排序不等式

定理（排序不等式）. 设 $x_1, x_2, \cdots, x_n$, $y_1, y_2, \cdots, y_n$ 是任意实数，满足：

$$x_1 \le x_2 \le \cdots \le x_n \quad , \quad y_1 \le y_2 \le \cdots \le y_n .$$

$\sigma: \{1,2,\cdots,n\} \longrightarrow \{1,2,\cdots,n\}$ 是一个双射（意即：σ 是一个 n 级置换）. 则有：

$$\cancel{x_1 y_n + x_2 y_{n-1} + \cdots + x_{n-1} y_2 + x_n y_1 \le}$$

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_{n-1} y_2 + x_n y_1 \le x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \le x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

proof: 先证明右边的不等式. 利用数学归纳法.

当 $n=1$ 时, $\sigma: \{1\} \longrightarrow \{1\}$, $1 \longmapsto 1$. $\therefore x_1 y_{\sigma(1)} = x_1 y_1 \le x_1 y_1$.

当 $n=2$ 时, $\sigma: \{1,2\} \longrightarrow \{1,2\}$, $1 \longmapsto 1$, $2 \longmapsto 2$. $x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} = x_1 y_1 + x_2 y_2 \le x_1 y_1 + x_2 y_2$.

$\sigma: \{1,2\} \longrightarrow \{1,2\}$, $1 \longmapsto 2$, $2 \longmapsto 1$.

$\because y_2 - y_1 \ge 0 \quad x_2 - x_1 \ge 0$ ~~$\because y_2 \ge y_1, x_2 \ge x_1$~~

$\therefore (y_2 - y_1)(x_2 - x_1) \ge 0$

$\therefore x_2 y_2 + x_1 y_1 - x_1 y_2 - x_2 y_1 \ge 0$.

$\therefore x_1 y_2 + x_2 y_1 \le x_1 y_1 + x_2 y_2$

$\therefore x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} = x_1 y_2 + x_2 y_1 \le x_1 y_1 + x_2 y_2$.

假设对于 $n=k$ ($k \in N_+, k \ge 2$) 时，对任意的置换 $\sigma: \{1,2,\cdots,k\} \longrightarrow \{1,2,\cdots,k\}$, 有：

$$x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_k y_{\sigma(k)} \le x_1 y_1 + x_2 y_2 + \cdots + x_k y_k .$$

则对于 $n=k+1$ 时，分两种情况讨论：

① $\sigma(k+1) = k+1$. 则：$\sigma(1), \sigma(2), \cdots, \sigma(k)$ 是 $1, 2, \cdots, k$ 的一个排列. 由归纳假设，有：

$$x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_k y_{\sigma(k)} + x_{k+1} y_{\sigma(k+1)} = x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_k y_{\sigma(k)} + x_{k+1} y_{k+1}$$

$$\le x_1 y_1 + x_2 y_2 + \cdots + x_k y_k + x_{k+1} y_{k+1} .$$

② $\sigma(k+1) \ne k+1$. 则 $\sigma(k+1) \in \{1,2,\cdots,k\}$. 设 $t \in \{1,2,\cdots,k\}$ s.t. $\sigma(t) = k+1$.

则：$\sigma(1), \cdots, \sigma(t-1), \sigma(t+1), \cdots, \sigma(k), \sigma(k+1)$ 是 $1, 2, \cdots, k$ 的一个排列. ~~即~~

（意即：$\{\sigma(i)\}_{\substack{i=1 \\ i \ne t}}^{k+1}$ 是 $1,2,\cdots,k$ 的一个排列）.

$\because (x_{k+1} - x_t)(y_{k+1} - y_{\sigma(k+1)}) \geqslant 0$

$\therefore x_{k+1} y_{k+1} + x_t y_{\sigma(k+1)} \geqslant x_t y_{k+1} + x_{k+1} y_{\sigma(k+1)} = x_t y_{\sigma(t)} + x_{k+1} y_{\sigma(k+1)}$

$\therefore x_1 y_{\sigma(1)} + \cdots + x_{t-1} y_{\sigma(t-1)} + x_t y_{\sigma(t)} + x_{t+1} y_{\sigma(t+1)} + \cdots + x_k y_{\sigma(k)} + x_{k+1} y_{\sigma(k+1)}$

$\leqslant x_1 y_{\sigma(1)} + \cdots + x_{t-1} y_{\sigma(t-1)} + x_t y_{\sigma(k+1)} + x_{t+1} y_{\sigma(t+1)} + \cdots + x_k y_{\sigma(k)} + x_{k+1} y_{k+1}$

$\leq x_1 y_1 + \cdots + x_k y_k + x_{k+1} y_{k+1}$

$\therefore$ 对 $\forall n \in \mathbb{N}_+$，有：$x_1 y_{\sigma(1)} + \cdots + x_n y_{\sigma(n)} \leq x_1 y_1 + \cdots + x_n y_n$.

$\because y_1 \leq y_2 \leq \cdots \leq y_n \qquad \therefore -y_1 \geqslant -y_2 \geqslant \cdots \geqslant -y_n. \qquad -y_n \leq -y_{n-1} \leq \cdots \leq -y_2 \leq -y_1.$

$\because x_1 \leq x_2 \leq \cdots \leq x_n, \quad -y_n \leq -y_{n-1} \leq \cdots \leq -y_2 \leq -y_1.$

~~$\therefore x_1(-y_{\sigma(n)}) + x_2(-y_{\sigma(n-1)}) + \cdots + x_{n-1}(-y_{\sigma(2)}) + x_n(-y_{\sigma(1)})$~~

~~$\leq x_1(-y_n) + x_2(-y_{n-1}) + \cdots + x_{n-1}(-y_2) + x_n(-y_1)$~~

~~$\therefore -(x_1$~~

$\therefore x_1(-y_{\sigma(1)}) + x_2(-y_{\sigma(2)}) + \cdots + x_{n-1}(-y_{\sigma(n-1)}) + x_n(-y_{\sigma(n)})$

$\leq x_1(-y_n) + x_2(-y_{n-1}) + \cdots + x_{n-1}(-y_2) + x_n(-y_1)$

$\therefore -(x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_{n-1} y_{\sigma(n-1)} + x_n y_{\sigma(n)}) \leqslant -(x_1 y_n + x_2 y_{n-1} + \cdots + x_{n-1} y_2 + x_n y_1)$

$\therefore x_1 y_n + x_2 y_{n-1} + \cdots + x_{n-1} y_2 + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_{n-1} y_{\sigma(n-1)} + x_n y_{\sigma(n)}$

$\therefore x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \leq x_1 y_1 + \cdots + x_n y_n. \qquad \square$

以下我们来试着深入讨论以上证明过程中的一些细节.

①当 $n=3$ 时，情况会如何？

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}$

$1\longmapsto 1$

$2\longmapsto 2$

$3\longmapsto 3$

$$x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_1+x_2y_2+x_3y_3$$
$$\leq x_1y_1+x_2y_2+x_3y_3.$$

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}$

$1\longmapsto 1$

$2\longmapsto 3$

$3\longmapsto 2$

$\because (x_3-x_2)(y_3-y_2)\geqslant 0 \quad \therefore x_2y_2+x_3y_3\geqslant x_2y_3+x_3y_2.$

$$\therefore x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_1+x_2y_3+x_3y_2$$
$$\leq x_1y_1+x_2y_2+x_3y_3$$

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}.$

$1\longmapsto 2$

$2\longmapsto 1$

$3\longmapsto 3$

$\because (x_2-x_1)(y_2-y_1)\geqslant 0 \quad \therefore x_1y_1+x_2y_2\geqslant x_1y_2+x_2y_1$

$$\therefore x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_2+x_2y_1+x_3y_3$$
$$\leq x_1y_1+x_2y_2+x_3y_3$$

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}$

$1\longmapsto 2$

$2\longmapsto 3$

$3\longmapsto 1$

$\because (x_3-x_2)(y_3-y_1)\geqslant 0$

$\therefore x_2y_1+x_3y_3\geqslant x_2y_3+x_3y_1.$

$$\therefore x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_2+x_2y_3+x_3y_1\leq x_1y_2+x_2y_1+x_3y_3$$
$$\leq x_1y_1+x_2y_2+x_3y_3$$

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}$

$1\longmapsto 3$

$2\longmapsto 1$

$3\longmapsto 2$

$\because (x_3-x_1)(y_3-y_2)\geqslant 0$

$\therefore x_3y_3+x_1y_2\geqslant x_1y_3+x_3y_2$

$$\therefore x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_3+x_2y_1+x_3y_2\leq x_1y_2+x_2y_1+x_3y_3$$
$$\leq x_1y_1+x_2y_2+x_3y_3$$

$\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}$

$1\longmapsto 3$

$2\longmapsto 2$

$3\longmapsto 1$

$\because (x_3-x_1)(y_3-y_1)\geqslant 0$

$\therefore x_1y_1+x_3y_3\geqslant x_1y_3+x_3y_1$

$$\therefore x_1y_{\sigma(1)}+x_2y_{\sigma(2)}+x_3y_{\sigma(3)}=x_1y_3+x_2y_2+x_3y_1$$
$$\leq x_1y_1+x_2y_2+x_3y_3.$$

② 可否用数学归纳法直接证明 反序和≤乱序和?

proof: 当$n=1$时, $\sigma: \{1\} \longrightarrow \{1\}$, $1 \longmapsto 1$ $\therefore x_1y_1 = x_1 y_{\sigma(1)}$ $\therefore x_1y_1 \le x_1 y_{\sigma(1)}$.

当$n=2$时, $\sigma: \{1,2\} \longrightarrow \{1,2\}$, $1 \longmapsto 1$, $2 \longmapsto 2$

$(x_2-x_1)(y_2-y_1) \geq 0$

$\therefore x_2y_2 + x_1y_1 \geq x_1y_2 + x_2y_1$

$\therefore x_1y_2 + x_2y_1 \le x_1y_{\sigma(1)} + x_2y_{\sigma(2)}$

$\sigma: \{1,2\} \longrightarrow \{1,2\}$. $1 \longmapsto 2$, $2 \longmapsto 1$

$\therefore x_1y_2 + x_2y_1 \le x_1y_{\sigma(1)} + x_2y_{\sigma(2)}$

假设$n=k$ ($k\in\mathbb{N}_+$, $k\geq 2$)时不等式成立, 即: 对$\forall$ ~~级~~ k级置换 $\sigma: \{1,2,\cdots,k\} \longrightarrow \{1,2,\cdots,k\}$,

有: $x_1y_k + x_2y_{k-1} + \cdots + x_{k-1}y_2 + x_ky_1 \le x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x_ky_{\sigma(k)}$.

则当$n=k+1$时, 分两种情况讨论:

~~①. $\sigma(k+1)=1$, 则 $\sigma(1), \sigma(2), \cdots, \sigma(k)$~~

① $\sigma(1)=k+1$. 则: $\sigma(2), \sigma(3), \cdots, \sigma(k+1)$ 是 $1,2,\cdots,k$ 的一个排列. 由归纳假设,

有: ~~$x_1y_{k+1} + x_2y_k + \cdots + x_ky_2 + x_{k+1}y_1 = x_1y_{\sigma(1)} + x_2y_k + \cdots + x_ky_2 + x_{k+1}y_1$~~

$x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x_ky_{\sigma(k)} + x_{k+1}y_{\sigma(k+1)}$

$= x_1y_{k+1} + x_2y_{\sigma(2)} + \cdots + x_ky_{\sigma(k)} + x_{k+1}y_{\sigma(k+1)}$

$\geq x_1y_{k+1} + x_2y_k + \cdots + x_ky_2 + x_{k+1}y_1$.

② $\sigma(1) \neq k+1$. 则 $\sigma(k+1) \in \{1,2,\cdots,k\}$. 设 $t\in\{2,3,\cdots,k,k+1\}$, s.t. $\sigma(t)=k+1$

则: $\sigma(1), \cdots, \sigma(t-1), \sigma(t+1), \cdots, \sigma(k), \sigma(k+1)$ 是 $1,2,\cdots,k$ 的一个排列.

$\because x_1 \le x_t$, $y_{k+1} \geq y_{\sigma(1)}$ $\therefore (x_1-x_t)(y_{k+1}-y_{\sigma(1)}) \le 0$

$\therefore x_1y_{k+1} + x_ty_{\sigma(1)} \le x_1y_{\sigma(1)} + x_ty_{k+1} = x_1y_{\sigma(1)} + x_ty_{\sigma(t)}$

~~$\therefore x_1y_{\sigma(1)} + \cdots + x_{t-1}y_{\sigma(t-1)} + x_ty_{\sigma(t)}$~~ ~~$\therefore x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x$~~

$\therefore x_1y_{\sigma(1)} + x_2y_{\sigma(2)} + \cdots + x_{t-1}y_{\sigma(t-1)} + x_ty_{\sigma(t)} + x_{t+1}y_{\sigma(t+1)} + \cdots + x_{k+1}y_{\sigma(k+1)}$

$\geq x_1y_{k+1} + x_2y_{\sigma(2)} + \cdots + x_{t-1}y_{\sigma(t-1)} + x_ty_{\sigma(1)} + x_{t+1}y_{\sigma(t+1)} + \cdots + x_{k+1}y_{\sigma(k+1)}$

$\geq x_1y_{k+1} + x_2y_k + \cdots + x_{k+1}y_1$

∴ 对 $\forall n \in \mathbb{N}_+$，~~$x_1 y_{k+1} + x_2 y_k + \cdots + x_{k+1} y_1 \leq x_1 y_{\sigma(1)}$~~

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 y_{\sigma(1)} + x_2 y_{\sigma(2)} + \cdots + x_n y_{\sigma(n)} \quad \square$$