Lemma：R是一个任意的非零环，~~则有~~ R[X]是一个环，则有：

$$\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^k = 1_R X^k,\quad \forall k\in\mathbb{Z}_{\geqslant 0}$$

Proof：当 $k=0$ 时，

$$左边=\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^0 = 1_{R[X]} = 1_R+\sum_{n\geqslant 1}0X^n$$

$$右边 = 1_R X^0 = 1_R = 1_R+\sum_{n\geqslant 1}0X^n \qquad \therefore 左边=右边$$

当 $k=1$ 时，

$$左边=\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^1 = 0+1_R X+\sum_{n\geqslant 2}0X^n = 1_R X = 1_R X^1 = 右边$$

当 $k=2$ 时，

$$左边=\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^2 = 1_R X^2 = 右边.$$

假设对于 $k\in\mathbb{Z}_{\geqslant 0}$，$k\geqslant 2$，等号成立。则对于 $k+1$，有：

$$\begin{aligned}
左边 &= \cancel{0+1_R}\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^{k+1}\\
&=\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^k\cdot\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)\\
&= 1_R X^k\cdot\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)\\
&= 1_R X^{k+1} = 右边.
\end{aligned}$$

$\therefore$ 对 $\forall k\in\mathbb{Z}_{\geqslant 0}$，$\left(0+1_R X+\sum_{n\geqslant 2}0X^n\right)^k = 1_R X^k$. $\square$

Lemma：R是一个任意的非零环，$R[X]$是一个环，则有：

对$\forall k\in\mathbb{Z}_{\geq 0}$，$\forall r\in R$，有：$rX^k=\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(0+1_R X+\sum_{n\geq 2}0X^n\right)^k$

Proof：右边 $=\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(0+1_R X+\sum_{n\geq 2}0X^n\right)^k$

$$=\left(r+\sum_{n\geq 1}0X^n\right)\cdot 1_R X^k$$

$$=rX^k=\text{左边}. \quad \square$$

Lemma：R是一个任意的非零环，$R[X]$是一个环，则有：

对$\forall \sum_{i=0}^{n}r_iX^i\in R[X]$，有：$\sum_{i=0}^{n}r_iX^i=r_0+r_1X^1+r_2X^2+\cdots+r_{n-1}X^{n-1}+r_nX^n$.

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Proof：右边 $=r_0+r_1X^1+r_2X^2+\cdots+r_{n-1}X^{n-1}+r_nX^n$

$$\begin{aligned}
&=\left(r_0+0X^1+0X^2+\cdots+0X^{n-1}+0X^n\right)\\
&+\left(0+r_1X^1+0X^2+\cdots+0X^{n-1}+0X^n\right)\\
&+\left(0+0X^1+r_2X^2+\cdots+0X^{n-1}+0X^n\right)\\
&+\cdots\\
&+\left(0+0X^1+0X^2+\cdots+r_{n-1}X^{n-1}+0X^n\right)\\
&+\left(0+0X^1+0X^2+\cdots+0X^{n-1}+r_nX^n\right)\\
&=\sum_{i=0}^{n}r_iX^i=\text{左边}. \quad \square
\end{aligned}$$

Lemma: R是一个任意的非零环，则有：$f: R \longrightarrow R[X]$ 是环同态，
$r \longmapsto r$
且是单同态.

Proof: $\because$ R是非零环 $\therefore$ $R[X]$是环

对$\forall r \in R$，$f(r) = r = r + \sum_{n\geq 1} 0X^n \in R[X]$ $\therefore f(R) \subseteq R[X]$

对$\forall r_1, r_2 \in R$，

若$r_1 = r_2$，则有：$f(r_1) = r_1 = r_1 + \sum_{n\geq 1} 0X^n = r_2 + \sum_{n\geq 1} 0X^n = r_2 = f(r_2)$

$\therefore$ f是映射.

若$f(r_1) = f(r_2)$，则有：$r_1 + \sum_{n\geq 1} 0X^n = r_2 + \sum_{n\geq 1} 0X^n$ $\therefore r_1 = r_2$

$\therefore$ f是单射.

$$f(r_1 + r_2) = (r_1 + r_2) + \sum_{n\geq 1} 0X^n = \left(r_1 + \sum_{n\geq 1} 0X^n\right) + \left(r_2 + \sum_{n\geq 1} 0X^n\right) = f(r_1) + f(r_2)$$

$$f(r_1)\cdot f(r_2) = \left(r_1 + \sum_{n\geq 1} 0X^n\right)\cdot\left(r_2 + \sum_{n\geq 1} 0X^n\right) = r_1 r_2 + \sum_{n\geq 1} 0X^n = f(r_1 r_2)$$

$$f(1_R) = 1_R + \sum_{n\geq 1} 0X^n = 1_{R[X]}$$

$\therefore f: R \longrightarrow R[X]$ 是单同态. $\square$
$r \longmapsto r$

Lemma: R是一个任意的非零环，则有：$X \in R[X]$.

Proof: $X = 0 + 1_R X + \sum_{n\geq 2} 0X^n \in R[X]$ $\square$

定理(任意非零交换环R上的一元多项式环R[X]的泛性质)

R是一个任意的非零交换环，R[X]是一个交换环，

$f: R \longrightarrow R[X]$ 是一个环同态，$X \in R[X]$，则有:

$r \longmapsto r$

对∀交换环A，∀环同态 $\varphi: R \to A$，$\forall \alpha \in A$，存在唯一的环同态

$\theta: R[X] \longrightarrow A$，s.t. 下图交换:

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ R[X] \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

且有 $\theta(X) = \alpha$

Proof: 定义映射 $\theta: R[X] \longrightarrow A$

$$\sum_{i=0}^{n} r_i X^i \longmapsto \sum_{i=0}^{n} \varphi(r_i)\alpha^i \quad (\text{其中 } \alpha^0 = 1_A)$$

对 $\forall \sum_{i=0}^{n} r_i X^i \in R[X]$，有: $r_0, r_1, \cdots, r_n \in R$

$\because \varphi: R \to A$ 是环同态 $\quad \therefore \varphi(r_0) \in A,\ \varphi(r_1) \in A, \cdots, \varphi(r_n) \in A$

$\therefore \varphi(r_0)\alpha^0 = \varphi(r_0)\cdot 1_A = \varphi(r_0) \in A$

$\varphi(r_1)\alpha^1 = \varphi(r_1)\cdot \alpha \in A$

$\varphi(r_2)\alpha^2 = \varphi(r_2)\cdot \alpha^2 \in A$

$\cdots\cdots$

$\varphi(r_n)\alpha^n = \varphi(r_n)\cdot \alpha^n \in A$

$\therefore \theta\left(\sum_{i=0}^{n} r_i X^i\right) = \sum_{i=0}^{n} \varphi(r_i)\alpha^i \in A \qquad \therefore \theta(R[X]) \subseteq A$

对$\forall \sum_{i\geq 0} r_i X^i, \sum_{i\geq 0} r_i' X^i \in R[X]$. 若$\sum_{i\geq 0} r_i X^i = \sum_{i\geq 0} r_i' X^i$, 则

对$\forall i \in \mathbb{Z}_{\geq 0}$, 有: $r_i = r_i' \in R$ $\therefore$ 对$\forall i \in \mathbb{Z}_{\geq 0}$, 有 $\varphi(r_i) = \varphi(r_i') \in A$

$\therefore$ 对$\forall i \in \mathbb{Z}_{\geq 0}$, 有 $\varphi(r_i) a^i = \varphi(r_i') a^i$

$$\therefore \sum_{i\geq 0} \varphi(r_i) a^i = \sum_{i\geq 0} \varphi(r_i') a^i$$

$$\therefore \theta\left(\sum_{i\geq 0} r_i X^i\right) = \sum_{i\geq 0} \varphi(r_i) a^i = \sum_{i\geq 0} \varphi(r_i') a^i = \theta\left(\sum_{i\geq 0} r_i' X^i\right)$$

$\therefore \theta$是$R[X] \longrightarrow A$的映射.

对$\forall \sum_{i=0}^{n} r_i X^i, \sum_{j=0}^{m} r_j' X^j \in R[X]$, 有:

$$\theta\left(\sum_{i=0}^{n} r_i X^i + \sum_{j=0}^{m} r_j' X^j\right) = \theta\left(\sum_{i=0}^{\max\{n,m\}} (r_i + r_i') X^i\right)$$

$$= \sum_{i=0}^{\max\{n,m\}} \varphi(r_i + r_i') a^i = \sum_{i=0}^{\max\{n,m\}} (\varphi(r_i) + \varphi(r_i')) a^i$$

$$= \sum_{i=0}^{\max\{n,m\}} (\varphi(r_i) a^i + \varphi(r_i') a^i) = \sum_{i=0}^{\max\{n,m\}} \varphi(r_i) a^i + \sum_{i=0}^{\max\{n,m\}} \varphi(r_i') a^i$$

$$= \sum_{i=0}^{n} \varphi(r_i) a^i + \sum_{i=0}^{m} \varphi(r_i') a^i = \sum_{i=0}^{n} \varphi(r_i) a^i + \sum_{j=0}^{m} \varphi(r_j') a^j$$

$$= \theta\left(\sum_{i=0}^{n} r_i X^i\right) + \cancel{\theta\left(\sum_{j=0}^{m} \varphi(r\right.} \theta\left(\sum_{j=0}^{m} r_j' X^j\right)$$

$$\theta\left(\sum_{i=0}^{n} r_i X^i\right) \cdot \theta\left(\sum_{j=0}^{m} r_j' X^j\right) = \left(\sum_{i=0}^{n} \varphi(r_i) a^i\right) \cdot \left(\sum_{j=0}^{m} \varphi(r_j') a^j\right)$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi(r_i) a^i \cdot \varphi(r_j') a^j = \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi(r_i) \varphi(r_j') a^{i+j}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{m} \varphi(r_i r_j') a^{i+j} = \sum_{i=0}^{n+m} \left(\sum_{\substack{h,k\geq 0 \\ h+k=i \\ h\leq n, k\leq m}} \varphi(r_h r_k')\right) a^i$$

?

$$= \theta\left(\sum_{i=0}^{n+m}\left(\sum_{\substack{h,k\geq 0\\ h+k=i\\ h\leq n,\, k\leq m}} r_h r_k'\right)X^i\right) = \theta\left(\left(\sum_{i=0}^{n} r_i X^i\right)\cdot\left(\sum_{j=0}^{m} r_j' X^j\right)\right)$$

$$\theta(1_{R[X]}) = \theta\left(1_R + \sum_{n\geq 1} 0 X^n\right) = \varphi(1_R)\cdot a^0 + \sum_{n\geq 1}\varphi(0)a^n$$

$$= \varphi(1_R) + \sum_{n\geq 1} 0_A\cdot a^n = \varphi(1_R) = 1_A$$

$\therefore \theta: R[X] \longrightarrow A$ 是环同态.

$\because f: R \longrightarrow R[X]$ 是环同态, $\theta: R[X] \longrightarrow A$ 是环同态
$\quad r \longmapsto r$

$\therefore \theta\circ f: R \longrightarrow A$ 是环同态 $\quad \because \varphi: R \longrightarrow A$ 是环同态

对 $\forall r\in R$, $(\theta\circ f)(r) = \theta(f(r)) = \theta\left(r + \sum_{n\geq 1} 0X^n\right)$

$$= \varphi(r)a^0 + \sum_{n\geq 1}\varphi(0)a^n = \varphi(r)\cdot 1_A + \sum_{n\geq 1} 0_A\cdot a^n = \varphi(r)\cdot 1_A = \varphi(r)$$

$\therefore \theta\circ f = \varphi \quad \therefore$ 下图交换:

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ R[X] \arrow[rr, "\theta"'] & & A \end{tikzcd}$$

$$\theta(X) = \theta\left(0 + 1_R X + \sum_{n\geq 2} 0X^n\right) = \varphi(0)\cdot a^0 + \varphi(1_R)\cdot a^1 + \sum_{n\geq 2}\varphi(0)a^n$$

$= 0_A\cdot 1_A + 1_A\cdot a + \sum_{n\geq 2} 0_A\cdot a^n = 1_A\cdot a = a$ 存在性得证.

假设存在环同态 $\theta_1: R[X] \longrightarrow A$, s.t. 下图交换:
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ R[X] \arrow[rr, "\theta_1"'] & & A \end{tikzcd}$$
, 且有 $\theta_1(X) = a$

还存在环同态 $\theta_2: R[X] \longrightarrow A$, s.t. 下图交换:
$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "\varphi"] & \\ R[X] \arrow[rr, "\theta_2"'] & & A \end{tikzcd}$$
, 且有 $\theta_2(X) = a$.

对$\forall\, r\in R$，有：

$$\theta_1\left(r+\sum_{n\geq 1}0X^n\right)=\theta_1\left(f(r)\right)=(\theta_1\circ f)(r)=\varphi(r)=(\theta_2\circ f)(r)$$

$$=\theta_2(f(r))=\theta_2\left(r+\sum_{n\geq 1}0X^n\right)$$

对$\forall\, r\in R$，$\forall\, k\in\mathbb{Z}_{\geq 0}$，有：

$$\theta_1\left(rX^k\right)=\theta_1\left(\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(0+1_R X+\sum_{n\geq 2}0X^n\right)^k\right)$$

$$=\theta_1\left(r+\sum_{n\geq 1}0X^n\right)\cdot\theta_1\left(\left(0+1_R X+\sum_{n\geq 2}0X^n\right)^k\right)$$

$$=\theta_1\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(\theta_1\left(0+1_R X+\sum_{n\geq 2}0X^n\right)\right)^k$$

$$=\theta_2\left(r+\sum_{n\geq 1}0X^n\right)\cdot a^k=\theta_2\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(\theta_2\left(0+1_R X+\sum_{n\geq 2}0X^n\right)\right)^k$$

$$=\theta_2\left(\left(r+\sum_{n\geq 1}0X^n\right)\cdot\left(0+1_R X+\sum_{n\geq 2}0X^n\right)^k\right)=\theta_2\left(rX^k\right)$$

则有：对 $\forall \sum_{i=0}^{n} r_i X^i \in R[X]$，有：

$$\theta_1\left(\sum_{i=0}^{n} r_i X^i\right) = \theta_1\left(r_0 + r_1X^1 + r_2X^2 + \cdots + r_nX^n\right)$$

$$\begin{aligned}
= \theta_1\Big(& (r_0 + 0X^1 + 0X^2 + \cdots + 0X^{n-1} + 0X^n) \\
& + (0 + r_1X^1 + 0X^2 + \cdots + 0X^{n-1} + 0X^n) \\
& + (0 + 0X^1 + r_2X^2 + \cdots + 0X^{n-1} + 0X^n) \\
& + \cdots \\
& + (0 + 0X^1 + 0X^2 + \cdots + r_{n-1}X^{n-1} + 0X^n) \\
& + (0 + 0X^1 + 0X^2 + \cdots + 0X^{n-1} + r_nX^n)\Big)
\end{aligned}$$

$$= \theta_1(r_0) + \theta_1(r_1X^1) + \theta_1(r_2X^2) + \cdots + \theta_1(r_{n-1}X^{n-1}) + \theta_1(r_nX^n)$$

$$= \theta_2(r_0) + \theta_2(r_1X^1) + \theta_2(r_2X^2) + \cdots + \theta_2(r_{n-1}X^{n-1}) + \theta_2(r_nX^n)$$

$$\cancel{= \theta_2\big((r_0 + 0X^1 + 0X^2 + \cdots + 0X^{n-1} + 0X^n)}$$

$$\cancel{+ (0 + r_1X^1 + 0X^2 + \cdots + 0X^{n-1} + 0X^n)}$$

$$= \theta_2\left(r_0 + r_1X^1 + r_2X^2 + \cdots + r_{n-1}X^{n-1} + r_nX^n\right)$$

R[X]中的加法

$$= \theta_2\left(\sum_{i=0}^{n} r_i X^i\right)$$

$\therefore \theta_1 = \theta_2$. 唯一性得证. $\square$