复数三角不等式取等号的充要条件.

Lemma：对 $\forall z_1, z_2 \in \mathbb{C}^*$，有：

$$|z_1+z_2| = |z_1| + |z_2| \iff \exists \lambda \in \mathbb{R}, \lambda>0，使得 z_2 = \lambda z_1$$

proof：对 $\forall z_1, z_2 \in \mathbb{C}^*$.

$(\Leftarrow)$：$\because \exists \lambda \in \mathbb{R}, \lambda > 0$，s.t. $z_2 = \lambda z_1$

$\therefore |z_1+z_2| = |z_1+\lambda z_1| = |(1+\lambda)z_1| = |1+\lambda|\cdot|z_1| = (1+\lambda)|z_1|$

$= |z_1| + \lambda|z_1| = |z_1| + |\lambda|\cdot|z_1| = |z_1| + |\lambda z_1| = |z_1| + |z_2|$

$(\Rightarrow)$：$\because |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\cdot\overline{z_2})$

$|z_1+z_2|^2 = (|z_1|+|z_2|)^2 = |z_1|^2 + |z_2|^2 + 2\cdot|z_1|\cdot|z_2|$.

$\therefore \mathrm{Re}(z_1\cdot\overline{z_2}) = |z_1|\cdot|z_2| \geqslant 0$

设 $z_1 = a_1+b_1 i$，$z_2 = a_2+b_2 i$ $(a_1, b_1, a_2, b_2 \in \mathbb{R})$.

$\therefore z_1\cdot\overline{z_2} = (a_1+b_1 i)(a_2-b_2 i) = (a_1a_2+b_1b_2) + (a_2b_1 - a_1b_2)i$

$\therefore a_1a_2+b_1b_2 = \sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}$

$\therefore (a_1a_2+b_1b_2)^2 = (a_1^2+b_1^2)(a_2^2+b_2^2)$ 且 $a_1a_2+b_1b_2 \geqslant 0$

$\therefore (a_1b_2-a_2b_1)^2 = 0$ 且 $a_1a_2+b_1b_2 \geqslant 0$

$\therefore a_1b_2 = a_2b_1$ 且 $a_1a_2+b_1b_2 \geqslant 0$.

① 若 $a_1 \neq 0$，此时有下面两种可能.

(i) $b_2 = 0$. $\therefore a_1b_2 = 0$ $\therefore a_2b_1 = 0$ $\therefore a_2 = 0$ 或 $b_1 = 0$.

若 $a_2 = 0$，则 $a_2 = 0 = b_2$ $\therefore z_2 = 0$. 矛盾. $\therefore a_2 \neq 0$. $\therefore b_1 = 0$.

$\therefore b_1 = 0 = b_2$，$a_1 \neq 0$，$a_2 \neq 0$.

$\because a_1a_2 \geqslant 0$ $\therefore a_1a_2 > 0$. $\therefore (a_1>0$ 且 $a_2>0)$ 或 $(a_1<0$ 且 $a_2<0)$

$\therefore \frac{a_2}{a_1} > 0$. 设 $\lambda = \frac{a_2}{a_1}$ $\therefore \lambda>0, \lambda\in\mathbb{R}$.

$\because z_1 = a_1 \neq 0$，$z_2 = a_2$. $\therefore \frac{z_2}{z_1} = \frac{a_2}{a_1} = \lambda$ $\therefore z_2 = \lambda z_1$，$\lambda>0, \lambda\in\mathbb{R}$. ∎

(ii) $b_2\neq 0$.

$\because a_1\neq 0$ 且 $b_2\neq 0$ $\therefore a_1b_2\neq 0$ $\therefore a_2b_1=a_1b_2\neq 0$ $\therefore a_2\neq 0$ 且 $b_1\neq 0$

$\because a_1b_2=a_2b_1$ $\therefore$ 按正负分类讨论如下：

a_1	b_2	a_2	b_1	$a_1a_2+b_1b_2$	可行与否	$\frac{a_2}{a_1}$	$\frac{b_2}{b_1}$
+	+	+	+	+	可行	+	+
+	+	−	−	−	不可行		
+	−	+	−	+	可行	+	+
+	−	−	+	−	不可行		
−	+	+	−	−	不可行.		
−	+	−	+	+	可行.	+	+
−	−	+	+	−	不可行.		
−	−	−	−	+	可行	+	+

$\because a_1b_2=a_2b_1$ $\therefore \frac{a_2}{a_1}=\frac{b_2}{b_1}$ 设 $\lambda=\frac{a_2}{a_1}=\frac{b_2}{b_1}$ $\therefore \lambda>0$. $\lambda\in R$

$\therefore a_2=\lambda a_1$, $b_2=\lambda b_1$ $\therefore z_2=a_2+b_2i=\lambda a_1+\lambda b_1 i=\lambda(a_1+b_1i)=\lambda z_1$

$\therefore z_2=\lambda z_1$, $\lambda>0$, $\lambda\in R$

② 若 $a_1=0$. 则有：$a_1b_2=0$ $\therefore a_2b_1=a_1b_2=0$ $\therefore a_2=0$ 或 $b_1=0$.

若 $b_1=0$, 则有：$b_1=0=a_1$ $\therefore z_1=0$. 矛盾 $\therefore b_1\neq 0$ $\therefore a_2=0$

$\therefore a_1=0=a_2$, $b_1\neq 0$.

假设 $b_2=0$. 则 $b_2=0=a_2$. $\therefore z_2=0$. 矛盾. $\therefore b_2\neq 0$

$\therefore a_1=0=a_2$, $b_1\neq 0$, $b_2\neq 0$. $\therefore z_1=b_1i$, $z_2=b_2i$.

$\because b_1b_2\geqslant 0$ $\therefore b_1b_2>0$ $\therefore (b_1>0$ 且 $b_2>0)$ 或 $(b_1<0$ 且 $b_2<0)$

$\therefore \frac{b_2}{b_1}>0$. 设 $\lambda=\frac{b_2}{b_1}$ $\therefore \lambda>0$ 且 $\lambda\in R$. $z_2=b_2i=\lambda b_1 i=\lambda z_1$ □

Lemma：对 $\forall z_1, z_2 \in \mathbb{C}^*$，有：

$$|z_1 - z_2| = |z_1| + |z_2| \iff \exists \lambda \in \mathbb{R},\ \lambda < 0，使得\ z_2 = \lambda z_1$$

Proof：对 $\forall z_1, z_2 \in \mathbb{C}^*$．有：$|z_2| = |-z_2|$，$z_2 \in \mathbb{C}^* \iff -z_2 \in \mathbb{C}^*$

$\therefore |z_1 - z_2| = |z_1| + |z_2|$

$\iff |z_1 + (-z_2)| = |z_1| + |-z_2|$

$\iff \exists t \in \mathbb{R},\ t > 0,\ s.t.\ -z_2 = t z_1$

$\iff \exists t \in \mathbb{R},\ t > 0,\ s.t.\ z_2 = (-t) z_1$

$\iff \exists \lambda \in \mathbb{R},\ \lambda < 0,\ s.t.\ z_2 = \lambda z_1$ $\square$

Lemma：对 $\forall z_1, z_2 \in \mathbb{C}^*$，有：

$$\big||z_1| - |z_2|\big| = |z_1 + z_2| \iff \exists \lambda \in \mathbb{R},\ \lambda < 0,\ s.t.\ z_2 = \lambda z_1$$

Proof：对 $\forall z_1, z_2 \in \mathbb{C}^*$

$(\Leftarrow)$：$\because \exists \lambda \in \mathbb{R}, \lambda < 0, s.t.\ z_2 = \lambda z_1$

$$\therefore \big||z_1| - |z_2|\big| = \big||z_1| - |\lambda z_1|\big| = \big||z_1| - |\lambda||z_1|\big| = \big||z_1| + \lambda|z_1|\big|$$

$$= \big|(1+\lambda)|z_1|\big| = |1+\lambda||z_1| = |(1+\lambda)z_1| = |z_1 + \lambda z_1| = |z_1 + z_2|$$

$(\Rightarrow)$：$\because \big||z_1| - |z_2|\big| = |z_1 + z_2|$

$\therefore |z_1| - |z_2| = |z_1 + z_2|$ 或 $|z_2| - |z_1| = |z_1 + z_2|$

①若 $|z_1| - |z_2| = |z_1 + z_2|$，则有：$|z_1| = |z_1 + z_2| + |z_2|$

$\therefore |(z_1 + z_2) - z_2| = |z_1 + z_2| + |z_2|$

若 $z_1 + z_2 = 0$，则有：$z_2 = -z_1 = (-1)\cdot z_1$．$-1 \in \mathbb{R}$，$-1 < 0$．结论得证．

若 $z_1 + z_2 \neq 0$，则有：$\exists t \in \mathbb{R},\ t < 0,\ s.t.\ z_2 = t(z_1 + z_2)$

$\therefore z_2 = t z_1 + t z_2$． $\therefore (1-t) z_2 = t z_1$ $\therefore z_2 = \dfrac{t}{1-t} z_1$．

令 $\lambda=\frac{t}{1-t}$. $\because t<0$ $\therefore -t>0$ $\therefore 1-t>1>0$ $\therefore \lambda<0,\ \lambda\in\mathbb{R}$

$\therefore z_2=\lambda z_1,\ \lambda<0,\ \lambda\in\mathbb{R}$. 结论得证.

② 若 $|z_2|-|z_1|=|z_1+z_2|$, 则有: $|z_2|=|z_2+z_1|+|z_1|$

$\therefore |(z_2+z_1)-z_1|=|z_2+z_1|+|z_1|$

若 $z_2+z_1=0$, 则有: $z_2=-z_1=(-1)z_1$, $-1\in\mathbb{R}$, $-1<0$. 结论得证.

若 $z_2+z_1\neq 0$, 则有: $\exists\,\alpha\in\mathbb{R},\ \alpha<0$, s.t. $z_1=\alpha(z_2+z_1)$

$\therefore z_1=\alpha z_2+\alpha z_1$ $\therefore \alpha z_2=(1-\alpha)z_1$, $z_2=\frac{1-\alpha}{\alpha}z_1$.

令 $\lambda=\frac{1-\alpha}{\alpha}<0$ $\therefore \lambda\in\mathbb{R}$ 且 $\lambda<0$, $z_2=\lambda z_1$. 结论得证.

综上, $\exists\,\lambda\in\mathbb{R},\ \lambda<0$, s.t. $z_2=\lambda z_1$ $\square$

Lemma: 对 $\forall\, z_1, z_2\in\mathbb{C}^*$, 有:

$$\big||z_1|-|z_2|\big|=|z_1-z_2| \iff \exists\,\lambda\in\mathbb{R},\ \lambda>0,\ \text{s.t. } z_2=\lambda z_1$$

Proof: 对 $\forall\, z_1, z_2\in\mathbb{C}^*$, 有: $z_2\in\mathbb{C}^*\iff -z_2\in\mathbb{C}^*$

$\therefore \big||z_1|-|z_2|\big|=|z_1-z_2| \iff \big||z_1|-|z_2|\big|=|z_1+(-z_2)|$

$\iff \big||z_1|-|-z_2|\big|=|z_1+(-z_2)|$

$\iff \exists\, t\in\mathbb{R},\ t<0$, s.t. $-z_2=tz_1$

$\iff \exists\, t\in\mathbb{R},\ t<0$, s.t. $z_2=(-t)z_1$

$\iff \exists\,\lambda\in\mathbb{R},\ \lambda>0$, s.t. $z_2=\lambda z_1$ $\square$