人大附中高中数学练习册—必修三(2023年6月修订) P51. 4题

已知：$\dfrac{1}{\sin x}+\dfrac{1}{\cos x}=2\sqrt{2}$，$x\in(0,\pi)$. 求：$\sin\left(2x+\dfrac{\pi}{3}\right)$.

解：$\because \dfrac{1}{\sin x}+\dfrac{1}{\cos x}=2\sqrt{2}$ $\quad\therefore \cos x\neq 0$ $\quad\therefore x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)$

$\therefore \dfrac{\sin x+\cos x}{\sin x\cos x}=2\sqrt{2}$ $\quad\therefore \sin x+\cos x=2\sqrt{2}\sin x\cos x$.

设 $t=\sin x\cos x$. $\quad\therefore (\sin x+\cos x)^2=8t^2$ $\quad\therefore 1+2t=8t^2$

$8t^2-2t-1=0$，解得：$t_1=\dfrac{1}{2}$，$t_2=-\dfrac{1}{4}$

当 $\sin x\cos x=\dfrac{1}{2}$ 时. $\begin{cases}\sin x\cos x=\dfrac{1}{2}\\ \sin x+\cos x=\sqrt{2}\end{cases} \Rightarrow \begin{cases}\sin x=\dfrac{\sqrt{2}}{2}\\ \cos x=\dfrac{\sqrt{2}}{2}\end{cases}$ $\quad\therefore \sin\left(2x+\dfrac{\pi}{3}\right)=\dfrac{1}{2}$ $\quad x=\dfrac{\pi}{4}$

当 $\sin x\cos x=-\dfrac{1}{4}$ 时. $\because \sin x\cos x<0$ $\quad\therefore x\in\left(\dfrac{\pi}{2},\pi\right)$ $\quad\therefore \cos x<0$ 且 $\sin x>0$.

$\therefore \cos x-\sin x<0$

$\therefore (\cos x-\sin x)^2=1-2\times\left(-\dfrac{1}{4}\right)=1+\dfrac{1}{2}=\dfrac{3}{2}$ $\quad\therefore \cos x-\sin x=-\dfrac{\sqrt{6}}{2}$

$\because \cos x+\sin x=2\sqrt{2}\sin x\cos x=-\dfrac{\sqrt{2}}{2}$

$\therefore \sin\left(2x+\dfrac{\pi}{3}\right)=\dfrac{1}{2}\sin 2x+\dfrac{\sqrt{3}}{2}\cos 2x=\sin x\cos x+\dfrac{\sqrt{3}}{2}(\cos^2x-\sin^2x)=\dfrac{1}{2}$

此时 $\cos x=-\dfrac{\sqrt{6}+\sqrt{2}}{4}$. $\quad \sin x=\dfrac{\sqrt{6}-\sqrt{2}}{4}$

$\sin 2x=2\sin x\cos x=-\dfrac{1}{2}$, $\quad \cos 2x=2\cos^2x-1=\dfrac{\sqrt{3}}{2}$.

$\because x\in\left(\dfrac{\pi}{2},\pi\right)$ $\quad\therefore 2x\in(\pi,2\pi)$ $\quad\therefore 2x=\dfrac{11}{6}\pi$. $\quad\therefore x=\dfrac{11}{12}\pi$.

综上，$x=\dfrac{\pi}{4}$ 或 $\dfrac{11}{12}\pi$. $\quad\sin\left(2x+\dfrac{\pi}{3}\right)=\dfrac{1}{2}$.