Titu 4.7.2 前半部分笔记

定理（平面上任意两点的距离，用重心坐标表示）. P_1, P_2 是平面上的任意两个点. $\triangle ABC$ 是同一平面上的一个任意的三角形. 边 BC, CA, AB 的长度分别为 α, β, γ ($\alpha, \beta, \gamma \in \mathbb{R}_+$) O 是 $\triangle ABC$ 的外接圆圆心. 以 O 为原点建立复平面，点 P_1, P_2 的复数坐标分别为：z_{p_1}, z_{p_2}, 点 A, B, C 的复数坐标分别为 a, b, c. $\triangle ABC$ 的外接圆半径为 R.

$z_{p_1} = \alpha_1 a + \beta_1 b + \gamma_1 c$, $\alpha_1, \beta_1, \gamma_1 \in \mathbb{R}$ 且 $\alpha_1 + \beta_1 + \gamma_1 = 1$.

$z_{p_2} = \alpha_2 a + \beta_2 b + \gamma_2 c$, $\alpha_2, \beta_2, \gamma_2 \in \mathbb{R}$ 且 $\alpha_2 + \beta_2 + \gamma_2 = 1$. 则有：

$$|P_1P_2|^2 = -\sum_{cyc} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2$$

Proof: $|P_1P_2|^2 = |z_{p_1} - z_{p_2}|^2 = |z_{p_2} - z_{p_1}|^2 = |(\alpha_2-\alpha_1)a + (\beta_2-\beta_1)b + (\gamma_2-\gamma_1)c|^2$

$$= \big((\alpha_2-\alpha_1)a + (\beta_2-\beta_1)b + (\gamma_2-\gamma_1)c\big) \underset{\text{这是实积}}{\cdot} \big((\alpha_2-\alpha_1)a + (\beta_2-\beta_1)b + (\gamma_2-\gamma_1)c\big)$$

$$= \sum_{cyc} (\alpha_2-\alpha_1)^2 \underset{\text{这是实积}}{(a \cdot a)} + 2\sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)(a \cdot b)$$

$$= \sum_{cyc} (\alpha_2-\alpha_1)^2 |a|^2 + 2\sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= \sum_{cyc} (\alpha_2-\alpha_1)^2 R^2 + 2\sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)R^2 - \sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\gamma^2$$

$$= R^2\left(\sum_{cyc} (\alpha_2-\alpha_1)^2 + 2\sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\right) - \sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\gamma^2$$

$$= R^2(\alpha_2-\alpha_1+\beta_2-\beta_1+\gamma_2-\gamma_1)^2 - \sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\gamma^2$$

$$= -\sum_{cyc} (\alpha_2-\alpha_1)(\beta_2-\beta_1)\gamma^2 \qquad \square$$

定理（两个塞瓦点之间的距离）$\triangle ABC$是一个任意的三角形，点 A_1, A_2 在边 BC上，点 B_1, B_2 在边 CA上，点 C_1, C_2 在边 AB上. $\frac{BA_1}{A_1C}=\frac{p_1}{n_1}$，$\frac{CB_1}{B_1A}=\frac{m_1}{p_1}$，$\frac{AC_1}{C_1B}=\frac{n_1}{m_1}$ $(p_1, n_1, m_1 \in \mathbb{R}_+)$

$\frac{BA_2}{A_2C}=\frac{p_2}{n_2}$，$\frac{CB_2}{B_2A}=\frac{m_2}{p_2}$，$\frac{AC_2}{C_2B}=\frac{n_2}{m_2}$ $(p_2, n_2, m_2 \in \mathbb{R}_+)$. 直线$AA_1$，$BB_1$，$CC_1$相交于点 P_1，直线 AA_2，BB_2，CC_2 相交于点 P_2. 边 BC，CA，AB 的长度分别为 α, β, γ $(\alpha, \beta, \gamma \in \mathbb{R}_+)$

记 $S_1 = m_1+n_1+p_1$，$S_2 = m_2+n_2+p_2$. 则有：

$$|P_1P_2|^2 = \frac{1}{S_1^2S_2^2}\left(S_1S_2\sum_{cyc}(n_1p_2+p_1n_2)\alpha^2 - S_1^2\sum_{cyc}n_2p_2\alpha^2 - S_2^2\sum_{cyc}n_1p_1\alpha^2\right)$$

Proof：以 $\triangle ABC$ 的外接圆圆心为原点建立复平面. 设点 A, B, C 的复数坐标分别为 a, b, c.

$$\therefore z_{P_1} = \frac{m_1a+n_1b+p_1c}{m_1+n_1+p_1} = \frac{m_1}{S_1}a + \frac{n_1}{S_1}b + \frac{p_1}{S_1}c. \quad \frac{m_1}{S_1}+\frac{n_1}{S_1}+\frac{p_1}{S_1}=1$$

$$z_{P_2} = \frac{m_2a+n_2b+p_2c}{m_2+n_2+p_2} = \frac{m_2}{S_2}a + \frac{n_2}{S_2}b + \frac{p_2}{S_2}c. \quad \frac{m_2}{S_2}+\frac{n_2}{S_2}+\frac{p_2}{S_2}=1$$

$$\therefore \text{由上一定理得：} |P_1P_2|^2 = -\sum_{cyc}\left(\frac{m_2}{S_2}-\frac{m_1}{S_1}\right)\left(\frac{n_2}{S_2}-\frac{n_1}{S_1}\right)\gamma^2$$

$$\cancel{= -\Big(\Big(\frac{m_2}{S_2}-\frac{m_1}{S_1}\Big)\Big(\frac{n_2}{S_2}-\frac{n_1}{S_1}\Big)\gamma^2 + \Big(\frac{n_2}{S_2}-\frac{n_1}{S_1}\Big)}$$

$$= -\left(\left(\frac{m_2}{S_2}-\frac{m_1}{S_1}\right)\left(\frac{n_2}{S_2}-\frac{n_1}{S_1}\right)\gamma^2 + \left(\frac{n_2}{S_2}-\frac{n_1}{S_1}\right)\left(\frac{p_2}{S_2}-\frac{p_1}{S_1}\right)\alpha^2 + \left(\frac{p_2}{S_2}-\frac{p_1}{S_1}\right)\left(\frac{m_2}{S_2}-\frac{m_1}{S_1}\right)\beta^2\right)$$

$$= -\Big(\frac{1}{S_1^2S_2^2}(m_2S_1-m_1S_2)(n_2S_1-n_1S_2)\gamma^2 + \frac{1}{S_1^2S_2^2}(n_2S_1-n_1S_2)(p_2S_1-p_1S_2)\alpha^2$$
$$+ \frac{1}{S_1^2S_2^2}(p_2S_1-p_1S_2)(m_2S_1-m_1S_2)\beta^2\Big)$$

$$= -\frac{1}{S_1^2S_2^2}\Big((m_2n_2S_1^2+m_1n_1S_2^2-(m_1n_2+m_2n_1)S_1S_2)\gamma^2 + (n_2p_2S_1^2+n_1p_1S_2^2-(n_1p_2+n_2p_1)S_1S_2)\alpha^2$$
$$+ (m_2p_2S_1^2+m_1p_1S_2^2-(m_1p_2+m_2p_1)S_1S_2)\beta^2\Big)$$

$$= \frac{1}{S_1^2S_2^2}\Big(S_1S_2\big((m_1n_2+m_2n_1)\gamma^2+(n_1p_2+n_2p_1)\alpha^2+(p_1m_2+p_2m_1)\beta^2\big) - S_1^2(m_2n_2\gamma^2+n_2p_2\alpha^2+p_2m_2\beta^2)$$
$$-S_2^2(m_1n_1\gamma^2+n_1p_1\alpha^2+p_1m_1\beta^2)\Big)$$

$$= \frac{1}{S_1^2 S_2^2}\left(S_1 S_2 \sum_{cyc} (n_1 p_2 + n_2 p_1)\alpha^2 - S_1^2 \sum_{cyc} n_2 p_2 \alpha^2 - S_2^2 \sum_{cyc} n_1 p_1 \alpha^2 \right) \qquad \square$$

$$\left(m_1 \to n_1 \to p_1 \ , \ m_2 \to n_2 \to p_2 \right)$$