奥数教程第七版高中第一分册 P28. 例7

求函数 $f(x)=x^2+x\sqrt{x^2-1}$ 的值域.

解: $\because x^2-1\geqslant 0$ $\therefore (x+1)(x-1)\geqslant 0$ $\therefore x\in(-\infty,-1]\cup[1,+\infty)$

$\therefore f(x)$ 的定义域是: $(-\infty,-1]\cup[1,+\infty)$.

对 $\forall x_1, x_2\in[1,+\infty)$, $x_1<x_2$. 有: $1\leqslant x_1<x_2$.

$\therefore 1\leqslant x_1^2<x_2^2$. $\therefore 0\leqslant x_1^2-1<x_2^2-1$ $\therefore 0\leqslant\sqrt{x_1^2-1}<\sqrt{x_2^2-1}$

$\because 0<x_1<x_2$ $\therefore 0\leqslant x_1\sqrt{x_1^2-1}<x_2\sqrt{x_2^2-1}$

$\therefore 1\leqslant x_1^2+x_1\sqrt{x_1^2-1}<x_2^2+x_2\sqrt{x_2^2-1}$

$\therefore f(x_1)<f(x_2)$ $\therefore f(x)$ 在 $[1,+\infty)$ 上严格单调递增.

$\because f(1)=1$, $\lim\limits_{x\to+\infty}f(x)=+\infty$ $\therefore$ 当 $x\in[1,+\infty)$ 时, $f(x)\in[1,+\infty)$

当 $x\in(-\infty,-1]$ 时, $x^2\geqslant 1$ $\therefore x^2-1\geqslant 0$, $\sqrt{x^2-1}\geqslant 0$.

$\therefore -\sqrt{x^2-1}\leqslant 0$. $\because x\leqslant -1$ $\therefore x-\sqrt{x^2-1}\leqslant -1<0$.

$$\therefore f(x)=x^2+x\sqrt{x^2-1}=x\left(x+\sqrt{x^2-1}\right)=\frac{x(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})}{x-\sqrt{x^2-1}}=\frac{x}{x-\sqrt{x^2-1}}$$

$$=\frac{1}{1-\frac{\sqrt{x^2-1}}{x}}=\frac{1}{1+\frac{\sqrt{x^2-1}}{-x}}=\frac{1}{1+\frac{\sqrt{x^2-1}}{|x|}}=\frac{1}{1+\frac{\sqrt{x^2-1}}{\sqrt{x^2}}}$$

$$=\frac{1}{1+\sqrt{1-\frac{1}{x^2}}}$$

$\because x\in(-\infty,-1]$ $\therefore x^2\in[1,+\infty)$, $\sqrt{1-\frac{1}{x^2}}\in[0,1)$, $1+\sqrt{1-\frac{1}{x^2}}\in[1,2)$

$$\therefore f(x)=\frac{1}{1+\sqrt{1-\frac{1}{x^2}}}\in\left(\frac{1}{2},1\right]$$

$\therefore$ 当 $x\in(-\infty,-1]\cup[1,+\infty)$ 时, $f(x)\in\left(\frac{1}{2},+\infty\right)$. $f(x)$ 的值域为 $\left(\frac{1}{2},+\infty\right)$.