Hermite's identity

定义 (Fractional part) 对 $\forall x\in\mathbb{R}$，定义：

$$\mathrm{frac}(x) := x-\lfloor x\rfloor =: \{x\}$$

Lemma：对 $\forall x\in\mathbb{R}$，有：$\mathrm{frac}(x)\in[0,1)$

Proof：$\because x\in\mathbb{R}$ $\therefore x-1<\lfloor x\rfloor\leqslant x$ $\therefore -x+1>-\lfloor x\rfloor\geqslant -x$

$\therefore -x\leqslant -\lfloor x\rfloor<-x+1$ $\therefore x+(-x)\leqslant x-\lfloor x\rfloor< x+(-x)+1$

$\equiv$ $\therefore 0\leqslant \mathrm{frac}(x)<1$

$\therefore \mathrm{frac}(x)\in[0,1)$

对 $\forall \lambda\in[0,1)$，有：$\lambda\in\mathbb{R}$，$0\in\mathbb{Z}$，$0\leqslant\lambda<0+1$

$\therefore \lfloor\lambda\rfloor=0$

$\therefore \lambda\in\mathbb{R}$，且有 $\mathrm{frac}(\lambda)=\lambda-\lfloor\lambda\rfloor=\lambda-0=\lambda$

$\therefore$ 函数 $\mathrm{frac}(x)$ 的值域为 $[0,1)$

$\therefore$ 映射 $\mathrm{frac}:\mathbb{R}\longrightarrow[0,1)$ 是满射. $\square$

$$x\longmapsto \mathrm{frac}(x)$$

Lemma：对 $\forall x\in\mathbb{R}$，$\forall n\in\mathbb{Z}$，有：$\mathrm{frac}(x+n)=\mathrm{frac}(x)$

Proof：$\because x\in\mathbb{R}$，$n\in\mathbb{Z}$ $\therefore \lfloor x+n\rfloor=\lfloor x\rfloor+n$

$\therefore \mathrm{frac}(x+n)=(x+n)-\lfloor x+n\rfloor = x+n-(\lfloor x\rfloor+n)=x+n-\lfloor x\rfloor-n$

$=x-\lfloor x\rfloor=\mathrm{frac}(x)$ $\square$

Lemma: 对$\forall x\in\mathbb{R}$, 有: $\mathrm{frac}(x)+\mathrm{frac}(-x)=\begin{cases}0 & \text{当} x\in\mathbb{Z}\text{时}\\ 1 & \text{当} x\in\mathbb{R}\setminus\mathbb{Z}\text{时}\end{cases}$

Proof: 当$x\in\mathbb{Z}$时, $\lfloor x\rfloor+\lfloor -x\rfloor=0$

$\therefore \mathrm{frac}(x)+\mathrm{frac}(-x)= x-\lfloor x\rfloor+(-x)-\lfloor -x\rfloor$

$=(x+(-x))-(\lfloor x\rfloor+\lfloor -x\rfloor)=0-0=0$

当$x\in\mathbb{R}\setminus\mathbb{Z}$时, $\lfloor x\rfloor+\lfloor -x\rfloor=-1$

$\therefore \mathrm{frac}(x)+\mathrm{frac}(-x)= x-\lfloor x\rfloor+(-x)-\lfloor -x\rfloor$

$=(x+(-x))-(\lfloor x\rfloor+\lfloor -x\rfloor)=0-(-1)=1$ $\square$

Lemma: 对$\forall x\in\mathbb{R}$, 有: $\mathrm{frac}(\mathrm{frac}(x))=\mathrm{frac}(x)$

Proof: $\because x\in\mathbb{R}$ $\therefore \mathrm{frac}(x)\in[0,1)$ $\therefore 0\leqslant \mathrm{frac}(x)<0+1$

$\therefore \lfloor \mathrm{frac}(x)\rfloor=0$

$\therefore \mathrm{frac}(\mathrm{frac}(x))=\mathrm{frac}(x)-\lfloor \mathrm{frac}(x)\rfloor=\mathrm{frac}(x)-0=\mathrm{frac}(x)$

$\square$

定理 (Hermite's identity) 对$\forall x\in\mathbb{R}$, $\forall n\in\mathbb{Z}_{\geqslant 1}$, 有:

$$\sum_{k=0}^{n-1}\left\lfloor x+\frac{k}{n}\right\rfloor=\lfloor nx\rfloor$$

Proof: 令$f(t)=\sum_{k=0}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor-\lfloor nt\rfloor$, $t\in\mathbb{R}$.

$\because n\in\mathbb{Z}_{\geqslant 1}$ $\therefore \frac{1}{n}\in\mathbb{R}_{>0}$

对$\forall t\in\mathbb{R}$，有：$f(t+\frac{1}{n})=\sum\limits_{k=0}^{n-1}\lfloor t+\frac{1}{n}+\frac{k}{n}\rfloor-\lfloor n(t+\frac{1}{n})\rfloor$

$$=\sum_{k=0}^{n-1}\left\lfloor t+\frac{k+1}{n}\right\rfloor-\lfloor nt+1\rfloor=\sum_{k=1}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor+\left\lfloor t+\frac{n}{n}\right\rfloor-\lfloor nt+1\rfloor$$

$$=\sum_{k=1}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor+\lfloor t+1\rfloor-\lfloor nt+1\rfloor$$

$$=\sum_{k=1}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor+(\lfloor t\rfloor+1)-(\lfloor nt\rfloor+1)$$

$$=\sum_{k=1}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor+\left\lfloor t+\frac{0}{n}\right\rfloor-\lfloor nt\rfloor$$

$$=\sum_{k=0}^{n-1}\left\lfloor t+\frac{k}{n}\right\rfloor-\lfloor nt\rfloor=f(t)$$ $\therefore$ 函数$f(t)$的周期是$\frac{1}{n}$.

$\because f(0)=\sum\limits_{k=0}^{n-1}\lfloor 0+\frac{k}{n}\rfloor-\lfloor n\cdot 0\rfloor=\sum\limits_{k=0}^{n-1}\lfloor\frac{k}{n}\rfloor-\lfloor 0\rfloor=\sum\limits_{k=0}^{n-1}0-0=0$

$f(\frac{1}{n})=\sum\limits_{k=0}^{n-1}\lfloor\frac{1}{n}+\frac{k}{n}\rfloor-\lfloor n\cdot\frac{1}{n}\rfloor=\sum\limits_{k=0}^{n-1}\lfloor\frac{k+1}{n}\rfloor-\lfloor 1\rfloor=0$

对$\forall t\in(0,\frac{1}{n})$，有：

$f(t)=\sum\limits_{k=0}^{n-1}\lfloor t+\frac{k}{n}\rfloor-\lfloor nt\rfloor=0$

$\therefore$ 对$\forall t\in[0,\frac{1}{n}]$，有：$f(t)=0$

$\therefore$ 对$\forall t\in\mathbb{R}$，有：$f(t)=0$. $\therefore f(x)=0$.

$\therefore\sum\limits_{k=0}^{n-1}\lfloor x+\frac{k}{n}\rfloor-\lfloor nx\rfloor=0$

$\therefore\sum\limits_{k=0}^{n-1}\lfloor x+\frac{k}{n}\rfloor=\lfloor nx\rfloor$ $\square$

定理（Hermite's identity，$\lceil\,\rceil$形式）对$\forall x\in\mathbb{R}$，$\forall n\in\mathbb{Z}_{\geq 1}$，有：

$$\sum_{k=0}^{n-1}\left\lceil x-\frac{k}{n}\right\rceil=\lceil nx\rceil$$

Proof: $\lceil nx\rceil=-\lfloor -nx\rfloor=-\lfloor n(-x)\rfloor=-\sum_{k=0}^{n-1}\left\lfloor -x+\frac{k}{n}\right\rfloor$

$=\sum_{k=0}^{n-1}-\left\lfloor -x+\frac{k}{n}\right\rfloor=\sum_{k=0}^{n-1}\left\lceil x-\frac{k}{n}\right\rceil$ $\square$