

定理 (商环的性质, R/I 满足相应的性质) R 是环, I 是 R 的理想. 已经证明了 R/I 是环, 商同态 $\varphi: R \rightarrow R/I$ 是满同态,
 $x \mapsto x+I$

且有 $I = \ker(\varphi)$. 则有: 对 \forall 环 A , \forall 环同态 $\psi: R \rightarrow A$ 满足 $I \subseteq \ker(\psi)$,
 存在唯一的环同态 $\theta: R/I \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ & \downarrow \varphi & \downarrow \psi \\ R/I & \xrightarrow{\theta} & A \end{array}$$

$$\text{Proof: } \because R/I = \{x+I \mid x \in R\}$$

$$\therefore \text{定义映射 } \theta: R/I \rightarrow A$$

$$x+I \mapsto \psi(x)$$

$\forall x+I \in R/I$ (其中 $x \in R$), $\because x \in R$, $\psi: R \rightarrow A$ 是环同态

$$\therefore \psi(x) \in A \quad \therefore \theta(x+I) = \psi(x) \in A \quad \therefore \theta(R/I) \subseteq A$$

$\forall x+I, y+I \in R/I$ (其中 $x, y \in R$), 若 $x+I = y+I$, 则有:

$$x \equiv_I y \quad \therefore x-y \in I \quad \because I \subseteq \ker(\psi) \quad \therefore x-y \in \ker(\psi)$$

$$\therefore x-y \in R \text{ 且 } \psi(x-y) = 0_A$$

$$\therefore \psi(x-y) = \psi(x+(-y)) = \psi(x) + \psi(-y) = \psi(x) + (-\psi(y))$$

$$\therefore \psi(x) + (-\psi(y)) = 0_A$$

$$\begin{aligned} \therefore \psi(x) &= \psi(x) + 0_A = \psi(x) + ((-\psi(y)) + \psi(y)) = (\psi(x) + (-\psi(y))) + \psi(y) \\ &= 0_A + \psi(y) = \psi(y) \end{aligned}$$

$$\therefore \theta(x+I) = \varphi(x) = \varphi(y) = \theta(y+I)$$

$\therefore \theta : R/I \rightarrow A$ 是一个映射.

对 $\forall x+I, y+I \in R/I$ (其中 $x, y \in R$) ,

$$\begin{aligned}\theta((x+I)+(y+I)) &= \theta((x+y)+I) = \varphi(x+y) = \varphi(x)+\varphi(y) \\ &= \theta(x+I)+\theta(y+I)\end{aligned}$$

$$\begin{aligned}\theta((x+I) \cdot (y+I)) &= \theta(xy+I) = \varphi(xy) = \varphi(x) \cdot \varphi(y) \\ &= \theta(x+I) \cdot \theta(y+I)\end{aligned}$$

$$\theta(I_{R/I}) = \theta(I_R + I) = \varphi(I_R) = I_A$$

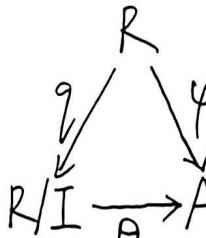
$\therefore \theta : R/I \rightarrow A$ 是一个环同态.

$\because \varphi : R \rightarrow R/I$ 是满同态, $\theta : R/I \rightarrow A$ 是环同态

$\therefore \theta \circ \varphi : R \rightarrow A$ 是环同态 $\because \varphi : R \rightarrow R/I$ 是环同态.

对 $\forall x \in R$, $(\theta \circ \varphi)(x) = \theta(\varphi(x)) = \theta(x+I) = \varphi(x)$

$\therefore \theta \circ \varphi = \varphi$

\therefore 下图交换:  存在性得证.

假设存在环同态 $\theta_1 : R/I \rightarrow A$, 环同态 $\theta_2 : R/I \rightarrow A$, s.t.
下图交换:

$$\begin{array}{ccc} & R & \\ q \swarrow & & \searrow \varphi \\ R/I & \xrightarrow{\theta_1} & A \\ & & \end{array} \quad \begin{array}{ccc} & R & \\ q \swarrow & & \searrow \varphi \\ R/I & \xrightarrow{\theta_2} & A \\ & & \end{array}$$

则对 $\forall x+I \in R/I$ (其中 $x \in R$), 有:

$$\begin{aligned} \theta_1(x+I) &= \theta_1(q(x)) = (\theta_1 \circ q)(x) = \varphi(x) = (\theta_2 \circ q)(x) \\ &= \theta_2(q(x)) = \theta_2(x+I) \end{aligned}$$

$\therefore \theta_1 = \theta_2$ \therefore 唯一性得证. □