Lemma: R是环，I是R的理想，则有:

$$I=R \Longleftrightarrow I\cap R^\times \neq \emptyset$$

Proof: $\because$ I是R的理想 $\therefore I\subseteq R$ 且 $I\neq\emptyset$

$\because R^\times\subseteq R$ 且 $1_R\in R^\times$ $\therefore R^\times\subseteq R$ 且 $R^\times\neq\emptyset$

($\Rightarrow$): $\because I=R$ $\therefore 1_R\in R=I$ $\therefore 1_R\in I$

$\because$ R是环 $\therefore 1_R\in R^\times$ $\therefore 1_R\in I\cap R^\times$

$\therefore I\cap R^\times\neq\emptyset$

($\Leftarrow$): $\because I\cap R^\times\neq\emptyset$ $\therefore \exists x\in I\cap R^\times$ $\therefore x\in I$

$\because x\in R^\times$ $\therefore x^{-1}\in R^\times$ $\therefore x^{-1}\in R$

$\because$ I是R的理想 $\therefore x^{-1}I\subseteq I$

$\therefore x^{-1}x\in x^{-1}I\subseteq I$ $\because x^{-1}x=1_R$ $\therefore 1_R\in I$

$\therefore I=R$ $\square$

Lemma: R是交换环，$x\in R$，则有:

$x\in R^\times \Longleftrightarrow$ 对$\forall r\in R$，$x\mid r \Longleftrightarrow x\mid 1_R \Longleftrightarrow (x)=R$

Proof: ①$\Rightarrow$②: 对$\forall r\in R$,

$\because x\in R^\times$ $\therefore \exists x^{-1}\in R^\times$, s.t. $x^{-1}x=1_R=xx^{-1}$

$\therefore r=r\cdot 1_R=r(x^{-1}x)=(rx^{-1})x$

$\because r\in R, x\in R, x^{-1}\in R$ $\therefore rx^{-1}\in R$ $\therefore x\mid r$

②$\Rightarrow$③ $\because$ 对$\forall r\in R$, $x\mid r$ 又$\because 1_R\in R$ $\therefore x\mid 1_R$

③ $\Rightarrow$ ④：$\because (x) = xR = \{xr : r\in R\} \subseteq R$

对 $\forall \alpha \in R$，$\because x \mid 1_R$ $\therefore \exists d\in R$, s.t. $1_R = dx$

$\because$ R是交换环 $\therefore 1_R = dx = xd$ $\therefore x\in R^\times$ 且 $d = x^{-1}$

$\because d\in R$ 且 $\alpha\in R$ $\therefore d\alpha\in R$

$\therefore \alpha = 1_R\cdot\alpha = (xd)\alpha = x(d\alpha) \in (x)$

$\therefore R\subseteq (x)$ $\therefore (x) = R$

④ $\Rightarrow$ ① $\because 1_R \in R = (x) = xR = \{xr : r\in R\}$

$\therefore \exists \lambda\in R$, s.t. $1_R = x\lambda$

$\because$ R是交换环 $\therefore 1_R = x\lambda = \lambda x$

$\therefore x\in R^\times$ 且 $\lambda = x^{-1}$ $\square$

定义（素理想）R是交换环，I是R的真理想，如果

对 $\forall x, y\in R$，$xy\in I \Rightarrow x\in I$ 或 $y\in I$

则称I是R的素理想

定义（极大理想）R是交换环，I是R的真理想，如果不存在严格包含I的理想，则称I是R的极大理想.

定义（真理想）R是环，I是R的理想，若 $I\neq R$，则称I是R的真理想.

Lemma：R是整环，$p\in R$，则有：

$p$是素元 $\Leftrightarrow$ $(p)$是非零素理想.

Proof: $\because$ R是整环 $\therefore$ R是交换环.

$\because p\in R$ $\therefore (p)=pR=\{pr: r\in R\}$是R的理想.

($\Rightarrow$): $\because$ p是素元 $\therefore p\neq 0_R$ 且 $p\notin R^{\times}$

$\because p\notin R^{\times}$ $\therefore (p)\neq R$ $\therefore$ (p)是R的真理想

对$\forall x,y\in R$, 若 $xy\in(p)$, 则有: $\exists r\in R$, s.t. $xy=pr$

$\because xy\in R$, $p\in R$, $r\in R$, $xy=rp$ $\therefore p\mid xy$

$\because$ p是素元 $\therefore p\mid x$ 或 $p\mid y$

若 $p\mid x$, 则 $\exists d_1\in R$, s.t. $x=d_1p=pd_1\in(p)$ $\therefore x\in(p)$

若 $p\mid y$, 则 $\exists d_2\in R$, s.t. $y=d_2p=pd_2\in(p)$ $\therefore y\in(p)$

$\therefore x\in(p)$ 或 $y\in(p)$

$\therefore$ (p)是R的素理想.

假设 $(p)=\{0_R\}$, 则有: $p=p\cdot 1_R\in(p)=\{0_R\}$ $\therefore p=0_R$

矛盾. $\therefore (p)\neq\{0_R\}$

$\therefore$ (p)是R的非零素理想.

($\Leftarrow$): 假设 $p=0_R$, 则有:

对$\forall r\in R$, $pr=0_R\cdot r=0_R$ $\therefore (p)=\{0_R\}$ $\therefore$ (p)是R的零理想

矛盾. $\therefore p\neq 0_R$

假设 $p\in R^{\times}$, 则有: $(p)=R$. $\because$ (p)是R的素理想

$\therefore$ (p)是R的真理想 $\therefore (p)\neq R$ 矛盾. $\therefore p\notin R^{\times}$

对$\forall a,b\in R$, 若 $p\mid ab$, 则有: $\exists d\in R$, s.t. $ab=dp$

$\therefore ab = dp = pd \in (p)$

$\because (p)$是$R$的素理想，$a, b \in R$，$ab \in (p)$

$\therefore a \in (p)$或$b \in (p)$

若$a \in (p)$，则$\exists d_1 \in R$，s.t. $a = pd_1 = d_1 p$ $\quad \therefore p \mid a$

若$b \in (p)$，则$\exists d_2 \in R$，s.t. $b = pd_2 = d_2 p$ $\quad \therefore p \mid b$

$\therefore p \mid a$或$p \mid b$ $\quad \therefore p$是素元. $\quad \square$

Lemma（主理想环的Noether性质）$R$是主理想环，$(I_n)_{n=1}^{\infty}$是$R$的一列理想，满足$I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq I_5 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$ 则$\exists N \in \mathbb{Z}_{\geq 1}$，s.t. 对$\forall n \in \mathbb{Z}_{\geq 1}$且$n \geq N$，都有$I_n = I_N$.

Proof: 令$I = \bigcup\limits_{n=1}^{\infty} I_n$.

$\because$对$\forall n \in \mathbb{Z}_{\geq 1}$，$I_n$是$R$的理想 $\quad \therefore$对$\forall n \in \mathbb{Z}_{\geq 1}$，$I_n \subseteq R$且$I_n \neq \emptyset$

$\therefore I = \bigcup\limits_{n=1}^{\infty} I_n \subseteq R$

$\because I_1 \neq \emptyset$，$I_1 \subseteq \bigcup\limits_{n=1}^{\infty} I_n = I$ $\quad \therefore I \neq \emptyset$ $\quad \therefore I$是$R$的非空子集.

对$\forall x, y \in I$，有：$\because x \in I = \bigcup\limits_{n=1}^{\infty} I_n$ $\quad \therefore \exists n_1 \in \mathbb{Z}_{\geq 1}$，s.t. $x \in I_{n_1}$

$\because y \in I = \bigcup\limits_{n=1}^{\infty} I_n$ $\quad \therefore \exists n_2 \in \mathbb{Z}_{\geq 1}$，s.t. $y \in I_{n_2}$

任取$n_3 \in \mathbb{Z}_{\geq 1}$，s.t. $n_3 > n_1$且$n_3 > n_2$.

$\because n_3 > n_1$ $\quad \therefore I_{n_1} \subseteq I_{n_3}$ $\quad \therefore x \in I_{n_3}$

$\because n_3 > n_2$ $\quad \therefore I_{n_2} \subseteq I_{n_3}$ $\quad \therefore y \in I_{n_3}$

$\because I_{n_3}$ 是 $R$ 的理想， $x,y\in I_{n_3}$ $\therefore x+y\in I_{n_3}\subseteq \bigcup_{n=1}^{\infty} I_n = I$

对 $\forall r\in R$，

任取 $rI$ 中的一元： $rx$ (其中 $x\in I$)． $\because x\in I=\bigcup_{n=1}^{\infty} I_n$

$\therefore \exists n_4\in\mathbb{Z}_{\geqslant 1}$, s.t. $x\in I_{n_4}$ $\therefore rx\in rI_{n_4}\subseteq I_{n_4}\subseteq I$ $\therefore rI\subseteq I$

任取 $Ir$ 中的一元：$xr$ (其中 $x\in I$) $\because x\in I=\bigcup_{n=1}^{\infty} I_n$

$\therefore \exists n_5\in\mathbb{Z}_{\geqslant 1}$, s.t. $x\in I_{n_5}$ $\therefore xr\in I_{n_5}r\subseteq I_{n_5}\subseteq I$ $\therefore Ir\subseteq I$

$\therefore$ $I$ 是 $R$ 的理想．

$\because$ $R$ 是主理想环 $\therefore$ $I$ 是 $R$ 的一个主理想．

$\therefore \exists \lambda\in R$，s.t. $I=(\lambda)=\lambda R=\{\lambda r: r\in R\}$．

$\because \lambda=\lambda\cdot 1_R\in(\lambda)=I=\bigcup_{n=1}^{\infty} I_n$ $\therefore \exists N\in\mathbb{Z}_{\geqslant 1}$, s.t. $\lambda\in I_N$

对 $\forall r\in R$，$\lambda r=r\lambda\in rI_N\subseteq I_N$ $\therefore (\lambda)\subseteq I_N$ $\therefore I\subseteq I_N$

对 $\forall n\in\mathbb{Z}_{\geqslant 1}$ 且 $n\geqslant N$，有：$\because n\geqslant N$ $\therefore I_N\subseteq I_n\subseteq I\subseteq I_N$

$\therefore I_n=I_N$ $\square$