初等数论 (5)

Lemma: $a, b, c, d \in \mathbb{Z}$, $a \neq 0$, $c \neq 0$, $a \mid b$, $c \mid d$, 则有:

$$ac \mid bd$$

Proof: $\because a \mid b$ $\quad \therefore \exists \lambda \in \mathbb{Z}$, s.t. $b = a\lambda$

$\because c \mid d$ $\quad \therefore \exists \mu \in \mathbb{Z}$, s.t. $d = c\mu$

$$\therefore bd = (a\lambda)(c\mu) = ((a\lambda)c)\mu = (a(\lambda c))\mu = (a(c\lambda))\mu$$

$$= ((ac)\lambda)\mu = (ac)(\lambda\mu)$$

$\because a, c \in \mathbb{Z}$ $\quad \therefore ac \in \mathbb{Z}$ $\quad \because a \neq 0$ 且 $c \neq 0$ $\quad \therefore ac \neq 0$ $\quad \therefore ac \in \mathbb{Z}$ 且 $ac \neq 0$

$\because b, d \in \mathbb{Z}$ $\quad \therefore bd \in \mathbb{Z}$ $\quad \because \lambda, \mu \in \mathbb{Z}$ $\quad \therefore \lambda\mu \in \mathbb{Z}$

$\therefore ac \mid bd$ $\quad \square$

Lemma (最小公倍数的性质) $x_1, x_2 \in \mathbb{Z}$, 则有:

~~$[a_1, a_2] =$~~ $\operatorname{lcm}(x_1, x_2) = \operatorname{lcm}(x_2, x_1) = \operatorname{lcm}(-x_1, x_2) = \operatorname{lcm}(x_1, -x_2) = \operatorname{lcm}(-x_1, -x_2)$

$= \operatorname{lcm}(|x_1|, |x_2|)$

Proof: 如果 x_1, x_2 中有 0, 则:

$\operatorname{lcm}(x_1, x_2) = 0$, $\operatorname{lcm}(x_2, x_1) = 0$, $\operatorname{lcm}(-x_1, x_2) = 0$, $\operatorname{lcm}(x_1, -x_2) = 0$,

$\operatorname{lcm}(-x_1, -x_2) = 0$, $\operatorname{lcm}(|x_1|, |x_2|) = 0$

$$\therefore \operatorname{lcm}(x_1, x_2) = \operatorname{lcm}(x_2, x_1) = \operatorname{lcm}(-x_1, x_2) = \operatorname{lcm}(x_1, -x_2) = \operatorname{lcm}(-x_1, -x_2) = \operatorname{lcm}(|x_1|, |x_2|) = 0$$

下设 x_1, x_2 中没有 0, 即 $x_1 \neq 0$ 且 $x_2 \neq 0$

$$\therefore \operatorname{lcm}(x_1, x_2) = \min\{m \in \mathbb{Z}_{\geqslant 1} : x_1 \mid m \text{ 且 } x_2 \mid m\} = \min\{m \in \mathbb{Z}_{\geqslant 1} : x_2 \mid m \text{ 且 } x_1 \mid m\} = \operatorname{lcm}(x_2, x_1)$$

$\operatorname{lcm}(x_1,x_2)=\min\{m\in\mathbb{Z}_{\geqslant 1} : x_1|m \text{ 且 } x_2|m\}=\min\{m\in\mathbb{Z}_{\geqslant 1} : -x_1|m \text{ 且 } x_2|m\}=\operatorname{lcm}(-x_1,x_2)$

$\operatorname{lcm}(x_1,x_2)=\min\{m\in\mathbb{Z}_{\geqslant 1} : x_1|m \text{ 且 } x_2|m\}=\min\{m\in\mathbb{Z}_{\geqslant 1} : x_1|m \text{ 且 } -x_2|m\}=\operatorname{lcm}(x_1,-x_2)$

$\operatorname{lcm}(x_1,x_2)=\operatorname{lcm}(-x_1,x_2)=\operatorname{lcm}(-x_1,-x_2)$

$\therefore \operatorname{lcm}(x_1,x_2)=\operatorname{lcm}(x_2,x_1)=\operatorname{lcm}(-x_1,x_2)=\operatorname{lcm}(x_1,-x_2)=\operatorname{lcm}(-x_1,-x_2)$

$=\operatorname{lcm}(|x_1|,|x_2|)$ □

Lemma（最小公倍数的性质）$n\in\mathbb{Z}_{\geqslant 1}$，$x_1,x_2,\cdots,x_n\in\mathbb{Z}$，$x_2|x_1, x_3|x_1,\cdots,x_n|x_1$. 则有：

$$\operatorname{lcm}(x_1,\cdots,x_n)=x_1$$

Proof：分两种情况讨论：

① $x_1=0$. ~~要~~ 此时有：$\operatorname{lcm}(x_1,\cdots,x_n)=\operatorname{lcm}(0,x_2,\cdots,x_n)=0=x_1$

② $x_1\neq 0$. 此时有：$\because x_2|x_1, x_3|x_1,\cdots,x_n|x_1 \quad \therefore x_2\neq 0, x_3\neq 0,\cdots,x_n\neq 0$

$\therefore x_1,x_2,\cdots,x_n$ 全都不为0 $\quad \therefore |x_1|\in\mathbb{Z}_{\geqslant 1}$

$\because x_1|x_1, x_2|x_1,\cdots,x_n|x_1 \quad \therefore x_1\,|\,|x_1|, x_2\,|\,|x_1|,\cdots,x_n\,|\,|x_1|$

$\because |x_1|$ 是 x_1 的最小的正倍数 ~~即 $x_1|$~~

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)=\min\{m\in\mathbb{Z}_{\geqslant 1} : x_1|m \text{ 且 } x_2|m \text{ 且}\cdots\text{且 } x_n|m\}=|x_1|$

（把 x_1 的全体倍数按从小到大排列，有：

$$\cdots<-4|x_1|<-3|x_1|<-2|x_1|<-|x_1|<0<|x_1|<2|x_1|<3|x_1|<4|x_1|<\cdots$$

$x_1,\cdots,x_n$ 的公倍数一定是 x_1 的倍数

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)=\min\{m\in\mathbb{Z}_{\geqslant 1} : x_1|m \text{ 且 } x_2|m \text{ 且}\cdots\text{且 } x_n|m\}\geqslant |x_1|$.

$\because |x_1|\in\mathbb{Z}_{\geqslant 1}$，$x_1\,|\,|x_1|, x_2\,|\,|x_1|,\cdots,x_n\,|\,|x_1|$

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)\leqslant |x_1| \qquad \therefore \operatorname{lcm}(x_1,\cdots,x_n)=|x_1|$ ） □

Lemma (最小公倍数的性质) $n\in\mathbb{Z}_{\geqslant 1}$, $x_1,\cdots,x_n\in\mathbb{Z}$, $d\in\mathbb{Z}$, $d\mid x_1$, 则有:

$$\operatorname{lcm}(x_1,\cdots,x_n)=\operatorname{lcm}(x_1,\cdots,x_n,d)$$

Proof: 分两种情况讨论:

① $x_1,\cdots,x_n$ 中有0. 此时 $\operatorname{lcm}(x_1,\cdots,x_n)=0$, $\operatorname{lcm}(x_1,\cdots,x_n,d)=0$

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)=0=\operatorname{lcm}(x_1,\cdots,x_n,d)$

② $x_1,\cdots,x_n$ 中没有0, 即: $x_1\neq 0,\cdots,x_n\neq 0$. $\because d\mid x_1$ $\therefore d\neq 0$

$\therefore x_1,\cdots,x_n,d$ 全都不为0

对 $\forall m\in\mathbb{Z}$, 有:

若 $x_1\mid m$, 则 $\because d\mid x_1$, $x_1\mid m$ $\therefore d\mid m$ $\therefore x_1\mid m$ 且 $d\mid m$.

若 $x_1\mid m$ 且 $d\mid m$, 则 $x_1\mid m$

$\therefore$ 对 $\forall m\in\mathbb{Z}$, 有: $x_1\mid m \Longleftrightarrow x_1\mid m$ 且 $d\mid m$

$$\begin{aligned}\therefore \operatorname{lcm}(x_1,\cdots,x_n)&=\min\{m\in\mathbb{Z}_{\geqslant 1}: x_1\mid m \text{ 且 } x_2\mid m \text{ 且}\cdots\text{且 } x_n\mid m\}\\ &=\min\{m\in\mathbb{Z}_{\geqslant 1}: (x_1\mid m \text{ 且 } d\mid m) \text{ 且 } x_2\mid m \text{ 且}\cdots\text{且 } x_n\mid m\}\\ &=\min\{m\in\mathbb{Z}_{\geqslant 1}: x_1\mid m \text{ 且 } x_2\mid m \text{ 且}\cdots\text{且 } x_n\mid m \text{ 且 } d\mid m\}\\ &=\operatorname{lcm}(x_1,\cdots,x_n,d) \qquad \square\end{aligned}$$

Lemma (最小公倍数的数乘性质) $n\in\mathbb{Z}_{\geqslant 1}$, $x_1,\cdots,x_n\in\mathbb{Z}$, $\lambda\in\mathbb{Z}_{\geqslant 0}$, 则有:

$$\operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n)=\lambda\cdot\operatorname{lcm}(x_1,\cdots,x_n)$$

Proof: 当 $\lambda=0$ 时, $\operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n)=\operatorname{lcm}(0,\cdots,0)=0=0\cdot\operatorname{lcm}(x_1,\cdots,x_n)=\lambda\cdot\operatorname{lcm}(x_1,\cdots,x_n)$

下设 $\lambda\in\mathbb{Z}_{\geqslant 1}$

如果 $x_1,\cdots,x_n$ 中有 0，则有：$\lambda x_1,\cdots,\lambda x_n$ 中有 0.

$\therefore \operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n) = 0 = \lambda\cdot 0 = \lambda\cdot\operatorname{lcm}(x_1,\cdots,x_n)$

下设 $x_1,\cdots,x_n$ 中没有 0. $\quad\therefore x_1\neq 0,\cdots,x_n\neq 0 \quad\therefore x_1,\cdots,x_n$ 全不为 0

$\because \lambda\in\mathbb{Z}_{\geqslant 1} \quad\therefore \lambda x_1\neq 0,\cdots,\lambda x_n\neq 0 \quad\therefore \lambda x_1,\cdots,\lambda x_n$ 全不为 0

设 $A=\operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n)\in\mathbb{Z}_{\geqslant 1}$，$B=\operatorname{lcm}(x_1,\cdots,x_n)\in\mathbb{Z}_{\geqslant 1}$

对 $\forall j=1,\cdots,n$. 有：

$\because \lambda x_j \mid A \quad\therefore \lambda x_j\mid A. \quad\because \lambda\mid\lambda x_j,\ \lambda x_j\mid A \quad\therefore \lambda\mid A$

$\because A\in\mathbb{Z}_{\geqslant 1},\ \lambda x_j\in\mathbb{Z},\ \lambda x_j\neq 0,\ \lambda\in\mathbb{Z}_{\geqslant 1},\ \lambda\neq 0,\ \lambda\mid\lambda x_j,\ \lambda\mid A,\ \lambda x_j\mid A$

$\therefore \dfrac{\lambda x_j}{\lambda}\,\Big|\,\dfrac{A}{\lambda} \quad\therefore x_j\,\Big|\,\dfrac{A}{\lambda}$

$\because A\in\mathbb{Z}_{\geqslant 1},\ \lambda\in\mathbb{Z}_{\geqslant 1},\ \lambda\mid A \quad\therefore \dfrac{A}{\lambda}\in\mathbb{Z}_{\geqslant 1}$

$\therefore \operatorname{lcm}(x_1,\cdots,x_n)\leqslant\dfrac{A}{\lambda} \quad\therefore B\leqslant\dfrac{A}{\lambda}. \quad\because\lambda>0 \quad\therefore \lambda B\leqslant\lambda\cdot\dfrac{A}{\lambda}=A$

对 $\forall k=1,\cdots,n$，有：

$\because x_k\mid\operatorname{lcm}(x_1,\cdots,x_n) \quad\therefore x_k\mid B$

$\because B\in\mathbb{Z}_{\geqslant 1},\ x_k\in\mathbb{Z},\ x_k\neq 0,\ \lambda\in\mathbb{Z}_{\geqslant 1},\ \lambda\neq 0,\ x_k\mid B$

$\therefore \lambda x_k\mid\lambda B$

$\because\lambda\in\mathbb{Z}_{\geqslant 1},\ B\in\mathbb{Z}_{\geqslant 1} \quad\therefore \lambda B\in\mathbb{Z}_{\geqslant 1}$

$\therefore \operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n)\leqslant\lambda B \quad\therefore A\leqslant\lambda B\leqslant A \quad\therefore A=\lambda B$

$\therefore \operatorname{lcm}(\lambda x_1,\cdots,\lambda x_n)=\lambda\cdot\operatorname{lcm}(x_1,\cdots,x_n) \quad\square$