The Approximation Ratio of the 2-Opt Heuristic for the Metric Traveling Salesman Problem

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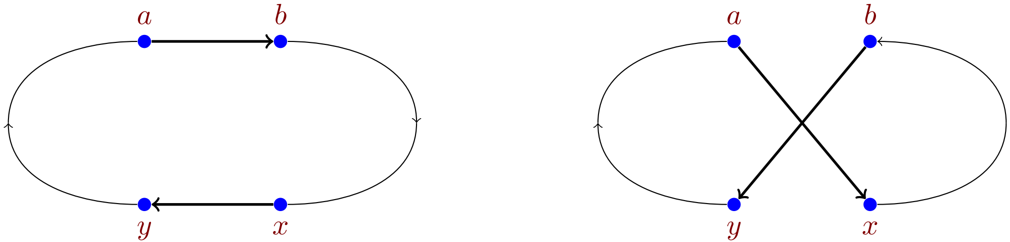

Figure: A TSP tour (left) and the tour obtained after replacing the edges (a, b) and (x, y) with the edges (a, x) and (b, y) (right). The orientation of the tour segment between the vertices b and x has been reversed in the new tour.

$$c(a, x) + c(b, y) \; < \; c(a, b) + c(x, y)$$

Upper Bound	Lower Bound	Reference
	$\sqrt{n/8}$	1987 AMUC
$4\sqrt{n}$		1999 SIAM J. Comput
$2\sqrt{n}$		2013 Networks

This leaves a gap of factor 8 between the upper bound $2\sqrt{2n}$ and the lower bound $\sqrt{n/8}$.The main result of this paper determines the exact approximation ratio of the 2-Opt.

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The length of a 2-optimal tour in a metric TSP instance with n cities is at most $\sqrt{n/2}$ times the length of a shortest tour and this bound is tight.

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The approximation ratio of the 2-Opt on metric TSP is at most $\sqrt{\frac{n}{2}}$.

Let T' be a 2-optimal tour. As usual, we assume that it is directed. Now consider for each edge (u, v) of T' the set

$$S_{r_1,r_2}(u, v) = \{(x, y) \in [0, 1) \times [0, 1) \mid d(x, i_{r_1}(u)) + d(y, i_{r_2}(v)) < c(u, v)\}.$$

$$\begin{aligned} c(u_1, u_2) + c(v_1, v_2) &\leq d(i_{r_1}(u_1), i_{r_1}(u_2)) + d(i_{r_2}(v_1), i_{r_2}(v_2)) \\ &\leq d(i_{r_1}(u_1), x) + d(x, i_{r_1}(u_2)) + d(i_{r_2}(v_1), y) + d(y, i_{r_2}(v_2)) \\ &< c(u_1, v_1) + c(u_2, v_2) \end{aligned}$$

$$\frac{\sum_{e \in E(T')} c(e)}{n} \leq \sqrt{\frac{\sum_{e \in E(T')} c(e)^2}{n}} \leq \frac{1}{\sqrt{2n}}$$

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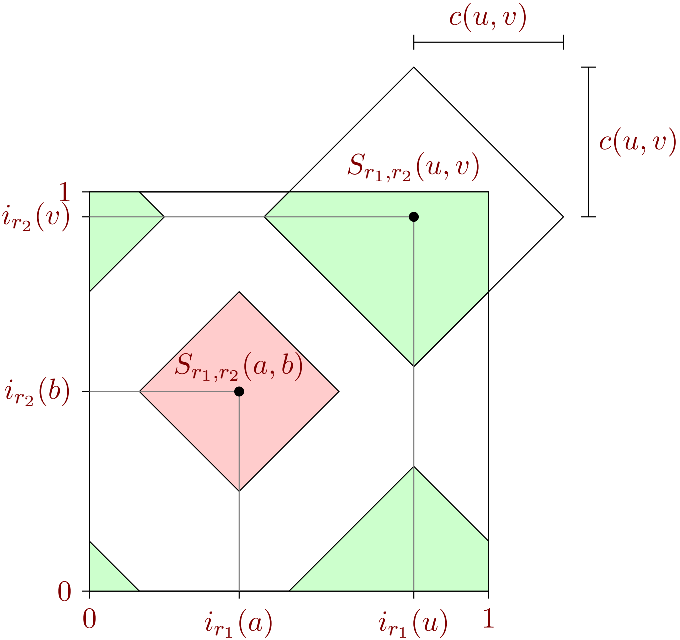
$c(u,v)$
$c(u,v)$
$S_{r_1,r_2}(u,v)$
1
$i_{r_2}(v)$
$S_{r_1,r_2}(a,b)$
$i_{r_2}(b)$
0
0
$i_{r_1}(a)$
$i_{r_1}(u)$
1

Theorem

The approximation ratio of the 2-Opt on metric TSP is at least $\sqrt{\frac{n}{2}}$.

Let G be a complete graph on $n := 2 \cdot k^2$ nodes with vertex set $V(G) := \{v_{i,j} \mid 1 \leq i, j \leq k\} \cup \{w_{i,j} \mid 1 \leq i, j \leq k\}$. For each i with $1 \leq i \leq k$, we call $V_i := \{v_{i,j} \mid 1 \leq j \leq k\}$ and $W_i := \{w_{i,j} \mid 1 \leq j \leq k\}$ a section of $V(G)$ and the v-vertices and w-vertices the two halves of $V(G)$.

We define a distance function $c : E(G) \to \mathbb{R}_{\geq 0}$ as follows:

$$c(v_{i,j}, w_{i',j'}) = 1 \qquad \text{for all } 1 \leq i, i', j, j' \leq k$$

$$c(v_{i,j}, v_{i',j'}) = \begin{cases} 0 & i = i' \\ 2 & i \neq i' \end{cases} \qquad \text{for all } 1 \leq j, j' \leq k$$

$$c(w_{i,j}, w_{i',j'}) = \begin{cases} 0 & i = i' \\ 2 & i \neq i' \end{cases} \qquad \text{for all } 1 \leq j, j' \leq k$$

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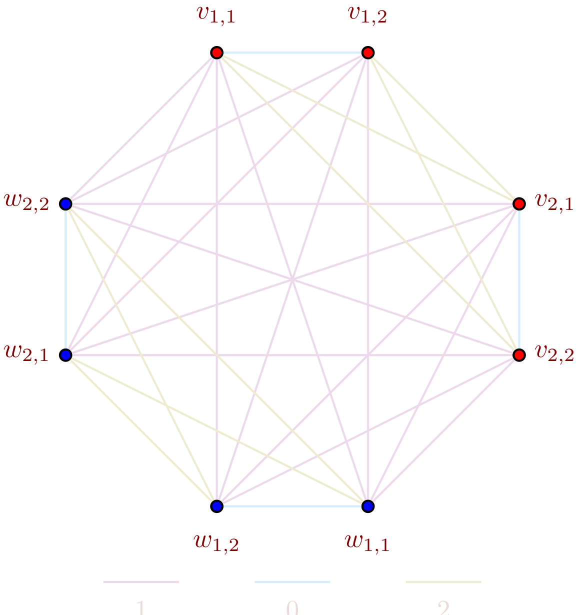
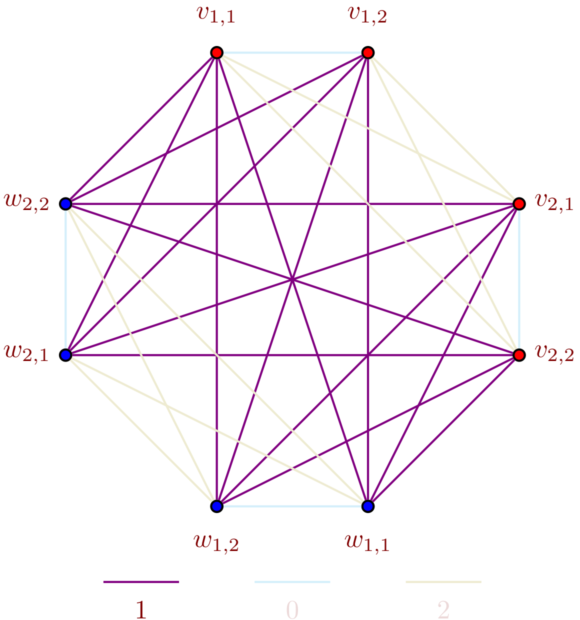
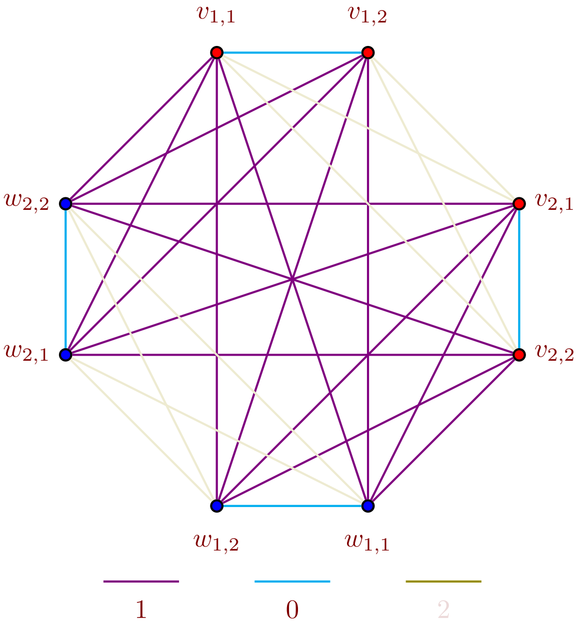
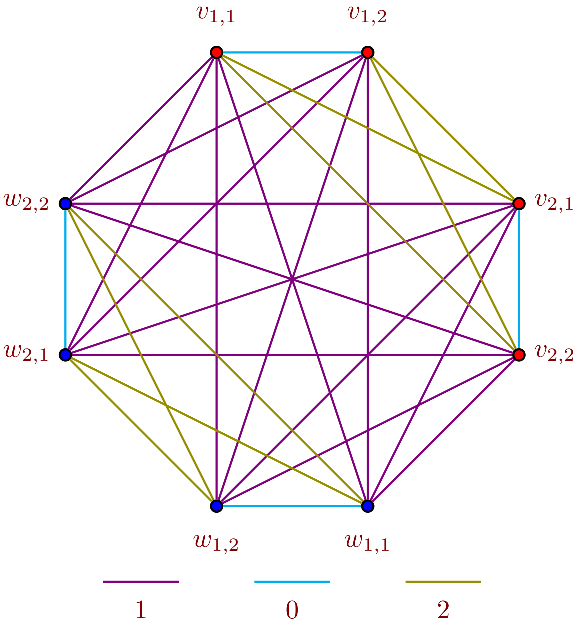

$v_{1,1}$
$v_{1,2}$
$w_{2,2}$
$v_{2,1}$
$w_{2,1}$
$v_{2,2}$
$w_{1,2}$
$w_{1,1}$
1
0
2

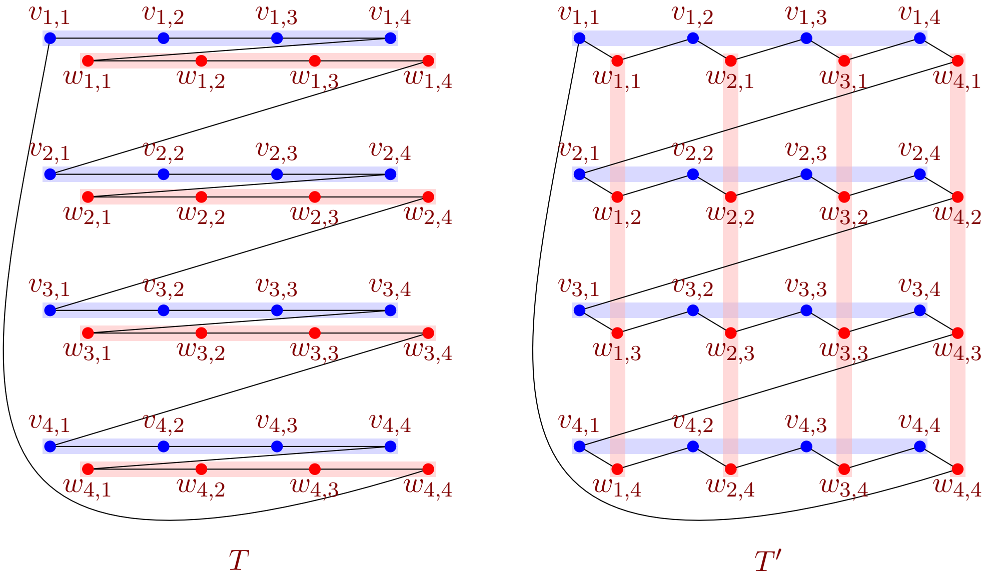

$v_{1,1}$
$v_{1,2}$
$w_{2,2}$
$v_{2,1}$
$w_{2,1}$
$v_{2,2}$
$w_{1,2}$
$w_{1,1}$
1
0
2


$v_{1,1}$
$v_{1,2}$
$w_{2,2}$
$v_{2,1}$
$w_{2,1}$
$v_{2,2}$
$w_{1,2}$
$w_{1,1}$
1
0
2


$v_{1,1}$
$v_{1,2}$
$w_{2,2}$
$v_{2,1}$
$w_{2,1}$
$v_{2,2}$
$w_{1,2}$
$w_{1,1}$
1
0
2

$$
\begin{aligned}
E(T) = {} & \{(v_{i,j}, v_{i,j+1}) \mid 1 \le i \le k, 1 \le j < k\} \cup \\
& \{(w_{i,j}, w_{i,j+1}) \mid 1 \le i \le k, 1 \le j < k\} \cup \\
& \{(v_{i,k}, w_{i,1}) \mid 1 \le i \le k\} \cup \\
& \{(w_{i,k}, v_{i+1,1}) \mid 1 \le i < k\} \cup \\
& \{(w_{k,k}, v_{1,1})\}
\end{aligned}
$$

$$
\begin{aligned}
E(T') = {} & \{(v_{i,j}, w_{j,i}) \mid 1 \le i, j \le k\} \cup \\
& \{(w_{j,i}, v_{i,j+1}) \mid 1 \le i \le k, 1 \le j < k\} \cup \\
& \{(w_{k,i}, v_{i+1,1}) \mid 1 \le i < k\} \cup \\
& \{(w_{k,k}, v_{1,1})\}
\end{aligned}
$$


$v_{1,1}$ $v_{1,2}$ $v_{1,3}$ $v_{1,4}$
$w_{1,1}$ $w_{1,2}$ $w_{1,3}$ $w_{1,4}$
$v_{2,1}$ $v_{2,2}$ $v_{2,3}$ $v_{2,4}$
$w_{2,1}$ $w_{2,2}$ $w_{2,3}$ $w_{2,4}$
$v_{3,1}$ $v_{3,2}$ $v_{3,3}$ $v_{3,4}$
$w_{3,1}$ $w_{3,2}$ $w_{3,3}$ $w_{3,4}$
$v_{4,1}$ $v_{4,2}$ $v_{4,3}$ $v_{4,4}$
$w_{4,1}$ $w_{4,2}$ $w_{4,3}$ $w_{4,4}$
T
$v_{1,1}$ $v_{1,2}$ $v_{1,3}$ $v_{1,4}$
$w_{1,1}$ $w_{2,1}$ $w_{3,1}$ $w_{4,1}$
$v_{2,1}$ $v_{2,2}$ $v_{2,3}$ $v_{2,4}$
$w_{1,2}$ $w_{2,2}$ $w_{3,2}$ $w_{4,2}$
$v_{3,1}$ $v_{3,2}$ $v_{3,3}$ $v_{3,4}$
$w_{1,3}$ $w_{2,3}$ $w_{3,3}$ $w_{4,3}$
$v_{4,1}$ $v_{4,2}$ $v_{4,3}$ $v_{4,4}$
$w_{1,4}$ $w_{2,4}$ $w_{3,4}$ $w_{4,4}$
T'