

Lemma (整环R上的分式域的泛性质) R是一个任意的整环，

F是一个域， $f: R \rightarrow F$ 是一个单同态，则以下两个命题等价：

- ① F的所有元素都能表成 $f(x)(f(y))^{-1}$ 的形式，其中 $x, y \in R$ 且 $y \neq 0_R$
② 对 \forall 交换环 A， \forall 环同态 $\varphi: R \rightarrow A$ 满足 $\varphi(R \setminus \{0_R\}) \subseteq A^\times$ ，存在唯一的环同态 $\theta: F \rightarrow A$ ，s.t. 下图交换：

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow \varphi \\ F & \xrightarrow{\theta} & A \end{array}$$

Proof: ① \Rightarrow ②: 对 \forall 交换环 A， \forall 环同态 $\varphi: R \rightarrow A$ 满足

$$\varphi(R \setminus \{0_R\}) \subseteq A^\times,$$

对 $\forall \lambda \in F$, $\because \exists x, y \in R$ 且 $y \neq 0_R$, s.t. $\lambda = f(x)(f(y))^{-1}$

(注意: $\because R$ 是整环, F 是域, $f: R \rightarrow F$ 是单同态 $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$)

$\because y \in R$ 且 $y \neq 0_R$ $\therefore f(y) \in F^\times$ $\therefore (f(y))^{-1} \in F^\times$, $(f(y))^{-1} \in F$

$\therefore x \in R$ $\therefore f(x) \in F$ $\therefore f(x)(f(y))^{-1} \in F$)

\therefore 定义映射 $\theta: F \longrightarrow A$

$$\lambda = f(x)(f(y))^{-1} \mapsto \varphi(x)(\varphi(y))^{-1}$$

$\therefore y \in R$ 且 $y \neq 0_R$ $\therefore \varphi(y) \in A^\times$ $\therefore (\varphi(y))^{-1} \in A^\times$, $(\varphi(y))^{-1} \in A$

$\therefore x \in R$ $\therefore \varphi(x) \in A$ $\therefore \varphi(x)(\varphi(y))^{-1} \in A$

$\therefore \theta(\lambda) = \varphi(x)(\varphi(y))^{-1} \in A$ $\therefore \theta(F) \subseteq A$

$\forall \lambda_1, \lambda_2 \in F$, 若 $\lambda_1 = \lambda_2$, 则有:

$$\because \lambda_1 \in F \quad \exists x_1, y_1 \in R \text{ 且 } y_1 \neq 0_R, \text{ s.t. } \lambda_1 = f(x_1)(f(y_1))^{-1}$$

$$\because \lambda_2 \in F \quad \exists x_2, y_2 \in R \text{ 且 } y_2 \neq 0_R, \text{ s.t. } \lambda_2 = f(x_2)(f(y_2))^{-1}$$

$$\because \lambda_1 = \lambda_2 \quad \therefore f(x_1)(f(y_1))^{-1} = f(x_2)(f(y_2))^{-1}$$

$$\therefore f(x_1)f(y_2) = f(x_2)f(y_1) \quad \therefore f(x_1y_2) = f(x_2y_1)$$

$$\therefore x_1y_2 = x_2y_1 \quad \therefore \varphi(x_1y_2) = \varphi(x_2y_1) \quad \therefore \varphi(x_1)\varphi(y_2) = \varphi(x_2)\varphi(y_1)$$

$$\because y_1 \in R \text{ 且 } y_1 \neq 0_R \quad \therefore \varphi(y_1) \in A^X \quad \therefore (\varphi(y_1))^{-1} \in A$$

$$\because y_2 \in R \text{ 且 } y_2 \neq 0_R \quad \therefore \varphi(y_2) \in A^X \quad \therefore (\varphi(y_2))^{-1} \in A$$

$$\therefore \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1}$$

$$\therefore \theta(\lambda_1) = \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1} = \theta(\lambda_2)$$

$\therefore \theta: F \rightarrow A$ 是一个映射.

$\forall \lambda_1, \lambda_2 \in F$.

$$\because \lambda_1 \in F \quad \exists x_1, y_1 \in R \text{ 且 } y_1 \neq 0_R, \text{ s.t. } \lambda_1 = f(x_1)(f(y_1))^{-1}$$

$$\because \lambda_2 \in F \quad \exists x_2, y_2 \in R \text{ 且 } y_2 \neq 0_R, \text{ s.t. } \lambda_2 = f(x_2)(f(y_2))^{-1}$$

R 是整环, F 是域, $f: R \rightarrow F$ 是单同态 $\therefore f(R \setminus \{0_R\}) \subseteq F^X$

$$\because y_1 \in R \setminus \{0_R\} \text{ 且 } y_2 \in R \setminus \{0_R\} \quad \therefore f(y_1) \in F^X \text{ 且 } f(y_2) \in F^X$$

$$\therefore (f(y_1)f(y_2))^{-1} = (f(y_2))^{-1}(f(y_1))^{-1} = (f(y_1))^{-1}(f(y_2))^{-1}$$

$$\therefore \lambda_1 + \lambda_2 = f(x_1)(f(y_1))^{-1} + f(x_2)(f(y_2))^{-1}$$

$$= f(x_1)f(y_2)(f(y_1))^{-1}(f(y_2))^{-1} + f(x_2)f(y_1)(f(y_1))^{-1}(f(y_2))^{-1}$$

$$= (f(x_1)f(y_2) + f(x_2)f(y_1))(f(y_1))^{-1}(f(y_2))^{-1}$$

$$= f(x_1y_2 + x_2y_1)(f(y_1y_2))^{-1}$$

$$\therefore \theta(\lambda_1 + \lambda_2) = \varphi(x_1y_2 + x_2y_1)(\varphi(y_1y_2))^{-1}$$

$$= (\varphi(x_1)\varphi(y_2) + \varphi(x_2)\varphi(y_1))(\varphi(y_1))^{-1}(\varphi(y_2))^{-1}$$

$$= \varphi(x_1)(\varphi(y_1))^{-1} + \varphi(x_2)(\varphi(y_2))^{-1}$$

$$= \theta(\lambda_1) + \theta(\lambda_2)$$

$$\because \lambda_1\lambda_2 = f(x_1)(f(y_1))^{-1}f(x_2)(f(y_2))^{-1}$$

$$= f(x_1x_2)(f(y_1y_2))^{-1}$$

$$\therefore \theta(\lambda_1\lambda_2) = \varphi(x_1x_2)(\varphi(y_1y_2))^{-1}$$

$$= \varphi(x_1)\varphi(x_2)(\varphi(y_2))^{-1}(\varphi(y_1))^{-1}$$

$$= \varphi(x_1)(\varphi(y_1))^{-1} \cdot \varphi(x_2)(\varphi(y_2))^{-1}$$

$$= \theta(\lambda_1) \cdot \theta(\lambda_2)$$

$\because R$ 是整环 $\therefore R$ 是非零环 $\therefore I_R \neq 0_R$

$$\therefore I_R \in R, I_R \in R \text{ 且 } I_R \neq 0_R, f(I_R)(f(I_R))^{-1} = I_F \cdot I_F^{-1} = I_F \cdot I_F = I_F$$

$$\therefore I_F \in F \text{ 且 } I_F = f(I_R)(f(I_R))^{-1}$$

$$\therefore \theta(I_F) = \varphi(I_R)(\varphi(I_R))^{-1} = I_A \cdot I_A^{-1} = I_A \cdot I_A = I_A$$

$\therefore \theta: F \rightarrow A$ 是一个环同态

$\because f: R \rightarrow F$ 是一个单同态, $\theta: F \rightarrow A$ 是一个环同态

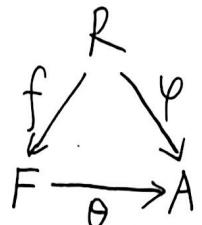
$\therefore \theta \circ f: R \rightarrow A$ 是一个环同态. $\varphi: R \rightarrow A$ 是一个环同态

对 $\forall x \in R$, 有: $f(x) \in F$

$\therefore x \in R, 1_R \in R, 1_R \neq 0_R, f(x)(f(1_R))^{-1} = f(x) \cdot |_F^{-1} = f(x) \cdot |_F = f(x)$

$\therefore f(x) = f(x)(f(1_R))^{-1}$

$$\begin{aligned}\therefore (\theta \circ f)(x) &= \theta(f(x)) = \varphi(x)(\varphi(1_R))^{-1} = \varphi(x) \cdot |_A^{-1} = \varphi(x) \cdot |_A \\ &= \varphi(x)\end{aligned}$$

$\therefore \theta \circ f = \varphi$ \therefore 下图交换:  存在性得证.

假设存在环同态 $\theta_1: F \rightarrow A$ s.t. 下图交换



假设还存在环同态 $\theta_2: F \rightarrow A$ s.t. 下图交换



则有: 对 $\forall \lambda \in F$, $\exists x, y \in R$ 且 $y \neq 0_R$, s.t. $\lambda = f(x)(f(y))^{-1}$

$\because R$ 是整环, F 是域, $f: R \rightarrow F$ 是单同态 $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$

$\therefore y \in R$ 且 $y \neq 0_R$ $\therefore f(y) \in F^\times$ $\therefore (f(y))^{-1} \in F^\times$ 且 $(f(y))^{-1} \in F$

$$\therefore \theta_1(\lambda) = \theta_1(f(x)(f(y))^{-1}) = \theta_1(f(x)) \theta_1((f(y))^{-1})$$

$$= (\theta_1 \circ f)(x) \cdot (\theta_1(f(y)))^{-1} = \varphi(x)(\varphi(y))^{-1}$$

$$= \theta_2(f(x)) \cdot (\theta_2(f(y)))^{-1} = \theta_2(f(x)) \cdot \theta_2((f(y))^{-1})$$

$$= \theta_2(f(x)(f(y))^{-1}) = \theta_2(\lambda)$$

$\therefore \theta_1 = \theta_2 \quad \therefore$ 唯一性得证.

$$\textcircled{2} \Rightarrow \textcircled{1}: \text{定义 } K = \left\{ f(x)(f(y))^{-1} \mid x, y \in R \text{ 且 } y \neq 0_R \right\}$$

$\because R$ 是整环, F 是域, $f: R \rightarrow F$ 是单同态 $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$

对 $\forall x, y \in R$ 且 $y \neq 0_R$, $\because y \in R$ 且 $y \neq 0_R \quad \therefore y \in R \setminus \{0_R\} \quad \therefore f(y) \in F^\times$

$\therefore (f(y))^{-1} \in F^\times, (f(y))^{-1} \in F \quad \therefore x \in R \quad \therefore f(x) \in F$

$\therefore f(x)(f(y))^{-1} \in F \quad \therefore K \subseteq F$

对 $\forall x \in R$, $\because R$ 是整环 $\therefore R$ 是非零环 $\therefore 1_R \in R$ 且 $1_R \neq 0_R$

$\therefore f(x)(f(1_R))^{-1} \in K \quad \therefore f(x)(f(1_R))^{-1} = f(x) \cdot 1_F^{-1} = f(x) \cdot 1_F = f(x)$

$\therefore f(x) \in K \quad \therefore f(R) \subseteq K \subseteq F$.

对 $\forall 0_R \in R, \forall y \in R$ 且 $y \neq 0_R$, 有: $f(0_R)(f(y))^{-1} \in K$

$\therefore f(0_R)(f(y))^{-1} = 0_F \cdot (f(y))^{-1} = 0_F \quad \therefore 0_F \in K$

对于 $1_R \in R, 1_R \in R$ 且 $1_R \neq 0_R$, 有: $f(1_R)(f(1_R))^{-1} \in K$

$\therefore f(1_R)(f(1_R))^{-1} = 1_F \cdot 1_F^{-1} = 1_F \cdot 1_F = 1_F \quad \therefore 1_F \in K$

$\therefore F$ 是域, $K \subseteq F$, K 是 F 的一个非空子集, $0_F \in K, 1_F \in K$

对 $\forall f(x_1)(f(y_1))^{-1} \in K, f(x_2)(f(y_2))^{-1} \in K$ (其中 $x_1, y_1, x_2, y_2 \in R$ 且 $y_1 \neq 0_R, y_2 \neq 0_R$), 有:

$$\begin{aligned}
& f(x_1)(f(y_1))^{-1} - f(x_2)(f(y_2))^{-1} \\
&= f(x_1)f(y_2)(f(y_2))^{-1}(f(y_1))^{-1} - f(x_2)f(y_1)(f(y_1))^{-1}(f(y_2))^{-1} \\
&= f(x_1y_2)(f(y_1)f(y_2))^{-1} - f(x_2y_1)(f(y_1)f(y_2))^{-1} \\
&= \left(f(x_1y_2) - f(x_2y_1) \right) (f(y_1y_2))^{-1} \\
&= f(x_1y_2 - x_2y_1)(f(y_1y_2))^{-1} \in K
\end{aligned}$$

($\because R$ 是整环, $y_1, y_2 \in R$, $y_1 \neq 0_R$ 且 $y_2 \neq 0_R$ $\therefore y_1y_2 \in R$ 且 $y_1y_2 \neq 0_R$)

对 $\forall f(x_1)(f(y_1))^{-1} \in K$, $\forall f(x_2)(f(y_2))^{-1} \in K \setminus \{0_F\}$, 有:

$$\therefore f(x_2)(f(y_2))^{-1} \in K \setminus \{0_F\} \quad \therefore f(x_2)(f(y_2))^{-1} \neq 0_F$$

假设 $x_2 = 0_R$, 则有 $f(x_2) = f(0_R) = 0_F$

$$\therefore f(x_2)(f(y_2))^{-1} = 0_F \cdot (f(y_2))^{-1} = 0_F \quad \text{矛盾.} \quad \therefore x_2 \neq 0_R$$

$$\therefore x_2 \in R \text{ 且 } x_2 \neq 0_R \quad \therefore f(x_2) \in F^\times \quad \therefore (f(x_2))^{-1} \in F^\times \text{ 且 } (f(x_2))^{-1} \in F$$

$$\therefore f(x_1)(f(y_1))^{-1} \cdot \left(f(x_2)(f(y_2))^{-1} \right)^{-1}$$

$$= f(x_1)(f(y_1))^{-1} \cdot f(y_2)(f(x_2))^{-1}$$

$$= f(x_1)f(y_2)(f(y_1))^{-1}(f(x_2))^{-1}$$

$$= f(x_1y_2)(f(x_2y_1))^{-1} \in K$$

($\because R$ 是整环, $x_2, y_1 \in R$, $x_2 \neq 0_R$ 且 $y_1 \neq 0_R$ $\therefore x_2y_1 \neq 0_R$ 且 $x_2y_1 \in R$)

$\therefore K$ 是 F 的子域

$\because F$ 是域 $\therefore F$ 是交换环 $\therefore f: R \rightarrow F$ 是单同态 且有 $f(R \setminus \{0_R\}) \subseteq F^\times$

\therefore 存在唯一的环同态 $\alpha: F \rightarrow F$ s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \downarrow f \\ F & \xrightarrow{\alpha} & F \end{array}$$

$\therefore id_F: F \rightarrow F$ 是环同构 $x \mapsto x$ $\therefore id_F: F \rightarrow F$ 是环同态

$\therefore id_F \circ f = f$ \therefore 下图交换: $\therefore \alpha = id_F$

$$\begin{array}{ccc} & R & \\ f \swarrow & & \downarrow f \\ F & \xrightarrow{id_F} & F \end{array}$$

$\therefore K$ 是 F 的子域 $\therefore K$ 是域 $\therefore K$ 是交换环

~~全重~~ 定义映射 $\psi: R \rightarrow K$
 $x \mapsto f(x)$

对 $\forall x \in R$, $\psi(x) = f(x) \in f(R) \subseteq K \quad \therefore \psi(R) \subseteq K$

对 $\forall x_1, x_2 \in R$, 若 $x_1 = x_2$, 则有: $\psi(x_1) = f(x_1) = f(x_2) = \psi(x_2)$

$\therefore \psi: R \rightarrow K$ 是一个映射.

对 $\forall x_1, x_2 \in R$,

$$\psi(x_1 + x_2) = f(x_1 + x_2) = f(x_1) + f(x_2) = \psi(x_1) + \psi(x_2)$$

$$\psi(x_1 x_2) = f(x_1 x_2) = f(x_1) f(x_2) = \psi(x_1) \psi(x_2)$$

$\psi(1_R) = f(1_R) = 1_F$ 就是 K 的乘法幺元

$\therefore \psi: R \rightarrow K$ 是一个环同态

对 $\forall x \in R \setminus \{0_R\}$, 有: $\psi(x) = f(x) \in F^\times = F \setminus \{0_F\}$

$\therefore \psi(x) \in K$ 且 $\psi(x) \neq 0_F \quad \therefore \psi(x) \in K \setminus \{0_F\} = K^\times \quad \therefore \psi(R \setminus \{0_R\}) \subseteq K^\times$

∴ 存在唯一的环同态 $\beta: F \rightarrow K$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & \downarrow \psi & \\ F & \xrightarrow{\beta} & K \end{array}$$

∴ F 是域, K 是 F 的子域 ∴ $\iota: K \rightarrow F$ 是单同态.
 $x \mapsto x$

∴ $\beta: F \rightarrow K$ 是一个环同态, $\iota: K \rightarrow F$ 是一个单同态
 $x \mapsto x$

∴ $\iota \circ \beta: F \rightarrow F$ 是一个环同态

∴ $\psi: R \rightarrow K$ 是一个环同态, $\iota: K \rightarrow F$ 是一个单同态
 $x \mapsto x$

∴ $\iota \circ \psi: R \rightarrow F$ 是一个环同态. ∵ $f: R \rightarrow F$ 是一个单同态

对 $\forall x \in R$, 有: $(\iota \circ \psi)(x) = \iota(\psi(x)) = \iota(f(x)) = f(x)$

∴ $\iota \circ \psi = f$

∴ $(\iota \circ \beta) \circ f = \iota \circ (\beta \circ f) = \iota \circ \psi = f$

∴ 下图交换: $\begin{array}{ccc} & R & \\ f \swarrow & \downarrow f & \\ F & \xrightarrow{\iota \circ \beta} & F \end{array}$ ∴ $\iota \circ \beta = \alpha = \text{id}_F$

∴ 对 $\forall x \in F$, 有: $x = \text{id}_F(x) = (\iota \circ \beta)(x) = \iota(\beta(x)) = \beta(x) \in K$

∴ $F \subseteq K$ ∴ $F = K = \{f(x)(f(y))^{-1} \mid x, y \in R \text{ 且 } y \neq 0_R\}$

① 得证. \square