人大附中高中数学练习册 必修三(2023年6月修订) P48 13题推广.

求出所有的角 α,β，使得：① $\tan(\alpha+2\beta)=-\sqrt{3}$　② $\tan\frac{\alpha}{2}\tan\beta=2-\sqrt{3}$.

解：$\because \tan(\alpha+2\beta)=-\sqrt{3}$

$$\therefore -\sqrt{3}=\tan(\alpha+2\beta)=\tan\left(2\left(\frac{\alpha}{2}+\beta\right)\right)=\frac{2\tan\left(\frac{\alpha}{2}+\beta\right)}{1-\tan^2\left(\frac{\alpha}{2}+\beta\right)}$$

$$\therefore \sqrt{3}\left(\tan^2\left(\frac{\alpha}{2}+\beta\right)-1\right)=2\tan\left(\frac{\alpha}{2}+\beta\right)$$

$$\tan^2\left(\frac{\alpha}{2}+\beta\right)-\frac{2}{\sqrt{3}}\tan\left(\frac{\alpha}{2}+\beta\right)-1=0.$$

解这个一元二次方程，得：$\tan\left(\frac{\alpha}{2}+\beta\right)=\sqrt{3}$ 或 $-\frac{\sqrt{3}}{3}$,

① 当 $\tan\left(\frac{\alpha}{2}+\beta\right)=\sqrt{3}$ 时

$$\sqrt{3}=\tan\left(\frac{\alpha}{2}+\beta\right)=\frac{\tan\frac{\alpha}{2}+\tan\beta}{1-\tan\frac{\alpha}{2}\tan\beta}=\frac{\tan\frac{\alpha}{2}+\tan\beta}{1-(2-\sqrt{3})}=\frac{\tan\frac{\alpha}{2}+\tan\beta}{\sqrt{3}-1}$$

$$\therefore \tan\frac{\alpha}{2}+\tan\beta=3-\sqrt{3}$$

$\therefore$ 有方程组：$\begin{cases}\tan\frac{\alpha}{2}+\tan\beta=3-\sqrt{3}\\ \tan\frac{\alpha}{2}\tan\beta=2-\sqrt{3}\end{cases}$　$\therefore \tan^2\beta+(\sqrt{3}-3)\tan\beta+2-\sqrt{3}=0$.

解得：$\begin{cases}\tan\beta=1\\ \tan\frac{\alpha}{2}=2-\sqrt{3}\end{cases}$ 或 $\begin{cases}\tan\beta=2-\sqrt{3}\\ \tan\frac{\alpha}{2}=1\end{cases}$

代回验证：当 $\begin{cases}\tan\beta=1\\ \tan\frac{\alpha}{2}=2-\sqrt{3}\end{cases}$ 时，$\tan\frac{\alpha}{2}\tan\beta=2-\sqrt{3}$.　$\tan\left(\frac{\alpha}{2}+\beta\right)=\frac{\tan\frac{\alpha}{2}+\tan\beta}{1-\tan\frac{\alpha}{2}\tan\beta}=\sqrt{3}$

$$\tan(\alpha+2\beta)=\tan\left(2\left(\frac{\alpha}{2}+\beta\right)\right)=\frac{2\tan\left(\frac{\alpha}{2}+\beta\right)}{1-\tan^2\left(\frac{\alpha}{2}+\beta\right)}=-\sqrt{3}$$ 符合题意.

~~当~~ $\cancel{\begin{cases}\tan\beta=2-\sqrt{3}\\ \tan\frac{\alpha}{2}=1\end{cases}}$ ~~时~~，$\cancel{\tan\frac{\alpha}{2}\tan\beta=2-\sqrt{3}}$.

当 $\begin{cases}\tan\beta=2-\sqrt{3}\\ \tan\frac{\alpha}{2}=1\end{cases}$ 时，$\tan\frac{\alpha}{2}\tan\beta=2-\sqrt{3}$.　$\tan\left(\frac{\alpha}{2}+\beta\right)=\frac{\tan\frac{\alpha}{2}+\tan\beta}{1-\tan\frac{\alpha}{2}\tan\beta}=\sqrt{3}$.

$$\tan(\alpha+2\beta)=\tan\left(2\cdot\left(\frac{\alpha}{2}+\beta\right)\right)=\frac{2\tan\left(\frac{\alpha}{2}+\beta\right)}{1-\tan^2\left(\frac{\alpha}{2}+\beta\right)}=-\sqrt{3}.$$ 符合题意.

∴当 $\begin{cases} \tan\beta = 1 \\ \tan\frac{\alpha}{2} = 2-\sqrt{3} \end{cases}$ 时，符合题意. 此时 $\tan\alpha = \tan\left(2\cdot\frac{\alpha}{2}\right) = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = \dfrac{\sqrt{3}}{3}$

$$\therefore \begin{cases} \beta = \frac{\pi}{4} + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{6} + k_2\pi, \ k_2\in\mathbb{Z} \end{cases}.$$

当 $\begin{cases} \tan\beta = 2-\sqrt{3} \\ \tan\frac{\alpha}{2} = 1 \end{cases}$ 时，符合题意. 此时 $\begin{cases} \beta = \arctan(2-\sqrt{3}) + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{2} + 2k_2\pi, \ k_2\in\mathbb{Z} \end{cases}.$ $\left(\tan\frac{\pi}{12} = 2-\sqrt{3}\right)$

综上，$\begin{cases} \beta = \frac{\pi}{4} + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{6} + k_2\pi, \ k_2\in\mathbb{Z} \end{cases}$ 或 $\cancel{\beta = \arctan(2-\sqrt{3}) + k\pi, \ k\in}$ $\begin{cases} \beta = \frac{\pi}{12} + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{2} + 2k_2\pi, \ k_2\in\mathbb{Z} \end{cases}$

② 当 $\tan\left(\frac{\alpha}{2}+\beta\right) = -\frac{\sqrt{3}}{3}$ 时，

$$-\frac{\sqrt{3}}{3} = \tan\left(\frac{\alpha}{2}+\beta\right) = \frac{\tan\frac{\alpha}{2} + \tan\beta}{1-\tan\frac{\alpha}{2}\tan\beta} = \frac{\tan\frac{\alpha}{2} + \tan\beta}{\sqrt{3}-1} \quad \therefore \tan\frac{\alpha}{2} + \tan\beta = \frac{\sqrt{3}}{3} - 1$$

$\because \begin{cases} \tan\frac{\alpha}{2} + \tan\beta = \frac{\sqrt{3}}{3} - 1 \\ \tan\frac{\alpha}{2}\tan\beta = 2-\sqrt{3} \end{cases}$ $\therefore \tan^2\beta + \left(1-\frac{\sqrt{3}}{3}\right)\tan\beta + 2-\sqrt{3} = 0.$

$\Delta = \frac{10}{3}(\sqrt{3}-2) < 0.$ 一元二次方程无实数解.

∴ 这种情况不符合题意.

综上，满足条件的全部角 α, β 为：

$$\begin{cases} \beta = \frac{\pi}{4} + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{6} + k_2\pi, \ k_2\in\mathbb{Z} \end{cases} \quad 或 \quad \begin{cases} \beta = \frac{\pi}{12} + k_1\pi, \ k_1\in\mathbb{Z} \\ \alpha = \frac{\pi}{2} + 2k_2\pi, \ k_2\in\mathbb{Z} \end{cases}.$$