LADR 2C 习题(2)

P50 18题

Lemma：任意 n $(n\geq 1, n\in\mathbb{N}_+)$ 维向量空间都可以分解为 n 个 1 维向量空间的直和

proof：设 V 是域 $\mathbb{F}$ 上的 n $(n\in\mathbb{N}_+)$ 维向量空间，取 V 的一个基：

$\alpha_1, \alpha_2, \cdots, \alpha_n$. 则有：向量组 $\alpha_1, \cdots, \alpha_n$ 线性无关 且 $V=\mathrm{span}(\alpha_1, \cdots, \alpha_n)$.

设 $V_1=\mathrm{span}(\alpha_1)$, $V_2=\mathrm{span}(\alpha_2)$, $\cdots$, $V_n=\mathrm{span}(\alpha_n)$.

则有：$V_1, \cdots, V_n$ 是 V 的 n 个子空间.

$\because$ 向量组 $\alpha_1, \cdots, \alpha_n$ 线性无关 $\therefore$ $\alpha_1\neq 0, \cdots, \alpha_n\neq 0$

$\therefore$ 向量组 α_1 线性无关，$\cdots$，向量组 α_n 线性无关

$\therefore$ $\dim V_1=1, \cdots, \dim V_n=1$

$\therefore$ $V_1, \cdots, V_n$ 是 V 的 n 个 1 维子空间. $\therefore$ $V_1+\cdots+V_n\subseteq V$

对 $\forall\beta\in V$. $\because$ $\beta\in V=\mathrm{span}(\alpha_1, \cdots, \alpha_n)$ $\therefore$ $\exists k_1, \cdots, k_n\in\mathbb{F}$, s.t. $\beta=k_1\alpha_1+\cdots+k_n\alpha_n$.

$\because$ $k_1\alpha_1\in\mathrm{span}(\alpha_1)=V_1, \cdots, k_n\alpha_n\in\mathrm{span}(\alpha_n)=V_n$

$\therefore$ $\beta\in V_1+\cdots+V_n$ $\therefore$ $V\subseteq V_1+\cdots+V_n$ $\therefore$ $V=V_1+\cdots+V_n$.

设 $0=v_1+\cdots+v_n$，其中 $v_1\in V_1, \cdots, v_n\in V_n$

$\because$ $v_1\in V_1=\mathrm{span}(\alpha_1)$ $\therefore$ $\exists l_1\in\mathbb{F}$, s.t. $v_1=l_1\alpha_1$

$\cdots$

$\because$ $v_n\in V_n=\mathrm{span}(\alpha_n)$ $\therefore$ $\exists l_n\in\mathbb{F}$, s.t. $v_n=l_n\alpha_n$

$\therefore$ $0=v_1+\cdots+v_n=l_1\alpha_1+\cdots+l_n\alpha_n$ $\because$ 向量组 $\alpha_1, \cdots, \alpha_n$ 线性无关

$\therefore$ $l_1=\cdots=l_n=0$ $\therefore$ $v_1=0, \cdots, v_n=0$

$\therefore$ 0 表示成 $v_1+\cdots+v_n$ (其中 $v_1\in V_1, \cdots, v_n\in V_n$) 的方式唯一，为 $v_1=0, \cdots, v_n=0$

$\therefore$ $V=V_1\oplus\cdots\oplus V_n$ $\square$

P50 20题

Lemma（三个有限维子空间的和的维数公式）. V是域F上的向量空间, V_1, V_2, V_3是V的有限维子空间，则有：

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1\cap V_2)+\dim(V_2\cap V_3)+\dim(V_3\cap V_1)}{3} - \frac{\dim((V_1+V_2)\cap V_3)+\dim((V_2+V_3)\cap V_1)+\dim((V_3+V_1)\cap V_2)}{3}$$

proof: $\because$ V_1, V_2, V_3是V的有限维子空间

$\therefore$ $V_1+V_2+V_3$是V的有限维子空间，且有：

$$V_1+V_2+V_3 = (V_1+V_2)+V_3 = V_1+(V_2+V_3) = (V_2+V_3)+V_1$$

$$= V_2+(V_3+V_1) = (V_3+V_1)+V_2$$

$\because$ $V_1\cap V_2 \subseteq V_1$ $\therefore$ $V_1\cap V_2$是V_1的子空间 $\therefore$ $V_1\cap V_2$是V的有限维子空间

$\because$ $V_2\cap V_3 \subseteq V_2$ $\therefore$ $V_2\cap V_3$是V_2的子空间 $\therefore$ $V_2\cap V_3$是V的有限维子空间

$\because$ $V_3\cap V_1 \subseteq V_3$ $\therefore$ $V_3\cap V_1$是V_3的子空间 $\therefore$ $V_3\cap V_1$是V的有限维子空间

$\because$ $(V_1+V_2)\cap V_3 \subseteq V_3$ $\therefore$ $(V_1+V_2)\cap V_3$是V_3的子空间

$\therefore$ $(V_1+V_2)\cap V_3$是V的有限维子空间

$\because$ $(V_2+V_3)\cap V_1 \subseteq V_1$ $\therefore$ $(V_2+V_3)\cap V_1$是V_1的子空间

$\therefore$ $(V_2+V_3)\cap V_1$是V的有限维子空间

$\because$ $(V_3+V_1)\cap V_2 \subseteq V_2$ $\therefore$ $(V_3+V_1)\cap V_2$是V_2的子空间

$\therefore$ $(V_3+V_1)\cap V_2$是V的有限维子空间

$\because V_1, V_2, V_3$ 是 V 的有限维子空间

$\therefore V_1+V_2,\ V_2+V_3,\ V_3+V_1$ 是 V 的有限维子空间

$\therefore \dim(V_1+V_2+V_3) = \dim((V_1+V_2)+V_3) = \dim(V_1+V_2) + \dim V_3 - \dim((V_1+V_2)\cap V_3)$

$= \dim V_1 + \dim V_2 - \dim(V_1\cap V_2) + \dim V_3 - \dim((V_1+V_2)\cap V_3)$

$= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1\cap V_2) - \dim((V_1+V_2)\cap V_3)$

$\dim(V_1+V_2+V_3) = \dim((V_2+V_3)+V_1) = \dim(V_2+V_3) + \dim V_1 - \dim((V_2+V_3)\cap V_1)$

$= \dim V_2 + \dim V_3 - \dim(V_2\cap V_3) + \dim V_1 - \dim((V_2+V_3)\cap V_1)$

$= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2\cap V_3) - \dim((V_2+V_3)\cap V_1)$

$\dim(V_1+V_2+V_3) = \dim((V_3+V_1)+V_2) = \dim(V_3+V_1) + \dim V_2 - \dim((V_3+V_1)\cap V_2)$

$= \dim V_3 + \dim V_1 - \dim(V_3\cap V_1) + \dim V_2 - \dim((V_3+V_1)\cap V_2)$

$= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_3\cap V_1) - \dim((V_3+V_1)\cap V_2)$

$$\therefore \dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \frac{\dim(V_1\cap V_2) + \dim(V_2\cap V_3) + \dim(V_3\cap V_1)}{3} - \frac{\dim((V_1+V_2)\cap V_3) + \dim((V_2+V_3)\cap V_1) + \dim((V_3+V_1)\cap V_2)}{3} \quad \square$$