定理（商环的泛性质，同构唯一性部分）R是环，I是R的理想，Q是一个环，$f: R \to Q$是一个满足$I \subseteq \ker(f)$的环同态，满足：对$\forall$环A，$\forall$环同态$\varphi: R \to A$满足$I \subseteq \ker(\varphi)$，存在唯一的环同态$\theta: Q \to A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ Q \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

Q'是另一个环，$f': R \to Q'$是另一个满足$I \subseteq \ker(f')$的环同态，满足：对$\forall$环A，$\forall$环同态$\varphi: R \to A$满足$I \subseteq \ker(\varphi)$，存在唯一的环同态$\theta': Q' \to A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "\varphi"] & \\ Q' \arrow[rr, "\theta'"'] & & A \end{tikzcd}$$

则存在唯一的映射$\Theta: Q \to Q'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Q \arrow[rr, "\Theta"'] & & Q' \end{tikzcd}$$

且$\Theta: Q \to Q'$是环同构.

Proof: $\because$ Q'是环，$f': R \to Q'$是环同态，满足$I \subseteq \ker(f')$

$\therefore$ 存在唯一的环同态 $\Psi: Q \to Q'$，s.t. 下图交换

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Q \arrow[rr, "\Psi"'] & & Q' \end{tikzcd}$$

$\because$ Q是环，$f: R \to Q$是环同态，满足$I \subseteq \ker(f)$

$\therefore$ 存在唯一的环同态 $\Phi: Q' \to Q$，s.t. 下图交换

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ Q' \arrow[rr, "\Phi"'] & & Q \end{tikzcd}$$

$\because$ $\Psi: Q \to Q'$是环同态，$\Phi: Q' \to Q$是环同态

$\therefore$ $\Phi \circ \Psi: Q \to Q$是环同态，$\Psi \circ \Phi: Q' \to Q'$是环同态.

$\because$ Q是环，$f: R \to Q$是环同态，满足$I \subseteq \ker(f)$

$\therefore$ 存在唯一的环同态 $\alpha: Q \to Q$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ Q \arrow[rr, "\alpha"'] & & Q \end{tikzcd}$$

$\because$ Q是环 $\therefore$ $id_Q: Q \to Q$是环同构

$\therefore$ $id_Q: Q \to Q$是环同态

$\because$ $id_Q \circ f = f$ $\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ Q \arrow[rr, "id_Q"'] & & Q \end{tikzcd}$$

$\therefore$ $id_Q = \alpha$

$\because$ $(\Phi \circ \Psi) \circ f = \Phi \circ (\Psi \circ f) = \Phi \circ f' = f$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ Q \arrow[rr, "\Phi \circ \Psi"'] & & Q \end{tikzcd}$$

$\therefore$ $\Phi \circ \Psi = \alpha = id_Q$

$\because$ Q'是环，$f': R \to Q'$是环同态，满足$I \subseteq \ker(f')$

$\therefore$ 存在唯一的环同态 $\beta: Q' \to Q'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ Q' \arrow[rr, "\beta"'] & & Q' \end{tikzcd}$$

$\because Q'$是环 $\quad \therefore id_{Q'}: Q' \to Q'$是环同构 $\quad \therefore id_{Q'}: Q' \to Q'$是环同态

$\because id_{Q'} \circ f' = f'$ $\quad \therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ Q' \arrow[rr, "id_{Q'}"'] & & Q' \end{tikzcd}$$

$\therefore id_{Q'} = \beta$

$\because (\Phi \circ \Psi) \circ f' = \Phi \circ (\Psi \circ f') = \Phi \circ f = f'$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ Q' \arrow[rr, "\Phi \circ \Psi"'] & & Q' \end{tikzcd}$$

$\therefore \Phi \circ \Psi = \beta = id_{Q'}$

$\because Q$和Q'是环，$\Phi: Q \to Q'$是环同态，$\Psi: Q' \to Q$是环同态，

$\Psi \circ \Phi = id_Q$，$\Phi \circ \Psi = id_{Q'}$

$\therefore \Phi: Q \to Q'$是可逆映射 $\quad \therefore \Phi: Q \to Q'$是双射

$\therefore \Phi: Q \to Q'$是环同构. 存在性得证.

假设存在映射$\Theta_1: Q \to Q'$, s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Q \arrow[rr, "\Theta_1"'] & & Q' \end{tikzcd}$$

且$\Theta_1: Q \to Q'$是环同构

假设还存在映射$\Theta_2: Q \to Q'$, s.t. 下图交换

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Q \arrow[rr, "\Theta_2"'] & & Q' \end{tikzcd}$$

且$\Theta_2: Q \to Q'$是环同构

$\because Q'$是环，$f': R \to Q'$是环同态，满足$I \subseteq \ker(f')$

$\therefore$存在唯一的环同态$\Phi: Q \to Q'$, s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ Q \arrow[rr, "\Phi"'] & & Q' \end{tikzcd}$$

$\therefore \Theta_1 = \Phi = \Theta_2$ $\quad \therefore$ 唯一性得证. $\square$