Lemma: R是环, 对$\forall x, y \in R$, 有: $(-x)+(-y) = -(x+y)$

Proof: $\because (x+y)+((-x)+(-y)) = (x+(-x))+(y+(-y))$

$= 0_R + 0_R = 0_R$

$\therefore -(x+y) = (-x)+(-y)$ $\square$

Lemma: R是环, 对$\forall x, y, z \in R$, 有:

$$(x-y)z = xz - yz \quad , \quad z(x-y) = zx - zy$$

Proof: $(x-y)z = (x+(-y))z = xz + (-y)z = xz + (-yz)$

$= xz - yz$

$z(x-y) = z(x+(-y)) = zx + z(-y) = zx + (-zy)$

$= zx - zy$ $\square$

定义（商环）R是环，I是R的理想，定义

$$R/I := R/\equiv_I = \{x+I \mid x\in R\}$$

在R/I上定义

▷ 加法：对$\forall\ x+I,\ y+I\in R/I$，

$$(x+I)+(y+I) := (x+y)+I$$

▷ 乘法：对$\forall\ x+I,\ y+I\in R/I$，

$$(x+I)\cdot(y+I) := xy+I$$

▷ 加法零元：$0_{R/I} := I = 0_R + I$

▷ 乘法幺元：$1_{R/I} := 1_R + I$

Lemma：$(R/I,\ +,\ \cdot,\ 0_{R/I},\ 1_{R/I})$是环.

Proof：对$\forall\ x+I,\ y+I\in R/I$（其中$x,y\in R$）

$\because x,y\in R \quad \therefore x+y\in R \quad \therefore (x+y)+I\in R/I$

$\therefore (x+I)+(y+I) = (x+y)+I \in R/I$

$\because x,y\in R \quad \therefore xy\in R \quad \therefore xy+I\in R/I$

$\therefore (x+I)\cdot(y+I) = xy+I\in R/I$

对 $\forall\, x+I,\ x'+I,\ y+I,\ y'+I \in R/I$,

若 $x+I = x'+I$ 且 $y+I = y'+I$, 则有:

$\because x+I = x'+I \quad \therefore x \equiv_I x' \quad \therefore x-x' \in I$

$\because y+I = y'+I \quad \therefore y \equiv_I y' \quad \therefore y-y' \in I$

$\therefore (x-x')+(y-y') \in I$

$\because (x-x')+(y-y') = x+(-x')+y+(-y') = (x+y)+((-x')+(-y'))$

$= (x+y)+(-(x'+y')) = (x+y)-(x'+y')$

$\therefore (x+y)-(x'+y') \in I \quad \therefore (x+y) \equiv_I (x'+y')$

$\therefore (x+y)+I = (x'+y')+I$

$\therefore (x+I)+(y+I) = (x+y)+I = (x'+y')+I$

$= (x'+I)+(y'+I)$

$\therefore R/I$ 中的加法运算良定义.

$\because x \in R,\ y-y' \in I \quad \therefore x(y-y') \in xI \subseteq I$

$\because y' \in R,\ x-x' \in I \quad \therefore (x-x')y' \in Iy' \subseteq I$

$\therefore x(y-y')+(x-x')y' \in I$

$\because x(y-y')+(x-x')y' = xy-xy'+xy'-x'y' = xy-x'y'$

$\therefore xy-x'y' \in I \quad \therefore xy \equiv_I x'y' \quad \therefore xy+I = x'y'+I$

$\therefore (x+I)(y+I) = xy+I = x'y'+I = (x'+I)(y'+I)$

$\therefore$ R/I 中的乘法运算良定义.

对 $\forall\, x+I,\ y+I,\ z+I \in R/I$,

$$\big((x+I)+(y+I)\big)+(z+I) = \big((x+y)+I\big)+(z+I)$$

$$= \big((x+y)+z\big)+I = \big(x+(y+z)\big)+I$$

$$= (x+I)+\big((y+z)+I\big) = (x+I)+\big((y+I)+(z+I)\big)$$

$\therefore$ R/I 中的加法满足结合律.

$$\big((x+I)(y+I)\big)(z+I) = (xy+I)(z+I) = (xy)z+I$$

$$= x(yz)+I = (x+I)(yz+I) = (x+I)\big((y+I)(z+I)\big)$$

$\therefore$ R/I 中的乘法满足结合律.

$$\big((x+I)+(y+I)\big)(z+I) = \big((x+y)+I\big)(z+I)$$

$$= (x+y)z+I = (xz+yz)+I = (xz+I)+(yz+I)$$

$$= (x+I)(z+I)+(y+I)(z+I)$$

$$(z+I)\big((x+I)+(y+I)\big) = (z+I)\big((x+y)+I\big)$$

$$= z(x+y)+I = (zx+zy)+I = (zx+I)+(zy+I)$$

$= (z+I)(x+I)+(z+I)(y+I)$ $\therefore$ R/I 中有乘法对加法的分配律.

对$\forall\ x+I,\ y+I\in R/I$，有：

$(x+I)+(y+I)=(x+y)+I=(y+x)+I=(y+I)+(x+I)$

$\therefore R/I$中的加法满足交换律.

对$\forall\lambda\in I$，有：$\lambda\in I\subseteq R$ $\quad\therefore\lambda\in R$

$\therefore\lambda=0_R+\lambda\in 0_R+I$ $\quad\therefore I\subseteq 0_R+I$

任取0_R+I中的一元：$0_R+\beta$（其中$\beta\in I$）

$\because 0_R+\beta=\beta\in I$ $\quad\therefore 0_R+I\subseteq I$ $\quad\therefore 0_R+I=I$.

对$\forall\ x+I\in R/I$，有：

$(x+I)+(0_R+I)=(x+0_R)+I=x+I$

$(0_R+I)+(x+I)=(0_R+x)+I=x+I$

$\therefore 0_R+I$是R/I的加法零元 $\quad\therefore 0_{R/I}=0_R+I=I$

$\because x+I\in R/I$ $\quad\therefore x\in R$ $\quad\therefore -x\in R$ $\quad\therefore(-x)+I\in R/I$

$\because(x+I)+((-x)+I)=(x+(-x))+I=0_R+I=0_{R/I}$

$\therefore -(x+I)=(-x)+I$ $\quad\therefore(-x)+I$是$x+I$的加法逆元.

$\because(x+I)(1_R+I)=x\cdot 1_R+I=x+I$

$(1_R+I)(x+I)=1_R\cdot x+I=x+I$

$\therefore 1_R+I$是R/I的乘法幺元. $\quad\therefore 1_{R/I}=1_R+I$.

$\therefore(R/I,+,\cdot,0_{R/I},1_{R/I})$是环. $\quad\square$

Lemma: R是交换环，I是R的理想，则有: R/I是交换环.

Proof: 对$\forall\ x+I,\ y+I \in R/I$（其中$x, y \in R$），有:

$$(x+I)(y+I) = xy+I = yx+I = (y+I)(x+I)$$

$\therefore R/I$是交换环. $\square$