3.3 多项式环

Lemma：R是非零环，则有：R是交换环 $\Leftrightarrow$ $R[X]$是交换环

Proof：($\Rightarrow$)：对$\forall \sum_{n\geq 0} a_n X^n$，$\sum_{n\geq 0} b_n X^n \in R[X]$，有：

$$\left(\sum_{n\geq 0} a_n X^n\right)\cdot\left(\sum_{n\geq 0} b_n X^n\right) = \sum_{n\geq 0}\left(\sum_{\substack{h,k\geq 0\\ h+k=n}} a_h b_k\right)X^n = \sum_{n\geq 0}\left(\sum_{\substack{h,k\geq 0\\ h+k=n}} b_k a_h\right)X^n$$

$$= \sum_{n\geq 0}\left(\sum_{\substack{k,h\geq 0\\ k+h=n}} b_k a_h\right)X^n = \left(\sum_{n\geq 0} b_n X^n\right)\cdot\left(\sum_{n\geq 0} a_n X^n\right)$$

$\therefore$ $R[X]$是交换环.

($\Leftarrow$)：对$\forall a, b \in R$，有：

常数多项式 $a = a + \sum_{n\geq 1} 0X^n \in R[X]$

常数多项式 $b = b + \sum_{n\geq 1} 0X^n \in R[X]$

$\because$ $R[X]$是交换环 $\therefore$ $\left(a+\sum_{n\geq 1} 0X^n\right)\cdot\left(b+\sum_{n\geq 1} 0X^n\right) = \left(b+\sum_{n\geq 1} 0X^n\right)\left(a+\sum_{n\geq 1} 0X^n\right)$

$\because$ $\left(a+\sum_{n\geq 1} 0X^n\right)\cdot\left(b+\sum_{n\geq 1} 0X^n\right) = ab + \sum_{n\geq 1}\left(\sum_{\substack{h,k\geq 0\\ h+k=n}} a_h b_k\right)X^n = ab + \sum_{n\geq 1} 0X^n = ab$

$$\left(b+\sum_{n\geq 1} 0X^n\right)\cdot\left(a+\sum_{n\geq 1} 0X^n\right) = ba + \sum_{n\geq 1}\left(\sum_{\substack{h,k\geq 0\\ h+k=n}} b_h a_k\right)X^n = ba + \sum_{n\geq 1} 0X^n = ba$$

$\therefore$ $ab = ba$

$\therefore$ R是交换环. $\square$

Lemma：R是整环，则对$\forall$非零的$f, g \in R[X]$都有

$$\deg(fg) = \deg(f) + \deg(g)$$

Proof：对$\forall$非零的$f, g \in R[X]$，

$\because f$是非零多项式　$\therefore f$的系数不全为0　$\therefore f$的~~首项~~系数非零的最高次项存在　$\therefore f$的首项系数非零．同理可证g的首项系数非零．

设$f = a_n X^n + \sum_{k=0}^{n-1} a_k X^k$，$n \in \mathbb{Z}_{\geq 0}$，$a_n \neq 0$

$g = b_m X^m + \sum_{l=0}^{m-1} b_l X^l$，$m \in \mathbb{Z}_{\geq 0}$　（空和规定为0），$b_m \neq 0$

$$\therefore fg = a_n b_m X^{n+m} + \sum_{k=0}^{n+m-1} c_k X^k$$

$\because a_n \neq 0$且$b_m \neq 0$，R是整环　$\therefore a_n b_m \neq 0$

$$\therefore \deg(fg) = n+m = \deg(f) + \deg(g) \qquad \square$$

Lemma：R是非零环，则有：R是整环 $\Leftrightarrow$ $R[X]$是整环．

Proof：($\Rightarrow$)：$\because R$是非零环　$\therefore 1 \neq 0$

$\therefore 1 = 1 + \sum_{n \geq 1} 0 X^n \in R[X]$　$\therefore R[X]$是非零环．

$\because R$是整环　$\therefore R$是交换环　$\therefore R[X]$是交换环．

对$\forall f, g \in R[X]$，若$f \neq 0$且$g \neq 0$，则有：

$\because f$是非零多项式　$\therefore f$的系数不全为0　$\therefore f$的系数非零的最高次项存在．

$\therefore f$的首项系数非零．同理可证g的首项系数非零．

设 $f = a_nX^n + \sum_{k=0}^{n-1} a_kX^k$ ，$n \in \mathbb{Z}_{\geq 0}$ ，$a_n \neq 0$

$g = b_mX^m + \sum_{l=0}^{m-1} b_lX^l$ ，$m \in \mathbb{Z}_{\geq 0}$ ，$b_m \neq 0$

$\therefore fg = a_nb_mX^{n+m} + \sum_{k=0}^{n+m-1} c_kX^k$

$\because a_n \neq 0$ 且 $b_m \neq 0$ ，R是整环 $\therefore a_nb_m \neq 0$

$\therefore fg \neq 0$ $\therefore R[X]$是整环.

($\Leftarrow$): $\because R[X]$是整环 $\therefore R[X]$是非零交换环

假设 R是零环，则有：$R = \{0\}$ $\therefore R[X]$中只有系数全为0的多项式.

$\therefore R[X]$中只有零多项式 $\therefore R[X]$是零环. 矛盾. $\therefore R$是非零环.

$\because R[X]$是交换环 $\therefore R$是交换环.

对$\forall a, b \in R$，若 $a \neq 0$ 且 $b \neq 0$，则有：

$a = a + \sum_{n \geq 1} 0X^n \in R[X] \setminus \{0\}$ ，$b = b + \sum_{n \geq 1} 0X^n \in R[X] \setminus \{0\}$

$\because R[X]$是整环 $\therefore \left(a + \sum_{n \geq 1} 0X^n\right) \cdot \left(b + \sum_{n \geq 1} 0X^n\right) \neq 0$

$\because \left(a + \sum_{n \geq 1} 0X^n\right) \cdot \left(b + \sum_{n \geq 1} 0X^n\right) = ab + \sum_{n \geq 1} 0X^n$ $\therefore ab + \sum_{n \geq 1} 0X^n \neq 0$

$\therefore ab \neq 0$ $\therefore R$是整环. $\square$

Remark: 非零多项式 $\Leftrightarrow$ 系数不全为0的多项式. (这是定义).

Lemma：R是整环，则有：$R[X]^\times = R^\times$.

Proof：对$\forall a\in R^\times$，有：

$\because a\in R^\times \quad \therefore \exists a^{-1}\in R$，s.t. $a^{-1}a=1=aa^{-1}$. 且$a^{-1}\in R^\times$

假设$a=0$，则有：$1=aa^{-1}=0\cdot a^{-1}=0 \quad \therefore R$是零环 $\therefore$ 矛盾.

$\therefore a\neq 0$. 同理可证：$a^{-1}\neq 0$

$$\because \left(a+\sum_{n\geq 1}0X^n\right)\cdot\left(a^{-1}+\sum_{n\geq 1}0X^n\right) = aa^{-1}+\sum_{n\geq 1}0X^n = 1+\sum_{n\geq 1}0X^n$$

$$\left(a^{-1}+\sum_{n\geq 1}0X^n\right)\cdot\left(a+\sum_{n\geq 1}0X^n\right) = a^{-1}a+\sum_{n\geq 1}0X^n = 1+\sum_{n\geq 1}0X^n$$

$\therefore a = a+\sum_{n\geq 1}0X^n \in R[X]^\times \quad \therefore R^\times \subseteq R[X]^\times$

对$\forall f\in R[X]^\times$，有：

$\because f\in R[X]^\times \quad \therefore \exists f^{-1}\in R[X]$，s.t. $f^{-1}f=1_{R[X]}=ff^{-1}$

且有$f^{-1}\in R[X]^\times$.

假设$f=0$，则有：$1=ff^{-1}=0\cdot f^{-1}=0 \quad \therefore R[X]$是零环

$\because R$是整环 $\quad \therefore R$是非零环 $\quad \therefore R[X]$是整环

$\therefore R[X]$是非零环，矛盾. $\quad \therefore f\neq 0$. 同理可证 $f^{-1}\neq 0$

$$\therefore \quad 0 = \deg(1_{R[X]}) = \deg(ff^{-1}) = \deg(f)+\deg(f^{-1})$$

$\because f\neq 0 \quad \therefore \deg(f)\in\mathbb{Z}_{\geq 0}$. $\quad \because f^{-1}\neq 0 \quad \therefore \deg(f^{-1})\in\mathbb{Z}_{\geq 0}$

$\therefore \deg(f)+\deg(f^{-1})=0 \quad \therefore \deg(f)=0$ 且 $\deg(f^{-1})=0$.

$\therefore f = a \in R$ 且 $f^{-1} = b \in R$

~~$\therefore ab = ff^{-1} = I$, $ba =$~~

$\therefore ab = ab + \sum_{n \geq 1} 0 X^n = \left(a + \sum_{n\geq 1} 0X^n\right) \cdot \left(b + \sum_{n \geq 1} 0 X^n\right) = ff^{-1} = 1_{R[X]} = 1_R$

$ba = ba + \sum_{n \geq 1} 0 X^n = \left(b + \sum_{n\geq 1} 0X^n\right) \cdot \left(a + \sum_{n \geq 1} 0 X^n\right) = f^{-1}f = 1_{R[X]} = 1_R$

$\therefore f = a \in R^{\times}$ $\quad \therefore R[X]^{\times} \subseteq R^{\times}$

$\therefore R[X]^{\times} = R^{\times}$ $\quad \square$