| < \delta_k \text{ and } |u_{k-1} - v_{k-1}| < \epsilon_{k-1} \Rightarrow |u_k - v_k| < \epsilon_k.$$

But, given ϵ_{k-1}, there exist δ_{k-1} and ϵ_{k-2} such that,

$$|u_{k-1}^* - v_{k-1}^*| < \delta_{k-1} \text{ and } |u_{k-2} - v_{k-2}| < \epsilon_{k-2} \Rightarrow |u_{k-1} - v_{k-1}| < \epsilon_{k-1}.$$

$$\vdots$$

But, given ϵ_2, there exist δ_2 and ϵ_1 such that,

$$|u_2^* - v_2^*| < \delta_2 \text{ and } |u_1 - v_1| < \epsilon_1 \Rightarrow |u_2 - v_2| < \epsilon_2.$$

At each step we redefine ϵ_i to equal $\min\{\epsilon_i, \epsilon_k\}$. Choosing $\delta = \min\{\delta_i, \epsilon_1\}$ and $\epsilon = \epsilon_k$ then

$$|u_i^* - v_i^*| < \delta \text{ and } |u_1 - v_1| < \delta \Rightarrow |u_i - v_i| < \epsilon.$$

Thus the u vertices can be made arbitrarily close to the v vertices if the u^* vertices are close to the v^* vertices and u_1 is close to v_1. □

Now it is shown that a k-cycle can be attached to the u^* vertices.

Theorem 7 *Let $k \geq 3$ be an odd number. For all natural numbers a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = k$, there exists δ such that the union of any a_i points in the δ-ball around C_i, $1 \leq i \leq 3$, can have a k-cycle attached.*

Proof: We use corollary 3 to find a path $v_1^{short}, \ldots, v_k^{short}$ attached to a_i points at C_i, for $1 \leq i \leq 3$. Lemma 6 says there is a path arbitrarily close to this for u_i^* in the appropriate δ-ball. Given ϵ, choose δ so that

$$|u_i^* - v_i^*| < \delta \text{ and } |u_i^{short} - v_i^{short}| < \epsilon \text{ for } 1 \leq i \leq k.$$

Then for all u_i^* in the appropriate δ-ball,

$$|u_k^{short} - u_1^{short}| = |(u_k^{short} - v_k^{short}) - (u_1^{short} - v_1^{short}) + (v_k^{short} - v_1^{short})|$$

$$\leq \epsilon + \epsilon + 0.99.$$

In other words there is an attached path $u_1^{short}, \ldots, u_k^{short}$ for which

$$|u_k^{short} - u_1^{short}| \leq 0.99 + 2\epsilon.$$

Next a path which is close to $v_1^{long}, \ldots, v_k^{long}$ is found. Given ϵ, choose δ so that

$$|u_i^* - v_i^*| < \delta \text{ and } |u_i^{long} - v_i^{long}| < \epsilon \text{ for } 1 \leq i \leq k.$$

Then for all u_i^* in the appropriate δ-ball,

$$|v_k^{long} - v_1^{long}| = |(v_k^{long} - u_k^{long}) - (v_1^{long} - u_1^{long}) + (u_k^{long} - u_1^{long})|.$$

$$1.01 \leq \epsilon + \epsilon + |(u_k^{long} - u_1^{long})|$$

In other words there is an attached path $u_1^{long}, \ldots, u_k^{long}$ for which

$$|u_k^{long} - u_1^{long}| \geq 1.01 - 2\epsilon.$$

Choose $\epsilon < 0.005$. Since one attachment can be obtained from the next by a continuous operation, there is an attached path where $|u_k - u_1| = 1$. In other words a cycle can be attached. □

1.3.3 Attaching a k-cycle to k Points in 4 Regions

Of course if an odd cycle can be attached to k points placed inside δ-balls around C_1, C_2, and C_3, then by symmetry any three of the center points C_1, C_2, C_3, and C_4 can be used. But what if the points are distributed inside δ-balls around all four of the center points? We generalize only to the case where the fourth region, the δ-ball around C_4 contains just a single vertex. Like earlier proofs, short and long paths are attached directly to the C_i. The situation is treated like the three region situation except a spoke vertex is replaced by a vertex distance one from C_4. Then close approximations to short and long paths are attached to points in the δ-balls around the C_i. Continuity proves the existence of a path where the distance between the first vertex and the last is one. In other words, there is an attached cycle.

Theorem 8 *Let $k \geq 5$ be an odd number. For all natural numbers a_1, a_2, a_3, a_4 such that $a_1 + a_2 + a_3 + a_4 = k$ and $a_4 = 1$, there exists δ such that the union of any a_i points in the δ-ball around C_i, $1 \leq i \leq 4$ can have a k-cycle attached.*

Proof: Without loss of generality, assume $a_1 \geq 2$. Let

$$v_1^{short}, v_2^{short}, \ldots, v_j^{short}, v_{j+1}^{short}, v_{j+2}^{short}, \ldots, v_k^{short}$$

be a path attached to $a_1 + 1$ points at C_1, a_2 points at C_2, and a_3 points at C_3 with $v_j^{short} = v_{j+2}^{short} = T_1^{short}$ and $v_{j+1}^{short} = S_1^{short}$. Replacing v_{j+1}^{short} in this cycle with a vertex distance one from C_4 and T_1^{short} gives a "short" path attached to a_i points at C_i, for $1 \leq i \leq 4$. Similarly there exists a "long" path attached to a_i points at C_i, for $1 \leq i \leq 4$. Analogous to the the proof of theorem 7, appropriate δ can be chosen so that "short" and "long" paths can be attached to the union of any a_i foundation points in the δ-ball around C_i, for $1 \leq i \leq 4$. Appropriate δ can be chosen so that any union of a_i vertices in the δ-ball around C_i, for $1 \leq i \leq 4$, can have short and long paths attached which closely approximate these paths. By continuity, an odd cycle can be attached. The appendix contains the coordinates of those points (rounded to five decimal places). □

1.3.4 Attaching a k-cycle to k Points in 2 Regions

So an odd cycle can be attached to k points placed inside δ-balls around any 3 or all 4 of C_1, C_2, C_3, and C_4. But what if the points are distributed between δ-balls around just two of the center points? The crucial step is still just attaching a triangle. Once it is shown that a triangle can be attached to the center points, the previous arguments show that a k cycle can be attached.

Theorem 9 *Let $k \geq 3$ be an odd number. For all natural numbers a_1, a_2 such that $a_1 + a_2 = k$, there exists δ such that the union of any a_i points in the δ-ball around C_i, $1 \leq i \leq 2$, can have a k-cycle attached.*

Proof: Without loss of generality, assume $a_1 \geq 2$. We attach a 3-cycle to two vertices at C_1 and one vertex at C_2, using the same notation as before for the triangle points, only here triangle points with subscripts 1 or 2 correspond to C_1 while those with subscript 3 correspond to C_2. Using the points listed in the appendix (rounded to five decimal places), two 3-vertex paths are attached to C_1, C_1 and C_2. In the first path, $T_1^{short}, T_2^{short}, T_3^{short}$, the distance from T_1^{short} to T_3^{short} is less than 1. In the second path $T_1^{long}, T_2^{long}, T_3^{long}$, the distance from T_1^{long} to T_3^{long} is greater than 1. Since one path is obtained from the other by continuously sliding the starting vertex, there must be a path for which the distance between the first and last vertices is exactly one. This is an attached 3-cycle.

Now we attach the k-cycle. Let a_1' and a_1'' be natural numbers such that $a_1'+a_1'' = a_1$. We treat C_1 as if it were two separate vertices C_1' and C_1'' and use the machinery from section 1.3.2 to find δ such that any a_1' points in the δ-ball around C_1', a_1'' points in the δ-ball around C_1'', and a_2 points in the δ-ball around C_2 can have a k-cycle attached. In other words any a_1 points in the δ-ball around C_1 and a_2 points in the δ-ball around C_2 can have a k-cycle attached. □

This allows the attachment of k-cycles if the center points are distance 0.9 from each other, like C_1 and C_2. The other situation, for example C_1 and C_3, is handled below in analogous fashion.

Theorem 10 *Let $k \geq 3$ be an odd number. For all natural numbers a_1, a_3 such that $a_1 + a_3 = k$, there exists δ such that the union of any a_1 points in the δ-ball around C_1 and a_3 points in the δ-ball around C_3 can have a k-cycle attached.*

Proof: The proof is identical to the previous one. The appendix contains the coordinates of the special points (rounded to five decimal places). □

1.3.5 Removing coincidences

If two vertices from a graph are placed at the same coordinates in the plane, small cycles may inadvertantly be created. We must ensure that no vertices coincide. For δ small enough, the regions containing the foundation vertices are disjoint from the regions containing cycle vertices. Furthermore, the foundation vertices can be placed anywhere in the δ-balls, so we choose distinct locations for all of them. It's possible, however, for cycle vertices to coincide. In small graphs it can be verified computationally that this doesn't occur. For larger graphs we develop procedures to remove these coincidences. If vertices from two different attached cycles coincide, one foundation vertex is moved slightly causing all vertices of the one attached cycle to move slightly while no vertices of the other cycle move. "Slightly" means not enough to introduce any new coincidences. If vertices from the same cycle coincide, a modificaton of this method is used.

Theorem 11 *If there is an embedding of G with $m \geq 1$ pairs of coincident vertices, then there is an embedding with fewer than m pairs of coincident vertices.*

Proof: Given an embedding of G with coincident vertices u and w, we shift some of the vertices of G. Let

$$\epsilon_1 = \min\{\delta - d(u^*, C) \mid u^* \text{ a foundation vertex in the } \delta\text{-ball around center point } C\}$$

$$\epsilon_2 = \min\{d(u, w) \mid \text{ non coincident vertices } u, w \in V(G)\}$$

$$\epsilon = \min\{\epsilon_1, \epsilon_2/2\}$$

Given $\epsilon > 0$ find δ', $0 < \delta' < \epsilon$ such that if a foundation vertex is moved a distance less than δ', then no vertex moves a distance ϵ or greater. Since foundation vertices are not moved more than ϵ_1 they remain inside the appropriate δ-ball, thus all k-cycles can still be attached. Since all non-coincident pairs of vertices are at least ϵ_2 apart, then movement by less than $\epsilon_2/2$ does not create new coincidences.

Assume u and w are on different cycles. Let's call them u_1 and w_1. Let u_j be a vertex such that no w_i is attached to the foundation vertex u_j^*. Moving u_{j+1} along the unit circle centered at u_{j+1}^* causes each vertex in

$$u_{j+1}, u_{j+2}, \ldots, u_k, u_1, \ldots, u_{j-1}$$

to move to maintain unit distance from its foundation vertex and from the preceding cycle vertex. We move u_{j+1} so that no vertex has moved more than ϵ and so that there is a point unit distance from u_{j-1} and u_{j+1} and distance less than δ' from u_j. This point is the new location of u_j. Now we move u_j^* that same distance so it is unit distance from the new u_j. Of course moving u_j^* may shift vertices of cycles attached to it by distances less than ϵ, but no new coincidences are introduced. Since u_1 moves and w_1 does not, at least one coincidence is removed.

Assume u and w are on the same cycle. Let's call them u_1 and u_i. We choose a cycle vertex u_j different from the coincident vertices and apply the procedure just described. The only foundation vertex that moves is u_j^*. The only point in the ϵ-ball around the coincident vertices which is distance one from u_1^* and u_i^* is the original location of those points. Since u_1 and u_i moved while u_1^* and u_i^* did not they no longer coincide. As before no new coincidences are introduced. □

1.4 Girth 5 and 6, 4-Chromatic Unit Distance Graphs

In [5], Tutte gives the following construction of girth 5, $(l+1)$-chromatic graphs for arbitrary l. Start with $(k-1)l+1$ foundation vertices and $\binom{(k-1)l+1}{k}$ copies of a k-vertex, l-chromatic, girth ≥ 5 graph H. To every k-subset of the foundation vertices, attach a copy of H. No l-coloring of this graph is proper since by the pigeonhole principle at least k of the foundation vertices get the same color, leaving only $l-1$ colors to color the attached copy of H. It is clear that the graph is $(l+1)$-colorable, and that if H has no 3-cycles or 4-cycles, then neither does this new graph. When H is a 5-cycle the resultant graph is a girth 5, 4-chromatic graph we call the "Tutte" graph.

This idea can be taken a step further by attaching copies of a girth 6, l-chromatic graph H. This results in an $(l+1)$-chromatic graph with no 3-cycles, 4-cycles, or 5-cycles. Attaching higher girth, l-chromatic graphs still gives just a girth 6, $(l+1)$-chromatic graph.

Wormald uses this construction to show the existence of a girth 5, 4-chromatic unit distance graph in the plane. He embeds the Tutte graph in the plane by positioning 13 foundation vertices evenly around a circle and showing that a 5-cycle can be attached to all 5-subsets of the foundation vertices. Furthermore, when the cycles are attached, it is verified that no coincidences occur.

Using the machinery of the previous section, we can give an alternate embedding. We place four foundation vertices in the δ-ball around C_i for $1 \leq i \leq 3$ and one foundation vertex in the δ-ball around C_4. Since the vertices of any 5-set of foundation vertices are in at least 2 δ-balls, the embedding lemmas allow the attachments of the 5-cycles and removal of any coincidences. Note if there were 5 foundation vertices in a single δ-ball, then a 5-cycle could not be attached.

Wormald suggests that the girth 6, 4-chromatic Tutte construction (attaching 7-cycles to all 7-subsets of 19 foundation vertices) could be properly embedded in the plane as a unit distance graph. This is obvious given the previous embedding. Simply place six foundation vertices in the δ-ball around C_i for $1 \leq i \leq 3$ and one foundation vertex in the δ-ball around C_4. Again all the cycles can be attached and the coincidences removed.

Other girth 6, 4-chromatic unit distance graphs may be obtained by attaching larger odd cycles to the appropriate number of foundation vertices. These graphs are all larger than the 352,735 vertex girth 6, 4-chromatic Tutte graph described earlier, but the embedding procedure is analagous.

1.5 Girth 9, 4-Chromatic Unit Distance Graphs

1.5.1 Graph Description

The graphs in this section consist of foundation vertices with attached copies of a k-cycle. For notation define the graph $G_{n,H,S}$ for $n \in \mathbb{N}$, H a graph on $k \leq n$ vertices, and $S \subseteq \binom{[n]}{k}$ a k-uniform hypergraph, to be the graph with foundation vertices $\{u_1^*, u_2^*, ..., u_n^*\}$ with copies of H attached to those subsets of the foundation vertices in the family S. Using this notation, the Tutte graph is $G_{13,5-cycle,\binom{[13]}{5}}$. The smallest of the girth 6, 4-chromatic graphs described in the previous section is $G_{19,7-cycle,\binom{[19]}{7}}$.

Can attaching higher length odd cycles to larger sets of foundation vertices bump up the girth while keeping the chromatic number at least four? Not if the k-cycles are attached to *all* k-sets of the foundation vertices. This is because some k-sets intersect in at least two elements. Depending on how the cycles $\{v_1, v_2, v_3, ..., v_k\}$ and $\{w_1, w_2, w_3, ..., w_k\}$ are attached, it is possible to get a 6-cycle. To avoid such a situation, we require that sets in S do not intersect in more than one element. This allows us to construct girth 9 graphs.

By attaching k-cycles judiciously to *some* of the k-subsets, a higher girth is achieved. Rather than attach cycles to arbitrary subsets, cycles are attached to specified arithmetic progressions (AP's) of foundation vertices. We restrict allowable AP's in two ways. First, the set of allowable common differences, D, is chosen so that no arithmetic progressions with different common differences intersect in more than one element. Second, given D, the set S is constructed so that arithmetic progressions with the same common difference do not overlap.

The distance between any two points in a k-term arithmetic progression is a multiple less than k of the common difference d. To prevent two AP's from intersecting in two places, we must ensure that

$$ad_1 \neq bd_2$$

for all a, b less than k and distinct common differences d_1, d_2 in D. Formally, let D_j denote the set of allowable common differences less than or equal to j. We define D_j recursively:

$$D_j = \begin{cases} D_{j-1} \cup \{j\}, \text{ if for all } d \in D_{j-1} \text{ and } a, b \in [k-1], \ ad \neq bj \\ D_{j-1}, \text{ otherwise.} \end{cases}$$

Then the allowable set of common differences is:

$$D = \bigcup_{j=1}^{\infty} D_j.$$

How sparse is D? If too many numbers are in D, then the graph will have short cycles. If too few numbers are in D, then the graph will not be 4-chromatic. The following claim gives an idea of the density of D.

Claim 12 *For all d, at least one of $\{d, 2d, 3d, \ldots, k!d\}$ is in D.*

Proof: If $k!d \in D$ then we are done. If not, then there exist $a, b \in [k-1]$ and $d_1 \in D$ with $d_1 < k!d$ such that

$$ad_1 = bk!d.$$

Solving for d_1,

$$d_1 = \frac{bk!d}{a}.$$

By definition it is in D and less than $k!d$. Since $k!$ is divisible by a, d_1 is an integral multiple of d. □

Given D, we construct the S, set of arithmetic progressions. Formally, let

$$S = S(n, k, D) = \{ \quad \{a, a+d, \quad a+2d, \dots, a+(k-1)d\} : d \in D,$$
$$a \equiv 1, 2, \dots, d \pmod{kd}, \text{ for } a+(k-1)d \leq n\}.$$

For example if $D = \{1, 3, 4, 5, \dots\}$ then $S(17, 3, D)$ is:

$$\begin{array}{cccc}
\{1,2,3\} & \{1,4,7\} & \{1,5,9\} & \{1,6,11\} \\
\{4,5,6\} & \{2,5,8\} & \{2,6,10\} & \{2,7,12\} \\
\{7,8,9\} & \{3,6,9\} & \{3,7,11\} & \{3,8,13\} \\
\{10,11,12\} & \{10,13,16\} & \{4,8,12\} & \{4,9,14\} \\
\{13,14,15\} & \{11,14,17\} & & \{5,10,15\}
\end{array}$$

Now we check the chromatic number and the girth of G for appropriate k and n, and verify that G is a unit distance graph.

1.5.2 $\chi(G) = 4$

We use van der Waerden's theorem [32] to show that for some n, $G_{n,k-cycle,S}$ is 4-chromatic.

Theorem 13 *(van der Waerden) For all $k, l \in \mathbb{N}$, there exists n such that any l-coloring of the integers from 1 to n contains a k-term monochromatic arithmetic progression.*

Theorem 14 *There exists n such that $\chi(G_{n,k-cycle,S}) = 4$*

Proof: By van der Waerden's theorem there exists n such that any 3-coloring of the integers from 1 to n contains a monochromatic arithmetic progression of length $(2k-1)k!$. Let d be the common difference of this AP. By claim 12 there exists $d' \in D$, such that d' is a multiple of d less than or equal to $k!d$. Hence there is a $2k-1$ term monochromatic arithmetic progression of foundation vertices

$$u_a, u_{a+d'}, u_{a+2d'}, \dots, u_{a+(2k-2)d'}$$

with $d' \in D$. One of the first k of these indices is congruent to some element in $\{1, 2, \ldots, k\}$ (mod kd). The vertex with this index and the $k-1$ vertices after it (in the AP with common difference d') form a set in S. This set has a k-cycle attached. But if all of these foundation vertices are the same color, there are only 2 colors remaining to color the odd cycle. This is not enough. Thus at least 4 colors are necessary to color $G_{n,k-cycle,S}$. Of course, 4 colors are sufficient since one can be used for the foundation vertices leaving three for the attached odd cycles. □

1.5.3 girth(G) ≥ 9

Theorem 15 *For odd $k \geq 9$, girth($G_{n,k-cycle,S}$) ≥ 9.*

Proof: A cycle containing no foundation vertices is a k-cycle. All other cycles consist of foundation vertices separated by at least 2 vertices of an attached cycle. Only one foundation vertex in a cycle is not possible. A cycle has only two foundation vertices if the arithmetic progressions of the two attached cycles intersect in two places. Our choices for D and S prevent this. A cycle with at least 3 foundation vertices has at least 9 vertices. So the girth is at least $\min\{9, k\}$. □

Just like the Tutte construction, this method generalizes to arbitrary chromatic number. By attaching girth 9, $(l-1)$-chromatic graphs to appropriate arithmetic progressions of foundation vertices we obtain girth 9, l-chromatic graphs. However, the 4-colorable ones seem like the only reasonable candidates for unit distance graphs.

1.5.4 G is a Unit Distance Graph

Theorem 16 *There exist girth 9, 4-chromatic unit distance graphs.*

Proof: We know from the previous theorems that for appropriate choices of k and n, $G_{n,k-cycle,S}$ is a 4-chromatic graph with girth at least 9. Given odd $k \geq 9$, let n_0 be the smallest such n. We show that $G_{n_0,k-cycle,S}$ is a unit distance graph using an embedding procedure similar to that used for the Tutte graphs. By the choice of n_0, there is a 3-coloring of the foundation vertices from 1 to $n_0 - 1$ such that no monochromatic set has an odd cycle attached. We place all the foundation vertices with color i in the δ-ball around C_i for $1 \leq i \leq 3$. We place vertex n_0 in the δ-ball around C_4. Since the vertices with a k-cycle attached are always in at least 2 δ-balls, the embedding lemmas allow the attachments of all cycles and removal of any coincidences. (Technically if the girth is more than 9, we add a 9-cycle to get a girth 9 graph.) □

1.6 Girth 12, 4-Chromatic Unit Distance Graphs

1.6.1 Graph Description

To achieve girth 12 we alter the set D of allowable common differences. This changes what sets are in S (i.e. what sets of foundation vertices get odd cycles attached). It's not enough for the sets in S to have intersection of size zero or one. We require also that no three sets in S intersect pairwise. To achieve this, k-cycles are attached only to specified arithmetic progressions whose common difference is an mth power. Formally, we let $D = \{x^m : x \in \mathbb{N}\}$, with m to be determined in section 1.6.3. Then we check the chromatic number and the girth of G for appropriate k, m, and n, and also verify that G is a unit distance graph.

1.6.2 $\chi(G) = 4$

We use a generalization of van der Waerden's theorem [32] to show that for n large enough $(G_{n,k-cycle,S})$ is 4-chromatic. We pause for a brief history. Szemerédi [30] showed that any subset of $\mathbb{N}$ of positive upper density contains arbitrarily long arithmetic progressions. Fürstenberg [12] discovered a proof of this using ergodic theory. Bergelson and Hindman [2] generalized the Fürstenberg result further to get the *Polynomial Szemerédi Theorem.*

Theorem 17 *(Bergelson, Hindman) Let $p_i(x)$ be polynomials with integer coefficients such that $p_i(0) = 0$. Then any subset of $\mathbb{N}$ with positive upper density contains a set of the form*

$$\{a + p_1(x), a + p_2(x), \ldots, a + p_k(x)\}$$

for some $x \in \mathbb{N}$.

CONSEQUENCE: Given any r-coloring of $\mathbb{N}$, there exist arbitrarily long monochromatic arithmetic progressions whose common difference is an mth power.

Theorem 18 *There exists n such that $\chi(G_{n,k-cycle,S}) = 4$.*

Proof: By the Polynomial Szemerédi Theorem there exists n such that any 3-coloring of the integers from 1 to n contains a $(2k-1)$-term monochromatic arithmetic progression of foundation vertices

$$u_a, u_{a+d}, u_{a+2d}, \ldots, u_{a+(2k-2)d}$$

where d is an mth power. One of the first k of these indices is congruent to some element in $\{1, 2, \ldots, k\}$ (mod kd). The vertex with this index and the $k-1$ vertices after it form a set in S. This set has a k-cycle attached. But if all of these foundation vertices are the same color, there are only 2 colors remaining to color the odd cycle. This is not enough. Thus at least 4 colors are necessary to color $G_{n,k-cycle,S}$. Of course, 4 colors are sufficient since one can be used for the foundation vertices leaving three for the attached odd cycles. □

1.6.3 girth(G) ≥ 12

We use a theorem of Faltings [10] to construct the set of allowable common differences. It deals with integer solutions to equations of the form

$$ax^m + by^m + cz^m = 0.$$

Before we state that theorem, we discuss some preliminaries. A solution (x_0, y_0, z_0) is *primitive* if $gcd\{x_0, y_0, z_0\} = 1$. A solution (x_0, y_0, z_0) is *trivial* if $x_0, y_0, z_0 \in \{-1, 0, 1\}$. Notice that if (x_0, y_0, z_0) is a solution then any integer multiple is also a solution:

$$ax^m + by^m + cz^m = 0 \quad \Rightarrow$$

$$j^m(ax^m + by^m + cz^m) = 0 \quad \Rightarrow$$

$$a(jx)^m + b(j)y^m + c(jz)^m = 0.$$

So if an equation has one solution it has infinitely many. However, for appropriate choice of m, it has only a finite number of primitive solutions. For a better choice of m all primitive solutions are also trivial. For the final choice of m, all equations

$$ax^m + by^m + cz^m = 0 \text{ with } a, b, c \in \{-k, ..., k\} \text{ not all zero}$$

have no nontrivial primitive solutions. This allows us to construct the set of allowable common differences and the set of arithmetic progressions to which the odd cycles are attached.

Theorem 19 *(Faltings) A nonsingular projective curve of genus at least two over a number field has only finitely many points with coordinates in the number field.*

CONSEQUENCE: Given $a, b, c \in \mathbb{Z}$, not all zero, the equation $ax^m + by^m + cz^m = 0$ has only finitely many primitive solutions for $m \geq 4$.

Lemma 20 *Given $a, b, c \in \mathbb{Z}$, not all zero, there exists m such that $ax^m + by^m + cz^m = 0$ has no nontrivial primitive solutions.*

Proof: Faltings theorem says for $m \geq 4$, $ax^m + by^m + cz^m = 0$ has finitely many primitive solutions. Given a, b, c let w be the integer of largest absolute value in any primitive solution to $ax^4 + by^4 + cz^4 = 0$. Choose $l = l(a, b, c)$ so that $2^l > w$. We need the following claim to complete the proof:

Claim 21 *$ax^{4l} + by^{4l} + cz^{4l} = 0$ has no primitive solutions except possibly $x, y, z \in \{-1, 0, 1\}$.*

Proof: Assume $ax_0^{4l} + by_0^{4l} + cz_0^{4l} = 0$ with $gcd\{x_0, y_0, z_0\} = 1$. Then $a(x_0^l)^4 + b(y_0^l)^4 + c(z_0^l)^4 = 0$ shows that x_0^l, y_0^l, z_0^l is a primitive solution to $ax^4 + by^4 + cz^4 = 0$. By definition of w,

$$|x_0^l|, |y_0^l|, |z_0^l| \leq |w| < 2^l,$$

so $x_0, y_0, z_0 \in \{-1, 0, 1\}$. □

So $m(a, b, c) = 4l$ satifies the statement of the lemma. □

Corollary 22 *Given k, there exists m' such that none of the equations*

$$ax^{m'} + by^{m'} + cz^{m'} = 0 \text{ with } a, b, c \in \{-k, ..., k\} \text{ not all zero}$$

has nontrivial primitive solutions.

Proof: Given a, b, c, there exixts $m = m(a, b, c)$ such that $ax^m + by^m + cz^m = 0$ has no nontrivial primitive solutions. The same holds with for any exponent which is a multiple of m. Hence

$$m' = \prod_{\{(a,b,c) | a,b,c \in \{-k,...,k\} \text{ , not all zero }\}} m(a, b, c)$$

suffices. □

Given $m' = m'(k)$, define $D = \{x^{m'} : x \in \mathbb{N}\}$. This is the set of allowable common differences needed to construct the set S of arithmetic progressions. Each arithmetic progression in S corresponds to a set of foundation vertices with an attached cycle.

Theorem 23 *For odd $k \geq 13$ $girth(G_{n,k-cycle,S}) \geq 12$.*

Proof: A cycle containing no foundation vertices is a k-cycle. All other cycles consist of foundation vertices separated by at least 2 vertices of an attached cycle. So a cycle with at least 4 foundation vertices has at least 12 vertices.

A cycle has 3 foundation vertices if the arithmetic progressions of the three attached cycles intersect pairwise. Let a_i be the starting point and d_i be the common difference, $1 \leq i \leq 3$, for the three arithmetic progressions. The pairwise intersections of the AP's implies the existence of constants $c_1, c_2, ..., c_6$ between 0 and $k - 1$ such that

$$\begin{aligned} a_1 + c_1 d_1 &= a_2 + c_2 d_2 \\ a_2 + c_3 d_2 &= a_3 + c_4 d_3 \\ a_3 + c_5 d_3 &= a_1 + c_6 d_1. \end{aligned}$$

Thus,

$$a_1 + a_2 + a_3 + c_1 d_1 + c_3 d_2 + c_5 d_3 = a_1 + a_2 + a_3 + c_6 d_1 + c_2 d_2 + c_4 d_3$$

or,

$$(c_1 - c_6)d_1 + (c_3 - c_2)d_2 + (c_5 - c_4)d_3 = 0.$$

Since the common differences are all m'th powers and the three foundation vertices are distinct, this is an equation of the form $ax^{m'} + by^{m'} + cz^{m'} = 0$, with nonzero integer coefficients between $-k$ and k. It has only trivial primitive solutions. Thus any solution has all the d_i equal, yet in the construction of S, arithmetic progressions with the same common difference do not intersect.

A cycle has only 2 foundation vertices if the arithmetic progressions of the two attached cycles intersect in two places. Let a_i be the starting point and d_i be the common difference, $1 \leq i \leq 2$, for the two arithmetic progressions. The intersection of the AP's implies the existence of constants c_1, c_2, c_3, c_4 between 0 and $k-1$ such that

$$\begin{aligned} a_1 + c_1 d_1 &= a_2 + c_2 d_2 \\ a_1 + c_3 d_1 &= a_2 + c_4 d_2. \end{aligned}$$

Thus,

$$(c_1 - c_3)d_1 + (c_4 - c_2)d_2 = 0.$$

Since the common differences are all m'th powers and the two foundation vertices are distinct, this is an equation of the form $ax^m + by^m = 0$, with nonzero integer coefficients between $-k$ and k. As in the previous case, there are no nontrivial primitive solutions. The d_i must be equal, yet in the construction of S, arithmetic progressions with the same common difference do not intersect.

A cycle with only one foundation vertex is not possible. Therefore the girth is at least $\min\{12, k\}$. □

Just like the Tutte and girth 9 constructions, this method generalizes to arbitrary chromatic number. By attaching girth 12, $(l-1)$-chromatic graphs to appropriate arithmetic progressions of foundation vertices we get girth 12, l-chromatic graphs. Again, the 4-colorable ones seem like the only reasonable candidates for unit distance graphs.

1.6.4 G is a Unit Distance Graph

Theorem 24 *There exist girth 12, 4-chromatic unit distance graphs.*

Proof: By the previous theorems we know that for appropriate choices of k and n, $(G_{n,k-cycle,S})$ is a 4-chromatic graph with girth at least 12. Given odd $k \geq 13$, let n_0 be the smallest such n. We show that $(G_{n_0,k-cycle,S})$ is a unit distance graph using an embedding procedure similar to that used for the Tutte graphs and the girth 9 graphs. By the choice of n_0, there is a 3-coloring of the foundation vertices from 1 to $n_0 - 1$ such that no monochromatic set has an odd cycle attached. We place all the foundation vertices with color i in the δ-ball around C_i for $1 \leq i \leq 3$. We place vertex n_0 in the δ-ball around C_4. Since the vertices with a k-cycle attached are always in at least 2 δ-balls, the embedding lemmas allow the attachments of all cycles and removal of any coincidences. (Technically if the girth is more than 12, we add a 12-cycle to get a girth 12 graph.) □

1.7 Girth k, 4-Chromatic Unit Distance Graphs

1.7.1 Graph Description

In the introduction we gave a little history of the arbitrary girth, arbitrary chromatic number hypergraph problem. Only arbitrary girth, 4-chromatic hypergraphs are needed.

Theorem 25 *(Erdős-Hajnal) For all integers $k \geq 2$ and $g \geq 2$ and $l \geq 2$ there exist k-uniform, girth g, l-chromatic hypergraphs.*

This theorem gives the desired generalization of the girth 9 and girth 12 constructions. Instead of attaching cycles to arithmetic progressions, we attach cycles to the edges (hyperedges) of a hypergraph. Given k and g, let $\mathcal{H}$ be a k-uniform, girth g, 4-chromatic hypergraph. Let $n = |V(\mathcal{H})|$. Then $G_{n,k-cycle,\mathcal{H}}$ is the desired graph.

1.7.2 $\chi(G) = 4$

Theorem 26 $\chi(G_{n,k-cycle,\mathcal{H}}) = 4$

Proof: Since $\mathcal{H}$ is 4-chromatic, any 3-coloring of the foundation vertices contains a monochromatic hyperedge. In other words any 3-coloring of the foundation vertices has a monochromatic set with an odd cycle attached. That cycle can not be colored with the remaining two colors, so $\chi(G_{n,k-cycle,\mathcal{H}}) \geq 4$. With four colors, one can be used for the foundation vertices leaving three for the odd cycles. Thus $\chi(G_{n,k-cycle,\mathcal{H}}) = 4$. □

1.7.3 girth$(G) = k$

Theorem 27 *girth*$(G_{n,k-cycle,\mathcal{H}}) = k$

Proof: We show that girth$(G_{n,k-cycle,\mathcal{H}}) \geq \min\{k, 3g\}$ and choose $g \geq k/3$. The only cycles containing no foundation vertices are the attached k-cycles. All other cycles consist of foundation vertices separated by at least two vertices of attached cycles. Since any two consecutive foundation vertices are in the shadow of an attached cycle in G (i.e. appear in a hyperedge of $\mathcal{H}$), then the consecutive foundation vertices form a cycle (i.e. hypercycle) in $\mathcal{H}$. So if the girth of $\mathcal{H}$ is g, the length of the cycle in $G_{n,k-cycle,\mathcal{H}}$ is at least $3g$. Thus all cycles of $G_{n,k-cycle,\mathcal{H}}$ are either k-cycles or $\geq 3g$ cycles. □

1.7.4 G is a Unit Distance Graph

Theorem 28 *For all $k \geq 3$, there exist girth k, 4-chromatic, unit distance graphs.*

Proof: Assume k is odd. Let $\mathcal{H}$ be a k-uniform, 4-chromatic hypergraph with girth $\geq k/3$ having the fewest vertices. Let $n_0 = |V(\mathcal{H})|$. Then $G_{n_0,k-cycle,\mathcal{H}}$ is a girth k, 4-chromatic graph. We use embedding procedures similar to those used for the Tutte, girth 9, and girth 12 graphs. By the choice of n_0, there is a 3-coloring of the foundation vertices from 1 to $n_0 - 1$ such that no hyperedge is monochromatic, in other words, so that no monochromatic set has an odd cycle attached. We place all the foundation vertices with color i in the δ-ball around C_i for $1 \leq i \leq 3$, and place vertex n_0 in the δ-ball around C_4. Since the vertices with a k-cycle attached are always in at least 2 δ-balls, the embedding lemmas allow the attachments of all cycles and removal of any coincidences. For even k, a k-cycle is added to a girth $> k$, 4-chromatic unit distance graph. □

1.8 Small, Girth 4, 4-Chromatic Unit Distance Graphs

1.8.1 A 56 Vertex, Girth 4, 4-Chromatic Unit Distance Graph

We now investigate the minimum number of vertices in a "high" girth 4-chromatic unit distance graph. Mycielski [21] describes a method to construct triangle-free graphs with arbitrary chromatic number l: Start with a triangle-free $(l-1)$-chromatic graph G. For each vertex $v_i \in V(G)$ add a vertex w_i adjacent to all vertices in the neighborhood of v_i. Next, add a vertex z adjacent to all the new vertices. The chromatic number of this graph is l, and it is still triangle-free. Unfortunately, the resultant graph probably does not embed in the plane. Notice that if a vertex of G has degree 3 or more, then the "Mycielskian" of G contains a $K_{2,3}$. The plane contains no unit distance $K_{2,3}$ subgraph, so the starting graph G must have maximum degree at most two for the Mycielskian to be a unit distance graph. The only candidates for the unit distance version of the Mycielski construction are unions of paths and cycles. Mycielskians of odd cycles do not embed in the plane however, so the Mycielski construction does not give a 4-chromatic unit distance graph. The Mycielskian of at least one even cycle does embed, though.

The graph H in Figure 1.2 is the Mycielskian of the 10-cycle. It can be shown with basic geometry and algebra that H can be embedded in the plane, but R. Hochberg [14] pointed out a nicer proof which shows *why* this is so: H is a subgraph of the projection of the 5-cube along a diagonal onto the plane. The coordinates of the vertices v_1, v_3, v_5, v_7, v_9 are the fifth roots of unity, while the edges are all unit length since they are translations of these unit vectors. Alas this graph is only 3-chromatic. We attach 5-cycles to make it 4-chromatic.

Claim 29 *A 5-cycle can be attached to* $R = \{v_1, v_3, v_5, v_7, v_9\}$.

Proof: Center a regular pentagon of side length one at the origin and rotate it until the distance from one of its vertices to v_1 is 1. Then the respective distances from the other vertices of the pentagon to the other vertices in the set are 1. □

Claim 30 *A 5-cycle can be attached to* $S = \{w_1, w_3, w_6, w_8, z\}$.

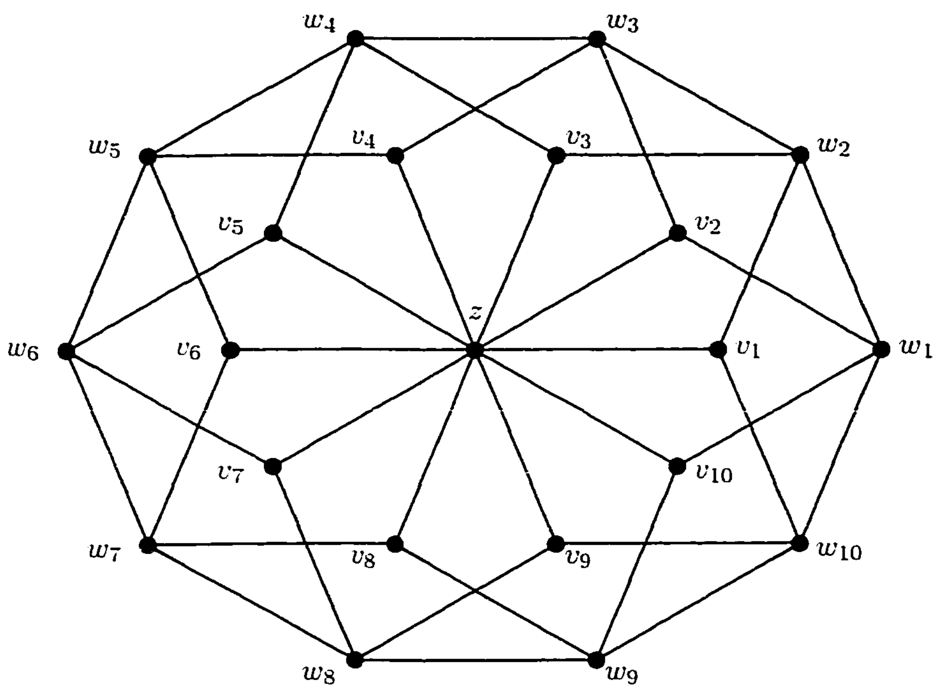

Figure 1.2: H IS THE MYCIELSKIAN OF C_{10}.

Proof: The proof relies on the intermediate value theorem and continuity methods used earlier. Described a little less formally, we try to attach a 5-cycle to the five vertices in S so that the cycle edges and the connecting edges are all length one. In fact we try it twice. The problem with the attachments is that in the first, one of the edges in the cycle is too short, in the second, it is too long. Since one configuration is obtained from the next by a continuous transformation, there exists an attachment where that same edge has length one. Thus S can have a 5-cycle attached (see Figure 1.3). □

Claim 29 allows us to attach a 5-cycle to $\{v_1, v_3, v_5, v_7, v_9\}$. Similarly we can attach a 5-cycle to $\{v_2, v_4, v_6, v_8, v_{10}\}$. Call this new graph H'. In a proper 3-coloring of H' the vertices in $\{v_1, v_3, v_5, v_7, v_9\}$ can not get the same color since that leaves only two colors for the attached 5-cycle. The same holds for $\{v_2, v_4, v_6, v_8, v_{10}\}$. This is enough to rule out most of the 3-colorings of H'. In fact, except for the vertices of the attached 5-cycles, the coloring of H' is completely determined up to symmetries. This coloring is shown in Figure 1.4 (the attached 5-cycles are not shown in the figure). Note that there are numerous ways to color the attached 5-cycles, but their attachment forces the rest of the graph to have a unique coloring up to permutation of the colors and rotation

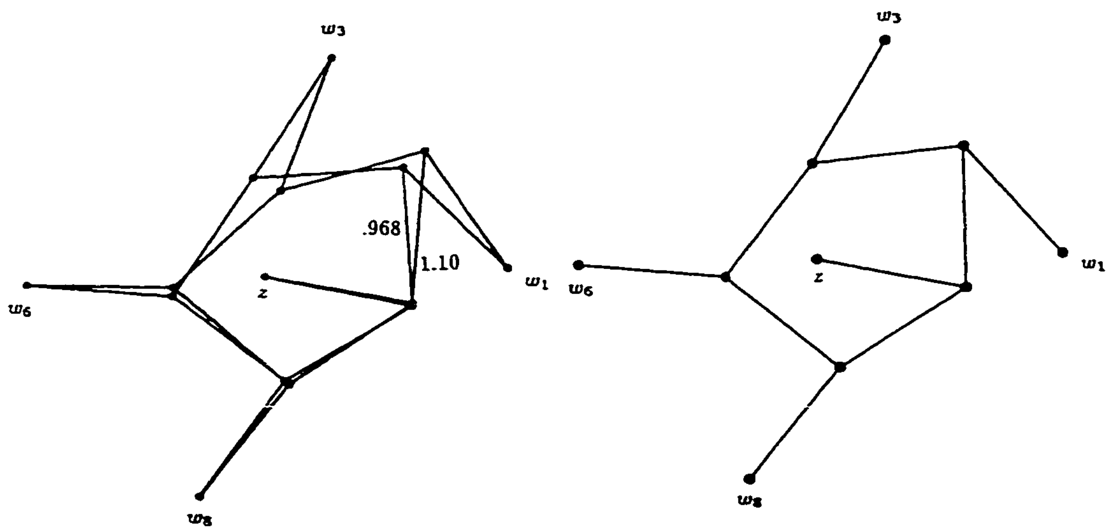

Figure 1.3: THE "SHORT ATTACHMENT" AND THE "LONG ATTACHMENT" ARE SHOWN TOGETHER ON THE LEFT. THE "JUST RIGHT" ATTACHMENT IS ON THE RIGHT. (ALL UNLABELED EDGES ARE UNIT LENGTH.)

of the graph.

In particular, in every three coloring of H', one of the sets $\{w_j, w_{j+2}, w_{j+5}, w_{j+7}, z\}$ (addition modulo 10) for $1 \leq j \leq 5$ is monochromatic, where z and the w_i are as in Figure 1.2. By attaching 5-cycles to all five of these sets, all 3-colorings are excluded. The result is a four chromatic graph. Moreover since H is triangle-free, this new graph is also triangle-free. Approximation of the coordinates of the vertices ensures there are no coincident vertices.

Counting everything, H has 21 vertices, then two 5-cycles are added, then five more. The result is a triangle-free, 4-chromatic graph on 56 vertices (Figure 1.5).

1.8.2 A 40 Vertex, Girth 4, 4-Chromatic, Unit Distance Graph

A similar approach starts with the Mycielskian of the 5-cycle. This 11-vertex graph is the smallest triangle-free 4-chromatic graph. Since this is not a unit distance graph, we modify it by taking the "central" vertex adjacent to the 5 "new" vertices, and replacing it with five vertices each adjacent to a pair of "new" vertices as shown in Figure 1.6. This graph is called H.

H is 3-chromatic, but all the 3-colorings share a property. In every 3-coloring, one

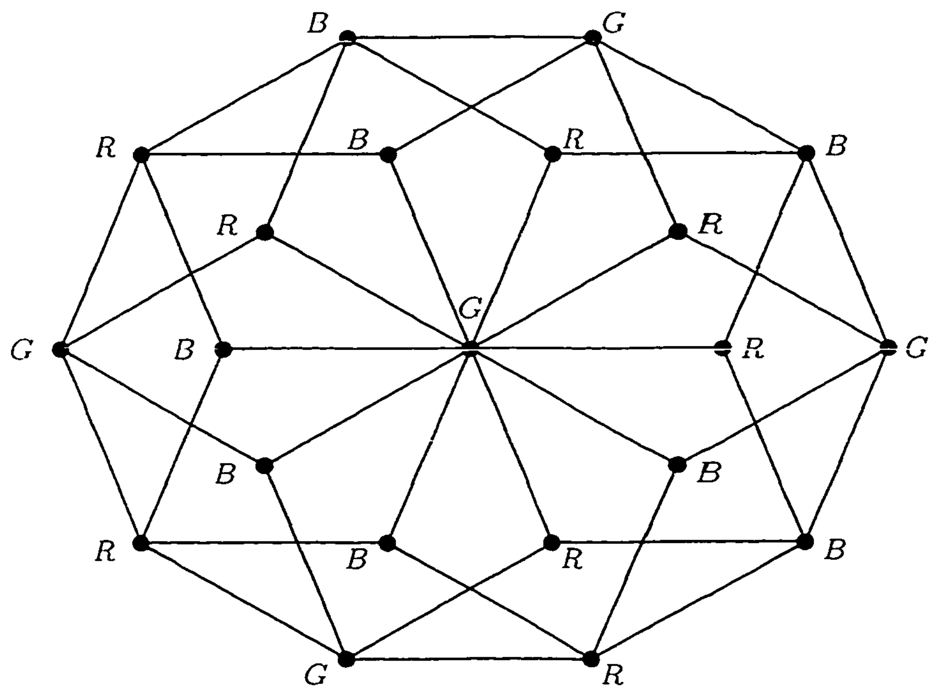

Figure 1.4: THE VERTICES OF H MUST HAVE THIS COLORING UP TO SYMMETRIES WHEN THE TWO 5-CYCLES ARE ATTACHED. THE ATTACHED 5-CYCLES ARE NOT SHOWN.

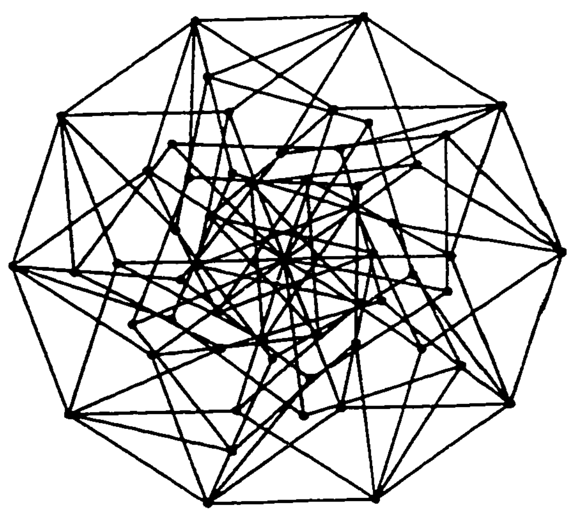

Figure 1.5: A 4-CHROMATIC UNIT DISTANCE GRAPH WITH NO 3-CYCLES

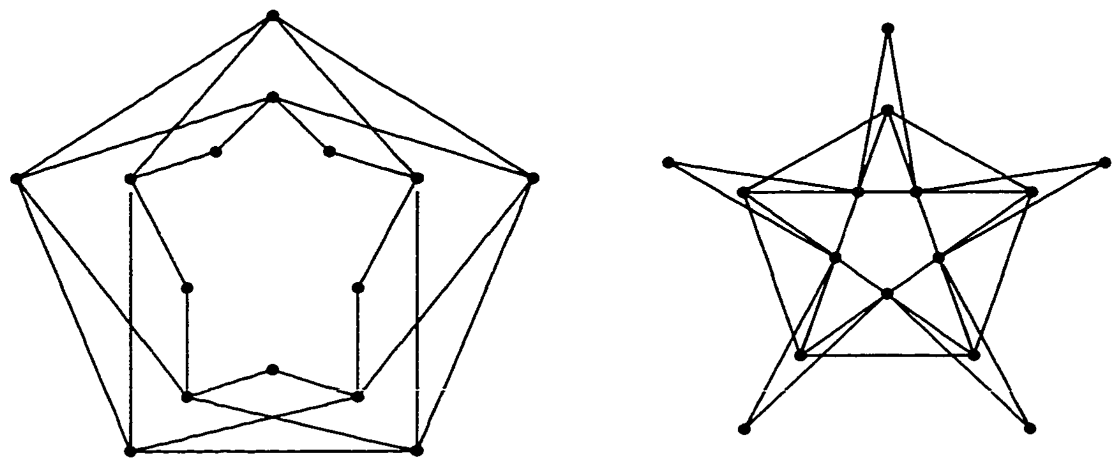

Figure 1.6: A VISUALLY APPEALING DRAWING OF H IS ON THE LEFT. A UNIT DISTANCE DRAWING OF H IS ON THE RIGHT (IN WHICH ONLY VERTICES UNIT DISTANCE APART ARE ADJACENT, DESPITE OVERLAPPING EDGES IN THE DRAWING GIVING THE ILLUSION OTHERWISE).

of the sets $\{v_{1+i}, v_{6+i}, v_{11+(i+1)}, v_{11+(i+2)}, v_{11+(i+3)}\}$, for $0 \leq i \leq 4$ (where parentheses indicate addition modulo 5), is monochromatic. By attaching 5-cycles to all such sets, all 3-colorings are excluded. Thus, the resultant graph is 4-chromatic and still triangle-free. It remains to be shown that it is a unit distance graph.

Claim 31 *A 5-cycle can be attached to* $T = \{v_1, v_6, v_{12}, v_{13}, v_{14}\}$

Proof: We try to attach a 5-cycle w_1, w_2, w_3, w_4, w_5 so that the cycle edges and all the connecting edges are length 1. As in the previous claim, this can be done. □

By attaching 5-cycles to T and its rotations, and ensuring no coincidences exist, a graph with the desired properties is obtained. Since H had 15 vertices and five 5-cycles were attached, the result is a triangle-free, 4-chromatic unit distance graph on 40 vertices (Figure 1.7).

1.9 Related Questions and Answers

In joint work with R. Hochberg [15], the upper bounds on the sizes of the smallest 4-chromatic unit distance graphs with girths 4 and 5 were lowered even more. A 23

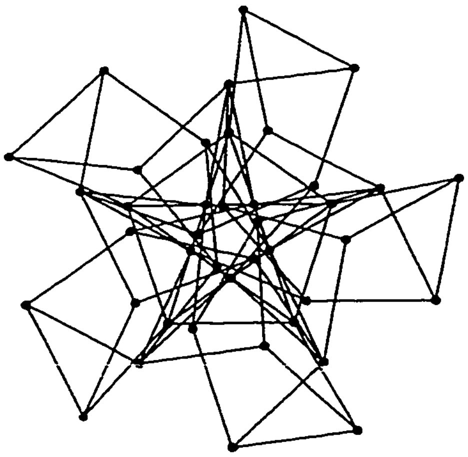

Figure 1.7: A 4-CHROMATIC UNIT DISTANCE GRAPH WITH NO 3-CYCLES

vertex, girth 4, 4-chromatic unit distance graph was found. A 45 vertex, girth 5, 4-chromatic unit distance graph was found. The constructions involved a generalized version of cycle attachment.

Chapter 2

The Choice Numbers of $K_{5,q}$ and $K_{6,q}$

2.1 Introduction

In a generalization of the idea of chromatic number of a graph, Erdős, Rubin, and Taylor [9], and separately Vizing [31], formulated the idea of *choice number* of a graph. Rather than having a list of k colors to use for the entire graph, each vertex has its own list of k colors. The obvious coloring rules apply: Each vertex must be colored with a color on its list, and no two adjacent vertices may be colored the same. Of course if every vertex has the same list of colors, this is just normal graph coloring. Since this need not be the case, the question instead is: Can the graph be colored no matter how k colors are assigned to each vertex?

For example, $K_{3,3}$ with the lists of length 2 shown below can not be properly colored. However, any assignment of lists of length 3 to each vertex does permit a proper coloring. Therefore $K_{3,3}$ has choice number 3.

$K_{7,7}$ has choice number 4, because it can be properly colored given any assignment of lists of length 4 to each vertex, but there are lists of length 3 which do not permit

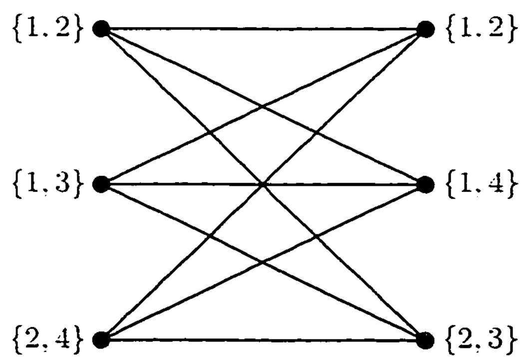

Figure 2.1: $K_{3,3}$ CAN NOT BE COLORED USING THE LIST ASSIGNMENT SHOWN.

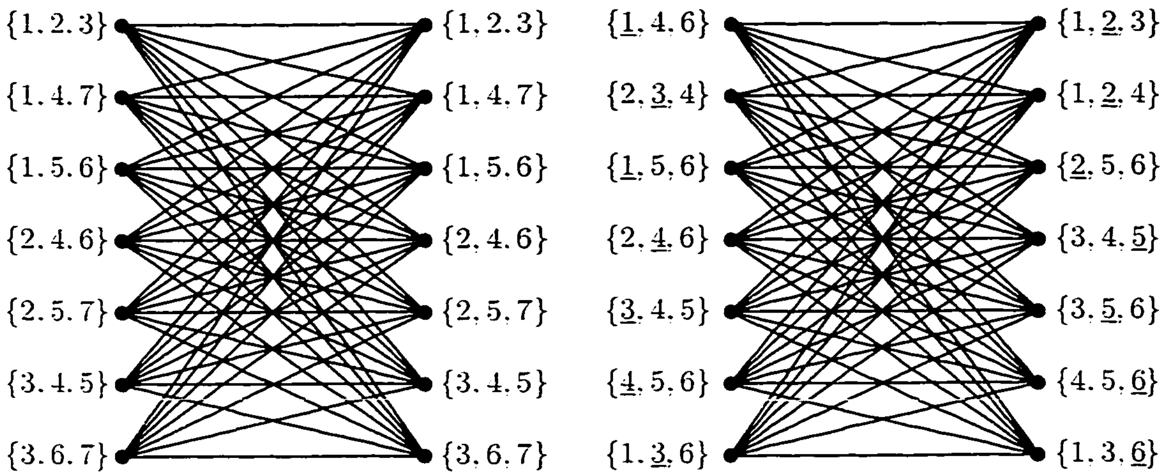

Figure 2.2: $K_{7,7}$ CAN NOT BE COLORED USING THE LISTS OF LENGTH 3 ON THE VERTICES OF THE GRAPH ON THE LEFT. NOTICE THAT 7 COLORS TOTAL ARE USED ON THOSE LISTS. IF FEWER THAN 7 COLORS ARE USED, THEN $K_{7,7}$ CAN BE PROPERLY COLORED. FOR EXAMPLE, WITH THE LISTS ON THE VERTICES OF THE GRAPH ON THE RIGHT, THE UNDERLINED NUMBERS GIVE A PROPER COLORING. NOTICE THAT ONLY 6 COLORS TOTAL ARE USED ON THOSE LISTS.

a proper coloring. For example, with the lists of length 3 for the $K_{7,7}$ on the left in Figure 2.2, the graph can not be properly colored. Notice that 7 colors are used to make up the lists for the vertices. If only 6 colors are used, like in the $K_{7,7}$ on the right in Figure 2.2, then no matter what lists of length 3 are assigned to the vertices, the graph can be properly colored. We say $K_{7,7}$ has *restricted choice number* 3 when only 6 colors are allowed.

Erdős, Rubin, and Taylor characterized those graphs which can be colored given a list of length two for each vertex (i.e. graphs with choice number 2), but trying to find which graphs could be colored with just 3 choices for each vertex is difficult. Even classifying the complete bipartite graphs with this property proved hard. Erdős, Rubin, and Taylor; Mahadev, Roberts, and Santhanakrishnan; Füredi; and Shende and Tesman classified some as shown in Table 2.1. The second to last piece was a lengthy case by case analysis by Tesman and Shende showing that $K_{5,q}$ has choice number 3 for $q \leq 12$. We give a short proof of that result, and fill in the last piece of the classification by determining which $K_{6,q}$ graphs have choice number 3.

We exhibit conditions under which a coloring problem with r total colors can be reduced to a problem with $r-1$ total colors on the lists. This gives a good upper bound on the number of colors needed, cuts down the number of cases, and answers a question posed in [16]. We also develop a transformation from a $K_{m,n}$ coloring problem to a $K_{m',n'}$ coloring problem where n' is bigger than n, but m' is smaller than m. This new problem is often one whose answer is already known. This allows for easy analysis of several cases.

2.2 Choice Numbers

Basic graph theory notation and definitions can be found in [4]. Let $G = (V, E)$ be an n vertex graph. We consider list colorings for which all the lists are the same length. Formally, let $L = \langle L(v_1), L(v_2), L(v_3), \ldots, L(v_n)\rangle$ be a list assignment to the vertices of G such that $L(v_i) \subseteq \mathbb{N}$ and $|L(v_i)| = k$. An L-coloring of G is a system of representatives for L, i.e. an n-tuple $(r_1, r_2, \ldots, r_n)$, such that for each i, $1 \leq i \leq n$: $r_i \in L(v_i)$ and $r_i \neq r_j$ if $\{v_i, v_j\} \in E$. If there exists an L-coloring of G for all list assignments L, then G is *k-choosable* (where k is the number of colors on each list). The smallest k for which G is k-choosable is called the *choice number*, $c(G)$, of G.

We finish the classification of 3-choosable complete bipartite graphs by showing that $K_{6,q}$ is 3-choosable for $q \leq 10$, and give a shorter proof of the $K_{5,q}$ bound. Table 2.1 summarizes the classification.

p	*3-choosable*		*not 3-choosable*	
1	all $q \geq 1$,	[9]		
2	all $q \geq 1$,	[9]		
3	all $q \leq 26$,	[9]	$q \geq 27$,	[9]
4	all $q \leq 18$,	[22]	$q \geq 19$,	[22]
5	all $q \leq 12$,	[28]	$q \geq 13$,	[11]
6	all $q \leq 10$		$q \geq 11$,	[22]
≥ 7			all $q \geq p$	[22]

Table 2.1: Summary of 3-choosability for $K_{p,q}$.

2.3 Restricted Choice Numbers

Rather than having $\mathbb{N}$ as the set of possible colors for the lists, only the set containing the first r natural numbers will be used, $[r] = \{1, 2, 3, \dots, r\}$. As in [16], the *restricted choice number*, $c_r(G)$ is the smallest k for which there is a proper L-coloring of G for any list assignment L whose lists are k-subsets of $[r]$. Notice that

$$\chi(G) \leq c_r(G) \leq c_{r+1}(G) \leq c(G)$$

for all natural numbers $r \geq \chi(G)$. In [16], the *thwart number*, $thw(G)$, of a graph is defined to be the smallest number r_0 such that $c_{r_0}(G) = c(G)$. Since $c_{r_0}(G) \leq k$ implies G is k-choosable, only list assignments using at most r_0 colors need to be considered to determine k-choosability.

For the n-vertex complete graph, $\chi(G) = c_n(G) = c(G) = n$, so clearly there are cases which require at least n colors. In special situations we can do much better. If a list assignment has r colors, two of which do not appear on a list together, they can be combined into a single color. If the graph can be colored with the modified lists of $r-1$ colors, then it can be colored with the original lists. This is illustrated in figure 2.3. The technicalities are worked out in Lemma 32 and Corollary 33. A bound for the thwart number is also obtained, answering a question in [16].

Lemma 32 *Let G be a graph such that $c_{r-1}(G) \leq k$ and $\binom{k}{2}|V| < \binom{r}{2}$, then $c(G) \leq k$.*

Proof: Let L be any list assignment in which $L(v_i) \subset [r]$ and $|L(v_i)| = k$. If $|\bigcup L(v_i)| < r$, by assumption there is an L-coloring since $c_{r-1}(G) \leq k$. If $|\bigcup L(v_i)| = r$ and $\binom{k}{2}|V| < \binom{r}{2}$, then there exists a pair of colors which do not appear together on any $L(v_i)$. Merging the two colors reduces this problem to one in which each list is a subset of $[r-1]$. Since $c_{r-1}(G) \leq k$, there is a proper coloring using these merged lists. This can easily be extended back to a proper L-coloring, showing $c_r(G) \leq k$. Induction gives $c_p(G) \leq k$ for all $p \geq r-1$. Hence $c(G) \leq k$. □

Corollary 33 *Let G be a graph such that $c_{r-1}(G) \leq k$. Let $r - 1 = a(k-1) + b$, with $a, b \in \mathbb{N}$, $0 \leq b < k-1$. If $b > 0$ and $\binom{k}{2}|V| - [r(k-1-b)]/2 < \binom{r}{2}$ then $c(G) \leq k$.*

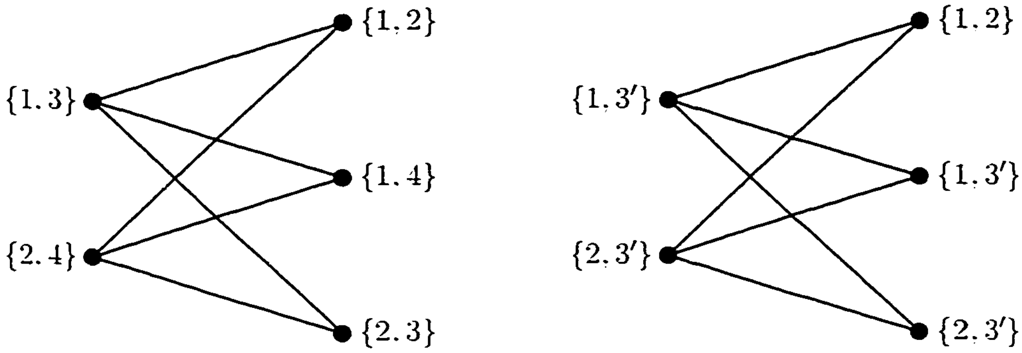

Figure 2.3: SINCE THE COLORS 3 AND 4 DO NOT APPEAR TOGETHER ON ANY LIST OF THE GRAPH ON THE LEFT, THEY CAN BE COMBINED INTO A SINGLE COLOR 3′ AND THE GRAPH WILL STILL HAVE LISTS OF LENGTH 2. IF THE GRAPH ON THE RIGHT (WITH THE COMBINED COLOR) CAN BE PROPERLY COLORED, SO CAN THE GRAPH ON THE LEFT (WITH THE ORIGINAL LIST ASSIGNMENT).

Proof: Assume $b > 0$. A color i_0 must appear on at least $a+1$ lists for it to appear with every other color on some list. There are $r-1$ other colors, but at least $(a+1)(k-1)$ pairs of the form $\{i_0, j\}$. At least $k-1-b$ must be duplicates. Summing over all colors, and dividing by two since each pair is counted twice, there are at least $[r(k-1-b)]/2$ duplicate color pairs. Thus, given r and k satisfying the above, any list assignment will already have fewer than r colors or will have two which do not appear together on any list (and so can be merged). As above, this implies $c_r(G) \leq k$. Induction gives $c_p(G) \leq k$ for all $p \geq r-1$. Hence $c(G) \leq k$. □

Corollary 34 $thw(G) \leq c(G)\sqrt{|V|}$.

Proof: Assume $r_0 = \lceil c(G)\sqrt{|V|}\rceil$. Let $k = c_{r_0-1}(G)$. Then

$$\binom{k}{2}|V| = \binom{c_{r_0-1}(G)}{2}|V| \leq \binom{c(G)}{2}|V| \leq \binom{r_0}{2}|V|.$$

By lemma 32, $c(G) \leq k$. But $c_{r_0-1}(G)$ and $c(G)$ are lower and upper bounds respectively for the restricted choice number $c_r(G)$ for all $r \geq r_0 - 1$. Therefore $c_r(G) = k$, for all $r \geq r_0 - 1$ and $thw(G) \leq r_0 - 1$. □

2.4 Choice Numbers of Complete Bipartite Graphs

2.4.1 Preliminaries

The notation is modified for ease in dealing with the 3-choosability of complete bipartite graphs $G = (\mathcal{A}, \mathcal{B}, E) = K_{m,n}$. Let $A = \langle a_1, a_2, \ldots, a_m \rangle$ and $B = \langle b_1, b_2, \ldots, b_n \rangle$ be list assignments for the vertices of $\mathcal{A}$ and $\mathcal{B}$ respectively, with $|a_i| = |b_j| = 3$, for $1 \leq i \leq m$, $1 \leq j \leq n$. Then disjoint systems of representatives for A and B constitute a list coloring of G, denoted an AB-coloring. Define A_i (or B_i) to be the collection of sets in A (or B) containing the element i. Let $\nu(A)$ (or $\nu(B)$) be the maximum number of sets from A (or B) that have nonempty intersection, i.e. $\nu(A) := \max_j |A_j|$.

Lemma 35 *Let A, B be list assignments for the two parts of a complete bipartite graph G. For fixed i_0, let G' be the complete bipartite graph with list assignments A', B' as follows, where α, β, γ are new colors:*

$$\{\alpha, \beta, \gamma\} \in A',$$

$$a \in A \text{ but } a \notin A_{i_0} \Longrightarrow a \in A',$$

$$b \in B_{i_0} \Longrightarrow (b \backslash i_0) \cup \{\alpha\},\ (b \backslash i_0) \cup \{\beta\},\ (b \backslash i_0) \cup \{\gamma\} \in B', \text{ and}$$

$$b \in B \text{ but } b \notin B_{i_0} \Longrightarrow b \in B'.$$

If there is an $A'B'$-coloring, then there is an AB-coloring.

This lemma just makes the statement, "If there is a proper coloring where color i_0 is used as a representative of some of the lists in A, then there is a proper coloring." The problem of AB-coloring $K_{m,n}$ reduces to $A'B'$-coloring $K_{m-j+1,n+2\ell}$, where $j = |A_{i_0}|$ and $\ell = |B_{i_0}|$. The existence of the $A'B'$-coloring is often implied by the results in Table 2.1.

Proof: Given an $A'B'$-coloring, we produce an AB-coloring. Lists $a \notin A_{i_0}$, $b \notin B_{i_0}$ use the color from the $A'B'$-coloring. Lists $a \in A_{i_0}$ use the color i_0. Since one of the colors α, β, γ, say α, must be used to represent $\{\alpha, \beta, \gamma\} \in A'$ in a proper $A'B'$-coloring, then

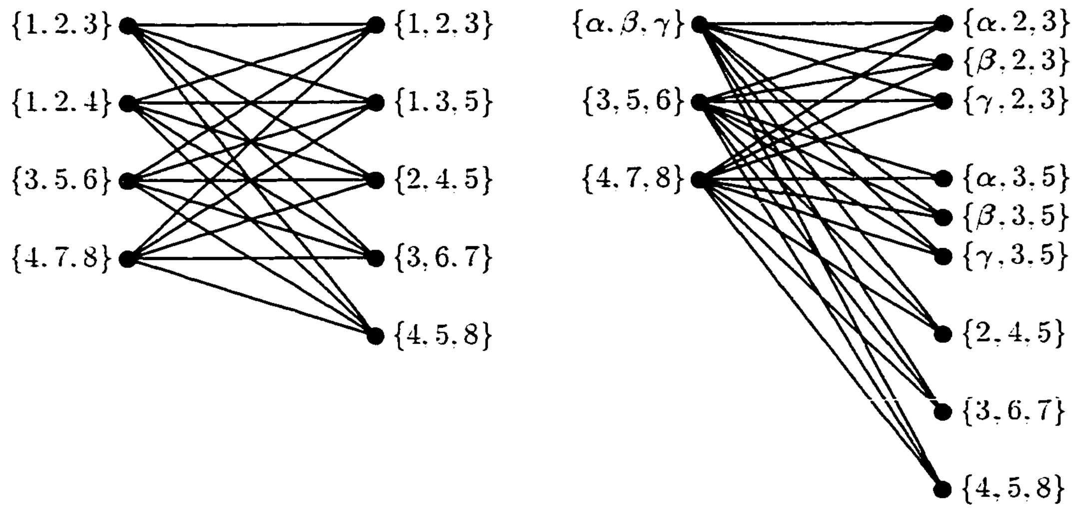

Figure 2.4: THE TRANSFORMATION IN SIMPLE TERMS SAYS THAT THE COLOR 1 IS USED TO COLOR VERTICES ON THE LEFT PART OF THE BIPARTITE GRAPH ON THE LEFT (WHICH MEANS IT CAN'T BE USED TO COLOR ANY OF THE VERTICES ON THE RIGHT PART OF THAT GRAPH). WE CAN COLOR THE REST OF THAT GRAPH AS LONG AS THE BIPARTITE GRAPH ON THE RIGHT CAN BE COLORED WITH THE LIST ASSIGNMENT SHOWN.

for lists $b \in B_{i_0}$, a color other than α must be used to represent the list $(b \backslash i_0) \cup \{\alpha\}$ in B'. This color is used for b. This is a proper AB-coloring. □

We'll need the following basic lemmas. They appear throughout the literature. A brief explanation/rationale follows each.

Lemma 36 *A complete bipartite graph $G = (\mathcal{A}, \mathcal{B}, E)$ with list assignments A and B is not 3-choosable if and only if every SR of A contains some $b \in B$.*

If some SR of A doesn't contain any list $b \in B$, then each list in B contains a color (not in SR) which can be used to color its corresponding vertex.

Lemma 37 *If A has more than $|\mathcal{B}|$ SR's with pairwise intersection at most two, then G is 3-choosable.*

If there is not a coloring, then every SR must contain some $b \in B$. By assumption, no two of these SR's contain the same 3-element set, so there are at most $|\mathcal{B}|$ SR's with pairwise intersection at most two.

Lemma 38 *If A has an SR with just two colors, then G is 3-choosable.*

Such an SR can not contain a 3-element $b \in B$.

2.4.2 Choice Number of $K_{5,12}$

Theorem 39 *$K_{5,q}$ is 3-choosable for $q \leq 12$.*

Proof: It is enough to show that $K_{5,12}$ is 3-choosable since by inclusion this implies the result for $q < 12$. Let A and B be list assignments for $K_{5,12}$. By corollary 33, it is sufficient to consider list assignments using at most 10 colors. One of the following cases holds:

Case I: $\nu(A) \geq 3$ (i.e. there is a color which appears on at least 3 of of the 5 lists in A).

Clearly if $\nu(A) \geq 4$ then there is an SR for A with one or two colors. By lemma 38, there is a coloring of G. If $\nu(A) = 3$ then some color appears on three of the 5 lists in A. We may assume the color $1 \in a_1 \cap a_2 \cap a_3$. If $a_4 \cap a_5 \neq \emptyset$ then again there is an SR of size less than three for A, and hence a coloring of G. Otherwise, by lemma 36, $\nu(B) \geq 9$ since all nine 3-element SR's for A which contain the color 1 must appear as lists of B. By lemma 35, this reduces to $K_{\leq 11, \leq 4}$, which is 3-choosable as stated in Table 2.1.

Case II: $\nu(A) = 2$.

Assume A and B are list assignments which do not permit a proper coloring. We start with an analogue of lemma 32. If two colors don't appear together on a list, they can be merged, so we may assume each pair of colors appears on some list together. If k colors total are used for A and B, it follows that a color which appears only once on lists of A must appear at least $(k-3)/2$ times on lists of B in order to appear with every other color on a list.

If a color i_0 appears twice on lists of A how many times must it appear on lists of B? The answer depends on the configuration of the three lists not containing i_0. If only five colors are used (the minimum) then A has at least seven 3-element SR's containing the color i_0. By lemma 36, each of these appears as a list in B. If nine colors are used (the maximum) for the lists in $A \backslash A_{i_0}$, then there are twenty-seven 4-element SR's containing the color i_0. By lemma 36, each of these has a subset which is a list in B. Since there are only 12 lists in B, at least eight must contain the color i_0. There are other configurations of $A \backslash A_{i_0}$, but in each case, lemma 36 can be used to show that every color which appears twice in A appears at least five times in B.

Say A consists of 6 colors which appear two times and 3 colors which appear one time. Each of the 6 "doubles" must appear at least five times in B, while each of the 3 "singles" must appear three times in B. This equals 39 colors (counting multiplicities) to fill 36 positions on the lists in B (3 colors per list, 12 lists). Impossible. The other configurations of A yield similar results. There must be a proper coloring.

These are the only two cases (at most 10 colors are used total, so $\nu(A) \geq 2$). Therefore $K_{5,12}$ is 3-choosable. □

2.4.3 Choice Number of $K_{6,10}$

Theorem 40 *$K_{6,q}$ is 3-choosable for $q \leq 10$.*

Proof: It is enough to show that $K_{6,10}$ is 3-choosable, since by inclusion this implies the result for $q < 10$. Let A and B represent list assignments for $K_{6,10}$. By corollary 33 it is sufficient to consider cases using at most 9 colors. One of the following cases holds:

Case I: $\nu(A) \geq 4$ (i.e. there is a color which appears on at least 4 of the 6 lists in A).

Clearly, if $\nu(A) \geq 5$ then A can be colored with one or two colors, so by lemma 38 there is a coloring of G. If $\nu(A) = 4$ then if $\nu(B) \leq 8$, by lemma 35 we can reduce this to a $K_{3,q}$ with $q \leq 26$ which is known to be 3-choosable. Otherwise if $\nu(B) \geq 9$, then B can be colored with one or two colors, so again by lemma 38 there is a coloring of G.

Case II: $\nu(A) = 3$.

Assume $|A_1| = 3$. in other words the color 1 appears on 3 of the lists of A. If $|B_1| \leq 4$ or $|B_1| \geq 6$. using lemma 2 (and its dual version, swapping the roles of A and B) the assignment can be reduced to a 3-choosable case, either $K_{4,q}$ with $q \leq 18$, or $K_{12,q}$ with $q \leq 5$. On the other hand, if $|B_1| = 5$, we may assume the color $1 \in (\bigcap_{i=1}^{3} a_i) \cap (\bigcap_{j=1}^{5} b_j)$, but not on any other lists. What could a_4, a_5, and a_6 look like if there were no AB-coloring? Let j, k, and l be the number of minimal SR's for $\{a_4, a_5, a_6\}$ containing one, two, and three elements respectively. Then $j = 0$. otherwise there is a 2-element SR for A. and by lemma 36. $k \leq 5$ and $l \leq 20 - 3k$. So a_4, a_5, a_6 could only look like $\{234\}, \{256\}, \{378\}$. We now verify that even in this case a proper AB-coloring is possible. By lemma 36. B must contain $\{123\}, \{127\}, \{128\}, \{135\}, \{136\}$ and also $\{457\}, \{458\}, \{467\}, \{468\}$. This leaves one last set in B, say $\{xyz\}$. If that last set contains a 1 or 4, then we have a 2-element SR for B, and hence a coloring. If not, then applying lemma 36 from B to A, A must contain $\{14x\}, \{14y\}, \{14z\}$. But this implies that 4 appears in four of the a_i (it's also in $\{234\}$), contradicting $\nu(A) = 3$.

Case IIIA: $\nu(A) = 2$ and there exist i, j such that $|a_i \cap a_j| \geq 2$, $i \neq j$.

Notice that $\nu(A) = 2$ implies $|\bigcup a_i| = 9$ with each color appearing on exactly two lists of A. If $a_i = a_j$ then the assignment reduces to $K_{5,10}$, so we assume $|a_i \cap a_j| = 2$. In fact, we may assume $a_1 = \{12x\}, a_2 = \{12y\}$. Notice that if the color 1 is used to color a_1 and a_2, then we can use the color 2 for the sets in B_2 without penalty. Lemma 35 combined with this observation provides a way to reduce this AB-coloring problem for $K_{6,10}$ to an $A'B'$-coloring problem for $K_{5,10+2\ell_1-\ell_2}$, where $\ell_1 = |B_1 \backslash B_2|$ and $\ell_2 = |B_2|$ (or the analogous situation with colors 1 and 2 switched). We conclude, by the previous arguments, that there is a proper AB-coloring, except perhaps if $|B_1 \cup B_2| \geq 8$ or if $|B_1| = |B_2| = 3$ and $|B_1 \cap B_2| = 0$.

In the first instance we construct a proper coloring as follows. Assume $|B_1 \cup B_2| \geq 8$. From each list in B not containing the colors 1 or 2, any element except x or y is chosen as a representative. For the rest of B, the colors 1 and 2 are used. Since any list a_i contains at most two of these, A has a system of representatives disjoint from the colors used for B. Hence there is an AB-coloring.

In the second instance we will use a counting argument. Assume $|B_1| = |B_2| = 3$ and $|B_1 \cap B_2| = 0$. Using nine colors, there are $\binom{7}{3} = 35$ ways to choose the color 1 and three additional colors as possible representatives for A, with the color 2 and the remaining four colors for B. Is it possible that a partition does not give rise to a coloring? A partition is bad if one of the lists in B is contained in the colors assigned to A, or if one of the lists in A is contained in the colors assigned to B. The former could happen to at most $3\binom{5}{1} + 4\binom{4}{0} = 19$ partitions. The latter can happen to at most $4\binom{4}{1} = 16$ partitions. So there are 35 partitions and at most 35 bad ones. Fewer than 35 bad partitions leaves a good one from which we get a proper AB-coloring. If exactly 35 bad partitions exist, then in the sum there is no double counting of bad partitions.

Notice for example, if two lists in B_1 had pairwise intersection of size 2 or more, say $\{1xy\}$ and $\{1xz\}$, then the partition with colors 1, x, y, and z for A contains both of these. In other words there is double counting of this bad partition (which means there is at least one good one). So lists in B_1 can only have pairwise intersection of size 1. Furthermore each list in B_1 must intersect every list in $A \backslash A_1$, while lists in $A \backslash A_1$ must have pairwise intersection of size 0 or 1. To satisfy the first condition, we may assume $B_1 = \langle\{134\}, \{156\}, \{178\}\rangle$. To satisfy the others, $A \backslash A_1 \cong \langle\{357\}, \{368\}, \{458\}, \{467\}\rangle$. This forces $a_1 = a_2 = \{129\}$, contradicting $|a_i \cap a_j| = 2$.

Case IIIB: $\nu(A) = 2$ and $\forall i, j\ |a_i \cap a_j| \leq 1$.

As in case IIIA, each of nine colors appears on exactly two lists a_i. Since $|a_i \cap a_j| \leq 1$, the intersection graph of the a_i is a six vertex three-regular graph. There are two such graphs (the complements of which are the only six vertex two-regular graphs: a 6-cycle and disjoint 3-cycles). In both, there are more than ten 3- or 4-element vertex covers which intersect pairwise in fewer than three edges. These correspond to more than ten SR's with pairwise intersection at most 2. By lemma 37, this implies the existence of a proper AB-coloring in each instance.

Graph 1: K_6 minus a Hamilton cycle is the intersection graph of $A = \langle\{123\}, \{456\}, \{178\}, \{249\}, \{358\}, \{679\}\rangle$, which has the following eleven SR's: 159, 268, 347, 257, 1256, 1346, 1457, 1489, 2367, 2478, 2589.

Graph 2: $K_{3,3}$ is the intersection graph of $A = \langle\{123\}, \{456\}, \{789\}, \{147\}, \{258\}, \{369\}\rangle$, which has the following eleven SR's: 159, 168, 267, 249, 348, 357, 1269, 1358, 2347, 3459, 2468. □

Chapter 3

Disjoint Non-Crossing Edges in the Plane

3.1 Introduction

The following question was posed by Alon and Erdős [1]. Given a two dimensional drawing of a graph, try to find k edges no two of which share a common vertex or cross. Certainly a huge complete graph no matter how it is drawn contains large "geometric matchings." But what if the graph is not the complete graph? How many edges must a graph contain to guarantee that it contains a large "geometric matching," and in particular one of size 3? Alon and Erdős showed that any graph G with $6n - 5$ edges contains 3 edges which form a matching in G and which do not cross in the two dimensional drawing of G. We improve this bound to $3.6n + 3.4$.

The lemmas used by Alon and Erdős are needed for the new bound as well, so they are included with some terminology changed around to facilitate the proof of the improved bound.

3.2 Definitions

Let G be a planar representation of a graph in $\mathbb{R}^2$ on n vertices in general position. We define a coordinate system so that no two vertices have the same x-coordinate. The ordering of the x-coordinates:

$$x_1 < x_2 < ... < x_n$$

defines an ordering of the vertices:

$$v_1 < v_2 < ... < v_n.$$

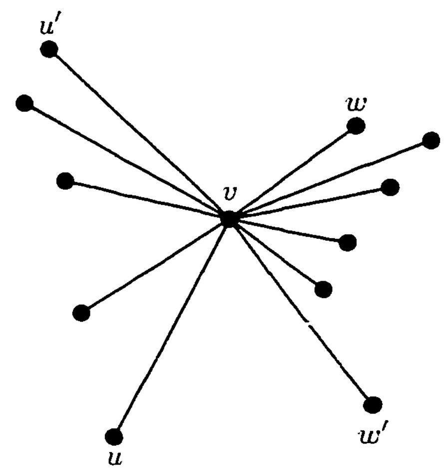

Figure 3.1: IN THE PORTION OF THE GRAPH SHOWN ABOVE, THE EDGES uv AND vw ARE COUNTERCLOCKWISE EXTREME AT v. THE EDGES $u'v$ AND vw' ARE CLOCKWISE EXTREME AT v.

With this ordering, an edge $\{u, v\}$ is thought of as the (directed) edge uv if $u < v$. In other words, uv denotes an edge if and only if $u < v$ and $\{u, v\}$ is an edge in G. If uv is an edge, we say it *begins* at u and *ends* at v.

Two edges in G are called *disjoint* if they are vertex disjoint and do not cross. Ultimately, we are looking for a set of three pairwise disjoint edges in G, and in particular an upper bound on the number of edges G must have in order to guarantee that such a set exists.

Let v be a vertex. Consider the set of all edges beginning at v. If this set is not empty, then there is an edge, say vw, in this set whose slope is maximum (where the slope of an edge is the slope of the corresponding segment in $\mathbb{R}^2$). Similarly if there are any edges ending at v, then there is an edge, say uv, whose slope is maximum for all edges ending at v. Such edges are called *counterclockwise extreme* at v. Edges of minimum slope starting or ending at v are called *clockwise extreme* at v. A *counterclockwise reduction* of G consists of removing all vertices which are counterclockwise extreme in G, while a *clockwise reduction* consists of removing all edges which are clockwise extreme.

The notation $e_k(n)$ represents the minimum number of edges such that all n-vertex graphs G containing $e_k(n)$ edges, contain k disjoint edges.

3.3 Assorted Lemmas

Lemma 41 *(Pannwitz. Hopf [17]. Kupitz [19]) A graph on n points in general position (no three on a line) in $\mathbb{R}^2$ with at least $n+1$ edges has 2 disjoint edges. This is the best possible.*

In other words. $e_2(n) = n + 1$.

Lemma 42 *(Alon. Erdős) Let G be a graph on n vertices in general position in $\mathbb{R}^2$ coordinatized so that no two vertices have the same x-coordinate. Let G^* be the counterclockwise reduction of the clockwise reduction of the counterclockwise reduction of G. If G^* is not the empty graph. then G has three pairwise disjoint edges.*

Since a vertex has at most two counterclockwise extreme edges and two clockwise extreme edges. at most $6n$ edges are removed from G to get G^*. Therefore. if G has $6n + 1$ edges it will contain three pairwise disjoint edges. Alon and Erdős did a little more work to reduce this to $6n - 5$. Perles [27] reduced the bound to $4n + 3$. It will be shown that the bound can be reduced to $3.6n + 3.4$ edges. In other words we'll show that $e_3(n) \leq 3.6n + 3.4$.

3.4 The Upper Bound

Theorem 43 *If G is a geometric graph with at least $3.6n + 3.4$ edges. then it must contain 3 pairwise disjoint edges.*

Proof: By lemma 41 there exist two disjoint edges. We choose a line which strictly separates them and whose slope is not the same as the slope of the line through any pair of vertices of G. This line will be the y-axis of our coordinate system. and by this choice. no two vertices will have the same x-coordinates. This gives an ordering of the vertices.

Let u be the first vertex which is the end of an edge. Define

$$A = \{v : v < u\}\ ,\ |A| = a.$$

Similarly, let w be the last vertex which is the beginning of an edge. Define

$$B = \{v : v > w\} \text{ , } |B| = b.$$

Note that A and B do not intersect since A and u are to the left of the y-axis while B and w are to the right. Let

$$C = V(G) - A - B.$$

then

$$|C| = c = n - a - b.$$

These vertex sets are illustrated in Figure 3.2.

Since no vertex in A is the end of an edge, each vertex in A has at most one clockwise extreme edge and at most one counterclockwise extreme edge. Similarly, since no vertex in B is the beginning of an edge, each vertex in B has at most one clockwise extreme edge and at most one counterclockwise extreme edge. Vertices in C, however, may have two of each. Even so, at most a+b+2c edges are removed at each step in the counterclockwise reduction, the clockwise reduction, and the counterclockwise reduction taking G to the reduced graph G^* as described in lemma 42. At most $3a + 3b + 6c$ edges are removed. By the lemma, if G^* is not empty then G has 3 pairwise disjoint edges. Thus, if $e_3(n)$ is the number of edges needed in G to guarantee 3 pairwise vertex-disjoint non-crossing edges,

$$e_3(n) \leq (3a + 3b + 6c) + 1 = 3n + 3c + 1 \text{ (since } a + b + c = n). \tag{3.1}$$

Changing viewpoint a little, let

$$A' = A \cup \{u\} \text{ , } |A'| = a + 1.$$

$$B' = B \cup \{w\} \text{ , } |B'| = b + 1.$$

$$C' = C - \{u, w\} \text{ , } |C'| = c - 2.$$

By lemma 41, we may assume that there are at most $|A'| + |C'| = a + c - 1$ edges in the graph induced by the vertices in $A' \cup C'$ (otherwise we have 2 disjoint edges, and we can find a third with both vertices in B' in which case we are done). Similarly, we may assume there are at most $|B'| + |C'| = b + c - 1$ edges in the graph induced by the

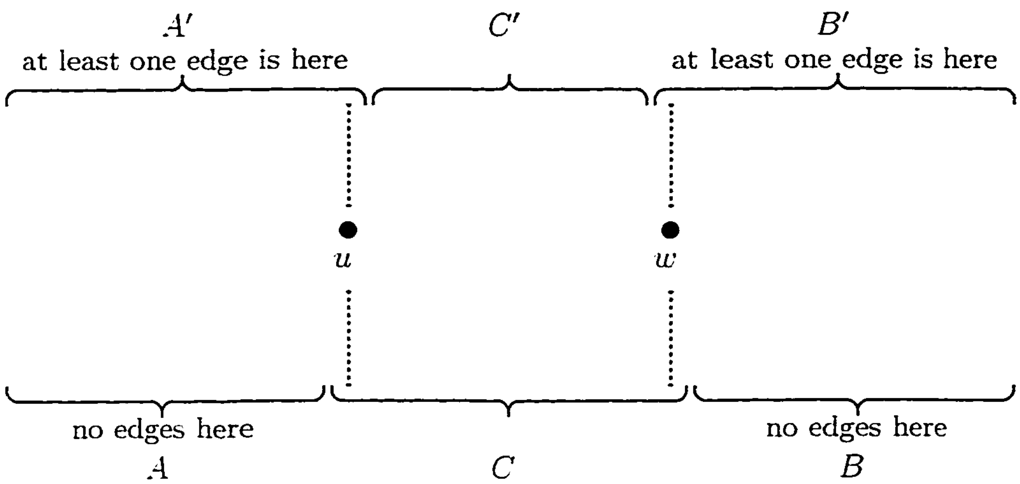

Figure 3.2: SINCE u IS THE FIRST VERTEX WHICH IS THE RIGHT END POINT OF AN EDGE, IT IS ON THE BORDER OF A AND C (OR A' AND C'). SINCE w IS THE LAST VERTEX WHICH IS THE LEFT END POINT OF AN EDGE, IT IS ON THE BORDER OF B AND C (OR B' AND C').

vertices in $B' \cup C'$. Removal of these edges from G leaves only edges from A' to B'. At most $3(|A'| + |B'|) = 3(a + 1) + 3(b + 1)$ edges are removed as the three reductions are performed on this graph as described in lemma 42. If an edge remains, then the original graph G has 3 pairwise disjoint edges. Hence,

$$e_3(n) \leq (a + c - 1) + (b + c - 1) + 3(a + 1) + 3(b + 1) + 1 = 4n - 2c + 5. \qquad (3.2)$$

Some manipulation of inequalities 3.1 and 3.2 establishes the upper bound on how many edges an n-vertex geometric graph must have to guarantee it contains 3 pairwise disjoint edges. If $c \leq n/5 + 0.8$ then by inequality 3.1,

$$e_3(n) \leq 3n + 3c + 1 \leq 3.6n + 3.4.$$

If $c \geq n/5 + 0.8$ then by inequality 3.2,

$$e_3(n) \leq 4n - 2c + 5 \leq 3.6n + 3.4.$$

Therefore, $e_3(n) \leq 3.6n + 3.4$. □

3.5 Related Questions and Answers

The original question Alon and Erdős were investigating was whether $e_k(n) = O(n)$. They proved it for $k = 3$ showing $e_3(n) \leq 6n - 5$. We merely improved their bound. Later, Pach and Töröcsik [26] used Dilworth's theorem to show that indeed $e_k(n) = O(n)$. Goddard, Katchalski, and Kleitman [13] improved our result by obtaining $e_3(n) \leq 3n + c$.

Appendix

Vertices used to show a cycle can be attached to vertices at the points C_1, C_2 and C_3:

T_1^{short}	$(0.99635, 0.08533)$
T_2^{short}	$(0.98269, 1.08524)$
T_3^{short}	$(1.84978, 0.58709)$
T_4^{short}	$(0.99800, 0.06319)$
S_1^{short}	$(0.57208, -0.82020)$
S_2^{short}	$(0.65177, 0.14158)$
S_3^{short}	$(1.64588, 1.56608)$
S_{14}^{short}	$(1.60981, -0.70439)$ (distance one from C_4 and T_1^{short})
S_{24}^{short}	$(1.77788, 0.47888)$ (distance one from C_4 and T_2^{short})
S_{34}^{short}	$(1.81111, -0.41216)$ (distance one from C_4 and T_3^{short})
T_1^{long}	$(0.99541, 0.09567)$
T_2^{long}	$(0.98069, 1.09556)$
T_3^{long}	$(1.85956, 0.61850)$
T_4^{long}	$(0.99280, 0.11977)$
S_1^{long}	$(0.58056, -0.81422)$
S_2^{long}	$(0.65971, 0.14848)$
S_3^{long}	$(1.62357, 1.59025)$
S_{14}^{long}	$(1.65414, -0.65671)$ (distance one from C_4 and T_1^{long})
S_{24}^{long}	$(1.77374, 0.48640)$ (distance one from C_4 and T_2^{long})
S_{34}^{long}	$(1.82462, -0.38089)$ (distance one from C_4 and T_3^{long})
T_1	$(0.99591, 0.09038)$
T_2	$(0.98173, 1.09028)$
T_3	$(1.85476, 0.60261)$
S_1	$(0.57623, -0.81729)$
S_2	$(0.65565, 0.14494)$
S_3	$(1.63492, 1.57815)$
S_{14}	$(1.63230, -0.68098)$ (distance one from C_4 and T_1)
S_{24}	$(1.77587, 0.48255)$ (distance one from C_4 and T_2)
S_{34}	$(1.81794, -0.39671)$ (distance one from C_4 and T_3)

Vertices used to show a cycle can be attached to vertices at the points C_1, C_2:

T_1^{short}	$(-0.06194, 0.99808)$
T_2^{short}	$(0.83339, 0.55268)$
T_3^{short}	$(0.75995, 1.54998)$
T_4^{short}	$(0.83339, 0.55268)$
S_1^{short}	$(0.83339, 0.55268)$
S_2^{short}	$(-0.06194, 0.99808)$
S_3^{short}	$(0.94288, 0.56685)$
T_1^{long}	$(-0.08916, 0.99602)$
T_2^{long}	$(0.81800, 0.57522)$
T_3^{long}	$(0.74037, 1.57220)$
T_4^{long}	$(0.81800, 0.57522)$
S_1^{long}	$(0.81800, 0.57522)$
S_2^{long}	$(-0.08916, 0.99602$
S_3^{long}	$(0.95233, 0.59493)$
T_1	$(-0.07551, 0.99715)$
T_2	$(0.82580, 0.56397)$
T_3	$(0.75029, 1.56111)$
S_1	$(0.82580, 0.56397)$
S_2	$(-0.07551, 0.99715)$
S_3	$(0.94768, 0.58079)$

References

[1] N. Alon and P. Erdős. *Disjoint Edges in Geometric Graphs*. **Discrete-Comput-Geom.** Volume 4, 287-290, 1989.

[2] Vitaly Bergelson and Neil Hindman *Some Topological Semicommutative van der Waerden type Theorems and their Combinatorial Consequences*. **J. London Math Soc (2)**. Volume 45, number 3, 385-403, 1992.

[3] B. Bollobás. **Combinatorics: Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability** Cambridge University Press, New York, 1986.

[4] B. Bollobás. **Graph Theory: An Introductory Course** Springer Verlag, New York, 1979.

[5] Blanche Descartes (W.T. Tutte). *Advanced Problems and Solutions: k-chromatic Graphs without Triangles*. **American Mathematical Monthly**. Volume 61, 352, 1954.

[6] P. Erdős. *Graph Theory and Probability*. **Canadian Journal of Mathematics**, Volume 11, 34-38, 1959.

[7] P. Erdős. *Unsolved Problems* **Congressus Numerantium XV – Proceedings of the 5th British Comb Conf. 1975**, 681, 1976.

[8] P. Erdős. A. Hajnal. *On Chromatic Numbers of Graphs and Set Systems*. **Acta Math. Acad. Sci. Hungar.**, Volume 17, 61-99, 1966.

[9] P. Erdős. A.L. Rubin. H. Taylor. *Choosability in Graphs*. **Congressus Numerantium**. 26:125-157, 1979.

[10] Gerhard Faltings. *Endlichkeitssätze für abelische Varietäten über Zahlkörpern* **Inventiones Mathematicae**. Volume 73, 349-366, 1983.

[11] Z. Füredi. Cited in: D. West. *Open Problems no. 15*. **Newsletter of the SIAM Activity Group on Discrete Mathematics**. 4(4):8-11, Summer 1994.

[12] Furstenberg. *Ergodic Behavior of Diagonal Measures and a Theorem of Szemerédi on Arithmetic Progressions*. **J. Anal. Math.** Volume 31, 204-256, 1977.

[13] W. Goddard, M. Katchalski, D. J. Kleitman. *Forcing Disjoint Segments in the Plane*. manuscript, 1994.

[14] R. Hochberg. Personal communication, 1994.

[15] R. Hochberg, P. O'Donnell. *Some 4-Chromatic Unit Distance Graphs Without Short Cycles*. **Geombinatorics**. Volume 5, 137-141, 1996.

[16] D.G. Hoffman. P.D. Johnson. E.B. Wantland, *Restricted Choice Numbers*, manuscript. 1994.

[17] H. Hopf, E. Pannwitz. Cited in: P. Erdős *On Some Problems of Elementary and Combinatorial Geometry*, **Ann. Math. Pura Appl.**, Volume 53, 102, 1975.

[18] John B. Kelly and L.M. Kelly, *Paths and Circuits in Critical Graphs*, **Amer. J. Math.**,Volume 76, 786-792, 1954.

[19] Y. Kupitz. *Extremal Problems in Combinatorial Geometry*, **M. Sc. Thesis, Lecture Notes Series, No. 53, Aarhus University**, 1979.

[20] L. Lovász. *On the Chromatic Number of Finite Set-Systems*, **Acta Mathematica Academiae Scientiarium Hungaricae**, Volume 19. 59-67, 1968.

[21] J. Mycielski. *Sur le coloriage des graphes*, **Colloquium Mathematicum**, Volume 3. 161-162. 1955.

[22] N.V.R. Mahadev, Fred S. Roberts, Prakash Santhanakrishnan. *3-choosable complete bipartite graphs*. **Technical Report 91-62, DIMACS**, September 1991.

[23] J. Nešetřil, V. Rödl, *A Short Proof of the Existence of Highly Chromatic Hypergraphs Without Short Cycles*, **J.C.T Series B**, Volume 27, 225-227, 1979.

[24] Paul O'Donnell, *A Triangle-Free 4-Chromatic Graph in the Plane*, **Geombinatorics IV(1)**, 23-29, 1994.

[25] Paul O'Donnell, *A 40 Vertex 4-Chromatic Triangle-Free Unit Distance Graph*, **Geombinatorics V(1)**, 31-34, 1995.

[26] J. Pach, J. Törőcsik, it Some Geometric Applications of Dilworth's Theorem, **Discrete Comput. Geom.**, Volume 12, 1-7, 1994.

[27] M. Perles. Personal communication, 1990.

[28] Anil M. Shende, Barry Tesman, *3-Choosability of $K_{5,q}$*. **Computer Science Technical Report #94-9, Bucknell University**, December 1994.

[29] Alexander Soifer, *Chromatic Number of the Plane: A Historical Essay*, **Geombinatorics**, Volume 2, Issue 1, 8-12 1992.

[30] E. Szemerédi, *On Sets of Integers Containing k Elements in Arithmetic Progression*. **Acta Arith.**, Volume 27, 199-245. 1975.

[31] V.G. Vizing, *Coloring the Vertices of a Graph in Prescribed Colors (Russian)*, **Discret. Analiz. No. 29 Metody Diskret. Anal. V. Teorii Kodov i Shem**, 101:3-10, 1976.

[32] B.L. van der Waerden, *Beweis einer Baudetschen Vermutung*, **Nieuw Arch. Wisk.**, Volume 15, 212-216, 1927.

[33] Nicholas Wormald, *A 4-Chromatic Graph With a Special Plane Drawing*, **Austral. Math. Soc. (Series A)**, Volume 28, 1-8, 1979.

Vita

Paul Michael O'Donnell

1989	B.A. in Mathematics and Computer Science, Drew University.
1989-96	Teaching Assistant, Mathematics Department, Rutgers University.
1993	R. Hochberg and P. O'Donnell, A Large Independent Set in the Unit Distance Graph, *Geombinatorics*, Volume 2, Issue 4.
1994	P. O'Donnell, A Triangle-Free Unit Distance Graph in the Plane, *Geombinatorics*, Volume 4, Issue 1.
1995	P. O'Donnell, A 40-Vertex 4-Chromatic Triangle-Free Unit Distance Graph in the Plane, *Geombinatorics*, Volume 5, Issue 1.
1996	R. Hochberg and P. O'Donnell, Some 4-Chromatic Unit Distance Graphs Without Short Cycles, *Geombinatorics*, Volume 5, Issue 4.
1996-98	Instructor, Mathematics Department, Rutgers University.
1999-	Assistant Professor, Computer Science Department, Drew University.
1999	Ph.D. in Mathematics, Rutgers University.