Lemma：X是任意集合，$A, B \subseteq X$，则有：

$$A \cap (X \setminus B) = \phi \iff A \subseteq B$$

Proof：($\Rightarrow$)：对$\forall x \in A$，假设 $x \notin B$，则有：$x \in A \subseteq X$ $\therefore x \in X \setminus B$

$\therefore x \in A \cap (X \setminus B) = \phi$ 矛盾． $\therefore x \in B$ $\therefore A \subseteq B$

($\Leftarrow$)：假设 $A \cap (X \setminus B) \neq \phi$，则有：$\exists \lambda \in A \cap (X \setminus B)$．

$\therefore \lambda \in A$且 $\lambda \in X \setminus B$

$\therefore \lambda \in A$且 $\lambda \in X$且 $\lambda \notin B$．

$\because \lambda \in A \subseteq B$ $\therefore \lambda \in B$ 矛盾．

$\therefore A \cap (X \setminus B) = \phi$． $\square$

Lemma：A, B是任意的集合，则有：$A \setminus (A \setminus B) = A \cap B$

Proof：$A \setminus (A \setminus B) = \{x : x \in A$且 $x \notin A \setminus B\}$

$= \{x : x \in A$且 $(x \notin A$或$x \in B)\}$

$= \{x : x \in A$且 $x \in B\} = A \cap B$． $\square$

Lemma：A, B, C是任意的集合，则有：$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

Proof：对$\forall x \in A \setminus (B \setminus C)$，有：$x \in A$且 $x \notin B \setminus C$

$\because x \notin B \setminus C$ $\therefore x \notin B$或$x \in C$．

若$x \notin B$，则有：$x \in A$且$x \notin B$ $\therefore x \in A \setminus B$ $\therefore x \in$右 ~~$\therefore$左$\subseteq$右~~

若$x \in C$，则有：$x \in A$且$x \in C$ $\therefore x \in A \cap C$ $\therefore x \in$右 ~~左$\subseteq$右~~．

$\therefore$左$\subseteq$右．

对 $\forall x \in$ 右，有：$x \in A \setminus B$ 或 $x \in A \cap C$.

若 $x \in A \setminus B$，则有：$x \in A$ 且 $x \notin B$ $\therefore x \in A$ 且 $x \notin B \setminus C$ $\therefore x \in A \setminus (B \setminus C)$

$\therefore x \in$ 左 ~~右⊆左~~

若 $x \in A \cap C$，则有：$x \in A$ 且 $x \in C$ $\therefore x \in A$ 且 $x \notin B \setminus C$ $\therefore x \in A \setminus (B \setminus C)$

$\therefore x \in$ 左 ~~右⊆左~~

$\therefore$ 右 $\subseteq$ 左.

$\therefore$ 左 $=$ 右. $\square$

内部和闭包

Lemma: (X,d) 是度量空间，$A, B \subseteq X$. 若 $A \subseteq B$，则有:

$$\operatorname{int} A \subseteq \operatorname{int} B$$

Proof: 对 $\forall x \in \operatorname{int} A$，有:

$\because x \in \operatorname{int} A$ $\quad \therefore \exists r \in \mathbb{R}_{>0}$, s.t. $B(x;r) \subseteq A$

$\because B(x;r) \subseteq A$, $A \subseteq B$ $\quad \therefore B(x;r) \subseteq B$

$\because r \in \mathbb{R}_{>0}$, $B(x;r) \subseteq B$ $\quad \therefore x \in \operatorname{int} B$ $\quad \therefore \operatorname{int} A \subseteq \operatorname{int} B$ $\quad \square$

Lemma: (X,d) 是度量空间，$A, B \subseteq X$. 若 $A \subseteq B$，则有:

$$\operatorname{cl} A \subseteq \operatorname{cl} B$$

Proof: 对 $\forall x \in \operatorname{cl} A$，有:

$\because x \in \operatorname{cl} A$ $\quad \therefore x \in X$.

对 $\forall r \in \mathbb{R}_{>0}$, $\because x \in \operatorname{cl} A$ $\quad \therefore B(x;r) \cap A \neq \emptyset$

$\because A \subseteq B$ $\quad \therefore B(x;r) \cap B \neq \emptyset$ $\quad \therefore x \in \operatorname{cl} B$ $\quad \therefore \operatorname{cl} A \subseteq \operatorname{cl} B$ $\quad \square$

Lemma: (X,d) 是度量空间，$A_i \subseteq X$ $(i \in I)$. 则有:

$$\bigcup_{i \in I} \operatorname{int} A_i \subseteq \operatorname{int}\left(\bigcup_{i \in I} A_i\right)$$

Proof: 对 $\forall i \in I$，有: $A_i \subseteq \bigcup_{j \in I} A_j$ $\quad \therefore \operatorname{int} A_i \subseteq \operatorname{int}\left(\bigcup_{j \in I} A_j\right)$

$\therefore \bigcup_{i \in I} \operatorname{int} A_i \subseteq \operatorname{int}\left(\bigcup_{j \in I} A_j\right)$ $\quad \therefore \bigcup_{i \in I} \operatorname{int} A_i \subseteq \operatorname{int}\left(\bigcup_{i \in I} A_i\right)$ $\quad \square$

Lemma：(X,d)是度量空间，$A_i \subseteq X\ (i\in I)$．则有：

$$\bigcup_{i\in I} \mathrm{cl}A_i \subseteq \mathrm{cl}\Big(\bigcup_{i\in I} A_i\Big)$$

Proof：对$\forall x \in \bigcup_{i\in I}\mathrm{cl}A_i$，有：

$\because x\in \bigcup_{i\in I}\mathrm{cl}A_i \quad \therefore \exists \lambda\in I$，s.t. $x\in \mathrm{cl}A_\lambda$

$\because$ 对$\forall i\in I$，$A_i\subseteq X \quad \therefore \bigcup_{i\in I}A_i\subseteq X$

对$\forall r\in \mathbb{R}_{>0}$，有：$\because x\in \mathrm{cl}A_\lambda \quad \therefore B(x;r)\cap A_\lambda \neq \emptyset$

$\therefore B(x;r)\cap\Big(\bigcup_{i\in I}A_i\Big)\neq\emptyset \quad \therefore x\in \mathrm{cl}\Big(\bigcup_{i\in I}A_i\Big)$

$\therefore \bigcup_{i\in I}\mathrm{cl}A_i \subseteq \mathrm{cl}\Big(\bigcup_{i\in I}A_i\Big)$ $\square$

Lemma：(X,d)是度量空间，$A_i\subseteq X\ (i=1,2,\cdots,n)\ (n\in\mathbb{Z}_{\geq 1})$．则有：

$$\bigcup_{i=1}^{n}\mathrm{cl}A_i = \mathrm{cl}\Big(\bigcup_{i=1}^{n}A_i\Big)$$

Proof：由上一引理，有：$\bigcup_{i=1}^{n}\mathrm{cl}A_i \subseteq \mathrm{cl}\Big(\bigcup_{i=1}^{n}A_i\Big)$

对$\forall x\in \mathrm{cl}\Big(\bigcup_{i=1}^{n}A_i\Big)$，有：假设$x\notin \bigcup_{i=1}^{n}\mathrm{cl}A_i$，则有：

$x\notin \mathrm{cl}A_1$ 且 $x\notin \mathrm{cl}A_2$ 且…且 $x\notin \mathrm{cl}A_n$

$\because (X,d)$是度量空间，$A_1\subseteq X$，$x\notin \mathrm{cl}A_1$

$\therefore \exists r_1\in\mathbb{R}_{>0}$，s.t. $B(x;r_1)\cap A_1=\emptyset$

$\because (X,d)$是度量空间，$A_2\subseteq X$，$x\notin \mathrm{cl}A_2$

$\therefore \exists r_2\in\mathbb{R}_{>0}$，s.t. $B(x;r_2)\cap A_2=\emptyset$

$\because (X,d)$是度量空间，$A_n \subseteq X$，$x \notin \mathrm{cl}A_n$

$\therefore \exists r_n \in \mathbb{R}_{>0}$，s.t. $B(x;r_n) \cap A_n = \varnothing$

$\because r_1, r_2, \cdots, r_n \in \mathbb{R}_{>0}$ $\therefore \min\{r_1, r_2, \cdots, r_n\} \in \mathbb{R}_{>0}$

任取 $\varepsilon \in \mathbb{R}_{>0}$，满足 $\varepsilon < \min\{r_1, r_2, \cdots, r_n\}$. 则有: $B(x;\varepsilon) \cap (\bigcup_{i=1}^{n} A_i) = \varnothing$

(假设 $B(x;\varepsilon) \cap (\bigcup_{i=1}^{n} A_i) \neq \varnothing$，则有: $\exists \zeta \in B(x;\varepsilon) \cap (\bigcup_{i=1}^{n} A_i)$

$\therefore \zeta \in B(x;\varepsilon)$ 且 $\zeta \in \bigcup_{i=1}^{n} A_i$ $\because \zeta \in \bigcup_{i=1}^{n} A_i$ $\therefore \exists k \in \{1, \cdots, n\}$, s.t. $\zeta \in A_k$

~~$\therefore \exists k \in \{1,$~~

$\because \zeta \in B(x;\varepsilon)$ $\therefore \zeta \in X$ 且有 $d(x,\zeta) < \varepsilon$

$\therefore \zeta \in X$ 且 $d(x,\zeta) < \varepsilon < \min\{r_1, r_2, \cdots, r_n\} \leqslant r_k$ $\therefore \zeta \in B(x;r_k)$

$\therefore \zeta \in B(x;r_k) \cap A_k = \varnothing$ 矛盾. $\therefore B(x;\varepsilon) \cap (\bigcup_{i=1}^{n} A_i) = \varnothing$)

$\therefore x \notin \mathrm{cl}(\bigcup_{i=1}^{n} A_i)$ 矛盾. $\therefore x \in \bigcup_{i=1}^{n} \mathrm{cl}A_i$

$\therefore \mathrm{cl}(\bigcup_{i=1}^{n} A_i) \subseteq \bigcup_{i=1}^{n} \mathrm{cl}A_i$

$\therefore \bigcup_{i=1}^{n} \mathrm{cl}A_i = \mathrm{cl}(\bigcup_{i=1}^{n} A_i)$ $\square$

Lemma (内部和闭包的对偶) (X,d)是度量空间，$A \subseteq X$，则有:

① $\mathrm{cl}A = X \setminus \mathrm{int}(X \setminus A)$

② $\mathrm{int}A = X \setminus \mathrm{cl}(X \setminus A)$

③ $\partial A = \mathrm{cl}A \setminus \mathrm{int}A$

Proof: ① 对$\forall x \in \mathrm{cl}A$，有: $x \in X$. 假设 $x \in \mathrm{int}(X \setminus A)$，则有:

$\exists r \in \mathbb{R}_{>0}$，s.t. $B(x;r) \subseteq X \setminus A$ $\therefore B(x;r) \cap A = \varnothing$ $\therefore x \notin \mathrm{cl}A$. 矛盾.

$\therefore x \notin \mathrm{int}(X \setminus A)$ $\therefore x \in X \setminus \mathrm{int}(X \setminus A)$ $\therefore \mathrm{cl} A \subseteq X \setminus \mathrm{int}(X \setminus A)$

对$\forall x \in X \setminus \mathrm{int}(X \setminus A)$，有：$x \in X$且有$x \notin \mathrm{int}(X \setminus A)$

假设$x \notin \mathrm{cl} A$，则有：$\exists r \in \mathbb{R}_{>0}$，s.t. $B(x;r) \cap A = \emptyset$.

$\therefore B(x;r) \subseteq X \setminus A$ $\therefore x \in \mathrm{int}(X \setminus A)$ 矛盾. $\therefore x \in \mathrm{cl} A$

$\therefore X \setminus \mathrm{int}(X \setminus A) \subseteq \mathrm{cl} A$ $\therefore \mathrm{cl} A = X \setminus \mathrm{int}(X \setminus A)$

② 对$\forall x \in \mathrm{int} A$，有：$x \in X$.

$\because x \in \mathrm{int} A$ $\therefore \exists r \in \mathbb{R}_{>0}$，s.t. $B(x;r) \subseteq A$ $\therefore B(x;r) \cap (X \setminus A) = \emptyset$

$\therefore x \notin \mathrm{cl}(X \setminus A)$ $\therefore x \in X \setminus \mathrm{cl}(X \setminus A)$ $\therefore \mathrm{int} A \subseteq X \setminus \mathrm{cl}(X \setminus A)$

对$\forall x \in X \setminus \mathrm{cl}(X \setminus A)$，有：$x \in X$且$x \notin \mathrm{cl}(X \setminus A)$

$\because x \notin \mathrm{cl}(X \setminus A)$ $\therefore \exists r \in \mathbb{R}_{>0}$，s.t. $B(x;r) \cap (X \setminus A) = \emptyset$

$\therefore B(x;r) \subseteq A$ $\therefore x \in \mathrm{int} A$ $\therefore X \setminus \mathrm{cl}(X \setminus A) \subseteq \mathrm{int} A$

$\therefore \mathrm{int} A = X \setminus \mathrm{cl}(X \setminus A)$

③ $\mathrm{cl} A \setminus \mathrm{int} A = \mathrm{cl} A \setminus (X \setminus \mathrm{cl}(X \setminus A)) = (\mathrm{cl} A \setminus X) \cup (\mathrm{cl} A \cap \mathrm{cl}(X \setminus A))$

$= \emptyset \cup (\mathrm{cl} A \cap \mathrm{cl}(X \setminus A)) = \mathrm{cl} A \cap \mathrm{cl}(X \setminus A) = \partial A$ $\square$