内部，闭包

Lemma：(X,d)是度量空间，$A_i \subseteq X\ (i\in I)$．则有：

$$\operatorname{int}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \operatorname{int} A_i$$

Proof：对$\forall x\in \operatorname{int}\left(\bigcap_{i\in I} A_i\right)$，有：$x\in X$

对$\forall j\in I$，$\because x\in \operatorname{int}\left(\bigcap_{i\in I} A_i\right)$ $\therefore \exists r\in \mathbb{R}_{>0}$，s.t. $B(x;r)\subseteq \bigcap_{i\in I} A_i$

$\because B(x;r)\subseteq \bigcap_{i\in I} A_i \subseteq A_j$ $\therefore x\in \operatorname{int} A_j$

$\therefore x\in \bigcap_{j\in I} \operatorname{int} A_j \doteq \bigcap_{i\in I} \operatorname{int} A_i$

$\therefore \operatorname{int}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \operatorname{int} A_i$ $\square$

Lemma：(X,d)是度量空间，$A_i\subseteq X\ (i=1,2,\cdots,n)\ (n\in \mathbb{Z}_{\geqslant 1})$．则有：

$$\operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} \operatorname{int} A_i$$

Proof：由上一引理，有：$\operatorname{int}\left(\bigcap_{i=1}^{n} A_i\right) \subseteq \bigcap_{i=1}^{n} \operatorname{int} A_i$

对$\forall x\in \bigcap_{i=1}^{n} \operatorname{int} A_i$，有：$x\in \operatorname{int} A_1$ 且 $x\in \operatorname{int} A_2$ 且$\cdots$且 $x\in \operatorname{int} A_n$

$\because (X,d)$是度量空间，$A_1\subseteq X$，$x\in \operatorname{int} A_1$

$\therefore \exists r_1\in \mathbb{R}_{>0}$，s.t. $B(x;r_1)\subseteq A_1$

$\because (X,d)$是度量空间，$A_2\subseteq X$，$x\in \operatorname{int} A_2$

$\therefore \exists r_2\in \mathbb{R}_{>0}$，s.t. $B(x;r_2)\subseteq A_2$

……

$\because (X,d)$是度量空间，$A_n\subseteq X$，$x\in \operatorname{int} A_n$

$\therefore \exists r_n\in \mathbb{R}_{>0}$，s.t. $B(x;r_n)\subseteq A_n$

$\because r_1, r_2, \cdots, r_n \in \mathbb{R}_{>0}$ $\quad \therefore \min\{r_1, r_2, \cdots, r_n\} \in \mathbb{R}_{>0}$

任取 $\varepsilon \in \mathbb{R}_{>0}$， s.t. $\varepsilon < \min\{r_1, r_2, \cdots, r_n\}$

对 $\forall k = 1, 2, \cdots, n$，有：$B(x;\varepsilon) \subseteq B(x; r_k) \subseteq A_k$

$\therefore B(x;\varepsilon) \subseteq \bigcap\limits_{i=1}^{n} A_i$ $\qquad \therefore x \in \operatorname{int}\left(\bigcap\limits_{i=1}^{n} A_i\right)$

$\therefore \bigcap\limits_{i=1}^{n} \operatorname{int} A_i \subseteq \operatorname{int}\left(\bigcap\limits_{i=1}^{n} A_i\right)$

$\therefore \operatorname{int}\left(\bigcap\limits_{i=1}^{n} A_i\right) = \bigcap\limits_{i=1}^{n} \operatorname{int} A_i$ $\quad \square$

Lemma（内部和闭包的幂等性）(X,d) 是度量空间，$A \subseteq X$，则有：

① $\operatorname{int}(\operatorname{int} A) = \operatorname{int} A$

② $\operatorname{cl}(\operatorname{cl} A) = \operatorname{cl} A$

Proof: $\because (X,d)$ 是度量空间，$A \subseteq X$ $\quad \therefore \operatorname{int} A$ 是 (X,d) 的开集

$\therefore \operatorname{int} A = \operatorname{int}(\operatorname{int} A)$

$\because (X,d)$ 是度量空间，$A \subseteq X$ $\quad \therefore \operatorname{cl} A$ 是 (X,d) 的闭集

$\therefore \operatorname{cl} A = \operatorname{cl}(\operatorname{cl} A)$ $\quad \square$

Lemma: (X,d) 是度量空间，$A_i \subseteq X\ (i \in I)$，则有：

$$\operatorname{cl}\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} \operatorname{cl} A_i$$

Proof: 对 $\forall j \in I$，$\because \bigcap\limits_{i\in I} A_i \subseteq A_j$ $\quad \therefore \operatorname{cl}\left(\bigcap\limits_{i\in I} A_i\right) \subseteq \operatorname{cl} A_j$

$\therefore \operatorname{cl}\left(\bigcap\limits_{i\in I} A_i\right) \subseteq \bigcap\limits_{j\in I} \operatorname{cl} A_j = \bigcap\limits_{i\in I} \operatorname{cl} A_i$ $\quad \square$

Remark：补充用定义的证法.

对 $\forall x\in cl\left(\bigcap_{i\in I}A_i\right)$，有：

对 $\forall j\in I$. 对 $\forall r\in\mathbb{R}_{>0}$， $\because x\in cl\left(\bigcap_{i\in I}A_i\right)$ $\therefore B(x;r)\cap\left(\bigcap_{i\in I}A_i\right)\neq\emptyset$

$\therefore B(x;r)\cap A_j\neq\emptyset$ $\therefore x\in clA_j$ $\therefore x\in\bigcap_{j\in I}clA_j$

$\therefore x\in\bigcap_{i\in I}clA_i$ $\therefore cl\left(\bigcap_{i\in I}A_i\right)\subseteq\bigcap_{i\in I}clA_i$

定义（两个度量空间的积空间）. (X_1,d_1) 和 (X_2,d_2) 是两个度量空间，

定义 $X_1\times X_2$ 上的度量为：

对 $\forall (x_1,x_2),(y_1,y_2)\in X_1\times X_2$， $d((x_1,x_2),(y_1,y_2))=d_1(x_1,y_1)+d_2(x_2,y_2)$

则有：$(X_1\times X_2,d)$ 是度量空间.

~~Proof~~

Lemma： $(X_1\times X_2,d)$ 是度量空间

proof：对 $\forall (x_1,x_2),(y_1,y_2)\in X_1\times X_2$，有：

① $d((x_1,x_2),(y_1,y_2))=\underbrace{d_1(x_1,y_1)}_{\in\mathbb{R}_{\geqslant 0}}+\underbrace{d_2(x_2,y_2)}_{\in\mathbb{R}_{\geqslant 0}}\in\mathbb{R}_{\geqslant 0}$

② $d((x_1,x_2),(y_1,y_2))=d_1(x_1,y_1)+d_2(x_2,y_2)=d_1(y_1,x_1)+d_2(y_2,x_2)$

d_1和d_2的对称性

$=d((y_1,y_2),(x_1,x_2))$

③ $d((x_1,x_2),(y_1,y_2))=0 \Leftrightarrow d_1(x_1,y_1)+d_2(x_2,y_2)=0$

$\Leftrightarrow d_1(x_1,y_1)=0$ 且 $d_2(x_2,y_2)=0$ $\Leftrightarrow x_1=y_1$ 且 $x_2=y_2$

$\Leftrightarrow (x_1,x_2)=(y_1,y_2)$

则对 $\forall (x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$（其中 $x_1, y_1, z_1 \in X_1$，$x_2, y_2, z_2 \in X_2$），有：

$$d((x_1,x_2),(y_1,y_2)) = d_1(x_1,y_1) + d_2(x_2,y_2)$$

$$\leq d_1(x_1,z_1) + d_1(z_1,y_1) + d_2(x_2,z_2) + d_2(z_2,y_2)$$

$$= \big(d_1(x_1,z_1) + d_2(x_2,z_2)\big) + \big(d_1(z_1,y_1) + d_2(z_2,y_2)\big)$$

$$= d((x_1,x_2),(z_1,z_2)) + d((z_1,z_2),(y_1,y_2))$$

$\therefore (X_1 \times X_2, d)$ 是度量空间. $\square$

Lemma（积空间上的另一种度量）(X_1, d_1) 和 (X_2, d_2) 是两个度量空间，定义 $X_1 \times X_2$ 上的度量 ρ：

对 $\forall (x_1,x_2), (y_1,y_2) \in X_1 \times X_2$，$\rho((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$.

则有：$(X_1 \times X_2, \rho)$ 是度量空间.

Proof：对 $\forall (x_1,x_2), (y_1,y_2) \in X_1 \times X_2$（其中 $x_1, y_1 \in X_1$，$x_2, y_2 \in X_2$），有：

① $\because x_1, y_1 \in X_1$ $\therefore d_1(x_1,y_1) \in \mathbb{R}_{\geq 0}$. $\because x_2, y_2 \in X_2$ $\therefore d_2(x_2,y_2) \in \mathbb{R}_{\geq 0}$

$\therefore \rho((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} \in \mathbb{R}_{\geq 0}$

② $\rho((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} = \max\{d_1(y_1,x_1), d_2(y_2,x_2)\}$

$= \rho((y_1,y_2),(x_1,x_2))$

③ $\rho((x_1,x_2),(y_1,y_2)) = 0 \iff \max\{d_1(x_1,y_1), d_2(x_2,y_2)\} = 0$

$\iff d_1(x_1,y_1) = 0$ 且 $d_2(x_2,y_2) = 0 \iff x_1 = y_1$ 且 $x_2 = y_2 \iff (x_1,x_2) = (y_1,y_2)$

④ 对 $\forall (x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$（其中 $x_1, y_1, z_1 \in X_1$，$x_2, y_2, z_2 \in X_2$），有

$$\rho((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

$\because x_1, y_1, z_1 \in X_1 \quad \therefore d_1(x_1, y_1) \leqslant d_1(x_1, z_1) + d_1(z_1, y_1)$

$$\begin{aligned} \therefore d_1(x_1, y_1) &\leqslant d_1(x_1, z_1) + d_1(z_1, y_1) \\ &\leqslant \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\ &= \rho((x_1, x_2), (z_1, z_2)) + \rho((z_1, z_2), (y_1, y_2)) \end{aligned}$$

$\because x_2, y_2, z_2 \in X_2$

$$\begin{aligned} \therefore d_2(x_2, y_2) &\leqslant d_2(x_2, z_2) + d_2(z_2, y_2) \\ &\leqslant \max\{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max\{d_1(z_1, y_1), d_2(z_2, y_2)\} \\ &= \rho((x_1, x_2), (z_1, z_2)) + \rho((z_1, z_2), (y_1, y_2)) \end{aligned}$$

$$\begin{aligned} \therefore \rho((x_1, x_2), (y_1, y_2)) &= \max\{d_1(x_1, y_1), d_2(x_2, y_2)\} \\ &\leqslant \rho((x_1, x_2), (z_1, z_2)) + \rho((z_1, z_2), (y_1, y_2)) \end{aligned}$$

$\therefore (X_1 \times X_2, \rho)$ 是度量空间．$\square$