Frac(R)的环结构

R是一个任意的整环，$\mathrm{Frac}(R)=\{[f,g] \mid (f,g)\in R^2 \text{且} g\neq 0\}$

定义（Frac(R)中的加法和乘法）对$\forall [f_1,g_1],[f_2,g_2]\in \mathrm{Frac}(R)$，定义它们的和与积为：

$$[f_1,g_1]+[f_2,g_2]:=[f_1g_2+f_2g_1,\ g_1g_2]$$

$$[f_1,g_1][f_2,g_2]:=[f_1f_2,\ g_1g_2]$$

Lemma：Frac(R)上的加法运算良定义

Proof：对$\forall [f_1,g_1],[f_2,g_2]\in \mathrm{Frac}(R)$，

$\because [f_1,g_1]\in \mathrm{Frac}(R)$ $\quad\therefore f_1\in R$ 且 $g_1\in R$ 且 $g_1\neq 0$

$\because [f_2,g_2]\in \mathrm{Frac}(R)$ $\quad\therefore f_2\in R$ 且 $g_2\in R$ 且 $g_2\neq 0$

$\because g_1\in R$ 且 $g_2\in R$ $\quad\therefore g_1g_2\in R$ $\quad\because g_1\neq 0$ 且 $g_2\neq 0$ $\quad\therefore g_1g_2\neq 0$

$\because f_1g_2+f_2g_1\in R$ $\quad\therefore (f_1g_2+f_2g_1,\ g_1g_2)\in \mathrm{Ratio}(R)$

$\therefore [f_1,g_1]+[f_2,g_2]=[f_1g_2+f_2g_1,\ g_1g_2]\in \mathrm{Frac}(R)$

对$\forall [f_1,g_1],[\tilde{f}_1,\tilde{g}_1],[f_2,g_2],[\tilde{f}_2,\tilde{g}_2]\in \mathrm{Frac}(R)$，

若$[f_1,g_1]=[\tilde{f}_1,\tilde{g}_1]$ 且 $[f_2,g_2]=[\tilde{f}_2,\tilde{g}_2]$，则有：

$\because [f_1,g_1]=[\tilde{f}_1,\tilde{g}_1]$ $\quad\therefore (f_1,g_1)\sim(\tilde{f}_1,\tilde{g}_1)$ $\quad\therefore f_1\tilde{g}_1=\tilde{f}_1g_1$

$\because [f_2,g_2]=[\tilde{f}_2,\tilde{g}_2]$ $\quad\therefore (f_2,g_2)\sim(\tilde{f}_2,\tilde{g}_2)$ $\quad\therefore f_2\tilde{g}_2=\tilde{f}_2g_2$

$\therefore (f_1 g_2 + f_2 g_1)(\tilde{g_1}\tilde{g_2}) = f_1 g_2 \tilde{g_1}\tilde{g_2} + f_2 g_1 \tilde{g_1}\tilde{g_2}$

$= (f_1\tilde{g_1})(g_2\tilde{g_2}) + (f_2\tilde{g_2})(g_1\tilde{g_1})$

$= (\tilde{f_1} g_1)(g_2\tilde{g_2}) + (\tilde{f_2} g_2)(g_1\tilde{g_1})$

$= (\tilde{f_1}\tilde{g_2})(g_1 g_2) + (\tilde{f_2}\tilde{g_1})(g_1 g_2)$

$= (\tilde{f_1}\tilde{g_2} + \tilde{f_2}\tilde{g_1})(g_1 g_2)$

$\therefore (f_1 g_2 + f_2 g_1, g_1 g_2) \sim (\tilde{f_1}\tilde{g_2} + \tilde{f_2}\tilde{g_1}, \tilde{g_1}\tilde{g_2})$

$\therefore [f_1 g_2 + f_2 g_1, g_1 g_2] = [\tilde{f_1}\tilde{g_2} + \tilde{f_2}\tilde{g_1}, \tilde{g_1}\tilde{g_2}]$

$\therefore [f_1, g_1] + [f_2, g_2] = [\tilde{f_1}, \tilde{g_1}] + [\tilde{f_2}, \tilde{g_2}]$

$\therefore$ Frac(R)上的加法运算良定义. $\square$

Lemma: Frac(R)上的乘法运算良定义

Proof: 对$\forall [f_1, g_1], [f_2, g_2] \in$ Frac(R),

$\because [f_1, g_1] \in \mathrm{Frac}(R)$ $\therefore f_1 \in R$ 且 $g_1 \in R$ 且 $g_1 \neq 0$

$\because [f_2, g_2] \in \mathrm{Frac}(R)$ $\therefore f_2 \in R$ 且 $g_2 \in R$ 且 $g_2 \neq 0$

$\because g_1 \in R$ 且 $g_2 \in R$ $\therefore g_1 g_2 \in R$ $\because g_1 \neq 0$ 且 $g_2 \neq 0$ $\therefore g_1 g_2 \neq 0$

$\because f_1 f_2 \in R$ $\therefore [f_1, g_1][f_2, g_2] = [f_1 f_2, g_1 g_2] \in \mathrm{Frac}(R)$

对 $\forall [f_1, g_1], [\tilde{f_1}, \tilde{g_1}], [f_2, g_2], [\tilde{f_2}, \tilde{g_2}] \in \mathrm{Frac}(R)$,

若 $[f_1, g_1] = [\tilde{f_1}, \tilde{g_1}]$ 且 $[f_2, g_2] = [\tilde{f_2}, \tilde{g_2}]$, 则有:

$\because [f_1, g_1] = [\tilde{f_1}, \tilde{g_1}] \quad \therefore (f_1, g_1) \sim (\tilde{f_1}, \tilde{g_1}) \quad \therefore f_1\tilde{g_1} = \tilde{f_1}g_1$

$\because [f_2, g_2] = [\tilde{f_2}, \tilde{g_2}] \quad \therefore (f_2, g_2) \sim (\tilde{f_2}, \tilde{g_2}) \quad \therefore f_2\tilde{g_2} = \tilde{f_2}g_2$

$\because (f_1f_2)(\tilde{g_1}\tilde{g_2}) = f_1f_2\tilde{g_1}\tilde{g_2} = (f_1\tilde{g_1})(f_2\tilde{g_2}) = (\tilde{f_1}g_1)(\tilde{f_2}g_2)$

$= \tilde{f_1}g_1\tilde{f_2}g_2 = (\tilde{f_1}\tilde{f_2})(g_1g_2)$

$\therefore (f_1f_2, g_1g_2) \sim (\tilde{f_1}\tilde{f_2}, \tilde{g_1}\tilde{g_2})$

$\therefore [f_1f_2, g_1g_2] = [\tilde{f_1}\tilde{f_2}, \tilde{g_1}\tilde{g_2}]$

$\therefore [f_1, g_1][f_2, g_2] = [\tilde{f_1}, \tilde{g_1}][\tilde{f_2}, \tilde{g_2}]$

$\therefore$ Frac(R)上的乘法运算良定义 $\square$

Lemma: 对 $\forall g_1, g_2 \in R$ 且 $g_1, g_2 \neq 0$, 有: $[0, g_1] = [0, g_2]$

~~$\because 0 \in R$~~ Proof: $\because 0 \in R,\ g_1 \in R,\ g_1 \neq 0 \quad \therefore [0, g_1] \in \mathrm{Frac}(R)$

$\because 0 \in R,\ g_2 \in R,\ g_2 \neq 0 \quad \therefore [0, g_2] \in \mathrm{Frac}(R)$

$\because 0g_2 = 0 = 0g_1 \quad \therefore (0, g_1) \sim (0, g_2) \quad \therefore [0, g_1] = [0, g_2]$

$\square$

定义（$Frac(R)$的加法零元）定义 $0_{Frac(R)}$ 为：

$$0_{Frac(R)} := [0, g] \quad , g \in R 且 g \neq 0$$

Lemma：对$\forall g \in R$且$g \neq 0$，有：$[1,1] = [g,g]$

Proof：$\because 1 \in R$，$g \in R$，R是交换环

$\therefore 1 \cdot g = g \cdot 1 \qquad \therefore (1,1) \sim (g,g)$

$\therefore [1,1] = [g,g] \qquad \square$

定义（$Frac(R)$的乘法幺元）定义 $1_{Frac(R)}$ 为：

$$1_{Frac(R)} := [1,1] = [g,g] \ , g \in R 且 g \neq 0$$

Lemma（同分母分式的加法）对$\forall [f_1, g], [f_2, g] \in Frac(R)$，有：

$$[f_1, g] + [f_2, g] = [f_1 + f_2, g]$$

Proof：$\because [f_1, g] \in Frac(R) \quad \therefore f_1 \in R$且$g \in R$且$g \neq 0$

$\because [f_2, g] \in Frac(R) \quad \therefore f_2 \in R$且$g \in R$且$g \neq 0$

$\therefore [f_1, g] + [f_2, g] = [f_1 g + f_2 g, g^2] = [(f_1 + f_2) g, g \cdot g]$

$= [f_1 + f_2, g]$

（$\because f_1 \in R$且$f_2 \in R \quad \therefore f_1 + f_2 \in R \quad \because g \in R$且$g \neq 0$

$\therefore [f_1 + f_2, g] \in Frac(R)$ ）

$\square$