Lemma：(X,d)是度量空间，$A\subseteq X$，$x\in X$，则有：

x是A的极限点 $\Longleftrightarrow$ $\exists A$中的由彼此互异的点组成的点列$\{x_n\}_{n\geq 1}$，使得

$$\lim_{n\to+\infty} x_n = x$$

Proof：($\Rightarrow$)：对于$1\in\mathbb{R}_{>0}$，$\because x$是A的极限点，

$\therefore \exists x_1\in B(x;1)\cap A$ 满足 $x_1\neq x$．　$\therefore x_1\in A$ 且 $d(x_1,x)<1$

$\because x_1\neq x$　$\therefore d(x_1,x)\in\mathbb{R}_{>0}$．

对于$\min\{\frac{1}{2},d(x_1,x)\}\in\mathbb{R}_{>0}$，$\because x$是$A$的极限点，

$\therefore \exists x_2\in B(x;\min\{\frac{1}{2},d(x_1,x)\})\cap A$ 满足 $x_2\neq x$．　$\therefore x_2\in A$ 且 $d(x_2,x)<\frac{1}{2}$

$\because d(x_2,x)<d(x_1,x)$　$\therefore x_2\neq x_1$

$\because x_2\neq x$　$\therefore d(x_2,x)\in\mathbb{R}_{>0}$

对于$\min\{\frac{1}{3},d(x_2,x)\}\in\mathbb{R}_{>0}$，$\because x$是$A$的极限点，

$\therefore \exists x_3\in B(x;\min\{\frac{1}{3},d(x_2,x)\})\cap A$ 满足 $x_3\neq x$　$\therefore x_3\in A$ 且 $d(x_3,x)<\frac{1}{3}$

$\because d(x_3,x)<d(x_2,x)<d(x_1,x)$　$\therefore x_1,x_2,x_3$ 两两互异

$\because x_3\neq x$　$\therefore d(x_3,x)\in\mathbb{R}_{>0}$

对于$\min\{\frac{1}{4},d(x_3,x)\}\in\mathbb{R}_{>0}$，$\because x$是$A$的极限点，

$\therefore \exists x_4\in B(x;\min\{\frac{1}{4},d(x_3,x)\})\cap A$ 且 $x_4\neq x$　$\therefore x_4\in A$ 且 $d(x_4,x)<\frac{1}{4}$

$\because d(x_4,x)<d(x_3,x)<d(x_2,x)<d(x_1,x)$　$\therefore x_1,x_2,x_3,x_4$ 两两互异

$\because x_4\neq x$　$\therefore d(x_4,x)\in\mathbb{R}_{>0}$

将上述证明过程继续下去，可得：

$\because x_{n-1}\neq x$　$\therefore d(x_{n-1},x)\in\mathbb{R}_{>0}$

对于$\min\{\frac{1}{n},d(x_{n-1},x)\}\in\mathbb{R}_{>0}$，$\because x$是$A$的极限点，

$\therefore \exists x_n\in B(x;\min\{\frac{1}{n},d(x_{n-1},x)\})\cap A$ 满足 $x_n\neq x$　$\therefore x_n\in A$ 且 $d(x_n,x)<\frac{1}{n}$

$\because d(x_n,x)<d(x_{n-1},x)<\cdots<d(x_4,x)<d(x_3,x)<d(x_2,x)<d(x_1,x)$

$\therefore x_1,x_2,\cdots,x_n$ 彼此互异

$\because x_n\neq x \quad \therefore d(x_n,x)\in\mathbb{R}_{>0}$

$\therefore \{x_n\}_{n\geq 1}$ 是 A 中的由彼此互异的点组成的点列，使得 $\lim\limits_{n\to+\infty}x_n=x$

$(\Leftarrow)$：对 $\forall\varepsilon\in\mathbb{R}_{>0}$

$\because \lim\limits_{n\to+\infty}x_n=x \quad \therefore \exists N\in\mathbb{Z}_{\geq 1}$，对 $\forall n\geq N$，有：$d(x_n,x)<\varepsilon$

$\therefore d(x_{N+1},x)<\varepsilon$ 且 $d(x_{N+2},x)<\varepsilon$

$\because$ ~~点~~点列 $\{x_n\}_{n\geq 1}$ 彼此互异 $\quad \therefore x_{N+1}\neq x_{N+2}$

$\therefore x_{N+1},x_{N+2}$ 中必有一个不等于 x. 不妨设 $x_{N+2}\neq x$

$\because x_{N+2}\in X$ 且 $d(x_{N+2},x)<\varepsilon \quad \therefore x_{N+2}\in B(x;\varepsilon)$

$\because x_{N+2}\in A \quad \therefore x_{N+2}\in B(x;\varepsilon)\cap A$ 且 $x_{N+2}\neq x$

$\therefore x$ 是 A 的极限点 $\quad \square$

Lemma：(X,d) 是度量空间，$A\subseteq X$，则有：$\mathrm{cl}A=A\cup\{x: x\text{是}A\text{的极限点}\}$

Proof：$\because (X,d)$ 是度量空间，$A\subseteq X$

$\therefore A\subseteq \mathrm{cl}A$

对 $\forall x\in\{x: x\text{是}A\text{的极限点}\}$，有：$x$ 是 A 的极限点 $\quad \therefore x\in X$

对 $\forall r\in\mathbb{R}_{>0}$，$\because x$ 是 A 的极限点 $\quad \therefore \exists a\in B(x;r)\cap A$ 满足 $a\neq x$

$\therefore B(x;r)\cap A\neq\emptyset \quad \therefore x\in\mathrm{cl}A$

$\therefore \{x: x\text{是}A\text{的极限点}\}\subseteq\mathrm{cl}A$

$\therefore A\cup\{x: x\text{是}A\text{的极限点}\}\subseteq\mathrm{cl}A$. $\quad \therefore$ 右 $\subseteq$ 左.

对 $\forall x\in$ 左，有：$x\in\mathrm{cl}A$. 分两种情况讨论：

① $x\in A$ $\quad\therefore x\in$右

② $x\notin A$. $\because x\in clA$ $\quad\therefore x\in X$ $\quad\therefore x\in X$且$x\notin A$.

对$\forall\varepsilon\in\mathbb{R}_{>0}$ $\quad\because x\in clA$ $\quad\therefore B(x;\varepsilon)\cap A\neq\emptyset$ $\quad\therefore\exists a\in B(x;\varepsilon)\cap A$

$\because a\in A$且$x\notin A$ $\quad\therefore a\neq x$

$\therefore a\in B(x;\varepsilon)\cap A$且有$a\neq x$

$\therefore$ x是A的极限点

$\therefore x\in$右 $\quad\therefore$左$\subseteq$右. $\quad\therefore$左$=$右 $\quad\square$

Lemma: (X,d)是度量空间, $A\subseteq X$, 则有:

A是(X,d)的闭集 $\Longleftrightarrow\{x: x$是$A$的极限点$\}\subseteq A$

Proof: A是(X,d)的闭集

$\Longleftrightarrow A=clA$

$\Longleftrightarrow A=A\cup\{x: x$是$A$的极限点$\}$

$\Longleftrightarrow\{x: x$是$A$的极限点$\}\subseteq A$ $\quad\square$