Lemma：R_1和R_2是环，$f: R_1 \longrightarrow R_2$是环同态，$I_1$是环$R_1$的理想，$I_2$是环$R_2$的理想，$f(I_1) \subseteq I_2$，$q_1: R_1 \longrightarrow R_1/I_1$
$x \longmapsto x+I_1$

是商同态，$q_2: R_2 \longrightarrow R_2/I_2$ 是商同态．则有：存在唯一的环同态
$x \longmapsto x+I_2$

$\bar{f}: R_1/I_1 \longrightarrow R_2/I_2$ 使得下图交换：

$$\begin{tikzcd}
R_1 \arrow[r, "f"] \arrow[d, "q_1"'] & R_2 \arrow[d, "q_2"] \\
R_1/I_1 \arrow[r, "\bar{f}"'] & R_2/I_2
\end{tikzcd}$$

Proof：$\because R_1$是环，I_1是环R_1的理想

$\therefore R_1/I_1$是环，$q_1: R_1 \longrightarrow R_1/I_1$ 是满同态．
$x \longmapsto x+I_1$

$\because R_2$是环，I_2是环R_2的理想

$\therefore R_2/I_2$是环，$q_2: R_2 \longrightarrow R_2/I_2$ 是满同态．
$x \longmapsto x+I_2$

定义映射 $\bar{f}: R_1/I_1 \longrightarrow R_2/I_2$
$x+I_1 \longmapsto f(x)+I_2$

对$\forall\, x+I_1 \in R_1/I_1$（其中$x \in R_1$），$\because x \in R_1$，$f: R_1 \to R_2$是环同态

$\therefore f(x) \in R_2$ $\quad\therefore f(x)+I_2 \in R_2/I_2$．

$\because \bar{f}(x+I_1) = f(x)+I_2 \in R_2/I_2 \quad \therefore \bar{f}(R_1/I_1) \subseteq R_2/I_2$

对$\forall\ x+I_1,\ y+I_1 \in R_1/I_1$ (其中 $x, y\in R_1$). 若 $x+I_1 = y+I_1$, 则有:

$x \equiv_{I_1} y \quad \therefore x-y \in I_1 \quad \therefore f(x-y) \in I_2$

$\because f(x-y) = f(x+(-y)) = f(x)+f(-y) = f(x)+(-f(y)) = f(x)-f(y)$

$\therefore f(x)-f(y) \in I_2 \quad \therefore f(x) \equiv_{I_2} f(y) \quad \therefore f(x)+I_2 = f(y)+I_2$

$\therefore \bar{f}(x+I_1) = f(x)+I_2 = f(y)+I_2 = \bar{f}(y+I_1)$

$\therefore \bar{f}: R_1/I_1 \longrightarrow R_2/I_2$ 是一个映射

对$\forall\ x+I_1,\ y+I_1 \in R_1/I_1$ (其中 $x, y\in R_1$),

$$\bar{f}((x+I_1)+(y+I_1)) = \bar{f}((x+y)+I_1) = f(x+y)+I_2$$
$$= (f(x)+f(y))+I_2 = (f(x)+I_2)+(f(y)+I_2)$$
$$= \bar{f}(x+I_1)+\bar{f}(y+I_1)$$

$$\bar{f}((x+I_1)\cdot(y+I_1)) = \bar{f}(xy+I_1) = f(xy)+I_2$$
$$= f(x)\cdot f(y)+I_2 = (f(x)+I_2)\cdot(f(y)+I_2)$$
$$= \bar{f}(x+I_1)\cdot\bar{f}(y+I_1)$$

$$\bar{f}(1_{R_1/I_1}) = \bar{f}(1_{R_1}+I_1) = f(1_{R_1})+I_2 = 1_{R_2}+I_2 = 1_{R_2/I_2}$$

$\therefore \bar{f}: R_1/I_1 \rightarrow R_2/I_2$ 是一个环同态.

$\because f: R_1 \to R_2$ 是环同态，$q_2: R_2 \to R_2/I_2$ 是满同态

$\therefore q_2 \circ f: R_1 \to R_2/I_2$ 是环同态

$\because q_1: R_1 \to R_1/I_1$ 是满同态，$\bar{f}: R_1/I_1 \to R_2/I_2$ 是环同态

$\therefore \bar{f} \circ q_1: R_1 \to R_2/I_2$ 是环同态.

对 $\forall\, x \in R_1$，$(q_2 \circ f)(x) = q_2(f(x)) = f(x) + I_2$

$$= \bar{f}(x + I_1) = \bar{f}(q_1(x)) = (\bar{f} \circ q_1)(x)$$

$\therefore q_2 \circ f = \bar{f} \circ q_1$ $\therefore$ 下图交换：

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "q_1"'] & R_2 \arrow[d, "q_2"] \\ R_1/I_1 \arrow[r, "\bar{f}"'] & R_2/I_2 \end{tikzcd}$$

存在性得证.

假设存在两个环同态 $\alpha: R_1/I_1 \to R_2/I_2$，$\beta: R_1/I_1 \to R_2/I_2$

使得下图交换：

$$\begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "q_1"'] & R_2 \arrow[d, "q_2"] \\ R_1/I_1 \arrow[r, "\alpha"'] & R_2/I_2 \end{tikzcd} \qquad \begin{tikzcd} R_1 \arrow[r, "f"] \arrow[d, "q_1"'] & R_2 \arrow[d, "q_2"] \\ R_1/I_1 \arrow[r, "\beta"'] & R_2/I_2 \end{tikzcd}$$

则对 $\forall\, x + I_1 \in R_1/I_1$（其中 $x \in R_1$），有：

$$\alpha(x + I_1) = \alpha(q_1(x)) = (\alpha \circ q_1)(x) = (q_2 \circ f)(x)$$

$$= (\beta \circ q_1)(x) = \beta(q_1(x)) = \beta(x + I_1)$$

$\therefore \alpha = \beta$. 唯一性得证.