Titu 4.6.3 − 4.6.4笔记

定理（Nagel点到外心的距离）$\triangle ABC$是一个任意的三角形，O是$\triangle ABC$的外接圆圆心，N是$\triangle ABC$的Nagel点．R是$\triangle ABC$的外接圆半径，r是$\triangle ABC$的内切圆半径．则有：

$$ON = R - 2r$$

proof：以$\triangle ABC$的外接圆圆心O为原点建立复平面．设点A，B，C的复数坐标分别为a，b，c．设BC，CA，AB三条边的长度分别为α，β，γ（$\alpha,\beta,\gamma\in\mathbb{R}_+$）．

设$s=\frac{1}{2}(\alpha+\beta+\gamma)$．则有：$z_N=\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c$

$$\therefore \underset{\text{点O}}{ON}^2=|z_N-\underset{\text{复数“零”}}{0}|^2=|z_N|^2=\underset{\text{这是实积}}{z_N\cdot z_N}=\left(\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c\right)\underset{\text{这是实积}}{\cdot}\left(\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c\right)$$

$$=\sum_{cyc}\left(1-\frac{\alpha}{s}\right)^2(a\cdot a)+2\sum_{cyc}\left(1-\frac{\alpha}{s}\right)\left(1-\frac{\beta}{s}\right)(a\cdot b)$$

$$=R^2\sum_{cyc}\left(1-\frac{\alpha}{s}\right)^2+2\sum_{cyc}\left(1-\frac{\alpha}{s}\right)\left(1-\frac{\beta}{s}\right)\left(R^2-\frac{\gamma^2}{2}\right)$$

$$=R^2\sum_{cyc}\left(1-\frac{2}{s}\alpha+\frac{\alpha^2}{s^2}\right)+2\sum_{cyc}\left(R^2-\frac{R^2}{s}\alpha-\frac{R^2}{s}\beta+\frac{R^2}{s^2}\alpha\beta-\frac{\gamma^2}{2}+\frac{\gamma^2\alpha}{2s}+\frac{\gamma^2\beta}{2s}-\frac{\alpha\beta\gamma^2}{2s^2}\right)$$

$$=R^2\left(3-\frac{2}{s}\sum_{cyc}\alpha+\frac{1}{s^2}\sum_{cyc}\alpha^2\right)+$$
$$2\left(3R^2-\frac{R^2}{s}\sum_{cyc}\alpha-\frac{R^2}{s}\sum_{cyc}\alpha+\frac{R^2}{s^2}\sum_{cyc}\alpha\beta-\frac{1}{2}\sum_{cyc}\alpha^2+\frac{1}{2s}\sum_{cyc}\gamma^2(2s-\gamma)-\frac{\alpha\beta\gamma}{2s^2}\sum_{cyc}\alpha\right)$$

$$=R^2\left(3-\frac{2}{s}\sum_{cyc}\alpha+\frac{1}{s^2}\sum_{cyc}\alpha^2\right)+$$
$$2\left(3R^2-\frac{2R^2}{s}\sum_{cyc}\alpha+\frac{R^2}{s^2}\sum_{cyc}\alpha\beta-\frac{1}{2}\sum_{cyc}\alpha^2+\sum_{cyc}\alpha^2-\frac{1}{2s}\sum_{cyc}\alpha^3-\frac{\alpha\beta\gamma}{2s^2}\sum_{cyc}\alpha\right)$$

$$=R^2\left(3-\frac{2}{s}\cdot 2s+\frac{1}{s^2}\cdot 2(s^2-r^2-4Rr)\right)+$$
$$2\left(3R^2-\frac{2R^2}{s}\cdot 2s+\frac{R^2}{s^2}(s^2+r^2+4Rr)+\frac{1}{2}\cdot 2(s^2-r^2-4Rr)-\frac{1}{2s}\cdot 2s(s^2-3r^2-6Rr)-\frac{\alpha\beta\gamma}{2s^2}\cdot 2s\right)$$

$$=(R-2r)^2$$

$\because R-2r\geq 0$ $\quad\therefore ON=\underset{\text{绝对值}}{|ON|}=\sqrt{ON^2}=\sqrt{(R-2r)^2}=\underset{\text{绝对值}}{|R-2r|}=R-2r$

□

定理（Feuerbach's theorem，不是完整的，仅含内切圆）$\triangle ABC$是一个任意的三角形．则有：$\triangle ABC$的内切圆与$\triangle ABC$的九点圆相切．

proof：设O为$\triangle ABC$的外接圆圆心，I为$\triangle ABC$的内切圆圆心．O_9为$\triangle ABC$的九点圆圆心，N为$\triangle ABC$的Nagel点．

以O为原点建立复平面，设点A，B，C的复数坐标分别为a，b，c．设边BC，CA，AB的长度分别为α，β，γ（$\alpha,\beta,\gamma\in\mathbb{R}_+$），$s=\frac{1}{2}(\alpha+\beta+\gamma)$．则有：

$$z_I=\frac{\alpha}{2s}a+\frac{\beta}{2s}b+\frac{\gamma}{2s}c\ ,\quad z_{O_9}=\frac{1}{2}(a+b+c)\ ,\quad z_N=\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c.$$

$$\therefore IO_9=\left|z_I-z_{O_9}\right|=\left|\left(\frac{\alpha}{2s}-\frac{1}{2}\right)a+\left(\frac{\beta}{2s}-\frac{1}{2}\right)b+\left(\frac{\gamma}{2s}-\frac{1}{2}\right)c\right|$$

$$=\left|\frac{1}{2}\left(\frac{\alpha}{s}-1\right)a+\frac{1}{2}\left(\frac{\beta}{s}-1\right)b+\frac{1}{2}\left(\frac{\gamma}{s}-1\right)c\right|$$

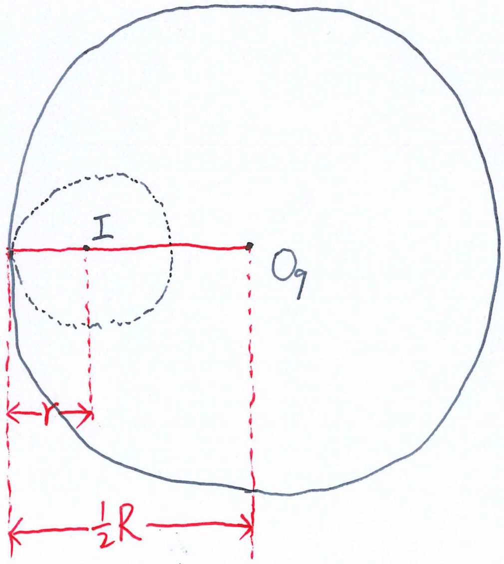
$$=\left|\left(-\frac{1}{2}\right)\left(1-\frac{\alpha}{s}\right)a+\left(-\frac{1}{2}\right)\left(1-\frac{\beta}{s}\right)b+\left(-\frac{1}{2}\right)\left(1-\frac{\gamma}{s}\right)c\right|$$

$$=\left|\left(-\frac{1}{2}\right)\left(\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c\right)\right|$$

$$=\left|-\frac{1}{2}\right|\cdot\left|\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c\right|=\frac{1}{2}\left|z_N\right|=\frac{1}{2}\left|z_N-\underset{\text{复数“零”}}{0}\right|=\frac{1}{2}ON$$

$$\therefore IO_9=\frac{1}{2}ON=\frac{1}{2}(R-2r)=\frac{R}{2}-r$$

．有如下的示意图：



$\therefore\triangle ABC$的内切圆 内切于 $\triangle ABC$的九点圆． $\square$

定义（Feuerbach点）．$\triangle ABC$的内切圆与$\triangle ABC$的九点圆相内切的切点（记作φ）称为$\triangle ABC$的费尔巴哈点．

定理（三角形外心与垂心的距离）$\triangle ABC$是一个任意的三角形，O是$\triangle ABC$的外接圆圆心，H是$\triangle ABC$的垂心．则有：

$$OH^2 = 9R^2 + 2r^2 + 8Rr - 2S^2$$

Proof：以$\triangle ABC$的外接圆圆心O为原点建立复平面．则有：$z_H = a+b+c$

$$\therefore OH^2 = |z_H - 0|^2 = |z_H|^2 = \underset{\text{这是实积}}{z_H \cdot z_H} = \underset{\text{这是实积}}{(a+b+c)\cdot(a+b+c)}$$

$$= \sum_{cyc}(a\cdot a) + 2\sum_{cyc}(a\cdot b) = \sum_{cyc}|a|^2 + 2\sum_{cyc}\left(R^2 - \frac{\alpha^2}{2}\right)$$

$$= 3R^2 + 2\left(3R^2 - \frac{1}{2}\sum_{cyc}\alpha^2\right) = 9R^2 - \sum_{cyc}\alpha^2 = 9R^2 - 2S^2 + 2r^2 + 8Rr \quad \square$$

定理（三角形外心与重心的距离）$\triangle ABC$是一个任意的三角形，O是$\triangle ABC$的外心，G是$\triangle ABC$的重心，则有：

$$OG^2 = R^2 + \frac{2}{9}r^2 + \frac{8}{9}Rr - \frac{2}{9}S^2$$

Proof：以$\triangle ABC$的外心O为原点建立复平面，则有：$z_G = \dfrac{a+b+c}{3}$

$$\therefore OG^2 = |z_G - 0|^2 = |z_G|^2 = \underset{\text{这是实积}}{z_G \cdot z_G} = \frac{1}{9}\underset{\text{这是实积}}{\left((a+b+c)\cdot(a+b+c)\right)}$$

$$= R^2 + \frac{2}{9}r^2 + \frac{8}{9}Rr - \frac{2}{9}S^2 \quad \square$$

定理（三角形外心与九点圆圆心的距离）$\triangle ABC$是一个任意的三角形，O是$\triangle ABC$的外心，O_9是$\triangle ABC$的九点圆的圆心．则有：

$$OO_9^2 = \frac{9}{4}R^2 + \frac{1}{2}r^2 + 2Rr - \frac{1}{2}S^2$$

Proof：以$\triangle ABC$的外心O为原点建立复平面，则有：$z_{O_9} = \frac{1}{2}(a+b+c)$

$$\therefore OO_9^2 = |z_{O_9} - 0|^2 = |z_{O_9}|^2 = \underset{\text{这是实积}}{z_{O_9} \cdot z_{O_9}} = \frac{1}{4}\underset{\text{这是实积}}{\left((a+b+c)\cdot(a+b+c)\right)}$$

$$= \frac{9}{4}R^2 + \frac{1}{2}r^2 + 2Rr - \frac{1}{2}S^2 \quad \square$$

定理（三角形三边平方和的不等式）$\triangle ABC$是一个任意的三角形，则有：

$$\alpha^2+\beta^2+\gamma^2\leqslant 9R^2$$

等号成立当且仅当$\triangle ABC$是等边三角形.

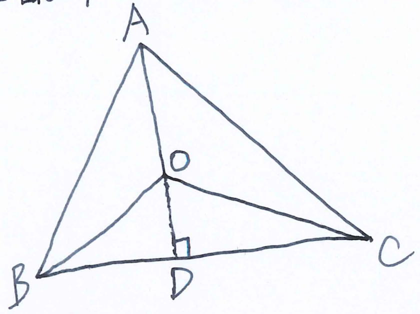
Proof: $9R^2-(\alpha^2+\beta^2+\gamma^2)=9R^2-2s^2+2r^2+8Rr=9R^2+2r^2+8Rr-2s^2$

$=OH^2\geqslant 0$

$\therefore \alpha^2+\beta^2+\gamma^2\leqslant 9R^2$

$\alpha^2+\beta^2+\gamma^2=9R^2 \Longleftrightarrow OH^2=\underset{\text{实数“零”}}{0} \Longleftrightarrow OH=\underset{\text{实数“零”}}{0} \Longleftrightarrow$ $\triangle ABC$的外心与垂心重合.

$\because$ $\triangle ABC$的外心与垂心重合　$\therefore$ O既是$\triangle ABC$的外心，又是$\triangle ABC$的垂心



$\because$ O是$\triangle ABC$的外心　$\therefore OB=OC$　$\because \begin{cases} OB=OC \\ OD=OD \end{cases}$　$\therefore Rt\triangle OBD\cong Rt\triangle OCD$

$\therefore BD=CD$　$\because \begin{cases} AD=AD \\ \angle ADB=\angle ADC=\frac{\pi}{2} \\ BD=CD \end{cases}$　$\therefore \triangle ADB\cong\triangle ADC$　$\therefore \angle BAD=\angle CAD$

$\therefore AD$平分$\angle BAC$. 同理可证 BO平分$\angle ABC$，CO平分$\angle ACB$

$\therefore$ O是$\triangle ABC$的内心.　$\therefore$ $\triangle ABC$的外心与内心重合　$\therefore$ $\triangle ABC$是等边三角形.

$\triangle ABC$是等边三角形 $\Rightarrow$ $\triangle ABC$的外心与垂心重合　显然！

$\therefore$ $\triangle ABC$的外心与垂心重合 $\Longleftrightarrow$ $\triangle ABC$是等边三角形　□