Lemma: R是非零环，则有：$R^\times \subseteq R\setminus\{0_R\}$.

Proof: 对$\forall x \in R^\times$ $\because R^\times \subseteq R$ $\therefore x \in R$

假设 $x = 0_R$，则有：

$\because x \in R^\times$ $\therefore \exists x^{-1} \in R$, s.t. $x^{-1}\cdot x = 1_R = x\cdot x^{-1}$

$\therefore 1_R = x^{-1}\cdot x = x^{-1}\cdot 0_R = 0_R$

$\because$ R是非零环 $\therefore 1_R \neq 0_R$ 矛盾 $\therefore x \neq 0_R$

$\therefore x \notin \{0_R\}$ $\therefore x \in R\setminus\{0_R\}$ $\therefore R^\times \subseteq R\setminus\{0_R\}$ $\square$

Frac(R)是域

Lemma: $(\mathrm{Frac}(R), +, \cdot, [0,1], [1,1])$是域

Proof: 对$\forall [f_1,g_1], [f_2,g_2] \in \mathrm{Frac}(R)$,

$$[f_1,g_1]+[f_2,g_2]=[f_1g_2+f_2g_1, g_1g_2]$$

$$=[f_2g_1+f_1g_2, g_2g_1]=[f_2,g_2]+[f_1,g_1]$$ ∴加法交换律成立

$$[f_1,g_1]\cdot[f_2,g_2]=[f_1f_2, g_1g_2]=[f_2f_1, g_2g_1]$$

$$=[f_2,g_2][f_1,g_1]$$ ∴乘法交换律成立.

对$\forall [f_1,g_1], [f_2,g_2], [f_3,g_3] \in \mathrm{Frac}(R)$

$$([f_1,g_1]+[f_2,g_2])+[f_3,g_3]$$

$$=[f_1g_2+f_2g_1, g_1g_2]+[f_3,g_3]$$

$$=[(f_1g_2+f_2g_1)g_3+f_3(g_1g_2), (g_1g_2)g_3]$$

$$=[f_1g_2g_3+f_2g_1g_3+f_3g_1g_2, g_1g_2g_3]$$

$$=[f_1(g_2g_3)+(f_2g_3+f_3g_2)g_1, g_1(g_2g_3)]$$

$$=[f_1, g_1] + [f_2 g_3 + f_3 g_2, g_2 g_3]$$

$$=[f_1, g_1] + \left([f_2, g_2] + [f_3, g_3]\right)$$ ∴ 加法结合律成立.

$$\left([f_1, g_1] \cdot [f_2, g_2]\right) \cdot [f_3, g_3] = [f_1 f_2, g_1 g_2] \cdot [f_3, g_3]$$

$$=\left[(f_1 f_2) f_3, (g_1 g_2) g_3\right] = \left[f_1 (f_2 f_3), g_1 (g_2 g_3)\right]$$

$$=[f_1, g_1] \cdot [f_2 f_3, g_2 g_3] = [f_1, g_1] \cdot \left([f_2, g_2] \cdot [f_3, g_3]\right)$$

∴ 乘法结合律成立

$$\left([f_1, g_1] + [f_2, g_2]\right) \cdot [f_3, g_3] = [f_1 g_2 + f_2 g_1, g_1 g_2] \cdot [f_3, g_3]$$

$$=\left[(f_1 g_2 + f_2 g_1) \cdot f_3, (g_1 g_2) g_3\right] = \left[(f_1 g_2 + f_2 g_1) \cdot f_3 g_3, (g_1 g_2) g_3^2\right]$$

$$=\left[(f_1 f_3)(g_2 g_3) + (f_2 f_3)(g_1 g_3), (g_1 g_3)(g_2 g_3)\right]$$

$$=[f_1 f_3, g_1 g_3] + [f_2 f_3, g_2 g_3]$$

$$=[f_1, g_1] \cdot [f_3, g_3] + [f_2, g_2] \cdot [f_3, g_3]$$

$$[f_3, g_3]\cdot\left([f_1, g_1]+[f_2, g_2]\right) = [f_3, g_3]\cdot[f_1 g_2 + f_2 g_1, g_1 g_2]$$

$$= [f_3(f_1 g_2 + f_2 g_1), g_3(g_1 g_2)] \cancel{= [f_3 g_3}$$

$$= [f_3 g_3(f_1 g_2 + f_2 g_1), g_3^2(g_1 g_2)]$$

$$= [(f_3 f_1)(g_3 g_2) + (f_3 f_2)(g_3 g_1), (g_3 g_1)(g_3 g_2)]$$

$$= [f_3 f_1, g_3 g_1] + [f_3 f_2, g_3 g_2]$$

$$= [f_3, g_3]\cdot[f_1, g_1] + [f_3, g_3]\cdot[f_2, g_2]$$

$\therefore$ 乘法对加法满足分配律.

对 $\forall [f, g] \in Frac(R)$,

$$[f, g] + [0, 1] = [f\cdot 1 + 0\cdot g, g\cdot 1] = [f, g]$$

$$[0, 1] + [f, g] = [0\cdot g + f\cdot 1, 1\cdot g] = [f, g]$$

$$\therefore [f, g] + [0, 1] = [f, g] = [0, 1] + [f, g]$$

$\therefore [0, 1]$ 是环 $Frac(R)$ 的加法零元.

$$[f, g]\cdot[1, 1] = [f\cdot 1, g\cdot 1] = [f, g]$$

$$[1, 1]\cdot[f, g] = [1\cdot f, 1\cdot g] = [f, g]$$

$\therefore [f,g]\cdot[1,1]=[f,g]=[1,1]\cdot[f,g]$

$\therefore [1,1]$是环$\mathrm{Frac}(R)$的乘法幺元

$\because [f,g]\in\mathrm{Frac}(R)$ $\therefore f\in R$且$g\in R$且$g\neq 0$

$\therefore -f\in R$ 且有 $f+(-f)=0$

$\therefore [-f,g]\in\mathrm{Frac}(R)$

$\because [f,g]+[-f,g]=[f+(-f),g]=[0,g]=[0,1]$

$\therefore -[f,g]=[-f,g]\in\mathrm{Frac}(R)$

$\therefore (\mathrm{Frac}(R),+,\cdot,[0,1],[1,1])$是交换环

假设$[0,1]=[1,1]$，则有$(0,1)\sim(1,1)$

$\therefore 0\cdot 1=1\cdot 1$ $\therefore 0=1$ $\therefore R$是零环

$\because R$是整环 $\therefore R$是非零环 矛盾 $\therefore [0,1]\neq[1,1]$

$\therefore (\mathrm{Frac}(R),+,\cdot,[0,1],[1,1])$是非零交换环

$\therefore \mathrm{Frac}(R)$是非零环 $\therefore (\mathrm{Frac}(R))^{\times}\subseteq \mathrm{Frac}(R)\setminus\{[0,1]\}$

对 $\forall [f,g] \in \mathrm{Frac}(R) \setminus \{[0,1]\}$,

~~假设 $f=0$, 则有~~ $\because [f,g] \in \mathrm{Frac}(R)$ $\therefore f \in R$ 且 $g \in R$ 且 $g \neq 0$

假设 $f=0$, 则有: $[f,g] = [0,g] = [0,1]$. 矛盾.

$\therefore f \neq 0$ $\quad \therefore [g,f] \in \mathrm{Frac}(R)$

$\because R$ 是整环, $f \in R$ 且 $f \neq 0$, $g \in R$ 且 $g \neq 0$

$\therefore fg \in R$ 且 $fg \neq 0$

$\therefore [f,g] \cdot [g,f] = [fg, gf] = [fg, fg] = [1,1]$

$[g,f] \cdot [f,g] = [gf, fg] = [fg, fg] = [1,1]$

$\therefore [f,g] \cdot [g,f] = [1,1] = [g,f] \cdot [f,g]$

$\therefore [f,g] \in (\mathrm{Frac}(R))^\times$

$\therefore \mathrm{Frac}(R) \setminus \{[0,1]\} \subseteq (\mathrm{Frac}(R))^\times$

$\therefore (\mathrm{Frac}(R))^\times = \mathrm{Frac}(R) \setminus \{[0,1]\}$

$\therefore$ 环 $\mathrm{Frac}(R)$ 是除环

$\therefore (\mathrm{Frac}(R), +, \cdot, [0,1], [1,1])$ 是域. $\square$

定义 (分式域) R 是一个任意的整环, 则 $\mathrm{Frac}(R)$ 是域, 称为整环 R 的分式域.

Lemma：对$\forall [f,g]\in Frac(R)$，有：

① $[f,g]=[0,1] \Longleftrightarrow f=0$

② $[f,g]=[1,1] \Longleftrightarrow f=g$

Proof：~~由~~ $\because [f,g]\in Frac(R)$ $\quad \therefore f\in R$ 且 $g\in R$ 且 $g\neq 0$

①($\Leftarrow$)：$\because f=0$ $\quad \therefore [f,g]=[0,g]=[0,1]$

①($\Rightarrow$)：$\because [f,g]=[0,1]$ $\quad \therefore (f,g)\sim(0,1)$

$\therefore f\cdot 1=0\cdot g$ $\quad \therefore f=0$

②($\Leftarrow$)：$\because f=g$ $\quad \therefore [f,g]=[g,g]=[1,1]$

②($\Rightarrow$)：$\because [f,g]=[1,1]$ $\quad \therefore (f,g)\sim(1,1)$

$\therefore f\cdot 1=1\cdot g$ $\quad \therefore f=g$ $\quad \square$

Lemma：对$\forall f_1, f_2\in R$，有：

$$[f_1,1]=[f_2,1] \Longleftrightarrow f_1=f_2$$

Proof：$\because$ R是整环 $\quad \therefore$ R是非零环 $\quad \therefore 1\neq 0$ $\quad \therefore 1\in R$ 且 $1\neq 0$

$\because f_1\in R$ $\quad \therefore [f_1,1]\in Frac(R)$

$\because f_2\in R$ $\quad \therefore [f_2,1]\in Frac(R)$

$\therefore [f_1,1]=[f_2,1] \Longleftrightarrow (f_1,1)\sim(f_2,1) \Longleftrightarrow f_1\cdot 1=f_2\cdot 1$

$\Longleftrightarrow f_1=f_2$ $\quad \square$

Lemma: $\varphi: R \longrightarrow \mathrm{Frac}(R)$ 是单同态

$f \longmapsto [f,1]$

Proof: 对 $\forall f \in R$ ∵ R 是整环 ∴ R 是非零环 ∴ $1 \neq 0$

∴ $1 \in R$ 且 $1 \neq 0$ ∴ $\varphi(f) = [f,1] \in \mathrm{Frac}(R)$

∴ $\varphi(R) \subseteq \mathrm{Frac}(R)$

对 $\forall f_1, f_2 \in R$

若 $f_1 = f_2$，则有：$f_1 \cdot 1 = f_1 = f_2 = f_2 \cdot 1$

∴ $(f_1, 1) \sim (f_2, 1)$ ∴ $[f_1, 1] = [f_2, 1]$

∴ $\varphi(f_1) = [f_1, 1] = [f_2, 1] = \varphi(f_2)$ ∴ φ 是 $R \to \mathrm{Frac}(R)$ 的映射

若 $\varphi(f_1) = \varphi(f_2)$，则有：$[f_1, 1] = [f_2, 1]$

∴ $(f_1, 1) \sim (f_2, 1)$ ∴ $f_1 \cdot 1 = f_2 \cdot 1$ ∴ $f_1 = f_2$

∴ φ 是 $R \to \mathrm{Frac}(R)$ 的单射

对 $\forall f, g \in R$，有：

$$\varphi(f+g) = [f+g, 1] = [f,1] + [g,1] = \varphi(f) + \varphi(g)$$

$$\varphi(fg) = [fg, 1] = [fg, 1\cdot 1] = [f,1]\cdot[g,1] = \varphi(f)\cdot\varphi(g)$$

$\varphi(1) = [1,1] = 1_{\mathrm{Frac}(R)}$ ∴ φ 是 $R \to \mathrm{Frac}(R)$ 的单同态. $\square$