定理（商环的泛性质，R/I 满足相应的泛性质）R是环，I是R的理想. 已经证明了 R/I是环，商同态 $q: R \longrightarrow R/I$ 是满同态，

$$x \longmapsto x+I$$

且有 $I = \ker(q)$. 则有：对$\forall$环A，$\forall$环同态 $\varphi: R \to A$ 满足 $I \subseteq \ker(\varphi)$，存在唯一的环同态 $\theta: R/I \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd}
 & R \arrow[dl, "q"'] \arrow[dr, "\varphi"] & \\
R/I \arrow[rr, "\theta"'] & & A
\end{tikzcd}$$

Proof: $\because R/I = \{x+I \mid x \in R\}$

$\therefore$ 定义映射 $\theta: R/I \longrightarrow A$

$$x+I \longmapsto \varphi(x)$$

对$\forall\ x+I \in R/I$ （其中 $x \in R$），$\because x \in R$，$\varphi: R \to A$是环同态

$\therefore \varphi(x) \in A \quad \therefore \theta(x+I) = \varphi(x) \in A \quad \therefore \theta(R/I) \subseteq A$

对$\forall\ x+I, y+I \in R/I$ （其中$x, y \in R$），若 $x+I = y+I$，则有：

$x \equiv_I y \quad \therefore x-y \in I \quad \because I \subseteq \ker(\varphi) \quad \therefore x-y \in \ker(\varphi)$

$\therefore x-y \in R$ 且 $\varphi(x-y) = 0_A$

$\because \varphi(x-y) = \varphi(x+(-y)) = \varphi(x) + \varphi(-y) = \varphi(x) + (-\varphi(y))$

$\therefore \varphi(x) + (-\varphi(y)) = 0_A$

$\therefore \varphi(x) = \varphi(x) + 0_A = \varphi(x) + \big((-\varphi(y)) + \varphi(y)\big) = \big(\varphi(x) + (-\varphi(y))\big) + \varphi(y)$

$= 0_A + \varphi(y) = \varphi(y)$

$\therefore \theta(x+I) = \varphi(x) = \varphi(y) = \theta(y+I)$

$\therefore \theta : R/I \longrightarrow A$ 是一个映射、

对$\forall\, x+I,\ y+I \in R/I$ （其中 $x, y \in R$），

$$\theta\big((x+I)+(y+I)\big) = \theta\big((x+y)+I\big) = \varphi(x+y) = \varphi(x)+\varphi(y)$$

$$= \theta(x+I) + \theta(y+I)$$

$$\theta\big((x+I)\cdot(y+I)\big) = \theta(xy+I) = \varphi(xy) = \varphi(x)\cdot\varphi(y)$$

$$= \theta(x+I)\cdot\theta(y+I)$$

$$\theta(1_{R/I}) = \theta(1_R + I) = \varphi(1_R) = 1_A$$

$\therefore \theta : R/I \longrightarrow A$ 是一个环同态.

$\because q : R \rightarrow R/I$ 是满同态， $\theta : R/I \longrightarrow A$ 是环同态

$\therefore \theta \circ q : R \longrightarrow A$ 是环同态 $\quad \because \varphi : R \rightarrow A$ 是环同态.

对$\forall\, x \in R$， $(\theta\circ q)(x) = \theta(q(x)) = \theta(x+I) = \varphi(x)$

$\therefore \theta \circ q = \varphi$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "q"'] \arrow[dr, "\varphi"] & \\ R/I \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

存在性得证.

假设存在环同态 $\theta_1: R/I \to A$，环同态 $\theta_2: R/I \to A$，s.t.

下图交换：

$$
\begin{tikzcd}
& R \arrow[dl, "q"'] \arrow[dr, "\varphi"] & \\
R/I \arrow[rr, "\theta_1"'] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \arrow[dl, "q"'] \arrow[dr, "\varphi"] & \\
R/I \arrow[rr, "\theta_2"'] & & A
\end{tikzcd}
$$

则对 $\forall\, x+I \in R/I$（其中 $x \in R$），有：

$$
\begin{aligned}
\theta_1(x+I) &= \theta_1(q(x)) = (\theta_1 \circ q)(x) = \varphi(x) = (\theta_2 \circ q)(x) \\
&= \theta_2(q(x)) = \theta_2(x+I)
\end{aligned}
$$

$\therefore \theta_1 = \theta_2$ $\therefore$ 唯一性得证. $\square$