Lemma：R是环，F是除环，$\varphi: R\longrightarrow F$是环同态，则有：

映射 $\varphi: R\to F$是单射 $\Longleftrightarrow \varphi(R\setminus\{0_R\})\subseteq F^{\times}$

Proof：($\Rightarrow$)：对 $\forall x\in R\setminus\{0_R\}$. 假设 $\varphi(x)=0_F$，则

$\because \varphi: R\longrightarrow F$是环同态 $\therefore \varphi(0_R)=0_F$ $\therefore \varphi(x)=0_F=\varphi(0_R)$

$\because \varphi: R\longrightarrow F$是单射 $\therefore x=0_R$ $\because x\in R\setminus\{0_R\}$ $\therefore x\neq 0_R$ 矛盾.

$\therefore \varphi(x)\neq 0_F$ $\therefore \varphi(x)\in F\setminus\{0_F\}=F^{\times}$

$\therefore \varphi(R\setminus\{0_R\})\subseteq F^{\times}$

($\Leftarrow$)：对 $\forall x_1, x_2\in R$. 若 $\varphi(x_1)=\varphi(x_2)$，则有：

$\because x_1, x_2\in R$ $\therefore x_1-x_2\in R$. 假设 $x_1-x_2\neq 0_R$，则有：

$x_1-x_2\in R\setminus\{0_R\}$. $\therefore \varphi(x_1-x_2)\in F^{\times}$

$\because F$是除环 $\therefore F^{\times}=F\setminus\{0_F\}$ $\therefore \varphi(x_1-x_2)\in F^{\times}=F\setminus\{0_F\}$

$\therefore \varphi(x_1-x_2)\in F$且 $\varphi(x_1-x_2)\neq 0_F$

$\therefore 0_F\neq\varphi(x_1-x_2)=\varphi(x_1+(-x_2))=\varphi(x_1)+\varphi(-x_2)$

$=\varphi(x_2)+\varphi(-x_2)=\varphi(x_2)+(-\varphi(x_2))=0_F$ 矛盾.

$\therefore x_1-x_2=0_R$

$\therefore x_2=0_R+x_2=(x_1-x_2)+x_2=(x_1+(-x_2))+x_2=x_1+((-x_2)+x_2)$

$=x_1+0_R=x_1$

$\therefore x_1=x_2$ $\therefore$映射 $\varphi: R\to F$是单射. $\square$

推论：R是整环，F是域，$\varphi: R \to F$是环同态，则有：

映射$\varphi: R \to F$是单射 $\Longleftrightarrow$ $\varphi(R \setminus \{0_R\}) \subseteq F^\times$

Proof：由上一引理立得. $\square$

Lemma（分式域的泛性质）R是一个任意的整环

F是一个域，$f: R \to F$是一个单同态，满足：

对$\forall$交换环A，$\forall$环同态$\varphi: R \to A$满足$\varphi(R\setminus\{0\}) \subseteq A^{\times}$，存在唯一的环同态$\theta: F \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ F \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

F'是另一个域，$f': R \to F'$是另一个单同态，满足：
对$\forall$交换环A，$\forall$环同态$\varphi: R \to A$满足$\varphi(R\setminus\{0\}) \subseteq A^{\times}$，存在唯一的环同态$\theta': F' \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "\varphi"] & \\ F' \arrow[rr, "\theta'"'] & & A \end{tikzcd}$$

则存在唯一的映射$\Theta: F \longrightarrow F'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Theta"'] & & F' \end{tikzcd}$$

且$\Theta: F \longrightarrow F'$是域同构.

Proof: $\because$ F'是域 $\therefore$ F'是交换环

$\because$ R是整环，F'是域，$f': R \to F'$是单同态

$\therefore$ $f'(R\setminus\{0\}) \subseteq (F')^\times$

$\therefore$ 存在唯一的环同态 $\Phi: F \longrightarrow F'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Phi"'] & & F' \end{tikzcd}$$

$\because$ F是域 $\therefore$ F是交换环

$\because$ R是整环，F是域，$f: R \to F$是单同态

$\therefore$ $f(R\setminus\{0\}) \subseteq F^\times$

$\therefore$ 存在唯一的环同态 $\Psi: F' \longrightarrow F$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ F' \arrow[rr, "\Psi"'] & & F \end{tikzcd}$$

$\because$ $\Phi: F \longrightarrow F'$是环同态，$\Psi: F' \longrightarrow F$是环同态

$\therefore$ $\Psi \circ \Phi: F \longrightarrow F$是环同态，$\Phi \circ \Psi: F' \longrightarrow F'$是环同态.

$\because$ F是交换环，$f: R \to F$是环同态且满足 $f(R\setminus\{0\}) \subseteq F^\times$

$\therefore$ 存在唯一的环同态 $\alpha: F \longrightarrow F$ s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ F \arrow[rr, "\alpha"'] & & F \end{tikzcd}$$

$\because$ $id_F: F \longrightarrow F$是环同构　　$\therefore$ $id_F: F \longrightarrow F$是环同态

$\because$ $id_F \circ f = f$　　$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ F \arrow[rr, "id_F"'] & & F \end{tikzcd}$$

$\therefore$ $\alpha = id_F$

$\because$ $(\Phi \circ \Psi) \circ f = \Phi \circ (\Psi \circ f) = \Phi \circ f' = f$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ F \arrow[rr, "\Phi \circ \Psi"'] & & F \end{tikzcd}$$

$\therefore$ $\Phi \circ \Psi = \alpha = id_F$

$\because$ F'是交换环，$f': R \longrightarrow F'$是环同态且满足 $f'(R \setminus \{0\}) \subseteq (F')^{\times}$

$\therefore$ 存在唯一的环同态 $\beta: F' \longrightarrow F'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ F' \arrow[rr, "\beta"'] & & F' \end{tikzcd}$$

$\because$ $id_{F'}: F' \longrightarrow F'$是环同构　　$\therefore$ $id_{F'}: F' \longrightarrow F'$是环同态

$\because$ $id_{F'} \circ f' = f'$　　$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ F' \arrow[rr, "id_{F'}"'] & & F' \end{tikzcd}$$

$\therefore$ $\beta = id_{F'}$

$\because$ $(\Psi \circ \Phi) \circ f' = \Psi \circ (\Phi \circ f') = \Psi \circ f = f'$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ F' \arrow[rr, "\Psi \circ \Phi"'] & & F' \end{tikzcd}$$

$\therefore$ $\Psi \circ \Phi = \beta = id_{F'}$

$\because$ $\Psi: F \longrightarrow F'$是环同态，$\Phi: F' \longrightarrow F$是环同态，$\Phi \circ \Psi = id_F$，$\Psi \circ \Phi = id_{F'}$　　$\therefore$ $\Psi: F \longrightarrow F'$是可逆映射　　$\therefore$ $\Psi: F \longrightarrow F'$是双射

$\therefore \Phi: F \to F'$ 是环同构. $\therefore \Phi: F \to F'$ 是域同构.

$\therefore \Phi: F \to F'$ 是域同构，且下图交换：
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Phi"'] & & F' \end{tikzcd}$$
存在性得证.

假设存在映射 $\Theta_1: F \to F'$, s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Theta_1"'] & & F' \end{tikzcd}$$

且 $\Theta_1: F \to F'$ 是域同构

假设还存在映射 $\Theta_2: F \to F'$ s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Theta_2"'] & & F' \end{tikzcd}$$

且 $\Theta_2: F \to F'$ 是域同构. 则有:

对于交换环 F'，环同态 $f': R \to F'$ 满足 $f'(R \setminus \{0\}) \subseteq (F')^\times$，存在唯一的环同态 $\Phi: F \to F'$, s.t. 下图交换：
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F \arrow[rr, "\Phi"'] & & F' \end{tikzcd}$$

$\therefore \Theta_1 = \Phi = \Theta_2$ $\therefore$ 唯一性得证. $\square$