Lemma：R是环，I是R的理想，定义两个集合：

$$S=\{A \mid A\text{是环}R\text{的子环，且}I\subseteq A\}$$

$$T=\{B \mid B\text{是环}R/I\text{的子环}\}$$

定义映射 $f: S \longrightarrow T$　则有：$f: S\to T$是一个保持包含关系的双射

$A \longmapsto A/I$.

Proof：∵ R是环，I是R的理想　∴ R/I是环.

∵ R是环R的子环，且 $I\subseteq R$　∴ $R\in S$　∴ $S\neq\emptyset$

∵ R/I是环R/I的子环　∴ $R/I\in T$　∴ $T\neq\emptyset$

对 $\forall A\in S$，有：A是环R的子环，且 $I\subseteq A$

∴ $(A,+,\cdot,0_R,1_R)$是环

∵ I是R的理想　∴ $I\neq\emptyset$　∴ I是环A的非空子集

对 $\forall x,y\in I$，∵ I是R的理想　∴ $x+y\in I$

对 $\forall a\in A$，∵ $a\in A\subseteq R$　∴ $a\in R$

∵ I是R的理想　∴ $aI\subseteq I$ 且 $Ia\subseteq I$

∴ I是环A的理想　∴ A/I是环.

∵ R是环，I是R的理想，A是R的子环，且 $I\subseteq A\subseteq R$

∴ A/I是环R/I的子环

∴ $f(A)=A/I$是环R/I的子环.　∴ $f(A)\in T$

∴ $f(S)\subseteq T$

对$\forall A_1, A_2 \in S$

若$A_1 = A_2$，则有：$\because A_1 \in S$ $\therefore f(A_1) = A_1/I$是环$R/I$的子环

$\because A_2 \in S$ $\therefore f(A_2) = A_2/I$是环$R/I$的子环

任取A_1/I中的一元：$x+I$（其中$x \in A_1$） $\because x \in A_1 = A_2$

$\therefore x+I \in A_2/I$ $\therefore A_1/I \subseteq A_2/I$

任取A_2/I中的一元：$x+I$（其中$x \in A_2$） $\because x \in A_2 = A_1$

$\therefore x+I \in A_1/I$ $\therefore A_2/I \subseteq A_1/I$ $\therefore A_1/I = A_2/I$

$\therefore f(A_1) = A_1/I = A_2/I = f(A_2)$ $\therefore f: S \to T$是一个映射

若$f(A_1) = f(A_2)$，则有：$A_1/I = A_2/I$

对$\forall x \in A_1$，有：$x+I \in A_1/I = A_2/I$

$\therefore \exists \lambda \in A_2$, s.t. $x+I = \lambda + I$ $\therefore x \equiv_I \lambda$

$\therefore x - \lambda \in I$ $\because A_2 \in S$ $\therefore A_2$是环R的子环，且$I \subseteq A_2$

$\therefore x - \lambda \in A_2$ $\because \lambda \in A_2$ $\therefore (x-\lambda)+\lambda \in A_2$

$\because (x-\lambda)+\lambda = (x+(-\lambda))+\lambda = x+((-\lambda)+\lambda) = x+0_R = x$

$\therefore x \in A_2$ $\therefore A_1 \subseteq A_2$

对$\forall x \in A_2$，有：$x+I \in A_2/I = A_1/I$

$\therefore \exists \mu \in A_1$, s.t. $x+I = \mu + I$ $\therefore x \equiv_I \mu$

$\therefore x - \mu \in I$ $\because A_1 \in S$ $\therefore A_1$是环R的子环，且$I \subseteq A_1$

$\therefore x - \mu \in A_1$ $\because \mu \in A_1$ $\therefore (x-\mu)+\mu \in A_1$

$\because (x-\mu)+\mu = (x+(-\mu))+\mu = x+((-\mu)+\mu) = x+0_R = x$

$\therefore x \in A_1$ $\therefore A_2 \subseteq A_1$ $\therefore A_1 = A_2$ $\therefore f: S \to T$是一个单射

对$\forall B\in T$，有：B是环R/I的子环

$\because R$是环，I是R的理想，B是环R/I的子环

$\therefore B$能表示成A/I的形式，其中A是R的子环且有$I\subseteq A\subseteq R$

$\therefore A\in S$，且有$f(A)=A/I=B$　$\therefore f: S\to T$是一个满射

$\therefore f: S\to T$是一个双射.

对$\forall A_1, A_2\in S$，

若$A_1\subseteq A_2$，则有：$\because A_1\in S$　$\therefore f(A_1)=A_1/I$是环R/I的子环

$\because A_2\in S$　$\therefore f(A_2)=A_2/I$是环R/I的子环

任取A_1/I中的一元：$x+I$（其中$x\in A_1$）

$\therefore x\in A_1\subseteq A_2$　$\therefore x\in A_2$　$\therefore x+I\in A_2/I$

$\therefore A_1/I\subseteq A_2/I$　$\therefore f(A_1)\subseteq f(A_2)$

若$f(A_1)\subseteq f(A_2)$，则有：$A_1/I\subseteq A_2/I$

对$\forall x\in A_1$，有：$x+I\in A_1/I\subseteq A_2/I$　$\therefore x+I\in A_2/I$

$\therefore \exists \lambda\in A_2$，s.t. $x+I=\lambda+I$　$\therefore x\equiv_I \lambda$　$\therefore x-\lambda\in I$

$\because A_2\in S$　$\therefore A_2$是环R的子环，且$I\subseteq A_2$　$\therefore x-\lambda\in A_2$

$\because x-\lambda\in A_2$，$\lambda\in A_2$，$A_2$是环$R$的子环　$\therefore (x-\lambda)+\lambda\in A_2$

$\because (x-\lambda)+\lambda=(x+(-\lambda))+\lambda=x+((-\lambda)+\lambda)=x+0_R=x$

$\therefore x\in A_2$　$\therefore A_1\subseteq A_2$

$\therefore A_1\subseteq A_2 \iff f(A_1)\subseteq f(A_2)$

$\therefore f: S\to T$是一个保持包含关系的双射.　$\square$

Lemma: R是环，I是R的理想，$A\subseteq R$，A在R的加法运算和加法取逆 $x\mapsto -x$ 之下封闭，$I\subseteq A$，则有：

$$A\text{是环}R\text{的理想} \iff A/I\text{是环}R/I\text{的理想}$$

Proof: $\because$ R是环，I是R的理想 $\therefore I\neq\emptyset$

$\because I\subseteq A$ $\therefore A\neq\emptyset$ $\therefore$ A是环R的非空子集

$\therefore A/I=\{x+I \mid x\in A\}\subseteq R/I$

$\because 0_R\in I\subseteq A$ $\therefore 0_R\in A$ $\therefore 0_R+I\in A/I$ $\therefore A/I\neq\emptyset$

$\therefore$ A/I是环R/I的非空子集 $\because 0_{R/I}=0_R+I\in A/I$ $\therefore 0_{R/I}\in A/I$

$\therefore$ A是环R的非空子集，且$0_R\in A$

A/I是环R/I的非空子集，且$0_{R/I}\in A/I$

$(\Rightarrow)$：对$\forall\, x+I,\ y+I\in A/I$ (其中 $x,y\in A$)

$\because$ A对R的加法运算封闭，$x,y\in A$ $\therefore x+y\in A$

$\therefore (x+y)+I\in A/I$

$\therefore (x+I)+(y+I)=(x+y)+I\in A/I$

对$\forall \lambda+I\in R/I$ (其中 $\lambda\in R$)

任取 $(\lambda+I)(A/I)$ 中的一元：$(\lambda+I)\alpha$ (其中 $\alpha\in A/I$)

$\because \alpha\in A/I$ $\therefore \exists\, x\in A$, s.t. $\alpha=x+I$

$\therefore (\lambda+I)\alpha=(\lambda+I)(x+I)=\lambda x+I$

$\because \lambda\in R$，A是环R的理想 $\therefore \lambda A\subseteq A$

$\therefore \lambda x\in\lambda A\subseteq A$ $\therefore \lambda x+I\in A/I$ $\therefore (\lambda+I)\alpha\in A/I$

$\therefore (\lambda+I)(A/I)\subseteq A/I$.

任取$(A/I)(\lambda+I)$中的一元：$\beta(\lambda+I)$（其中$\beta\in A/I$）

$\because \beta\in A/I \quad \therefore \exists y\in A$，s.t. $\beta=y+I$

$\therefore \beta(\lambda+I)=(y+I)(\lambda+I)=y\lambda+I$

$\because \lambda\in R$，A是环R的理想 $\quad \therefore A\lambda\subseteq A$

$\therefore y\lambda\in A\lambda\subseteq A \quad \therefore y\lambda+I\in A/I \quad \therefore \beta(\lambda+I)\in A/I$

$\therefore (A/I)(\lambda+I)\subseteq A/I$

$\therefore A/I$是环R/I的理想

$(\Leftarrow)$：对$\forall\, x,y\in A$，有：

$\because x,y\in A \quad \therefore x+I,\ y+I\in A/I$

$\because A/I$是环R/I的理想 $\quad \therefore (x+I)+(y+I)\in A/I$

$\because (x+I)+(y+I)=(x+y)+I \quad \therefore (x+y)+I\in A/I$

$\therefore \exists a\in A$，s.t. $(x+y)+I=a+I \quad \therefore (x+y)\equiv_I a$

$\therefore (x+y)-a\in I\subseteq A \quad \therefore (x+y)-a\in A$

$\because A$对R的加法运算封闭 $\quad \therefore \big((x+y)-a\big)+a\in A$

$\because \big((x+y)-a\big)+a=\big((x+y)+(-a)\big)+a=(x+y)+\big((-a)+a\big)=(x+y)+0_R=x+y$

$\therefore x+y\in A$

对$\forall\, r\in R$，

任取rA中的一元：ra（$a\in A$）

$\because r\in R \quad \therefore r+I\in R/I \quad \because a\in A \quad \therefore a+I\in A/I$

$\because A/I$是环R/I的理想

$\therefore (r+I)(a+I)\in(r+I)(A/I)\subseteq A/I \quad \therefore ra+I\in A/I$

$\therefore \exists \alpha \in A$, s.t. $ra + I = \alpha + I$ $\quad \therefore ra \equiv_I \alpha$ $\quad \therefore ra - \alpha \in I \subseteq A$

$\because$ A在R的加法运算之下封闭 $\quad \therefore (ra - \alpha) + \alpha \in A$

$\therefore (ra - \alpha) + \alpha = (ra + (-\alpha)) + \alpha = ra + ((-\alpha) + \alpha) = ra + 0_R = ra$

$\therefore ra \in A$ $\quad \therefore rA \subseteq A$

任取 Ar 中的一元：ar $(a \in A)$

$\because r \in R$ $\quad \therefore r + I \in R/I$ $\quad \because a \in A$ $\quad \therefore a + I \in A/I$

$\because A/I$ 是环 R/I 的理想

$\therefore (a+I)(r+I) \in (A/I)(r+I) \subseteq A/I$ $\quad \therefore ar + I \in A/I$

$\therefore \exists \beta \in A$, s.t. $ar + I = \beta + I$ $\quad \therefore ar \equiv_I \beta$ $\quad \therefore ar - \beta \in I \subseteq A$

$\because$ A在R的加法运算之下封闭 $\quad \therefore (ar - \beta) + \beta \in A$

$\because (ar - \beta) + \beta = (ar + (-\beta)) + \beta = ar + ((-\beta) + \beta) = ar + 0_R = ar$

$\therefore ar \in A$ $\quad \therefore Ar \subseteq A$

$\therefore$ A是环R的理想 $\quad \square$

定理：R是环，I是R的理想，$A \subseteq R$，A是R的子环，$I \subseteq A$，则有：

$$A\text{是环}R\text{的理想} \iff A/I\text{是环}R/I\text{的理想}$$

Proof: $\because$ A是R的子环

$\therefore$ A在R的加法运算和加法取逆 $x \mapsto -x$ 之下封闭

$\therefore$ 由上一引理立得. $\quad \square$