奥数教程第七版 高中第一分册 P48 8题.

设x, y是正实数，求 $x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}$ 的最小值.

解：分以下4种情况讨论：

① $x\geqslant 1$ 且 $y\geqslant 1$.

$\cancel{x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}}$

$\because |x-1|\geqslant 0$. $y\geqslant 1$

$\therefore \frac{|x-1|}{y}\geqslant 0$.

$\because |y-1|\geqslant 0$, $x\geqslant 1$ $\therefore \frac{|y-1|}{x}\geqslant 0$

$\therefore x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}\geqslant 1+1+0+0=2$.

当 $x=y=1$ 时，$x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=1+1+0+0=2$.

② $x\geqslant 1$ 且 $0<y<1$.

$\because 0<y<1$ $\therefore \frac{1}{y}>1$ $\because x\geqslant 1$ $\therefore \frac{y}{x}>0$. $\therefore \frac{y}{x}+\frac{1}{y}>1>0$.

$\because x-1\geqslant 0$ $\therefore \left(\frac{y}{x}+\frac{1}{y}\right)(x-1)\geqslant 0$.

$\therefore x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=x+y+\frac{x-1}{y}+\frac{1-y}{x}$

$=x+y+\frac{x}{y}-\frac{1}{y}+\frac{1}{x}-\frac{y}{x}=x+\frac{1}{x}+y+\frac{x}{y}-\frac{1}{y}-\frac{y}{x}$

$=x+\frac{1}{x}+\frac{xy-y}{x}+\frac{x-1}{y}=x+\frac{1}{x}+\frac{y}{x}(x-1)+\frac{1}{y}(x-1)$

$=x+\frac{1}{x}+\left(\frac{y}{x}+\frac{1}{y}\right)(x-1)\geqslant 2\sqrt{x\cdot\frac{1}{x}}+0=2$

当 $x=1$ 且 $0<y<1$ 时，

$x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=1+y+0+1-y=2$.

③ $0<x<1$ 且 $y\geqslant 1$.

$\because 0<x<1 \quad \therefore \frac{1}{x}>1$. $\because x>0$ 且 $y\geqslant 1>0 \quad \therefore \frac{x}{y}>0$

$\therefore \frac{x}{y}+\frac{1}{x}>1>0 \quad \because y-1\geqslant 0 \quad \therefore \left(\frac{x}{y}+\frac{1}{x}\right)(y-1)\geqslant 0$

$\because y\geqslant 1 \quad \therefore y+\frac{1}{y}\geqslant 2\sqrt{y\cdot\frac{1}{y}}=2$.

$\therefore x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=x+y+\frac{1-x}{y}+\frac{y-1}{x}$

$=x+y+\frac{1}{y}-\frac{x}{y}+\frac{1}{x}(y-1)=y+\frac{1}{y}+\frac{xy-x}{y}+\frac{1}{x}(y-1)$

$=y+\frac{1}{y}+\frac{x}{y}(y-1)+\frac{1}{x}(y-1)=y+\frac{1}{y}+\left(\frac{x}{y}+\frac{1}{x}\right)(y-1)\geqslant 2+0=2$.

当 $0<x<1$ 且 $y=1$ 时，

$x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=x+1+1-x+0=2$

④ $0<x<1$ 且 $0<y<1$.

$\because 0<y<1 \quad \therefore \frac{1}{y}>1$. $\because 1-x>0 \quad \therefore \frac{1-x}{y}>1-x$.

$\because 0<x<1 \quad \therefore \frac{1}{x}>1 \quad \because 1-y>0 \quad \therefore \frac{1-y}{x}>1-y$.

$\therefore x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}=x+y+\frac{1-x}{y}+\frac{1-y}{x}$

$>x+y+1-x+1-y=2$

综上，$x+y+\frac{|x-1|}{y}+\frac{|y-1|}{x}$ 的最小值为 2.

Remark：难点在于放缩的路径不容易选择.