Lemma：R是一个任意的环，则有：$\mathrm{id}_R: R \longrightarrow R$ 是环同构
$$x \longmapsto x$$

Proof：显然 $\mathrm{id}_R: R \longrightarrow R$ 是双射.
$$x \longmapsto x$$

对$\forall x, y \in R$，有：

$$\mathrm{id}_R(x+y) = x+y = \mathrm{id}_R(x) + \mathrm{id}_R(y)$$

$$\mathrm{id}_R(x \cdot y) = x \cdot y = \mathrm{id}_R(x) \cdot \mathrm{id}_R(y)$$

$$\mathrm{id}_R(1_R) = 1_R$$

$\therefore \mathrm{id}_R: R \longrightarrow R$ 是环同态

$\therefore \mathrm{id}_R: R \longrightarrow R$ 是环同构. $\square$

定理（任意环R上的一元多项式环的泛性质）

R是一个任意的非零环.

B是一个环，$f: R \longrightarrow B$是一个环同态，$b \in B$，满足：

对$\forall$环A，$\forall$环同态$\varphi: R \longrightarrow A$，$\forall a \in A$，$\exists$唯一的环同态

$\theta: B \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ B \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

且有$\theta(b) = a$.

B'是另一个环，$f': R \longrightarrow B'$是另一个环同态，$b' \in B'$，满足：

对$\forall$环A，$\forall$环同态$\varphi: R \longrightarrow A$，$\forall a \in A$，$\exists$唯一的环同态

$\theta': B' \longrightarrow A$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "\varphi"] & \\ B' \arrow[rr, "\theta'"'] & & A \end{tikzcd}$$

且有$\theta'(b') = a$

则存在唯一的映射$\Theta: B \longrightarrow B'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "\Theta"'] & & B' \end{tikzcd}$$

且 $\Theta(b)=b'$，$\Theta: B\to B'$ 是环同构.

Proof：对于环 B'，环同态 $f': R\to B'$，$b'\in B'$，$\exists$ 唯一的环同态

$\Phi: B\to B'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "\Phi"'] & & B' \end{tikzcd}$$

且有：$\Phi(b)=b'$

对于环 B，环同态 $f: R\to B$，$b\in B$，$\exists$ 唯一的环同态

$\Psi: B'\to B$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ B' \arrow[rr, "\Psi"'] & & B \end{tikzcd}$$

且有 $\Psi(b')=b$

$\because$ $\Phi: B\to B'$ 和 $\Psi: B'\to B$ 都是环同态

$\therefore$ $\Psi\circ\Phi: B\to B$ 和 $\Phi\circ\Psi: B'\to B'$ 都是环同态.

对于环 B，环同态 $f: R\to B$，$b\in B$，存在唯一的环同态

$\alpha: B\to B$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, "\alpha"'] & & B \end{tikzcd}$$

且有 $\alpha(b)=b$

$\because$ B是环 $\therefore$ $id_B: B \longrightarrow B$ 是环同构，也是环同态.
$x \longmapsto x$

$\because$ $id_B \circ f = f$ $\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, "id_B"'] & & B \end{tikzcd}$$

$\because$ $id_B(b) = b$ $\therefore$ $id_B = \alpha$

$\because$ $\Phi \circ f = f'$ 且有 $\Psi \circ f' = f$

$\therefore$ $(\Psi \circ \Phi) \circ f = \Psi \circ (\Phi \circ f) = \Psi \circ f' = f$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f"] & \\ B \arrow[rr, "\Psi \circ \Phi"'] & & B \end{tikzcd}$$

$\because$ $(\Psi \circ \Phi)(b) = \Psi(\Phi(b)) = \Psi(b') = b$

$\therefore$ $\Psi \circ \Phi = \alpha = id_B$

又对环B'，环同态$f': R \to B'$，$b' \in B'$，$\exists$唯一的环同态
$\beta: B' \longrightarrow B'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ B' \arrow[rr, "\beta"'] & & B' \end{tikzcd}$$

且有$\beta(b') = b'$

$\because$ B'是环 $\therefore$ $id_{B'}: B' \longrightarrow B'$ 是环同构，也是环同态.
$x \longmapsto x$

$\because$ $id_{B'} \circ f' = f'$ $\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ B' \arrow[rr, "id_{B'}"'] & & B' \end{tikzcd}$$

$\because id_{B'}(b') = b' \qquad \therefore id_{B'} = \beta$

$\because (\Phi \circ \Psi) \circ f' = \Phi \circ (\Psi \circ f') = \Phi \circ f = f'$

$\therefore$ 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f'"'] \arrow[dr, "f'"] & \\ B' \arrow[rr, "\Phi \circ \Psi"'] & & B' \end{tikzcd}$$

$\because (\Phi \circ \Psi)(b') = \Phi(\Psi(b')) = \Phi(b) = b'$

$\therefore \Phi \circ \Psi = \beta = id_{B'}$

$\because \Psi \circ \Phi = id_B$ 且 $\Phi \circ \Psi = id_{B'}$

$\therefore \Phi : B \longrightarrow B'$ 是可逆映射 $\qquad \therefore \Phi : B \longrightarrow B'$ 是双射.

$\because \Phi : B \longrightarrow B'$ 是环同态 $\qquad \therefore \Phi : B \longrightarrow B'$ 是环同构.

$\therefore \Phi : B \to B'$ 是一个映射，下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "\Phi"'] & & B' \end{tikzcd}$$

$\Phi(b) = b'$，$\Phi : B \to B'$ 是环同构 $\qquad \therefore$ 存在性得证.

假设存在映射 $\Theta_1 : B \longrightarrow B'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "\Theta_1"'] & & B' \end{tikzcd}$$

且 $\Theta_1(b) = b'$，$\Theta_1 : B \longrightarrow B'$ 是环同构.

存在另一个映射 $\Theta_2 : B \longrightarrow B'$，s.t. 下图交换：

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ B \arrow[rr, "\Theta_2"'] & & B' \end{tikzcd}$$

且 $\Theta_2(b)=b'$，$\Theta_2: B\to B'$ 是环同构．

$\therefore$ 对于环 B'，环同态 $f': R\to B'$，$b'\in B'$，$\exists$ 唯一的环同态 $\Psi: B\to B'$，s.t. 下图交换：

$$\begin{tikzcd}
 & R \arrow[dl, "f"'] \arrow[dr, bend left, "f'"] & \\
B \arrow[rr, "\Psi"'] & & B'
\end{tikzcd}$$

且有 $\Psi(b)=b'$

$\therefore \Theta_1=\Psi=\Theta_2$　　$\therefore$ 唯一性得证． $\square$