Lemma: R是环，对$\forall m\in\mathbb{Z}$，$\forall r\in R$，$\forall x\in R$，有:

① $r(mx) = m(rx)$

② $(mx)r = m(xr)$

Proof: ① 当$m\in\mathbb{Z}_{\geqslant 1}$时，

$$r(mx) = r(\underbrace{x+x+\cdots+x}_{m个x}) = \underbrace{rx+rx+\cdots+rx}_{m个rx} = m(rx)$$

当$m=0$时，

$$r(mx) = r(0\cdot x) = r\cdot 0_R = 0_R = 0\cdot(rx) = m(rx)$$

当$m\in\mathbb{Z}_{\leqslant -1}$时，$-m\in\mathbb{Z}_{\geqslant 1}$

$$\therefore r(mx) = r((-(-m))x) = r(-((-m)x)) = -(r((-m)x))$$

$$= -((-m)(rx)) = m(rx)$$

$\therefore$ 对$\forall m\in\mathbb{Z}$，有: $r(mx) = m(rx)$

②: Lecture_Notes_in_Algebra_WenWeiLi_20250225_20250717/第三章/3.1/2 笔记已证

$\square$

定义（环的子集的和）R是环，$A_1, A_2, \cdots, A_n$是R的非空子集，定义：

$$A_1+A_2+\cdots+A_n := \{a_1+a_2+\cdots+a_n \mid a_1\in A_1,\ a_2\in A_2,\ \cdots,\ a_n\in A_n\} \subseteq R$$

定义（环的子集的积）R是环，S和T是R的非空子集，定义：

$$ST := \left\{\sum_{i=1}^{n} x_i y_i \,\middle|\, n\in\mathbb{Z}_{\geqslant 1},\ 对\forall i=1,2,\cdots,n,\ x_i\in S,\ y_i\in T\right\} \subseteq R$$

定义（整数集与环的子集的积）R是环，X是R的非空子集，定义

$$\mathbb{Z}X := \left\{\sum_{i=1}^{n} m_i x_i \,\middle|\, n\in\mathbb{Z}_{\geqslant 1},\ 对\forall i=1,2,\cdots,n,\ m_i\in\mathbb{Z},\ x_i\in X\right\} \subseteq R$$

$$RX := \left\{\sum_{i=1}^{n} r_i x_i \,\middle|\, n\in\mathbb{Z}_{\geqslant 1},\ 对\forall i=1,2,\cdots,n,\ r_i\in R,\ x_i\in X\right\} \subseteq R$$

$$XR := \left\{\sum_{i=1}^{n} x_i r_i \,\middle|\, n\in\mathbb{Z}_{\geqslant 1},\ 对\forall i=1,2,\cdots,n,\ x_i\in X,\ r_i\in R\right\} \subseteq R$$

$$RXR := \left\{\sum_{i=1}^{n} r_i x_i r_i' \,\middle|\, n\in\mathbb{Z}_{\geqslant 1},\ 对\forall i=1,2,\cdots,n,\ r_i\in R,\ x_i\in X,\ r_i'\in R\right\} \subseteq R$$

$\because$ X是R的非空子集 $\quad\therefore \exists\, x_1\in X\subseteq R$

$\therefore$ $0\cdot x_1 = 0_R \in \mathbb{Z}X$ $\quad\therefore$ $\mathbb{Z}X$是R的非空子集

$\therefore$ $0_R\cdot x_1 = 0_R \in RX$ $\quad\therefore$ RX是R的非空子集

$\because$ $x_1\cdot 0_R = 0_R\in XR$ $\quad\therefore$ XR是R的非空子集

$\because$ $0_R\cdot x_1\cdot 0_R = 0_R\cdot 0_R = 0_R \in RXR$ $\quad\therefore$ RXR是R的非空子集

$\therefore$ $\mathbb{Z}X+RX+XR+RXR$是R的非空子集.

Lemma：R是环，X是R的非空子集，则有：$\mathbb{Z}X+RX+XR+RXR$是R的包含X的最小理想.

proof: 对$\forall x\in X$，$x = 0\cdot x + 1_R\cdot x + x\cdot 0_R + 0_R\cdot x\cdot 0_R$

$\therefore x\in \mathbb{Z}X + RX + XR + RXR$ $\quad\therefore X\subseteq \mathbb{Z}X+RX+XR+RXR$

对$\forall \alpha,\beta\in \mathbb{Z}X+RX+XR+RXR$，有：

$$\alpha = \sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} r_i' x_i' + \sum_{i=1}^{n} x_i'' r_i'' + \sum_{i=1}^{n} r_i''' x_i''' r_i''''$$

（4个求和的上限都是n，因为可以让 $m_i=0$，$r_i'=0_R$，$r_i''=0_R$，$r_i'''=r_i''''=0_R$）

$$\beta = \sum_{i=1}^{n} \hat{m}_i \hat{x}_i + \sum_{i=1}^{n} \hat{r}_i' \hat{x}_i' + \sum_{i=1}^{n} \hat{x}_i'' \hat{r}_i'' + \sum_{i=1}^{n} \hat{r}_i''' \hat{x}_i''' \hat{r}_i''''$$

$$\therefore \alpha+\beta = \left(\sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n}\hat{m}_i\hat{x}_i\right) + \left(\sum_{i=1}^{n} r_i'x_i' + \sum_{i=1}^{n}\hat{r}_i'\hat{x}_i'\right) + \left(\sum_{i=1}^{n} x_i''r_i'' + \sum_{i=1}^{n}\hat{x}_i''\hat{r}_i''\right)$$
$$+ \left(\sum_{i=1}^{n} r_i'''x_i'''r_i'''' + \sum_{i=1}^{n}\hat{r}_i'''\hat{x}_i'''\hat{r}_i''''\right) \in \mathbb{Z}X+RX+XR+RXR$$

对$\forall r\in R$，

任取 $r(\mathbb{Z}X+RX+XR+RXR)$ 中的一个元素 $r\alpha$（其中$\alpha\in\mathbb{Z}X+RX+XR+RXR$）

$$\therefore \alpha = \sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} r_i' x_i' + \sum_{i=1}^{n} x_i'' r_i'' + \sum_{i=1}^{n} r_i''' x_i''' r_i''''$$

$$\therefore r\alpha = r\left(\sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} r_i' x_i' + \sum_{i=1}^{n} x_i'' r_i'' + \sum_{i=1}^{n} r_i''' x_i''' r_i''''\right)$$
$$= \sum_{i=1}^{n} r(m_i x_i) + \sum_{i=1}^{n} r(r_i'x_i') + \sum_{i=1}^{n} r(x_i''r_i'') + \sum_{i=1}^{n} r(r_i'''x_i'''r_i'''')$$
$$= \sum_{i=1}^{n} m_i(rx_i) + \sum_{i=1}^{n} (rr_i')x_i' + \sum_{i=1}^{n} r x_i'' r_i'' + \sum_{i=1}^{n} (rr_i''')x_i'''r_i''''$$
$$= 0\cdot x_1 + \left(\sum_{i=1}^{n} (m_i r)x_i + \sum_{i=1}^{n}(rr_i')x_i'\right) + x_1\cdot 0_R + \left(\sum_{i=1}^{n} r x_i'' r_i'' + \sum_{i=1}^{n}(rr_i''')x_i'''r_i''''\right)$$
$$\in \mathbb{Z}X+RX+XR+RXR$$

$$\therefore r(\mathbb{Z}X+RX+XR+RXR)\subseteq \mathbb{Z}X+RX+XR+RXR$$

任取 $(\mathbb{Z}X+RX+XR+RXR)r$ 中的一个元素 αr（其中 $\alpha\in\mathbb{Z}X+RX+XR+RXR$）

$\therefore \alpha=\sum\limits_{i=1}^{n}m_ix_i+\sum\limits_{i=1}^{n}r_i'x_i'+\sum\limits_{i=1}^{n}x_i''r_i''+\sum\limits_{i=1}^{n}r_i'''x_i'''r_i''''$

$$\therefore \alpha r=\left(\sum_{i=1}^{n}m_ix_i+\sum_{i=1}^{n}r_i'x_i'+\sum_{i=1}^{n}x_i''r_i''+\sum_{i=1}^{n}r_i'''x_i'''r_i''''\right)r$$

$$=\sum_{i=1}^{n}(m_ix_i)r+\sum_{i=1}^{n}r_i'x_i'r+\sum_{i=1}^{n}x_i''(r_i''r)+\sum_{i=1}^{n}r_i'''x_i'''(r_i''''r)$$

$$=\sum_{i=1}^{n}x_i(m_ir)+\sum_{i=1}^{n}x_i''(r_i''r)+\sum_{i=1}^{n}r_i'x_i'r+\sum_{i=1}^{n}r_i'''x_i'''(r_i''''r)$$

$$=0\cdot x_1+0_R\cdot x_1+\left(\sum_{i=1}^{n}x_i(m_ir)+\sum_{i=1}^{n}x_i''(r_i''r)\right)+\left(\sum_{i=1}^{n}r_i'x_i'r+\sum_{i=1}^{n}r_i'''x_i'''(r_i''''r)\right)$$

$$\in\mathbb{Z}X+RX+XR+RXR$$

$\therefore (\mathbb{Z}X+RX+XR+RXR)r\subseteq\mathbb{Z}X+RX+XR+RXR$

$\therefore$ $\mathbb{Z}X+RX+XR+RXR$ 是 R 的理想.

$\therefore$ $\mathbb{Z}X+RX+XR+RXR$ 是 R 的包含 X 的理想.

~~设I是~~ ~~设D是R的包含X的~~

设 D 是 R 的一个任意的包含 X 的理想，则有：

D 是 R 的理想且 $X\subseteq D$

对 $\forall\alpha\in\mathbb{Z}X+RX+XR+RXR$，有：$\alpha=\sum\limits_{i=1}^{n}m_ix_i+\sum\limits_{i=1}^{n}r_i'x_i'+\sum\limits_{i=1}^{n}x_i''r_i''+\sum\limits_{i=1}^{n}r_i'''x_i'''r_i''''$

对 $\forall i=1,2,\cdots,n$，$\because x_i\in X\subseteq D$ $\quad\therefore m_ix_i\in D$（分 $m_i\in\mathbb{Z}_{\geqslant 1}$，$m_i=0$，$m_i\in\mathbb{Z}_{\leqslant -1}$ 讨论）

$\therefore \sum\limits_{i=1}^{n}m_ix_i\in D$

对 $\forall i=1,2,\cdots,n$；$\because x_i'\in X\subseteq D$ $\quad\therefore r_i'x_i'\in r_i'D\subseteq D$ $\quad\therefore \sum\limits_{i=1}^{n}r_i'x_i'\in D$

对 $\forall i=1,2,\cdots,n$ $\quad\because x_i''\in X\subseteq D$ $\quad\therefore x_i''r_i''\in Dr_i''\subseteq D$ $\quad\therefore \sum\limits_{i=1}^{n}x_i''r_i''\in D$

对 $\forall i=1,2,\cdots,n$ $\quad\because x_i'''\in X\subseteq D$ $\quad\therefore r_i'''x_i'''\in r_i'''D\subseteq D$ $\quad\therefore r_i'''x_i'''r_i''''\in Dr_i''''\subseteq D$

$\therefore \sum\limits_{i=1}^{n}r_i'''x_i'''r_i''''\in D$

$\therefore \alpha = \sum_{i=1}^{n} m_i x_i + \sum_{i=1}^{n} r_i' x_i' + \sum_{i=1}^{n} x_i'' r_i'' + \sum_{i=1}^{n} r_i''' x_i''' r_i'''' \in D$

$\therefore \mathbb{Z}X + RX + XR + RXR \subseteq D$.

$\therefore \mathbb{Z}X + RX + XR + RXR$ 是 R 的包含 X 的最小理想. $\square$

定义（环的子集生成的理想）R 是环，X 是 R 的非空子集，则称 R 的包含 X 的最小理想为 X 生成的理想，记作 $\langle X \rangle$.

由上面的引理得：$\langle X \rangle = \mathbb{Z}X + RX + XR + RXR$.