整环与分式域

定义（Ratio(R)）R是一个任意的整环，定义集合

$$\mathrm{Ratio}(R) := \{(f,g)\in R^2 : g\neq 0\} \quad (R^2 := R\times R)$$

在Ratio(R)上定义二元关系：

对$\forall (f_1,g_1),(f_2,g_2)\in \mathrm{Ratio}(R)$，$(f_1,g_1)\sim(f_2,g_2) \Leftrightarrow f_1g_2=f_2g_1$

Lemma：$\sim$是Ratio(R)上的等价关系.

Proof：对$\forall (f,g)\in\mathrm{Ratio}(R)$ $\because fg=fg$

$\therefore (f,g)\sim(f,g)$ $\quad\therefore$ 反身性成立

对$\forall (f_1,g_1),(f_2,g_2)\in\mathrm{Ratio}(R)$. 若$(f_1,g_1)\sim(f_2,g_2)$，则有：

$\because (f_1,g_1)\sim(f_2,g_2)$ $\quad\therefore f_1g_2=f_2g_1$ $\quad\therefore f_2g_1=f_1g_2$

$\therefore (f_2,g_2)\sim(f_1,g_1)$ $\quad\therefore$ 对称性成立.

对$\forall (f_1,g_1),(f_2,g_2),(f_3,g_3)\in\mathrm{Ratio}(R)$. 若$(f_1,g_1)\sim(f_2,g_2)$且$(f_2,g_2)\sim(f_3,g_3)$，则有：

$\because (f_1,g_1)\sim(f_2,g_2)$ $\quad\therefore f_1g_2=f_2g_1$

$\because (f_2,g_2)\sim(f_3,g_3)$ $\quad\therefore f_2g_3=f_3g_2$

$\therefore (f_1g_2)g_3 = (f_2g_1)g_3 = f_2(g_1g_3) = f_2(g_3g_1) = (f_2g_3)g_1$

$= (f_3g_2)g_1$

$\therefore (f_1 g_3) g_2 = f_1 (g_3 g_2) = f_1 (g_2 g_3) = (f_1 g_2) g_3$

$= (f_3 g_2) g_1 = f_3 (g_2 g_1) = f_3 (g_1 g_2) = (f_3 g_1) g_2$

$\therefore (f_1 g_3) g_2 = (f_3 g_1) g_2$ $\because g_2 \neq 0$ $\therefore f_1 g_3 = f_3 g_1$

$\therefore (f_1, g_1) \sim (f_3, g_3)$ $\therefore$ 传递性成立.

$\therefore \sim$ 是 $Ratio(R)$ 上的等价关系. $\square$

定义 ($Frac(R)$) R 是一个任意的整环，定义商集

$$Frac(R) := Ratio(R) / \sim$$

对 $\forall (f, g) \in Ratio(R)$，记包含 (f, g) 的等价类为 $[f, g]$.

$\therefore [f, g] \subseteq Ratio(R)$，$[f, g] \in Frac(R)$

$$\therefore Frac(R) = \{ [f, g] \mid (f, g) \in Ratio(R) \}$$

$$= \{ [f, g] \mid (f, g) \in R^2 \text{ 且 } g \neq 0 \}$$

$$[f, g] = \{ (\lambda, \mu) \in Ratio(R) : (\lambda, \mu) \sim (f, g) \}$$

$$= \{ (\lambda, \mu) \in R^2 : \mu \neq 0 \text{ 且 } (\lambda, \mu) \sim (f, g) \}$$

$$= \{ (\lambda, \mu) \in R^2 : \mu \neq 0 \text{ 且 } \lambda g = f \mu \}$$

Lemma（分式的基本性质）对 $\forall (f,g) \in Ratio(R)$，$\forall h \in R\setminus\{0\}$，有：

$$[f,g] = [fh, gh]$$

Proof：$\because (f,g) \in Ratio(R)$ $\therefore (f,g) \in R^2$ 且 $g \neq 0$

$\therefore f \in R$ 且 $g \in R$ 且 $g \neq 0$

$\because f \in R$ 且 $h \in R$ $\therefore fh \in R$ $\because g \in R$ 且 $h \in R$ $\therefore gh \in R$

$\because R$ 是整环，$g \neq 0$ 且 $h \neq 0$ $\therefore gh \neq 0$

$\therefore (fh, gh) \in R^2$ 且 $gh \neq 0$ $\therefore (fh, gh) \in Ratio(R)$

$\therefore (f,g) \in Ratio(R)$ 且 $(fh, gh) \in Ratio(R)$

$\because f(gh) = f(hg) = (fh)g$ $\therefore (f,g) \sim (fh, gh)$

$\therefore [f,g] = [fh, gh]$ $\square$