Notes for Probability and Stochastic Process

©Author: Zhan Cheng E-Mail: zhanc113003@gmail.com Textbook: ISBN 978-7-5635-4537-7

Chapter 1:Events and Their Probabilities

1.1 Experiment, Sample Space and Random Event

1.1.1 Basic Definations

Random experiment has three characteristics: Reapeatability, Predictability, Uncertainty. And it can be denoted by E , here is an example:

$$E_1 : \text{Determintation of the sex of a newborn child.}$$

Each possible outcome is called an ***sample point*** or ***elementary event*** ω . The set of all possible outcomes of E is known as the ***sample space*** Ω . Here is an example about Ω_1 corresponding to E_1:

$$\text{For } E_1\text{: } \Omega_1 = \{g, b\}, \text{ where the outcome } g \text{ means that the child is a girl and } b \text{ that it is a boy}$$

Any subset of the Ω is known as an ***random event*** or ***event*** $A, B, C, \cdots$. We say that A occurs when the outcome of the experiment lies in A. Those events must occur in the experiment are called the ***inevitable events*** S. Those could not happen anytime are said to be ***impossible events*** $\emptyset$.

1.1.2 Events as Sets

The relationships and operations between random events could be described in term of ***set theory***.

Let Ω be the sample space of the random experiment E and $A, B, A_i(i = 1, 2, \cdots)$ be the random events of E.

1. $A \subset B : \text{if event A occurs, then B occurs}$
2. $A \cup B : \text{either A or B occurs}$
3. $A \cap B$ or $AB : \text{both A and B occur}$
4. $A - B : \text{A occurs but B does not occur}$

Common operations of the events are not listed, but there are two important formulas to remember:

$$A - B = A - AB = A\overline{B}$$
$$A \cup B = A \cup \overline{A}B$$

1.2 Probabilities Defined on Events

1.2.1 Classical Probability

A random experiment E is ***classical*** if:

(i) : E contains only different limited basic events.

(ii) : all outcomes are equally likely to occur.

If $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n\}$, we define the probability of event A as

$$P(A) = \frac{\#A}{\#\Omega}$$

where $\#$A means the number of all possible outcomes of event A, $\#\Omega$ means the number of all possible outcomes of sample space !

and $P(\emptyset) = 0$

For classical random experiment E, the probability has the following properties:

(1) for every event $A, P(A) \geq 0,$

(2)$P(\Omega) = 1,$

(3)for every finite sequence of n disjoint events $A_1, A_2, \cdots, A_n,$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

We just skip that proof.

1.2.2 Geometric Probability

A random experiment E is ***geometric*** if:

(i) : the sample space is a measurable region, i.e. $0 < L(\Omega) < \infty$, and

(ii) : the probability of every event has nothing to do with its position and shape.

In this case, we define the probability of event A as

$$P(A) = \frac{L(A)}{L(\Omega)}$$

and $P(\emptyset) = 0$

For geometrical random experiment E, the probability has the following properties:

(1) for every event $A, P(A) \geq 0,$

(2)$P(\Omega) = 1,$

(3)for every finite sequence of n disjoint events $A_1, A_2, \cdots, A_n,$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

We skip that proof as well.

1.2.3 The Frequency Interpretation of Probability

Let $f_n(A)$ be the times that A occurs. The ration

$$F_n(A) = \frac{f_n(A)}{n}$$

is said to be the ***frequency*** of event A in the n trials.

If n is large enough, the probability of event A will be approximated by $F_n(A)$

For a random experiment E, the frequency has the following properties:

(1) for every event $A, F_n(A) \geq 0,$

(2)$F_n(\Omega) = 1,$

(3)for every finite sequence of n disjoint events $A_1, A_2, \cdots, A_n,$

$$F_n(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} F_n(A_i).$$

We skip the part of σ-algebra because it is too hard😳

1.3 Conditional Probabilities

1.3.1 The Defination of Conditional Probability

Given two events A and B with $P(B) > 0$, the ***conditional probability of*** A ***given*** B is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$\text{If } P(B) > 0, \text{ then } P(A|B) \text{ is also a probability, that is}$

$(1) \text{ for every event } A, P(A|B) \geq 0,$

$(2) P(\Omega|B) = 1,$

$(3) \text{for every finite sequence of countable disjoint events } A_1, A_2, \cdots, A_n,$

$$P(\bigcup_{i=1}^{\infty} A_i|B) = \sum_{i=1}^{\infty} P(A_i|B).$$

Some fomulas:

$$P(\overline{A}|B) = 1 - P(A|B)$$
$$\text{if } A \subset B, \text{then } P(C - A|B) = P(C|B) - P(A|B) \text{ and } P(A|B) \leq P(C|B)$$
$$P(A \cup C|B) = P(A|B) + P(C|B) - P(AC|B)$$

1.3.2 The Multiplication Rule

Assume that $P(B) > 0$. Then

$$P(AB) = P(B) \cdot P(A|B)$$

Or if $P(A) > 0$, Then

$$P(AB) = P(A) \cdot P(B|A)$$

Suppose that $A_1, A_2, \cdots, A_n$ are events satisfying $P(A_1 A_2 \cdots A_{n-1}) > 0$.Then

$$P(A_1 A_2 \cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1A_2)\cdots P(A_n|A_1A_2\cdots A_{n-1})$$

1.3.3 Total Probability Formula

Let Ω denote the sample space of some experiment. n events $P(B_1 B_2 \cdots B_n)$ are said to form a ***partition*** of Ω if these events satisfy:

$(1) B_1, B_2, \cdots, B_n \text{ are disjoint and}$

$(2) \bigcup_{i=1}^{n} B_i = \Omega$

Suppose that the events $B_1, B_2, \cdots, B_n$ form a partition of the sample space Ω and $P(B_i) > 0$ for $i = 1, 2, \cdots, n$. Then, for every event A in Ω

$$P(A) = \sum_{i=1}^{n} P(B_i)P(A|B_i)$$

1.3.4 Baye's Theorem

Let the events $B_1, B_2, \cdots, B_n$ form a partition of the sample space Ω such that $P(B_i) > 0$ for $i = 1, 2, \cdots, n$, and let A be an event such that $P(A) > 0$. Then for $i = 1, 2, \cdots, n$,

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{n} P(B_j)P(A|B_j)}$$

1.4 Independence of Two Events

1.4.1 Independence of Two Events and Several Events

Two events A and B are ***independent*** if

$$P(AB) = P(A)P(B)$$

Three events A, B and C in the sample space S of a random experiment are said to be ***mutually independent*** if

$$\begin{gathered} P(AB) = P(A)P(B) \\ P(BC) = P(B)P(C) \\ P(AC) = P(A)P(C) \\ P(ABC) = P(A)P(B)P(C) \end{gathered}$$

Events A, B and C are said to be ***pairwise independent*** if the first three equations hold.

1.4.2 Bernoulli Trials

A ***Bernoulli experiment*** E is such kind of random experiment, the outcome of which can be classified in but one of two mutually exclusive and exhaustive ways, mainly, *success* and *failure*

A sequence of Bernoulli trials E_n occurs when a Bernoulli experiment is *performed serveral independent times so that the probability of success, say, p, remains the same from trial to trial*

The outcomes of E_n are the 2^n sequences of lenth n. The number of outcomes of E_n that contain a exactly k times is given by the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The probability that the outcome of an experiment that consists of n Bernoulli trials has k success and $n-k$ failures is given by:

$$P_n(k) = \binom{n}{k} p^k q^{n-k}$$