

定理 (商环的泛性质, 同构唯一性部分) R 是环, I 是 R 的理想, Q 是一个环, $f: R \rightarrow Q$ 是一个满足 $I \subseteq \ker(f)$ 的环同态, 满足: 对 \forall 环 A , \forall 环同态 $\varphi: R \rightarrow A$ 满足 $I \subseteq \ker(\varphi)$, 存在唯一的环同态 $\theta: Q \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow \varphi \\ Q & \xrightarrow{\theta} & A \end{array}$$

Q' 是另一个环, $f': R \rightarrow Q'$ 是另一个满足 $I \subseteq \ker(f')$ 的环同态, 满足: 对 \forall 环 A , \forall 环同态 $\varphi: R \rightarrow A$ 满足 $I \subseteq \ker(\varphi)$, 存在唯一的环同态 $\theta': Q' \rightarrow A$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f' \swarrow & & \searrow \varphi \\ Q' & \xrightarrow{\theta'} & A \end{array}$$

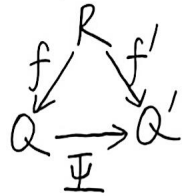
则存在唯一的映射 $\theta: Q \rightarrow Q'$, s.t. 下图交换:

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ Q & \xrightarrow{\theta} & Q' \end{array}$$

且 $\theta: Q \rightarrow Q'$ 是环同构.

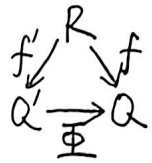
Proof: $\because Q'$ 是环, $f': R \rightarrow Q'$ 是环同态, 满足 $I \subseteq \ker(f')$

\therefore 存在唯一的环同态 $\Psi: Q \rightarrow Q'$, s.t. 下图交换



$\because Q$ 是环, $f: R \rightarrow Q$ 是环同态, 满足 $I \subseteq \ker(f)$

\therefore 存在唯一的环同态 $\Phi: Q' \rightarrow Q$, s.t. 下图交换



$\therefore \Psi: Q \rightarrow Q'$ 是环同态, $\Phi: Q' \rightarrow Q$ 是环同态

$\therefore \Phi \circ \Psi: Q \rightarrow Q$ 是环同态, $\Psi \circ \Phi: Q' \rightarrow Q'$ 是环同态.

$\because Q$ 是环, $f: R \rightarrow Q$ 是环同态, 满足 $I \subseteq \ker(f)$

\therefore 存在唯一的环同态 $\alpha: Q \rightarrow Q$, s.t. 下图交换:



$\because Q$ 是环 $\therefore \text{id}_Q: Q \rightarrow Q$ 是环同构

$\therefore \text{id}_Q: Q \rightarrow Q$ 是环同态

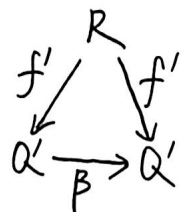
$\therefore \text{id}_Q \circ f = f \quad \therefore$ 下图交换: $\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f \\ Q & \xrightarrow{\text{id}_Q} & Q \end{array} \quad \therefore \text{id}_Q = \alpha$

$\therefore (\Phi \circ \Psi) \circ f = \Phi \circ (\Psi \circ f) = \Phi \circ f' = f$

\therefore 下图交换: ~~$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f \\ Q & \xrightarrow{\Psi \circ \Phi} & Q \end{array}$~~ $\therefore \Phi \circ \Psi = \alpha = \text{id}_Q$

$\because Q'$ 是环, $f': R \rightarrow Q'$ 是环同态, 满足 $I \subseteq \ker(f')$

\therefore 存在唯一的环同态 $\beta: Q' \rightarrow Q'$, s.t. 下图交换:



$\because Q'$ 是环 $\therefore \text{id}_{Q'} : Q' \rightarrow Q'$ 是环同构 $\therefore \text{id}_{Q'} : Q' \rightarrow Q'$ 是环同态

$\therefore \text{id}_{Q'} \circ f' = f'$ \therefore 下图交换 :

$$\begin{array}{ccc} & R & \\ f' \swarrow & & \searrow f' \\ Q' & \xrightarrow{\text{id}_{Q'}} & Q' \end{array} \quad \therefore \text{id}_{Q'} = \beta$$

$\therefore (\Psi \circ \Phi) \circ f' = \Psi \circ (\Phi \circ f') = \Psi \circ f = f'$

\therefore 下图交换 :

$$\begin{array}{ccc} & R & \\ f' \swarrow & & \searrow f' \\ Q' & \xrightarrow{\Psi \circ \Phi} & Q' \end{array} \quad \therefore \Psi \circ \Phi = \beta = \text{id}_{Q'}$$

$\because Q$ 和 Q' 是环, $\Psi : Q \rightarrow Q'$ 是环同态, $\Phi : Q' \rightarrow Q$ 是环同态,

$\Phi \circ \Psi = \text{id}_Q, \Psi \circ \Phi = \text{id}_{Q'}$

$\therefore \Psi : Q \rightarrow Q'$ 是可逆映射 $\therefore \Psi : Q \rightarrow Q'$ 是双射

$\therefore \Psi : Q \rightarrow Q'$ 是环同构. 存在性得证.

假设存在映射 $\Theta_1 : Q \rightarrow Q'$, s.t. 下图交换 :

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ Q & \xrightarrow{\Theta_1} & Q' \end{array}$$

且 $\Theta_1 : Q \rightarrow Q'$ 是环同构

假设还存在映射 $\Theta_2 : Q \rightarrow Q'$, s.t. 下图交换

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ Q & \xrightarrow{\Theta_2} & Q' \end{array}$$

且 $\Theta_2 : Q \rightarrow Q'$ 是环同构

$\because Q'$ 是环, $f' : R \rightarrow Q'$ 是环同态, 满足 $I \subseteq \ker(f')$

\therefore 存在唯一的环同态 $\Psi : Q \rightarrow Q'$, s.t. 下图交换 :

$$\begin{array}{ccc} & R & \\ f \swarrow & & \searrow f' \\ Q & \xrightarrow{\Psi} & Q' \end{array}$$

$\therefore \Theta_1 = \Psi = \Theta_2 \quad \therefore$ 唯一性得证. \square