

Lemma: R 是一个任意的环, 则有: $\text{id}_R : R \rightarrow R$ 是环同构
 $x \mapsto x$

Proof: 显然 $\text{id}_R : R \rightarrow R$ 是双射.
 $x \mapsto x$

$\forall x, y \in R$, 有:

$$\text{id}_R(x+y) = x+y = \text{id}_R(x) + \text{id}_R(y)$$

$$\text{id}_R(x \cdot y) = x \cdot y = \text{id}_R(x) \cdot \text{id}_R(y)$$

$$\text{id}_R(1_R) = 1_R$$

$\therefore \text{id}_R : R \rightarrow R$ 是环同态

$\therefore \text{id}_R : R \rightarrow R$ 是环同构. \square

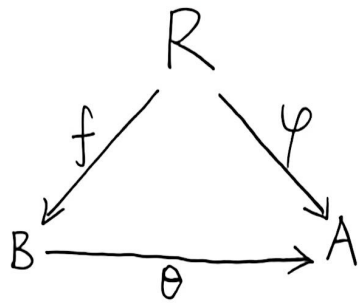
定理(任意环 R 上的一元多项式环的泛性质)

R 是一个任意的非零环.

B 是一个环, $f: R \rightarrow B$ 是一个环同态, $b \in B$, 满足:

对 \forall 环 A , \forall 环同态 $\varphi: R \rightarrow A$, $\forall a \in A$, \exists 唯一的环同态

$\theta: B \rightarrow A$, s.t. 下图交换:

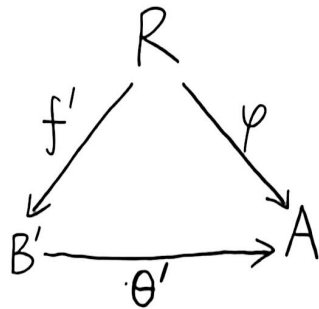


且有 $\theta(b) = a$.

B' 是另一个环, $f': R \rightarrow B'$ 是另一个环同态, $b' \in B'$, 满足:

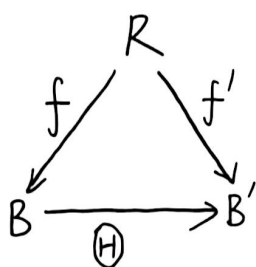
对 \forall 环 A , \forall 环同态 $\varphi: R \rightarrow A$, $\forall a \in A$, \exists 唯一的环同态

$\theta': B' \rightarrow A$, s.t. 下图交换:



且有 $\theta'(b') = a$

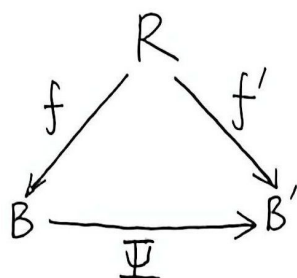
则存在唯一的映射 $\theta: B \rightarrow B'$, s.t. 下图交换:



且 $\Theta(b) = b'$, $\Theta: B \rightarrow B'$ 是环同构.

Proof: 对于环 B' , 环同态 $f': R \rightarrow B'$, $b' \in B'$, \exists 唯一的环同态

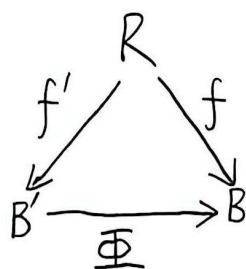
$\Psi: B \rightarrow B'$, s.t. 下图交换:



且有: $\Psi(b) = b'$

对于环 B , 环同态 $f: R \rightarrow B$, $b \in B$, \exists 唯一的环同态

$\Phi: B' \rightarrow B$, s.t. 下图交换:



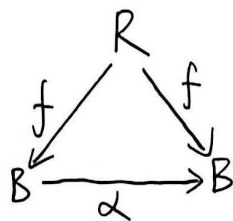
且有 $\Phi(b') = b$

$\therefore \Psi: B \rightarrow B'$ 和 $\Phi: B' \rightarrow B$ 都是环同态

$\therefore \Phi \circ \Psi: B \rightarrow B$ 和 $\Psi \circ \Phi: B' \rightarrow B'$ 都是环同态.

对于环 B , 环同态 $f: R \rightarrow B$, $b \in B$, 存在唯一的环同态

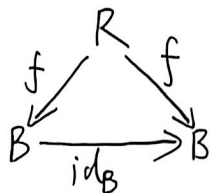
$\alpha: B \rightarrow B$, s.t. 下图交换:



且有 $\alpha(b) = b$

$\because B$ 是环 $\therefore \text{id}_B : B \rightarrow B$ 是环同构, 也是环同态.
 $x \mapsto x$

$\therefore \text{id}_B \circ f = f$ \therefore 下图交换:

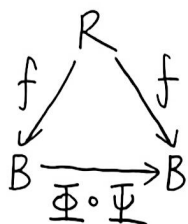


$\therefore \text{id}_B(b) = b$ $\therefore \text{id}_B = \alpha$

$\therefore \Psi \circ f = f'$ 且有 $\Phi \circ f' = f$

$\therefore (\Phi \circ \Psi) \circ f = \Phi \circ (\Psi \circ f) = \Phi \circ f' = f$

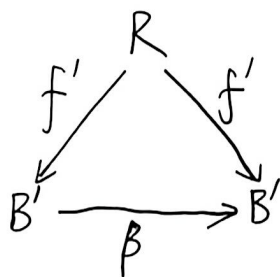
\therefore 下图交换:



$\therefore (\Phi \circ \Psi)(b) = \Phi(\Psi(b)) = \Phi(b') = b$

$\therefore \Phi \circ \Psi = \alpha = \text{id}_B$

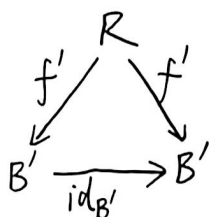
对于环 B' , 环同态 $f' : R \rightarrow B'$, $b' \in B'$, \exists 唯一的环同态 $\beta : B' \rightarrow B'$, s.t. 下图交换:



且有 $\beta(b') = b'$

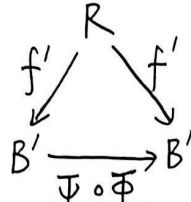
$\because B'$ 是环 $\therefore \text{id}_{B'} : B' \rightarrow B'$ 是环同构, 也是环同态.
 $x \mapsto x$

$\therefore \text{id}_{B'} \circ f' = f'$ \therefore 下图交换:



$$\therefore \text{id}_{B'}(b') = b' \quad \therefore \text{id}_{B'} = \beta$$

$$\therefore (\Psi \circ \Phi) \circ f' = \Psi \circ (\Phi \circ f') = \Psi \circ f = f'$$

$$\therefore \text{下图交换:}$$


$$\therefore (\Psi \circ \Phi)(b') = \Psi(\Phi(b')) = \Psi(b) = b'$$

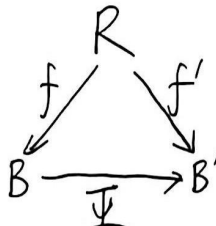
$$\therefore \Psi \circ \Phi = \beta = \text{id}_{B'}$$

$$\therefore \Phi \circ \Psi = \text{id}_B \text{ 且 } \Psi \circ \Phi = \text{id}_{B'}$$

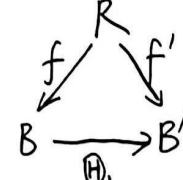
$$\therefore \Psi: B \rightarrow B' \text{ 是可逆映射} \quad \therefore \Psi: B \rightarrow B' \text{ 是双射.}$$

$$\therefore \Psi: B \rightarrow B' \text{ 是环同态} \quad \therefore \Psi: B \rightarrow B' \text{ 是环同构.}$$

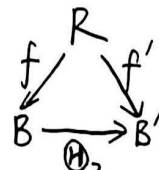
$$\therefore \Psi: B \rightarrow B' \text{ 是一个映射, 下图交换:}$$



$$\Psi(b) = b', \Psi: B \rightarrow B' \text{ 是环同构} \quad \therefore \text{存在性得证.}$$

$$\text{假设存在映射 } \Theta_1: B \rightarrow B', \text{ s.t. 下图交换:}$$


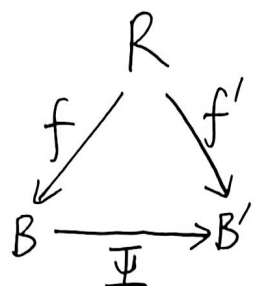
$$\text{且 } \Theta_1(b) = b', \Theta_1: B \rightarrow B' \text{ 是环同构.}$$

$$\text{存在另一个映射 } \Theta_2: B \rightarrow B', \text{ s.t. 下图交换:}$$


且 $\mathbb{H}_2(b) = b'$, $\mathbb{H}_2: B \rightarrow B'$ 是环同构.

\therefore 对于环 B' , 环同态 $f': R \rightarrow B'$, $b' \in B'$, \exists 唯一的环同态

$\underline{\Psi}: B \rightarrow B'$, s.t. 下图交换:



且有 $\underline{\Psi}(b) = b'$

$\therefore \mathbb{H}_1 = \underline{\Psi} = \mathbb{H}_2$

\therefore 唯一性得证.

