

Hermite's identity

定义 (Fractional part) $x \in \mathbb{R}$, 定义:

$$\text{frac}(x) := x - \lfloor x \rfloor =: \{x\}$$

Lemma: $x \in \mathbb{R}$, 有: $\text{frac}(x) \in [0, 1)$

Proof: $\because x \in \mathbb{R} \quad \therefore x-1 < \lfloor x \rfloor \leq x \quad \therefore -x+1 > -\lfloor x \rfloor \geq -x$
 $\therefore -x \leq -\lfloor x \rfloor < -x+1 \quad \therefore x+(-x) \leq x-\lfloor x \rfloor < x+(-x)+1$
 $\equiv \therefore 0 \leq \text{frac}(x) < 1$
 $\therefore \text{frac}(x) \in [0, 1)$

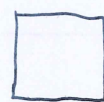
$x \in \mathbb{R}, \lambda \in [0, 1)$, 有: $\lambda \in \mathbb{R}, 0 \in \mathbb{Z}, 0 \leq \lambda < 0+1$

$$\therefore \lfloor \lambda \rfloor = 0$$

$$\therefore \lambda \in \mathbb{R}, \text{ 且有 } \text{frac}(\lambda) = \lambda - \lfloor \lambda \rfloor = \lambda - 0 = \lambda$$

\therefore 函数 $\text{frac}(x)$ 的值域为 $[0, 1)$

\therefore 映射 $\text{frac}: \mathbb{R} \rightarrow [0, 1)$ 是满射.



$$x \mapsto \text{frac}(x)$$

Lemma: $x \in \mathbb{R}, \forall n \in \mathbb{Z}$, 有: $\text{frac}(x+n) = \text{frac}(x)$

Proof: $\because x \in \mathbb{R}, n \in \mathbb{Z} \quad \therefore \lfloor x+n \rfloor = \lfloor x \rfloor + n$

$$\therefore \text{frac}(x+n) = (x+n) - \lfloor x+n \rfloor = x+n - (\lfloor x \rfloor + n) = x+n - \lfloor x \rfloor - n$$
$$= x - \lfloor x \rfloor = \text{frac}(x) \quad \square$$

Lemma: 对 $\forall x \in \mathbb{R}$, 有: $\text{frac}(x) + \text{frac}(-x) = \begin{cases} 0 & \text{当 } x \in \mathbb{Z} \text{ 时} \\ 1 & \text{当 } x \in \mathbb{R} \setminus \mathbb{Z} \text{ 时} \end{cases}$

Proof: 当 $x \in \mathbb{Z}$ 时, $\lfloor x \rfloor + \lfloor -x \rfloor = 0$

$$\therefore \text{frac}(x) + \text{frac}(-x) = x - \lfloor x \rfloor + (-x) - \lfloor -x \rfloor$$

$$= (x + (-x)) - (\lfloor x \rfloor + \lfloor -x \rfloor) = 0 - 0 = 0$$

当 $x \in \mathbb{R} \setminus \mathbb{Z}$ 时, $\lfloor x \rfloor + \lfloor -x \rfloor = -1$

$$\therefore \text{frac}(x) + \text{frac}(-x) = x - \lfloor x \rfloor + (-x) - \lfloor -x \rfloor$$

$$= (x + (-x)) - (\lfloor x \rfloor + \lfloor -x \rfloor) = 0 - (-1) = 1 \quad \square$$

Lemma: 对 $\forall x \in \mathbb{R}$, 有: $\text{frac}(\text{frac}(x)) = \text{frac}(x)$

Proof: $\because x \in \mathbb{R} \quad \therefore \text{frac}(x) \in [0, 1) \quad \therefore 0 \leq \text{frac}(x) < 0+1$

$$\therefore \lfloor \text{frac}(x) \rfloor = 0$$

$$\therefore \text{frac}(\text{frac}(x)) = \text{frac}(x) - \lfloor \text{frac}(x) \rfloor = \text{frac}(x) - 0 = \text{frac}(x) \quad \square$$

定理 (Hermite's identity) 对 $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{Z}_{\geq 1}$, 有:

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor$$

Proof: 令 $f(t) = \sum_{k=0}^{n-1} \left\lfloor t + \frac{k}{n} \right\rfloor - \lfloor nt \rfloor$, $t \in \mathbb{R}$.

$$\because n \in \mathbb{Z}_{\geq 1} \quad \therefore \frac{1}{n} \in \mathbb{R}_{>0}$$

$$\begin{aligned}
 & \text{对 } \forall t \in \mathbb{R}, \text{ 有: } f(t + \frac{1}{n}) = \sum_{k=0}^{n-1} \lfloor t + \frac{1}{n} + \frac{k}{n} \rfloor - \lfloor n(t + \frac{1}{n}) \rfloor \\
 &= \sum_{k=0}^{n-1} \lfloor t + \frac{k+1}{n} \rfloor - \lfloor nt + 1 \rfloor = \sum_{k=1}^{n-1} \lfloor t + \frac{k}{n} \rfloor + \lfloor t + \frac{n}{n} \rfloor - \lfloor nt + 1 \rfloor \\
 &= \sum_{k=1}^{n-1} \lfloor t + \frac{k}{n} \rfloor + \lfloor t + 1 \rfloor - \lfloor nt + 1 \rfloor \\
 &= \sum_{k=1}^{n-1} \lfloor t + \frac{k}{n} \rfloor + (\lfloor t \rfloor + 1) - (\lfloor nt \rfloor + 1) \\
 &= \sum_{k=1}^{n-1} \lfloor t + \frac{k}{n} \rfloor + \lfloor t + \frac{0}{n} \rfloor - \lfloor nt \rfloor \\
 &= \sum_{k=0}^{n-1} \lfloor t + \frac{k}{n} \rfloor - \lfloor nt \rfloor = f(t) \quad \therefore \text{函数 } f(t) \text{ 的周期是 } \frac{1}{n}.
 \end{aligned}$$

$$\therefore f(0) = \sum_{k=0}^{n-1} \lfloor 0 + \frac{k}{n} \rfloor - \lfloor n \cdot 0 \rfloor = \sum_{k=0}^{n-1} \lfloor \frac{k}{n} \rfloor - \lfloor 0 \rfloor = \sum_{k=0}^{n-1} 0 - 0 = 0$$

$$f(\frac{1}{n}) = \sum_{k=0}^{n-1} \lfloor \frac{1}{n} + \frac{k}{n} \rfloor - \lfloor n \cdot \frac{1}{n} \rfloor = \sum_{k=0}^{n-1} \lfloor \frac{k+1}{n} \rfloor - \lfloor 1 \rfloor = 0$$

对 $\forall t \in (0, \frac{1}{n})$, 有:

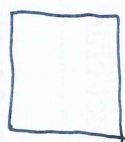
$$f(t) = \sum_{k=0}^{n-1} \lfloor t + \frac{k}{n} \rfloor - \lfloor nt \rfloor = 0$$

\therefore 对 $\forall t \in [0, \frac{1}{n}]$, 有: $f(t) = 0$

\therefore 对 $\forall t \in \mathbb{R}$, 有: $f(t) = 0$. $\therefore f(x) = 0$.

$$\therefore \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor - \lfloor nx \rfloor = 0$$

$$\therefore \sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor = \lfloor nx \rfloor$$

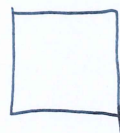


定理 (Hermite's identity, $\lceil \cdot \rceil$ 形式) 对 $\forall x \in \mathbb{R}, \forall n \in \mathbb{Z}_{\geq 1}$, 有:

$$\sum_{k=0}^{n-1} \left\lceil x - \frac{k}{n} \right\rceil = \lceil nx \rceil$$

Proof: $\lceil nx \rceil = -\lfloor -nx \rfloor = -\lfloor n(-x) \rfloor = -\sum_{k=0}^{n-1} \left\lfloor -x + \frac{k}{n} \right\rfloor$

$$= \sum_{k=0}^{n-1} -\left\lfloor -x + \frac{k}{n} \right\rfloor = \sum_{k=0}^{n-1} \left\lceil x - \frac{k}{n} \right\rceil$$



Group Meeting Checklist				
(1) Check agenda (小组会议议程)				☆
(2) Prepare and report work in group (汇报小组工作)				☆
(3) Group discussion and decision (小组讨论决策)				☆
(4) Check and assign tasks (检查并分配任务)				☆