Titu 2.1笔记.

有关复数的三角表示，我们已在之前的笔记中引入，这里不再赘述.

Lemma: $z_1 = r_1(\cos t_1 + i\sin t_1)$, $z_2 = r_2(\cos t_2 + i\sin t_2)$ $(r_1, r_2 > 0, t_1, t_2 \in \mathbb{R})$.

则有: $z_1 \cdot z_2 = r_1 r_2 (\cos(t_1+t_2) + i\sin(t_1+t_2))$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(t_1-t_2) + i\sin(t_1-t_2))$$

proof: $z_1 = r_1(\cos t_1 + i\sin t_1) = (r_1\cos t_1) + i\cdot(r_1\sin t_1)$.

$z_2 = r_2(\cos t_2 + i\sin t_2) = (r_2\cos t_2) + i(r_2\sin t_2)$

$\therefore z_1 z_2 = \left((r_1\cos t_1)(r_2\cos t_2) - (r_1\sin t_1)(r_2\sin t_2)\right) + \left((r_1\cos t_1)(r_2\sin t_2) + (r_2\cos t_2)(r_1\sin t_1)\right)i$

$= [r_1 r_2(\cos t_1\cos t_2 - \sin t_1\sin t_2)] + [r_1 r_2(\sin t_1\cos t_2 + \cos t_1\sin t_2)]i$

$= (r_1 r_2\cos(t_1+t_2)) + (r_1 r_2\sin(t_1+t_2))\cdot i$

$= r_1 r_2(\cos(t_1+t_2) + i\sin(t_1+t_2))$

$\because |z_2| = r_2 > 0$ $\therefore z_2 \neq 0$ $\therefore z_2$可以作分母.

$$\frac{z_1}{z_2} = \frac{r_1(\cos t_1 + i\sin t_1)}{r_2(\cos t_2 + i\sin t_2)} = \frac{r_1(\cos t_1 + i\sin t_1)(\cos t_2 - i\sin t_2)}{r_2(\cos t_2 + i\sin t_2)(\cos t_2 - i\sin t_2)}.$$

$$= \frac{r_1(\cos t_1\cos t_2 + \sin t_1\sin t_2 + (\sin t_1\cos t_2 - \cos t_1\sin t_2)i)}{r_2(\cos^2 t_2 + \sin^2 t_2)}$$

$$= \frac{r_1}{r_2}(\cos(t_1-t_2) + i\sin(t_1-t_2)). \quad \square$$

Lemma: $z = r(\cos t + i\sin t)$, $r > 0$, $t \in \mathbb{R}$. $\forall n \in \mathbb{Z}$. 则有:

$$z^n = r^n(\cos(nt) + i\sin(nt)).$$

proof: $n=1$时，显然. $n=2$时，上面引理已证. $\therefore$很容易由数学归纳法得，对$\forall n \in \mathbb{N}_+$，成立.

$\because |z| = r > 0$ $\therefore z \neq 0$ $\therefore z^0 = 1$. $\because r^0(\cos 0 + i\sin 0) = 1\cdot(1+0) = 1$.

$\therefore n=0$ 时成立.

对 $\forall n \in \mathbb{Z} \setminus \mathbb{N}$（即: n 为负整数），有: $-n \in \mathbb{N}_+$.

$$\therefore z^n = z^{-(-n)} = \frac{1}{z^{-n}} = \frac{1}{r^{-n}(\cos((-n)t) + i\sin((-n)t))}$$

$$= \frac{1}{r^{-n}(\cos(nt) - i\sin(nt))} = \frac{r^n(\cos(nt) + i\sin(nt))}{(\cos(nt) - i\sin(nt))(\cos(nt) + i\sin(nt))}$$

$$= \frac{r^n(\cos(nt) + i\sin(nt))}{\cos^2(nt) + \sin^2(nt)} = r^n(\cos(nt) + i\sin(nt))$$

$\therefore$ 对 $\forall n \in \mathbb{Z}$，引理成立. $\square$

为了加快推进重要理论的学习，习题和 ~~注~~ Titu 一书 2.2 节中有关单位根的内容之后再学.

需要学习初等数论的内容，原根等概念.