

定理(环的第四同构定理, 理想版本) R 是环, I 是 R 的理想, 已经证明了 R/I 是环, 商同态 $q: R \rightarrow R/I$ 是满同态.
 $x \mapsto x+I$

定义两个集合: $S = \{A \mid A \text{ 是环 } R \text{ 的理想, 且 } I \subseteq A\}$

$T = \{B \mid B \text{ 是环 } R/I \text{ 的理想}\}$

定义映射 $f: S \rightarrow T$, 则有: $f: S \rightarrow T$ 是一个保持包含关系的双射,
 $A \mapsto A/I$

且对 $\forall A \in S$, 有: $A = q^{-1}(f(A))$, 对 $\forall B \in T$, 有: $f^{-1}(B) = q^{-1}(B)$ $\rightarrow f^{-1}: T \rightarrow S$ 是 f 的逆映射

Proof: $\because R$ 是环, I 是 R 的理想 $\therefore R/I$ 是环

$\because R$ 是环 $\therefore R$ 是环 R 的理想, 且 $I \subseteq R \therefore R \in S$

$\therefore S \neq \emptyset$

$\because R/I$ 是环 $\therefore R/I$ 是环 R/I 的理想 $\therefore R/I \in T$

$\therefore T \neq \emptyset$

对 $\forall A \in S$, 有: A 是环 R 的理想, 且 $I \subseteq A$

$\because R$ 是环, I 是 R 的理想, A 是 R 的理想, 且 $I \subseteq A \subseteq R$

$\therefore A/I$ 是环 R/I 的理想.

$\therefore f(A) = A/I$ 是环 R/I 的理想 $\therefore f(A) \in T \therefore f(S) \subseteq T$

对 $\forall A_1, A_2 \in S$,

若 $A_1 = A_2$, 则有: $\because A_1 \in S \therefore f(A_1) = A_1/I$ 是环 R/I 的理想

$\because A_2 \in S \therefore f(A_2) = A_2/I$ 是环 R/I 的理想

任取 A_1/I 中的一元: $x+I$ (其中 $x \in A_1$) $\because x \in A_1 = A_2$

$$\because x+I \in A_2/I \quad \because A_1/I \subseteq A_2/I$$

$$\text{任取 } A_2/I \text{ 中的一元: } x+I \text{ (其中 } x \in A_2) \quad \because x \in A_2 = A_1$$

$$\because x+I \in A_1/I \quad \because A_2/I \subseteq A_1/I \quad \because A_1/I = A_2/I$$

$$\because f(A_1) = A_1/I = A_2/I = f(A_2) \quad \because f: S \rightarrow T \text{ 是一个映射.}$$

$$\text{若 } f(A_1) = f(A_2), \text{ 则有: } A_1/I = A_2/I$$

$$\text{对 } \forall x \in A_1, \text{ 有: } x+I \in A_1/I = A_2/I$$

$$\because \exists \lambda \in A_2, \text{ s.t. } x+I = \lambda+I \quad \because x \equiv_I \lambda$$

$$\because x - \lambda \in I \quad \because A_2 \in S \quad \because A_2 \text{ 是环 } R \text{ 的理想, 且 } I \subseteq A_2$$

$$\because x - \lambda \in A_2 \quad \because \lambda \in A_2, A_2 \text{ 是环 } R \text{ 的理想} \quad \because (x - \lambda) + \lambda \in A_2$$

$$\because (x - \lambda) + \lambda = (x + (-\lambda)) + \lambda = x + ((-\lambda) + \lambda) = x + 0_R = x$$

$$\because x \in A_2 \quad \because A_1 \subseteq A_2$$

$$\text{对 } \forall x \in A_2, \text{ 有: } x+I \in A_2/I = A_1/I$$

$$\because \exists \mu \in A_1, \text{ s.t. } x+I = \mu+I \quad \because x \equiv_I \mu$$

$$\because x - \mu \in I \quad \because A_1 \in S \quad \because A_1 \text{ 是环 } R \text{ 的理想, 且 } I \subseteq A_1$$

$$\because x - \mu \in A_1 \quad \because \mu \in A_1, A_1 \text{ 是环 } R \text{ 的理想} \quad \because (x - \mu) + \mu \in A_1$$

$$\because (x - \mu) + \mu = (x + (-\mu)) + \mu = x + ((-\mu) + \mu) = x + 0_R = x$$

$$\because x \in A_1 \quad \because A_2 \subseteq A_1 \quad \because A_1 = A_2$$

$$\because f: S \rightarrow T \text{ 是一个单射.}$$

$$\text{对 } \forall B \in T, \text{ 有: } B \text{ 是环 } R/I \text{ 的理想}$$

$$\because B \text{ 是环 } R/I \text{ 的理想} \quad \because B \text{ 能表示成 } A/I \text{ 的形式, 其中 } A \text{ 是 } R \text{ 的理想}$$

$$\text{且有 } I \subseteq A \subseteq R$$

$$\because A \in S \text{ 且 } B = A/I$$

$$\because f(A) = A/I = B \quad \because f: S \rightarrow T \text{ 是一个满射}$$

$$\because f: S \rightarrow T \text{ 是一个双射.}$$

$$\text{对 } \forall A_1, A_2 \in S,$$

$$\text{若 } A_1 \subseteq A_2, \text{ 则有: } \cancel{\because A_1 \in S} \quad \cancel{\because A_1 \text{ 是环 } R \text{ 的理想, 且}}$$

$$\because A_1 \in S \quad \because f(A_1) = A_1/I \text{ 是环 } R/I \text{ 的理想}$$

$$\because A_2 \in S \quad \because f(A_2) = A_2/I \text{ 是环 } R/I \text{ 的理想}$$

$$\text{任取 } A_1/I \text{ 中的一元: } x+I \text{ (其中 } x \in A_1 \text{)}$$

$$\because x \in A_1 \subseteq A_2 \quad \because x \in A_2 \quad \because x+I \in A_2/I$$

$$\because A_1/I \subseteq A_2/I \quad \because f(A_1) \subseteq f(A_2)$$

$$\text{若 } f(A_1) \subseteq f(A_2), \text{ 则有: } A_1/I \subseteq A_2/I$$

$$\text{对 } \forall x \in A_1, \text{ 有: } x+I \in A_1/I \subseteq A_2/I \quad \because x+I \in A_2/I$$

$$\because \exists \lambda \in A_2, \text{ s.t. } x+I = \lambda+I \quad \because x \equiv_I \lambda \quad \because x-\lambda \in I$$

$$\because A_2 \in S \quad \because A_2 \text{ 是环 } R \text{ 的理想, 且 } I \subseteq A_2 \quad \because x-\lambda \in A_2$$

$$\because x-\lambda \in A_2, \lambda \in A_2, A_2 \text{ 是环 } R \text{ 的理想} \quad \because (x-\lambda)+\lambda \in A_2$$

$$\because (x-\lambda)+\lambda = (x+(-\lambda))+\lambda = x+((- \lambda)+\lambda) = x+0_R = x$$

$$\because x \in A_2 \quad \because A_1 \subseteq A_2$$

$$\because A_1 \subseteq A_2 \iff f(A_1) \subseteq f(A_2)$$

$$\because f: S \rightarrow T \text{ 是一个保持包含关系的双射.}$$

对 $\forall A \in S$, 有:

$\because A \in S \quad \therefore A$ 是环 R 的理想, 且 $I \subseteq A \quad \therefore A \subseteq R$

$\because A \in S \quad \therefore f(A) = A/I$ 是环 R/I 的理想 $\therefore f(A) \subseteq R/I$

\because 商同态 $q: R \rightarrow R/I$ 是满同态
 $x \mapsto x+I$

$\therefore q^{-1}(f(A)) = \{x \in R \mid q(x) \in f(A)\} \subseteq R$

$\therefore A \subseteq R$ 且 $q^{-1}(f(A)) \subseteq R$

对 $\forall x \in A$, 有: $x \in R$

$\therefore q(x) = x+I \in A/I = f(A) \quad \therefore q(x) \in f(A) \quad \therefore x \in q^{-1}(f(A))$

$\therefore A \subseteq q^{-1}(f(A))$

对 $\forall x \in q^{-1}(f(A))$, 有: $x \in R$ 且 $q(x) \in f(A)$

$\therefore x+I \in A/I \quad \therefore \exists a \in A, \text{ s.t. } x+I = a+I \quad \therefore x \equiv_I a$

$\therefore x-a \in I \quad \because I \subseteq A \quad \therefore x-a \in A$

$\therefore x-a \in A, a \in A, A$ 是环 R 的理想 $\therefore (x-a)+a \in A$

$\therefore (x-a)+a = (x+(-a))+a = x+((-a)+a) = x+0_R = x$

$\therefore x \in A \quad \therefore q^{-1}(f(A)) \subseteq A \quad \therefore A = q^{-1}(f(A))$

对 $\forall B \in T$, 有: B 是环 R/I 的理想 $\therefore B \subseteq R/I$

$\because B$ 是环 R/I 的理想 $\therefore B$ 能表示成 A/I 的形式, 其中 A 是 R 的理想

且有 $I \subseteq A \subseteq R \quad \therefore A \in S$ 且 $B = A/I$

$\therefore f(A) = A/I = B$

$\therefore f: S \rightarrow T$ 是一个双射 $\therefore f: S \rightarrow T$ 是一个可逆映射.

\therefore 存在映射 $f^{-1}: T \rightarrow S$, s.t. $f^{-1}f = \text{id}_S$ 且 $ff^{-1} = \text{id}_T$

$$\therefore f^{-1}(B) = f^{-1}(f(A)) = (f^{-1} \circ f)(A) = \text{id}_S(A) = A$$

$\therefore A$ 是 R 的理想 $\therefore A \subseteq R$

$$\therefore B \subseteq R/I \quad \therefore \varphi^{-1}(B) = \{x \in R \mid \varphi(x) \in B\} \subseteq R$$

又 $\forall x \in A$, 有: $\exists x \in R$, 且 $\varphi(x) = x+I \in A/I = B$

$$\therefore x \in \varphi^{-1}(B) \quad \therefore A \subseteq \varphi^{-1}(B)$$

又 $\forall x \in \varphi^{-1}(B)$, 有: $x \in R$ 且 $\varphi(x) \in B \quad \therefore x+I \in B = A/I$

$$\therefore \exists a \in A, \text{ s.t. } x+I = a+I \quad \therefore x \equiv_I a$$

$$\therefore x-a \in I \quad \therefore I \subseteq A \quad \therefore x-a \in A$$

$$\therefore x-a \in A, a \in A, A \text{ 是 } R \text{ 的理想} \quad \therefore (x-a)+a \in A$$

$$\therefore (x-a)+a = (x+(-a))+a = x+((-a)+a) = x+0_R = x$$

$$\therefore x \in A \quad \therefore \varphi^{-1}(B) \subseteq A \quad \therefore A = \varphi^{-1}(B) \quad \therefore f^{-1}(B) = A = \varphi^{-1}(B)$$

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