

定理 (商环的泛性质, R/I 满足相应的泛性质) R 是环, I 是 R 的理想. 已经证明了 R/I 是环, 商同态 $q: R \rightarrow R/I$ 是满同态,
 $x \mapsto x+I$

且有 $I = \ker(q)$. 则有: 对 \forall 环 A , \forall 环同态 $\varphi: R \rightarrow A$ 满足 $I \subseteq \ker(\varphi)$, 存在唯一的环同态 $\theta: R/I \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ q \swarrow & & \searrow \varphi \\ R/I & \xrightarrow{\theta} & A \end{array}$$

Proof: $\because R/I = \{x+I \mid x \in R\}$

\therefore 定义映射 $\theta: R/I \rightarrow A$

$$x+I \mapsto \varphi(x)$$

对 $\forall x+I \in R/I$ (其中 $x \in R$), $\because x \in R$, $\varphi: R \rightarrow A$ 是环同态

$$\therefore \varphi(x) \in A \quad \therefore \theta(x+I) = \varphi(x) \in A \quad \therefore \theta(R/I) \subseteq A$$

对 $\forall x+I, y+I \in R/I$ (其中 $x, y \in R$), 若 $x+I = y+I$, 则有:

$$x \equiv_I y \quad \therefore x-y \in I \quad \because I \subseteq \ker(\varphi) \quad \therefore x-y \in \ker(\varphi)$$

$$\therefore x-y \in R \text{ 且 } \varphi(x-y) = 0_A$$

$$\therefore \varphi(x-y) = \varphi(x+(-y)) = \varphi(x) + \varphi(-y) = \varphi(x) + (-\varphi(y))$$

$$\therefore \varphi(x) + (-\varphi(y)) = 0_A$$

$$\begin{aligned} \therefore \varphi(x) &= \varphi(x) + 0_A = \varphi(x) + ((-\varphi(y)) + \varphi(y)) = (\varphi(x) + (-\varphi(y))) + \varphi(y) \\ &= 0_A + \varphi(y) = \varphi(y) \end{aligned}$$

$$\therefore \theta(x+I) = \varphi(x) = \varphi(y) = \theta(y+I)$$

$\therefore \theta: R/I \rightarrow A$ 是一个映射.

对 $\forall x+I, y+I \in R/I$ (其中 $x, y \in R$),

$$\begin{aligned} \theta((x+I) + (y+I)) &= \theta((x+y)+I) = \varphi(x+y) = \varphi(x) + \varphi(y) \\ &= \theta(x+I) + \theta(y+I) \end{aligned}$$

$$\begin{aligned} \theta((x+I) \cdot (y+I)) &= \theta(xy+I) = \varphi(xy) = \varphi(x) \cdot \varphi(y) \\ &= \theta(x+I) \cdot \theta(y+I) \end{aligned}$$

$$\theta(1_{R/I}) = \theta(1_R + I) = \varphi(1_R) = 1_A$$

$\therefore \theta: R/I \rightarrow A$ 是一个环同态.

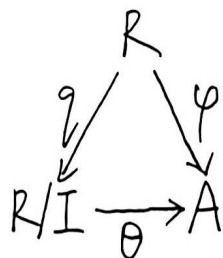
$\therefore q: R \rightarrow R/I$ 是满同态, $\theta: R/I \rightarrow A$ 是环同态

$\therefore \theta \circ q: R \rightarrow A$ 是环同态 $\therefore \varphi: R \rightarrow A$ 是环同态.

$$\text{对 } \forall x \in R, (\theta \circ q)(x) = \theta(q(x)) = \theta(x+I) = \varphi(x)$$

$$\therefore \theta \circ q = \varphi$$

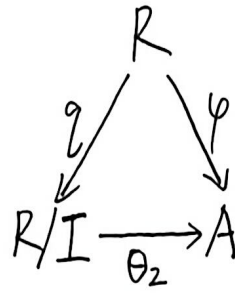
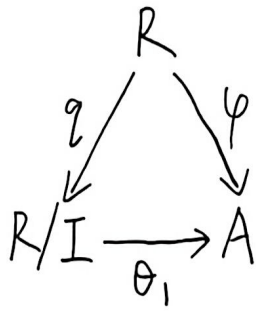
\therefore 下图交换:



存在性得证.

假设存在环同态 $\theta_1: R/I \rightarrow A$, 环同态 $\theta_2: R/I \rightarrow A$, s.t.

下图交换:



则对 $\forall x+I \in R/I$ (其中 $x \in R$), 有:

$$\begin{aligned}\theta_1(x+I) &= \theta_1(q(x)) = (\theta_1 \circ q)(x) = \varphi(x) = (\theta_2 \circ q)(x) \\ &= \theta_2(q(x)) = \theta_2(x+I)\end{aligned}$$

$\therefore \theta_1 = \theta_2$ \therefore 唯一性得证. \square