2.5 等价关系与商集

例（过原点的直线表示为商集）考虑空间 $\mathbb{R}^3$. 在扣掉原点 $(0,0,0)$ 得到的子集 $\mathbb{R}^3\setminus\{(0,0,0)\}$ 上定义二元关系：

$$(x,y,z)\sim(x',y',z') \iff \exists t\in\mathbb{R},\ t\neq 0,\ 使得 (x',y',z')=(tx,ty,tz).$$

首先证明这是集合 $\mathbb{R}^3\setminus\{(0,0,0)\}$ 上的等价关系.

对 $\forall (x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0)\}$，$\exists\, 1\in\mathbb{R}$，$1\neq 0$，使得 $(x,y,z)=(1x,1y,1z)$

$\therefore (x,y,z)\sim(x,y,z)$. 反身性成立.

对 $\forall (x,y,z),(x',y',z')\in\mathbb{R}^3\setminus\{(0,0,0)\}$，$(x,y,z)\sim(x',y',z')$，有：

$\exists t\in\mathbb{R}$，$t\neq 0$，s.t. $(x',y',z')=(tx,ty,tz)$.

$\because t\neq 0$ $\quad\therefore \frac{1}{t}\neq 0$ $\quad\therefore \frac{1}{t}\in\mathbb{R}$，$\frac{1}{t}\neq 0$，且有 $(x,y,z)=(\frac{1}{t}x',\frac{1}{t}y',\frac{1}{t}z')$

$\therefore (x',y',z')\sim(x,y,z)$ $\quad\therefore$ 对称性成立.

对 $\forall (x,y,z),(x',y',z'),(\alpha,\beta,\gamma)\in\mathbb{R}^3\setminus\{(0,0,0)\}$，若 $(x,y,z)\sim(x',y',z')$，$(x',y',z')\sim(\alpha,\beta,\gamma)$. 则有：

$\because (x,y,z)\sim(x',y',z')$ $\quad\therefore \exists t_1\in\mathbb{R}$，$t_1\neq 0$，使得 $(x',y',z')=(t_1x,t_1y,t_1z)$

$\because (x',y',z')\sim(\alpha,\beta,\gamma)$ $\quad\therefore \exists t_2\in\mathbb{R}$，$t_2\neq 0$，使得 $(\alpha,\beta,\gamma)=(t_2x',t_2y',t_2z')$

$\because t_1\in\mathbb{R}$ 且 $t_2\in\mathbb{R}$ $\quad\therefore t_2t_1\in\mathbb{R}$. $\quad\because t_1\neq 0$ 且 $t_2\neq 0$ $\quad\therefore t_2t_1\neq 0$.

$\because t_2t_1\in\mathbb{R}$，$t_2t_1\neq 0$，$(\alpha,\beta,\gamma)=((t_2t_1)x,(t_2t_1)y,(t_2t_1)z)$

$\therefore (x,y,z)\sim(\alpha,\beta,\gamma)$ $\quad\therefore$ 传递性成立.

$\therefore \sim$ 是集合 $\mathbb{R}^3\setminus\{(0,0,0)\}$ 上的等价关系.

接下来我们考查 $\mathbb{R}^3\setminus\{(0,0,0)\}$ 中任一元的等价类.

对 $\forall (x,y,z)\in\mathbb{R}^3\setminus\{(0,0,0)\}$，$C_{(x,y,z)}:=\{(\alpha,\beta,\gamma)\in\mathbb{R}^3\setminus\{(0,0,0)\}:(\alpha,\beta,\gamma)\sim(x,y,z)\}$.

$\therefore C_{(x,y,z)}=\{(\alpha,\beta,\gamma)\in\mathbb{R}^3\setminus\{(0,0,0)\}:(x,y,z)\sim(\alpha,\beta,\gamma)\}$

$=\{(\alpha,\beta,\gamma)\in\mathbb{R}^3\setminus\{(0,0,0)\}:\exists t\in\mathbb{R},\ t\neq 0$，使得 $(\alpha,\beta,\gamma)=(tx,ty,tz)\}$

现在来证明，集合 $C_{(x,y,z)}$ 中的点~~就是~~就是 过 (x,y,z) 和原点的唯一直线上的点（不包括原点）

对 $\forall (\alpha,\beta,\gamma)\in C_{(x,y,z)}$，有：$(\alpha,\beta,\gamma)\in\mathbb{R}^3\setminus\{(0,0,0)\}$，且 $\exists t\in\mathbb{R}$，$t\neq 0$，使得

$(\alpha,\beta,\gamma)=(tx,ty,tz)$

设 $A(\alpha,\beta,\gamma)$，$B(x,y,z)$. $\therefore \overrightarrow{OA}=(\alpha,\beta,\gamma)$，$\overrightarrow{OB}=(x,y,z)$

$\therefore (\alpha,\beta,\gamma)=(tx,ty,tz)=t(x,y,z)$ $\therefore \overrightarrow{OA}=t\overrightarrow{OB}$

$\because \overrightarrow{OA}=t\overrightarrow{OB}$，$t\in\mathbb{R}$，$t\neq 0$ $\therefore A\in$(直线 $OB\setminus\{O\}$)

$\therefore$ 点 $(\alpha,\beta,\gamma)\in$ (过 (x,y,z) 和原点的唯一直线 $\setminus\{$原点$\}$)

反之，对 $\forall (\delta,\varepsilon,\varphi)\in$ (过 (x,y,z) 和原点的唯一直线 $\setminus\{$原点$\}$)，设 $D(\delta,\varepsilon,\varphi)$，$E(x,y,z)$.

$\because D\in$(直线 $OE\setminus\{O\}$) $\therefore \exists k\in\mathbb{R}$，$k\neq 0$，使得 $\overrightarrow{OD}=k\overrightarrow{OE}$

$\therefore \exists k\in\mathbb{R}$，$k\neq 0$，使得 $(\delta,\varepsilon,\varphi)=k(x,y,z)=(kx,ky,kz)$

$\because D\neq O$ $\therefore (\delta,\varepsilon,\varphi)\in\mathbb{R}^3\setminus\{(0,0,0)\}$

$\therefore (\delta,\varepsilon,\varphi)\in\mathbb{R}^3\setminus\{(0,0,0)\}$ 且 $\exists k\in\mathbb{R}$，$k\neq 0$，使得 $(\delta,\varepsilon,\varphi)=(kx,ky,kz)$

$\therefore (\delta,\varepsilon,\varphi)\in C_{(x,y,z)}$

$\therefore C_{(x,y,z)}=$ (过 (x,y,z) 和原点的唯一直线 $\setminus\{$原点$\}$). $\square$

$\therefore$ 有如下的双射

$$\left(\mathbb{R}^3 \setminus \{(0,0,0)\}\right)/\sim \xrightarrow{1:1} \{\ell \subseteq \mathbb{R}^3 : \ell \text{是过原点的直线}\}$$

$$\text{点}(x,y,z)\text{的等价类} \longmapsto \text{过}(x,y,z)\text{和原点的唯一直线} \setminus \{\text{原点}\}.$$

Lemma: A, B是任意非空集合, $f: A \to B$是一个映射, 在A上定义二元关系$\sim_f$为:

$$\text{对}\forall a, a' \in A, \quad a \sim_f a' \iff f(a) = f(a').$$

则有: $\sim_f$是A上的等价关系.

Proof: 对$\forall a \in A$, $\because f(a) = f(a)$ $\quad \therefore a \sim_f a$ $\quad \therefore$ 反身性成立.

对$\forall a, b \in A$. 若$a \sim_f b$, 则有: $f(a) = f(b)$ $\quad \therefore f(b) = f(a)$ $\quad \therefore b \sim_f a$

$\therefore$ 对称性成立.

对$\forall a, b, c \in A$. 若$a \sim_f b$且$b \sim_f c$, 则有:

$\because a \sim_f b$ $\quad \therefore f(a) = f(b)$ $\quad \because b \sim_f c$ $\quad \therefore f(b) = f(c)$

$\therefore f(a) = f(c)$ $\quad \therefore a \sim_f c$ $\quad \therefore$ 传递性成立.

$\therefore \sim_f$是A上的等价关系. $\square$

Remark: $\because A$是任意非空集合, $\sim_f$是A上的等价关系, $q: A \to A/\sim_f$是对应的商映射, B是任意非空集合, $f: A \to B$是一个映射, 且满足:

$$\text{对}\forall a, a' \in A, \quad a \sim_f a' \iff f(a) = f(a')$$

$\therefore$ 存在唯一的映射 $\bar{f}: (A/\sim_f) \to B$ 使得 $\bar{f} \circ q = f$

接下来我们证明$\bar{f}: (A/\sim_f) \to B$是单射.

已知: $A/\sim_f = \{C \subseteq A : C\text{是}A\text{的相对于}\sim_f\text{的等价类}\}$.

对$\forall C_1, C_2 \in A/\sim_f$. 若$\bar{f}(C_1)=\bar{f}(C_2)$. 则有:

任取$\varphi_1 \in C_1$, 有: $\varphi_1 \in A$ $\therefore \bar{f}(C_1)=f(\varphi_1)\in B$

任取$\varphi_2 \in C_2$, 有: $\varphi_2 \in A$ $\therefore \bar{f}(C_2)=f(\varphi_2)\in B$

$\because \bar{f}(C_1)=\bar{f}(C_2)$ $\therefore f(\varphi_1)=f(\varphi_2)$

$\because \varphi_1\in A$, $\varphi_2\in A$, $f(\varphi_1)=f(\varphi_2)$ $\therefore \varphi_1 \sim_f \varphi_2$ $\therefore \varphi_2\sim_f\varphi_1$.

对$\forall \lambda\in C_1$, 有: $\because C_1$是A的相对于$\sim_f$的等价类, $\lambda\in C_1$, $\varphi_1\in C_1$ $\therefore \lambda\sim_f\varphi_1$

$\because \lambda\in A$, $\varphi_1\in A$, $\varphi_2\in A$, $\lambda\sim_f\varphi_1$, $\varphi_1\sim_f\varphi_2$ $\therefore \lambda\sim_f\varphi_2$ $\therefore \varphi_2\sim_f\lambda$.

$\because C_2$是A的相对于$\sim_f$的等价类, $\varphi_2\in C_2$, $\lambda\in A$, $\varphi_2\sim_f\lambda$ $\therefore \lambda\in C_2$ $\therefore C_1\subseteq C_2$.

对$\forall \mu\in C_2$, 有: $\because C_2$是A的相对于$\sim_f$的等价类, $\mu\in C_2$, $\varphi_2\in C_2$ $\therefore \mu\sim_f\varphi_2$.

$\because \mu\in A$, $\varphi_2\in A$, $\varphi_1\in A$, $\mu\sim_f\varphi_2$, $\varphi_2\sim_f\varphi_1$. $\therefore \mu\sim_f\varphi_1$ $\therefore \varphi_1\sim_f\mu$.

$\because C_1$是A的相对于$\sim_f$的等价类, $\varphi_1\in C_1$, $\mu\in A$, $\varphi_1\sim_f\mu$ $\therefore \mu\in C_1$ $\therefore C_2\subseteq C_1$

$\therefore C_1=C_2$ $\therefore \bar{f}:(A/\sim_f)\longrightarrow B$是单射.

接下来我们证明$im(\bar{f})=im(f)$.

$\because$ A是任意非空集合, $A/\sim_f$是非空集合, B是任意非空集合.

$q: A\longrightarrow A/\sim_f$是商映射, $\bar{f}:(A/\sim_f)\longrightarrow B$是一个映射. q是满射.

$\therefore im(\bar{f}\circ q)=im(\bar{f})$ $\therefore im(\bar{f})=im(\bar{f}\circ q)=im(f)$

$\therefore \bar{f}:(A/\sim_f)\longrightarrow im(f)$ 是一个双射. $\therefore \bar{f}:(A/\sim_f)\xrightarrow{1:1} im(f)$ $\square$