奥数教程第七版高中第一分册. P45. 例6.

设 $x, y, z \in [0,1]$. 求 $M=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}$ 的最大值.

解: 分六种情况讨论. 利用 Cauchy 不等式.

① 当 $0 \leq x \leq y \leq z \leq 1$ 时, 有: $M=\sqrt{y-x}+\sqrt{z-y}+\sqrt{z-x}$

$\because \left[(\sqrt{y-x})^2+(\sqrt{z-y})^2\right](1^2+1^2) \geqslant (\sqrt{y-x}+\sqrt{z-y})^2$

$\therefore 2(z-x) \geqslant (\sqrt{y-x}+\sqrt{z-y})^2$

$\therefore \sqrt{2(z-x)} \geqslant |\sqrt{y-x}+\sqrt{z-y}|$, $\sqrt{2}\cdot\sqrt{z-x} \geqslant \sqrt{y-x}+\sqrt{z-y}$

"=" iff $\sqrt{y-x}=\sqrt{z-y}$ iff $2y=x+z$

$\therefore M=\sqrt{y-x}+\sqrt{z-y}+\sqrt{z-x} \leqslant \sqrt{2}\cdot\sqrt{z-x}+\sqrt{z-x}=(\sqrt{2}+1)\sqrt{z-x}$

$\leqslant (\sqrt{2}+1)\sqrt{1-0}=\sqrt{2}+1$

"=" iff $z=1$ 且 $x=0$ 且 $y=\frac{1}{2}$.

$\therefore M$ 的最大值为 $\sqrt{2}+1$, ~~当且仅当~~ 当 $x=0$, $y=\frac{1}{2}$, $z=1$ 时取到.

② 当 $0 \leq x \leq z \leq y \leq 1$ 时, 有: $M=\sqrt{y-x}+\sqrt{y-z}+\sqrt{z-x}$

$\because \left[(\sqrt{y-z})^2+(\sqrt{z-x})^2\right](1^2+1^2) \geqslant (\sqrt{y-z}+\sqrt{z-x})^2$

"=" iff $\sqrt{y-z}=\sqrt{z-x}$ iff $2z=x+y$

$\therefore 2(y-x) \geqslant (\sqrt{y-z}+\sqrt{z-x})^2$

$\sqrt{2}\sqrt{y-x} \geqslant \sqrt{y-z}+\sqrt{z-x}$

$\therefore M=\sqrt{y-x}+\sqrt{y-z}+\sqrt{z-x} \leqslant (\sqrt{2}+1)\sqrt{y-x} \leqslant \sqrt{2}+1$.

"=" iff $x=0$ 且 $y=1$ 且 $z=\frac{1}{2}$.

③ 当$0\le y\le x\le z\le 1$时，有：$M=\sqrt{x-y}+\sqrt{z-y}+\sqrt{z-x}$

$\because \left[(\sqrt{x-y})^2+(\sqrt{z-x})^2\right](1^2+1^2)\geqslant(\sqrt{x-y}+\sqrt{z-x})^2$

$\therefore\ 2(z-y)\geqslant(\sqrt{x-y}+\sqrt{z-x})^2$

$\therefore\ \sqrt{2}\sqrt{z-y}\geqslant\sqrt{x-y}+\sqrt{z-x}$

$\therefore M=\sqrt{x-y}+\sqrt{z-x}+\sqrt{z-y}\leqslant(\sqrt{2}+1)\sqrt{z-y}\leqslant\sqrt{2}+1$

"=" iff $z=1$ 且 $y=0$ 且 $x=\frac{1}{2}$

④ 当$0\le y\le z\le x\le 1$时，有：$M=\sqrt{x-y}+\sqrt{z-y}+\sqrt{x-z}$

$\because\left[(\sqrt{z-y})^2+(\sqrt{x-z})^2\right](1^2+1^2)\geqslant(\sqrt{z-y}+\sqrt{x-z})^2$

"=" iff $\sqrt{z-y}=\sqrt{x-z}$ iff $2z=x+y$

$\therefore\ 2(x-y)\geqslant(\sqrt{z-y}+\sqrt{x-z})^2$

$\therefore\ \sqrt{2}\sqrt{x-y}\geqslant\sqrt{z-y}+\sqrt{x-z}$

$\therefore M=\sqrt{x-y}+\sqrt{z-y}+\sqrt{x-z}\leqslant(\sqrt{2}+1)\sqrt{x-y}\leqslant\sqrt{2}+1$

"=" iff $x=1$ 且 $y=0$ 且 $z=\frac{1}{2}$

⑤ 当$0\le z\le x\le y\le 1$时，$M=\sqrt{y-x}+\sqrt{y-z}+\sqrt{x-z}$

$\because\left[(\sqrt{y-x})^2+(\sqrt{x-z})^2\right](1^2+1^2)\geqslant(\sqrt{y-x}+\sqrt{x-z})^2$

"=" iff $\sqrt{y-x}=\sqrt{x-z}$ iff $2x=y+z$

$\therefore\ 2(y-z)\geqslant(\sqrt{y-x}+\sqrt{x-z})^2$ $\quad\therefore\ \sqrt{2}\sqrt{y-z}\geqslant\sqrt{y-x}+\sqrt{x-z}$

$\therefore\ M=\sqrt{y-x}+\sqrt{x-z}+\sqrt{y-z}\leqslant(\sqrt{2}+1)\sqrt{y-z}\leqslant\sqrt{2}+1$

"=" iff $y=1$ 且 $z=0$ 且 $x=\frac{1}{2}$

⑥ 当 $0 \leq z \leq y \leq x \leq 1$ 时，$M=\sqrt{x-y}+\sqrt{y-z}+\sqrt{x-z}$

$\because \left[(\sqrt{x-y})^2+(\sqrt{y-z})^2\right](1^2+1^2) \geq (\sqrt{x-y}+\sqrt{y-z})^2$

$$2(x-z) \geq (\sqrt{x-y}+\sqrt{y-z})^2$$

$\therefore \sqrt{2}\sqrt{x-z} \geq \sqrt{x-y}+\sqrt{y-z}$

$\therefore M=\sqrt{x-y}+\sqrt{y-z}+\sqrt{x-z} \leq (\sqrt{2}+1)\sqrt{x-z} \leq \sqrt{2}+1$

"=" iff $x=1$ 且 $z=0$ 且 $y=\frac{1}{2}$.

综上，M 的最大值为 $\sqrt{2}+1$.

Remark：我们观察 $M=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}$,

把 x 换成 y，y 换成 x，得到：

$$\sqrt{|y-x|}+\sqrt{|x-z|}+\sqrt{|z-y|}=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}=M.$$

把 y 换成 z，z 换成 y，得到：

$$\sqrt{|x-z|}+\sqrt{|z-y|}+\sqrt{|y-x|}=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}=M$$

把 z 换成 x，x 换成 z，得到：

$$\sqrt{|z-y|}+\sqrt{|y-x|}+\sqrt{|x-z|}=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}=M.$$

$\therefore M=\sqrt{|x-y|}+\sqrt{|y-z|}+\sqrt{|z-x|}$ 是关于 x,y,z 的对称式.