Lemma: A, B, C是任意集合，$f: A \longrightarrow B$是一个映射，$g: B \rightarrow C$是一个映射，f是满射，则有：$\mathrm{im}(gf) = \mathrm{im}(g)$

Proof: $\because g: B \rightarrow C$是一个从$B$到$C$的映射 $\quad \therefore \mathrm{im}(g) \subseteq C$

$\because f: A \rightarrow B$是一个映射，$g: B \rightarrow C$是一个映射

$\therefore gf: A \longrightarrow C$是一个从$A$到$C$的映射 $\quad \therefore \mathrm{im}(gf) \subseteq C$

对$\forall \lambda \in \mathrm{im}(g)$，$\exists\, b \in B$，s.t. $g(b) = \lambda$.

$\because f: A \rightarrow B$是一个满射 $\quad \therefore \exists\, a \in A$，s.t. $f(a) = b$.

$\therefore a \in A$，且有：$(gf)(a) = g(f(a)) = g(b) = \lambda$

$\therefore \lambda \in \mathrm{im}(gf) \quad \therefore \mathrm{im}(g) \subseteq \mathrm{im}(gf)$

对$\forall \mu \in \mathrm{im}(gf)$，$\exists\, \alpha \in A$，s.t. ~~$(gf)(\mu)$~~

$(gf)(\alpha) = \mu$.

$\therefore \mu = (gf)(\alpha) = g(f(\alpha))$.

$\because \alpha \in A \quad \therefore f(\alpha) \in B$.

$\because \mu \in C$，$f(\alpha) \in B$，$g(f(\alpha)) = \mu \quad \therefore \mu \in \mathrm{im}(g)$

$\therefore \mathrm{im}(gf) \subseteq \mathrm{im}(g) \quad \therefore \mathrm{im}(gf) = \mathrm{im}(g) \quad \square$

Lemma：A, B, C是任意集合，$f: A \longrightarrow B$是一个映射，$g: B \longrightarrow C$是一个映射. g是单射. 则：对$\forall b \in B$，有：$f^{-1}(b) = (gf)^{-1}(g(b))$

Proof：对$\forall b \in B$，有：$f^{-1}(b) \subseteq A$

$\because g(b) \in C$，$gf: A \longrightarrow C$是一个映射 $\quad \therefore (gf)^{-1}(g(b)) \subseteq A$

对$\forall \lambda \in f^{-1}(b)$，有：$\lambda \in A$且$f(\lambda) = b$.

$\because (gf)(\lambda) = g(f(\lambda)) = g(b) \quad \therefore \lambda \in (gf)^{-1}(g(b))$

$\therefore f^{-1}(b) \subseteq (gf)^{-1}(g(b))$

对$\forall \mu \in (gf)^{-1}(g(b))$，有：$\mu \in A$且$(gf)(\mu) = g(b)$

$\therefore g(f(\mu)) = g(b) \quad \because g$是单射 $\quad \therefore f(\mu) = b \quad \therefore \mu \in f^{-1}(b)$

$\therefore (gf)^{-1}(g(b)) \subseteq f^{-1}(b) \quad \therefore f^{-1}(b) = (gf)^{-1}(g(b)) \quad \square$