Lemma：R是环，A是R的子环，则一定存在从A到R的单同态.

Proof：∵ A是R的子环　∴ $A \subseteq R$ 且 $0_R, 1_R \in A$.

∴ $0_A = 0_R$ 且 $1_A = 1_R$.

定义映射 $\iota: A \longrightarrow R$
$x \longmapsto x$

对 $\forall x \in A$，∵ $A \subseteq R$　∴ $x \in R$　∴ $\iota(x) = x \in R$

∴ $\iota(A) \subseteq R$

对 $\forall x_1, x_2 \in A$.

若 $x_1 = x_2$，则有：$\iota(x_1) = x_1 = x_2 = \iota(x_2)$　∴ $\iota: A \to R$ 是一个映射.

若 $\iota(x_1) = \iota(x_2)$，则有：$x_1 = x_2$　∴ $\iota: A \to R$ 是一个单射.

$\iota(x_1 + x_2) = x_1 + x_2 = \iota(x_1) + \iota(x_2)$

$\iota(x_1 x_2) = x_1 x_2 = \iota(x_1)\,\iota(x_2)$

$\iota(1_A) = 1_A = 1_R$

∴ $\iota: A \to R$ 是单同态.　$\square$

Lemma：R是整环，F是域，域F包含整环R作为子环. 则一定存在从R到F的单同态.

Proof：∵ F是域　∴ F是环. ∵ 域F包含整环R作为子环

∴ R是F的子环.

∴ 一定存在从R到F的单同态.　$\square$　(Remark：唯一性一般不能保证. 之后找不唯一的例子)

Lemma（整环 R 上的分式域的泛性质）R 是一个任意的整环，

已经证明了 $\mathrm{Frac}(R)$ 是域，$f: R \longrightarrow \mathrm{Frac}(R)$ 是单同态.

$$x \longmapsto [x,1]$$

F 是一个域，域 F 包含整环 R 作为子环，已经证明了存在从 R 到 F 的单同态. 设 $\varphi: R \to F$ 是任意一个从 R 到 F 的单同态.

F 的所有元素都能表成 $\varphi(x)(\varphi(y))^{-1}$ 的形式，其中 $x, y \in R$ 且 $y \neq 0$.

则有：存在唯一的环同构 $\Theta: \mathrm{Frac}(R) \longrightarrow F$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ \mathrm{Frac}(R) \arrow[rr, "\Theta"'] & & F \end{tikzcd}$$

Proof：$\because$ F 是域　$\therefore$ F 是交换环

$\because$ $\varphi: R \to F$ 是单同态　$\therefore$ $\varphi(R \backslash \{0_R\}) \subseteq F^{\times}$

$\therefore$ 存在唯一的环同态 $\Theta: \mathrm{Frac}(R) \longrightarrow F$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ \mathrm{Frac}(R) \arrow[rr, "\Theta"'] & & F \end{tikzcd}$$

对 $\forall [x_1, y_1], [x_2, y_2] \in \mathrm{Frac}(R)$，若 $\Theta([x_1, y_1]) = \Theta([x_2, y_2])$，

则有：

$\because [x_1, y_1], [x_2, y_2] \in Frac(R)$ $\therefore y_1, y_2 \in R$ 且 $y_1 \neq 0, y_2 \neq 0$

$\therefore y_1 y_2 \in R$ $\therefore [y_1 y_2, 1] \in Frac(R)$ $\therefore \Theta([y_1 y_2, 1]) \in F$

$\because \Theta([x_1, y_1]) = \Theta([x_2, y_2])$

$\therefore \Theta([x_1, y_1]) \cdot \Theta([y_1 y_2, 1]) = \Theta([x_2, y_2]) \cdot \Theta([y_1 y_2, 1])$

$\therefore \Theta([x_1, y_1] \cdot [y_1 y_2, 1]) = \Theta([x_2, y_2] \cdot [y_1 y_2, 1])$

$\therefore \Theta([x_1 y_1 y_2, y_1]) = \Theta([x_2 y_1 y_2, y_2])$

$\therefore \Theta([x_1 y_2, 1]) = \Theta([x_2 y_1, 1])$

$\therefore \Theta(f(x_1 y_2)) = \Theta(f(x_2 y_1))$

$\therefore \varphi(x_1 y_2) = \varphi(x_2 y_1)$

$\because \varphi: R \to F$ 是单同态 $\therefore x_1 y_2 = x_2 y_1$

$\therefore [x_1, y_1] = [x_2, y_2]$ $\therefore \Theta: Frac(R) \to F$ 是单射.

对 $\forall \lambda \in F$, $\exists x, y \in R$ 且 $y \neq 0$, s.t. $\lambda = \varphi(x)(\varphi(y))^{-1}$

$\because y \in R$ 且 $y \neq 0$ $\therefore y \in R \setminus \{0_R\}$ $\therefore \varphi(y) \in F^{\times}$

$\therefore [x, y] \in Frac(R)$, 且有:

$\Theta([x, y]) = \Theta([x \cdot 1, 1 \cdot y]) = \Theta([x, 1] \cdot [1, y])$

$= \Theta([x, 1]) \cdot \Theta([1, y])$

对 $\forall y \in R$ 且 $y \neq 0$，有：

$$1_F = \Theta(1_{Frac(R)}) = \Theta([y,y]) = \Theta([y,1]\cdot[1,y])$$

$$= \Theta([y,1])\cdot\Theta([1,y]) = \Theta(f(y))\cdot\Theta([1,y])$$

$$= \varphi(y)\cdot\Theta([1,y]) \qquad \because y \in R\setminus\{0_R\} \quad \therefore \varphi(y) \in F^{\times}$$

$$\therefore \Theta([1,y]) = (\varphi(y))^{-1}$$

$$\therefore \Theta([x,y]) = \Theta([x,1])\cdot\Theta([1,y])$$

$$= \Theta(f(x))\cdot(\varphi(y))^{-1} = \varphi(x)\cdot(\varphi(y))^{-1} = \lambda$$

$\therefore \Theta : Frac(R) \longrightarrow F$ 是满射.

$\therefore \Theta : Frac(R) \longrightarrow F$ 是双射.

$\therefore \Theta : Frac(R) \longrightarrow F$ 是环同构.

$\therefore$ 存在唯一的环同构 $\Theta : Frac(R) \longrightarrow F$，s.t. 下图交换：

$$\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\
Frac(R) \arrow[rr, "\Theta"'] & & F
\end{tikzcd}$$

$\square$