最小公倍数.

Lemma：对 $\forall x\in\mathbb{Z}$，有：$\operatorname{lcm}(x)=|x|$

Proof：当 $x=0$ 时，有：$\operatorname{lcm}(x)=\operatorname{lcm}(0)=0=|0|=|x|$

当 $x\neq 0$ 时，有：$\operatorname{lcm}(x)=\min\{t\in\mathbb{Z}_{\geqslant 1}:x\mid t\}=|x|$

$(\because x\neq 0,\ t\neq 0 \quad \text{且}\ x\mid t \quad \therefore |x|\leqslant|t|=t \quad \therefore t\geqslant|x|\ )$ $\square$

Lemma：$n\in\mathbb{Z}_{\geqslant 1}$，$x_1,\cdots,x_n\in\mathbb{Z}$，则有：

$$\operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_n))=\operatorname{lcm}(x_1,\cdots,x_n)$$

Proof：$\because x_1,\cdots,x_n\in\mathbb{Z} \quad \therefore \operatorname{lcm}(x_1,\cdots,x_n)\in\mathbb{Z}_{\geqslant 0}$

$\therefore \operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_n))=|\operatorname{lcm}(x_1,\cdots,x_n)|=\operatorname{lcm}(x_1,\cdots,x_n)$ $\square$

Lemma：$n\in\mathbb{Z}_{\geqslant 1}$，$x_1,\cdots,x_n\in\mathbb{Z}$，则有：

$$\operatorname{lcm}(x_1,\cdots,x_n)=\operatorname{lcm}(\operatorname{lcm}(x_1),x_2,\cdots,x_n)$$

Proof：当 $x_1,\cdots,x_n$ 中有某个 $x_i=0$ 时，$\operatorname{lcm}(x_1,\cdots,x_n)=0$.

此时 $\operatorname{lcm}(x_1)=|x_1|$，$x_2,\cdots,x_n$ 中也有0. $\therefore \operatorname{lcm}(\operatorname{lcm}(x_1),x_2,\cdots,x_n)=0$

$\therefore$ 左边 = 右边. 结论得证.

当 $x_1,\cdots,x_n$ 全不为0时，$\operatorname{lcm}(x_1)=|x_1|$，$x_2,\cdots,x_n$ 全不为0

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)\in\mathbb{Z}_{\geqslant 1}$，$\operatorname{lcm}(\operatorname{lcm}(x_1),x_2,\cdots,x_n)\in\mathbb{Z}_{\geqslant 1}$

$$\begin{aligned}\therefore \operatorname{lcm}(x_1,\cdots,x_n)&=\min\{t\in\mathbb{Z}_{\geqslant 1}:x_1\mid t\ \text{且}\ x_2\mid t\ \text{且}\cdots\text{且}\ x_n\mid t\}\\&=\min\{t\in\mathbb{Z}_{\geqslant 1}:|x_1|\mid t\ \text{且}\ x_2\mid t\ \text{且}\cdots\text{且}\ x_n\mid t\}\\&=\operatorname{lcm}(|x_1|,x_2,\cdots,x_n)=\operatorname{lcm}(\operatorname{lcm}(x_1),x_2,\cdots,x_n)\end{aligned}$$ $\square$

Lemma（最小公倍数前 k 个元的分组）$n\in\mathbb{Z}_{\geqslant1}$，$x_1,\cdots,x_n\in\mathbb{Z}$，$k\in\{1,\cdots,n\}$，则有：

$$\operatorname{lcm}(x_1,\cdots,x_n)=\operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_k),x_{k+1},\cdots,x_n)$$

Proof：当 $k=1$ 时，结论已证．　当 $k=n$ 时，结论已证．

当 $k\in\{2,\cdots,n-1\}$ 时，分情况讨论如下：

① $x_1,\cdots,x_k$ 中有 0．此时有：$\operatorname{lcm}(x_1,\cdots,x_k)=0$，$\operatorname{lcm}(x_1,\cdots,x_n)=0$

$\therefore$ 右边 $=\operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_k),x_{k+1},\cdots,x_n)=\operatorname{lcm}(0,x_{k+1},\cdots,x_n)=0=\operatorname{lcm}(x_1,\cdots,x_n)=$ 左边．

② $x_1,\cdots,x_k$ 中没有 0．此时再分两种情况讨论：

(i) $x_{k+1},\cdots,x_n$ 中有 0．此时有：

左边 $=\operatorname{lcm}(x_1,\cdots,x_n)=\operatorname{lcm}(x_1,\cdots,x_k,x_{k+1},\cdots,x_n)=0$

右边 $=\operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_k),x_{k+1},\cdots,x_n)=0$　　$\therefore$ 左边 $=$ 右边．

(ii) $x_{k+1},\cdots,x_n$ 中没有 0．此时 $x_1,\cdots,x_n$ 全不为 0．

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)\in\mathbb{Z}_{\geqslant1}$，$\operatorname{lcm}(x_1,\cdots,x_k)\in\mathbb{Z}_{\geqslant1}$

对 $\forall t\in\mathbb{Z}$．$\because x_1,\cdots,x_k\in\mathbb{Z}$，$x_1,\cdots,x_k$ 中没有 0

$\therefore x_1\mid t$ 且 $\cdots$ 且 $x_k\mid t \iff \operatorname{lcm}(x_1,\cdots,x_k)\mid t$

$\therefore$ ~~lcm~~ 左边 $=\operatorname{lcm}(x_1,\cdots,x_n)=\min\{t\in\mathbb{Z}_{\geqslant1}: x_1\mid t$ 且 $\cdots$ 且 $x_n\mid t\}$

$=\min\{t\in\mathbb{Z}_{\geqslant1}: x_1\mid t$ 且 $\cdots$ 且 $x_k\mid t$ 且 $x_{k+1}\mid t$ 且 $\cdots$ 且 $x_n\mid t\}$

$=\min\{t\in\mathbb{Z}_{\geqslant1}: \operatorname{lcm}(x_1,\cdots,x_k)\mid t$ 且 $x_{k+1}\mid t$ 且 $\cdots$ 且 $x_n\mid t\}$

$=\operatorname{lcm}(\operatorname{lcm}(x_1,\cdots,x_k),x_{k+1},\cdots,x_n)=$ 右边　　□

下面可能会把习题写成引理的形式.

P18 . 2题.

Lemma: $a_0, a_1, \cdots, a_{n-1} \in \mathbb{Z}$, 方程 $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ 有整数根 $x_0 \neq 0$,

则有: $x_0 \mid a_0$

Proof: $\because$ 方程 $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ 有整数根 x_0

$\therefore x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_1 x_0 + a_0 = 0$

$$\begin{aligned}\therefore a_0 &= -(x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_1 x_0) = -(x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_2 x_0^2 + a_1 x_0) \\ &= -x_0(x_0^{n-1} + a_{n-1}x_0^{n-2} + \cdots + a_2 x_0 + a_1) \\ &= x_0\left(-(x_0^{n-1} + a_{n-1}x_0^{n-2} + \cdots + a_2 x_0 + a_1)\right)\end{aligned}$$

$\because a_0, a_1, \cdots, a_{n-1} \in \mathbb{Z}$, $x_0 \in \mathbb{Z}$ $\quad\therefore x_0^{n-1} + a_{n-1}x_0^{n-2} + \cdots + a_2 x_0 + a_1 \in \mathbb{Z}$

$\therefore -(x_0^{n-1} + a_{n-1}x_0^{n-2} + \cdots + a_2 x_0 + a_1) \in \mathbb{Z}$

$\because a_0 \in \mathbb{Z}$, $x_0 \in \mathbb{Z}$, $x_0 \neq 0$

$\therefore x_0 \mid a_0$ $\quad\square$