Lemma：R是环，对$\forall i\in I$，R_i是R的子环，则有：

$\bigcap\limits_{i\in I} R_i$是$R$的子环.

Proof：对$\forall i\in I$，$\because R_i$是R的子环　$\therefore R_i\subseteq R$且$0_R, 1_R\in R_i$

$\therefore \bigcap\limits_{i\in I} R_i \subseteq R$

$\because$对$\forall i\in I$，$0_R, 1_R\in R_i$　　$\therefore 0_R, 1_R \in \bigcap\limits_{i\in I} R_i$

对$\forall\ x, y\in \bigcap\limits_{i\in I} R_i$，

$\cancel{\therefore x, y\in \bigcap_{i\in I}}$ 对$\forall i\in I$，$\because x, y\in \bigcap\limits_{i\in I} R_i$　　$\therefore x, y\in R_i$

$\because R_i$是R的子环　　$\therefore x+y\in R_i$，$x\cdot y\in R_i$

$\therefore x+y\in \bigcap\limits_{i\in I} R_i$，　$x\cdot y\in \bigcap\limits_{i\in I} R_i$

$\therefore \bigcap\limits_{i\in I} R_i$ 对R中的加法运算和乘法运算封闭.

对$\forall\ x\in \bigcap\limits_{i\in I} R_i$，

对$\forall i\in I$，$\because x\in \bigcap\limits_{i\in I} R_i$　　$\therefore x\in R_i$

$\because R_i$是R的子环　　$\therefore -x\in R_i$　　$\therefore -x\in \bigcap\limits_{i\in I} R_i$

$\therefore \bigcap\limits_{i\in I} R_i$对$R$中的加法取逆 $x\mapsto -x$ 封闭.

$\therefore \bigcap\limits_{i\in I} R_i$是$R$的子环　$\square$

Lemma：R是环，R_1和R_2是R的子环，则有：

$R_1 \cup R_2$是R的子环 $\Longleftrightarrow R_1 \subseteq R_2$或$R_2 \subseteq R_1$

Proof：($\Leftarrow$)：若$R_1 \subseteq R_2$，则$R_1 \cup R_2 = R_2$是R的子环

若$R_2 \subseteq R_1$，则$R_1 \cup R_2 = R_1$是R的子环.

($\Rightarrow$)：假设$R_1 \nsubseteq R_2$且$R_2 \nsubseteq R_1$，则有：

$\because R_1 \nsubseteq R_2 \quad \therefore \exists x \in R_1$, s.t. $x \notin R_2$

$\because R_2 \nsubseteq R_1 \quad \therefore \exists y \in R_2$, s.t. $y \notin R_1$

$\because x \in R_1 \subseteq R_1 \cup R_2 \quad \therefore x \in R_1 \cup R_2$

$\because y \in R_2 \subseteq R_1 \cup R_2 \quad \therefore y \in R_1 \cup R_2$

$\because R_1 \cup R_2$是R的子环 $\quad \therefore x+y \in R_1 \cup R_2$

$\therefore x+y \in R_1$或$x+y \in R_2$

若$x+y \in R_1$，则有：$\because x \in R_1$，R_1是R的子环 $\quad \therefore -x \in R_1$

$\because x+y \in R_1$，$-x \in R_1$，R_1是R的子环 $\quad \therefore (x+y)+(-x) \in R_1$

$\because (x+y)+(-x) = (y+x)+(-x) = y+(x+(-x)) = y+0_R = y$

$\therefore y \in R_1 \quad \because y \notin R_1 \quad \therefore$矛盾.

若$x+y \in R_2$，则有：$\because y \in R_2$，R_2是R的子环 $\quad \therefore -y \in R_2$

$\because x+y \in R_2$，$-y \in R_2$，R_2是R的子环 $\quad \therefore (x+y)+(-y) \in R_2$

$\because (x+y)+(-y) = x+(y+(-y)) = x+0_R = x \quad \therefore x \in R_2$

$\because x \notin R_2 \quad \therefore$矛盾. $\quad \therefore R_1 \subseteq R_2$或$R_2 \subseteq R_1$ $\quad \square$