Lemma（整环上的分式域的泛性质）R是一个任意的整环，已经证明了

$\mathrm{Frac}(R)$是域，$f: R \longrightarrow \mathrm{Frac}(R)$ 是单同态，则有：

$$x \longmapsto [x,1]$$

对$\forall$交换环 A，$\forall$环同态 $\varphi: R \to A$ 满足 $\varphi(R\setminus\{0\}) \subseteq A^{\times}$，存在唯一的环同态 $\theta: \mathrm{Frac}(R) \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\
\mathrm{Frac}(R) \arrow[rr, "\theta"'] & & A
\end{tikzcd}$$

Proof：对$\forall$交换环 A，$\forall$环同态 $\varphi: R \to A$ 满足 $\varphi(R\setminus\{0\}) \subseteq A^{\times}$，

$\because$ R是整环 $\therefore$ $\mathrm{Ratio}(R) = \{(x,y) \in R^2 : y \neq 0\}$

$\because$ R是整环 $\therefore$ R是非零环 $\therefore$ $1_R \neq 0_R$ $\therefore$ $(0_R, 1_R) \in \mathrm{Ratio}(R)$

$\therefore$ $\mathrm{Ratio}(R) \neq \emptyset$.

已经证明了：$\mathrm{Ratio}(R)$上的二元关系 $\sim$：对$\forall (x_1,y_1),(x_2,y_2) \in \mathrm{Ratio}(R)$，

$$(x_1,y_1) \sim (x_2,y_2) \iff x_1y_2 = x_2y_1$$

是$\mathrm{Ratio}(R)$上的等价关系.

$\because$ A是交换环 $\therefore$ A是非空集合.

$\therefore$ 定义映射 $\alpha: \mathrm{Ratio}(R) \longrightarrow A$

$$(x,y) \longmapsto \varphi(x)(\varphi(y))^{-1}$$

下面证明映射α的良定性.

对 $\forall (x,y) \in Ratio(R)$，有：$x \in R$ 且 $y \in R$ 且 $y \neq 0$

$\because x \in R$ $\therefore \varphi(x) \in A$ $\because y \in R$ 且 $y \neq 0$ $\therefore y \in R \setminus \{0\}$ $\therefore \varphi(y) \in A^{\times}$

$\therefore (\varphi(y))^{-1} \in A^{\times}$ $\therefore (\varphi(y))^{-1} \in A$

$\therefore \alpha((x,y)) = \varphi(x)(\varphi(y))^{-1} \in A$ $\therefore \alpha(Ratio(R)) \subseteq A$

对 $\forall (x_1,y_1),(x_2,y_2) \in Ratio(R)$，若 $(x_1,y_1) = (x_2,y_2)$，则有：

$x_1 = x_2$ 且 $y_1 = y_2$

$\because x_1 = x_2 \in R$，$\varphi: R \to A$ 是映射 $\therefore \varphi(x_1) = \varphi(x_2)$

$\because y_1 = y_2 \in R$，$\varphi: R \to A$ 是映射 $\therefore \varphi(y_1) = \varphi(y_2)$

$\because y_1 = y_2$，$y_1 \neq 0$，$y_2 \neq 0$ $\therefore \varphi(y_1) \in A^{\times}$，$\varphi(y_2) \in A^{\times}$

$\because \varphi(y_1) = \varphi(y_2)$ $\therefore (\varphi(y_1))^{-1} = (\varphi(y_2))^{-1}$

$\therefore \alpha((x_1,y_1)) = \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1} = \alpha((x_2,y_2))$

$\therefore \alpha: Ratio(R) \to A$ 是映射.

对 $\forall (x_1,y_1),(x_2,y_2) \in Ratio(R)$，若 $(x_1,y_1) \sim (x_2,y_2)$，则有：

$x_1 y_2 = x_2 y_1$.

$\because x_1 y_2 = x_2 y_1 \in R$ $\therefore \varphi(x_1 y_2) = \varphi(x_2 y_1)$

$\therefore \varphi(x_1)\varphi(y_2) = \varphi(x_2)\varphi(y_1)$

$\because y_1, y_2 \in R$ 且 $y_1, y_2 \neq 0$ $\therefore \varphi(y_1) \in A^{\times}$ 且 $\varphi(y_2) \in A^{\times}$

$\therefore \varphi(x_1)\varphi(y_2)(\varphi(y_2))^{-1}(\varphi(y_1))^{-1} = \varphi(x_2)\varphi(y_1)(\varphi(y_1))^{-1}(\varphi(y_2))^{-1}$

需要A是交换环.

$\therefore \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1}$

$\therefore \alpha((x_1, y_1)) = \alpha((x_2, y_2))$

$\therefore$ 存在唯一的映射 $\theta: \mathrm{Ratio}(R)/\sim \longrightarrow A$ 使得

$\theta \circ q = \alpha$. 其中 $q: \mathrm{Ratio}(R) \longrightarrow \mathrm{Ratio}(R)/\sim$ 是商映射.

$\because \mathrm{Frac}(R) = \mathrm{Ratio}(R)/\sim \quad \therefore \theta$ 是 $\mathrm{Frac}(R) \to A$ 的映射

$\because f: R \longrightarrow \mathrm{Frac}(R)$ 是单同态, $\theta: \mathrm{Frac}(R) \to A$ 是映射

$x \longmapsto [x, 1]$

$\therefore \theta \circ f: R \to A$ 是映射 $\quad \varphi: R \to A$ 是环同态.

对 $\forall x \in R$, 有: $(\theta \circ f)(x) = \theta(f(x)) = \theta([x, 1]) = \theta(q((x,1)))$

$= (\theta \circ q)((x,1)) = \alpha((x,1)) = \varphi(x)(\varphi(1))^{-1} = \varphi(x) \cdot 1_A^{-1}$

$= \varphi(x) \cdot 1_A = \varphi(x)$

$\therefore \theta \circ f = \varphi \quad \therefore$ 下图交换:

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ \mathrm{Frac}(R) \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

对 $\forall [x_1, y_1], [x_2, y_2] \in \mathrm{Frac}(R)$, 有:

$\theta([x_1, y_1] + [x_2, y_2]) = \theta([x_1 y_2 + x_2 y_1, y_1 y_2])$

$= \theta(q((x_1 y_2 + x_2 y_1, y_1 y_2))) = (\theta \circ q)((x_1 y_2 + x_2 y_1, y_1 y_2))$

$$= \alpha\big((x_1y_2+x_2y_1,\ y_1y_2)\big) = \varphi(x_1y_2+x_2y_1)\big(\varphi(y_1y_2)\big)^{-1}$$

$$= \big(\varphi(x_1y_2)+\varphi(x_2y_1)\big)\big(\varphi(y_1)\varphi(y_2)\big)^{-1}$$

$$= \big(\varphi(x_1)\varphi(y_2)+\varphi(x_2)\varphi(y_1)\big)\big(\varphi(y_2)\big)^{-1}\big(\varphi(y_1)\big)^{-1}$$

$$= \varphi(x_1)\big(\varphi(y_1)\big)^{-1} + \varphi(x_2)\big(\varphi(y_2)\big)^{-1}$$

$$= \alpha\big((x_1,y_1)\big) + \alpha\big((x_2,y_2)\big)$$

$$= (\theta\circ q)(x_1,y_1) + (\theta\circ q)\big((x_2,y_2)\big)$$

$$= \theta\big(q((x_1,y_1))\big) + \theta\big(q((x_2,y_2))\big)$$

$$= \theta\big([x_1,y_1]\big) + \theta\big([x_2,y_2]\big)$$

$$\theta\big([x_1,y_1]\cdot[x_2,y_2]\big) = \theta\big([x_1x_2,\ y_1y_2]\big)$$

$$= \theta\big(q((x_1x_2,\ y_1y_2))\big) = \alpha\big((x_1x_2,\ y_1y_2)\big)$$

$$= \varphi(x_1x_2)\big(\varphi(y_1y_2)\big)^{-1} = \varphi(x_1)\varphi(x_2)\big(\varphi(y_1)\varphi(y_2)\big)^{-1}$$

$$= \varphi(x_1)\varphi(x_2)\big(\varphi(y_2)\big)^{-1}\big(\varphi(y_1)\big)^{-1}$$

$$= \varphi(x_1)\big(\varphi(y_1)\big)^{-1} \underset{\substack{\downarrow \\ \text{A中的乘法}}}{\cdot} \varphi(x_2)\big(\varphi(y_2)\big)^{-1}$$

$$= \alpha\big((x_1,y_1)\big)\cdot\alpha\big((x_2,y_2)\big) = \theta\big(q((x_1,y_1))\big)\cdot\theta\big(q((x_2,y_2))\big)$$

$$= \theta\big([x_1,y_1]\big)\cdot\theta\big([x_2,y_2]\big)$$

$$\theta([1,1]) = \theta(q((1,1))) = \alpha((1,1))$$

$$= \varphi(1)(\varphi(1))^{-1} = 1_A \cdot 1_A^{-1} = 1_A \cdot 1_A = 1_A$$

$\therefore$ $\theta: \mathrm{Frac}(R) \longrightarrow A$ 是环同态. $\therefore$ 存在性得证.

假设存在环同态 $\theta_1: \mathrm{Frac}(R) \longrightarrow A$, s.t. 下图交换:

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ \mathrm{Frac}(R) \arrow[rr, "\theta_1"'] & & A \end{tikzcd}$$

假设还存在环同态 $\theta_2: \mathrm{Frac}(R) \longrightarrow A$, s.t. 下图交换:

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ \mathrm{Frac}(R) \arrow[rr, "\theta_2"'] & & A \end{tikzcd}$$

则有: 对 $\forall y \in R \setminus \{0\}$, 有: $\varphi(y) \in A^\times$

$\because$ $\theta_1: \mathrm{Frac}(R) \longrightarrow A$ 是环同态

$$\therefore 1_A = \theta_1([1,1]) = \theta_1([y,y]) = \theta_1([y,1]\cdot[1,y])$$

$$= \theta_1([y,1])\cdot\theta_1([1,y]) = \theta_1(f(y))\cdot\theta_1([1,y])$$

$$= \varphi(y)\cdot\theta_1([1,y]) \qquad \therefore \theta_1([1,y]) = (\varphi(y))^{-1}$$

$\because$ $\theta_2: \mathrm{Frac}(R) \longrightarrow A$ 是环同态

$$\therefore 1_A = \theta_2([1,1]) = \theta_2([y,y]) = \theta_2([y,1]\cdot[1,y])$$

$$= \theta_2([y,1])\cdot\theta_2([1,y]) = \theta_2(f(y))\cdot\theta_2([1,y])$$

$= \varphi(y) \cdot \theta_2([1, y])$ $\quad \therefore \theta_2([1, y]) = (\varphi(y))^{-1}$

$\therefore$ 对 $\forall [x, y] \in Frac(R)$，有：$x \in R$ 且 $y \in R$ 且 $y \neq 0$

$\therefore \theta_1([x, y]) = \theta_1([x, 1] \cdot [1, y]) = \theta_1([x, 1]) \cdot \theta_1([1, y])$

$= \theta_1(f(x)) \cdot \theta_1([1, y]) = \varphi(x) \cdot (\varphi(y))^{-1}$

$= \theta_2(f(x)) \cdot \theta_2([1, y]) = \theta_2([x, 1]) \cdot \theta_2([1, y])$

$= \theta_2([x, 1] \cdot [1, y]) = \theta_2([x, y])$

$\therefore \theta_1 = \theta_2$. 唯一性得证. $\square$