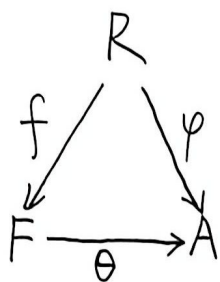


Lemma (整环 $R$ 上的分式域的泛性质)  $R$ 是一个任意的整环,

$F$ 是一个域,  $f: R \rightarrow F$ 是一个单同态, 则以下两个命题等价:

①  $F$ 的所有元素都能表成  $f(x)(f(y))^{-1}$  的形式, 其中  $x, y \in R$  且  $y \neq 0_R$

② 对 $\forall$ 交换环 $A$ ,  $\forall$ 环同态  $\varphi: R \rightarrow A$  满足  $\varphi(R \setminus \{0_R\}) \subseteq A^\times$ , 存在唯一的环同态  $\theta: F \rightarrow A$ , s.t. 下图交换:



Proof: ①  $\Rightarrow$  ②: 对 $\forall$ 交换环 $A$ ,  $\forall$ 环同态  $\varphi: R \rightarrow A$  满足

$$\varphi(R \setminus \{0_R\}) \subseteq A^\times,$$

对 $\forall \lambda \in F$ ,  $\because \exists x, y \in R$  且  $y \neq 0_R$ , s.t.  $\lambda = f(x)(f(y))^{-1}$

(注意:  $\because R$ 是整环,  $F$ 是域,  $f: R \rightarrow F$ 是单同态  $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$

$$\because y \in R \text{ 且 } y \neq 0_R \quad \therefore f(y) \in F^\times \quad \therefore (f(y))^{-1} \in F^\times, (f(y))^{-1} \in F$$

$$\because x \in R \quad \therefore f(x) \in F \quad \therefore f(x)(f(y))^{-1} \in F)$$

$\therefore$  定义映射  $\theta: F \longrightarrow A$

$$\lambda = f(x)(f(y))^{-1} \longmapsto \varphi(x)(\varphi(y))^{-1}$$

$$\because y \in R \text{ 且 } y \neq 0_R \quad \therefore \varphi(y) \in A^\times \quad \therefore (\varphi(y))^{-1} \in A^\times, (\varphi(y))^{-1} \in A$$

$$\because x \in R \quad \therefore \varphi(x) \in A \quad \therefore \varphi(x)(\varphi(y))^{-1} \in A$$

$$\therefore \theta(\lambda) = \varphi(x)(\varphi(y))^{-1} \in A \quad \therefore \theta(F) \subseteq A$$

对  $\forall \lambda_1, \lambda_2 \in F$ , 若  $\lambda_1 = \lambda_2$ , 则有:

$$\because \lambda_1 \in F \quad \because \exists x_1, y_1 \in R \text{ 且 } y_1 \neq 0_R, \text{ s.t. } \lambda_1 = f(x_1)(f(y_1))^{-1}$$

$$\because \lambda_2 \in F \quad \because \exists x_2, y_2 \in R \text{ 且 } y_2 \neq 0_R, \text{ s.t. } \lambda_2 = f(x_2)(f(y_2))^{-1}$$

$$\because \lambda_1 = \lambda_2 \quad \because f(x_1)(f(y_1))^{-1} = f(x_2)(f(y_2))^{-1}$$

$$\because f(x_1)f(y_2) = f(x_2)f(y_1) \quad \because f(x_1 y_2) = f(x_2 y_1)$$

$$\because x_1 y_2 = x_2 y_1 \quad \because \varphi(x_1 y_2) = \varphi(x_2 y_1) \quad \because \varphi(x_1)\varphi(y_2) = \varphi(x_2)\varphi(y_1)$$

$$\because y_1 \in R \text{ 且 } y_1 \neq 0_R \quad \because \varphi(y_1) \in A^\times \quad \because (\varphi(y_1))^{-1} \in A$$

$$\because y_2 \in R \text{ 且 } y_2 \neq 0_R \quad \because \varphi(y_2) \in A^\times \quad \because (\varphi(y_2))^{-1} \in A$$

$$\because \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1}$$

$$\because \theta(\lambda_1) = \varphi(x_1)(\varphi(y_1))^{-1} = \varphi(x_2)(\varphi(y_2))^{-1} = \theta(\lambda_2)$$

$\therefore \theta: F \rightarrow A$  是一个映射.

对  $\forall \lambda_1, \lambda_2 \in F$ .

$$\because \lambda_1 \in F \quad \because \exists x_1, y_1 \in R \text{ 且 } y_1 \neq 0_R, \text{ s.t. } \lambda_1 = f(x_1)(f(y_1))^{-1}$$

$$\because \lambda_2 \in F \quad \because \exists x_2, y_2 \in R \text{ 且 } y_2 \neq 0_R, \text{ s.t. } \lambda_2 = f(x_2)(f(y_2))^{-1}$$

$$\because R \text{ 是整环, } F \text{ 是域, } f: R \rightarrow F \text{ 是单同态} \quad \because f(R \setminus \{0_R\}) \subseteq F^\times$$

$$\because y_1 \in R \setminus \{0_R\} \text{ 且 } y_2 \in R \setminus \{0_R\} \quad \because f(y_1) \in F^\times \text{ 且 } f(y_2) \in F^\times$$

$$\because (f(y_1)f(y_2))^{-1} = (f(y_2))^{-1}(f(y_1))^{-1} = (f(y_1))^{-1}(f(y_2))^{-1}$$

$$\because \lambda_1 + \lambda_2 = f(x_1)(f(y_1))^{-1} + f(x_2)(f(y_2))^{-1}$$

$$= f(x_1)f(y_2)(f(y_1))^{-1}(f(y_2))^{-1} + f(x_2)f(y_1)(f(y_1))^{-1}(f(y_2))^{-1}$$

$$= (f(x_1)f(y_2) + f(x_2)f(y_1)) (f(y_1))^{-1}(f(y_2))^{-1}$$

$$= f(x_1y_2 + x_2y_1) (f(y_1y_2))^{-1}$$

$$\therefore \theta(\lambda_1 + \lambda_2) = \varphi(x_1y_2 + x_2y_1) (\varphi(y_1y_2))^{-1}$$

$$= (\varphi(x_1)\varphi(y_2) + \varphi(x_2)\varphi(y_1)) (\varphi(y_1))^{-1}(\varphi(y_2))^{-1}$$

$$= \varphi(x_1)(\varphi(y_1))^{-1} + \varphi(x_2)(\varphi(y_2))^{-1}$$

$$= \theta(\lambda_1) + \theta(\lambda_2)$$

$$\therefore \lambda_1\lambda_2 = f(x_1)(f(y_1))^{-1} f(x_2)(f(y_2))^{-1}$$

$$= f(x_1x_2) (f(y_1y_2))^{-1}$$

$$\therefore \theta(\lambda_1\lambda_2) = \varphi(x_1x_2) (\varphi(y_1y_2))^{-1}$$

$$= \varphi(x_1)\varphi(x_2) (\varphi(y_2))^{-1}(\varphi(y_1))^{-1}$$

$$= \varphi(x_1)(\varphi(y_1))^{-1} \cdot \varphi(x_2)(\varphi(y_2))^{-1}$$

$$= \theta(\lambda_1) \cdot \theta(\lambda_2)$$

$$\therefore R \text{ 是整环} \quad \therefore R \text{ 是非零环} \quad \therefore 1_R \neq 0_R$$

$$\therefore 1_R \in R, 1_R \in R \text{ 且 } 1_R \neq 0_R, f(1_R)(f(1_R))^{-1} = 1_F \cdot 1_F^{-1} = 1_F \cdot 1_F = 1_F$$

$$\therefore 1_F \in F \text{ 且 } 1_F = f(1_R)(f(1_R))^{-1}$$

$$\therefore \theta(1_F) = \varphi(1_R)(\varphi(1_R))^{-1} = 1_A \cdot 1_A^{-1} = 1_A \cdot 1_A = 1_A$$

$$\therefore \theta: F \rightarrow A \text{ 是一个环同态}$$

$\because f: R \rightarrow F$  是一个单同态,  $\theta: F \rightarrow A$  是一个环同态


$\therefore \theta \circ f: R \rightarrow A$  是一个环同态.  $\varphi: R \rightarrow A$  是一个环同态

对  $\forall x \in R$ , 有:  $f(x) \in F$

$\because x \in R, 1_R \in R, 1_R \neq 0_R, f(x)(f(1_R))^{-1} = f(x) \cdot 1_F^{-1} = f(x) \cdot 1_F = f(x)$

$$\therefore f(x) = f(x)(f(1_R))^{-1}$$

$$\begin{aligned} \therefore (\theta \circ f)(x) &= \theta(f(x)) = \varphi(x)(\varphi(1_R))^{-1} = \varphi(x) \cdot 1_A^{-1} = \varphi(x) \cdot 1_A \\ &= \varphi(x) \end{aligned}$$

$\therefore \theta \circ f = \varphi$   $\therefore$  下图交换:  存在性得证.

假设存在环同态  $\theta_1: F \rightarrow A$  s.t. 下图交换



假设还存在环同态  $\theta_2: F \rightarrow A$  s.t. 下图交换



则有: 对  $\forall \lambda \in F, \exists x, y \in R$  且  $y \neq 0_R$ , s.t.  $\lambda = f(x)(f(y))^{-1}$

$\because R$  是整环,  $F$  是域,  $f: R \rightarrow F$  是单同态  $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$

$\because y \in R$  且  $y \neq 0_R \therefore f(y) \in F^\times \therefore (f(y))^{-1} \in F^\times$  且  $(f(y))^{-1} \in F$

$$\therefore \theta_1(\lambda) = \theta_1(f(x)(f(y))^{-1}) = \theta_1(f(x)) \theta_1((f(y))^{-1})$$

$$= (\theta_1 \circ f)(x) \cdot (\theta_1(f(y)))^{-1} = \varphi(x) (\varphi(y))^{-1}$$

$$= \theta_2(f(x)) \cdot (\theta_2(f(y)))^{-1} = \theta_2(f(x)) \cdot \theta_2((f(y))^{-1})$$

$$= \theta_2(f(x)(f(y))^{-1}) = \theta_2(\lambda)$$

$\therefore \theta_1 = \theta_2 \quad \therefore$  唯一性得证.

②  $\Rightarrow$  ①: 定义  $K = \{f(x)(f(y))^{-1} \mid x, y \in R \text{ 且 } y \neq 0_R\}$

$\because R$  是整环,  $F$  是域,  $f: R \rightarrow F$  是单同态  $\therefore f(R \setminus \{0_R\}) \subseteq F^\times$

对  $\forall x, y \in R$  且  $y \neq 0_R$ ,  $\because y \in R$  且  $y \neq 0_R \quad \therefore y \in R \setminus \{0_R\} \quad \therefore f(y) \in F^\times$

$\therefore (f(y))^{-1} \in F^\times$ ,  $(f(y))^{-1} \in F \quad \because x \in R \quad \therefore f(x) \in F$

$\therefore f(x)(f(y))^{-1} \in F \quad \therefore K \subseteq F$

对  $\forall x \in R$ ,  $\because R$  是整环  $\therefore R$  是非零环  $\therefore 1_R \in R$  且  $1_R \neq 0_R$

$\therefore f(x)(f(1_R))^{-1} \in K \quad \because f(x)(f(1_R))^{-1} = f(x) \cdot 1_F^{-1} = f(x) \cdot 1_F = f(x)$

$\therefore f(x) \in K \quad \therefore f(R) \subseteq K \subseteq F$

对  $0_R \in R$ ,  $\forall y \in R$  且  $y \neq 0_R$ , 有:  $f(0_R)(f(y))^{-1} \in K$

$\therefore f(0_R)(f(y))^{-1} = 0_F \cdot (f(y))^{-1} = 0_F \quad \therefore 0_F \in K$

对  $1_R \in R$ ,  $1_R \in R$  且  $1_R \neq 0_R$ , 有:  $f(1_R)(f(1_R))^{-1} \in K$

$\therefore f(1_R)(f(1_R))^{-1} = 1_F \cdot 1_F^{-1} = 1_F \cdot 1_F = 1_F \quad \therefore 1_F \in K$

$\therefore F$  是域,  $K \subseteq F$ ,  $K$  是  $F$  的一个非空子集,  $0_F \in K$ ,  $1_F \in K$

对  $\forall f(x_1)(f(y_1))^{-1} \in K$ ,  $f(x_2)(f(y_2))^{-1} \in K$  (其中  $x_1, y_1, x_2, y_2 \in R$  且  $y_1 \neq 0_R, y_2 \neq 0_R$ ), 有:

$$\begin{aligned}
& f(x_1)(f(y_1))^{-1} - f(x_2)(f(y_2))^{-1} \\
&= f(x_1) f(y_2) (f(y_2))^{-1} (f(y_1))^{-1} - f(x_2) f(y_1) (f(y_1))^{-1} (f(y_2))^{-1} \\
&= f(x_1 y_2) (f(y_1) f(y_2))^{-1} - f(x_2 y_1) (f(y_1) f(y_2))^{-1} \\
&= (f(x_1 y_2) - f(x_2 y_1)) (f(y_1 y_2))^{-1} \\
&= f(x_1 y_2 - x_2 y_1) (f(y_1 y_2))^{-1} \in K
\end{aligned}$$

( $\because R$  是整环,  $y_1, y_2 \in R$ ,  $y_1 \neq 0_R$  且  $y_2 \neq 0_R \quad \therefore y_1 y_2 \in R$  且  $y_1 y_2 \neq 0_R$ )

对  $\forall f(x_1)(f(y_1))^{-1} \in K$ ,  $\forall f(x_2)(f(y_2))^{-1} \in K \setminus \{0_F\}$ , 有:

$$\because f(x_2)(f(y_2))^{-1} \in K \setminus \{0_F\} \quad \therefore f(x_2)(f(y_2))^{-1} \neq 0_F$$

假设  $x_2 = 0_R$ , 则有  $f(x_2) = f(0_R) = 0_F$

$$\therefore f(x_2)(f(y_2))^{-1} = 0_F \cdot (f(y_2))^{-1} = 0_F \quad \text{矛盾} \quad \therefore x_2 \neq 0_R$$

$$\therefore x_2 \in R \text{ 且 } x_2 \neq 0_R \quad \therefore f(x_2) \in F^\times \quad \therefore (f(x_2))^{-1} \in F^\times \text{ 且 } (f(x_2))^{-1} \in F$$

$$\therefore f(x_1)(f(y_1))^{-1} \cdot (f(x_2)(f(y_2))^{-1})^{-1}$$

$$= f(x_1)(f(y_1))^{-1} \cdot f(y_2)(f(x_2))^{-1}$$

$$= f(x_1) f(y_2) (f(y_1))^{-1} (f(x_2))^{-1}$$

$$= f(x_1 y_2) (f(x_2 y_1))^{-1} \in K$$

( $\because R$  是整环,  $x_2, y_1 \in R$ ,  $x_2 \neq 0_R$  且  $y_1 \neq 0_R \quad \therefore x_2 y_1 \neq 0_R$  且  $x_2 y_1 \in R$ )

$\therefore K$  是  $F$  的子域

$\because F$  是域  $\therefore F$  是交换环  $\therefore f: R \rightarrow F$  是单同态 且有  $f(R \setminus \{0_R\}) \subseteq F^\times$

$\therefore$  存在唯一的环同态  $\alpha: F \rightarrow F$  s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f \\ F & \xrightarrow{\alpha} & F \end{array}$$

$\therefore \text{id}_F: F \rightarrow F$  是环同构  
 $x \mapsto x$

$\therefore \text{id}_F: F \rightarrow F$  是环同态

$\therefore \text{id}_F \circ f = f$   $\therefore$  下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f \\ F & \xrightarrow{\text{id}_F} & F \end{array}$$

$\therefore \alpha = \text{id}_F$

$\therefore K$  是  $F$  的子域  $\therefore K$  是域  $\therefore K$  是交换环

~~定义~~ 定义映射  $\psi: R \rightarrow K$   
 $x \mapsto f(x)$

对  $\forall x \in R$ ,  $\psi(x) = f(x) \in f(R) \subseteq K \therefore \psi(R) \subseteq K$

对  $\forall x_1, x_2 \in R$ , 若  $x_1 = x_2$ , 则有:  $\psi(x_1) = f(x_1) = f(x_2) = \psi(x_2)$

$\therefore \psi: R \rightarrow K$  是一个映射.

对  $\forall x_1, x_2 \in R$ ,

$$\psi(x_1 + x_2) = f(x_1 + x_2) = f(x_1) + f(x_2) = \psi(x_1) + \psi(x_2)$$

$$\psi(x_1 x_2) = f(x_1 x_2) = f(x_1) f(x_2) = \psi(x_1) \psi(x_2)$$

$\psi(1_R) = f(1_R) = 1_F$  就是  $K$  的乘法元

$\therefore \psi: R \rightarrow K$  是一个环同态

对  $\forall x \in R \setminus \{0_R\}$ , 有:  $\psi(x) = f(x) \in F^\times = F \setminus \{0_F\}$

$\therefore \psi(x) \in K$  且  $\psi(x) \neq 0_F \therefore \psi(x) \in K \setminus \{0_F\} = K^\times \therefore \psi(R \setminus \{0_R\}) \subseteq K^\times$

$\therefore$  存在唯一的环同态  $\beta: F \rightarrow K$ , s.t. 下图交换:



$\therefore F$  是域,  $K$  是  $F$  的子域

$\therefore \iota: K \rightarrow F$  是单同态.  
 $x \mapsto x$

$\therefore \beta: F \rightarrow K$  是一个环同态,  $\iota: K \rightarrow F$  是一个单同态  
 $x \mapsto x$

$\therefore \iota \circ \beta: F \rightarrow F$  是一个环同态

$\therefore \psi: R \rightarrow K$  是一个环同态,  $\iota: K \rightarrow F$  是一个单同态  
 $x \mapsto x$

$\therefore \iota \circ \psi: R \rightarrow F$  是一个环同态.  $\therefore f: R \rightarrow F$  是一个单同态

对  $\forall x \in R$ , 有:  $(\iota \circ \psi)(x) = \iota(\psi(x)) = \iota(f(x)) = f(x)$

$$\therefore \iota \circ \psi = f$$

$$\therefore (\iota \circ \beta) \circ f = \iota \circ (\beta \circ f) = \iota \circ \psi = f$$

$\therefore$  下图交换:  $\therefore \iota \circ \beta = \alpha = \text{id}_F$

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f \\ F & \xrightarrow{\iota \circ \beta} & F \end{array}$$

$\therefore$  对  $\forall x \in F$ , 有:  $x = \text{id}_F(x) = (\iota \circ \beta)(x) = \iota(\beta(x)) = \beta(x) \in K$

$$\therefore F \subseteq K \quad \therefore F = K = \{f(x)(f(y))^{-1} \mid x, y \in R \text{ 且 } y \neq 0_R\}$$

①得证.  $\square$