

Lemma: R 是环, F 是除环, $\varphi: R \rightarrow F$ 是环同态, 则有:

$$\text{映射 } \varphi: R \rightarrow F \text{ 是单射} \Leftrightarrow \varphi(R \setminus \{0_R\}) \subseteq F^\times$$

Proof: (\Rightarrow) : 对 $\forall x \in R \setminus \{0_R\}$. 假设 $\varphi(x) = 0_F$, 则

$$\because \varphi: R \rightarrow F \text{ 是环同态} \quad \therefore \varphi(0_R) = 0_F \quad \therefore \varphi(x) = 0_F = \varphi(0_R)$$

$$\because \varphi: R \rightarrow F \text{ 是单射} \quad \therefore x = 0_R \quad \because x \in R \setminus \{0_R\} \quad \therefore x \neq 0_R \text{ 矛盾.}$$

$$\therefore \varphi(x) \neq 0_F \quad \therefore \varphi(x) \in F \setminus \{0_F\} = F^\times$$

$$\therefore \varphi(R \setminus \{0_R\}) \subseteq F^\times$$

(\Leftarrow) : 对 $\forall x_1, x_2 \in R$. 若 $\varphi(x_1) = \varphi(x_2)$, 则有:

$$\because x_1, x_2 \in R \quad \therefore x_1 - x_2 \in R. \quad \text{假设 } x_1 - x_2 \neq 0_R, \text{ 则有:}$$

$$x_1 - x_2 \in R \setminus \{0_R\} \quad \therefore \varphi(x_1 - x_2) \in F^\times$$

$$\because F \text{ 是除环} \quad \therefore F^\times = F \setminus \{0_F\} \quad \therefore \varphi(x_1 - x_2) \in F^\times = F \setminus \{0_F\}$$

$$\therefore \varphi(x_1 - x_2) \in F \text{ 且 } \varphi(x_1 - x_2) \neq 0_F$$

$$\because 0_F \neq \varphi(x_1 - x_2) = \varphi(x_1 + (-x_2)) = \varphi(x_1) + \varphi(-x_2)$$

$$= \varphi(x_2) + \varphi(-x_2) = \varphi(x_2) + (-\varphi(x_2)) = 0_F \quad \text{矛盾.}$$

$$\therefore x_1 - x_2 = 0_R$$

$$\therefore x_2 = 0_R + x_2 = (x_1 - x_2) + x_2 = (x_1 + (-x_2)) + x_2 = x_1 + ((-x_2) + x_2)$$

$$= x_1 + 0_R = x_1$$

$$\therefore x_1 = x_2 \quad \therefore \text{映射 } \varphi: R \rightarrow F \text{ 是单射.} \quad \square$$

推论: R 是整环, F 是域, $\varphi: R \rightarrow F$ 是环同态, 则有:

$$\text{映射 } \varphi: R \rightarrow F \text{ 是单射} \Leftrightarrow \varphi(R \setminus \{0_R\}) \subseteq F^\times$$

Proof: 由上一引理立得. \square

Lemma (分式域的泛性质) R 是一个任意的整环

F 是一个域, $f: R \rightarrow F$ 是一个单同态, 满足:

对 \forall 交换环 A , \forall 环同态 $\varphi: R \rightarrow A$ 满足 $\varphi(R \setminus \{0\}) \subseteq A^\times$, 存在唯一的环同态 $\theta: F \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow \varphi \\ F & \xrightarrow{\theta} & A \end{array}$$

F' 是另一个域, $f': R \rightarrow F'$ 是另一个单同态, 满足:

对 \forall 交换环 A , \forall 环同态 $\varphi: R \rightarrow A$ 满足 $\varphi(R \setminus \{0\}) \subseteq A^\times$, 存在唯一的环同态 $\theta': F' \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f' \swarrow & & \searrow \varphi \\ F' & \xrightarrow{\theta'} & A \end{array}$$

则存在唯一的映射 $\theta: F \rightarrow F'$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ F & \xrightarrow{\theta} & F' \end{array}$$

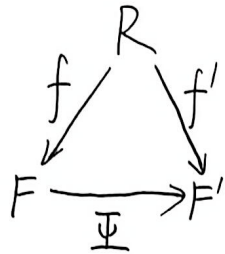
且 $\theta: F \rightarrow F'$ 是域同构.

Proof: $\because F'$ 是域 $\therefore F'$ 是交换环

$\because R$ 是整环, F' 是域, $f': R \rightarrow F'$ 是单同态

$$\therefore f'(R \setminus \{0\}) \subseteq (F')^\times$$

\therefore 存在唯一的环同态 $\psi: F \rightarrow F'$, s.t. 下图交换:

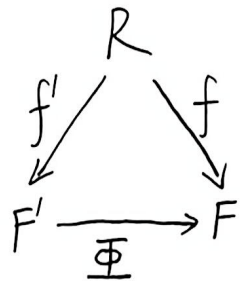


$\because F$ 是域 $\therefore F$ 是交换环

$\because R$ 是整环, F 是域, $f: R \rightarrow F$ 是单同态

$$\therefore f(R \setminus \{0\}) \subseteq F^\times$$

\therefore 存在唯一的环同态 $\psi: F' \rightarrow F$, s.t. 下图交换:

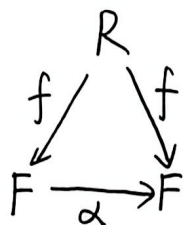


$\psi: F \rightarrow F'$ 是环同态, $\psi: F' \rightarrow F$ 是环同态

$\psi \circ \psi: F \rightarrow F$ 是环同态, $\psi \circ \psi: F' \rightarrow F'$ 是环同态.

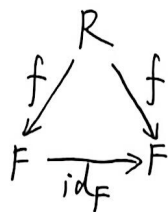
$\because F$ 是交换环, $f: R \rightarrow F$ 是环同态且满足 $f(R \setminus \{0\}) \subseteq F^\times$

\therefore 存在唯一的环同态 $\alpha: F \rightarrow F$ s.t. 下图交换:



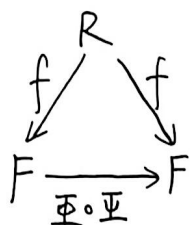
$\therefore \text{id}_F: F \rightarrow F$ 是环同构 $\therefore \text{id}_F: F \rightarrow F$ 是环同态

$\therefore \text{id}_F \circ f = f$ \therefore 下图交换: $\therefore \alpha = \text{id}_F$



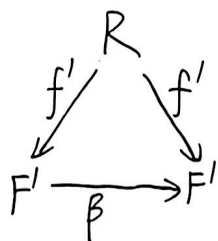
$$\therefore (\Phi \circ \Psi) \circ f = \Phi \circ (\Psi \circ f) = \Phi \circ f' = f$$

\therefore 下图交换: $\therefore \Phi \circ \Psi = \alpha = \text{id}_F$



$\therefore F'$ 是交换环, $f': R \rightarrow F'$ 是环同态且满足 $f'(R \setminus \{0\}) \subseteq (F')^\times$

\therefore 存在唯一的环同态 $\beta: F' \rightarrow F'$, s.t. 下图交换:



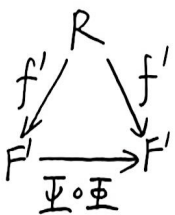
$\therefore \text{id}_{F'}: F' \rightarrow F'$ 是环同构 $\therefore \text{id}_{F'}: F' \rightarrow F'$ 是环同态

$\therefore \text{id}_{F'} \circ f' = f'$ \therefore 下图交换: $\therefore \beta = \text{id}_{F'}$



$$\therefore (\Psi \circ \Phi) \circ f' = \Psi \circ (\Phi \circ f') = \Psi \circ f = f'$$

\therefore 下图交换: $\therefore \Psi \circ \Phi = \beta = \text{id}_{F'}$

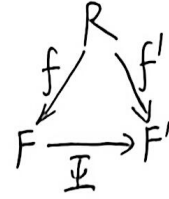


$\therefore \Psi: F \rightarrow F'$ 是环同态, $\Phi: F' \rightarrow F$ 是环同态, $\Phi \circ \Psi = \text{id}_F$,

$\Psi \circ \Phi = \text{id}_{F'}$ $\therefore \Psi: F \rightarrow F'$ 是可逆映射 $\therefore \Psi: F \rightarrow F'$ 是双射

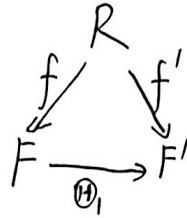
$\therefore \varphi: F \rightarrow F'$ 是环同构. $\therefore \varphi: F \rightarrow F'$ 是域同构.

$\therefore \varphi: F \rightarrow F'$ 是域同构, 且下图交换:



存在性得证.

假设存在映射 $\theta_1: F \rightarrow F'$, s.t. 下图交换:



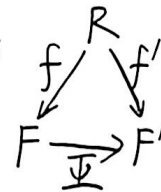
且 $\theta_1: F \rightarrow F'$ 是域同构

假设还存在映射 $\theta_2: F \rightarrow F'$ s.t. 下图交换:



且 $\theta_2: F \rightarrow F'$ 是域同构. 则有:

对于交换环 F' , 环同态 $f': R \rightarrow F'$ 满足 $f'(R \setminus \{0\}) \subseteq (F')^\times$, 存在唯一的环同态 $\varphi: F \rightarrow F'$, s.t. 下图交换:



$\therefore \theta_1 = \varphi = \theta_2$

\therefore 唯一性得证.

