CYCLES MODULO k

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One of the aims of this note is a proof of the following conjecture of Burr and Erdős [3]. For every odd natural number k there is a constant c_k such that for every natural number l every graph of order n with at least $c_k n$ edges contains a cycle of length l modulo k. We shall deduce this result with $c_k = k^{-1}\big((k+1)^k - 1\big)$ from the existence of certain semitopological configurations in graphs with sufficiently many edges (Theorems 1 and 1′). Up to now the following two special cases of the conjecture above have been settled: $l = 2$ by Erdős and Burr and $l = 0$ by N. Robertson (see [3]).

Our terminology is that of Harary [4], except instead of point and line we say *vertex* and *edge*. $V(G)$ is the *vertex set* of a graph G and $E(G)$ is its *edge set*. $\delta(G)$ is the *minimal degree* in G and $e(G)$ is the *number of edges*.

THEOREM 1. *Let d and s be natural numbers, $d \geqslant 2$. If $\delta(G) \geqslant (d^s-1)/(d-1)+d-1$ then G contains a vertex x_0, a path P not containing x_0 and d paths of length s joining x_0 to P, say $P_1, P_2, \ldots, P_d$, such that*

$$V(P_i) \cap V(P_j) = \{x_0\}, \qquad 1 \leqslant i < j \leqslant d,$$

$$|V(P_i) \cap V(P)| = 1, \qquad 1 \leqslant i \leqslant d.$$

Proof. The case $s = 1$ is trivial since if $Q = a_1 a_2 \ldots a_{l-1} a_l$ is a maximal path in G then $x_0 = a_1$ is joined to at least $\delta(G) \geqslant d$ vertices of the path $P = Q - \{a_1\}$. From now on we suppose that $s \geqslant 2$. Construct a tree $T(d, s)$ as follows. Let c_0 be its centre, let $d(c_0) = d$, and let the vertices at distance t from c_0 have degree $d+1$ for $1 \leqslant t \leqslant s-2$ and degree 1 for $t = s-1$. It is easily checked that for every vertex $a \in V(G)$ G contains a tree $T(d, s)$ with centre a. (This will also follow from a subsequent argument.) Pick $a \in V(G)$ and let Q be a maximal path in G with initial vertex a such that if b is the end vertex of Q then $G - \big(V(Q) - \{b\}\big)$ contains a tree $T = T(d, s)$ with centre b. Denote by $b_1, b_2, \ldots, b_d$ the vertices adjacent to b on T and denote by Q_i the $a - b_i$ path obtained from Q by adding to it the edge bb_i. The maximality of Q implies that the branch T_i of T at b containing b_i cannot be extended to a tree $T(d, s)$ in $G - \big(V(Q_i) - \{b_i\}\big) = G - V(Q)$.

Let $\{c_{ij}: 1 \leqslant j \leqslant d^{s-2}\}$ be the set of endvertices of T_i. Suppose that for a fixed i each vertex c_{ij} is joined to at least d^{s-1} vertices in $W_i = V(G) - V(Q) - V(T_i)$. Then we can successively select disjoint d element sets in W_i, say $U_{i1}, U_{i2}, \ldots,$ such that c_{ij} is joined to every vertex in U_{ij}, $1 \leqslant j \leqslant d^{s-2}$. Adding the edges $c_{ij}x$, $x \in U_{ij}$, $1 \leqslant j \leqslant d^{s-2}$, to the tree T_i we obtain a tree $T(d, s)$ with centre b_i in $G - \big(V(Q_i) - \{b_i\}\big)$. Consequently for each i, $1 \leqslant i \leqslant j$, there is an endvertex of T_i, say $c_i = c_{i1}$, that is joined to at most $d^{s-1} - 1$ vertices in W_i. As $d(c_i) \geqslant (d^s-1)/(d-1)+d-1$, c_i is joined to at least $(d^{s-1}-1)/(d-1)+d$ vertices in $V(Q) \cup \big(V(T_i) - \{c_i\}\big)$ so it is joined to at

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least d vertices in $V(Q)-\{b\}$. Then one can select successively vertices $e_1, e_2, \ldots, e_d$ in $V(Q)-\{b\}$ such that $e_i \neq e_j$, $1 \leqslant i \leqslant j \leqslant d$ and c_i is joined to e_i.

Put $P = Q-\{b\}$, $x_0 = b$ and let P_i be the $b-e_i$ path obtained from the b_i-c_i path on T_i by adding to it the edges bb_i and c_ie_i. The system $\{x_0, P, P_1, \ldots, P_d\}$ clearly satisfies the conditions of the theorem.

Remark. It is very likely that Theorem 1 is rather far from being best possible, i.e. the same conclusions could be drawn under considerably weaker assumptions. If at the end of the proof above we do not require $e_i \neq e_j$ then the conditions of Theorem 1 can be weakened slightly and we obtain the following result.

THEOREM 1$'$. *Let d and s be natural numbers, $d \geqslant 2$. If $\delta(G) \geqslant (d^s-1)/(d-1)$ then G contains a vertex x_0, a path P not containing x_0 and d paths of length s joining x_0 to P, say $P_1, \ldots, P_d$, such that*

$$V(P_i) \cap V(P_j) \subset V(P) \cup \{x_0\}, \qquad 1 \leqslant i < j \leqslant d$$

and

$$|V(P_i) \cap V(P)| = 1, \qquad 1 \leqslant i \leqslant d.$$

THEOREM 2. *Let $k \geqslant 3$ be an odd natural number and let G be a graph of order n. If*

$$\delta(G) \geqslant \frac{(k+1)^k-1}{k} \tag{1}$$

or

$$e(G) \geqslant \frac{(k+1)^k-k-1}{k}\, n \tag{2}$$

then for every natural number l G contains a cycle of length l module k.

Proof. It is easily checked that if (2) holds then G contains a subgraph H with $\delta(H) \geqslant ((k+1)^k-1)/k$. Therefore we may assume that (1) holds. Then the conditions of Theorem 1$'$ are satisfied with $d = k+1$ and $1 \leqslant s \leqslant k$. Let $\{x_0, P, P_1, \ldots, P_{k+s}\}$ be the system guaranteed by Theorem 1$'$. If P_i is an x_0-x_i path $(1 \leqslant i \leqslant k+1)$ then there exist x_i, x_j, $1 \leqslant i < j \leqslant k+1$, whose distances on P is 0 modulo k. Then the cycle formed by P_i, P_j and the x_i-x_j segment of P has length $2s$ modulo k. As $\{2, 4, \ldots, 2k\}$ is a complete set of residues modulo k, the proof is complete.

In conclusion let us say a few words about cycles modulo k for even k. The bipartite graphs show that a graph of order n with $[n^2/4]$ edges might not contain an odd cycle, so there is no constant c_k for an even k. On the other hand, it was shown by Bondy [1], [2] that a graph of order n with more than $[n^2/4]$ edges contains a cycle of length l for every l, $3 \leqslant l \leqslant [n+3/2]$. Thus if $n \geqslant 2k+1$ then every graph of order n with more than $[n^2/4]$ edges is such that for every integer l it contains a cycle of length l modulo k.

References

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