

Lemma: R 是环, A 是 R 的子环, 则一定存在从 A 到 R 的单同态.

Proof: $\because A$ 是 R 的子环 $\therefore A \subseteq R$ 且 $0_R, 1_R \in A$.

$$\therefore 0_A = 0_R \text{ 且 } 1_A = 1_R.$$

定义映射 $l: A \longrightarrow R$
 $x \longmapsto x$

$$\text{对 } \forall x \in A, \because A \subseteq R \quad \therefore x \in R \quad \therefore l(x) = x \in R$$

$$\therefore l(A) \subseteq R$$

$$\text{对 } \forall x_1, x_2 \in A.$$

若 $x_1 = x_2$, 则有: $l(x_1) = x_1 = x_2 = l(x_2) \quad \therefore l: A \rightarrow R$ 是一个映射.

若 $l(x_1) = l(x_2)$, 则有: $x_1 = x_2 \quad \therefore l: A \rightarrow R$ 是一个单射.

$$l(x_1 + x_2) = x_1 + x_2 = l(x_1) + l(x_2)$$

$$l(x_1 x_2) = x_1 x_2 = l(x_1) l(x_2)$$

$$l(1_A) = 1_A = 1_R$$

$\therefore l: A \rightarrow R$ 是单同态. \square

Lemma: R 是整环, F 是域, 域 F 包含整环 R 作为子环. 则一定存在从 R 到 F 的单同态.

Proof: $\because F$ 是域 $\therefore F$ 是环. \because 域 F 包含整环 R 作为子环

$\therefore R$ 是 F 的子环.

\therefore 一定存在从 R 到 F 的单同态. \square (Remark: 唯一性一般不能保证.
之后找不唯一的例子)

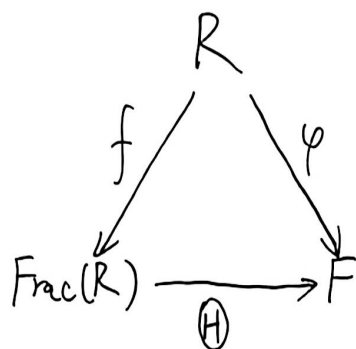
Lemma (整环 R 上的分式域的泛性质) R 是一个任意的整环, 已经证明了 $\text{Frac}(R)$ 是域, $f: R \rightarrow \text{Frac}(R)$ 是单同态.

$$x \mapsto [x, 1]$$

F 是一个域, 域 F 包含整环 R 作为子环, 已经证明了存在从 R 到 F 的单同态. 设 $\varphi: R \rightarrow F$ 是任意一个从 R 到 F 的单同态.

F 的所有元素都能表成 $\varphi(x)(\varphi(y))^{-1}$ 的形式, 其中 $x, y \in R$ 且 $y \neq 0$.

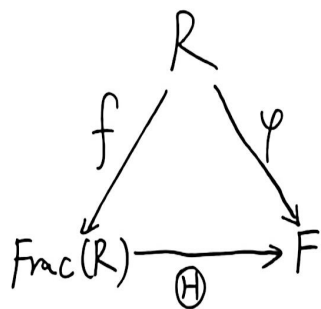
则有: 存在唯一的环同构 $\Theta: \text{Frac}(R) \rightarrow F$, s.t. 下图交换:



Proof: $\because F$ 是域 $\therefore F$ 是交换环

$\because \varphi: R \rightarrow F$ 是单同态 $\therefore \varphi(R \setminus \{0_R\}) \subseteq F^\times$

\therefore 存在唯一的环同态 $\Theta: \text{Frac}(R) \rightarrow F$, s.t. 下图交换:



对 $\forall [x_1, y_1], [x_2, y_2] \in \text{Frac}(R)$, 若 $\Theta([x_1, y_1]) = \Theta([x_2, y_2])$, 则有:

$$\because [x_1, y_1], [x_2, y_2] \in \text{Frac}(R) \quad \because y_1, y_2 \in R \text{ 且 } y_1 \neq 0, y_2 \neq 0$$

$$\because y_1, y_2 \in R \quad \because [y_1, y_2, 1] \in \text{Frac}(R) \quad \because \Theta([y_1, y_2, 1]) \in F$$

$$\because \Theta([x_1, y_1]) = \Theta([x_2, y_2])$$

$$\because \Theta([x_1, y_1]) \cdot \Theta([y_1, y_2, 1]) = \Theta([x_2, y_2]) \cdot \Theta([y_1, y_2, 1])$$

$$\because \Theta([x_1, y_1] \cdot [y_1, y_2, 1]) = \Theta([x_2, y_2] \cdot [y_1, y_2, 1])$$

$$\because \Theta([x_1 y_1 y_2, y_1]) = \Theta([x_2 y_1 y_2, y_2])$$

$$\because \Theta([x_1 y_2, 1]) = \Theta([x_2 y_1, 1])$$

$$\because \Theta(f(x_1, y_2)) = \Theta(f(x_2, y_1))$$

$$\because \varphi(x_1 y_2) = \varphi(x_2 y_1)$$

$$\because \varphi: R \rightarrow F \text{ 是单同态} \quad \because x_1 y_2 = x_2 y_1$$

$$\because [x_1, y_1] = [x_2, y_2] \quad \because \Theta: \text{Frac}(R) \rightarrow F \text{ 是单射.}$$

$$\text{对 } \forall \lambda \in F, \exists x, y \in R \text{ 且 } y \neq 0, \text{ s.t. } \lambda = \varphi(x)(\varphi(y))^{-1}$$

$$\because y \in R \text{ 且 } y \neq 0 \quad \because y \in R \setminus \{0_R\} \quad \because \varphi(y) \in F^\times$$

$$\because [x, y] \in \text{Frac}(R), \text{ 且有:}$$

$$\begin{aligned} \Theta([x, y]) &= \Theta([x \cdot 1, 1 \cdot y]) = \Theta([x, 1] \cdot [1, y]) \\ &= \Theta([x, 1]) \cdot \Theta([1, y]) \end{aligned}$$

对 $\forall y \in R$ 且 $y \neq 0$, 有:

$$1_F = \Theta(1_{\text{Frac}(R)}) = \Theta([y, y]) = \Theta([y, 1] \cdot [1, y])$$

$$= \Theta([y, 1]) \cdot \Theta([1, y]) = \Theta(f(y)) \cdot \Theta([1, y])$$

$$= \varphi(y) \cdot \Theta([1, y]) \quad \because y \in R \setminus \{0_R\} \quad \therefore \varphi(y) \in F^\times$$

$$\therefore \Theta([1, y]) = (\varphi(y))^{-1}$$

$$\therefore \Theta([x, y]) = \Theta([x, 1]) \cdot \Theta([1, y])$$

$$= \Theta(f(x)) \cdot (\varphi(y))^{-1} = \varphi(x) \cdot (\varphi(y))^{-1} = \lambda$$

$\therefore \Theta: \text{Frac}(R) \rightarrow F$ 是满射.

$\therefore \Theta: \text{Frac}(R) \rightarrow F$ 是双射.

$\therefore \Theta: \text{Frac}(R) \rightarrow F$ 是环同构.

\therefore 存在唯一的环同构 $\Theta: \text{Frac}(R) \rightarrow F$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow \varphi \\ \text{Frac}(R) & \xrightarrow{\Theta} & F \end{array}$$

