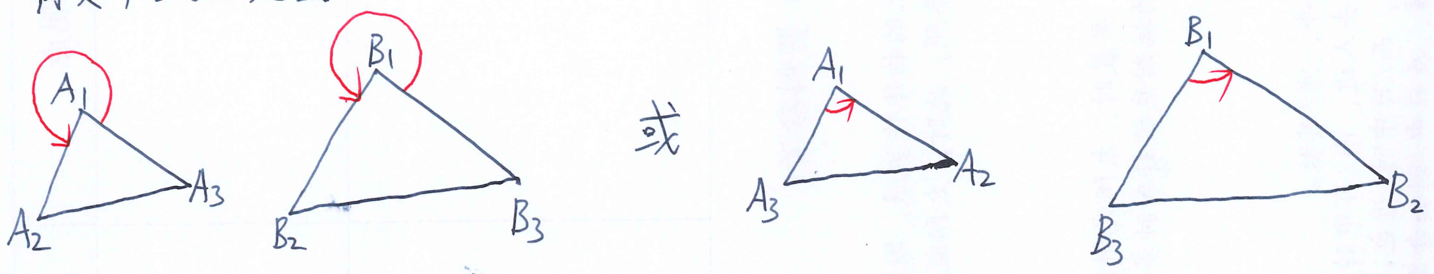
Lemma: $A_1(a_1)$, $A_2(a_2)$, $A_3(a_3)$, $B_1(b_1)$, $B_2(b_2)$, $B_3(b_3)$ 是六个彼此不同的点，则有:

$$\triangle A_1A_2A_3 \sim \triangle B_1B_2B_3 \text{ 且有相同的定向} \Longleftrightarrow \frac{a_2-a_1}{a_3-a_1}=\frac{b_2-b_1}{b_3-b_1}$$

proof: ($\Rightarrow$): ∵ $\triangle A_1A_2A_3$ 相似于 $\triangle B_1B_2B_3$ 且 $\triangle A_1A_2A_3$ 与 $\triangle B_1B_2B_3$ 有相同的定向

∴ 有如下的示意图:



$$\therefore \widehat{A_3A_1A_2}=\widehat{B_3B_1B_2} \text{ 且 } \frac{|A_1A_2|}{|B_1B_2|}=\frac{|A_1A_3|}{|B_1B_3|}$$

$$\therefore \widehat{A_3A_1A_2}=\widehat{B_3B_1B_2} \text{ 且 } \frac{|A_1A_2|}{|A_1A_3|}=\frac{|B_1B_2|}{|B_1B_3|}$$

$$\therefore \arg\left(\frac{a_2-a_1}{a_3-a_1}\right)=\arg\left(\frac{b_2-b_1}{b_3-b_1}\right) \text{ 且 } \frac{|a_2-a_1|}{|a_3-a_1|}=\frac{|b_2-b_1|}{|b_3-b_1|}$$

$$\therefore \arg\left(\frac{a_2-a_1}{a_3-a_1}\right)=\arg\left(\frac{b_2-b_1}{b_3-b_1}\right) \text{ 且 } \left|\frac{a_2-a_1}{a_3-a_1}\right|=\left|\frac{b_2-b_1}{b_3-b_1}\right|$$

$$\therefore \frac{a_2-a_1}{a_3-a_1}=\frac{b_2-b_1}{b_3-b_1}$$

($\Leftarrow$): $\because \dfrac{a_2-a_1}{a_3-a_1}=\dfrac{b_2-b_1}{b_3-b_1}$

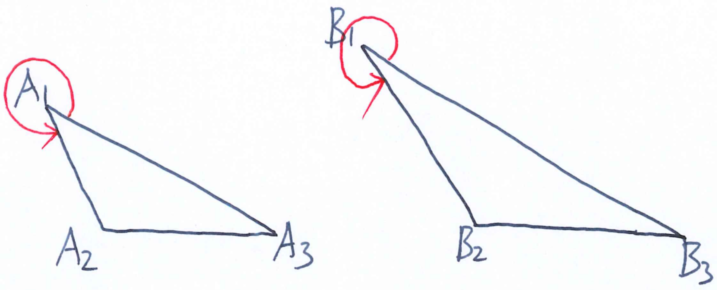
$$\therefore \left|\frac{a_2-a_1}{a_3-a_1}\right|=\left|\frac{b_2-b_1}{b_3-b_1}\right| \text{ 且 } \arg\left(\frac{a_2-a_1}{a_3-a_1}\right)=\arg\left(\frac{b_2-b_1}{b_3-b_1}\right)$$

$$\therefore \frac{|a_2-a_1|}{|a_3-a_1|}=\frac{|b_2-b_1|}{|b_3-b_1|} \text{ 且 } \widehat{A_3A_1A_2}=\widehat{B_3B_1B_2}$$

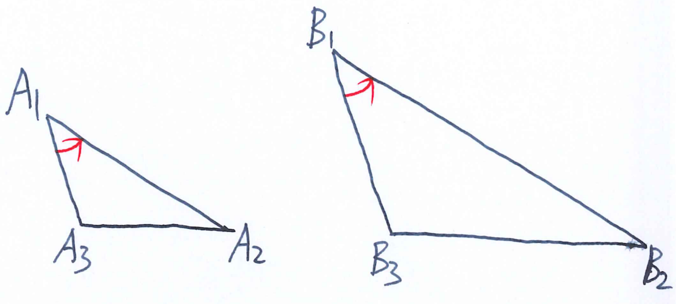
$\therefore \dfrac{|a_2-a_1|}{|b_2-b_1|} = \dfrac{|a_3-a_1|}{|b_3-b_1|}$ 且 $\widehat{A_3A_1A_2} = \widehat{B_3B_1B_2}$

$\therefore \dfrac{|A_1A_2|}{|B_1B_2|} = \dfrac{|A_1A_3|}{|B_1B_3|}$ 且 $\widehat{A_3A_1A_2} = \widehat{B_3B_1B_2}$

$\therefore$ 有如下的图示：



或



$\therefore \triangle A_1A_2A_3 \sim \triangle B_1B_2B_3$ 且 $\triangle A_1A_2A_3$ 和 $\triangle B_1B_2B_3$ 有相同的定向. $\square$

Lemma：$a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{C}$，且互不相等. 则有：

$$\frac{a_2-a_1}{a_3-a_1} = \frac{b_2-b_1}{b_3-b_1} \iff \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

Proof：($\Rightarrow$)：$\because \dfrac{a_2-a_1}{a_3-a_1} = \dfrac{b_2-b_1}{b_3-b_1}$ $\quad\therefore (a_2-a_1)(b_3-b_1) = (a_3-a_1)(b_2-b_1)$

$\therefore a_2b_3 - a_2b_1 - a_1b_3 + a_1b_1 = a_3b_2 - a_3b_1 - a_1b_2 + a_1b_1$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2b_3 + a_3b_1 + a_1b_2 - a_2b_1 - a_1b_3 - a_3b_2$$

$$= (a_2b_3 - a_2b_1 - a_1b_3) - (a_3b_2 - a_3b_1 - a_1b_2)$$

$$= 0$$

($\Leftarrow$)：$\because \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$ $\quad\therefore a_2b_3 - a_2b_1 - a_1b_3 = a_3b_2 - a_3b_1 - a_1b_2$

$\therefore a_2b_3 - a_2b_1 - a_1b_3 + a_1b_1 = a_3b_2 - a_3b_1 - a_1b_2 + a_1b_1$

$\therefore (a_2-a_1)(b_3-b_1) = (a_3-a_1)(b_2-b_1)$ $\quad\therefore \dfrac{a_2-a_1}{a_3-a_1} = \dfrac{b_2-b_1}{b_3-b_1}$ $\square$

Lemma: $A_1(a_1)$, $A_2(a_2)$, $A_3(a_3)$, $B_1(b_1)$, $B_2(b_2)$, $B_3(b_3)$是六个彼此不同的点，则有：

$\triangle A_1A_2A_3 \sim \triangle B_1B_2B_3$且有相反的定向 $\Leftrightarrow \dfrac{a_2-a_1}{a_3-a_1} = \dfrac{\overline{b_2}-\overline{b_1}}{\overline{b_3}-\overline{b_1}}$

proof: 将$\triangle B_1B_2B_3$沿实轴对称，得到$\triangle \overline{B_1}\,\overline{B_2}\,\overline{B_3}$.

$\therefore \triangle A_1A_2A_3 \sim \triangle B_1B_2B_3$且有相反的定向

$\Leftrightarrow \triangle A_1A_2A_3 \sim \triangle \overline{B_1}\,\overline{B_2}\,\overline{B_3}$且有相同的定向

$\Leftrightarrow \dfrac{a_2-a_1}{a_3-a_1} = \dfrac{\overline{b_2}-\overline{b_1}}{\overline{b_3}-\overline{b_1}}$ $\square$

Remark: 有如下的图示：