Dirichlet 卷积的乘性

Lemma：对 $\forall a, b, c \in \mathbb{Z}$，$\gcd(b, c) = 1$，有：

$$\gcd(\gcd(a, b), \gcd(a, c)) = 1$$

Proof：$\because \gcd(b, c) = 1$　$\therefore b$ 和 c 不能全为 0

分两种情况讨论：

① $a = 0$　此时有如下可能：

(i) $b = 0$ 且 $c \neq 0$.　$\because \gcd(0, c) = 1$

$\therefore \gcd(\gcd(a, b), \gcd(a, c)) = \gcd(0, 1) = 1$

(ii) $b \neq 0$ 且 $c = 0$　$\therefore \gcd(b, 0) = 1$　$\therefore \gcd(0, b) = 1$

$\therefore \gcd(\gcd(a, b), \gcd(a, c)) = \gcd(1, 0) = 1$

(iii) $b \neq 0$ 且 $c \neq 0$

$\therefore \gcd(\gcd(a, b), \gcd(a, c)) = \gcd(\gcd(0, b), \gcd(0, c))$

$= \gcd(|b|, |c|) = \gcd(b, c) = 1$

② $a \neq 0$. 设 $\gcd(\gcd(a, b), \gcd(a, c)) = \lambda$

$\because a \neq 0$　$\therefore \gcd(a, b) \in \mathbb{Z}_{\geqslant 1}$，$\gcd(a, c) \in \mathbb{Z}_{\geqslant 1}$　$\therefore \lambda \in \mathbb{Z}_{\geqslant 1}$

$\because \lambda \mid \gcd(a, b)$，$\gcd(a, b) \mid b$　$\therefore \lambda \mid b$

$\because \lambda \mid \gcd(a, c)$，$\gcd(a, c) \mid c$　$\therefore \lambda \mid c$　$\therefore \lambda$ 是 b 和 c 的公因数

$\therefore \gcd(b, c) \geqslant \lambda$　$\therefore 1 \leqslant \lambda \leqslant 1$　$\therefore \lambda = 1$　$\square$

Lemma: $a,b,c\in\mathbb{Z}$, $a\mid bc$, $\gcd(b,c)=1$, 则有: $\exists$ $a_1, a_2\in\mathbb{Z}$, 使得 $|a|=a_1a_2$, 且有: $a_1\mid b$, $a_2\mid c$, $\gcd(a_1,a_2)=1$

Proof: 令 $a_1=\gcd(a,b)$, $a_2=\gcd(a,c)$ $\quad\therefore a_1, a_2\in\mathbb{Z}_{\geq 0}$

$\because a\mid bc$

$\therefore |a|=\gcd(a,bc)=\gcd(a,b)\cdot\gcd(a,c)=a_1a_2$

$a_1=\gcd(a,b)\mid b$, $\quad a_2=\gcd(a,c)\mid c$,

$\gcd(a_1,a_2)=\gcd(\gcd(a,b),\gcd(a,c))=1$ $\quad$ 存在性得证. $\square$

Lemma (Dirichlet 卷积的乘性) 对 $\forall$ 算术函数 f, g, 若 f 和 g 都是乘性算术函数, 则有: $f*g$ 也是乘性算术函数

Proof: 对 $\forall a,b\in\mathbb{Z}_{\geq 1}$, 若 $\gcd(a,b)=1$, 则有:

$$(f*g)(ab)=\sum_{\substack{d\mid ab\\ d\in\mathbb{Z}_{\geq 1}}} f(d)\,g\left(\frac{ab}{d}\right)=\sum_{\substack{d\mid ab\\ d\in\mathbb{Z}_{\geq 1}\\ d=d_1d_2\\ d_1\mid a,\ d_2\mid b\\ \gcd(d_1,d_2)=1}} f(d)\,g\left(\frac{ab}{d}\right)$$

$$=\sum_{\substack{d_1,d_2\in\mathbb{Z}_{\geq 1}\\ d_1\mid a,\ d_2\mid b\\ \gcd(d_1,d_2)=1}} f(d_1d_2)\,g\left(\frac{ab}{d_1d_2}\right)=\sum_{\substack{d_1,d_2\in\mathbb{Z}_{\geq 1}\\ d_1\mid a,\ d_2\mid b,\\ \gcd(d_1,d_2)=1}} f(d_1)f(d_2)\,g\left(\frac{a}{d_1}\right)g\left(\frac{b}{d_2}\right)$$

(草稿纸上易证 $\gcd\left(\frac{a}{d_1},\frac{b}{d_2}\right)=1$)

$$= \sum_{\substack{d_1, d_2 \in \mathbb{Z}_{\geq 1} \\ d_1 | a,\ d_2 | b}} f(d_1) g\left(\frac{a}{d_1}\right) f(d_2) g\left(\frac{b}{d_2}\right)$$

$$= \sum_{\substack{d_1 | a \\ d_1 \in \mathbb{Z}_{\geq 1}}} f(d_1) g\left(\frac{a}{d_1}\right) \cdot \sum_{\substack{d_2 | b \\ d_2 \in \mathbb{Z}_{\geq 1}}} f(d_2) g\left(\frac{b}{d_2}\right)$$

$$= (f * g)(a) \cdot (f * g)(b)$$

$\therefore$ $f*g$ 是乘性算术函数. $\square$

Lemma: 对 $\forall\ a, b \in \mathbb{Z}_{\geq 1}$, 有: $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$

Proof: 分如下4种情况讨论:

① $a=1$ 且 $b=1$. $\varepsilon(ab) = \varepsilon(1) = 1$, $\varepsilon(a)\cdot\varepsilon(b) = \varepsilon(1)\cdot\varepsilon(1) = 1$

② $a=1$ 且 $b \geq 2$ $\varepsilon(ab) = \varepsilon(b)$, $\varepsilon(a)\cdot\varepsilon(b) = \varepsilon(1)\cdot\varepsilon(b) = \varepsilon(b)$

③ $a \geq 2$ 且 $b=1$. 同上.

④. $a \geq 2$ 且 $b \geq 2$. $\therefore ab \geq 4$ $\therefore \varepsilon(ab) = 0$. $\varepsilon(a)\cdot\varepsilon(b) = 0$ $\square$