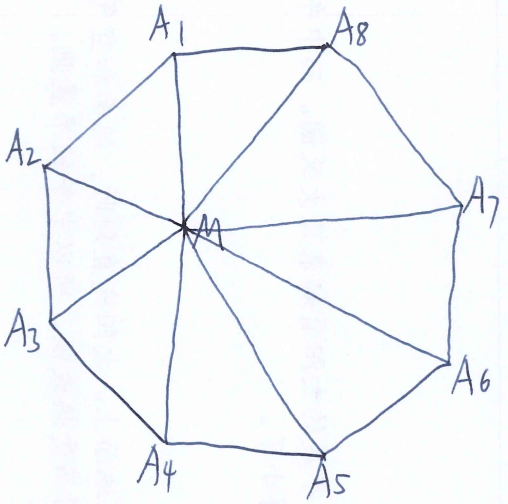
定义(凸多边形的定向为正) 设多边形$A_1A_2\cdots A_n$是一个凸多边形. 如果对凸多边形$A_1A_2\cdots A_n$内部任一点M, $\triangle MA_kA_{k+1}$ ($k=1,2,\cdots,n$; $A_{n+1}=A_1$) 都是正定向的. 则称凸多边形$A_1A_2\cdots A_n$ ~~$A_1A_2\cdots A_n$~~ 是一个正定向的凸多边形.

Remark: 凸多边形$A_1A_2\cdots A_n$的定向为正, 等价于它的顶点$A_1, A_2, \cdots, A_n$按逆时针顺序排列.

定理(凸多边形定向为正时的面积公式) 设多边形$A_1A_2\cdots A_n$是一个正定向的凸多边形. 顶点坐标$A_1(a_1)$, $A_2(a_2)$, $\cdots$, $A_n(a_n)$. 则有:

$$\text{area}[A_1A_2\cdots A_n]=\frac{1}{2}\operatorname{Im}\left(\overline{a_1}a_2+\overline{a_2}a_3+\cdots+\overline{a_{n-1}}a_n+\overline{a_n}a_1\right)$$

Proof: 在正定向的凸多边形$A_1A_2\cdots A_n$内部任取一点M. 则有:



设M点对应的复数为z.

$$\text{area}[A_1A_2\cdots A_n]=\cancel{\sum_{k=1} MA_kA_{k+1}}\ \sum_{k=1}^{n}\text{area}[MA_kA_{k+1}]=\sum_{k=1}^{n}\frac{1}{2}\operatorname{Im}\left(\bar{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z\right)$$

$$=\frac{1}{2}\sum_{k=1}^{n}\operatorname{Im}\left(\bar{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z\right)=\frac{1}{2}\operatorname{Im}\left(\sum_{k=1}^{n}\left(\bar{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z\right)\right)$$

$$=\frac{1}{2}\operatorname{Im}\left(\sum_{k=1}^{n}\bar{z}a_k+\sum_{k=1}^{n}\overline{a_k}a_{k+1}+\sum_{k=1}^{n}\overline{a_{k+1}}z\right)=\frac{1}{2}\operatorname{Im}\left(\bar{z}\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}\overline{a_k}a_{k+1}+z\sum_{k=1}^{n}\overline{a_k}\right)$$

$$=\frac{1}{2}\operatorname{Im}\left(\bar{z}\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}\overline{a_k}a_{k+1}+\overline{\bar{z}\sum_{k=1}^{n}a_k}\right)=\frac{1}{2}\left(\operatorname{Im}\left(\bar{z}\sum_{k=1}^{n}a_k+\overline{\bar{z}\sum_{k=1}^{n}a_k}\right)+\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)\right)$$

$$=\frac{1}{2}\left(0+\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)\right)=\frac{1}{2}\operatorname{Im}\left(\overline{a_1}a_2+\overline{a_2}a_3+\cdots+\overline{a_{n-1}}a_n+\overline{a_n}a_1\right)\quad\square$$