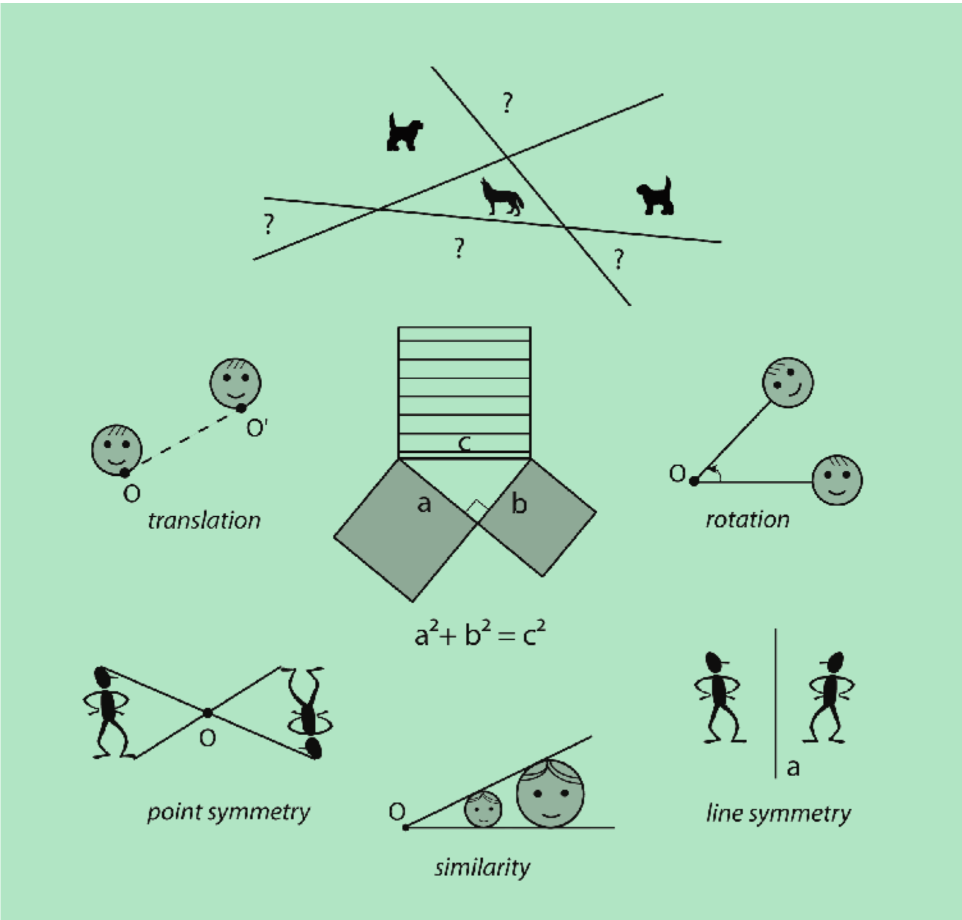
Israel M. Gelfand Tatiana Alekseyevskaya (Gelfand)

Geometry



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Israel M. Gelfand
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Contents

Preface

for the series of books written by Israel Gelfand for high-school students

> *In our century of rapid changes, it is impossible to know everything. The goal is to learn how to learn.* —Israel Gelfand

There are five books written by Israel Gelfand with co-authors for high-school students: *The Method of Coordinates*, *Functions and Graphs*, *Algebra*, *Trigonometry*, and *Geometry*.

Israel Gelfand was internationally known for his teaching skills and his remarkable ability to explain mathematical notions and concepts in an interesting, “fresh,” and easy-to-understand way to a varied group of people: from little kids, people who were not familiar with the subject at all, to students and specialists in the field. He could amazingly feel the level of the listener and adjust his explanations to this level.

In these books, Gelfand intended to cover the basics of mathematics in a clear and simple format suitable for independent study, while inviting students to learn and practice the material at their own pace. He wanted to raise students’ interest in mathematics and their ability to understand and learn. In his own words (from the preface in *The Method of Coordinates*), “The most important thing a student can get from the study of mathematics is the attainment of a higher intellectual level.”

Initially, the first two books, *Functions and Graphs* and *The Method of Coordinates*, were written for students of the Mathematical School by Correspondence organized by Gelfand in Moscow in 1964. They were quite popular among students. The later books were written for the Gelfand Correspondence Program in Mathematics (GCPM) [1], which Gelfand established at Rutgers University in 1991 and for which he specially wrote Assignments

[1] see http://www.israelmgelfand.com/egcpm.html

with me. He wanted us to write other books—*Calculus*, *Combinatorics*, *Arithmetic*, and *Geometry in Space*—but unfortunately, I was too busy writing the Assignments and the book *Geometry*, among other things, and these books were not written.

Gelfand viewed mathematics as one single and whole area of knowledge, and did not like when it was divided into separate narrow fields. Gelfand could often find unexpected connections between different areas of mathematics. In his talk entitled "The Unity of Mathematics," which he gave during the international conference in honor of his 90th birthday, he said "From my point of view, mathematics is a part of our culture like music, poetry and philosophy."

We hope that this series of books will help you learn the basics of mathematics from different approaches, and maybe help you discover for yourself some of their connections and the Unity of Mathematics.

Tatiana Alekseyevskaya (Gelfand)

Preface

Dear reader,

Geometry is the fifth and final book in the series written for high-school students by Israel Gelfand with his colleagues.

What is special about this book? Why and for whom was it written?

This book presents geometry in an unusual way. Instead of focusing on logic and axioms, it focuses on geometrical constructions and presents concepts in a visual form. It starts by introducing a few simple notions and then gradually builds upon them. Students are invited to draw figures and "move" them on the plane. We also introduce transformations—you can see them illustrated on the cover.

Israel Gelfand believed that geometry is the simplest model of spatial relationships in the world. Studying geometry will help students visualize objects and shapes on the plane and in space, and help them develop an intuitive understanding about how they change if they are moved. Rather than make students memorize theorems and practice logic, Gelfand wanted to raise students' interest in the subject and teach them skills such as geometrical vision, imagination, and creativity. These skills are very important in everyday life no matter what future path a student will choose.

Many books are written on calculus, but only a relative few on geometry. You can read more about the importance of geometry in our lives and a description of the structure of this book in the Introduction, which is presented in words as close as possible to Gelfand's own.

All the above makes this book suitable for a wide audience, from students with different backgrounds, to readers with a variety of interests, to educators and to mathematicians, who can appreciate this new way of presenting plane geometry in a simple form while adhering to its depth and rigor.

In Gelfand's approach, pictures play an essential role. He called the pictures "the main beauty of the book" which would distinguish this book from other geometry books. We were even considering the title *Geometry in Pictures*. Indeed, the book has 467 pictures.

Gelfand expected that this book might become quite revolutionary for mathematicians because of the new approach, which he called "algorithmic." Mathematicians would either be very negative about the book or would like it very much. He predicted that the book would only be accepted by the mathematical community years after its publication. Of course, one has to know that we started the book in 1989. Back then even the Internet was still a relatively new feature for the mainstream, and social media was only beginning to develop.

I am curious to see the response when this book is published.

About the process of writing *Geometry.*

Israel Gelfand first mentioned his idea of writing a high-school geometry book with me when we visited Boston in 1989. I was surprised because at that point, I had neither written a book nor even planned to. Israel told me that he chose to write this book specifically with me for several reasons. He said that I had a good intuition for understanding what he means, his ideas and vision of mathematics. He pointed out that I had good spatial imagination, could write the text clearly and logically, and could structurally organize it into a connected whole. It was important that the book have two levels. It had to be simple and attractive for students, yet be precise and exact at a deep mathematical level so that "no captious mathematician would be able to deprecate its rigor," as he said.

Of course, at that time, I did not take his words seriously. One day, within a few weeks of our stay in Boston, Israel simply asked me to take out a piece of paper and a pen, and to write down his thoughts for the book before he forgot them. This is how *Geometry* started. Every so often, at random times and places, Gelfand would ask me to write down separate sentences, problems, or small essays for the future book.

These small essays looked to me like an entertaining set of problems rather than the fundamentals of a subject. They were not even related to each other.

One of the first such essays was "The problem about wolves and dogs" illustrated on the cover (see Problem 16, Ch. I, Sec. 4). In this problem one has to count all bounded and unbounded domains formed by four intersecting lines on the plane. Gelfand was so fascinated with his idea of "placing dogs and wolves" in these domains that he tried to explain this problem to every mathematician he passed in the halls of Harvard or MIT or in attendance at dinners to which he was invited. Not many of them shared his excitement—especially during dinners. I think what fascinated Gelfand was that he had found a good metaphor for explaining on an intuitive level the mathematical notions of boundedness and unboundedness and the difference between them. "Wolves can run away as far as they want and will not come back" refers to students' intuition of being able to go infinitely far.

Maybe by coincidence, it was around that same time that he and I were engaged in joint mathematical research on combinatorial chambers. The word "chamber" connected perfectly the languages appropriate for students and for mathematicians. Indeed, in the problem above there are chambers for wolves. "Mathematicians also call them chambers but the word 'wolves' is not used." This problem presented a good example of how Gelfand managed to explain high-level mathematical notions to high-school students in a clear and simple way, while mixing them with a grain of humor.

In another essay, Gelfand wrote about what he considered one of the beauties of projective geometry—the Desargues configuration (see Sections 8, 9 in Chapter I). Because he wanted this essay to be well presented, he asked me to create a series of illustrations, consisting only of several points and lines, which had to explain in a clear step-by-step way a number of deep concepts of projective geometry.

After I collected a pile of notes and started organizing separate topics in a systematic way, the process of writing *Geometry* became quite interesting. I think this was one of Gelfand's educational "tricks"—once a student starts doing something in small, easy steps, he or she becomes involved in the process and is able to learn and create things without even noticing it. This is why Gelfand wanted students to draw figures themselves along with the book rather than rely on computers to draw and move objects for (or instead of) them.

The process of writing *Geometry* was slowed down by my involvement in writing Assignments for GCPM (Gelfand Correspondence Program in Mathematics) and by various life challenges. Some people asked Israel about this delay and even proposed that he write this book with them instead of with me, suggesting it would be "quicker and better." In response, Israel would repeat numerous times to them and to me that he "could not write this book with anyone else except Tanya."

This book was mostly completed around 2000. I planned to do the "final editing" within two to three months. Unfortunately, we both went through a number of injuries and health challenges which again delayed the book. On top of that, a great deal of technical work was needed to digitally draw all

467 pictures for the book. The good thing was that the complete manuscript was ready by September 2009—Gelfand's 96th birthday—and that he knew about it. Then the publication process at Springer started, and unfortunately, it took a decade to prepare the book for publication.

There is one more remark I want to make. Once, a mathematician who never worked with Gelfand asked me why most of Gelfand's papers were written jointly with co-authors and what Gelfand's role could be in so many such publications. As his co-author of this book, and from being present for many years during his collaborative work with people, I would compare Gelfand's role in his joint works with that of a composer or a conductor. Many times Gelfand himself said that, if he could have been born again, he would have become a composer. I would say that he actually was one.

Acknowledgements.

For me, Gelfand was more than just the co-author of this book. This is why I want to express here my gratitude for his trust in me and for his incredible ability to inspire and encourage, and have patience with, his students. I learned a lot from writing this book with him (which I hope to share one day), even though the process of working with Gelfand could not have been called "smooth and easy."

I would like to thank Andrei Alexeevsky for his numerous encouraging discussions, for practical help in clarifying a number of definitions, and for resolving a few logical conflicts that arose.

I thank Lyuba Pogost for creating digital images of most of the illustrations and Tatiana I. Gelfand for completing this important, time-consuming part of the job.

I thank Robert Wilson for reading the manuscript and for his valuable comments.

I want to thank Mark Saul, who did a thorough job of reading and making corrections to the manuscript. It was funny for me that a few places where he argued with the way material was presented were exactly the places where I made the same arguments to Gelfand, who did not agree with me at that time. This time I found good ways to compromise.

I am also very grateful to Fred Roberts, who always encouraged me in my work on this book.

It was also stimulating to hear from all the people who learned from the GCPM site about the coming *Geometry* book and asked me where to obtain it for their kids or themselves. I want to apologize to them and their children who grew up already while waiting for the book to go through the 10-year-long publication process. Ironically, one of the reasons for this delay was the use of computers for drawing and editing figures. For this book, the figures had to be preserved precisely and not arbitrarily altered to fit the formatting options provided by the typesetting software. You can read in the Introduction a comment about computers in Israel's own words.

Finally, I want to thank Samuel DiBella, who came to Springer in 2018 and devotedly moved the manuscript towards the finish line.

Tatiana Alekseyevskaya (Gelfand)

Introduction

Geometry is the simplest model of spatial relationships in our world

At the present time, geometry finds applications much more broadly than ever before. A geometric approach is used in a variety of fields that do not seem to be related to mathematics; for example, in medical diagnostics (tomography), surgical procedures, architecture, car design, and even in dental restorations.

In studying geometry, the main goal is to understand the structure of our space, learn how to see it, and find out how to orient ourselves in it. Mathematicians in ancient Greece, long before Euclid, proved geometric statements by drawing figures accompanied by the word "look." We have tried to follow this approach. This is why our book has so many pictures. We want to teach readers to see geometric objects (figures), to grasp relations between them, and to construct them by drawing, thus developing geometric intuition. Some examples of geometric objects considered in this book are points, straight lines, segments, triangles, quadrilaterals, and circles.

We obtain our knowledge of the geometry of our world from three sources: through words (verbal description), through seeing (visual perception), and through motion and touch (kinesthetic perception).

Verbal descriptions are good for making logical conclusions. For example, people often say, "Imagine a triangle. Inscribe a circle in it ...," or "Given a triangle with equal sides ...."

Visual perception enables us to see different things simultaneously and to grasp relations between them. We look at the picture or at the object and see at once what is presented in it.

We can also learn about the geometry of the world through movement. We must not underestimate the role of motion. The famous Russian physiologist, Sechenov, said, "Motion is the foundation of thinking." That is why it is important not only to look at the illustrations, but to actually practice drawing geometric constructions yourself and imagine moving geometric figures.

We want to make a remark about the use of computers. While reading this book, you can imagine using a computer doing the operations described in each chapter. You may actually solve some of the problems by drawings on the screen. The study of geometry on a computer has some advantages, similar to the use of calculators, which can save you time adding and multiplying numbers. However, in the case of geometry, you might reduce not only your work, but your understanding as well.

The number of ideas in mathematics is not large. Everything that is achieved is obtained from basic or fundamental concepts that are applied with some degree of variation. Mastering these basic concepts in one field of mathematics helps us to recognize and use them in other fields. We are so used to studying certain basic concepts for tests that we often do not notice their beauty. For example, the well-known statement that through two points one can draw one line, and that this line is unique, was a topic of thorough study by several mathematicians. In fact, every basic concept and idea was once a new and remarkable discovery!

In this book, we present, in simple form, the basic notions of geometry and some methods of studying geometric objects. This material does not require any specific knowledge or talents, but just interest, which we also have tried to raise. We start this book with simple notions and then build on them as if constructing a brick building. It is always a good idea to have a brick fixed securely before putting a new one on top; otherwise, the whole construction might collapse later.

You may find it useful and enjoyable to return to the book several times. This is not surprising; for example, in order to fully appreciate a piece of music, either classical or popular, one usually listens to it more than once.

There are different approaches to geometry.[2] The traditional approach is the axiomatic approach. This approach, started by Euclid and mastered by Hilbert, turned geometry into a logically perfect system. However, this approach is too difficult for the majority of students. Indeed, even a good student can hardly understand why it should be proven that vertical angles are equal (which is evident for him) and, at the same time, skip the statement that a straight line which crosses a triangle and does not pass through any vertex has common points with exactly two sides of the triangle. The proof of the first of these statements is one example of logical reasoning in geometry,

[2]One can classify the following approaches to geometry: axiomatic, analytic (with the help of coordinates), and the approach of group theory.

while the second statement is considered self-evident.

We consider geometry as the simplest model of spatial relations in our world. We use elements of every approach and pay most attention to constructions rather than to axioms. We introduce certain operations in each chapter and explore what we can construct using them. This is similar to how algorithms are described. Thus, we can call our approach to geometry "algorithmic."

Structure of this book and how to read it

This book consists of four chapters. In each chapter, we allow only certain procedures or operations to be available. It is important to remember that we cannot perform any other operations besides the specified ones. From chapter to chapter, we increase the number of available operations and thus the "usual" (i.e., Euclidean) geometry will be presented in the last chapter.[3]

In each chapter, using the defined operations and "instruments" to perform these operations, we draw objects and figures[4] on the plane. We study their properties, which sometimes give us other interesting notions and definitions.[5] We also study relations between the objects, to which we refer as "operations with figures" and "correspondences."

In **Chapter I**, we are allowed to use only a pencil and a straightedge. We assume that by using these instruments we can draw points and straight lines and perform the following operations with them:

(1) Draw the unique straight line that passes through two given points.

(2) Mark the point of intersection of two straight lines if these lines intersect.

As you will see, even by using only these two operations we can construct different geometric figures—in scientific language, *configurations*—and systems of straight lines which form strange and interesting patterns. Some of these configurations are so famous that they have their own names,

[3] Of course, we could have changed the order of chapters and started from Euclidean geometry. In that case, when introducing affine geometry we would have to "forget about length." We thought that this might not be easy to do, as it reminds us of the psychological exercise, "Don't think about a white elephant!"

[4] We will not define precisely what an object or a figure is, and leave this up to your intuition.

[5] We usually write these notions in italic when we use them for the first time. These words are defined and commented on also in the Glossary.

for example, the *Desargues configuration* and the *Pascal configuration*. The geometry based on only these operations corresponds to *projective geometry*.

In **Chapter II**, we add the possibility of drawing parallel straight lines by using a simple instrument such as, for instance, a rolling ruler. Thus, in Chapter II we can perform operations (1), (2) and:

(3) Given a point and a straight line, we can draw a straight line through the point that is parallel to the first line.

This operation enriches our knowledge and vision of different geometric figures. Though we still do not have any instrument to measure the lengths of segments, operation (3) enables us to compare segments lying on parallel lines and even (which is more difficult) to compare segments lying on the same straight line. The geometry described in Chapter II corresponds to *affine geometry*.

After Chapter II there is an **Appendix**, which contains some additional material from Chapter II for optional reading.

In **Chapter III**, in addition to the operations of Chapters I and II, we introduce the *area of a figure*. We choose a unit area, and require area, as a characteristic of a figure, to satisfy certain conditions. If we would like to imagine an "instrument" to measure area, we can think of a scale for weight measure.[6] Such geometry corresponds to *symplectic geometry*. Note that we still cannot measure the lengths of segments.

In **Chapter IV**, we use the operations of Chapters I and II, but not III, and the following operation:

(4) Given two points A and B, we can draw a circle with center at one of these points and radius AB.

An instrument used to draw a circle is a compass. Operation (4) allows us to define the length of a segment. We will be able to compare lengths of segments on different straight lines that are not necessarily parallel. The geometry considered in Chapter IV corresponds to *Euclidean geometry*.[7]

---

[6] If all objects are cut out of a sheet of some material of uniform thickness, then two figures will have the same area just when they have the same weight.

[7] Note that when "building" on affine geometry, we have a choice. We can define the length of a segment and obtain Euclidean geometry without necessarily defining area, or we can define area and obtain symplectic geometry without measuring lengths. Usually, if length measurement is introduced, as in Euclidean geometry, then the measurement of area is defined as well.

The book contains a **Glossary** of the terms which we define and use. Some of the definitions might appear different from the ones in the text. It is important to understand what the notions mean rather than memorize them.

There are many problems in this book. Some of them we solve. These are called Exercises. Others we recommend for you to solve on your own. A few problems are marked by an asterisk (*) and are more difficult.

We also highly recommend that you draw pictures for yourself, even the ones that are presented in the book. It is also useful to draw the pictures slightly differently than in the book, and observe how the statements or problems will look in the case of your own drawing. For example, if there is a quadrilateral in the text or in the problem, draw a trapezoid, a parallelogram, or a rhombus. If a triangle is considered, draw triangles with angles different from the ones in the illustration. Be careful, though, to follow the conditions which this triangle has to satisfy. For example, such conditions for the triangle may require that all angles of the triangle be acute, or that the triangle have a right angle, and so on.

Read the book at your own pace and return to it if you need to. The main "rule" is to enjoy reading and drawing figures, and to see the beauty of geometry as if you were playing at discovering a new world.

Chapter I

Points and Lines: A Look at Projective Geometry

1 Points and lines

1.1 What is a point and what is a line?

The most important objects on the plane are points and lines. In Fig. 1.1a, there is a picture of a point.

Fig. 1.1

Now look at it with the magnification 10 (see Fig. 1.1b). When magnified 100 times, a point will not look like a point at all, and when magnified 1000 times, even the sheet of paper will look different.

We will not define precisely what a point is. The Greek philosopher and mathematician Euclid, who lived about 2000 years ago, wrote in his famous *Elements* (which has greatly influenced all school curricula) that “a point is that which has no part.”

The stars in the sky look like points. Even when magnified 10, 100, 1000 times, the stars will still look like points. However, as we know, the stars are huge objects in the sky. Our Sun is one of these stars and its diameter is 109 times bigger than that of the Earth. Thus, whether we should consider a star as a point depends on the scale that we use to look at the sky.

I. M. Gelfand, T. Alekseyevskaya (Gelfand), *Geometry*,
https://doi.org/10.1007/978-1-0716-0299-7_1

The smaller the dot we use for a point, the closer it will represent an ideal point. However, there is a limit to this representation. Modern physics claims that at distances smaller than 10^{-31} cm, all the concepts of space change radically. One of the challenging problems in physics now is to understand how our world looks at distances smaller than 10^{-31} cm.

When we want to draw a line,[1] we have the same difficulties as when we want to draw a point. Indeed, you would hardly recognize a line when it is magnified 10 times.

However, there is another difficulty. Fig. 1.2 shows a line.

Fig. 1.2

If the sheet of paper were bigger, we could extend this line further. Thus we can say that this picture does not show an entire line but only the part of it that can be placed on this sheet of paper. There does not exist a sheet of paper on which you could draw a complete line, even if such a sheet were as large as the Earth, or the Solar System, or a galaxy.

Geometry deals with

- "ideal points" that cannot be subdivided; and
- "ideal lines" that have no thickness and extend infinitely in both directions.

We also assume that there exist lines, called *parallel lines*, that do not intersect no matter how far they are extended (see Fig. 1.3). Parallel lines are considered in more detail in Chapter II.

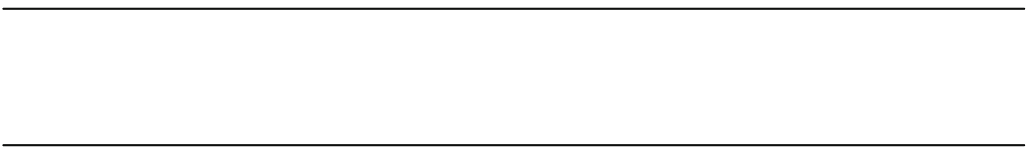


Fig. 1.3

[1] If not mentioned otherwise, we will always mean a straight line when saying "a line."

1.2 Operations available in Chapter I

In real life, we use a pen or a pencil to draw points. The sharper the pencil, the better the point.

To draw a line, we use an instrument called a straightedge. This is a ruler with no measuring marks on it. We can also draw points and lines which have some relation to each other. More precisely, given a point, we can draw a line passing through this point or even more than one line passing through it[2] (see Fig. 1.4).

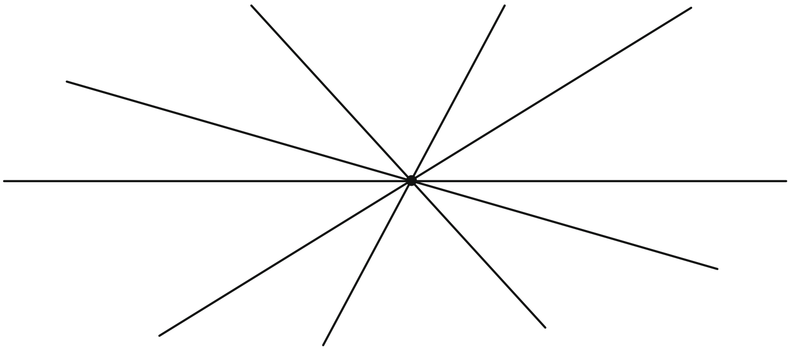


Fig. 1.4

Given a line, we can mark a point on it or several points on it[3] (see Fig. 1.5).

Fig. 1.5

We say that two lines *intersect* if there is a point lying on both of them; we call this point the *intersection point*, or simply the *intersection* of the lines.

Using a straightedge, one can draw a single line through any two points.[4]

[2]In fact, we can draw infinitely many lines passing through a point.

[3]We can mark infinitely many points lying on the line.

[4]Note, that in the real world the accuracy of this operation depends on the representation of points and also on the distance between these points.

If the points are too close, then the accuracy of the operation becomes very poor (see Fig. 1.6).

Fig. 1.6

We assume that we have "ideal geometric tools" that enable us to perform the following operations:

1. Draw the unique line through any two points.
2. Mark the point of intersection of two lines if these lines intersect.[5]

Remark 1. In geometry, the fact that operations 1 and 2 can be performed is expressed in a pair of statements called *axioms* that are usually formulated as follows:

1. Through any two points, one can draw a unique line.
2. Any two lines either have a unique point of intersection or do not intersect (are parallel).

Geometric constructions that are performed using only these two operations are called constructions in *projective geometry*.

1.3 Ray, segment, half-plane

Let us consider a line and mark a point on this line (see Fig. 1.7). This point divides the line into two parts. If we imagine moving along the line, then it is not possible to move from one part to the other without passing through the point or "jumping" over it. Each of these parts is called a *ray* or a *half-line*. We call the point that divides the line into two rays the *endpoint*; the endpoint belongs to both rays.

Fig. 1.7

[5]We suppose that if the lines are not parallel, but do not intersect on a particular sheet of paper, we can extend them and still find their intersection point.

Two points on a line divide it into three parts (Fig. 1.8).

Fig. 1.8

The part of the line between[6] the two points is called a *line segment* or simply a *segment*. Thus, a line with two points on it is divided into two rays and a segment. We call the two points *endpoints* of the segment; the endpoints belong to the segment. (Each one of them also belongs to one of the two rays.)

Notice that any picture representing a line looks like a segment; therefore, it is important to know what is under consideration.

PROBLEM 1. Into how many non-overlapping parts do 100 points divide a line? What are these parts? That is, how many of them are segments and how many are rays?

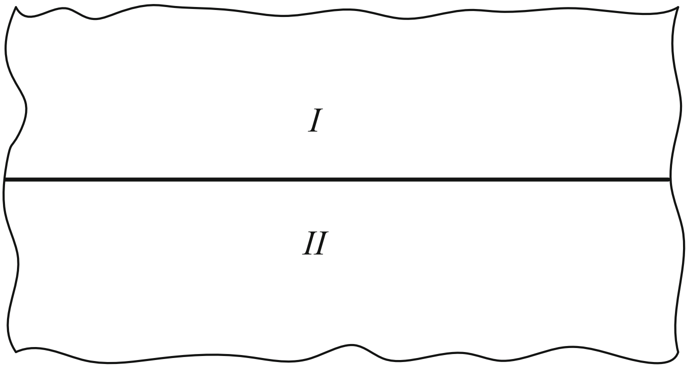


Fig. 1.9

A line divides the plane into two parts called *regions* or *domains* (see Fig. 1.9). It is not possible to move from one of these domains to another without crossing the line. Indeed, if you imagine the plane as a huge field and the line as a fence, and if we place one hostile dog in each domain, then they will be separated by the fence and will not be able to fight.

The domains into which the plane is divided by a line are called *half-planes*. The line separating the domains is called the *boundary* of each of these domains. The boundary is included in both domains.

[6]We will not define here the precise meaning of the term "between" because it requires high level mathematics. We rely on your intuitive understanding of this term.

PROBLEM 2. Into how many parts can two lines divide a plane? Consider all possible positions of two lines in the plane.

1.4 Constructions with a straightedge

Let us do some exercises using the operations defined in Section 1.2. We will denote points using capital letters and straight lines using lower case letters.

Exercise 1. Find the point of intersection of the two lines in Fig. 1.10.

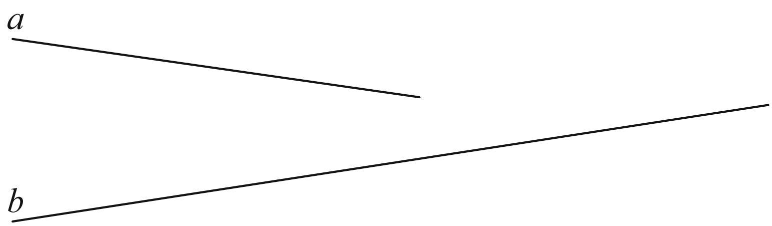


Fig. 1.10

Solution. We see in Fig. 1.10 that the line segments drawn do not actually intersect. We have to extend the line a. Now we can mark the point of intersection of the lines a and b; we can also denote it by a capital letter, for example, by A (see Fig. 1.11).

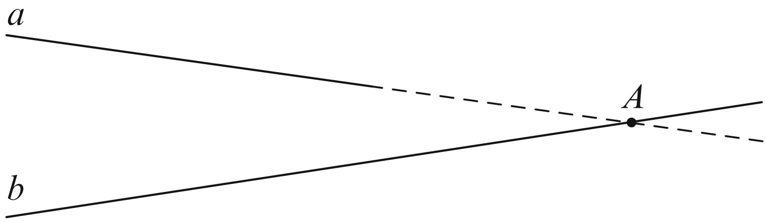


Fig. 1.11

PROBLEM 3. In Fig. 1.12, there are two lines a and b and a point A. Let C be the intersection point of the lines a and b. Draw a line through the points A and C.

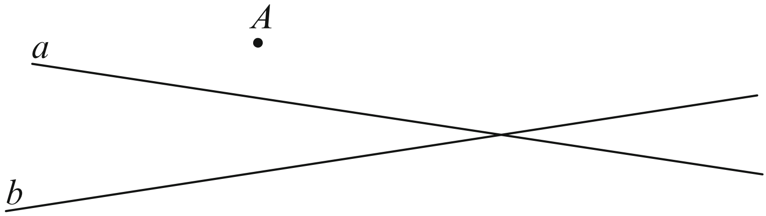


Fig. 1.12

Exercise 2. Suppose there are four lines in the plane, for example, a, b, c, and d. It is given that lines a and b intersect and lines c and d intersect. How many lines can one draw through the intersection point of lines a, b and the intersection point of lines c, d?

Solution. Let us draw two pairs of intersecting lines (see Fig. 1.13a).

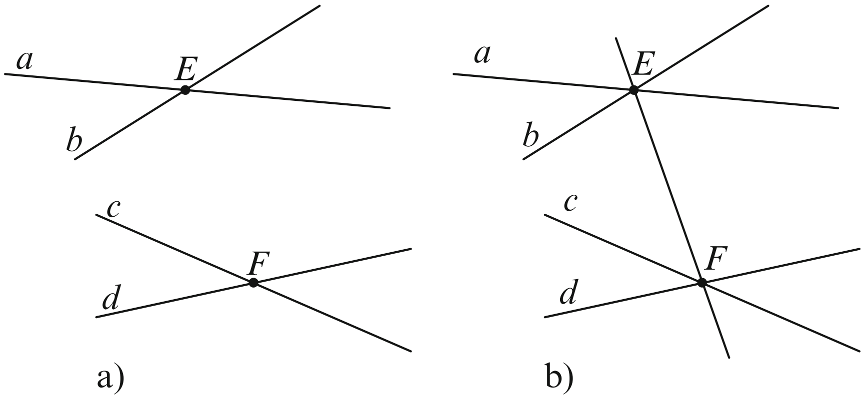


Fig. 1.13

Let us mark their intersection points and denote them by E and F. Now with the help of a straightedge we can draw a unique line EF[7] (see Fig. 1.13b).

[7]We may sometimes denote a line passing through the points E and F by EF.

Are there other possible cases? Yes: what if the point F of intersection of c and d coincides with the point E of intersection of a and b as in Fig. 1.14a? In this case one can draw infinitely many lines through the points E and F (see Fig. 1.14b).

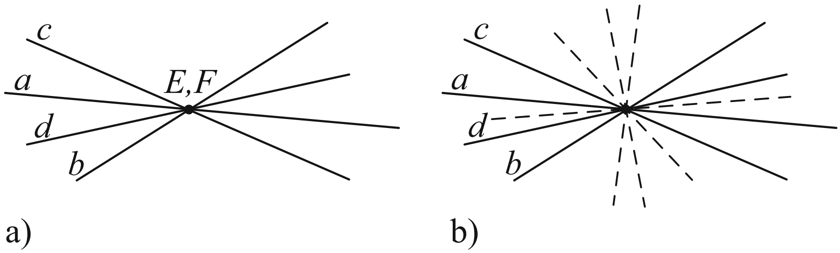


Fig. 1.14

Exercise 3. Draw a line through the point A and the point of intersection B of the two lines a and b in Fig. 1.15.

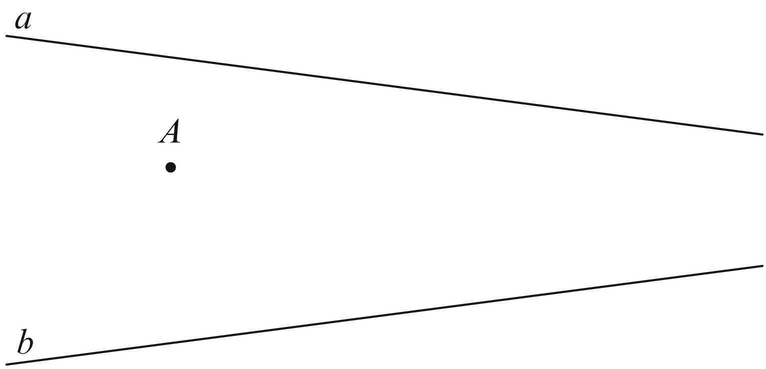


Fig. 1.15

Solution. As we can see, the point of intersection of these lines does not lie on the sheet of paper. However, there are some solutions to this problem:

- We can put a bigger sheet of paper underneath the sheet of paper with the figure and look again for the point of intersection.
- We can draw the line approximately, by estimating intuitively where the point B must lie. Do this yourself.

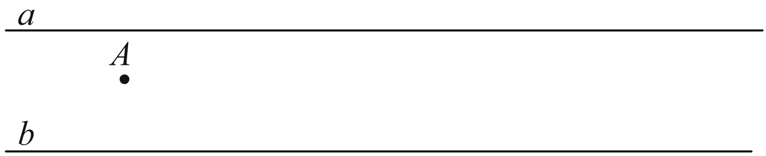


Fig. 1.16

Note that the first solution does not always work. For example, if lines a and b intersect 1 mile from here, they will look like Fig. 1.16. However, even in such a case, Exercise 3 can be solved precisely.

- We can give a precise geometric solution with the help of a configuration called the *Desargues configuration*; this solution is presented in Section 13.

PROBLEM 4. In Fig. 1.17 there are two pairs of intersecting lines a, b and c, d and a line e. Let F be the intersection point of a and b and G the intersection point of c and d. Mark the point H of intersection of the line e and the line FG. First do this by approximation, that is, guess the position of the point H. Then find the point H using the operations of this chapter; check your geometric intuition by comparing the results.

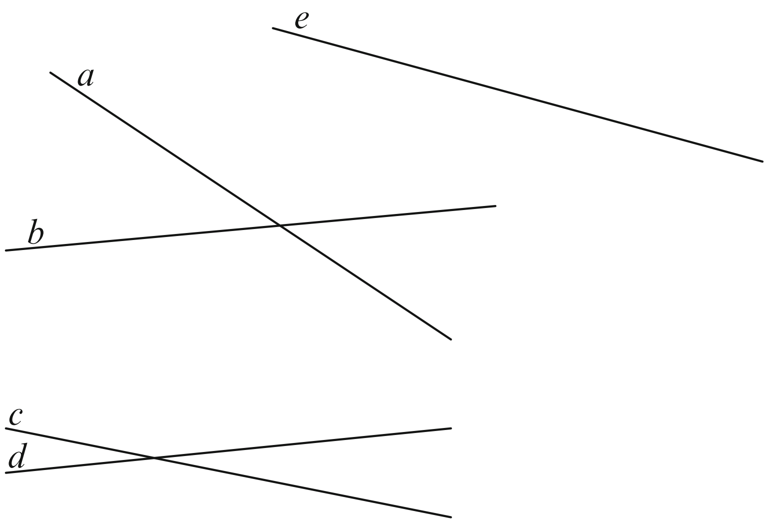


Fig. 1.17

2 Two lines and an angle

Two lines that do not intersect, no matter how far and in what direction we extend them, are called *parallel lines*.

Remark 2. In Chapter I we shall use the knowledge that parallel lines exist. However, we shall have no "tools" to construct them until Chapter II.

It is not possible to verify directly that two lines are parallel. How can one be sure that they will not intersect a mile from here?

In the usual geometry, which is called *Euclidean geometry*, we assume that, given a line and a point that is not on this line, there exists one and only one line passing through this point that is parallel to the given line.

In Problem 2 we posed the following question.

Question 1. Into how many parts do two lines divide a plane?

Answer. Two lines that intersect divide a plane into 4 parts. Two parallel lines divide a plane into 3 parts.

2.1 Notion of an angle

Two intersecting straight lines that divide the plane into four domains are shown in Fig. 1.18.

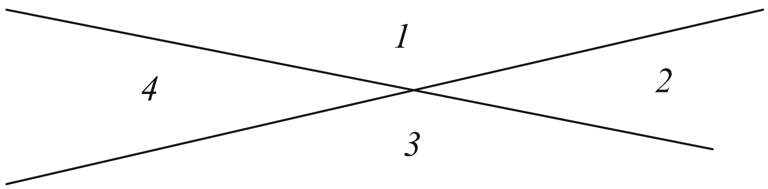


Fig. 1.18

Each of these domains is bounded by two half-lines with the same endpoint. We call such a domain an *angle*.

In fact, two rays with the same endpoint divide the plane into two parts (see Fig. 1.19).

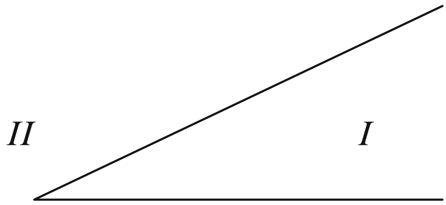


Fig. 1.19

In order to indicate which of these angles we have in mind, we usually draw an “arc” between the two rays that bound the domain under consideration (see Fig. 1.20). The point common to both rays is called the *vertex* of the angle. The rays are called the *sides* of the angle.

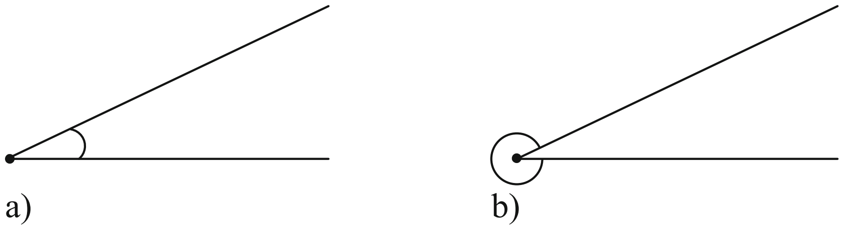


Fig. 1.20

Sometimes it is important to distinguish the angle between a and b and the angle between b and a. In this case we indicate the angle we mean with an arrow (see Fig. 1.21).

Fig. 1.21

PROBLEM 5. Draw two intersecting lines a and b. Indicate with an arrow the angles between b and a. How many angles did you mark?

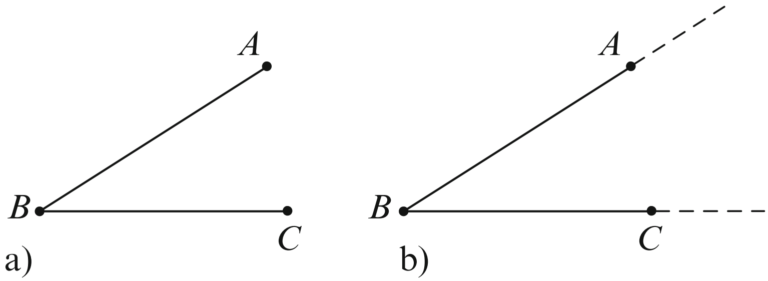


Fig. 1.22

Remark 3. In Fig. 1.22a there are two segments *AB* and *BC* that have the common point *B*. Though these segments do not divide the plane into parts, we can still define an angle. In this case we imagine the segments being extended to form rays (see Fig. 1.22b) and obtain the angle as before. The angle formed by two segments *AB* and *BC* is often denoted by $\angle ABC$. Note that the letter marking the vertex is always in the middle.

Remark 4. In Fig. 1.23a there are three segments *AO*, *BO*, *CO* that have a common point. We can see at once two angles $\angle AOB$ and $\angle BOC$. If we pretend for a moment that the segment *BO* is not there, we will see that $\angle AOC$ is also an angle (see Fig. 1.23b).

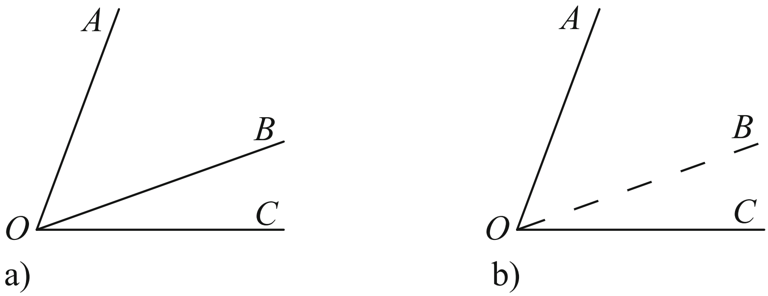


Fig. 1.23

Let us return to our discussion of the observation that two rays with the same endpoint divide the plane into two domains (Fig. 1.19). There is an important difference between these domains, which we can describe using the following definition.

Definition 1. A domain is called *convex* if the segment connecting any two points of this domain lies completely in this domain.

In Fig. 1.24 the shaded domain is convex and the unshaded one is not convex. (Note that the curved line indicates that the shaded area continues indefinitely.) Each of these two domains is an angle formed by the pair of rays. Whenever we are discussing an angle formed by a pair of rays, we will be choosing to refer to a convex angle, unless otherwise indicated.

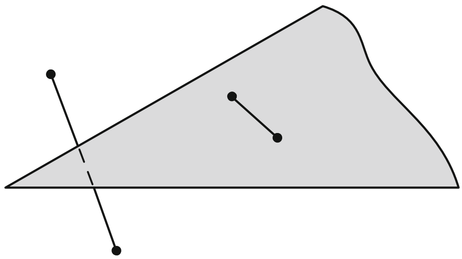


Fig. 1.24

There is only one case where both domains formed by a pair of rays with the same endpoint are convex (see Fig. 1.25).

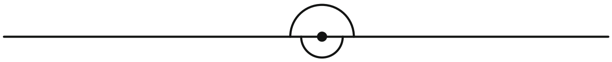


Fig. 1.25

In this case each ray is an extension of the other; that is, together they form a straight line. Each of these angles is called a *straight angle*. As you probably know, a straight angle contains 180 degrees, which is written as 180°. The instrument used to measure angles is called a *protractor*. We talk about angle measure in Chapter IV.

2.2 Some types of angles

Two convex angles that have one side in common and are positioned in such a way that the second side of one angle is an extension of the second side of the other angle are called *supplementary angles*.[8]

In Fig. 1.26 angles $\angle AOB$ and $\angle BOC$ are supplementary angles.

[8] In standard terminology, supplementary angles are defined as angles whose measures add up to 180°; but we cannot define angle measure using only the operations available in this chapter.

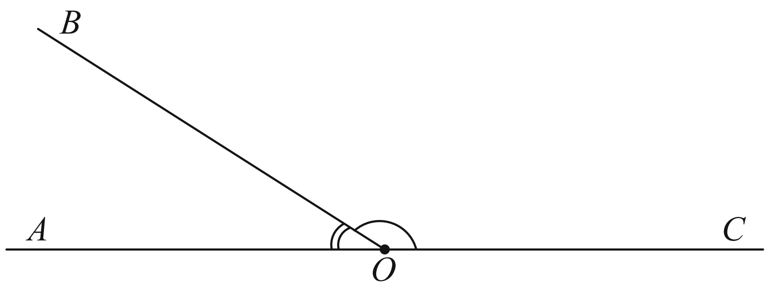


Fig. 1.26

A pair of convex angles in which the sides of one angle are extensions of the sides of the other angle are called *vertical angles*.

Two intersecting lines form two pairs of vertical angles, which in this context may also be called *opposite angles*. In Fig. 1.27 angles $\angle AOC$ and $\angle BOD$ are vertical, as are angles $\angle AOD$ and $\angle BOC$.

PROBLEM 6. Find four pairs of supplementary angles in Fig. 1.27, i.e., continue the list below:

1) $\angle AOC$ and $\angle COB$

2)

3)

4)

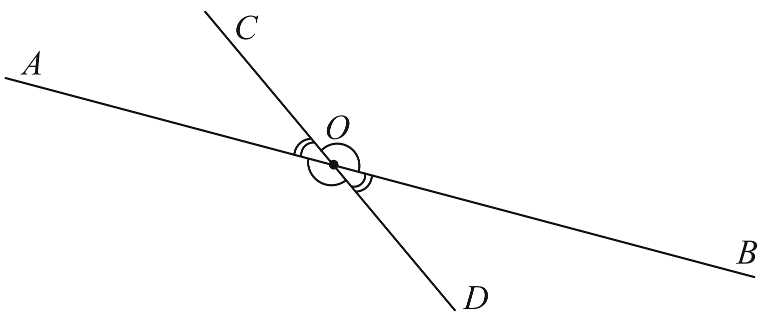


Fig. 1.27

Exercise 4. Fig. 1.28 shows two intersecting lines. How many different convex angles are there between the pairs of rays? Are there more than four angles in this figure?

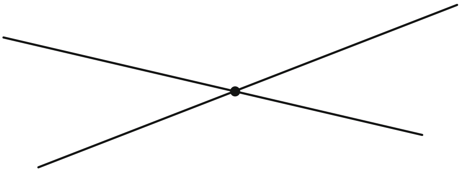


Fig. 1.28

Solution. In Fig. 1.28 we can see at once four angles. They are drawn separately below (see Fig. 1.29).

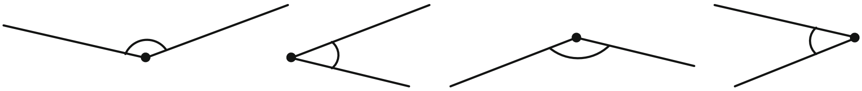


Fig. 1.29

Strictly speaking a straight angle is also convex. Therefore, there are four more convex angles in Fig. 1.28. (See Fig. 1.30.)

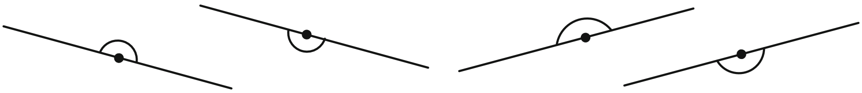


Fig. 1.30

However, there are still more angles in Fig. 1.28. These angles are not convex (see Fig. 1.31).

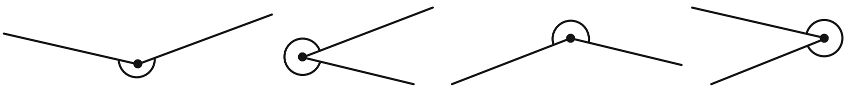


Fig. 1.31

Therefore, the answer to the question "How many angles are there in Fig. 1.28?" can vary. It can be 4, 8, and even 12. Usually it is clear from the context what angles to consider. It is useful to be able to see all of them.

One can also ask whether the figure formed by two coinciding rays with the same endpoint is an angle. There are two ways to answer this question:

1) we can say that such rays do not bound any domain and we call such an angle a *zero angle*; or 2) we can say that these rays bound the whole plane and we say that they form a *complete angle*.[9]

PROBLEM 7. How many convex angles, including straight angles, are formed by three straight lines that intersect in one point? Draw them.

If you like to count, you might want to count all the angles formed by these lines. We had no patience ourselves for doing that.

PROBLEM 8.

(a) How many convex angles, not including straight angles, are formed by four straight lines that intersect in one point?

(b) How many convex angles, not including straight angles, are formed by n straight lines that intersect in one point?

3 Three lines

3.1 Configurations of three lines

Definition 2. We say that some lines in the plane are in *general position* if no two of them are parallel and no three of them intersect in one point.

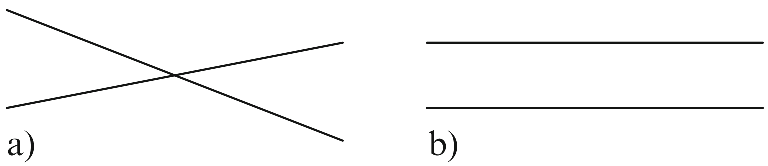


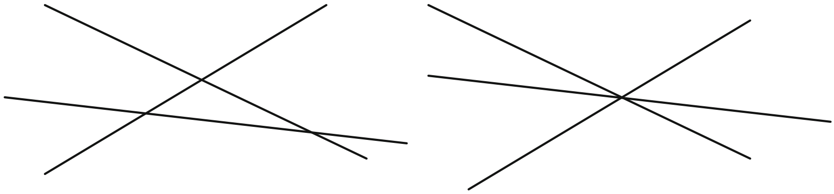
Fig. 1.32

[9]In earlier drafts, Israel Gelfand used an old term "perigon." We could not find many references to it in contemporary works. It is possibly derived from the name Pierre Herigone, a French mathematician, who introduced the notation $\angle$ for an angle and $\perp$ for perpendicularity.

Thus, two lines are in general position if they are not parallel.

In Fig. 1.32a there are two lines in general position and in Fig. 1.32b there are two lines that are not in general position.

In Fig. 1.33a there is an example of three lines in general position and in Fig. 1.33b an example of three lines which are not in general position.



a) b)

Fig. 1.33

Let us consider three lines. We can draw them in infinitely many ways (see some examples in Fig. 1.34). However, we can notice that it is possible to distinguish certain types within them, which we will call *configurations*. For example, figures a) and b) are examples of three lines in general position. In figures c) and d) three lines intersect in a single point; in figures e) and f) two lines are parallel and the third line intersects them.

We will not define precisely what a configuration is, but we will define how to distinguish between different configurations. In this chapter we consider only configurations of lines. A configuration may consist of two or more lines, and we may also refer to them as a set (or a collection) of lines.

Definition 3. We say that two sets of three or more lines are examples of the same *configuration* if it is possible to change one set of lines into the other by gradually moving[10] these lines on the plane while obeying the following rule:

- at no moment are we allowed to obtain a new intersection point or lose an existing one.

The notion of a configuration is more complicated, and the rule we have formulated is often not enough. However, in this book we will use the above definition, which is easy to understand on intuitive level.

[10]In this chapter we will be "moving the lines on the plane" based on common sense understanding of this phrase. In Chapter IV, moving a figure on the plane will obtain its precise meaning.

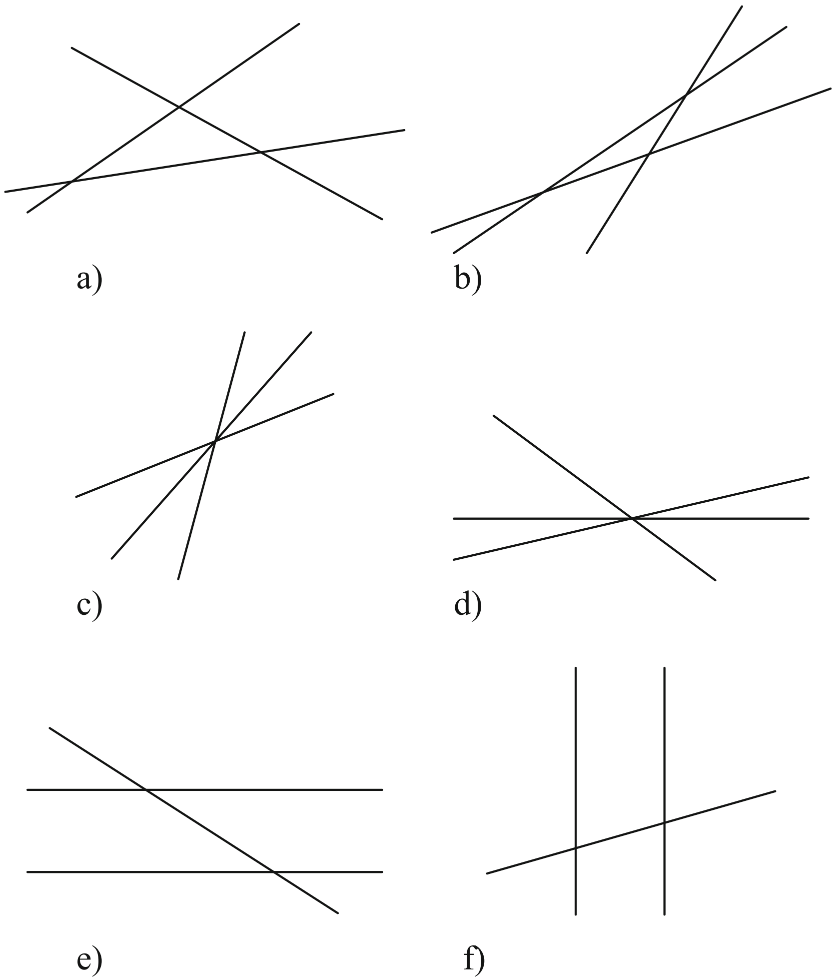


Fig. 1.34

Exercise 5. Find all possible configurations of three lines in general position.

Solution. Let us start with two lines. As we already know, two lines either intersect (see Fig. 1.32a) or they do not (see Fig. 1.32b).

Now let us add a third line. Since in Fig. 1.32b two lines are parallel (that is, they are not in general position), then no matter how we draw the third line we cannot obtain a figure in general position.

Let us add a third line to Fig. 1.32a so that it intersects both lines; in Fig. 1.35a it intersects these lines to the right of their intersection point and in Fig. 1.35b it intersects them to the left of their intersection point.

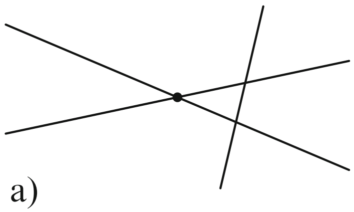


Fig. 1.35

Note that Fig. 1.35a and Fig. 1.35b are examples of the same configuration. Indeed, let us begin with the configuration in Fig. 1.35a and denote these lines as in Fig. 1.36a. If we turn all the lines around the point O, for example counterclockwise, we will obtain the configuration in Fig. 1.36b, which is the same as in Fig. 1.35b. The rule from Definition 3 is satisfied.

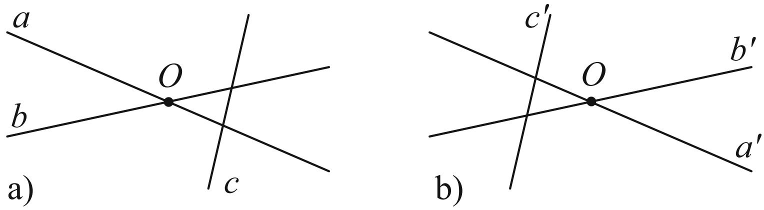


Fig. 1.36

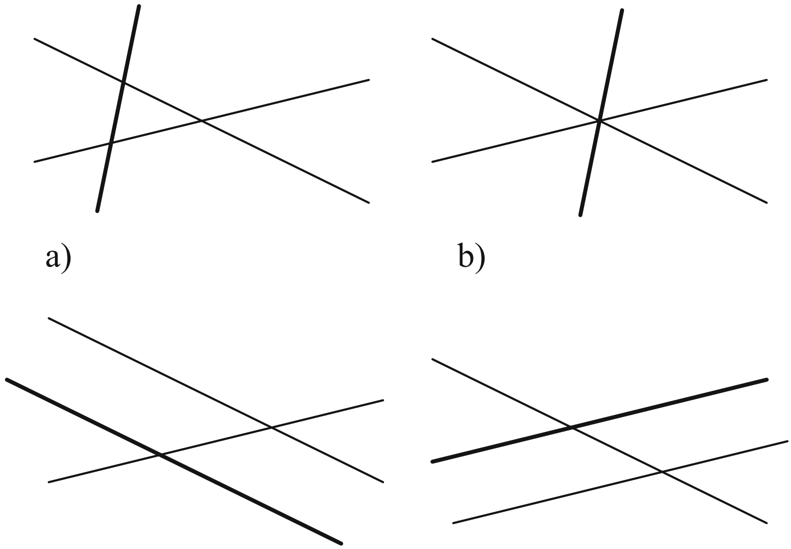
It is clear that if we add a third line to Fig. 1.32a in any other way we do not obtain three lines in general position (this is done in the next exercise). Therefore, there is only one configuration of three lines in general position.

Exercise 6. Find all possible configurations of three lines which are not in general position.

Solution. Let us start with configurations of two lines (see Fig. 1.32a, b). First, consider Fig. 1.32a. We can add a third line in the following ways:

- The third line can intersect both lines. In this case we obtain three lines in general position (see Fig. 1.37a).

- The third line can pass through the point of intersection of the two lines (see Fig. 1.37b).
- The third line can be parallel to one of the two lines (see Fig. 1.37c, d).

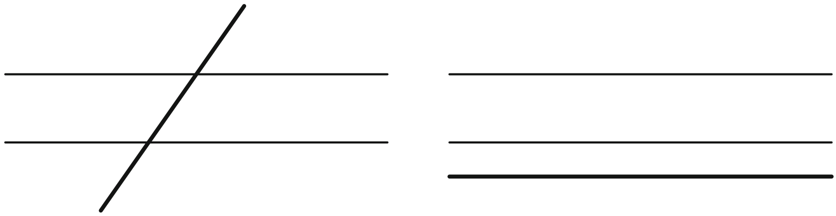


a) b)

c) d)

Fig. 1.37

Note that Fig. 1.37c and Fig. 1.37d present the same configuration. Check this.



a) b)

Fig. 1.38

Now let us add a third line to Fig. 1.32b:

- The third line can intersect the two parallel lines (see Fig. 1.38a).

- The third line can be parallel to these two lines (see Fig. 1.38b).

As we can easily check, Fig. 1.37c and Fig. 1.38a present examples of the same configuration.

Thus, the answer is: there are three different configurations of three lines that are not in general position (see Fig. 1.37b, Fig. 1.37c and Fig. 1.38b).

PROBLEM 9. Sketch together all (four) possible configurations of three lines.

Remark 5. A collection of lines in general position has the following interesting characteristic. If lines are in general position, then no matter how we move them it is always possible to make this move small enough that the lines still remain in general position. We say, therefore, that a general position is *stable*.

If lines are not in general position, it is possible to indicate how to move some of them, so that they will move into general position. In fact, these moves can be as small as desired. This means a figure that is not in general position is not stable.

3.2 Triangles

Several straight lines in the plane divide it into parts called *polygonal domains*. A polygonal domain can be *bounded* or *unbounded*. A polygonal domain is said to be *bounded* if its boundary consists only of a finite number of segments. Otherwise a polygonal domain is called *unbounded*. It is not possible for a ray to lie completely in a bounded polygonal domain.

PROBLEM 10.

(a) Into how many domains do three lines in general position divide the plane?

(b) How many of these domains are bounded?

PROBLEM 11. Draw three lines in general position.

(a) Mark all the unbounded domains which are angles.

(b) How many pairs of opposite angles have you obtained?

A *triangle* is a bounded domain formed by three lines in general position (see Fig. 1.39a).

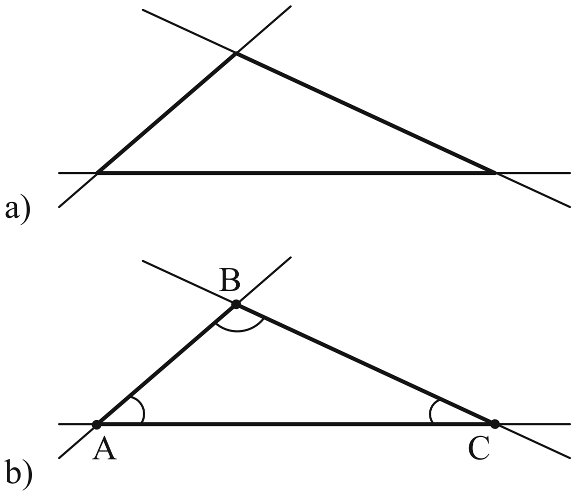


Fig. 1.39

A triangle has three *vertices*: points A, B, and C in Fig. 1.39b, and three *sides*: segments AB, BC, and CA. With a triangle we can also associate three angles called *interior angles*: $\angle BAC$, $\angle CBA$, $\angle ACB$.

The name "triangle" reflects the fact that a triangle has three interior angles, but it could have been also called a "trilateral," because it has three sides, or a "trivertex."

An *exterior angle* of a triangle is an angle formed by one side and an extension of another side.

For example, Fig. 1.40 shows a triangle with vertices A, B, C; this may be written for short as $\triangle ABC$. The angle $\angle BCD$ is an exterior angle of $\triangle ABC$.

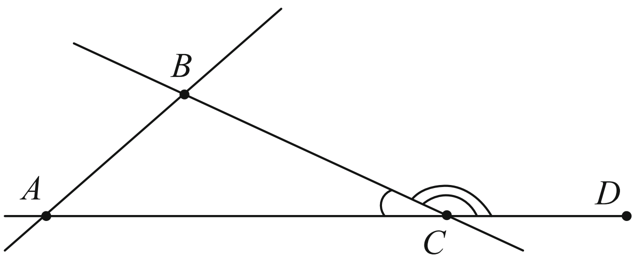


Fig. 1.40

Exercise 7. How many exterior angles does a triangle have?

Solution. See Fig. 1.41.

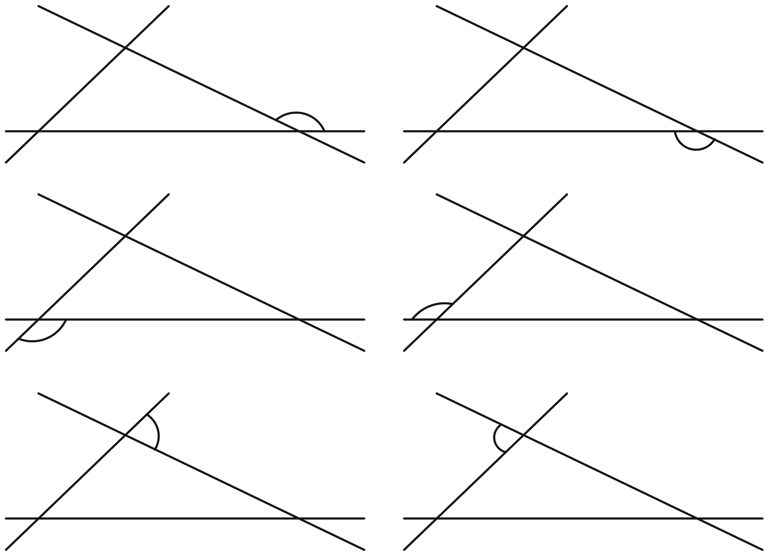


Fig. 1.41

All exterior angles of a triangle are presented together in Fig. 1.42.

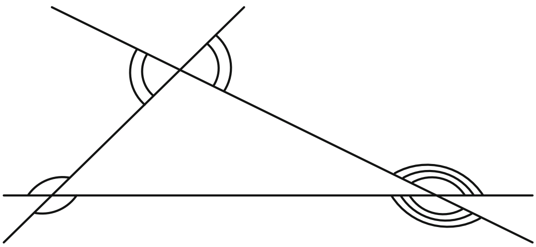


Fig. 1.42

PROBLEM 12. Draw three lines in general position.

(a) How many convex angles, excluding straight angles, are there?

(b) How many convex angles, including straight angles, are there?

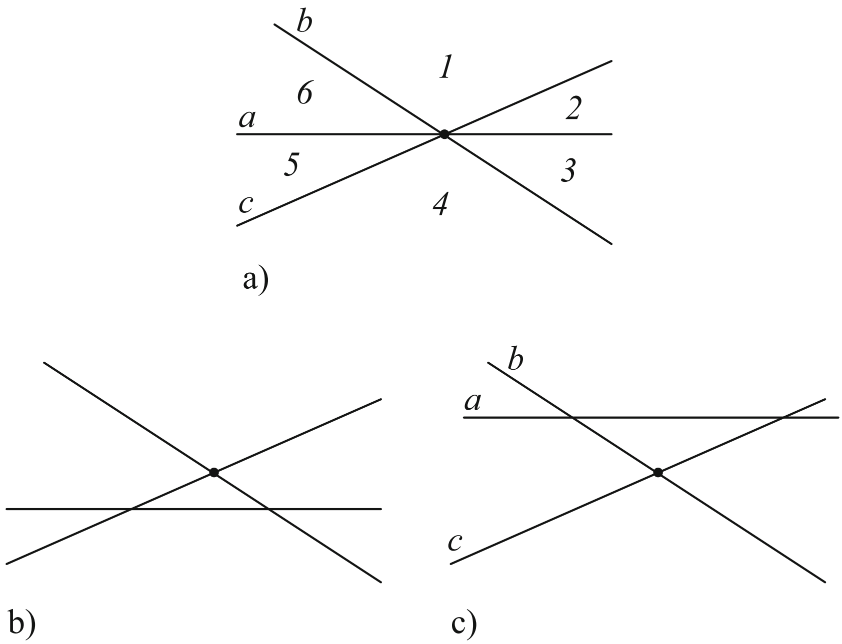


Fig. 1.43

PROBLEM 13. In Fig. 1.43a, there are three lines a, b, c that intersect in a single point. Six angles are labeled.

(a) The line a was moved down (see Fig. 1.43b). Indicate in this figure all six angles.

(b) The line a was moved up (see Fig. 1.43c). Indicate in this figure all six angles.

PROBLEM 14. How many

(a) points,

(b) segments,

(c) rays,

(d) straight lines,

(e) domains

are there in Fig. 1.44a and in Fig. 1.44b?

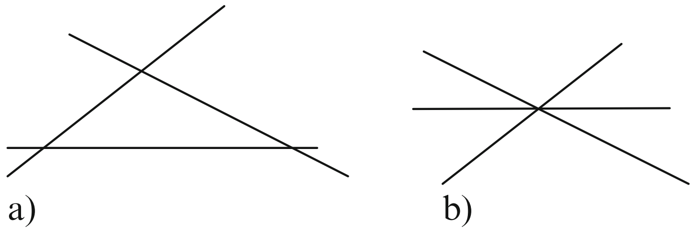


Fig. 1.44

4 Four lines. Quadrilaterals

Let us draw four lines in general position. First, consider three lines in general position (see Fig. 1.44a). We need to add the fourth line in such a way that it will intersect each of the three lines but will not pass through any of their points of intersection (see Fig. 1.45).

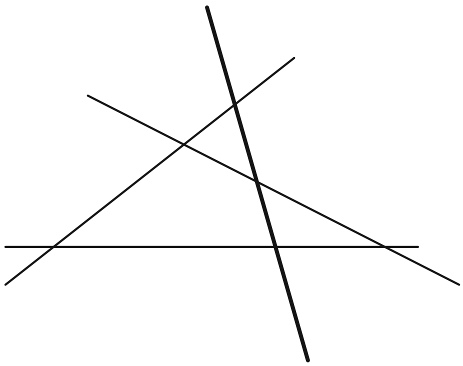


Fig. 1.45

PROBLEM 15. Are there other configurations of four lines in general position?

Hint. Draw three lines in general position (as you know, there is only one such configuration) and play with drawing the fourth line. Compare the configurations obtained.

PROBLEM 16. Imagine the following game. We have wild animals (wolves, for example) and domesticated animals (dogs). We have to place them in the plane, each animal in a domain so that they will not be able to fight. In the bounded domains we will, of course, place wolves so they will not run away. We can place dogs in the unbounded domains because they will return to be fed.

(a) How many wolves and dogs can we place on the plane, in which four lines in general position are drawn?

(b) Do all the domains for dogs have the same shape?

(c) Do all the domains for wolves have the same shape?

PROBLEM 17. Draw all possible configurations of four lines, including those that are not in general positions.

Hint. Consider all possible configurations of three lines (see Exercise 5, 6, or Problem 9), then add the fourth line. Recall that the fourth line can either be parallel to one of the lines, can pass through the point of their intersection, or both.

Definition 4. A *quadrilateral* is a domain bounded by four nonintersecting segments. These segments are called *sides* and the ends of the segments are called the *vertices* of the quadrilateral.

Notice that a quadrilateral is a bounded domain. Examples of quadrilaterals are in Fig. 1.46. As you can see, there are two types of quadrilaterals. Each quadrilateral in Fig. 1.46a, b has four sides, four vertices, and four interior angles (marked in Fig. 1.46).

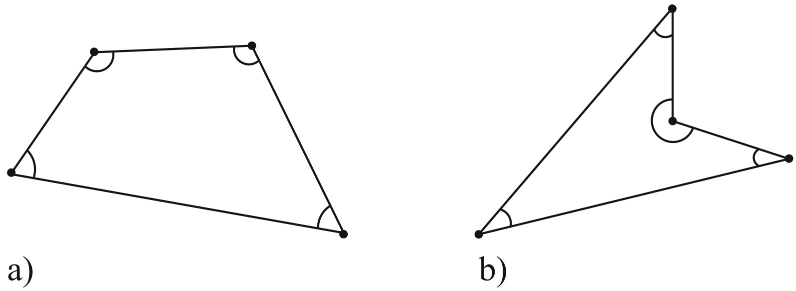


Fig. 1.46

A *diagonal* of a quadrilateral is a segment connecting two vertices that are not connected by a side. Every quadrilateral has two diagonals. Using diagonals, it is easy to check whether a quadrilateral is convex or not (see Definition 1): a quadrilateral that has both diagonals inside is convex.

One of the quadrilaterals in Fig. 1.46 is convex and the other one is not (see Fig. 1.47).

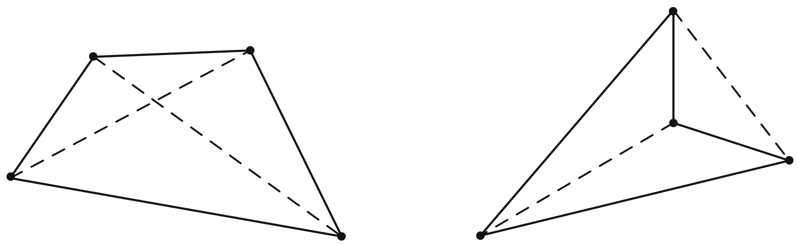


Fig. 1.47

Remark 6. Note that not every set of four points can be the vertices of a quadrilateral. Consider four points A, B, C, D and the quadrilateral $ABCD$ (see Fig. 1.48a). Let us see what happens if we fix three points and start

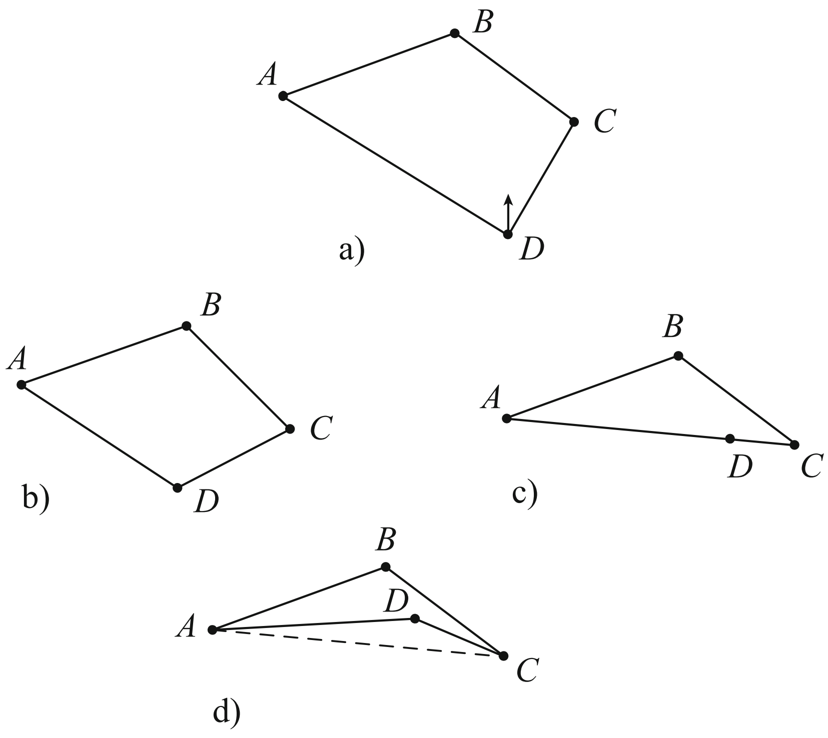


Fig. 1.48

moving the vertex D. In the positions a) and b) the figure $ABCD$ is a convex quadrilateral. In the position d) the figure $ABCD$ is a nonconvex quadrilateral. But at the moment when the point D is moved onto diagonal AC (see Fig. 1.48c) the figure $ABCD$ is no longer called a quadrilateral; it is simply a triangle ABC with the point D marked on side AC. Therefore, four points are vertices of a quadrilateral only if no three of them lie on a straight line.

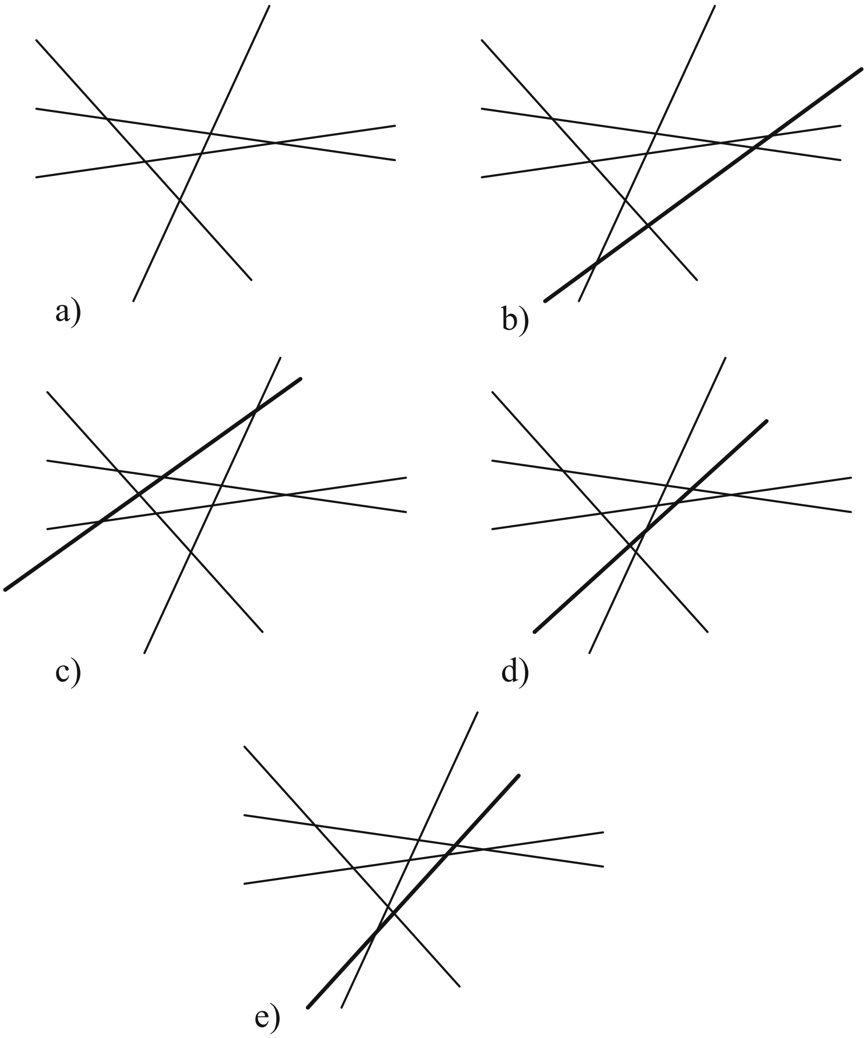


Fig. 1.49

PROBLEM 18.

(a) How many triangles and how many quadrilaterals are there in Fig. 1.45? Indicate them.

(b) How many of these triangles are chambers for wolves (see Problem 16)?

(c) How many of these quadrilaterals are chambers for wolves?

5 Five lines

We want to introduce to you a very interesting problem. Consider a configuration of five lines in general position. As we have already defined, general position means that no three of these lines intersect in one point and no two of them are parallel.

While there is only one configuration of four lines in general position (see Fig. 1.49a), configurations of five lines can be different (see for example Fig. 1.49b, c, d, e). Notice how many bounded domains are triangles in each of these configurations and where they are located. None of the configurations in Fig. 1.49b, c, d, e can be gradually moved into another according to the rule in Definition 3.

Five straight lines divide the plane into domains. Some of them are bounded (in terms of Problem 16 they are chambers for wolves[11]) and some are unbounded.

PROBLEM 19. What is the largest number of bounded domains that can be obtained in a configuration of five lines in general position?[12] Play with drawing different configurations and count this number.

6 Projection from a point onto a line

Consider a point O and a straight line a. Using point O we would like to make any point on the plane correspond to some point on line a. How can we do this? One useful way is described below.

[11] By the way, mathematicians also call these domains *chambers* but the word "wolves" is not used.

[12] This problem was the starting point for research in combinatorics published in some recent well-known papers. The problem was solved for any number n of straight lines in general position.

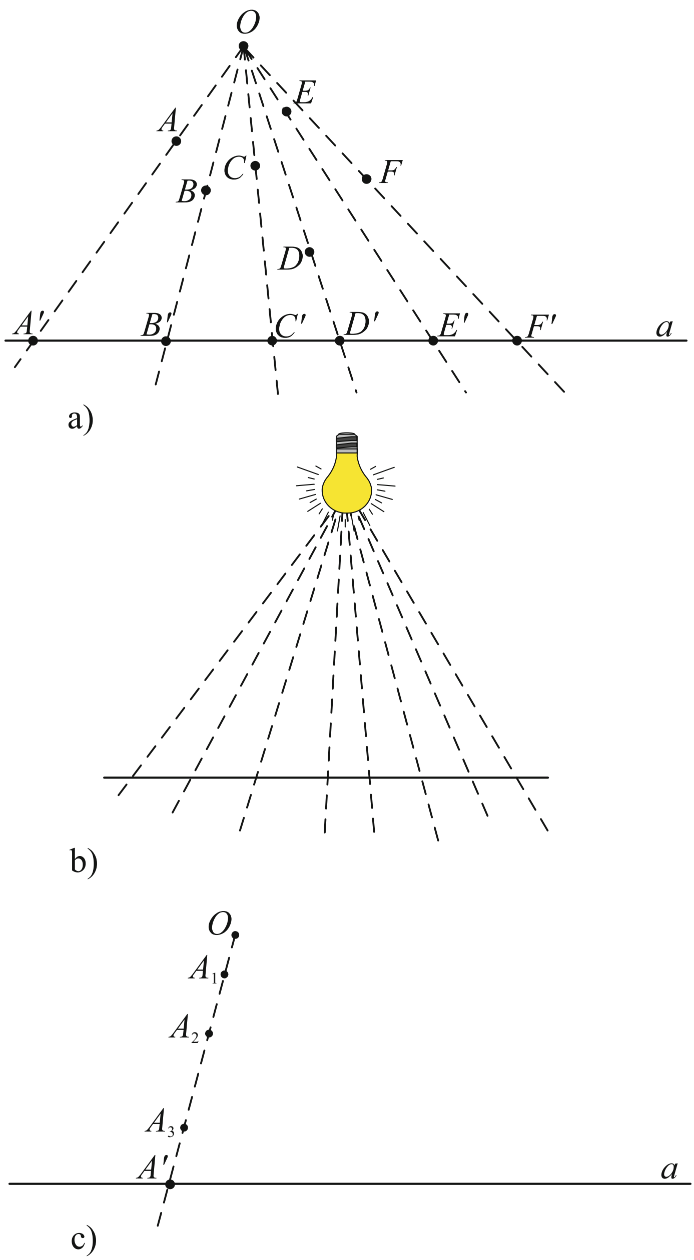


Fig. 1.50

Let us mark some points A, B, C, D, E, F on the plane. If we draw a ray from point O, we can make point A correspond to the point A' on the line a

(see Fig. 1.50a). Similarly, the point B' corresponds to the point B, etc.

We can also think of these rays as light rays coming from a source of light at O (see Fig. 1.50b). Then the points A', B', C', ... are shadows of the points A, B, C, ... correspondingly.

Note that in Fig. 1.50c the points A_1, A_2, A_3, as well as all other points lying on the segment OA', will have the same shadow, that is, point A'.

One can easily notice that not every point on the plane will have a corresponding point on the line a. Indeed, the points in Fig. 1.51a, as well as any point on line b, where b is parallel to a, will not have any projection (shadow) on the line a.

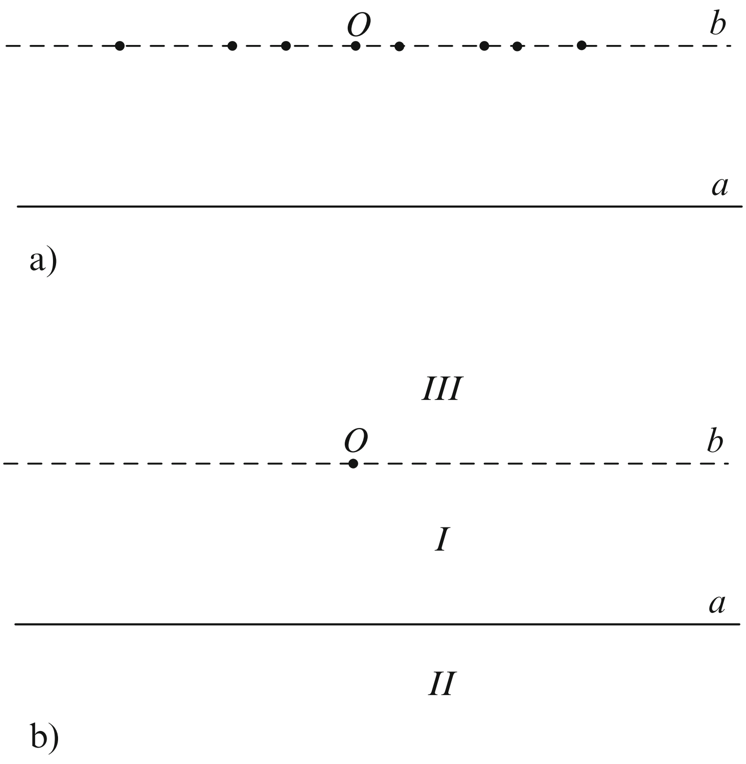


Fig. 1.51

As we know, the lines a and b divide the plane into three parts (see Fig. 1.51b). We have considered the points lying in part I.

Let us consider the points lying in part II (see Fig. 1.52a). If we draw rays from point O, we can see that any point from domain II will have a corresponding point on line a. Note that in this case such a correspondence is no longer a projection by light rays. Indeed, it is hard to imagine the shadow of an object being closer to the light than the object itself.

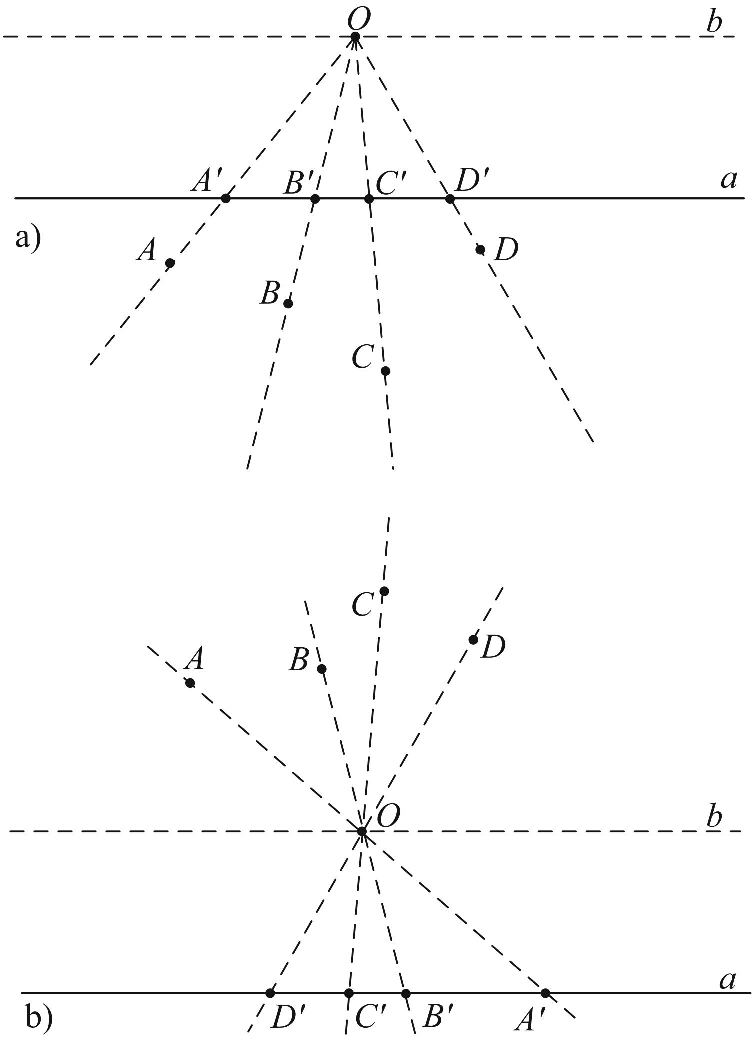


Fig. 1.52

Consider now points in domain III (see Fig. 1.52b). The rays that we have considered from point O no longer establish the correspondence. But mathematicians like to generalize. Instead of rays from point O, we draw lines passing through point O (see Fig. 1.52b). Thus, any point in domain III has a corresponding point on line a.

The correspondence between the points on the plane and the points on a certain line a established by lines passing through a certain point O we call *central projection* or *central perspective*. For a point A on the plane, the corresponding point A' on line a is called the *image* or *projection* of the point A.

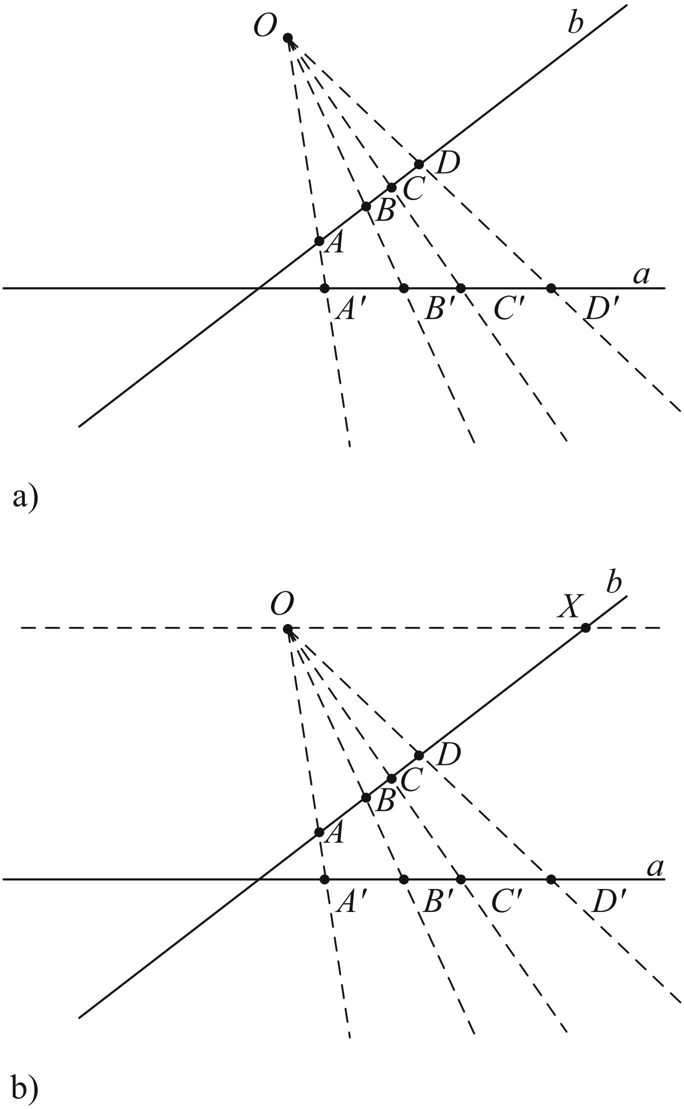


Fig. 1.53

Let us consider an interesting particular case of central perspective: projection of points lying on one line onto the other line.

Mark a point O and draw a line a. Consider a line b not parallel to a. Using central perspective from the point O we can find projections of points lying on line b onto line a. For example, the points A, B, C, D in Fig. 1.53a have the projections A', B', C', D'.

Exercise 8. Does every point on the line b in Fig. 1.53a have a projection on line a? If not, mark such an "exceptional" point.

Solution. As we know, all the points that lie on a line passing through point O and parallel to line a have no projections on the line a. Thus, the point X in Fig. 1.53b does not have a projection on line a. All other points on the line b have images (projections) on line a.

Exercise 9. Find the image of all the points lying on segment AB in Fig. 1.53a.

Solution. The image of segment AB is segment $A'B'$.

PROBLEM 20. Draw the projection from the point O of segment AB in Fig. 1.54a, b onto the line a.

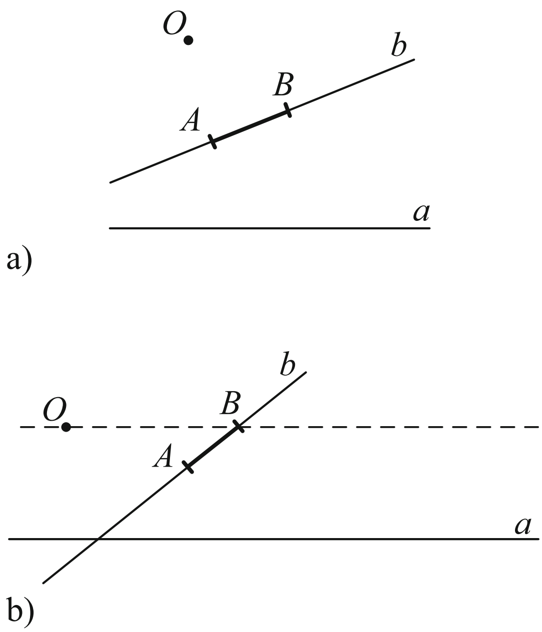


Fig. 1.54

Exercise 10. Find the projections of points A, B, C, D in Fig. 1.55a onto line a.

Solution. The projections of points A, B, C are A', B', C' correspondingly (see Fig. 1.55b). The projection of the point D coincides with point D itself.

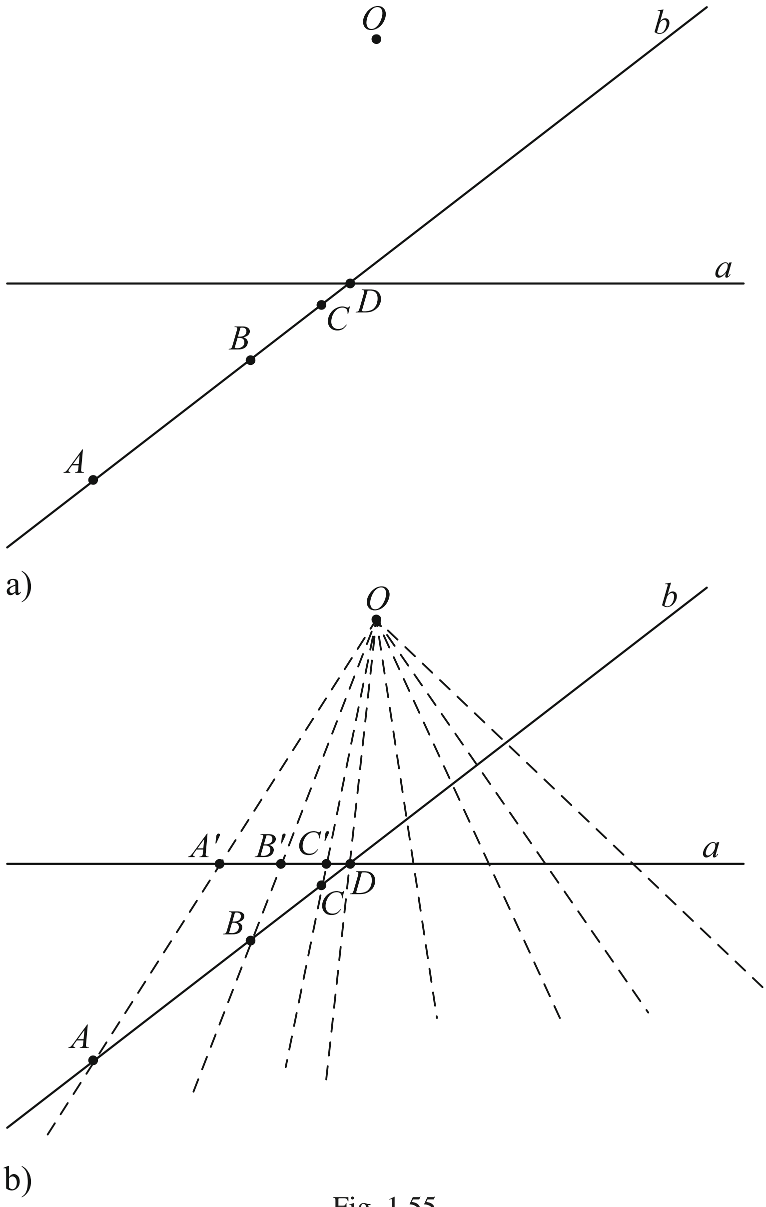


Fig. 1.55

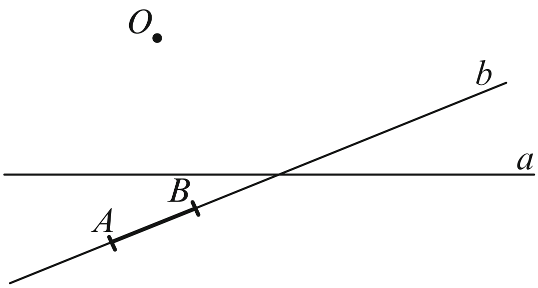


Fig. 1.56

PROBLEM 21. Find the projection of segment AB in Fig. 1.56 onto line a.

We have considered cases so far where the points lying on line b belong to the domains I or II (see Fig. 1.53–Fig. 1.56). Let the points A, B, C, lying on the line b, belong to the domain III. For example, in Fig. 1.57 there are points A, B, C on the line b; then the points A', B', C' are correspondingly their projections.

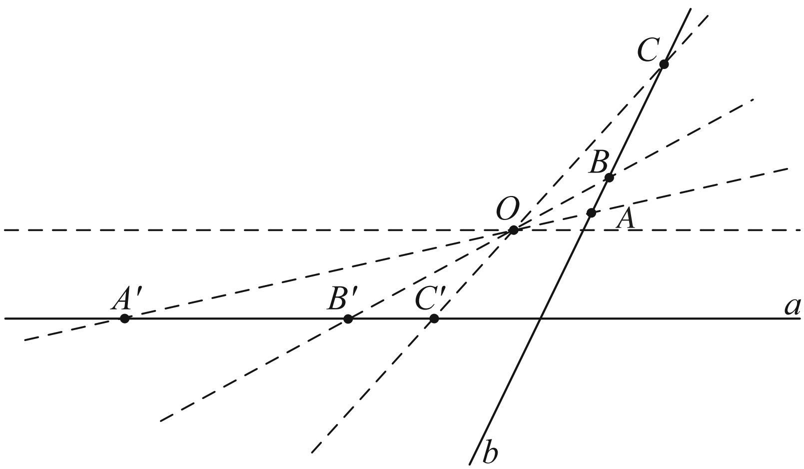


Fig. 1.57

PROBLEM 22. Find the projection onto line a of the segment AB in:

(a) Fig. 1.58a.

(b) Fig. 1.58b.

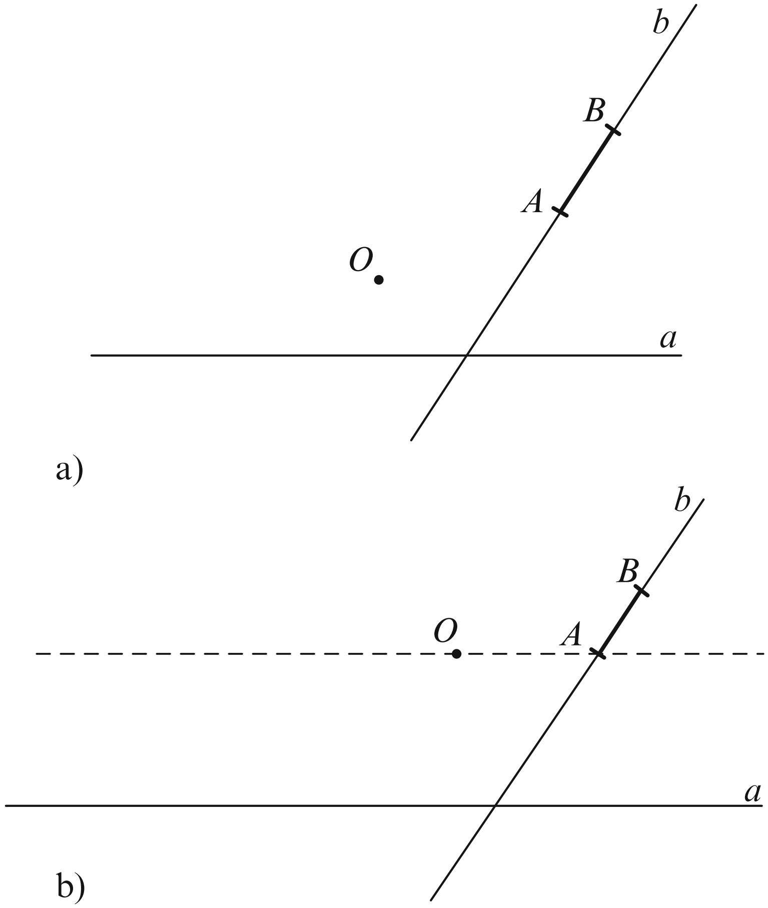


Fig. 1.58

PROBLEM 23.

(a) Find the projection onto line a of the segment AB in Fig. 1.59a.

(b) In Fig. 1.59b there is a scale of length marked on line a. Using the correspondence established by lines from point O, mark on line b the points corresponding to the numbers on line a. Label these points using the corresponding numbers.

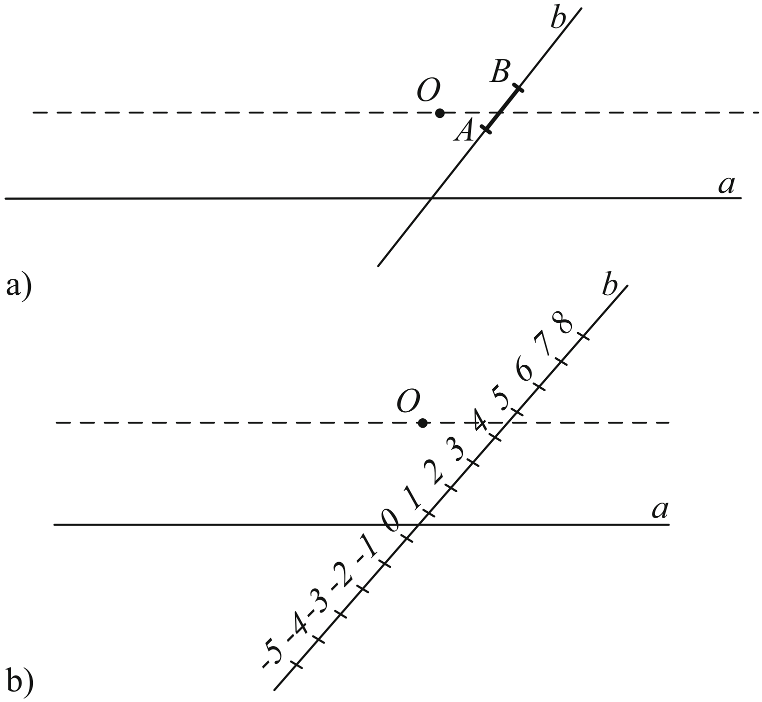


Fig. 1.59

Note that given a point O and two lines a and b, we can project a point lying on line b onto line a, and we can also project a point lying on line a onto line b. For example, in Fig. 1.60a points A, B lying on line a have projections A', B' on line b.

As we know, there is only one point on line a that does not have a projection onto line b, and this is the point X in Fig. 1.60b.

Exercise 11. Does every point on line b have a corresponding point on line a? If not, mark such an “exceptional” point.

Solution. The point Y in Fig. 1.60c does not have a projection on line a.

Thus, we can say that central perspective establishes a correspondence between the points of two lines a and b. For a point on line a we can indicate the corresponding point on line b and vice versa. This correspondence is not a *one-to-one correspondence*[13]: it is a one-to-one correspondence except for one point on each line.

[13] See Glossary.

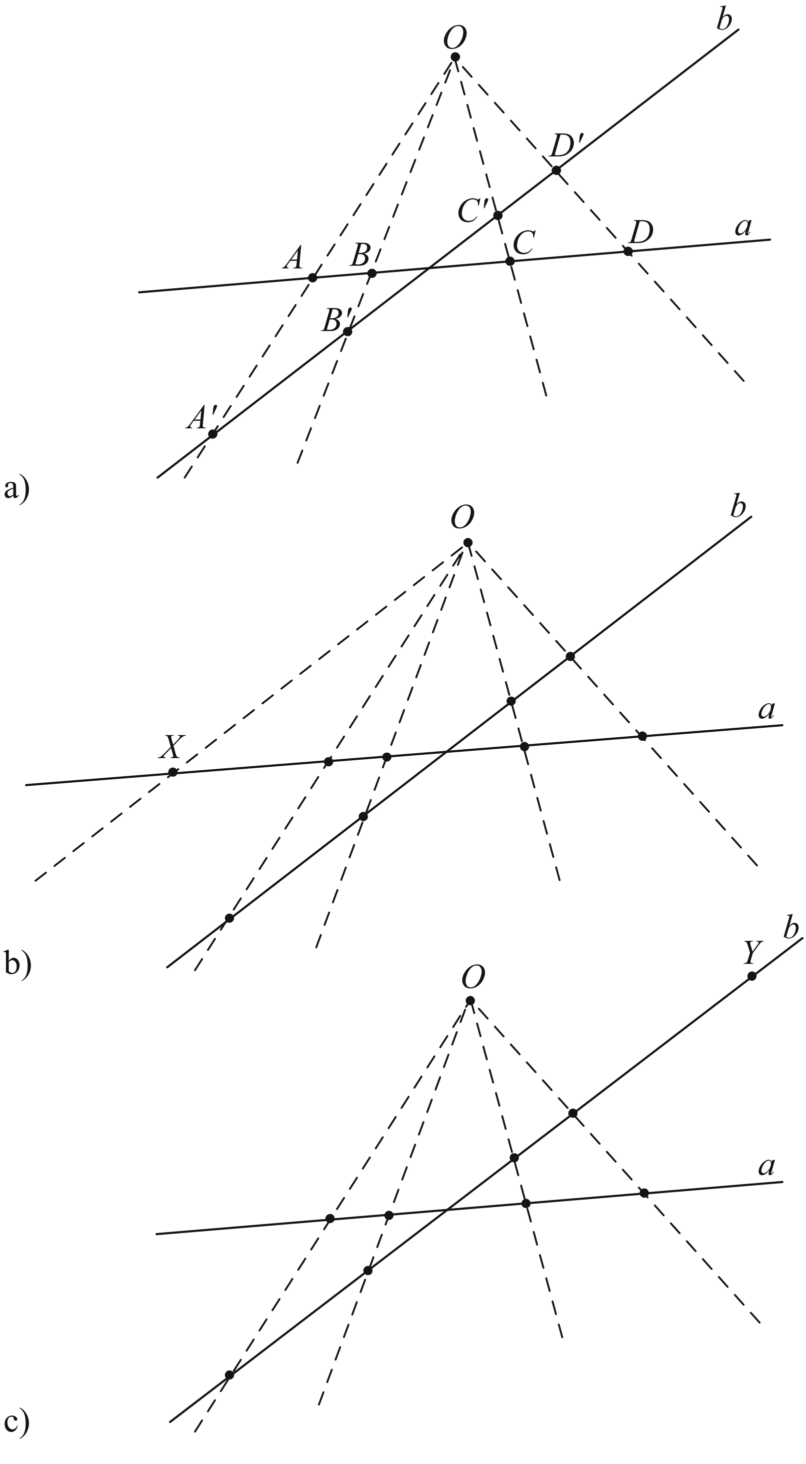


Fig. 1.60

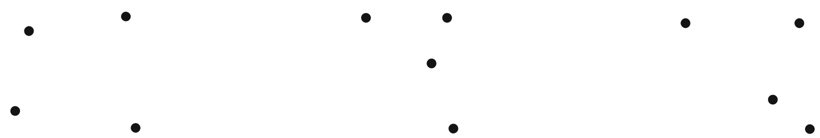
7 Dual configurations in projective geometry

Points in the plane are *in general position* if no three of them lie on the same line.

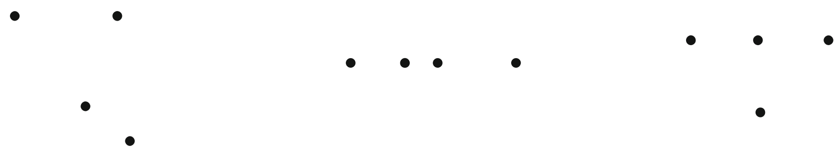
PROBLEM 24.

(a) Which of the points in Fig. 1.61 are in general position and which are not?

(b) Draw all possible lines through every pair of points in each of the figures in Fig. 1.61. Count the number of lines you have obtained in each case.



a) b) c)



d) e) f)

Fig. 1.61

In projective geometry, for any configuration there is a so-called *dual configuration*. Any configuration can be described using the following words: “a point,”, “a line,” “a point lying on the line,” “a line passing through the point.” Given a configuration, in order to obtain the dual configuration we need to replace these words according to the following table:

configuration	dual configuration
a point A	a line a
a line b	a point B
a point A lies on the line b	a line a passes through the point B
a line a passes though the point B	a point A lies on the line b

In Fig. 1.62a, b there is an example of two configurations that are dual to one another.

Fig. 1.62

As you can see from the table and the example, one can always change a configuration into its dual and vice versa.

Exercise 12. Consider three points A, B, C on the line d (see Fig. 1.63a). Draw the dual configuration.

Solution. Instead of each point we must have a line. That is, we need to draw three lines a, b, and c. Since the points A, B, C lie on the same line, the lines a, b, c should pass through the same point (see Fig. 1.63b).

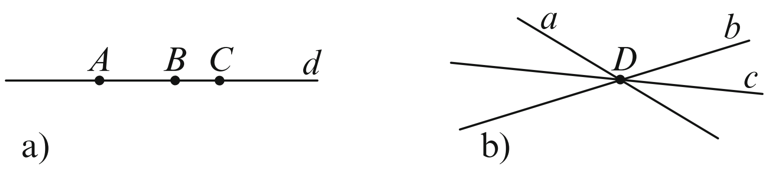


Fig. 1.63

Exercise 13. Let us consider three points in general position and draw all lines passing through any two of these points (see Fig. 1.64). Draw the dual configuration.

Solution. The dual configuration to this configuration looks exactly the same as the configuration in Fig. 1.64. This is why this configuration is called *dual to itself*. It has 3 points and 3 lines. We call it a *complete triangle*.

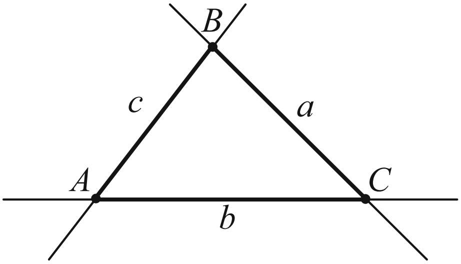


Fig. 1.64

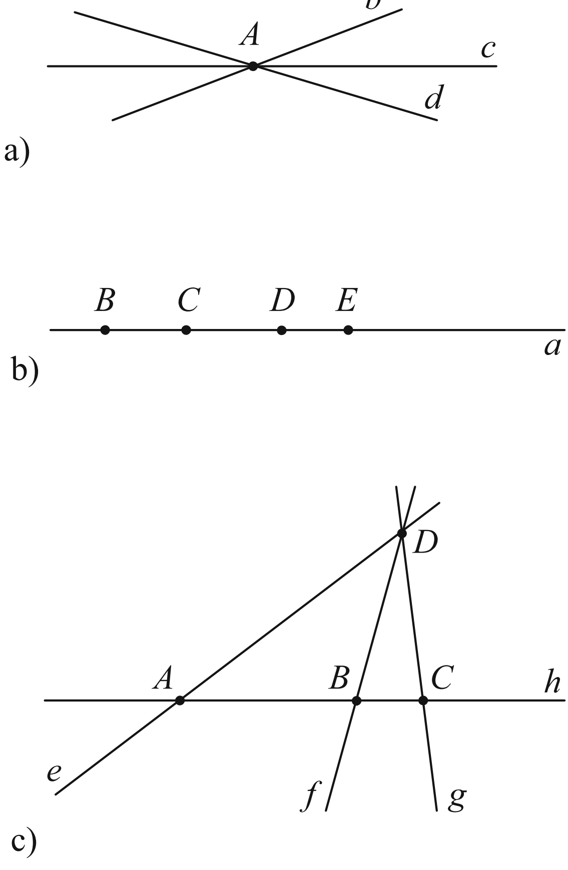


Fig. 1.65

PROBLEM 25. Draw the dual configurations to the configurations in Fig. 1.65a, b, and c. Label the corresponding elements.

PROBLEM 26. Draw the dual configurations to the configurations in Fig. 1.66a, b.

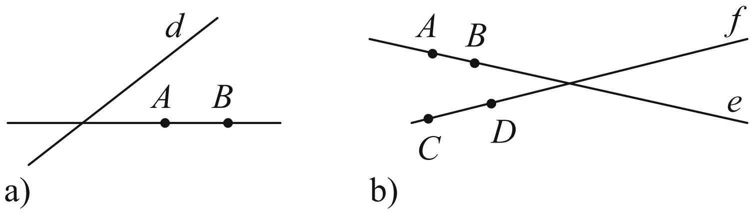


Fig. 1.66

Consider four points in general position (see Fig. 1.67a). Through every pair of these points let us draw all possible straight lines; these are the lines b_1, b_2, b_3, b_4, b_5, b_6 in Fig. 1.67b. This configuration is called a *complete quadrangle*. It consists of 4 points and 6 lines.

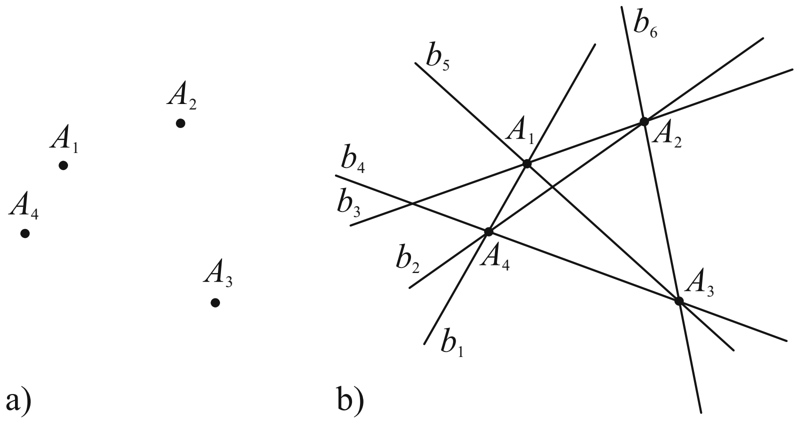


Fig. 1.67

Let us construct the configuration dual to the complete quadrangle. For this we draw four lines in general position and mark the intersection points. In Fig. 1.68 there are four lines a_1, a_2, a_3, a_4 and six intersection points B_1, B_2, B_3, B_4, B_5, B_6. This configuration is called a *complete quadrilateral*; it consists of 4 lines and 6 points.

Let us check the duality between these configurations. For example, in Fig. 1.68 three points B_1, B_2, B_4 lie on the same line a_4; correspondingly, in Fig. 1.67 three lines b_1, b_2, b_4 intersect at one point A_4.

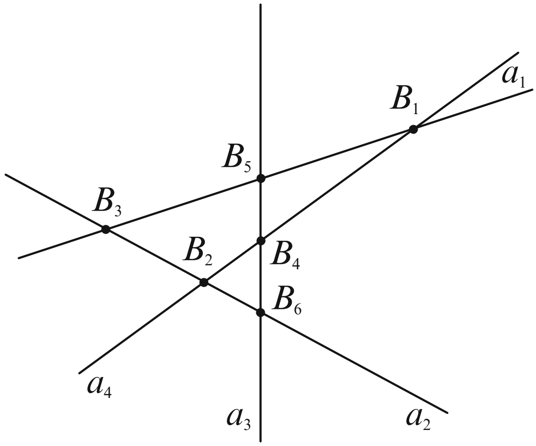


Fig. 1.68

Exercise 14. In Fig. 1.67 the lines b_4, b_5, and b_6 intersect at the point A_3. Write down the corresponding sentence for the dual configuration and find its illustration in Fig. 1.68.

Solution. The corresponding sentence for the dual configuration is: the points B_4, B_5, and B_6 lie on the line a_3. Indeed, this takes place in Fig. 1.68.

PROBLEM 27. In Fig. 1.67, points A_1 and A_4 lie on the line b_1. Write down the corresponding sentence for the dual configuration and find its illustration in Fig. 1.68.

PROBLEM 28. In Fig. 1.68 we drew four lines a_1, a_2, a_3, a_4 and then marked all the intersection points of these lines. But now we can draw more lines, that is, all the lines that pass through any two of the points $B_1, B_2, \ldots, B_6$.

(a) Draw them. How many have you obtained?

(b) Due to the duality between the two configurations there should be more points in Fig. 1.67. Mark them.

(c) Using lower case letters, name the additional lines in Fig. 1.68. Then name correspondingly the additional points in Fig. 1.67.

PROBLEM 29 (∗) Draw five lines in general position. How many points of intersection does this configuration have?

PROBLEM 30 (∗) Draw five points in general position. Through every pair of points draw a line. How many lines are there in this configuration?

8 Desargues configuration

In Section 7, we used only the following two operations:

1. mark the point of intersection between two lines;
2. draw the line through two points.

Let us draw two complete triangles ABC and $A'B'C'$. Thus, we have the points A, B, C, A', B', C' and the lines AB, BC, AC and $A'B'$, $B'C'$, $A'C'$. (See Fig. 1.69).

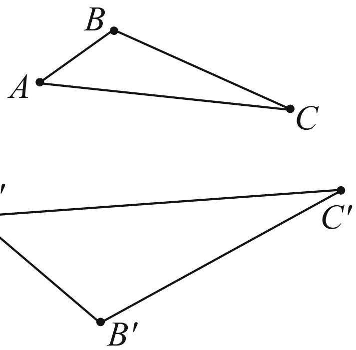


Fig. 1.69

What constructions can we make in this configuration using the two operations above? We can, for example, draw lines that pass through the corresponding vertices of the two triangles. Let us denote the line AA' also by e, the line BB' by f and the line CC' by g. In general these lines e, f, and g intersect and form a new triangle (see the shaded triangle in Fig. 1.70).

Definition 5. Two triangles ABC and $A'B'C'$ are called *Desargues triangles* if all the lines AA', BB', CC' intersect at one point. In this case the configuration is called a *Desargues configuration*.

An example of a Desargues configuration is in Fig. 1.71, where the point of intersection of the lines AA', BB', and CC' is the point O.

The Desargues configuration, studied by the French mathematician Desargues (1593–1662), is fundamental in projective geometry. However, projective geometry was developed only later. Nowadays it plays an important role in modern mathematics.

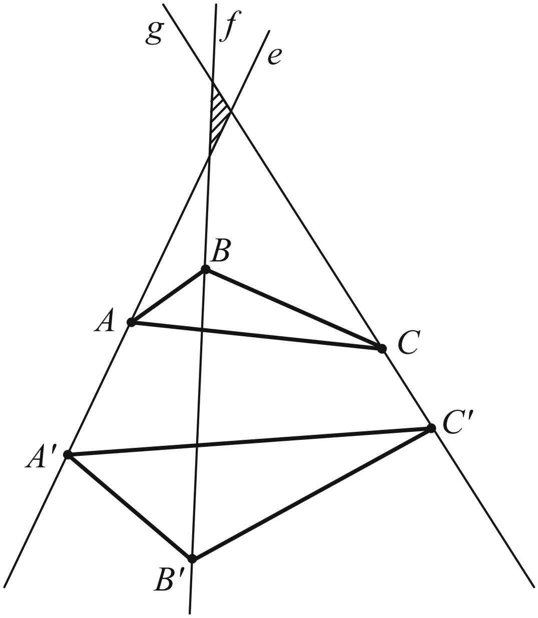


Fig. 1.70

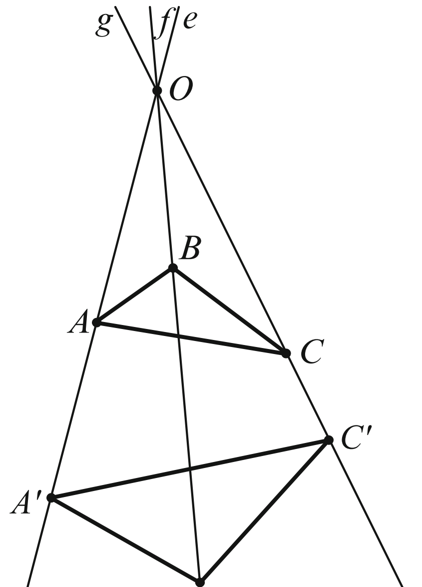


Fig. 1.71

Exercise 15. Fig. 1.72 shows a triangle ABC and two vertices A', B' of the second triangle. Complete triangle $A'B'C'$ in such a way that the lines AA', BB', and CC' will intersect at one point; that is, complete the figure to obtain a Desargues configuration. Where can the third vertex C' of the triangle $A'B'C'$ be?

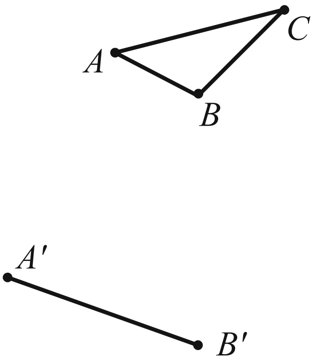


Fig. 1.72

Solution. As we know, in a Desargues configuration lines AA', BB', and CC' intersect at one point. Let us draw lines AA' and BB' (see Fig. 1.73a). Let O be point of the intersection.

The point C' must lie on the line OC (see Fig. 1.73b) and the points A', B', C' should form a triangle. There are many ways to choose point C'. Fig. 1.74 shows a few. Thus, the problem has many solutions.

PROBLEM 31. Take a new sheet of paper and draw a Desargues configuration from the beginning.

Hint. It might not be easy to guess how to draw two triangles in such a way that the lines AA', BB', and CC' will intersect at one point. Therefore, first draw three lines that intersect at one point, then mark points on them that will serve as vertices of the two triangles.

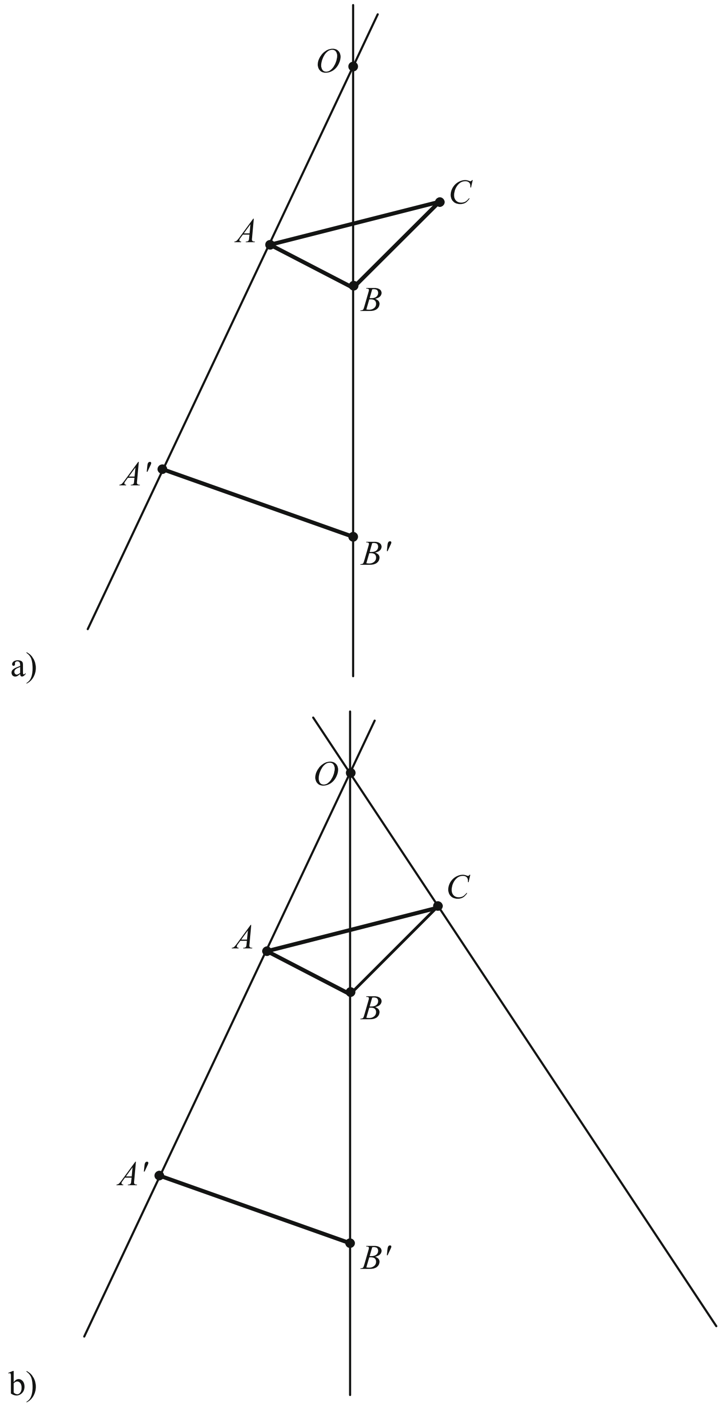


Fig. 1.73

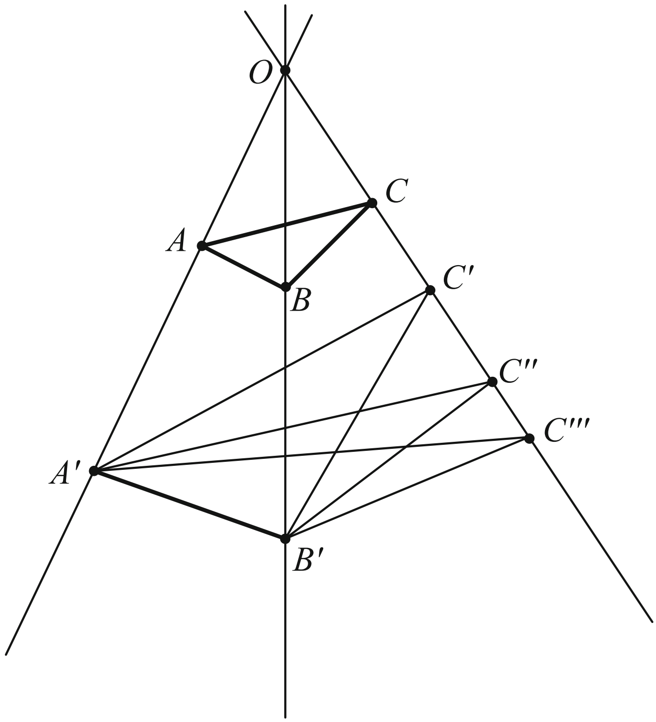


Fig. 1.74

9 Dual Desargues configuration

In the previous section we have considered two complete triangles ABC and $A'B'C'$, and we drew lines AA', BB', CC' through the corresponding vertices of these triangles. We can also make the dual construction. For this consider two triangles with sides abc and $a'b'c'$ (see Fig. 1.75a) and mark the points of intersection of the corresponding sides of these triangles. In Fig. 1.75b the point of intersection between lines a and a' is denoted by E, the point of intersection between b and b' by F, and the point of intersection between c and c' by G.

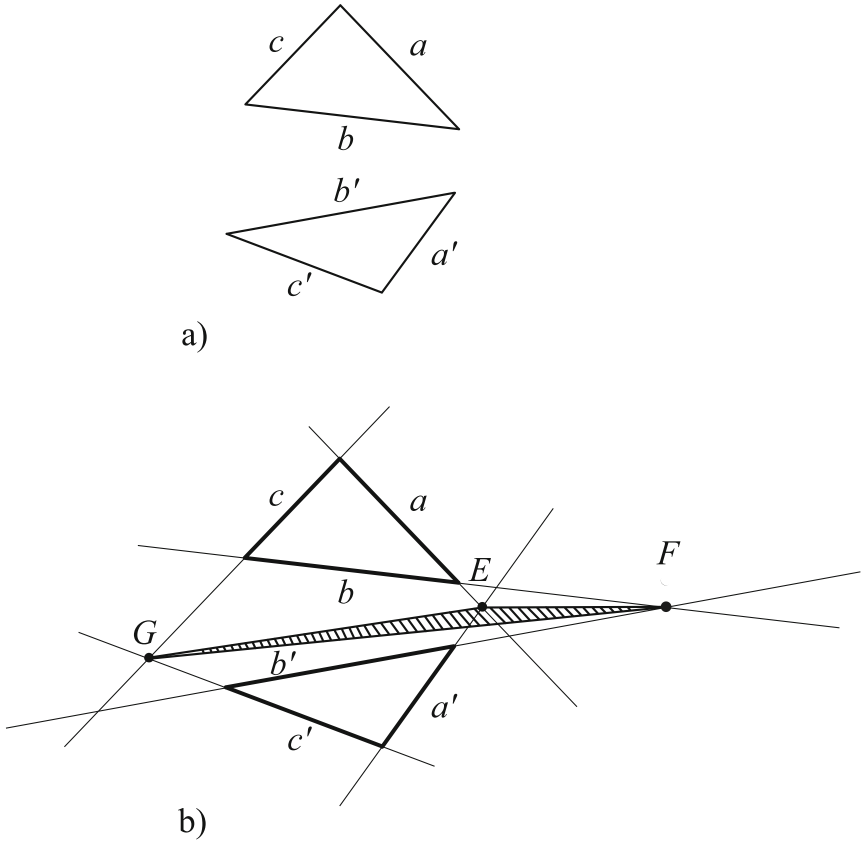


Fig. 1.75

Definition 6. Two triangles abc and $a'b'c'$ are called a *dual Desargues configuration* if the points of intersection of the lines a and a', the lines b and b', and the lines c and c' themselves lie on one straight line.

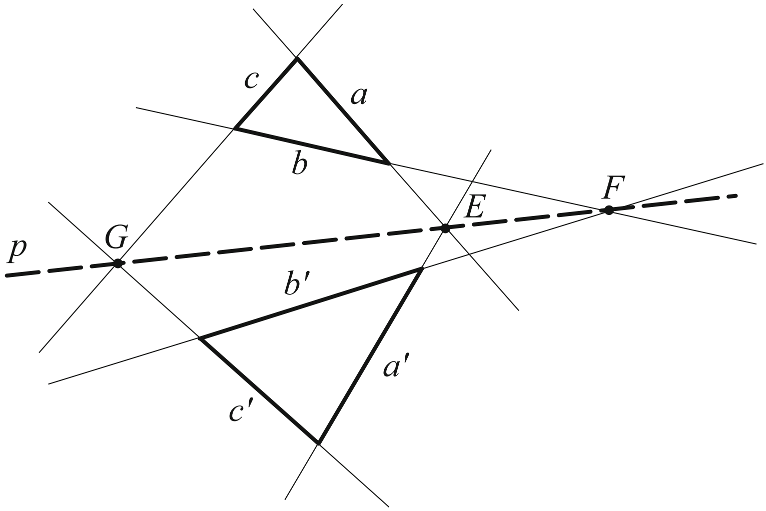


Fig. 1.76

An example of a dual Desargues configuration is shown in Fig. 1.76.

Exercise 16. Fig. 1.77 shows a triangle abc and two lines a' and c' on which the corresponding sides of the second triangle must lie. Draw a line b' in such a way that the lines a', b', c' form a triangle and the points of intersection of lines a and a', of lines b and b', and of lines c and c' lie on one line. In other words, complete the figure to obtain a dual Desargues configuration.

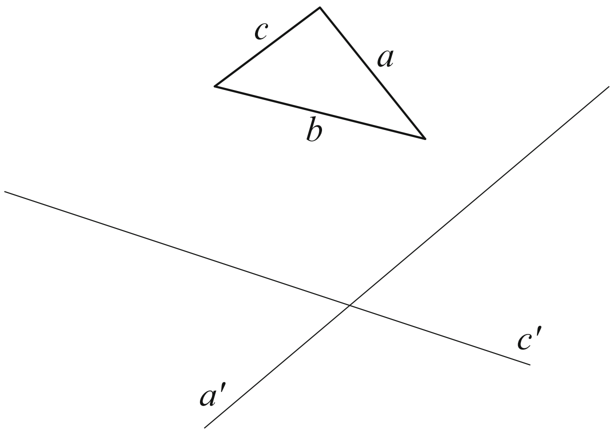


Fig. 1.77

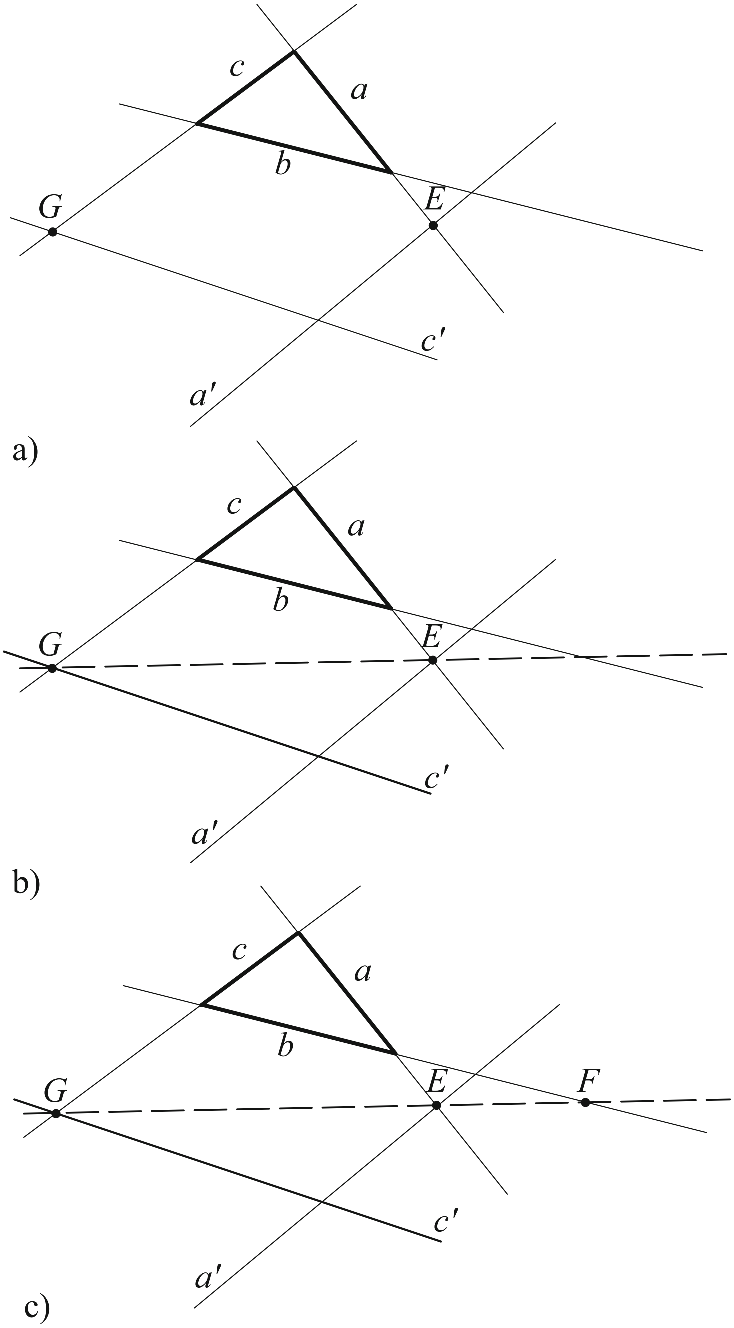


Fig. 1.78

Solution. Let us extend the lines a, b, c and mark the point E of intersection between a and a' and the point G of intersection between c and c' (see Fig. 1.78a). We can now draw the line EG (see Fig. 1.78b). The lines b and b' must intersect at point F, which must lie on line EG. Therefore, we can already mark the point F (see Fig. 1.78c).

Now we need to draw line b' through the point F so that lines a', b', and c' will form a triangle. There are many ways to do this; for example, as in Fig. 1.79a or in Fig. 1.79b. Thus, the problem has many solutions.

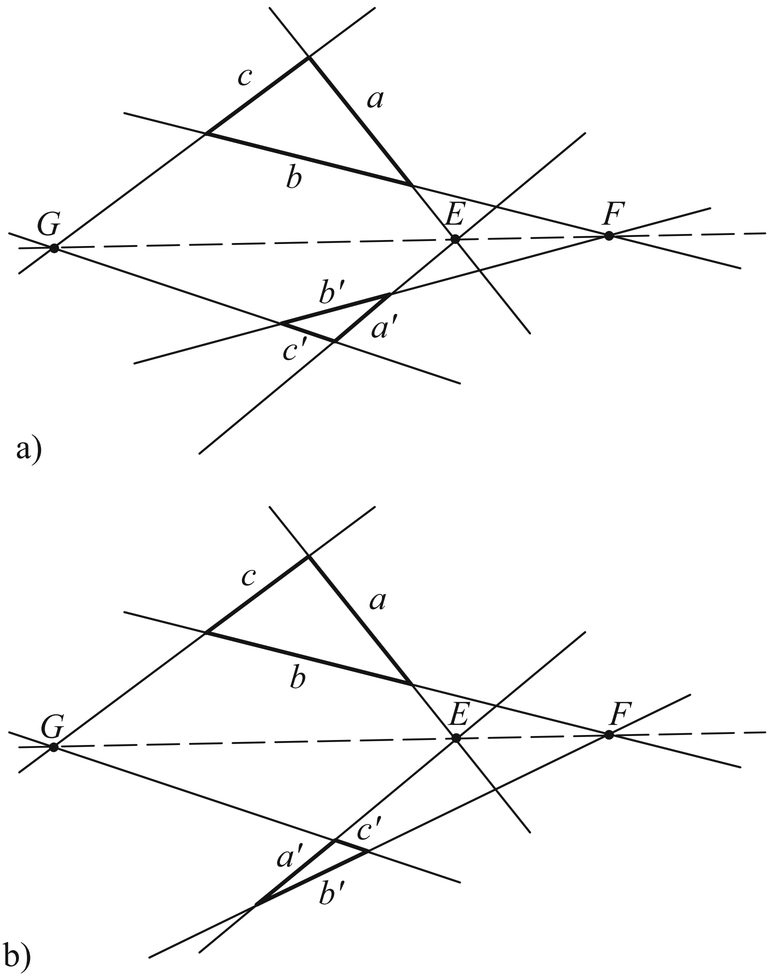


Fig. 1.79

PROBLEM 32. Take a new sheet of paper and draw a dual Desargues configuration from the beginning.

Hint. Start by drawing a triangle abc, then draw a line that intersects all three lines a, b and c. Then mark points E, F, G on this line and draw the lines that form the second triangle.

By now you should have two pictures: the Desargues configuration (see Fig. 1.71) and the dual Desargues configuration (see Fig. 1.76).

Let us look at the dual Desargues configuration (Fig. 1.76). Is it also a Desargues configuration? In order to check this let us denote vertices of the two triangles as A, B, C and A', B', C' correspondingly and draw the lines AA', BB' and CC'; that is, the lines that pass through the corresponding vertices of the two triangles.

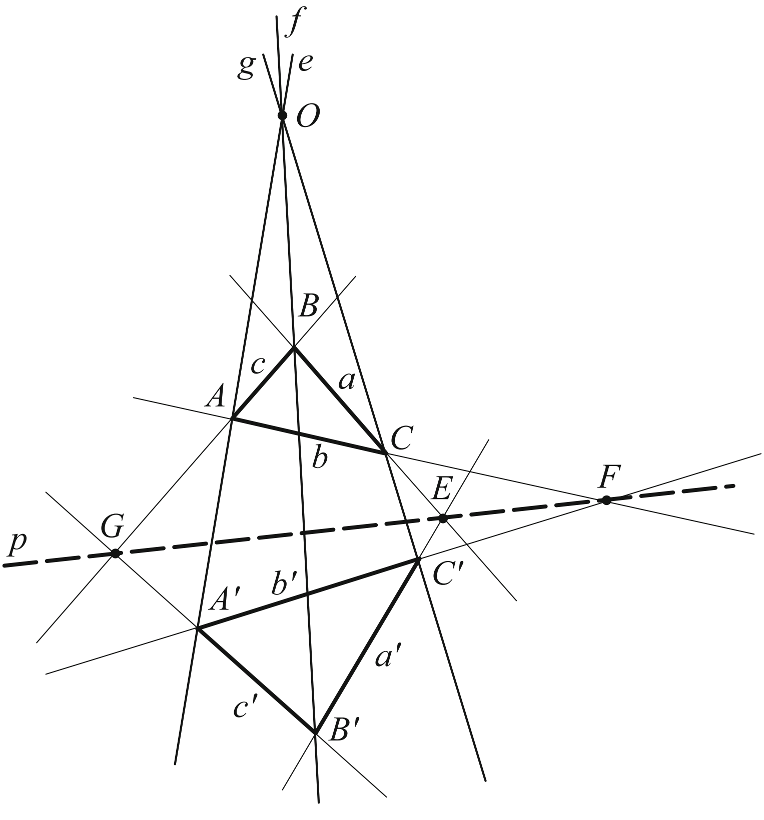


Fig. 1.80

As we can see (Fig. 1.80) these lines intersect at one point. This means that the configuration is a Desargues configuration. This is not by chance. The following statement holds.

Theorem 1 (Desargues' theorem) Given two triangles such that the lines passing through the corresponding vertices of these triangles intersect at one point, the points of intersection of the corresponding sides of these triangles lie on one line.

PROBLEM 33 ($*$)

(a) Draw a Desargues configuration and check whether it is also a dual Desargues configuration.

(b) Draw a different Desargues configuration and check again whether it is a dual Desargues configuration.

Theorem 2 (Converse of Desargues' theorem[14]) Given two triangles such that the points of intersection between their corresponding sides lie on one line, the three lines that pass through the corresponding vertices of these triangles intersect at one point.

Due to Theorems 1 and 2 any Desargues configuration is also a dual Desargues configuration and vice versa. Therefore, there is no need to distinguish between such configurations, and we can simply call it a Desargues configuration.

Remark 7. You have already checked Desargues' theorem on several pictures. However, it is not easy to prove this theorem. There are several different proofs. One proof uses length measure, which is not defined in Chapter I. There is a proof that we could present in this chapter. For this, however, we would have to consider a three-dimensional space and be able to find intersection points between lines and planes. The area of geometry that deals with space is called *solid geometry*, and is not the subject of this book. Therefore, we treat Desargues' theorem as an axiom.

Remark 8. In Section 2.1 of the Appendix to Chapter II, there is an interesting variation of the Desargues configuration for the case where the lines are parallel. In this chapter we have no tools to draw parallel lines.

[14] Two theorems are called *converse* to each other if what is given in the first theorem is a conclusion in the second one and vice versa.

10 Algebraic notation or "computer presentation" of configurations

We have considered configurations of points and lines. When drawing a configuration we used only the following two operations: 1) mark the point of intersection between two lines; 2) draw the line through two points.

Let us introduce some notations that will enable us to briefly record these operations.[15] We denote by $a \wedge b$ the point of intersection between the lines a and b. We denote by $A \vee B$ the straight line that passes through the points A and B.

Exercise 17. Using the notation above, write down the construction in Fig. 1.81a.

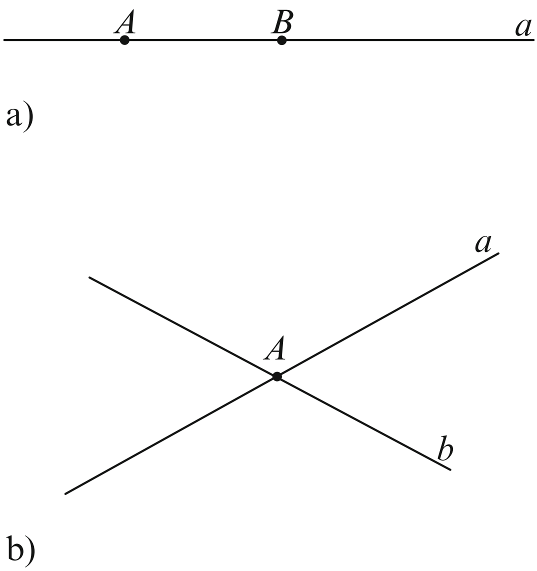


Fig. 1.81

Answer. $a = A \vee B$.

PROBLEM 34. Using the notation above, write down the construction in Fig. 1.81b.

PROBLEM 35. In Fig. 1.82 there are two points A, B and a straight line c. What is $x = (A \vee B) \wedge c$? Mark it on the figure.

[15]This algebraic way of presenting configurations is also needed to do geometric calculations on a computer.

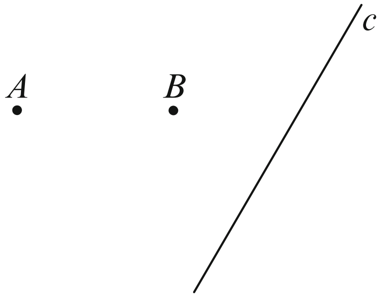


Fig. 1.82

PROBLEM 36. In Fig. 1.83a three points A, B, C are given. Draw $A \vee B$, $A \vee C$, $B \vee C$.

PROBLEM 37. In Fig. 1.83b there are four points A, B, C, D.

(a) Draw $A \vee B$, $A \vee C$, etc. Consider all possible pairs. How many did you obtain?

(b) Draw all possible points of the form $(A \vee B) \wedge (C \vee D)$. How many did you obtain?

(c) Compare your result with the complete quadrangle.

PROBLEM 38. In Fig. 1.83c there are four lines a, b, c, d.

(a) Draw all the points of the form $a \wedge b$.

(b) Draw all the lines of the form $(a \wedge b) \vee (c \wedge d)$.

(c) Compare your result with the complete quadrilateral.

PROBLEM 39 (∗) Using notations $\wedge$ and $\vee$, specify the vertices of the shaded triangle in Fig. 1.70.

PROBLEM 40 (∗) Using notations $\wedge$ and $\vee$, specify the sequence of constructions used in

(a) Problem 15,

(b) Problem 31,

(c) Problem 16,

(d) Problem 32.

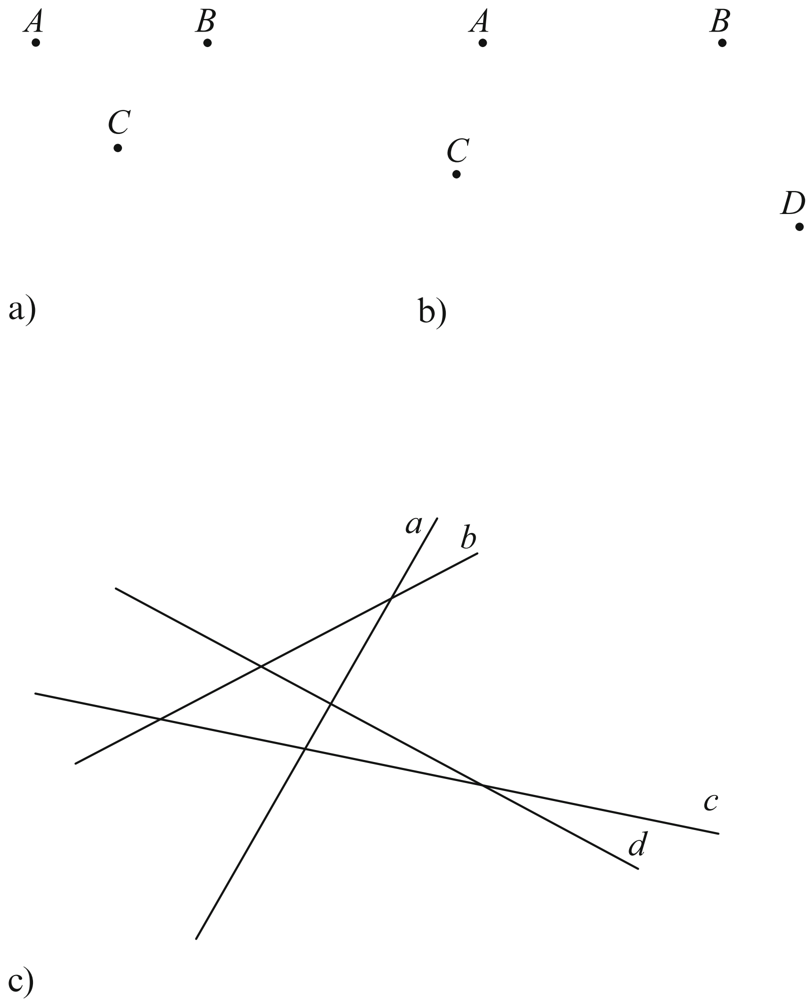


Fig. 1.83

11 Polygons and n straight lines

In Section 3.2 we mentioned that several lines divide the plane into polygonal domains. Note that the boundary of a polygonal domain consists of a finite number of rays and/or segments. If a polygonal domain is unbounded then its boundary will have exactly two rays and maybe also segments. If a polygonal domain is bounded then its boundary consists only of a finite number of segments.

A path consisting of a finite number of segments consecutively connected to each other is called a *broken line.* If the first and last points of a broken line coincide it is called a *closed broken line.* A broken line is said to be *non-self-intersecting* if it does not have intersection points except at the endpoints of its segments. A non-self-intersecting broken line is also called a *polygonal line.* Naturally, a closed non-self-intersecting broken line is called a *closed polygonal line.*

A *polygon* is a domain bounded by a closed polygonal line. Thus, a polygon is a bounded domain. A polygon has sides (segments of its bounding polygonal line) and vertices (ends of these segments).

Any polygon has a name depending on the number of its sides.[16] As we already know, a polygon with three sides is a triangle and a polygon with four sides is a quadrilateral. A polygon with five sides is called a *pentagon*, with six – a *hexagon*, with seven – a *heptagon*, with eight – an *octagon*. A polygon with n sides is called an *n-gon*.

PROBLEM 41.

(a) Draw an arbitrary pentagon. Mark its vertices. How many interior angles[17] does it have?

(b) Mark all the exterior angles of this pentagon. How many did you mark?

(c) Draw a heptagon and mark all its interior angles.

---

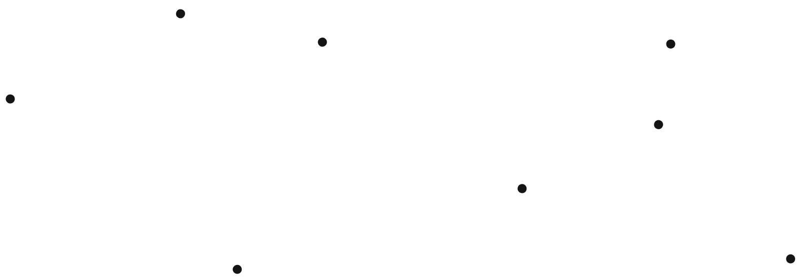
[16]The number of sides of a polygon coincides with the number of vertices of this polygon. Thus we can distinguish polygons by counting vertices as well.

[17]In Section 3.2 we have defined interior and exterior angles in a triangle. In a polygon they are defined similarly. See Glossary.

Exercise 18. Draw all possible polygons that you can construct using all the following points as its vertices:

(a) Fig. 1.84a;

(b) Fig. 1.84b.



a) b)

Fig. 1.84

Answer.

(a) We can draw only one quadrilateral (see Fig. 1.85).

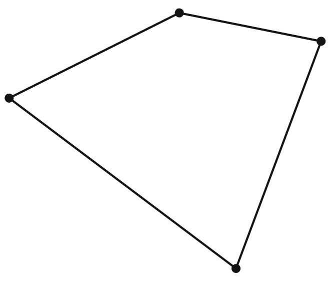
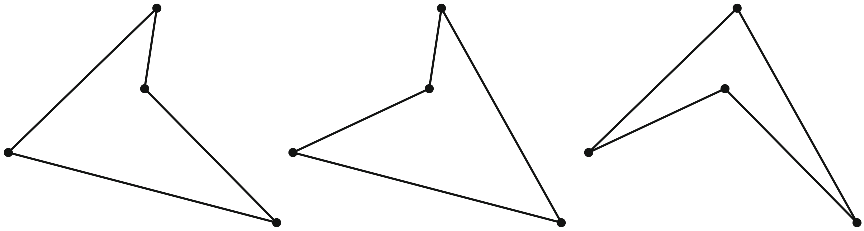


Fig. 1.85

(b) It is possible to draw three different quadrilaterals (see Fig. 1.86a, b, c).



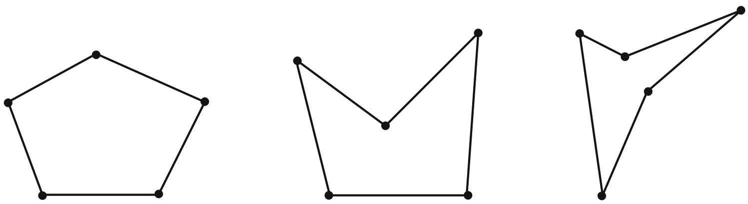
a) b) c)

Fig. 1.86

The quadrilateral in Fig. 1.85 and those in Fig. 1.86a, b, c belong to different types: the first one is convex (both of its diagonals lie inside), while the other three are not (one diagonal lies outside). We have already met such quadrilaterals in Section 4.

Exercise 19. Sketch pentagons with different numbers of diagonals lying outside. Consider all possibilities.

Answer. Three types of pentagons, belonging to different types according to the number of diagonals lying outside, are shown in Fig. 1.87. For a pentagon with yet more of its diagonals lying outside, see Fig. 1.49d.



a) b) c)

Fig. 1.87

PROBLEM 42. Draw all the diagonals in each of the pentagons in Fig. 1.87. How many diagonals lie outside the polygon in each case?

PROBLEM 43. As in Exercise 19, sketch hexagons with different numbers of diagonals lying outside. In each hexagon mark and count all such diagonals.

12 Convex polygons, convex hull of n points

Imagine that you have to choose between two lots for your property (see Fig. 1.88a, b). Which one is better? Lot b) might turn out to not be very convenient. For example, if you need to tie a rope between two poles as in Fig. 1.88b, some part of this rope will be hanging outside your lot and can bother your neighbour. This is why a lot is usually made convex (see Definition 1 in Section 2.1). In a convex lot, a rope connecting any two poles on this lot lies completely inside the lot.

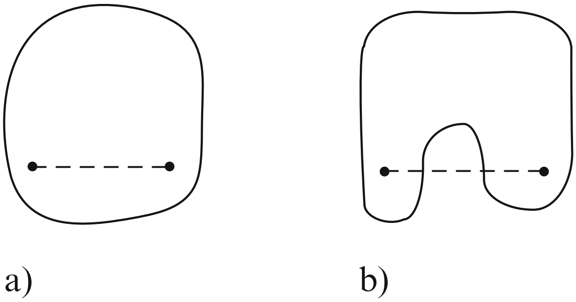


Fig. 1.88

As you see, the mathematical definition differs from the description above of a convex lot by replacing the word “rope” by the word “segment” and “pole” by “point.”

Exercise 20. Let us imagine four poles on the ground (see Fig. 1.89a,b). Draw the smallest convex lot that contains all the points.

Fig. 1.89

Answer. See Fig. 1.90.

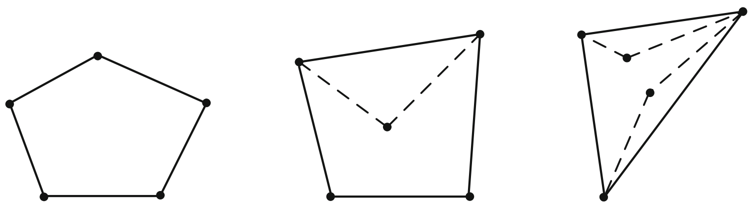
a) b)

Fig. 1.90

Given any finite number of points in the plane, the smallest convex domain that contains all these points is called the *convex hull* of these points. Note that some of the given points may not be vertices of the convex hull. The convex hull of a polygon is the convex hull of its vertices.

Exercise 21. We have considered different types of pentagons (see Fig. 1.87). Draw their convex hulls.

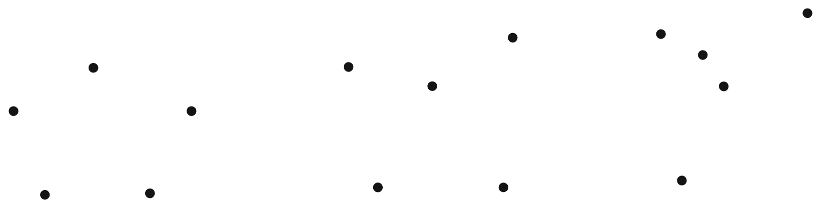
Answer. See Fig. 1.91.



a) b) c)

Fig. 1.91

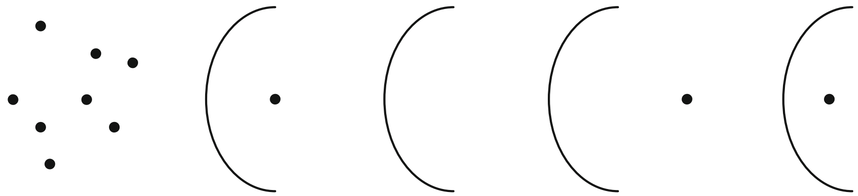
Notice that the convex hull of five points in each case is a different polygon; it might be a pentagon (Fig. 1.91a), a quadrilateral (Fig. 1.91b) or a triangle (Fig. 1.91c). These polygons also have different numbers of points lying inside.



a) b) c)

Fig. 1.92

PROBLEM 44. In Fig. 1.92a, b, c there are different configurations of five points in general position (i.e., no three points lie on the same straight line). Connect every two points by a segment. How many chambers (or bounded domains—see also footnote 11) are formed in each configuration?



a) b) c) d) e)

Fig. 1.93

PROBLEM 45. Draw the convex hull of the points in Fig. 1.93a–e.

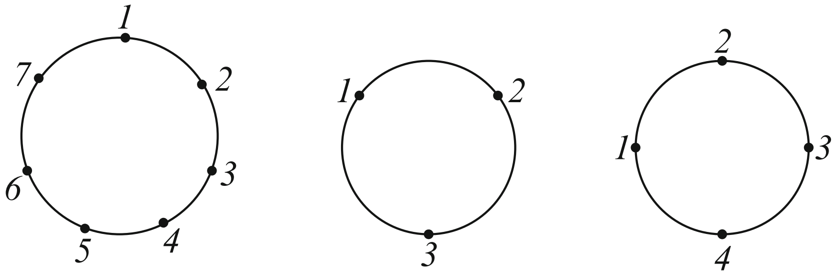
PROBLEM 46.

(a) How many diagonals does a convex hexagon have?

(b) Into how many chambers is a convex hexagon divided by its diagonals in the case where these diagonals are in general position?

PROBLEM 47 (∗) Make a table for the number of domains in the plane formed by $4, 5, \ldots$ straight lines that are in general position, and find a formula for the number of domains formed by n straight lines in general position.

PROBLEM 48. Let the points $1, 2, \ldots, n$ be given on a circle (see Fig. 1.94a for $n = 7$). How many pairs of points can we find if we count pairs such as $(1,2)$ and $(2,1)$ only once?



a) b) c)

Fig. 1.94

For example, for $n = 3$ (see Fig. 1.94b) we have the following pairs: $(1,2)$, $(2,3)$, $(3,1)$ and the answer is 3.

PROBLEM 49. Let the points $1, 2, \ldots, n$ be given on a circle. How many pairs of numbers are possible if pairs of "neighbors" are forbidden?

For example, for $n = 3$ the pairs $(1,2)$, $(2,3)$, $(3,1)$ are forbidden and the answer is 0. For $n = 4$ (see Fig. 1.94c) all the possible pairs are: $(1,3)$, $(2,4)$ and the answer is 2.

PROBLEM 50. How many diagonals are there in a convex n-gon?

PROBLEM 51. How many sides and diagonals together are there in a convex n-gon?

13 Solution of Exercise 3 with the help of a Desargues configuration

With the help of a Desargues configuration, Exercise 3 can be solved precisely. We repeat the statement of this exercise.

Exercise 22. Draw a line through the point A and the intersection point B of the two lines a and b, where the point B does not lie on the sheet of paper. An example is given in Fig. 1.95.

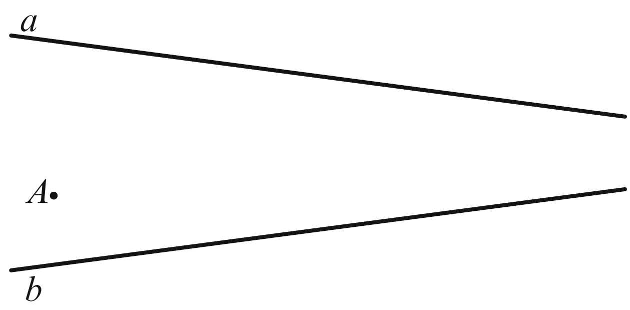


Fig. 1.95

Solution. We would like to think of the point B as being the point E and the point A as being the point G in the Desargues configuration presented in Fig. 1.80.

Let us choose a point O above the lines a, b, and draw a ray e from it intersecting the lines a and b (see Fig. 1.96a). Let us denote by P_1 the intersection point of e and line a and by V_1 the intersection point of e and line b. Points P_1 and V_1 will be vertices of the two triangles of the Desargues configuration we want to construct.

From point A we draw ray t_1 passing through point P_1 and ray t_2 passing through point V_1.

From point O we draw another ray f intersecting lines a and b as in Fig. 1.96b. We denote its points of intersection with rays t_1 and t_2 as P_2 and V_2 correspondingly. Points P_2 and V_2 are the second pair of vertices of the Desargues triangles we are constructing.

From point O we draw a third ray g intersecting lines a and b. We mark its points of intersection with lines a and b and denote them by P_3 and V_3 correspondingly (see Fig. 1.96b).

Finally, we draw ray t_3 passing through points P_2 and P_3 and ray t_4 passing through points V_2 and V_3 (see Fig. 1.97). We mark the intersection point of rays t_3 and t_4 and denote it by C. The line passing through points A and C is the line also passing through the intersection point B of the original lines a and b. Therefore, this is the line we had to construct (see Fig. 1.97).

PROBLEM 52. Using the notations $a \wedge b$ and $A \vee B$ from Section 10, write down the sequence of constructions in the above solution.

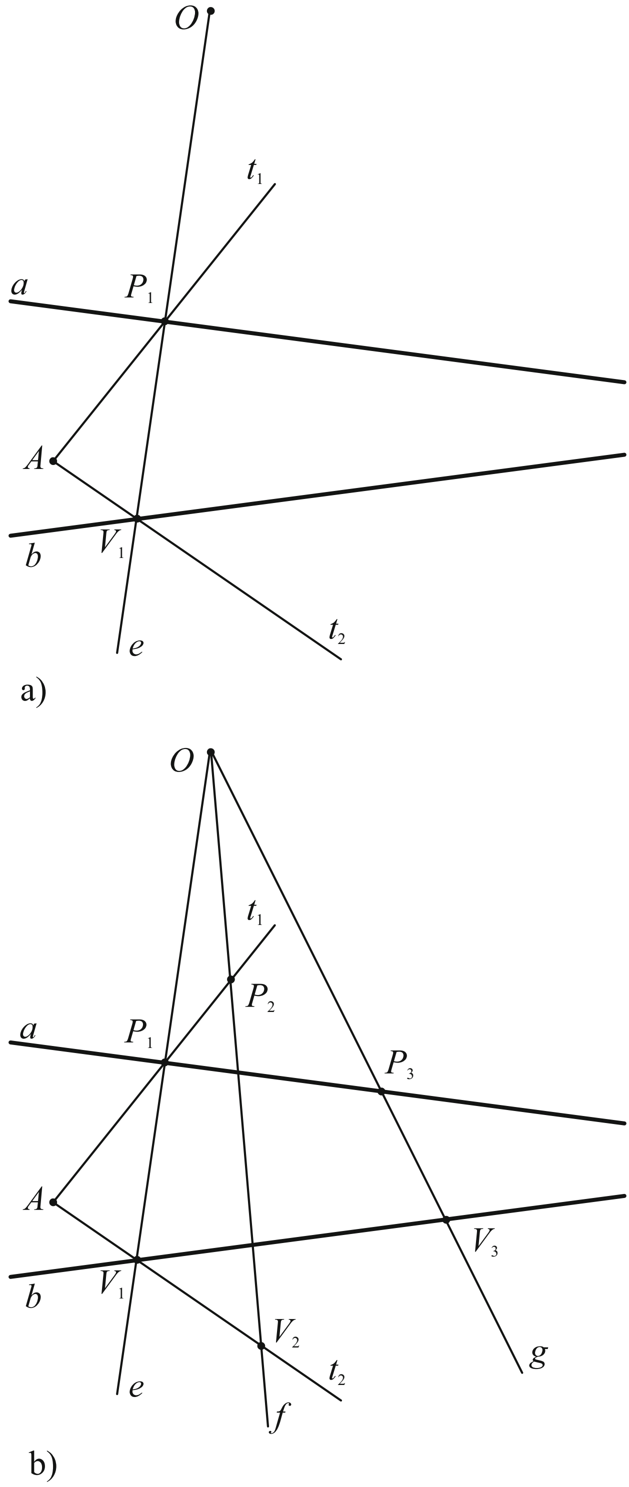


Fig. 1.96

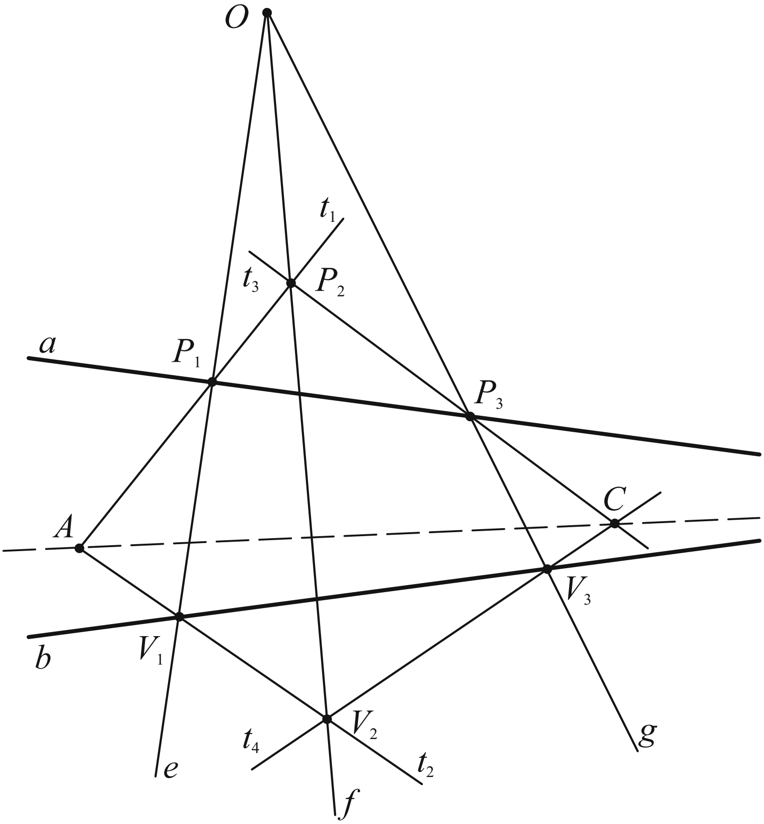


Fig. 1.97

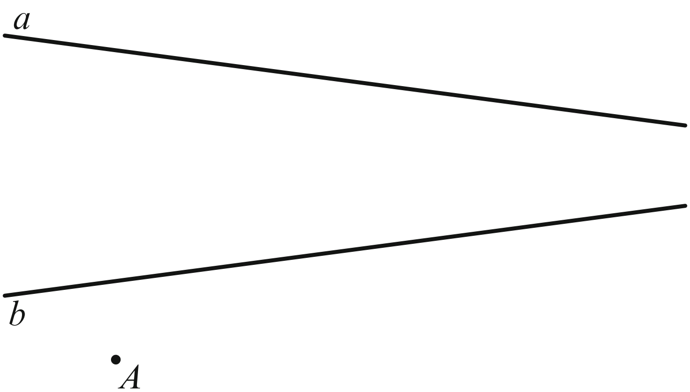


Fig. 1.98

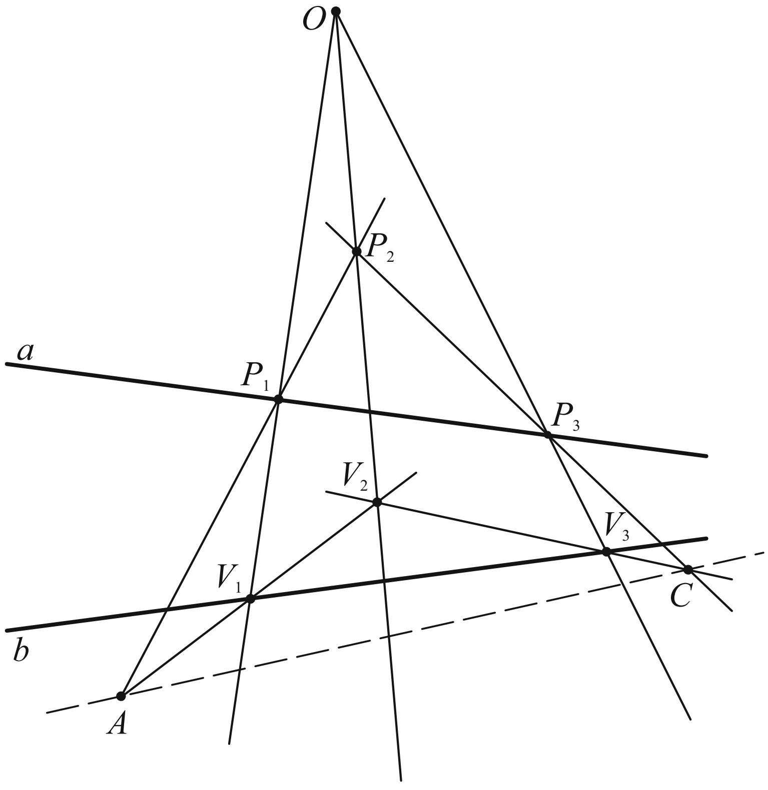


Fig. 1.99

PROBLEM 53. With the help of a Desargues configuration, draw a line through point A and the point of intersection B of the two lines a and b presented in Fig. 1.98.

If this problem is too challenging for you, use the answer presented in Fig. 1.99. Write down the sequence of constructions one has to make in order to obtain Fig. 1.99 from Fig. 1.98.

As we have mentioned in Section 1.4, Exercise 3 can be solved even for the lines presented in Fig. 1.16. In order to do this, one has to use a particular case of a Desargues configuration where the lines a and b are parallel (see Section 2.1 of the Appendix to Chapter II). We deal with parallel lines in Chapter II.

14 Overview of Chapter I

In Chapter I, we started with two simple geometric objects: points and lines on the plane. We used only two tools: a pencil and a straightedge. We introduced two actions that we called "operations available in Chapter I." By applying them to points and lines, we were able to define several objects and figures such as segments, rays, half-planes, angles, convex and nonconvex angles and figures, triangles, quadrilaterals, and polygons with n vertices.

In order to characterize a relation between several objects, we introduced the notion of a configuration of points and lines. Within the same configuration, we allowed the objects to be "moved" on the plane while obeying certain rules. We defined a dual configuration. The famous Desargues configuration was considered and its dual configuration was constructed.

We also introduced a different procedure, which we called a correspondence. A particular correspondence, called central perspective, established a relation between all the points on the plane and all the points on a line. This correspondence allowed us also to establish a relation between the points on two different lines. Central perspective is not a one-to-one correspondence.

Note that a configuration of points and lines and its dual configuration define a notion of duality in projective geometry. Duality establishes a one-to-one correspondence between points and lines.

Chapter II

Parallel Lines: A Look at Affine Geometry

PART I. Lines and segments

1 Parallel straight lines

Definition 1. Two lines in the plane are called *parallel* if they do not have a common point no matter how far they are extended.[1]

It is assumed that a line is parallel to itself.[2]

Consider a line a and a point A that does not lie on a. Through point A, draw a line b and start turning it clockwise around A (see Fig. 2.1a).

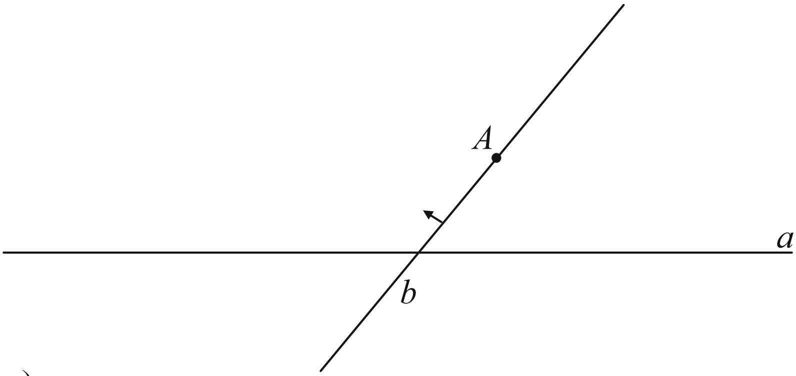
Let us see what will happen with the point of intersection of the lines a and b. As line b turns, this intersection point moves along line a further and further to the left. Then this point slips from the sheet of paper, but lines a and b still intersect at a point far to the left. As we continue turning line b, there will be a certain position (the darker line in Fig. 2.1b) of line b when it no longer intersects line a; i.e., lines a and b become parallel. Now if we turn line b just a little bit more, lines a and b will intersect at a point far to the right.[3] If we continue turning line b, the intersection point will move along

[1]This definition is good only for lines lying in the plane. Straight lines in space can be non-intersecting without being parallel (so-called *skew lines*).

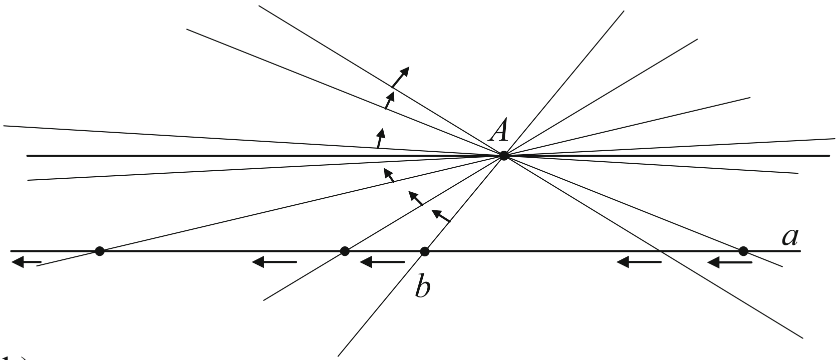
[2]See more about this in Section 3.3.

[3]It looks as if at a certain moment the intersection point between lines a and b instantaneously jumps from very far left on line a to very far right on a. Thus we can say that in some sense points far left on a are "close" to the points far right on a. This approach is used in projective geometry.

I. M. Gelfand, T. Alekseyevskaya (Gelfand), *Geometry*,
https://doi.org/10.1007/978-1-0716-0299-7_2



a)



b)

Fig. 2.1

line a to the left and will come back to the initial position.

Let a be a line and A be a point that is not on a. Among all the straight lines that pass through the point A, there exists only one line parallel to the line a.

This statement is called *Euclid's fifth postulate* (The word postulate is a synonym for "axiom").

In geometry (or any other branch of mathematics) certain statements, called *axioms*, are made because from our experience we believe that they are correct.[4] Based on axioms we can then prove statements, called propositions,

[4]It is not easy to check whether Euclid's postulate is true. Indeed, what if the lines will intersect five miles from here, or even in another galaxy?

lemmas and theorems. After some statement is proved we can use it to prove further, more complicated statements.

Remark 1. It is possible to construct other geometries in which Euclid's fifth postulate is not true. The first such geometry was constructed by the mathematicians Lobachevsky and Bolyai. Lobachevsky called his geometry the "imaginary geometry." About fifty years later the mathematicians Beltrami, Klein, and Poincare constructed a realization of Lobachevsky's geometry. This geometry describes our world on a cosmic scale. It has even been possible to check this experimentally. The properties of this geometry are crucial in Einstein's theory of relativity. However, in our everyday life Euclidean geometry is good enough. Thus in geometry Euclid's fifth postulate is usually accepted, and we shall accept it too.

2 Operations available in Chapter II

Recall that in Chapter I, using a pencil and a straightedge, we were allowed to perform the following operations:

(1) Draw a straight line that passes through any two given points.

(2) Mark the point of intersection of two straight lines if such a point exists.[5]

In Chapter II we add the following operation:[6]

(3) Draw a straight line that passes through a given point A and is parallel to a given line a.

Remark 2. According to Euclid's fifth postulate, such a line exists and, moreover, there exists exactly one such line.

The geometric figures and constructions that can be obtained using only these three operations are called constructions in *affine geometry*.

[5] We repeat here (see also footnote 5 of Chapter I) that if the lines are not parallel but do not intersect on a sheet of paper we assume that we can extend them and still find their intersection point.

[6] A simple instrument to draw parallel lines in everyday life is a rolling ruler. More precise devices called "drafting machines" are used by engineers and designers.

3 Properties of parallel lines

3.1 Transitivity of parallel lines

Let a straight line a be parallel to a straight line b and let line b be parallel to the straight line c (see Fig. 2.2). Will line a be parallel to c? The answer is "yes."

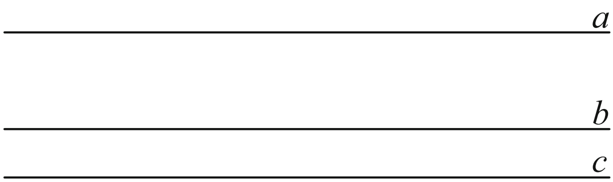


Fig. 2.2

This property is called *transitivity*. If lines a and b are parallel, this is sometimes denoted as $a \parallel b$. In this notation, transitivity can be written as follows: if $a \parallel b$ and $b \parallel c$, then $a \parallel c$.

We can prove that parallel lines satisfy the transitivity property. For this we need to show that lines a and c are parallel, i.e., they do not intersect no matter how far they are extended.[7] There are only two possibilities: these lines either intersect or they do not. Thus, we need to exclude the possibility that they might intersect somewhere. Let us denote the imaginary intersection point between the lines a and c by X (see Fig. 2.3).

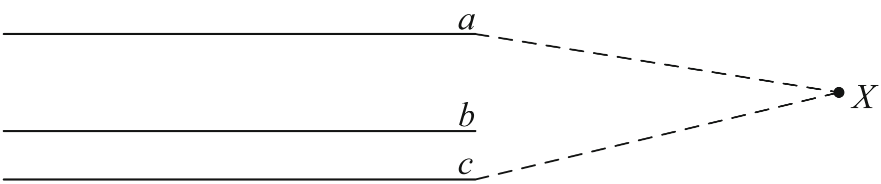


Fig. 2.3

[7]Note that, as we have mentioned before, this is not easy to verify experimentally. However, it is possible to *prove* that it is true.

But then there are two straight lines a and c that pass through the point X and are parallel to the line b. This contradicts Euclid's fifth postulate. Therefore, the first possibility cannot take place. Since there are only two possibilities, then the second one must hold; i.e., the lines a and c are parallel.[8] □

(We place the □ sign here in order to indicate that the proof is finished.)

3.2 Symmetry of parallel lines

There is another property of parallel lines called *symmetry*. If a straight line a is parallel to a straight line b, then the line b is parallel to the line a. This can be shortly written as follows: if $a \parallel b$, then $b \parallel a$.

Symmetry of parallel lines can be also proved: indeed, if the lines a and b do not have common points, then lines b and a also do not have common points. □

3.3 Reflexivity of parallel lines

According to our assumption (see Section 1), parallel lines also satisfy the property called *reflexivity*. This can be written as follows: $a \parallel a$.

Note that according to our definition, two lines are parallel if they do not have a common point no matter how far they are extended. When there is only one line, this definition is not really applicable. However, if there are two lines which are "very close" to each other and do not have a common point, they are parallel. We may imagine these parallel lines becoming "closer and closer" to each other and still remaining two lines without a common point, though we can no longer distinguish them as two lines. We recall that the "ideal line" has no thickness. Mathematicians like to have uniform rules. Therefore, they make an agreement, or an assumption, that a line is parallel to itself.

[8] The reasoning used in this paragraph is widely used by mathematicians and is called *proof by contradiction*.

4 Segments lying on parallel lines

4.1 Equality of segments lying on parallel lines

Suppose we have a segment AB on a straight line a and a point C that does not lie on a (see Fig. 2.4a). The following question arises: using only the three operations of affine geometry (from Section 2), is it possible to define what it means for a segment CD to be equal to segment AB?

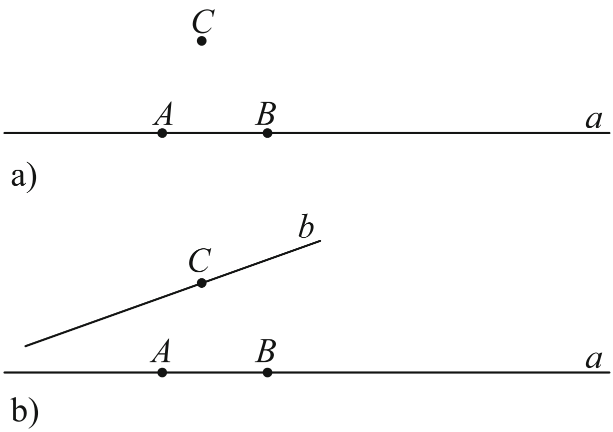


Fig. 2.4

We remind the reader that we have neither a ruler with length marks nor a compass.

Generally speaking, the answer is "no"[9] if the segment CD lies on an arbitrary straight line b that passes through point C (see Fig. 2.4b).

However, if CD is on a line b parallel to line a, the answer is "yes." In this case we can define equal segments on parallel lines.

For simplicity of notation, we will sometimes denote a segment by $\bar{a}$, $\bar{b}$, etc.

Definition 2. We say that two segments $\bar{a}$ and $\bar{b}$ lying on parallel straight lines are *equal* if it is possible to draw straight lines through the corresponding ends of the segments $\bar{a}$ and $\bar{b}$ in such a way that these new lines are parallel.

[9] This will be discussed in detail in Section 1 of the Appendix to Chapter II.

Remark 3. Given segments $\bar{a}$ and $\bar{b}$, note that there are two ways to draw a line through the corresponding ends of these segments (see Fig. 2.5 and Fig. 2.6.) That is why we have written "if it is possible" in Definition 2.

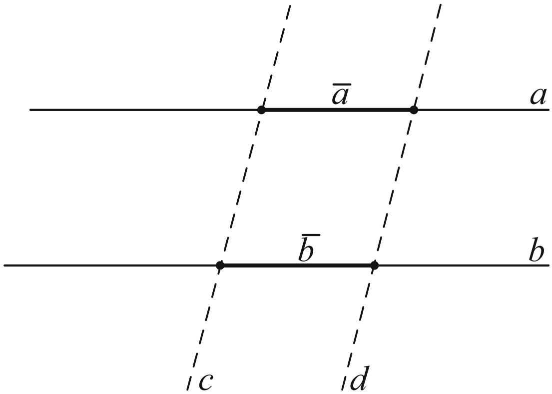


Fig. 2.5

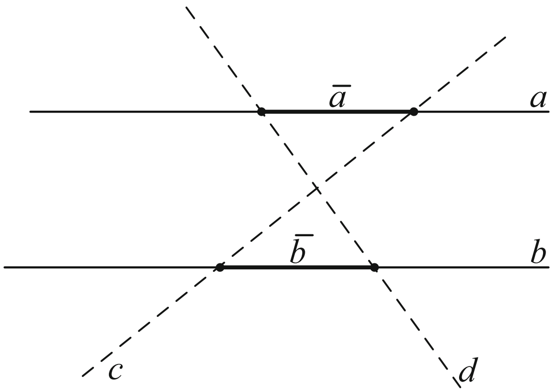


Fig. 2.6

Proposition 1. If two pairs of parallel lines intersect each other, then the opposite segments between these lines are equal to each other.

Proof. Indeed, consider two intersecting pairs of parallel lines (see Fig. 2.7a).

If we apply Definition 2 to segments $\bar{a}$ and $\bar{b}$ (see Fig. 2.7b), we can conclude that $\bar{a} = \bar{b}$. If we apply it to the segments $\bar{c}$ and $\bar{d}$ (see Fig. 2.7c), we obtain that $\bar{c} = \bar{d}$. □

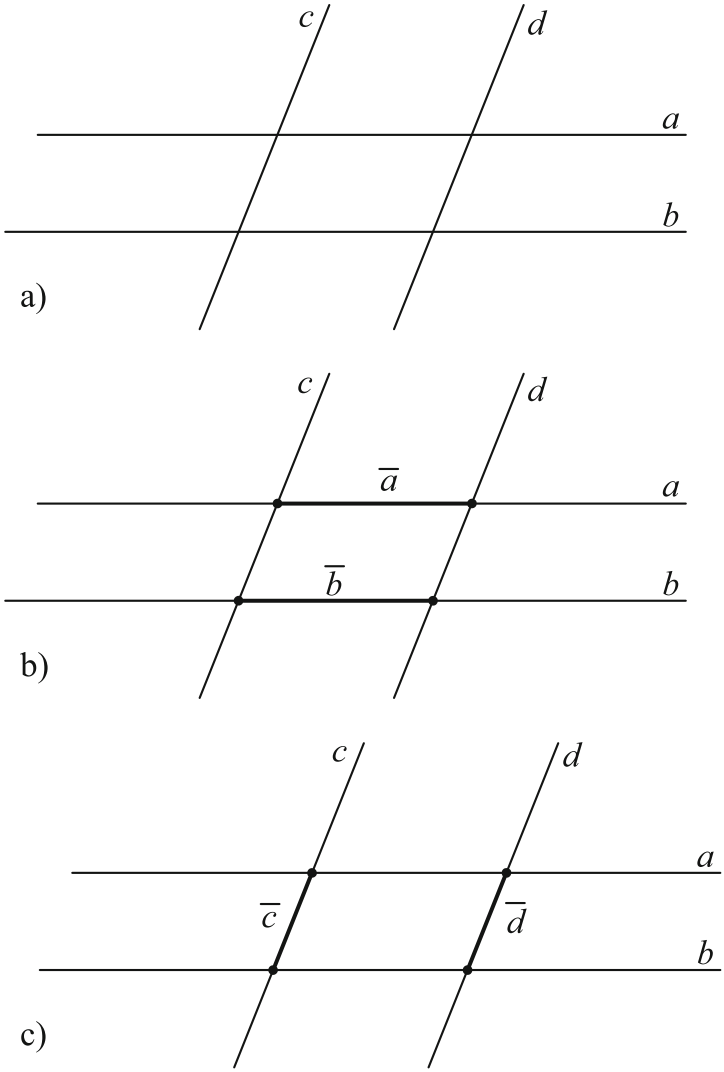


Fig. 2.7

The four segments formed by the intersection of two pairs of parallel lines form an interesting quadrilateral. In this quadrilateral, which is called a *parallelogram*, the opposite sides are parallel and equal. Parallelograms and their properties will be considered in detail in Section 5.

4.2 Construction of equal segments on parallel lines

We have defined equal segments on parallel lines in Definition 2. In this Section we will show how, by using only the operations (1), (2), (3) defined in Section 2, we can construct equal segments on parallel lines or on the same line.

Exercise 1. In Fig. 2.8a. there are two parallel lines a and b, a segment AB on line a and a point C on line b. Construct a segment CD on the line b such that $CD = AB$.

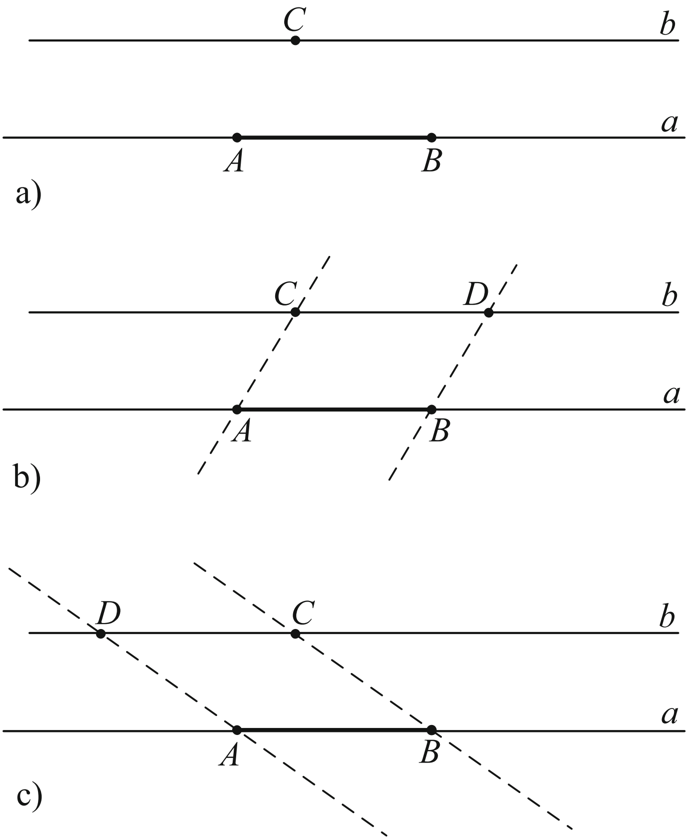


Fig. 2.8

Solution 1. Let us draw the straight line AC, then draw the straight line parallel to the line AC and passing through point B (see Fig. 2.8b.) On line b we have obtained segment CD equal to segment AB.

Solution 2. On line b we can construct another segment CD equal to the segment AB (see Fig. 2.8c.).

We have constructed equal segments on parallel lines. How can we construct equal segments on the same line?

Let both a segment AB and a point C lie on a straight line a (see Fig. 2.9a). Let us do the following.

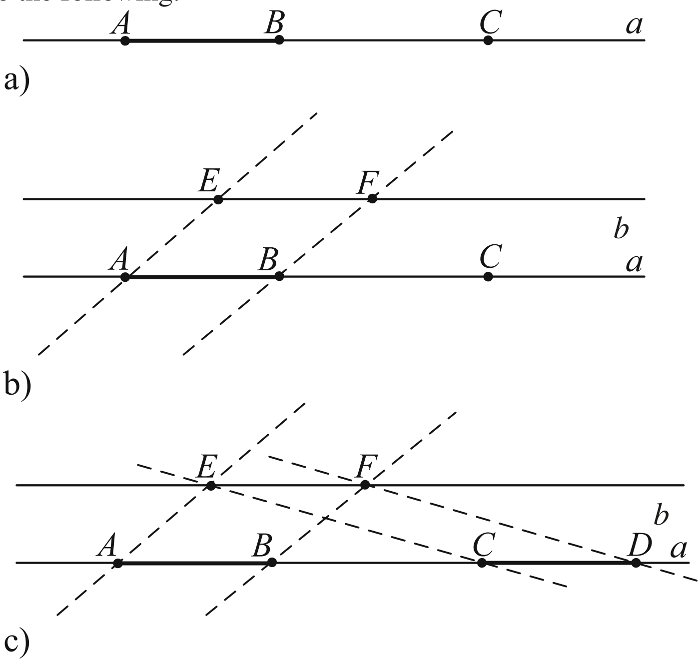


Fig. 2.9

We cannot apply the same construction as before, because there is no second line b parallel to a. Then let us draw any straight line parallel to line a and denote it as line b. Now mark an arbitrary point E on it. Then we can construct, as we have done above, segment EF equal to segment AB and lying on line b (see Fig. 2.9b). Thus, we have "translated" the segment AB to the segment EF on a line parallel to a. Now we can "translate" it back to the line a. For this we draw the straight line through the points E and C. Then draw the line parallel to the line EC and passing through point F (see Fig. 2.9c).

We have constructed segment CD on line a. By Definition 2, this segment CD is equal to EF and EF is equal to AB. Since the notion of equality is transitive,[10] we see that segment CD is equal to segment AB. We have constructed equal segments lying on the same line.

Remark 4. Note that when constructing segment CD in Fig. 2.9c we had choices: a choice of the point E on the line b, as well as a choice of the line b itself. However, it is possible to verify that point D (when lying to the right of the point C) and, therefore, the segment CD will always be in the same place.

PROBLEM 1. Draw several different pictures to illustrate this remark, choosing the line b and the point E on it differently. Check that the remark is true.

Definition 3. We say that two segments $\bar{a}$ and $\bar{b}$ lying on the same straight line l are *equal* if there is a segment $\bar{c}$ on a line parallel to l such that $\bar{a} = \bar{c}$ and $\bar{b} = \bar{c}$.

PROBLEM 2 (∗) Consider Fig. 2.9a again and construct another segment equal to segment AB from the point C.

Hint. This time point D should lie to the left of point C.

We have constructed equal segments on parallel lines. We can also compare segments lying on parallel lines (or on the same line). Do this yourself; i.e., solve the problem below.

PROBLEM 3.

(a) Let a segment $\bar{a}$ be given on a line a and a segment $\bar{b}$ be given on a line b parallel to a. How can you determine whether they are equal or not?

(b) Let two segments $\bar{a}$ and $\bar{b}$ be given on a line a. How can you determine whether they are equal or not?

Hint. See Fig. 2.10.

Properties of equal segments lying on parallel lines

The notion of equality satisfies the following properties, which are called *symmetry* and *transitivity*.[11]

[10] See "Properties of equal segments lying on parallel lines" below.

[11] Note that not every notion satisfies these properties. For example, "A likes B" does not imply that "B likes A." Or if "A likes B" and "B likes C," this does not mean that "A likes C." However, the equality of, for example, weights is transitive. Indeed, if the weight of A is equal to the weight of B and the weight of B is equal to the weight of C, then the weight of A is equal to the weight of C.

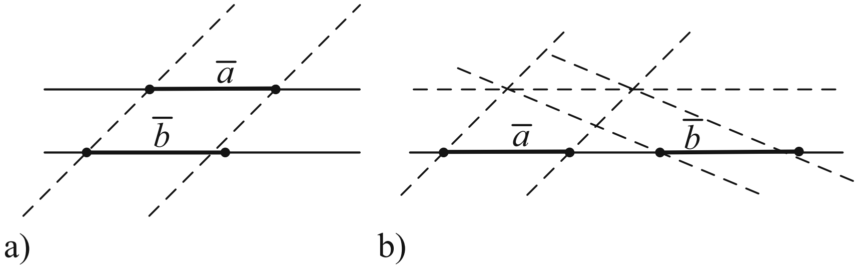


Fig. 2.10

Given two equal segments $\bar{a}$ and $\bar{b}$, symmetry means that if $\bar{a} = \bar{b}$ then $\bar{b} = \bar{a}$.

Let us prove this. Let a segment $\bar{a}$ lie on the line a and the segment $\bar{b}$ lie on a line b that is parallel to line a (see Fig. 2.11).

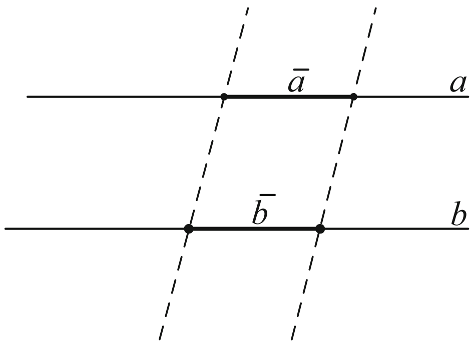


Fig. 2.11

If $\bar{a} = \bar{b}$ then by definition the lines connecting the ends of these segments are parallel. This also means that $\bar{b} = \bar{a}$. □

Transitivity of equal segments means that if $\bar{a} = \bar{b}$ and $\bar{b} = \bar{c}$, then $\bar{a} = \bar{c}$. The proof of the transitivity of equal segments is more difficult. It is given in Section 2.2 of the Appendix to Chapter II.[12]

It is also assumed that a segment is equal to itself, which can be written as $\bar{a} = \bar{a}$. This property is called the *reflexivity of equal segments*.

Compare the properties of equal segments with the properties of parallel lines.

We have defined equal segments on parallel lines and discussed their properties. Now let us consider how to construct segments of different lengths.

4.3 Construction of a segment of double length

Consider once more a line a with a segment $\bar{a}$ on it (see Fig. 2.12a). Using the construction methods from Section 4.2, we can now construct a segment of length $\bar{a}$ "next to" the segment $\bar{a}$, i.e., the new segment lies, for example, to the right of $\bar{a}$ and its end point coincides with the end point of $\bar{a}$ (see Fig. 2.12b). The segment combining these two segments (see Fig. 2.12c) has a length double that of the length of $\bar{a}$. We can denote it as $2\bar{a}$.

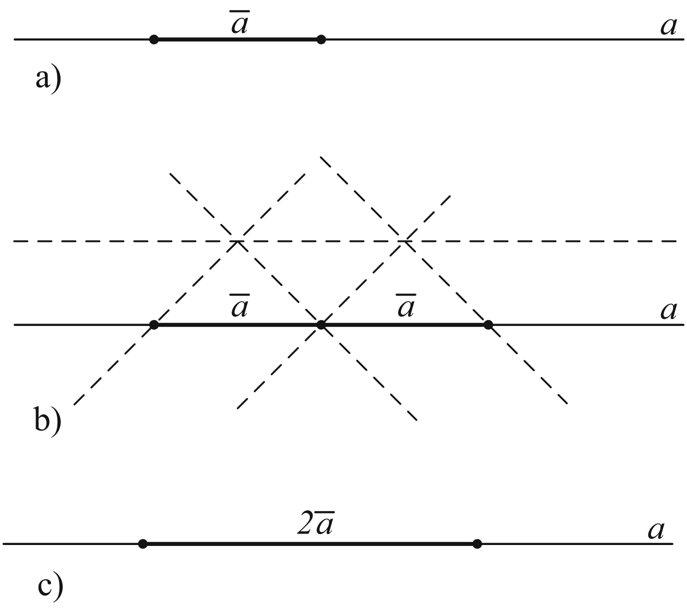


Fig. 2.12

[12]In fact, the transitivity of equal segments follows from a variation of the Desargues theorem described in Section 2.1 of the Appendix.

Similarly we can construct segments $3\bar{a}$, $4\bar{a}$, etc. See Section 3.2 of the Appendix to Chapter II where the arithmetic operations of multiplication and division on segments are introduced.

4.4 Division of a segment into equal parts

We have constructed above a segment $2\bar{a}$ for a given segment $\bar{a}$. We can also construct a segment $\frac{1}{2}\bar{a}$. For this we need to use the following proposition.

Proposition 2. *(Lemma)*[13] Suppose we have two straight lines a and b,[14] and on line a some equal segments are marked off. Then parallel lines passing through the ends of each of these segments mark off a second series of segments on line b. The segments in this second series on line b are equal to each other.

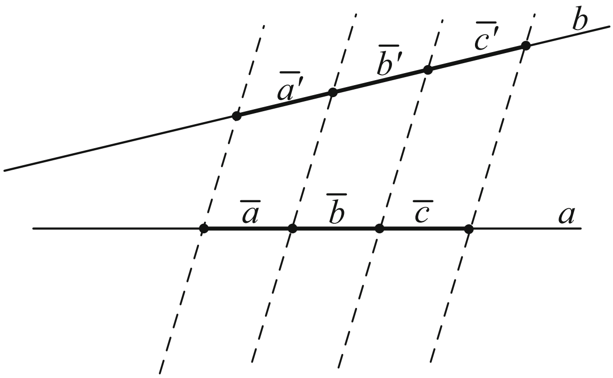


Fig. 2.13

Thus, given that $\bar{a} = \bar{b} = \bar{c}$ in Fig. 2.13 we obtain, according to the Lemma, $\bar{a'} = \bar{b'} = \bar{c'}$. We will postpone the proof of this Lemma to Section 5.3 since we need to prove some other propositions first.

[13] We designate this result as a lemma because of its importance.

[14] The lines a and b are not necessarily parallel. If these lines are parallel the statement will follow easily from Proposition 1.

Consider a line a with a segment $\bar{a}$ on it (see Fig. 2.14a). Through an endpoint of $\bar{a}$ draw any line b (different from a). On line b mark two equal segments $AB = BC$ with A being the intersection point of lines a and b as in Fig. 2.14b. Now draw a line through point C and the other endpoint of $\bar{a}$ and draw lines parallel to it through points A and B (see Fig. 2.14c). According to the Lemma, the segments formed by these parallel lines on the line a are equal to each other.

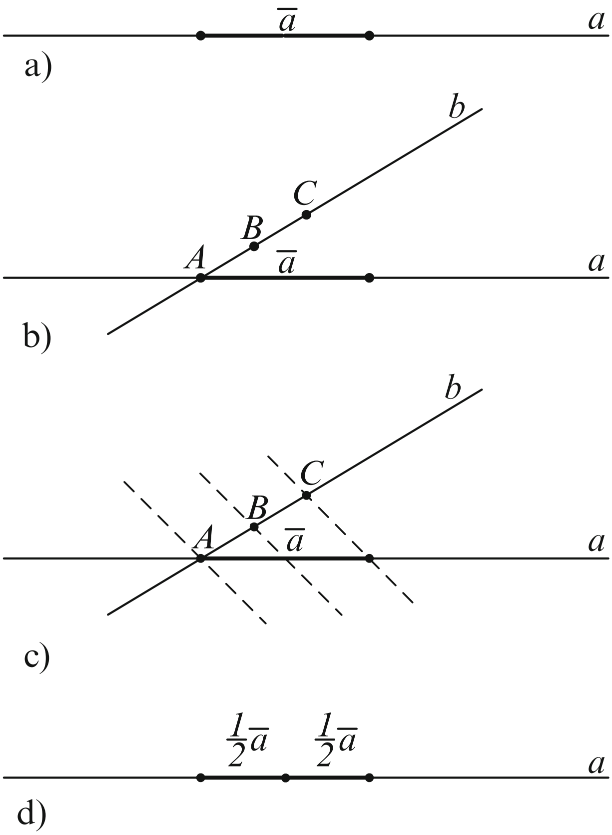


Fig. 2.14

Since we have divided a segment $\bar{a}$ into two equal parts, each of the two constructed segments is equal to $\frac{1}{2}\bar{a}$ (see Fig. 2.14d).

Definition 4. Given a segment AB, the point O on it such that $AO = OB$ is called the *midpoint* of the segment AB (see Fig. 2.15).

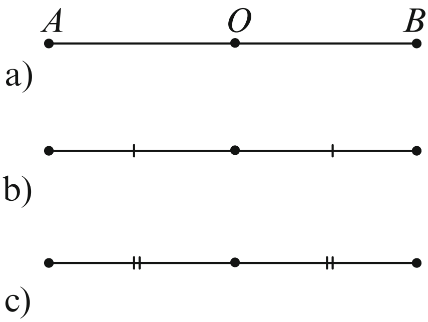


Fig. 2.15

Sometimes segments equal to each other are denoted as in Fig. 2.15b or Fig. 2.15c (of course, when it is clear what segments we mean).

PROBLEM 4. A segment $\bar{a}$ is given. Construct a segment $\frac{1}{3}\bar{a}$.

In Section 3 of the Appendix to Chapter II, we introduce arithmetic operations with segments: addition, subtraction, multiplication by a whole positive number, as well as division of a segment in a given ratio. This allows us to construct a segment of any rational length. These operations with segments make it possible also to introduce a *number axis*, and a coordinate system on the plane;[15] we show how to do this in Sections 4 and 5 of the same Appendix. This coordinate system is called an *affine coordinate system* and is different from the "usual" Cartesian coordinate system. Indeed, do not forget that we are allowed to perform only the three operations of affine geometry (see Section 2 of this chapter).

Let us summarize what we can do with segments in Chapter II. We defined equal segments on parallel lines and on the same line and showed how to construct them. We can also compare two segments on parallel lines or on the same line. We are able to construct a segment of any rational length. In Chapter II, however, we cannot construct equal segments lying on non-parallel lines and we cannot compare segments on non-parallel lines. These would be needed in order to define a Cartesian coordinate system. (We will be able to do such constructions in Chapter IV).

[15]Many mathematicians are fascinated by doing this and prefer to reduce geometry to algebra. We believe that geometry itself is at least as beautiful and interesting as algebra.

PART II. Figures

5 Parallelograms

5.1 Definition of a parallelogram

Definition 5. A quadrilateral whose opposite sides are parallel is called a *parallelogram.*

Below are examples of parallelograms in different positions:

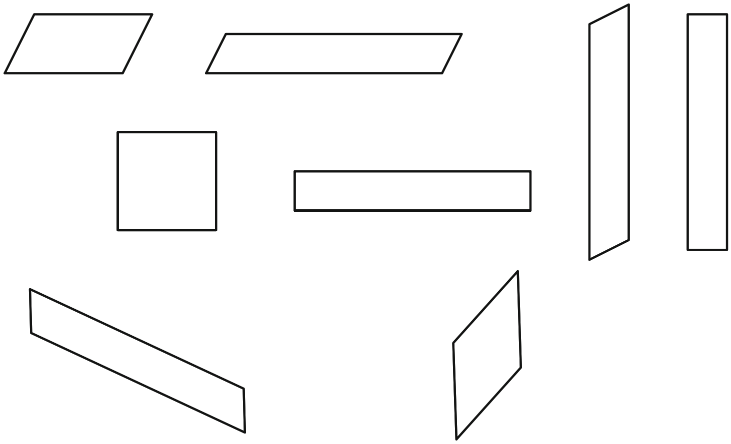
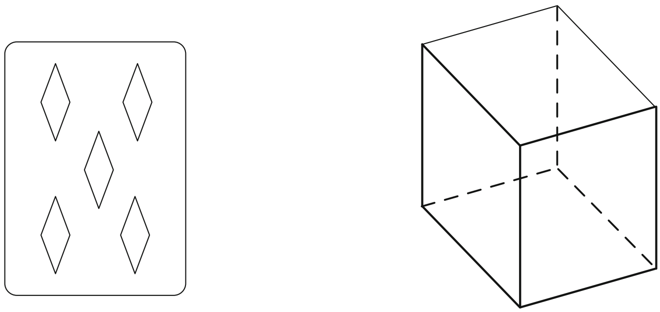


Fig. 2.16

We often meet parallelograms in our life (see Fig. 2.17a, b). Fig. 2.17b shows an image of a cube on the plane. The images of all 6 faces are parallelograms, but there is one more parallelogram. Can you find it?



a) b)

Fig. 2.17

Note, that the opposite sides of a parallelogram are equal. Indeed, this follows from Proposition 1.

Proposition 3. If a quadrilateral has two parallel sides that are equal to one another, then this quadrilateral is a parallelogram.

Proof. Indeed, consider a quadrilateral $BACD$ that has $AB \parallel CD$ and $AB = CD$. In order to prove that $BACD$ is a parallelogram, we need to prove that $AC \parallel BD$. We do not know for sure that $AC \parallel BD$. So, let us draw a straight line that is parallel to line AC and passes through point B. Let this line intersect line CD at point D' (see Fig. 2.18). We need to determine whether point D' can be different from point D the way it is drawn in the figure, or whether these two points must actually be the same.

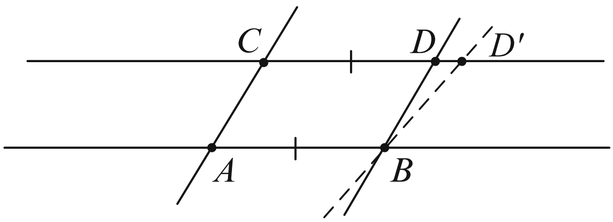


Fig. 2.18

Quadrilateral $BACD'$ is a parallelogram (since its opposite sides are parallel). As a result $AB = CD'$. But $AB = CD$; therefore, due to transitivity of equality, $CD = CD'$, and point D and point D' must coincide. Therefore, the line BD is parallel to AC. □

Thus, we have obtained two *equivalent* definitions of parallelogram:

1. A quadrilateral whose opposite sides are parallel to each other is a parallelogram.

2. A quadrilateral that has two opposite sides parallel and equal to one another is a parallelogram.

5.2 Properties of parallelograms

Suppose a parallelogram with sides a and b is intersected by two particular straight lines. One line passes through the midpoint of side a parallel to side b and the other line passes through the midpoint of side b parallel to side a (see Fig. 2.19). These two lines divide the given parallelogram into four quadrilaterals. What can we say about them?

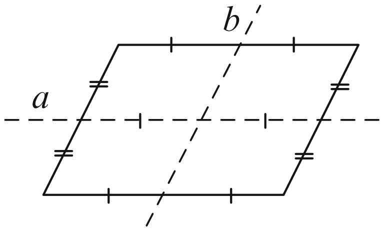


Fig. 2.19

Proposition 4. Consider a parallelogram and two lines intersecting each other within it: if each line passes through the midpoint of one side and is parallel to another side of the parallelogram, then the resulting quadrilaterals are parallelograms with corresponding sides equal to each other.

Proof. Indeed, in each of these four quadrilaterals (see Fig. 2.19) the opposite sides are parallel. Therefore, these quadrilaterals are parallelograms. From Proposition 1 the corresponding sides of these parallelograms are equal to each other. □

The point of intersection within a parallelogram of the two lines described in Proposition 4 is called the *center of the parallelogram*.

Below we will study some properties of parallelograms that are related to its center. Consider a parallelogram $ABCD$ with center O (Fig. 2.20a). Let us draw segments AO and HF as in Fig. 2.20b.

Proposition 5. Quadrilateral $AOFH$ in Fig. 2.20b is a parallelogram.

Proof. Indeed, we have $AH \parallel OF$ (by construction of center O) and $AH = OF$ by Proposition 4. Therefore, $AOFH$ is a parallelogram. □

Note that by similar reasoning we can conclude that quadrilateral $OCFH$ in Fig. 2.20c is also a parallelogram.

Proposition 6. In parallelogram $ABCD$ with center O, points A, O, and C lie on a single straight line.

In other words, this proposition means that the center O lies on line AC.

Proof. Consider once more Fig. 2.20. From Proposition 5, $AOFH$ is a parallelogram. Therefore $AO \parallel HF$. Since $OCFH$ is also a parallelogram, we have $OC \parallel HF$. But by Euclid's fifth postulate there can be only one straight line through point O and parallel to line HF. Thus, points A, O, C lie on a single line. □

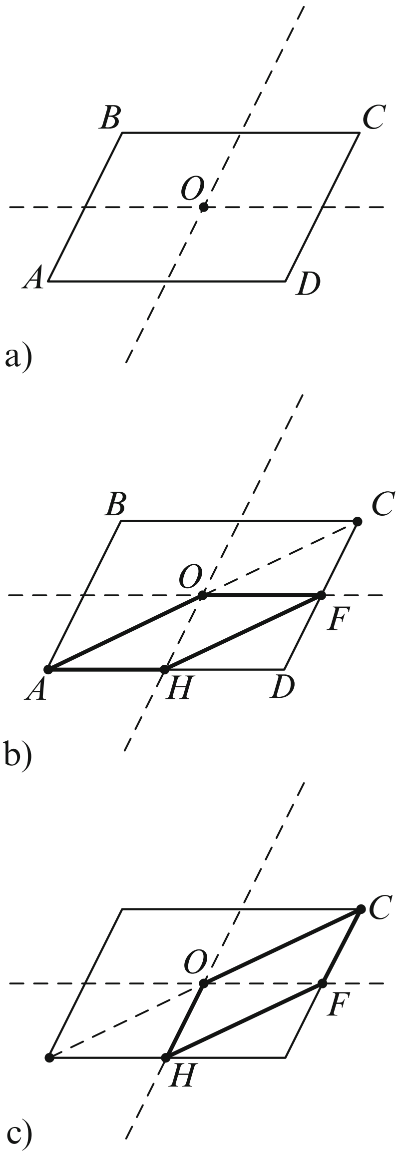


Fig. 2.20

In a parallelogram (see Fig. 2.20a), vertices A and C are usually called *opposite* vertices. Vertices B and D are also opposite vertices in this parallelogram.

A segment connecting two opposite vertices of a parallelogram is called a *diagonal* of the parallelogram. Thus, a parallelogram has two diagonals.

Theorem 1. In a parallelogram, each diagonal passes through the center of the parallelogram and is divided in half by it.

Proof. Consider parallelogram $ABCD$ with center O and diagonals AC and BD (see Fig. 2.21a).

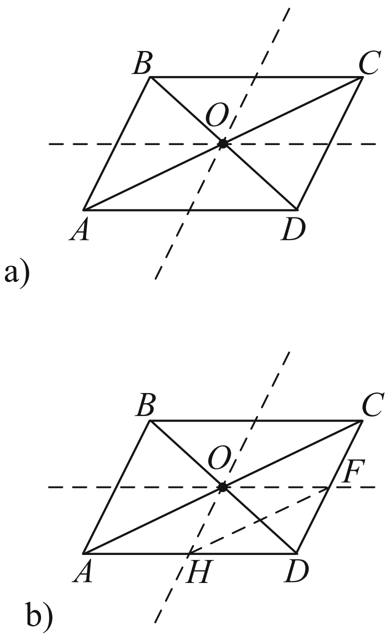


Fig. 2.21

We have already proved that points A, O, C lie on a single line and therefore diagonal AC passes through the center O. Now we need to prove that $AO = OC$. Indeed, let us again denote the points H, F as in Fig. 2.21b. Then in parallelogram $AOFH$ we have $AO = HF$ and in parallelogram $OCFH$ we have $OC = HF$. Therefore, $AO = OC$ (by transitivity of equal segments).

We have proved that diagonal AC is divided into two equal parts by point O. Similarly, we can prove that the other diagonal BD is also divided by the center O into two equal parts. $\square$

Note that we have obtained a new way to find the center of a parallelogram: we draw its diagonals, and their intersection point is the center.

Exercise 2. Given a point O (see Fig. 2.22a), draw a parallelogram such that point O is its center.

Solution. There are many ways to do this. One of them is the following. We draw two arbitrary straight lines through point O and mark equal segments $AO = OC$ on one line and two equal segments $BO = OD$ on the other line (see Fig. 2.22b). Then we connect points A, B, C, and D with segments. Quadrilateral $ABCD$ (see Fig. 2.22c) is a parallelogram (see Problem 5 in Section 5.4).

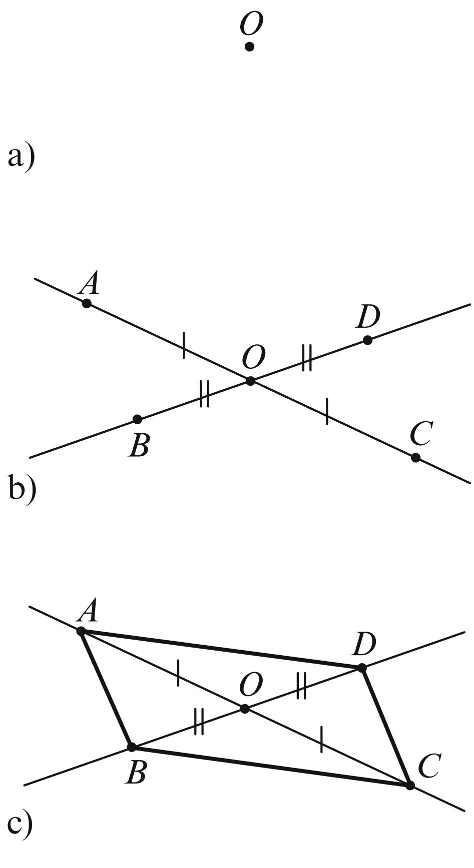


Fig. 2.22

5.3 Proof of the Lemma

Let us return to the Lemma introduced earlier as Proposition 2 in Section 4.4. In order to prove the Lemma, we will prove a simpler statement first.

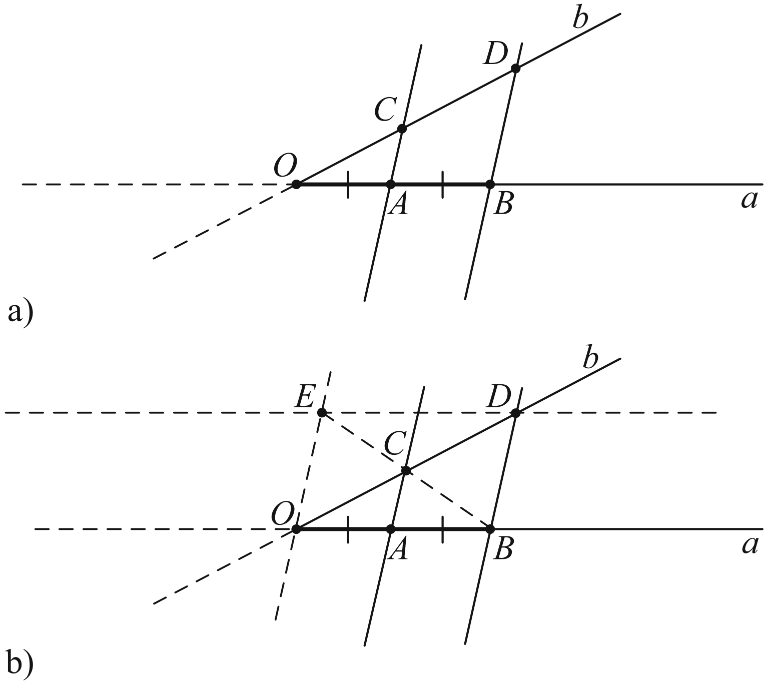


Fig. 2.23

Proposition 7. Consider a point O and two lines a and b passing through it. Suppose that on line a there are two equal segments $OA = AB$ (see Fig. 2.23a). Then if we draw parallel lines through points A and B they will mark equal segments $OC = CD$ on the second line (line b).

Proof. We need to prove that $OC = CD$. Let us draw the line through point D parallel to line a and the line through point O parallel to lines AC and BD. These two new lines intersect at point E. The figure $OBDE$ (in Fig. 2.23b) is a parallelogram (by Definition 5), and OD is one of its diagonals.

We already know that the center of parallelogram $OBDE$ lies on the line that passes through the midpoint of OB and is parallel to side BD, i.e., the center lies on AC. At the same time, diagonal OD passes through the center of the parallelogram. Therefore, point C must be the center.

By Theorem 1 a diagonal of a parallelogram is divided in half by its center. Therefore, $OC = CD$. □

Lemma. Suppose there are two straight lines a and b, and on line a there are equal segments marked off. Then parallel lines passing through the ends of each of these segments determine segments on line b. These segments on line b are equal to each other.

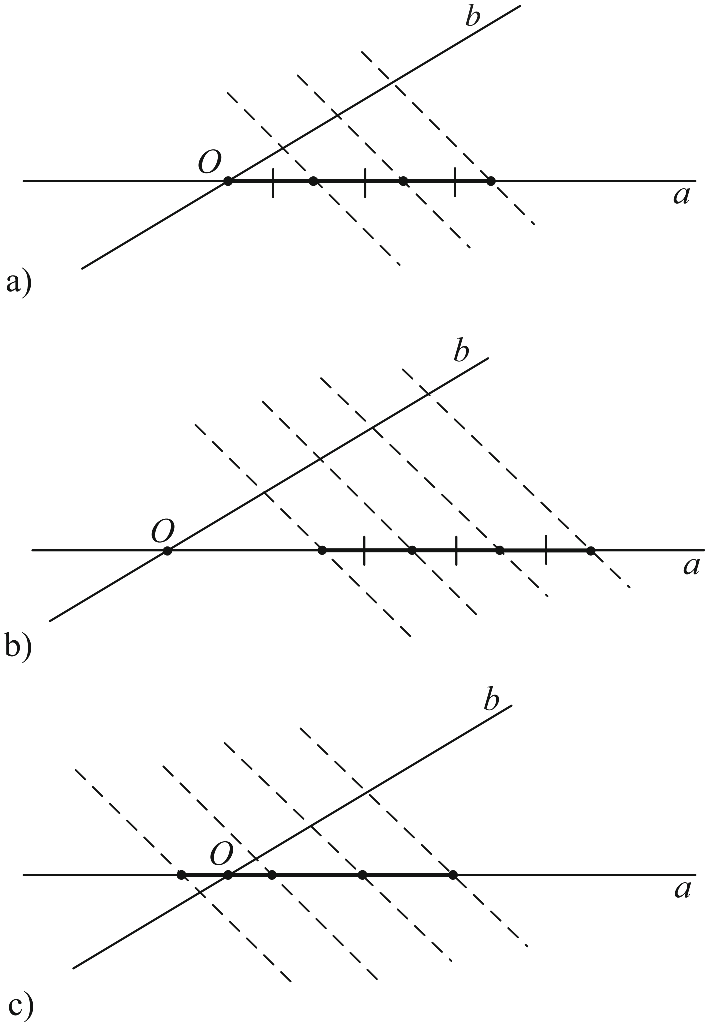


Fig. 2.24

Proof. First, note that there are different possible diagrams (see Fig. 2.24a, b, c). Notice where the position of the first segment in each of these figures is.

The case in Fig. 2.24a has already been considered in Proposition 7. We will consider only the case in Fig. 2.24b. The case in Fig. 2.24c is left for the reader.

Let us introduce some notations. (See Fig. 2.25a.) We have $AB = BC = CD$. Lines AA', BB', CC', DD' are parallel to each other. We need to prove that $A'B' = B'C' = C'D'$. Let us draw line c through point A' parallel to line a (see Fig. 2.25b).

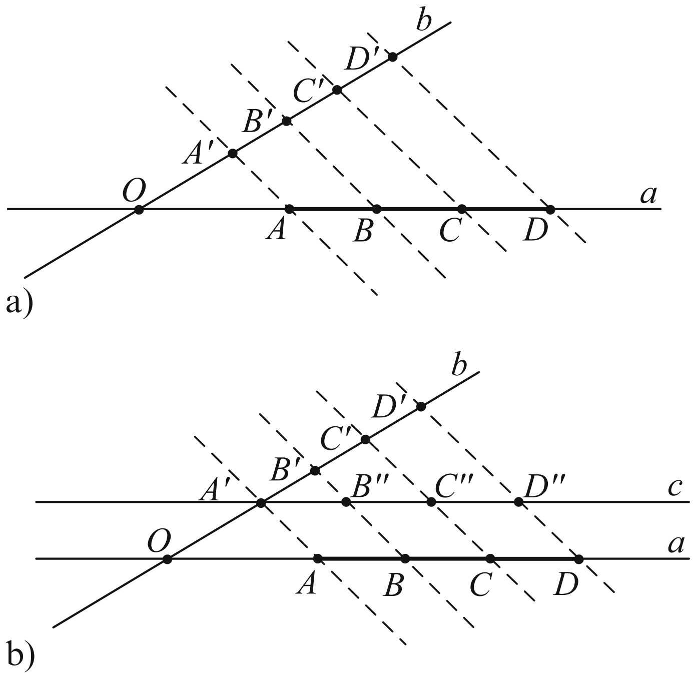


Fig. 2.25

From Proposition 1, we have $A'B'' = B''C'' = C''D''$. Now we can apply Proposition 7 to the lines c and b. We obtain $A'B' = B'C' = C'D'$. $\square$

5.4 More properties of parallelograms

The center of a parallelogram has the following property:

Proposition 8. Any straight line that passes through the center O of a parallelogram is divided by point O and the sides of the parallelogram into two equal segments (see Fig. 2.26a).

Proof. Fig. 2.26a shows a parallelogram $ABCD$ and a line a that passes through the center O.

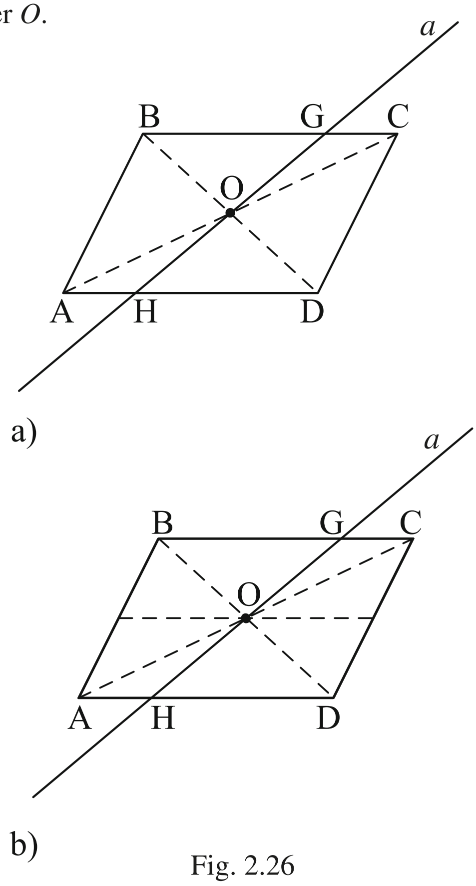


Fig. 2.26

We need to prove that segments GO and OH on this line are equal. We can apply the Lemma. Indeed, let us draw the line through the point O parallel to AD (see Fig. 2.26b). We have two intersecting lines a and AC, and on line AC there are two equal segments $AO = OC$ (by Theorem 1). There are parallel lines passing through the ends of these two segments (compare with Fig. 2.24c). Thus on line a the corresponding segments (i.e., HO and OG) are equal. □

Consider two parallel lines. We can draw many parallelograms that have two sides on these lines. For example, see Fig. 2.27a.

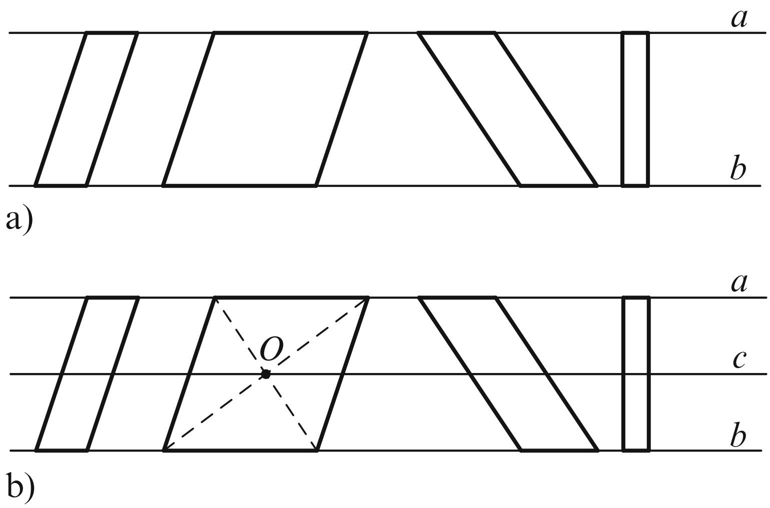


Fig. 2.27

Proposition 9. The centers of all parallelograms with opposite sides lying on two parallel lines lie on a single straight line.

Proof. Consider such parallelograms (see Fig. 2.27a). Let us draw diagonals in one of these parallelograms and let O be its center (see Fig. 2.27b). We know that each of these diagonals is divided into two equal parts by point O. Now let us draw the straight line c through point O parallel to lines a and b (Fig. 2.27b). By the Lemma, any other line intersected by the lines a, b, and c will be divided by them into equal segments. This means that a diagonal of any parallelogram in Fig. 2.27 is divided into two equal parts by line c. This means that the center of each of these parallelograms lies on line c. Thus, we have proved that the centers of all these parallelograms lie on a single straight line. □

We will call this line the *midline of two parallel lines*.

Exercise 3. Two parallel lines are given. Find a point on the midline of these parallel lines.

Solution. Consider two parallel lines (see Fig. 2.28a).

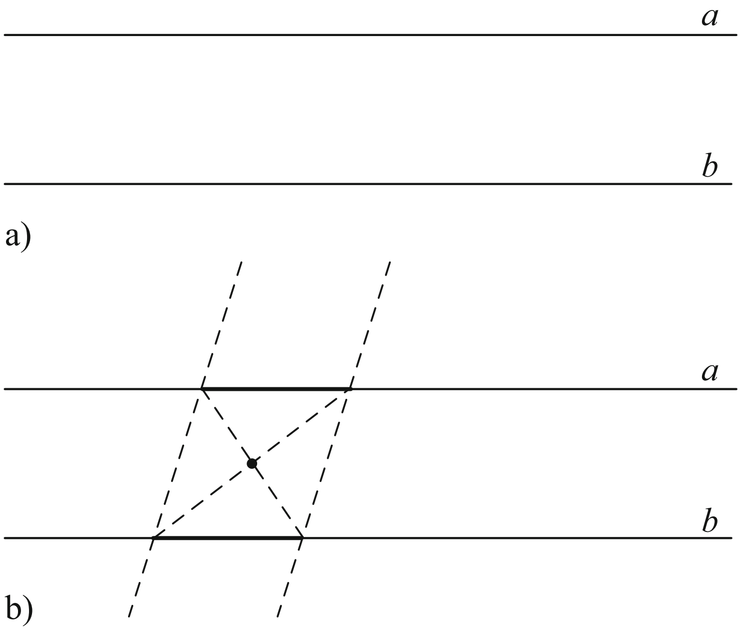


Fig. 2.28

Let us draw two parallel lines that intersect the given lines. We obtain a parallelogram with its sides on these parallel lines (Fig. 2.28b). The center of this parallelogram lies on the midline of the two given parallel lines.

PROBLEM 5 **(∗)** Prove that if diagonals of a quadrilateral are divided into two equal parts by their intersection, then this quadrilateral is a parallelogram.

PROBLEM 6 **(∗)** Fig. 2.29 shows two parallel lines a and b. On line a there are two points A and B. On line b there are two points C and D. Prove that if the point O of the intersection between AD and BC lies on the midline of parallel lines a and b, then $AB = CD$.

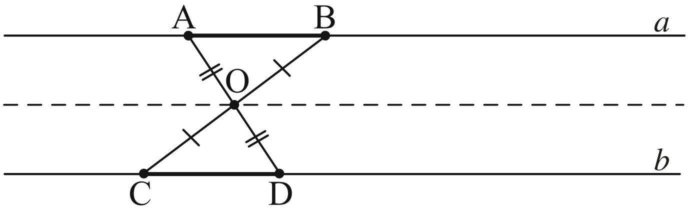


Fig. 2.29

Remark 5. Let a and b be two parallel lines. The midline of a and b has the following property. Let $\bar{a}$ be a segment on line a and $\bar{b}$ a segment on line b. If the lines connecting their endpoints intersect at the midline then $\bar{a} = \bar{b}$ (see the problem above). If this intersection point does not lie on the midline then it lies to one side of the midline, thus being "closer"[16] either to line a or to line b. If it lies closer, for example, to line a then segment $\bar{a}$ is smaller than segment $\bar{b}$ (see Fig. 2.30a). Otherwise, segment $\bar{a}$ is larger than segment $\bar{b}$ (see Fig. 2.30b).

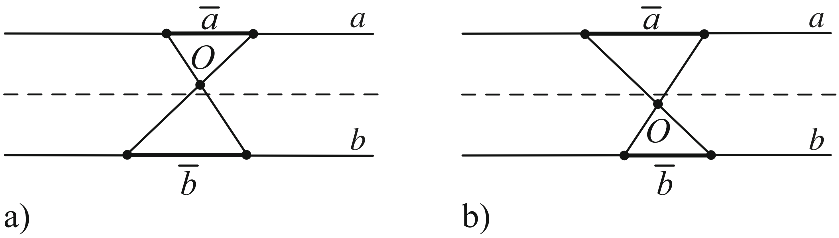


Fig. 2.30

6 Triangles

6.1 Bimedian of a triangle

Consider a triangle ABC[17] and the midpoint E of its side AB. Let us draw a straight line through the point E parallel to one of the other sides of this triangle (see Fig. 2.31a, b). The segment of this straight line that lies inside the triangle is called a *bimedian of the triangle*.

Proposition 10. Let EF be the bimedian of a triangle ABC such that E is the midpoint of AB and EF is parallel to AC (see Fig. 2.31a). The bimedian EF divides the side BC into two equal parts ($BF = FC$).

Proof. This follows from the Lemma. □

Therefore, a bimedian of a triangle connects midpoints of two sides of the triangle and is parallel to the third side of the triangle.

[16] Note that the word "closer" is not defined because we do not have length measure in Chapter II. We could have also said that the intersection lies "above" or "below" the midline or "between" the midline and line a. All these terms cannot be defined in this chapter and are left for your intuition. See also footnote 6 in Section 1.3, Chapter I.

[17] We will also use the notation $\triangle ABC$.

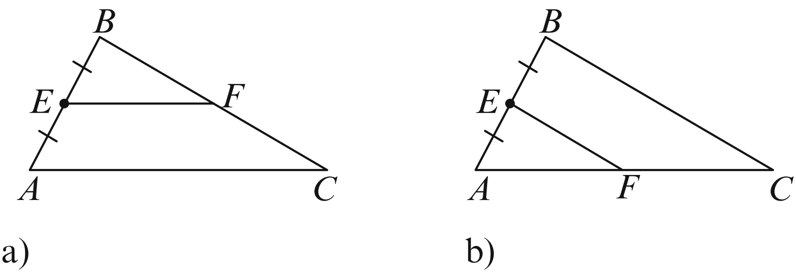


Fig. 2.31

Thus, there are two ways to draw a bimedian:

1. mark a midpoint on one side of a triangle and draw a line through this point parallel to another side of a triangle;
2. mark midpoints on two sides of a triangle and connect them.

There are three different bimedians in a triangle (see Fig. 2.32a, b, c).

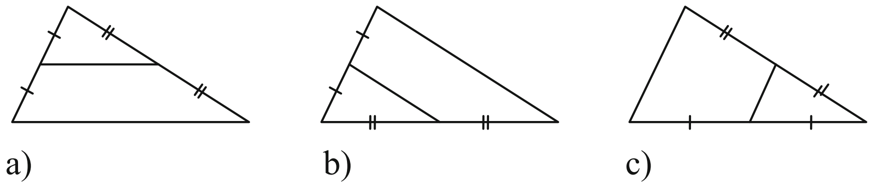


Fig. 2.32

Theorem 2. The length of a bimedian of a triangle is half of the length of the side to which this bimedian is parallel.

Proof. Consider triangle ABC and bimedian EF parallel to side AC (see Fig. 2.33). Let us draw segment FD parallel to AB. Segment FD is also a bimedian of $\triangle ABC$ and, therefore, $AD = DC$.

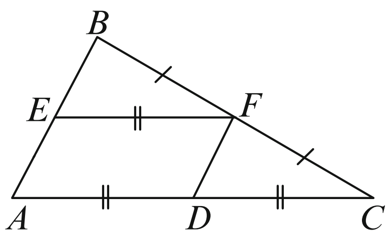


Fig. 2.33

Note that the figure $AEFD$ is a parallelogram (since its opposite sides are parallel) and therefore, $EF = AD$. Thus, $EF = \frac{1}{2}AC$. □

PROBLEM 7.

(a) Fig. 2.34a shows a triangle ABC with bimedians EF and FD, where $EF \parallel AC$ and $FD \parallel AB$. Indicate all segments in this figure that are equal to each other.

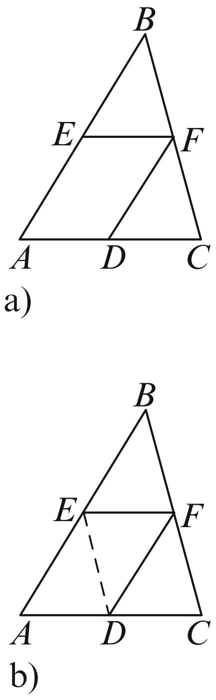


Fig. 2.34

(b) Find segments equal to the diagonal ED of parallelogram $AEFD$ in Fig. 2.34b.

PROBLEM 8. Consider a triangle ABC and let E be a point on side AB. Through point E draw the unique line parallel to AC. Let F be the intersection of this line with BC. Through point E draw line n parallel to BC, and through point F draw line m parallel to AB. When does point O at the intersection of lines m and n lie on segment AC (see Fig. 2.35a), when does point O lie above segment AC (see Fig. 2.35b), and when does point O lie below segment AC (see Fig. 2.35c)?

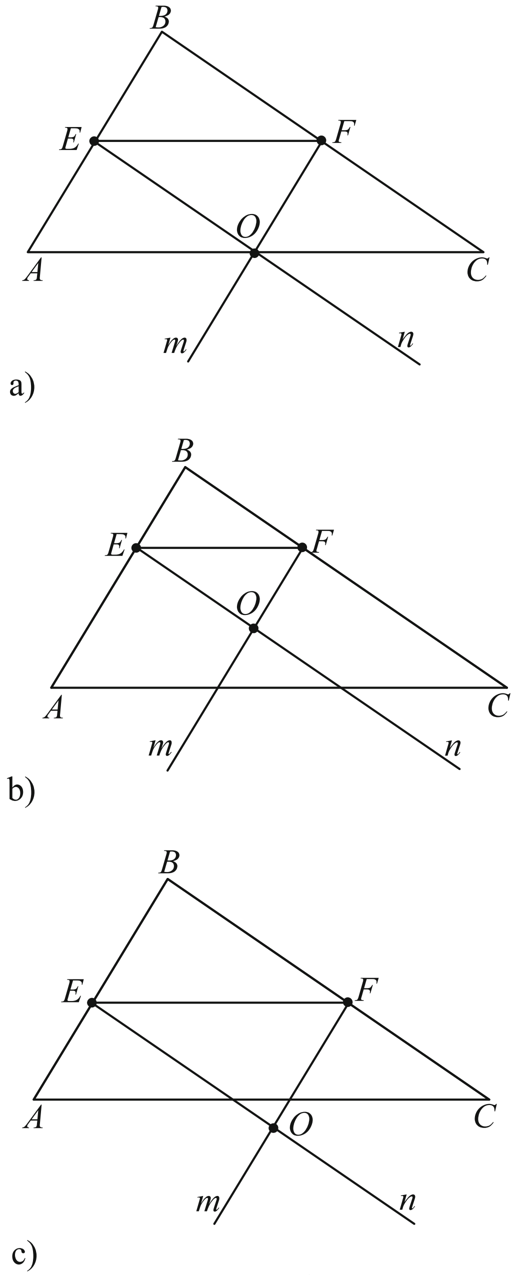


Fig. 2.35

PROBLEM 9. Consider a triangle ABC and let D be a point on side AB. Draw line DE (with E on BC) parallel to AC. Then draw line EF (with F on AC) parallel to AB, and then draw line FG (with G on AB) parallel to BC. What can you tell about the position of point G? When does it lie above point D (see Fig. 2.36a), when does it lie below point D (see Fig. 2.36b), and when does it coincide with point D (see Fig. 2.36c)?

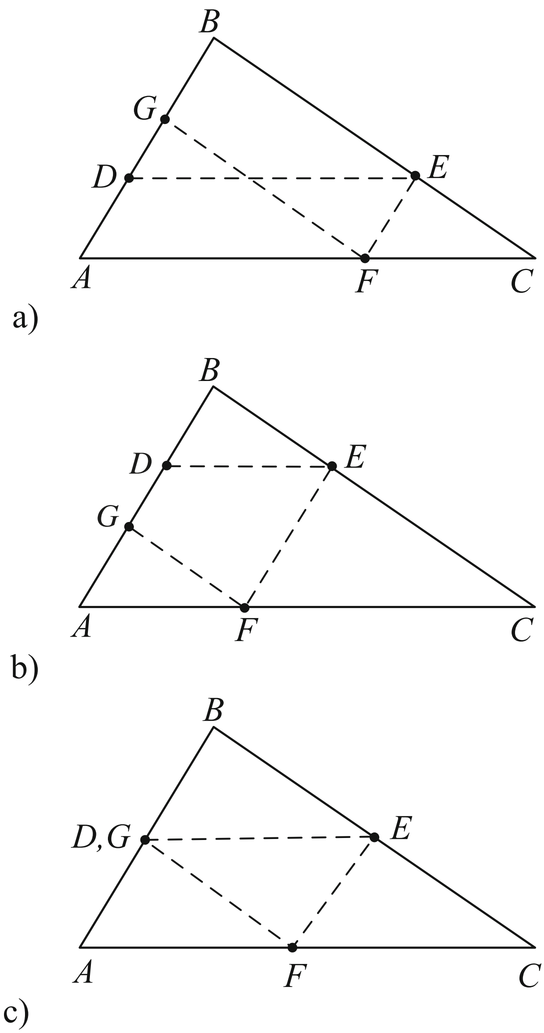
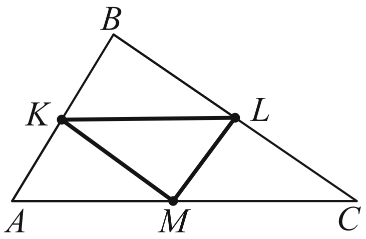


Fig. 2.36

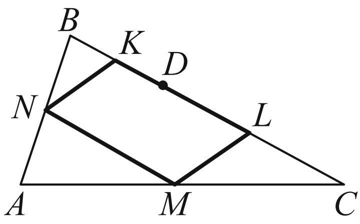
PROBLEM 10. A segment EF is given. Draw a triangle such that EF is its bimedian.

The next problem describes an interesting property of bimedians in a triangle.

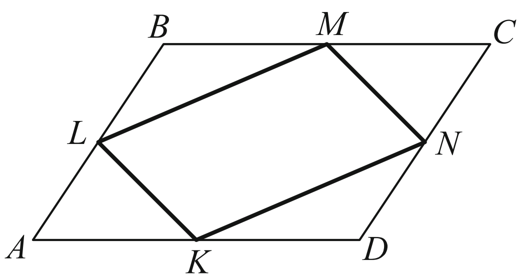
PROBLEM 11. Points K, L, and M are the midpoints of the sides of triangle ABC (see Fig. 2.37a). The three bimedians KL, LM, and MK divide the triangle into four triangles. Prove that any two of these four triangles have the same three sides. (That is, corresponding sides are equal. You can use three different colors to mark sides of one of the triangles. Then use the same color for any other segment that is equal to the side marked by this color.)



a)



b)



c)

Fig. 2.37

PROBLEM 12. In triangle ABC draw bimedian MN parallel to side BC. Choose a point D on side BC. Let K be the midpoint of segment BD and L be the midpoint of segment DC. Prove that quadrilateral $KLMN$ (see Fig. 2.37b) is a parallelogram.

PROBLEM 13. Let K, L, M, and N be the midpoints of the sides of parallelogram $ABCD$ (see Fig. 2.37c). Prove that quadrilateral $KLMN$ is itself a parallelogram.

Problems 11, 12, and 13 will be revisited in Chapter III.

We have considered the bimedian in a triangle. There are other interesting lines in a triangle.

6.2 Median of a triangle

A *median of a triangle* is a segment that connects a vertex of a triangle to the midpoint of the opposite side (see Fig. 2.38).

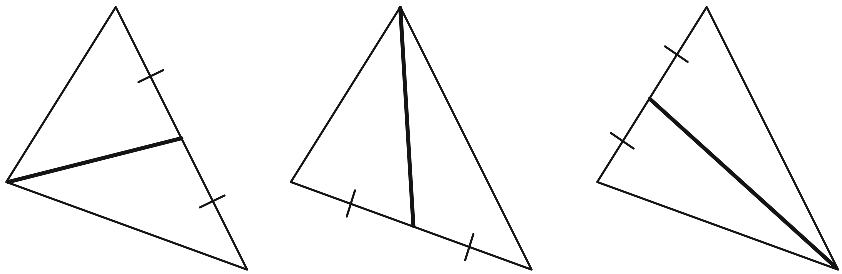


Fig. 2.38

The word "median" means "in the middle." It turns out that if we think of our triangle as being made of some substance, say chocolate, of uniform density, then a median divides the triangle into two parts of equal weight. Thus, it is a fair way to divide a triangular piece of chocolate or other valuable substance between two people (see Chapter III, Section 3, Proposition 6 for further explanation).

Exercise 4. Given a triangle, complete it to form a parallelogram, i.e., construct a parallelogram that will have two sides in common with the triangle.

Solution 1. Consider triangle ABC (see Fig. 2.39a). Let us draw a median from vertex B and then extend it by its own length beyond the side of the

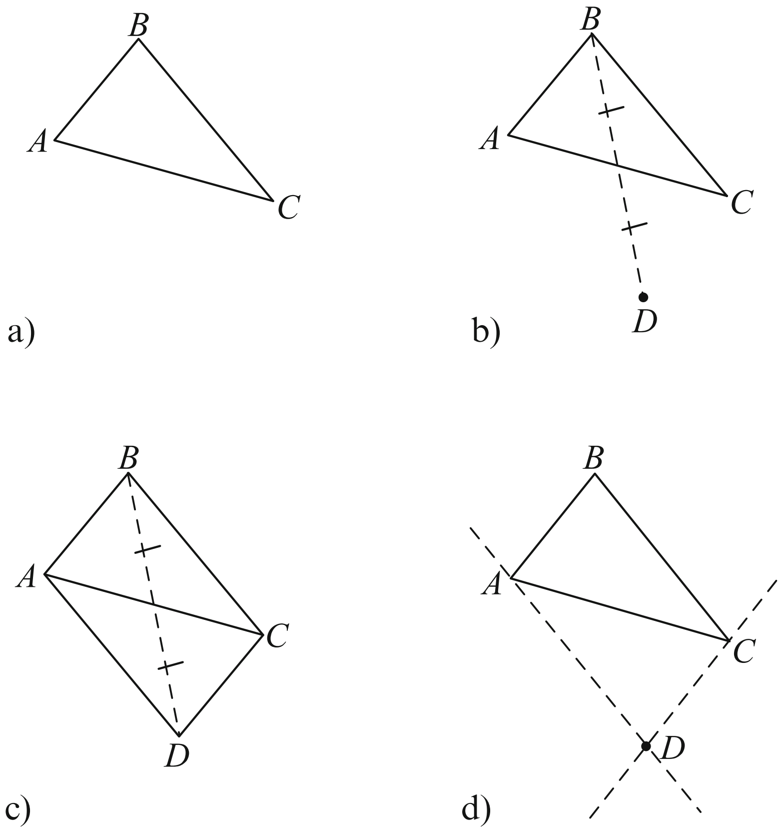


Fig. 2.39

triangle (see Fig. 2.39b). By connecting the end of this segment (point D) with the other two vertices of the triangle, we obtain parallelogram $ABCD$ (see Fig. 2.39c).

Solution 2. Consider a triangle ABC (see Fig. 2.39a). Through point C let us draw the straight line parallel to AB, and then through point A draw the line parallel to BC (see Fig. 2.39d). If we denote by D the intersection point of these additional lines, then figure $ABCD$ is a parallelogram.

PROBLEM 14. In Fig. 2.40 there should be a triangle with three medians in it. However, there is a mistake in this figure. Find it and draw the figure correctly.[18]

[18] If you have difficulties, return to this problem after you finish reading Section 6.2.

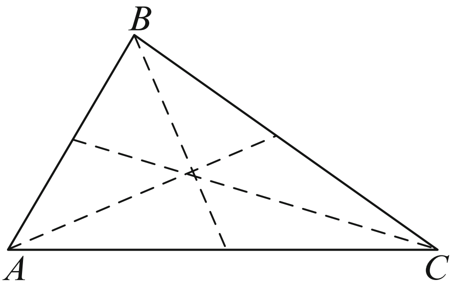


Fig. 2.40

Theorem 3. Any two medians of a triangle are divided by the point of their intersection in the ratio 2 : 1.

It is easy to prove this theorem by using some properties of a trapezoid. This is why we prove this theorem at the end of Section 7.

Fig. 2.41 shows a triangle ABC with two medians AF and CE intersecting at the point O. Theorem 3 states that $AO : OF = 2 : 1$ and $CO : OE = 2 : 1$.

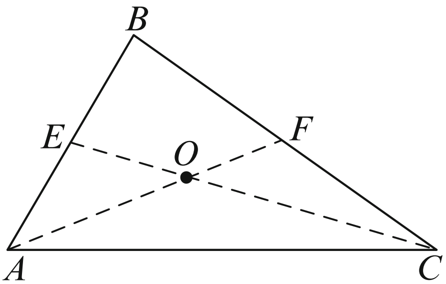


Fig. 2.41

Corollary 1. All the medians of a triangle intersect in a single point.

Proof. Consider a triangle ABC and one median in it, for example, AF in Fig. 2.41. Then, according to Theorem 3, the second median CE intersects AF at the point O such that the first median is divided in the ratio 2 : 1. But median AF is also divided in the ratio 2 : 1 at its point of intersection with the third median. Therefore, this intersection point is also the same point O. This means that all three medians intersect at the point O. □

7 Trapezoids

Definition 6. *A trapezoid* is a quadrilateral that has two opposite sides parallel.

In Fig. 2.42 there are examples of trapezoids.

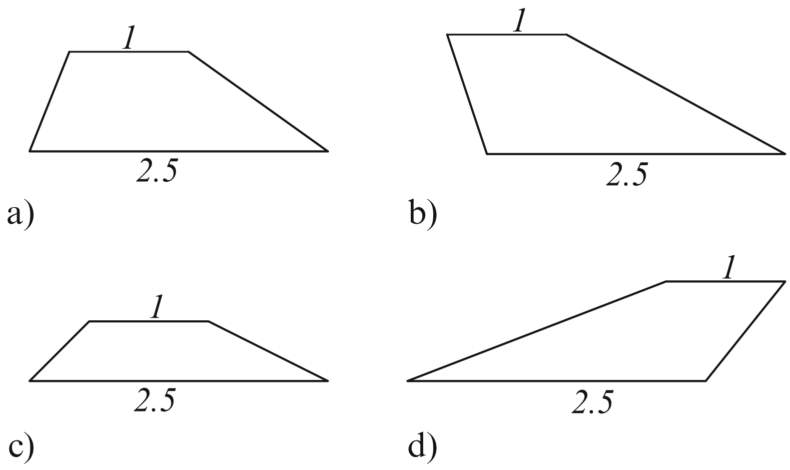


Fig. 2.42

The two parallel sides of a trapezoid are called its *bases* and the other two sides are called its *legs*.

Question 1. Is a parallelogram a trapezoid?

PROBLEM 15. Which of the figures in Fig. 2.43 is not a trapezoid?

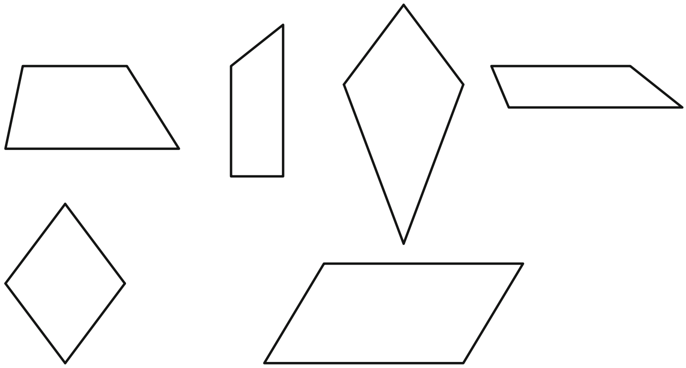


Fig. 2.43

Consider a trapezoid. Let us draw a straight line through the midpoint of one of its legs parallel to the base (see Fig. 2.44). The segment of this line connecting two legs of a trapezoid is called a *median of the trapezoid*.

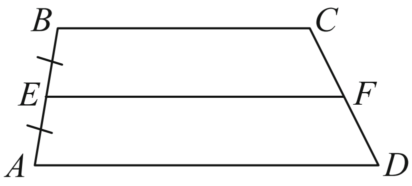


Fig. 2.44

PROBLEM 16. Let $ABCD$ be a trapezoid and let EF be its median drawn through point E on leg AB (see Fig. 2.44). Prove that the median EF divides the other leg of the trapezoid into two equal parts. In other words, a trapezoid has only one median.

Hint. Use the Lemma.

We can define the median of a trapezoid in two different ways:

1) The median of a trapezoid is the line that connects the midpoints of the two legs of the trapezoid.

2) The median of a trapezoid is the line that passes through the midpoint of a leg of the trapezoid and is parallel to the base of the trapezoid.

Exercise 5. Given a trapezoid with the bases 1 and 2.5 inches, find the length of the median.

Solution. At first glance, this problem seems unsolvable. Indeed, there are different trapezoids with bases a and b (see Fig. 2.42), and it is not clear which one to consider.

Let us start with the trapezoid in Fig. 2.42a. Draw its median. Let us also draw a diagonal. See Fig. 2.45, where the trapezoid is labeled $ABCD$, the median is EF, and the intersection of EF and the diagonal AC is point G.

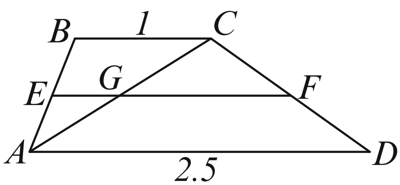


Fig. 2.45

We thus have two triangles ABC and ACD that are intersected by the segment EF. Note that EG is a bimedian in triangle ABC and GF is a bimedian in triangle ACD. By Theorem 2, $EG = 0.5$ (half of BC) and $GF = 1.25$ (half of AC). Thus, the length of the median EF is equal to $0.5 + 1.25 = 1.75$.

Consider for yourself other trapezoids in Fig. 2.42. Draw the medians in each of them and find their lengths.

PROBLEM 17. Consider a trapezoid and one of its diagonals. If we draw the median of this trapezoid it will divide this diagonal into two segments. What is the ratio of the lengths of these segments?

Proposition 11. The diagonals of a trapezoid are divided in half by its median.

Proof. Consider trapezoid $ABCD$, and draw its diagonals and the median EF (see Fig. 2.46). In triangle ACD we have $CF = FD$ and EF is parallel to AD (since EF is a median of trapezoid $ABCD$). This means that line EF is also a bimedian of $\triangle ACD$, and hence, $AG = GC$. Similarly, we can prove that $BH = HD$. □

PROBLEM 18. Consider trapezoid $ABCD$. Let us draw its diagonals and median EF (see Fig. 2.46). Median EF is divided by the diagonals into EG, GH, and HF. If the bases of $ABCD$ are 10 and 14, what is the length of each of these segments?

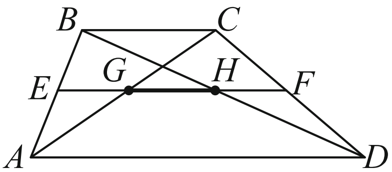


Fig. 2.46

PROBLEM 19. Consider trapezoid $ABCD$. Through point B let us draw the line parallel to CD and through point C the line parallel to AB (see the different cases in Fig. 2.47). Let the point of intersection of these lines be F. In the following examples, tell whether point F lies outside the trapezoid, inside it, or on side AD:

(a) $BC = 4$ and $AD = 6$;

(b) $BC = 4$ and $AD = 8$;

(c) $BC = 4$ and $AD = 10$.

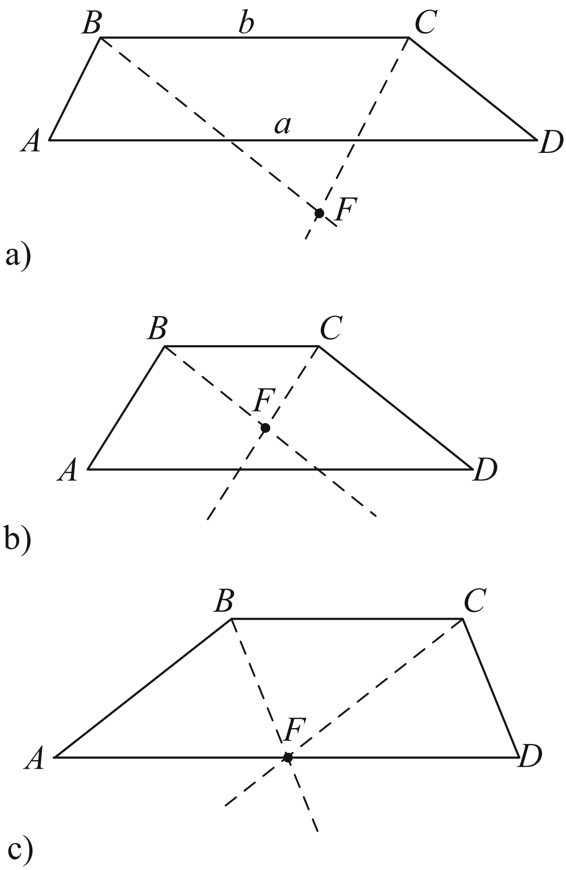


Fig. 2.47

PROBLEM 20 (∗) Consider trapezoid *ABCD* with bases *a* and *b*. Through point *B* let us draw the line parallel to *CD* and through point *C* the line parallel to *AB* (see Fig. 2.47). Let the point of intersection of these lines be *F*. For which lengths of *a* and *b* does the point *F* lie inside the trapezoid, for which does it lie outside it, and for which does it lie on the side *AD*?

PROBLEM 21. Consider two parallel lines and draw a trapezoid with its bases on these lines. Prove that if the intersection of the diagonals of this trapezoid lies on the midline of these parallel lines then this trapezoid is a parallelogram.

Theorem 4. The length of the median of a trapezoid is equal to half of the sum of the lengths of its bases.

Proof. Consider trapezoid $ABCD$ with bases a and b and median EF (see Fig. 2.48). Draw diagonal AC, intersecting EF at G. Note that EG is a bimedian in $\triangle ABC$ and GF is a bimedian in $\triangle ACD$.

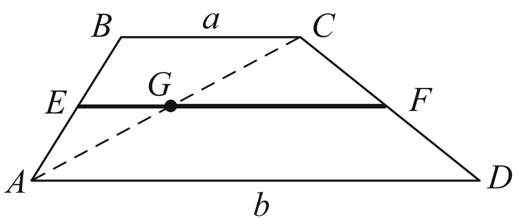


Fig. 2.48

Therefore, by Theorem 2, segment $EG = \frac{1}{2}a$ and segment $GF = \frac{1}{2}b$. Thus, $EF = \frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a+b)$. □

PROBLEM 22. Find the length x of the bold segment labeled X in Fig. 2.49.

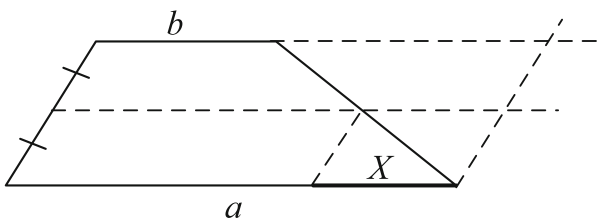


Fig. 2.49

Proof of Theorem 3. Now let us return to the study of triangles and prove Theorem 3. Here it is stated again.

Theorem 5. *(Restatement of Theorem 3)* Any two medians of a triangle divide each other at their point of intersection in the ratio 2 : 1.

Proof. Consider triangle ABC and two medians AF and CE intersecting at point O (see Fig. 2.50a).

Let us connect points E and F. Since $AE = EB$ and $CF = FB$, segment EF is a bimedian of $\triangle ABC$ and, therefore, $EF \parallel AC$. Thus quadrilateral $AEFC$ is a trapezoid. It also follows from Theorem 2 that $AC = 2EF$. We need to prove that $CO = 2OE$.

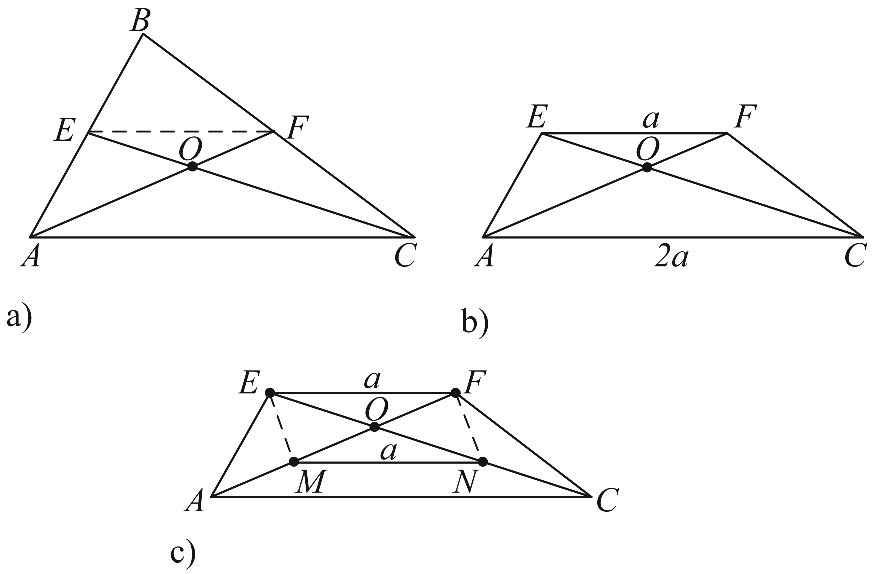


Fig. 2.50

Let us draw trapezoid $AEFC$ separately (see Fig. 2.50b). In triangle AOC, let us draw bimedian MN as in Fig. 2.50c. Now connect point E with M and point F with N. We have $EF \parallel MN$ from the transitivity of parallel lines (since $EF \parallel AC$ and $AC \parallel MN$). We also have $EF = MN$ since $EF = \frac{1}{2}AC$ (see above) and $MN = \frac{1}{2}AC$ (since MN is a bimedian in $\triangle AOC$). Thus, $EFNM$ is a parallelogram.

Then, from Theorem 1, $EO = ON$. But we also have $ON = NC$ (because NM is a bimedian in $\triangle AOC$). Therefore, we can conclude that $OC = 2OE$. This proves the theorem. □

PART III. Operations with figures

8 The Minkowsky addition of two figures

The famous mathematician Minkowsky (1864–1909) studied geometry and developed many new concepts in it. He successfully used geometry in number theory and physics (relativity theory). Minkowsky used the following geometric construction, which is now called *Minkowsky addition* or the *Minkowsky sum of two figures*.

Given two sets of points, let us connect each point of the first set with each point of the second set with line segments.

Definition 7. The Minkowsky sum of two figures (or two sets of points) is a figure that consists of the midpoints of all the segments connecting each point of one of the figures with each point of the other figure.

Consider two points A and B. The Minkowsky sum of these points is the midpoint C of the segment AB (see Fig. 2.51a).

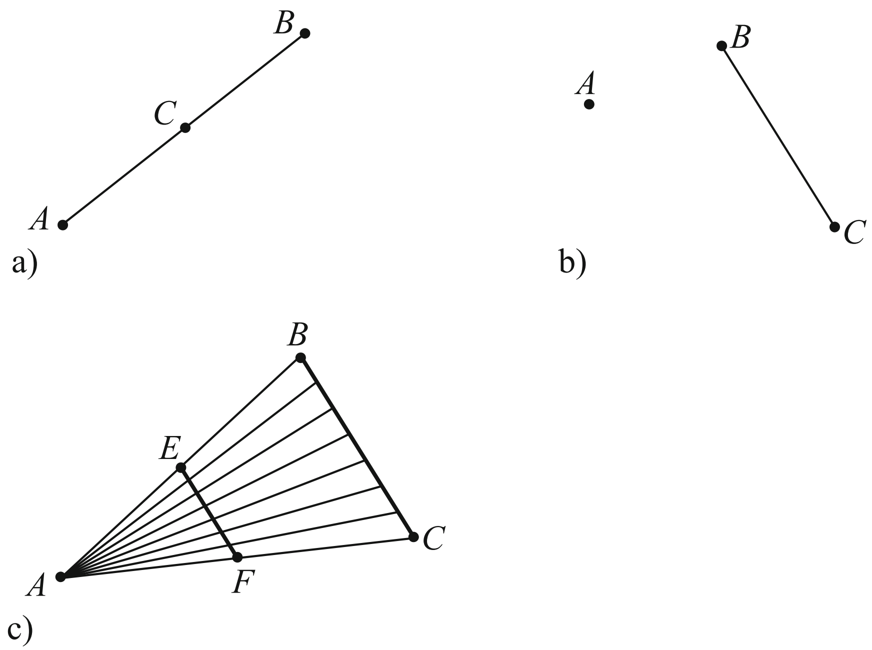


Fig. 2.51

Now let us consider a point A and a segment BC (see Fig. 2.51b). In order to find the Minkowsky sum of A and BC we must mark the midpoint of every segment connecting point A with a point of segment BC. The Minkowsky sum in this case is the segment EF (see Fig. 2.51c). Indeed, the midpoint of any segment connecting A with a point on BC lies on the bimedian EF of the triangle ABC.

Exercise 6. Construct the Minkowsky sum of the two segments AB and CD shown in Fig. 2.52a.

Solution. Let us first find the Minkowsky sum of point A and segment CD. To do this we connect point A with endpoints C and D and mark the midpoint E on segment AD and the midpoint H on segment AC (see Fig. 2.52b). Now

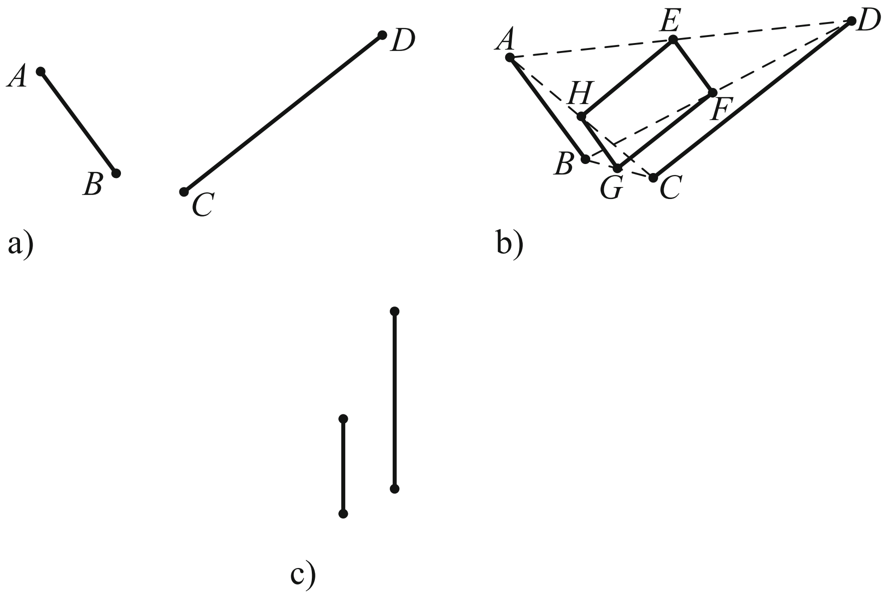


Fig. 2.52

find the Minkowsky sum of point B and segment CD. Similarly, we obtain midpoints F and G. It is possible to prove that quadrilateral $EFGH$ is the Minkowsky sum of segments AB and CD, but we will not do it here. We prefer to pay more attention to constructions and offer a few problems that will allow you to play with the Minkowsky sum.

PROBLEM 23. Find the Minkowsky sum of two segments lying on parallel lines, such as those in Fig. 2.52c.

Hint. Note that by connecting the endpoints of these segments we obtain a trapezoid.

PROBLEM 24. Find the Minkowsky sum of a triangle and a point, such as in Fig. 2.53a.

PROBLEM 25 ($*$) Find the Minkowsky sum of a triangle and a segment, such as in Fig. 2.53b.

9 Parallel projection

In Chapter I, we described the projection from one line to another established by rays originating from the same point. An example of such projection is the

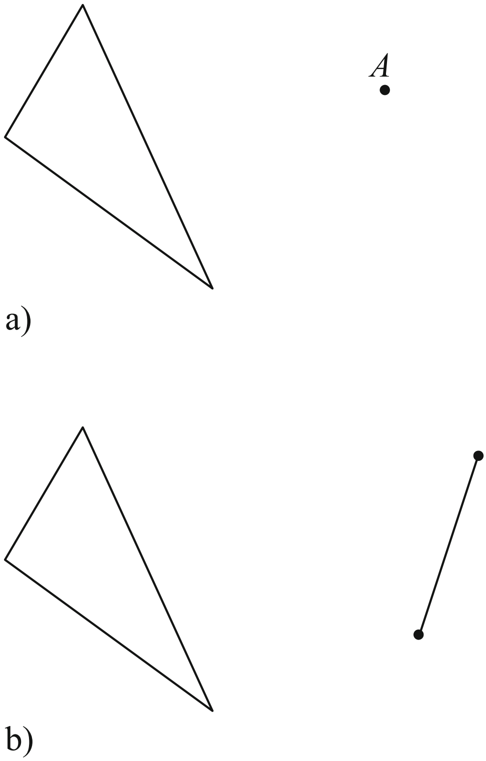


Fig. 2.53

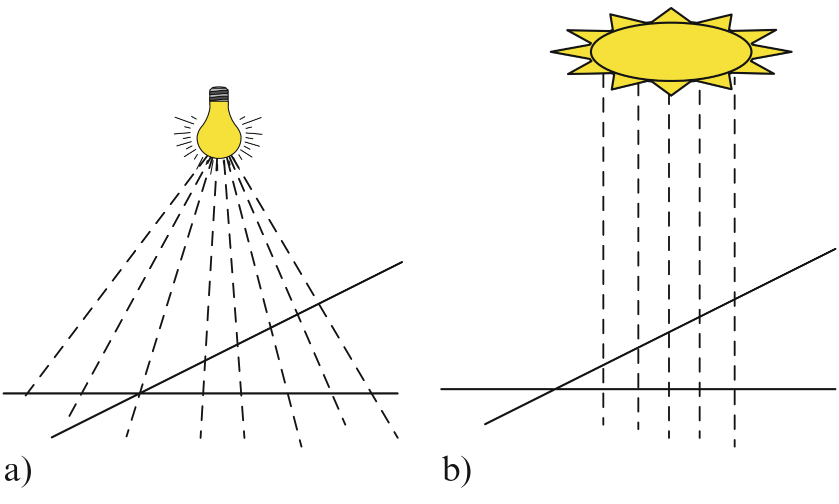


Fig. 2.54

shadow made by rays of light from a lamp or a candle (see Fig. 2.54a). If we move the source of light very far away (think of the sun's rays, for example), then the rays become nearly parallel with high accuracy. Projection from one line to the other, established by parallel rays, is called *parallel projection*. A shadow obtained by the Sun's rays is an example of parallel projection (see Fig. 2.54b).

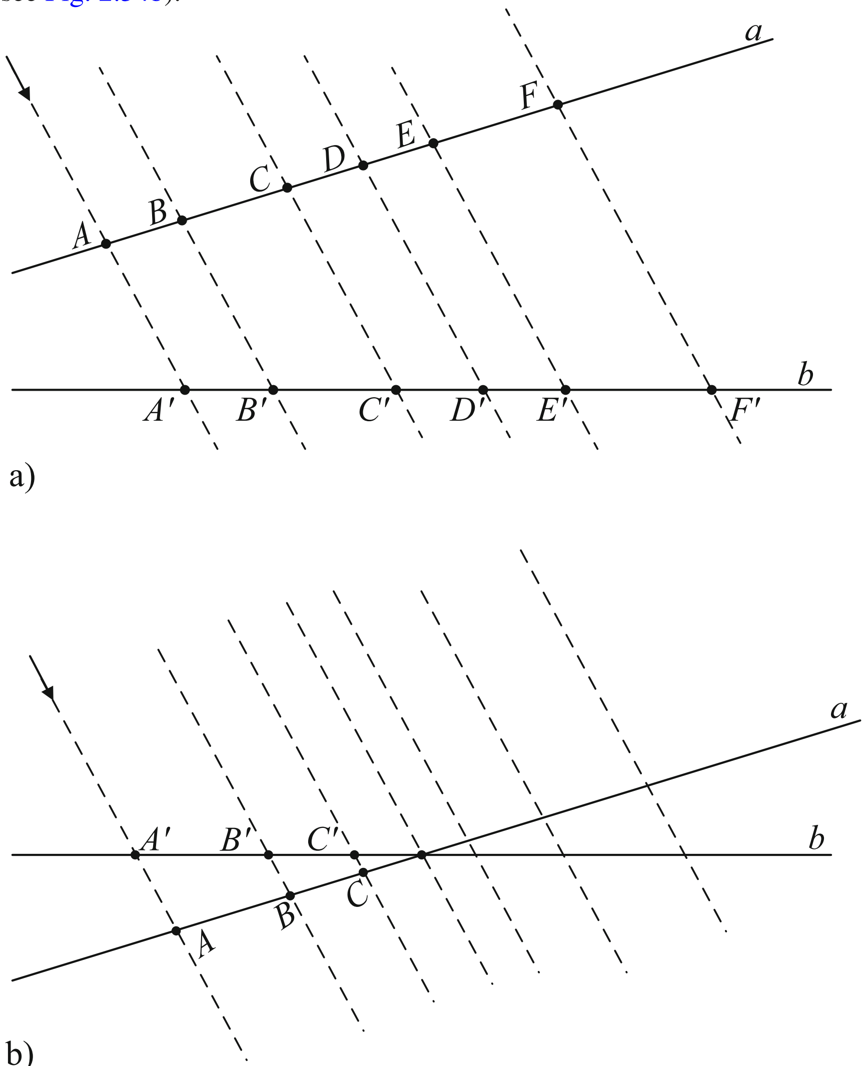


Fig. 2.55

Consider two lines a and b. Draw parallel rays intersecting these lines (see Fig. 2.55a). With the help of these rays we can project any point or any segment on line a onto line b. Points A', B', C', … are the projections of points $A, B, C, \ldots$ respectively. (Compare parallel projection, as in Fig. 2.55a, with projection from a point in Fig. 1.50 from Chapter I.)

We can extend lines a and b beyond their intersection and still define the projection of a point lying on line a onto line b (see Fig. 2.55b). In this case, however, the projection of a point is not the shadow of this point (see Fig. 1.52, and compare with the similar situation for projection from a point).

Remark 6. Unlike projection from a point, parallel projection has the following property: if two segments are equal, then their projections onto the same line are equal. This follows directly from the Lemma. When interpreting projection as a shadow, we may say that two identical objects have equal shadows under the sun's rays, while their shadows under a lamp are not necessarily equal.

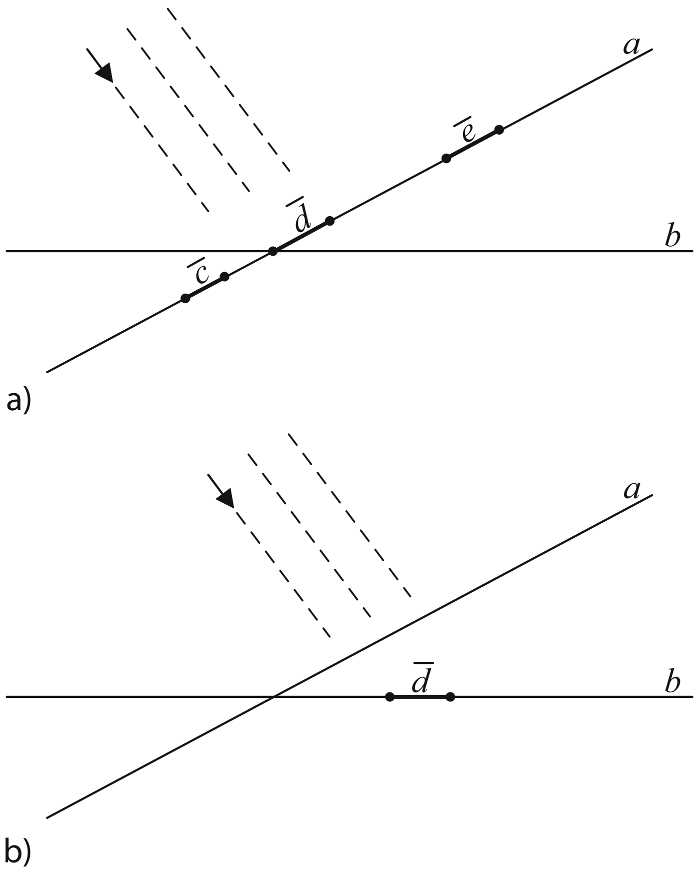


Fig. 2.56

PROBLEM 26.

(a) Draw parallel projections of segments $\bar{c}$, $\bar{d}$, $\bar{e}$ in Fig. 2.56a onto line b. The direction of the parallel rays is indicated by the arrow.

(b) Draw a segment on line a such that its projection by parallel rays is the segment $\bar{d}$ in Fig. 2.56b.

PROBLEM 27. Fig. 2.57a and Fig. 2.57b differ only by the arrows indicating the direction of parallel rays. For each of these figures draw projections of the segments on line a onto line b. Are the projections in Fig. 2.57a and in Fig. 2.57b the same?

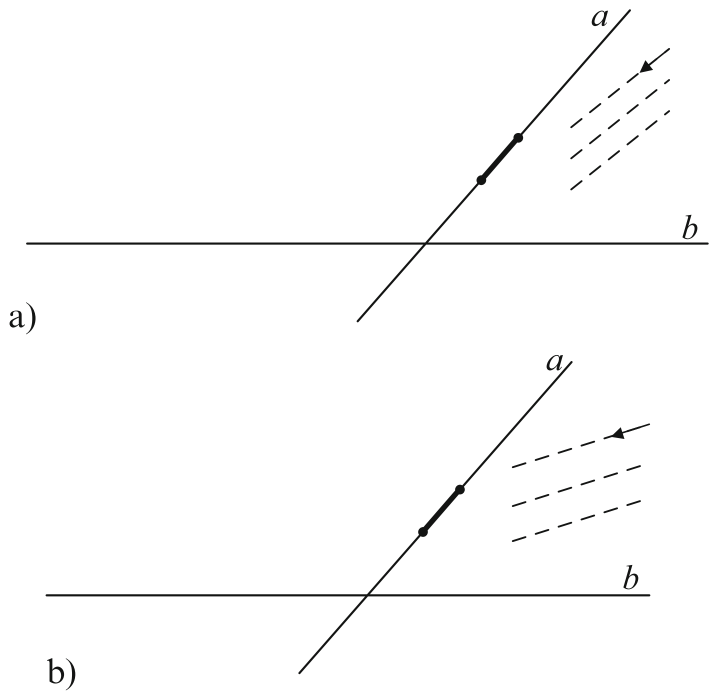


Fig. 2.57

You are probably already familiar with the answer to the problem above. Indeed, when the sun is up high, the shadows are short, and when the sun is close to the horizon, the shadows become very long (see Fig. 2.58 and also Section 1 of the Appendix to Chapter II).

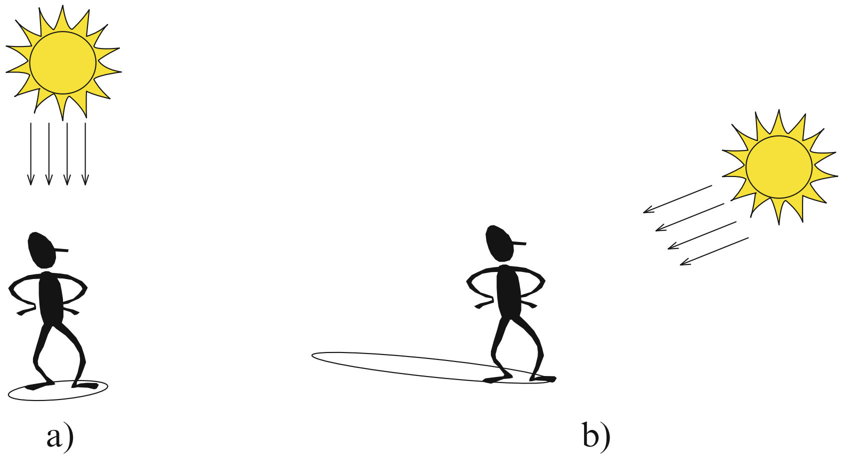


Fig. 2.58

10 Parallel translation

10.1 Parallel translation of a figure

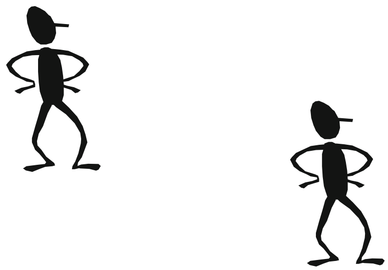


Fig. 2.59

Let us take a look at Fig. 2.59. It has two figures (boys) that are completely alike. In mathematics, we may say that the "first" figure and the "second" are obtained from each other by parallel translation. Let us describe this more precisely.

In order to define a *parallel translation* on the plane it is necessary to choose two points O and O' in a certain order. In Fig. 2.60a we indicate the order of the points by an arrow directed from the first point towards the second.[19]

[19]The directed segment OO' is an example of a vector (see Section 12).

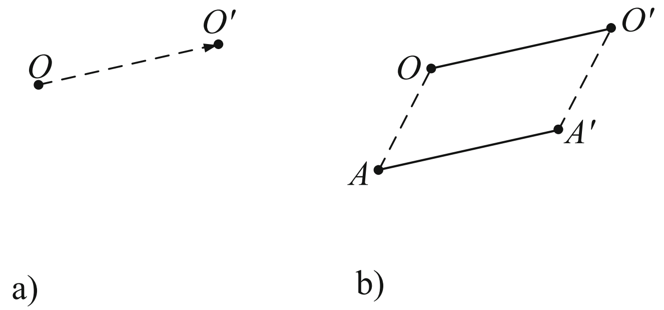


Fig. 2.60

Then the parallel translation defined by points O, O' moves any point A on the plane onto point A' according to the following rule: the point A' is such that quadrilateral $OO'A'A$ is a parallelogram (see Fig. 2.60b).

Point A' is sometimes called the *image of point* A under this parallel translation.

Exercise 7. Fig. 2.61a shows points A and B. Find where these points are translated to under the parallel translation defined by the points O, O'.

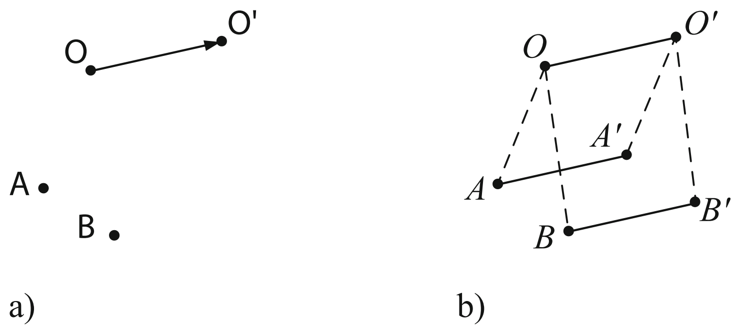


Fig. 2.61

Solution. In order to find images A' and B', we need to construct two parallelograms $OO'A'A$ and $OO'B'B$. This is done in figure Fig. 2.61b.

PROBLEM 28. In Fig. 2.62, points A, B, and C are given. Find where these points are translated to under the parallel translation defined by points O, O'.

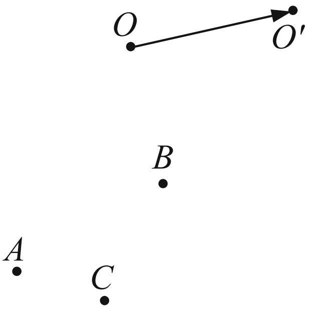


Fig. 2.62

In order to find the parallel translation of a figure, one has to find the parallel translations of every point of this figure. Note that if a figure consists of segments, then it is sufficient to find the parallel translations of each vertex in the figure, i.e., each endpoint of each of the segments. Then connect the obtained vertices accordingly. We are not going to prove this here.

Exercise 8. Find the images of all the points of the triangle $\triangle ABC$ shown in Fig. 2.63a under the parallel translation defined by points O, O'.

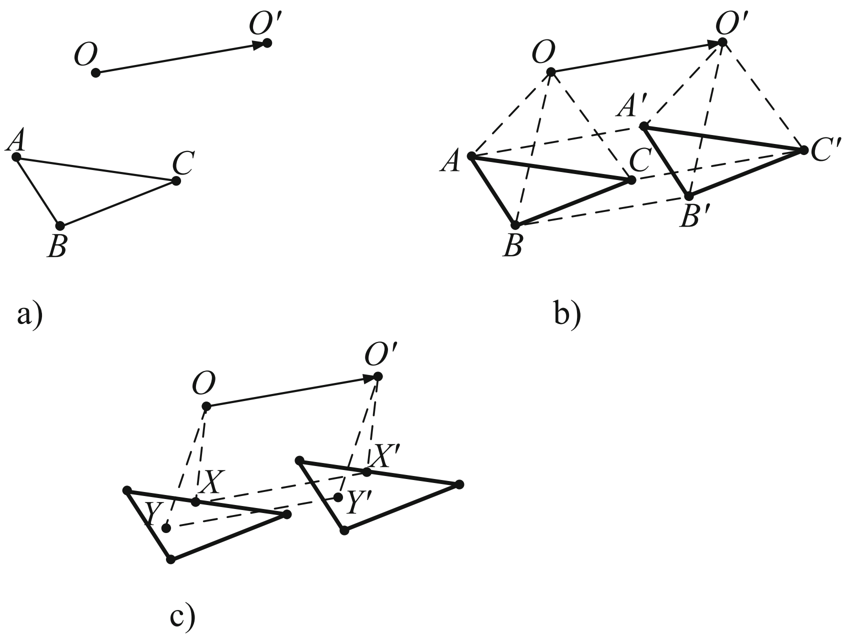


Fig. 2.63

Solution. It is sufficient to find where the vertices of these triangles are translated to by this parallel translation (see Fig. 2.63b). Then for any other point X on the boundary of $\triangle ABC$ or any point Y inside $\triangle ABC$ there will be a corresponding point X' or Y' of the triangle $A'B'C'$ (see Fig. 2.63c).

PROBLEM 29. Fig. 2.64 shows points A, A' and a point X. Find the image of point X under the parallel translation defined by points A, A'.

Fig. 2.64

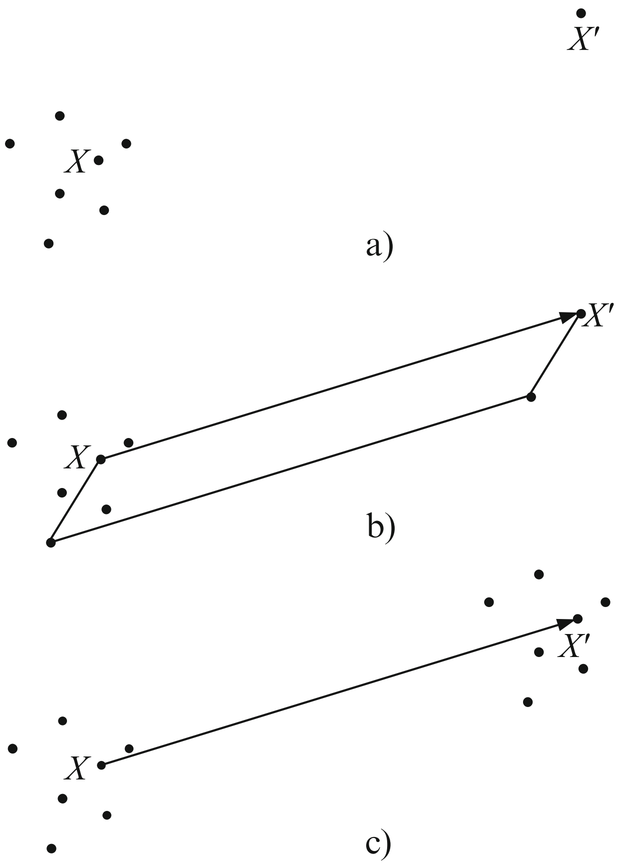


Fig. 2.65

Exercise 9. Fig. 2.65a shows two points X, X' and a group of points. Find where these points are translated to under the parallel translation defined by the points X, X'.

Solution. In order to translate one of the points, we construct a parallelogram (see Fig. 2.65b). Then we do the same for each of the other points (see the answer in Fig. 2.65c).

PROBLEM 30. Perform a parallel translation of the figure below (a triangle with point A inside) so that the point A' will be the image of point A.

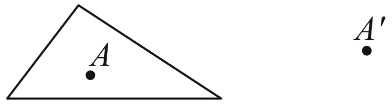


Fig. 2.66

PROBLEM 31. Fig. 2.67 presents a figure and its parallel translation. Point A' is the image of point A. Find the image B' of point B.

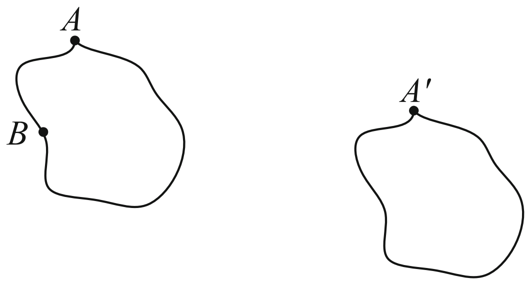


Fig. 2.67

PROBLEM 32. Fig. 2.68 shows four points: A, A', B, and B'. Can points A' and B' be images of the points A, B respectively under the same parallel translation?

Fig. 2.68

PROBLEM 33.

(a) Check that one of the two gloves in Fig. 2.69a is obtained from the other by the parallel translation defined by the points O, O'.

(b) Check that the two gloves in Fig. 2.69b cannot be obtained from one another by any parallel translation on the plane.

Fig. 2.69

PROBLEM 34. Fig. 2.70 shows several figures. Which of them can be obtained from another by a parallel translation and which cannot?

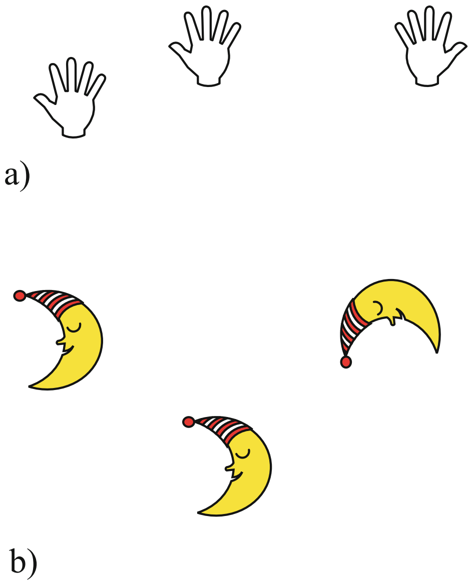


Fig. 2.70

10.2 Translation of the plane

According to our definition of a parallel translation, if we are given a pair of points O, O', we can find the image of any point of the plane under this parallel translation. Thus, besides finding where a certain figure is translated to under a parallel translation, we can find where an unbounded domain or even the whole plane is translated to.

PROBLEM 35. Consider the parallel translation defined by the points A, A' in Fig. 2.71.

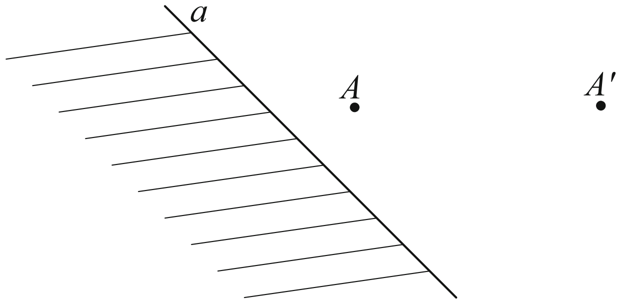


Fig. 2.71

Find where the half-plane to the left of the line a (shaded in Fig. 2.71) is translated to by this parallel translation.

Sum of the exterior angles of a polygon

With parallel translation on the plane as a tool, it is easy to prove the following important statement.

Proposition 12. The sum of the exterior angles[20] of a polygon is equal to a perigon.[21] (It is assumed that at each vertex only one corresponding exterior angle is considered.)

Proof. Fig. 2.72a shows a polygon. For each interior angle of this polygon let us mark one exterior angle,[22] as in Fig. 2.72b. Imagine "walking" counterclockwise along the boundary of the polygon. At each vertex measure the exterior angle from the extension of the side we were on to the new side.

[20]Exterior angles were defined in Chapter I.

[21]See Glossary.

[22]As we already know, one can associate two exterior angles with each interior angle of a polygon.

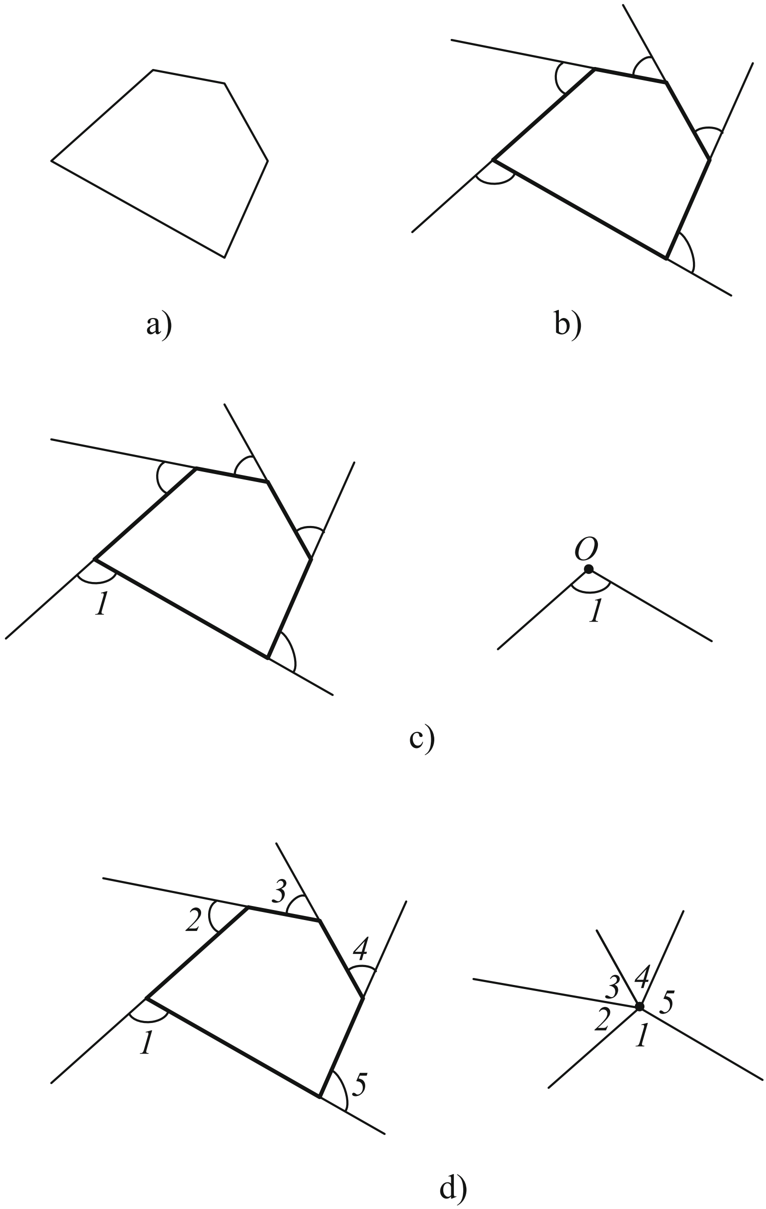


Fig. 2.72

Consider a point O outside of a polygon (see Fig. 2.72c). Let us make parallel translations of all the exterior angles of this polygon in such a way that the vertex of each angle will be translated onto the point O. For example, in Fig. 2.72c you can see one exterior angle (numbered 1) translated in such a way that its vertex moves onto the point O. Similarly, we translate the angle numbered by 2. Note that the angles 1 and 2 have one common side. Therefore they will be adjacent when the vertex of the angle 2 is translated onto the point O. We continue translating all the exterior angles one after another. Since every neighboring pair of these angles has a common side, they will all be sequentially adjacent at point O (see Fig. 2.72d). Therefore, these angles will not overlap or fall short. They fill in the whole plane. We have obtained a perigon. □

Defining the same parallel translation by indicating different pairs of points

Is it possible to define the same parallel translation of the plane as in Problem 35 by using different pairs of points instead of the points A, A'? The answer is "yes." Then what pairs of points can be chosen for this? The answer is given by the proposition below.

Proposition 13. Let points A, A' define a parallel translation of the plane. Two points B, B' define the same parallel translation of the plane as points A, A' if and only if $AA'B'B$ is a parallelogram.

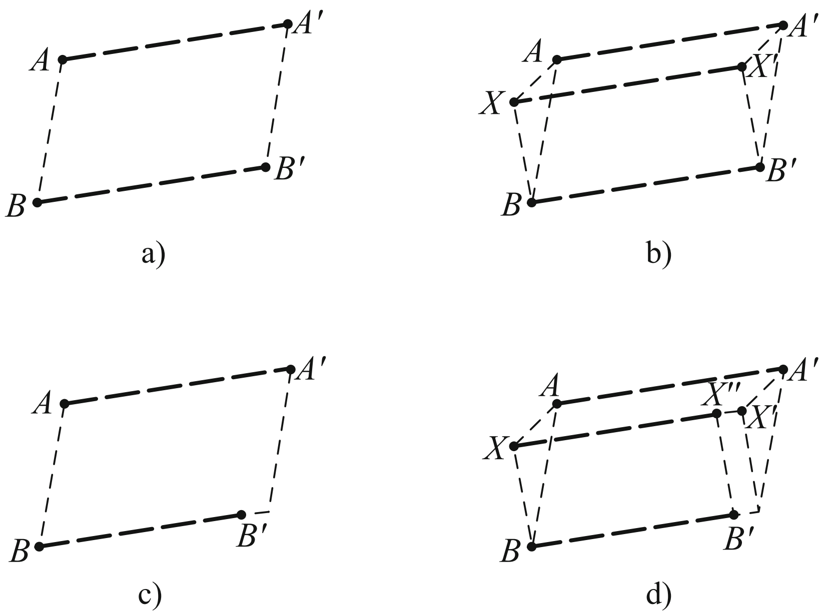


Fig. 2.73

Note that in using the notation $AA'B'B$ for a parallelogram, we are implicitly assuming that segments AA' and BB' have the same direction. You can find more details and the proof[23] of Proposition 13 in Section 2.3 of the Appendix to Chapter II. Let us illustrate this proposition.

Fig. 2.73a shows points A, A' and B, B', such that $AA'B'B$ is a parallelogram. Then $AA' = BB'$ and $AA' \parallel BB'$. As we can see from Fig. 2.73b, a point X is translated onto point X' both by the translation defined by A, A' and by the translation defined by B, B'.

Fig. 2.73c shows points A, A' and B, B' such that $AA' \parallel BB'$ but $AA' \neq BB'$. As we can see from Fig. 2.73d, a point X is translated onto the point X' by the translation defined by A, A'. However, by the translation defined by B, B' the point X is translated onto the point X'' different from X'.

Let us consider a simple example.

Exercise 10. Fig. 2.74a, b, c shows a figure (a small triangle) and pairs of points such that segments AA', BB', and CC' are parallel and equal. Given that these pairs of points define parallel translations, find where the figure will be translated to in each case.

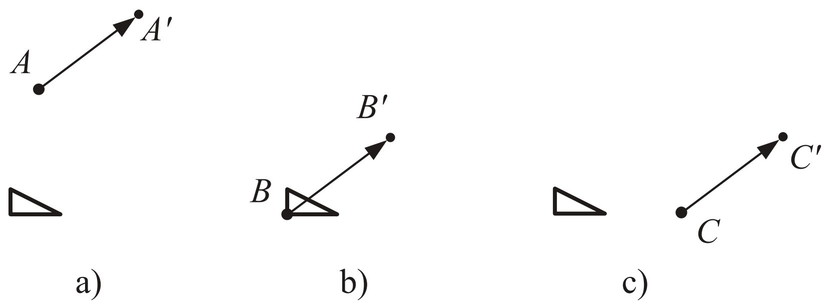


Fig. 2.74

[23] In Proposition 13 there are in fact two statements. The expression that “hides” them and make them look like one is *if and only if*. This expression has a precise mathematical meaning and is widely used.

Solution. The solution is presented in Fig. 2.75a, b, c. Each of these three diagrams shows the same starting position for the figure (at the lower left), and as you can see, each of these parallel translations translates the figure to the same place.

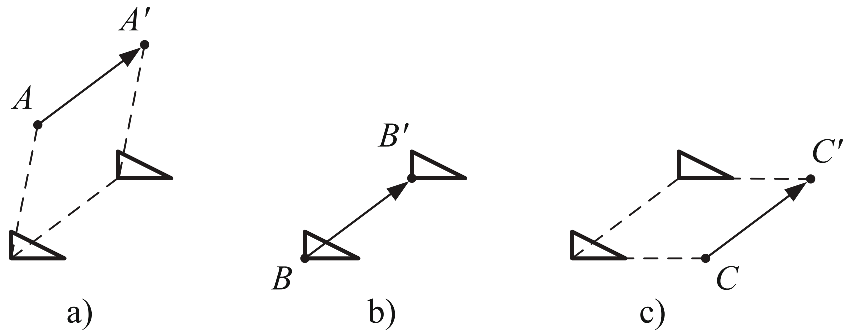


Fig. 2.75

10.3 Parallel translation on a line

We have considered parallel translation on the plane. We can also define a parallel translation on a straight line (also called a *parallel shift* or simply a *shift*), though there are not many different figures that we can draw on a line.

As on the plane, two points A, A' on a line define a parallel shift. A point B is shifted to the point B' by this parallel shift if the segment AB is equal to the segment $A'B'$. It is important that both segments are "directed" in the same way, e.g., if the point B is to the right of the point A, then the point B' must also be to the right of the point A'.

Exercise 11. Consider a line a with two points A and A' on it (see Fig. 2.76a). Given that these points A and A' define a parallel shift on the line, find the image of segment AB in Fig. 2.76b.

Solution. We need to mark a segment equal to segment AB to the right of the point A'. As we already know (from Section 4), in order to do this we need to draw an additional line parallel to line a. The construction of segment $A'B'$ is shown in Fig. 2.76c.

PROBLEM 36. It is given that points A, A' define a parallel shift on the line. Perform a parallel shift of segment BC in Fig. 2.77a and in Fig. 2.77b.

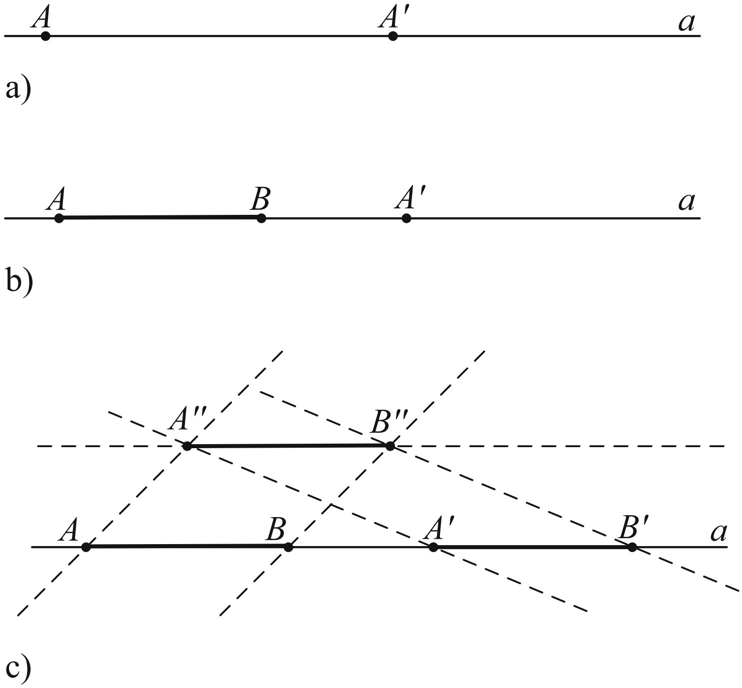


Fig. 2.76

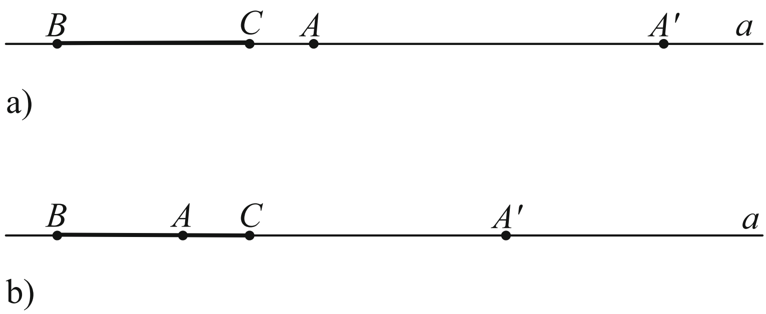


Fig. 2.77

11 Central symmetry on the plane

There are two types of symmetries on the plane:

(1) *symmetry with respect to a line*
(also called *reflection* or *line symmetry*);

(2) *symmetry with respect to a point*
(also called *central symmetry* or *point symmetry*).

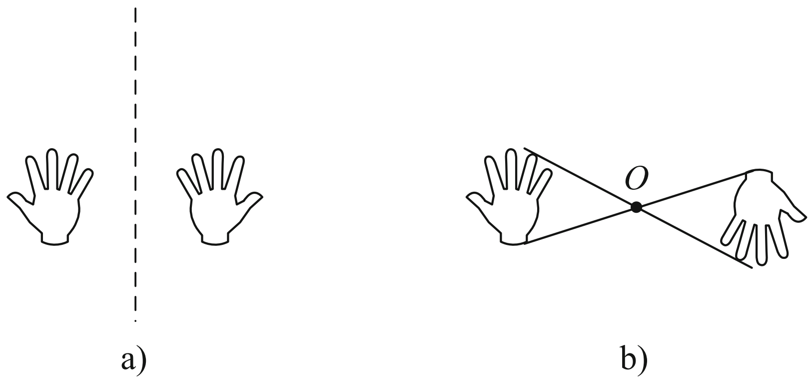


Fig. 2.78

In Fig. 2.78a there are two figures (gloves) symmetric with respect to a line, and in Fig. 2.78b there are two figures (gloves) symmetric with respect to a point. As you may notice, both gloves in Fig. 2.78b are for the left hand, while in Fig. 2.78a one glove is left-handed and the other is right-handed.

Symmetry with respect to a line will be studied in Chapter IV, since it requires the notion of perpendicular lines (see Fig. 2.79), which cannot be defined in this chapter.

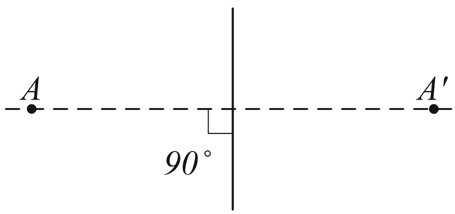


Fig. 2.79

Let us describe symmetry with respect to a point or *central symmetry*. First, we choose a point O, the *center of symmetry*. Consider a point A

(see Fig. 2.80a). The point A' symmetric to point A with respect to point O is shown in Fig. 2.80b. In order to find it, we draw line AO and construct segment OA' such that $OA' = AO$ and point A' lies on the line, but on the other side of point O. Point A' is called the *image of A* under central symmetry with respect to O.

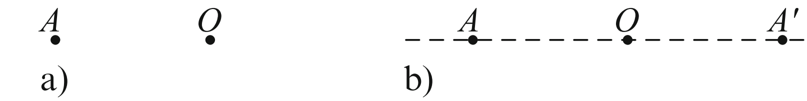


Fig. 2.80

Suppose we have a figure in the plane and a point O. In order to construct the figure symmetric with respect to point O, we need to find the symmetric point for each point of the initial figure with respect to O.

Exercise 12. Given a triangle $\triangle ABC$ and a point O (see Fig. 2.81a), construct the symmetric triangle $\triangle A'B'C'$ with respect to the point O.

Solution. Let us find the points A', B', and C' symmetric to the vertices A, B, and C of $\triangle ABC$ (see Fig. 2.81b). The triangle $\triangle A'B'C'$ is symmetric to the triangle $\triangle ABC$ with respect to O.

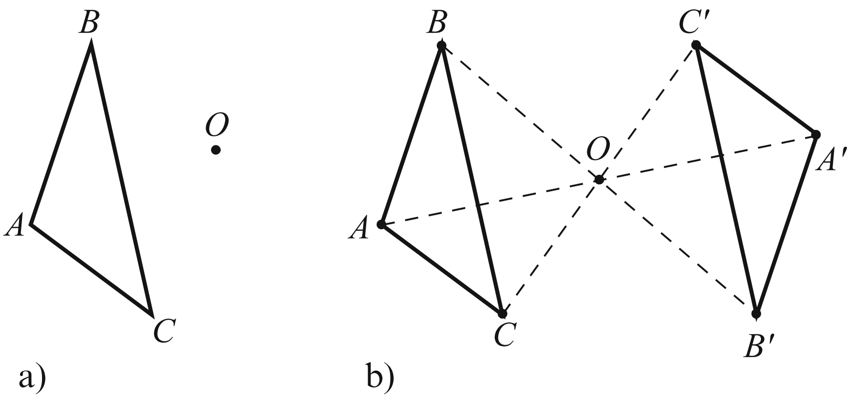


Fig. 2.81

Remark 7. Just as in parallel translation, in order to construct the symmetric triangle with respect to a point, it is sufficient to find the image of each vertex of the triangle.

Exercise 13. Fig. 2.82a shows two triangles that are symmetric with respect to a certain point O. Find the point O.

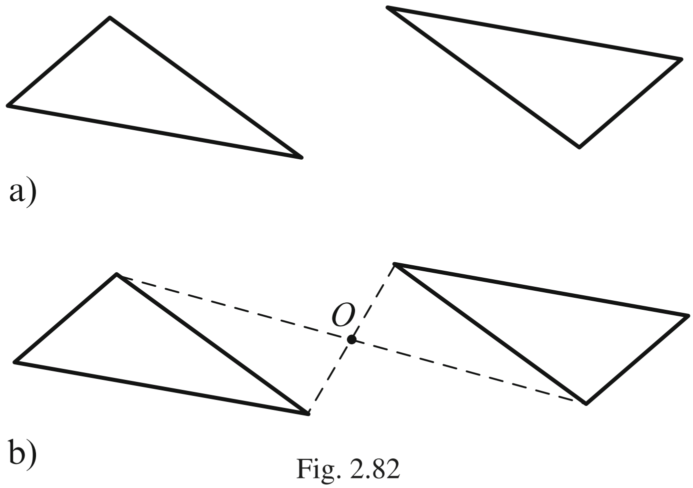


Fig. 2.82

Solution. Let us connect the vertices of these triangles so that these segments will intersect (see Fig. 2.82b). The point of their intersection is the point O.

PROBLEM 37. Fig. 2.83a and b show two cases of a quadrilateral and a point O. Draw a quadrilateral symmetric to the one given, with respect to O.

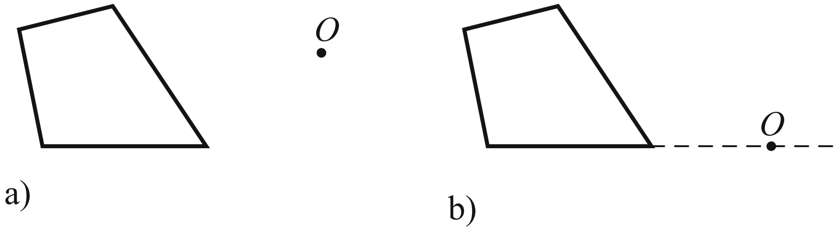


Fig. 2.83

PROBLEM 38. Draw a figure symmetric to the one in Fig. 2.84 (a triangle with a point A inside it) with respect to the point O. (Compare with Problem 30.)

PROBLEM 39. Consider a triangle ABC. Let O be the midpoint of BC. Through point O, draw line OP parallel to AC and line OQ parallel to AB (see Fig. 2.85).

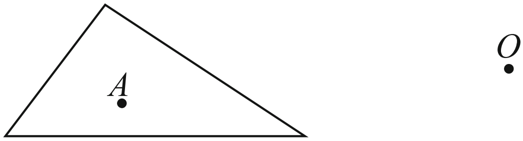


Fig. 2.84

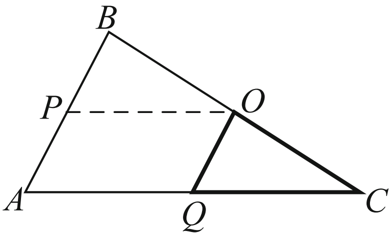


Fig. 2.85

(a) Construct the triangle symmetric to $\triangle OCQ$ with respect to the point O.

(b) What figure is formed if triangle $\triangle OCQ$ is cut out from $\triangle ABC$ and the triangle symmetric to $\triangle OCQ$ is added instead?

Consider a figure and a point O (see Fig. 2.86a). How can we describe the figure symmetric to it with respect to point O?

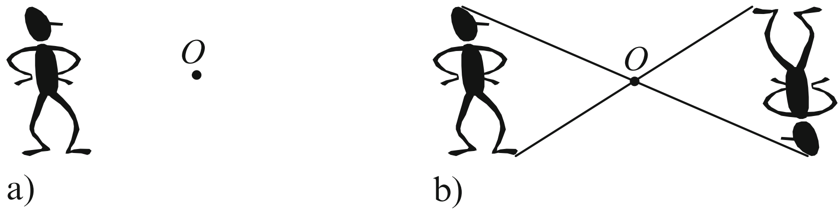
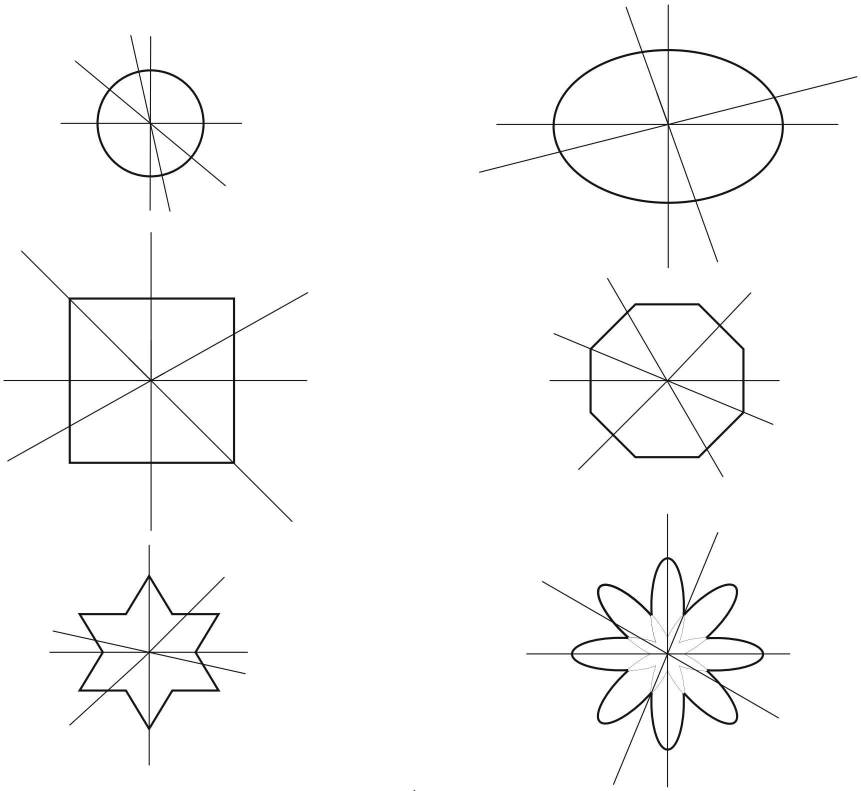
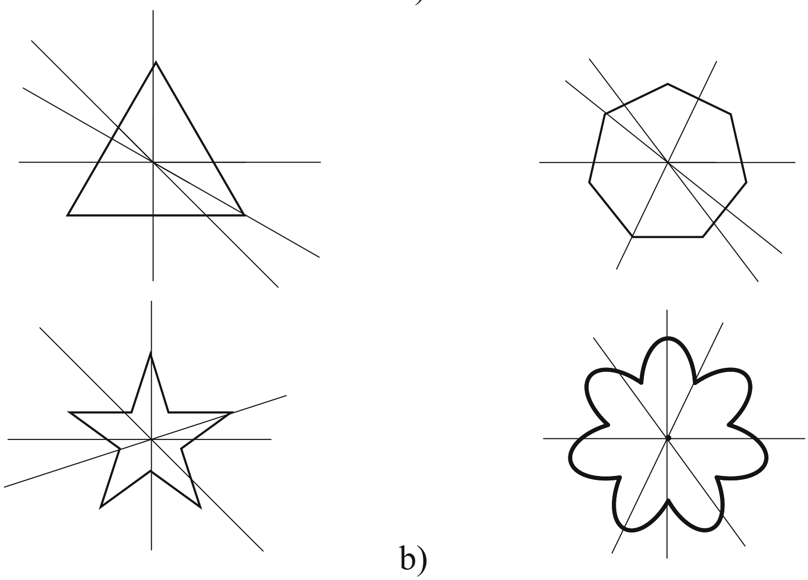


Fig. 2.86

Such a figure is shown in Fig. 2.86b. As you can see, the figure is turned upside down and is also facing the other way. Compare these figures with those arising from parallel translation (Fig. 2.59).



a)



b)

Fig. 2.87

A figure can be symmetric to itself with respect to a certain point. Such a figure is called a *centrally symmetric figure* (or a *self-symmetric* figure). Examples are given in Fig. 2.87a. Note that all regular polygons with an even number of sides are centrally symmetric figures, while those with an odd number of sides are not. In Fig. 2.87b there are examples of figures that are not self-symmetric.

Proposition 14. The center of a parallelogram (i.e., the point of intersection of its diagonals) is the center of symmetry of this parallelogram.

Proof. For any point A on a parallelogram, let us draw a line passing through the center O of the parallelogram. Then according to Proposition 8 a point symmetric to point A with respect to O also lies on the parallelogram. Therefore, point O is a center of symmetry of the parallelogram.

Note that the proposition also requires us to prove that this center of symmetry is unique. We leave this for you to prove. □

Remark 8. If we consider only points along a single line and not all those in a plane, we can define "central symmetry" on a line. We leave this as an exercise: define the point A' symmetric to point A with respect to point O in the case where all these points lie on a straight line.

PROBLEM 40. A segment AB and a point O on the same line are given.

(a) Construct the segment $A'B'$ symmetric with respect to the point O.

(b) Make a parallel shift of segment AB defined by point O and another point O' on the same line. Compare the new segment with segment $A'B'$ in part (a).

11.1 Sequences of parallel translations and central symmetries. The relation between central symmetry and parallel translation

Exercise 14. Consider a triangle with all its bimedians. Four new triangles are formed (see Fig. 2.88a).

(a) Which of these triangles can be obtained from each other by parallel translation?

(b) Which pairs of these triangles cannot be obtained from each other by parallel translation? Can any of these be obtained from each other with the help of central symmetry?

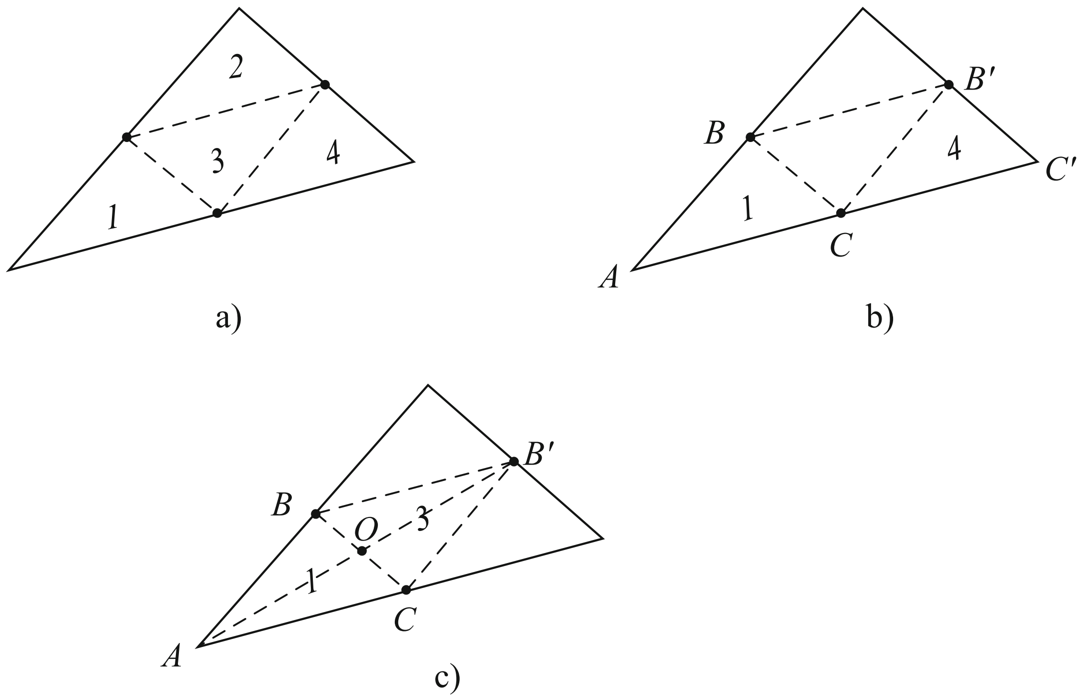


Fig. 2.88

Solution.

(a) It looks like the triangles $\triangle_1$, $\triangle_2$, $\triangle_4$ are parallel translations of one another. First, let us check that $\triangle_1$ is a parallel translation of $\triangle_4$.

We need to define the parallel translation by indicating two points. Let us denote the vertices of $\triangle_1$ by A, B, C. Consider the parallel translation defined by C, C' (see Fig. 2.88b). Then point B is translated to point B', because $ABB'C$ is a parallelogram (see Section 6.1). Since $AC = CC'$, point A is translated to point C.

Therefore, by the parallel translation C, C' the triangle ABC (or $\triangle_1$) is translated onto the triangle BCC' (or $\triangle_4$). Similarly, we can prove that triangles $\triangle_2$ and $\triangle_4$ are parallel translations of one another.

(b) Triangle $\triangle_3$ is not a parallel translation of any of the other triangles. However, $\triangle_3$ can be obtained from $\triangle_1$ by central symmetry. Indeed, the figure $ABB'C$ is a parallelogram and according to Proposition 14 its center O is the center of symmetry of $ABB'C$ (see Fig. 2.88c). This means that with respect to point O, the points A and B' are symmetric to each other as are the points B and C, i.e., $\triangle_1$ is centrally symmetric to $\triangle_3$ with respect to the point O.

PROBLEM 41. Refer again to Fig. 2.88a.

(a) Indicate the parallel translation which translates $\triangle_1$ onto $\triangle_2$.

(b) Indicate the parallel translation which translates $\triangle_2$ onto $\triangle_4$.

PROBLEM 42. Refer again to Fig. 2.88a.

(a) Indicate the center of symmetry for triangles $\triangle_2$ and $\triangle_3$.

(b) Indicate the center of symmetry for triangles $\triangle_3$ and $\triangle_4$.

PROBLEM 43. Fig. 2.89a shows a triangle with bimedians in it. Let us use the notations in Fig. 2.89b. Connect points A and B' and points B and C'. Take triangle ABC and find the triangle symmetric to it with respect to point O. Then take the latter triangle and obtain the triangle symmetric to it with respect to point O'. What triangle have you obtained? How else could it be obtained from triangle ABC?

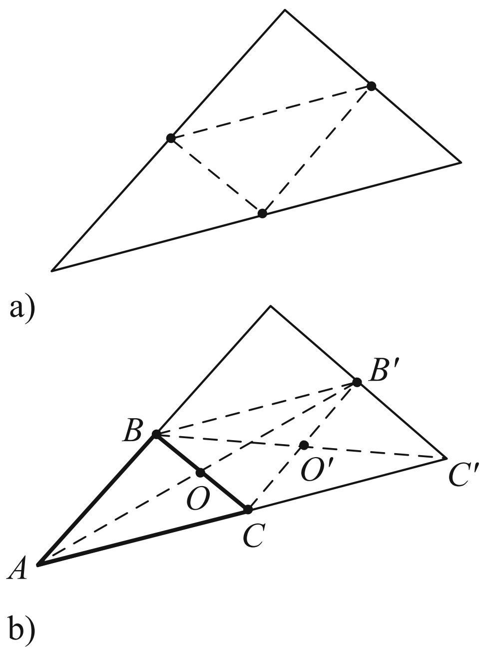
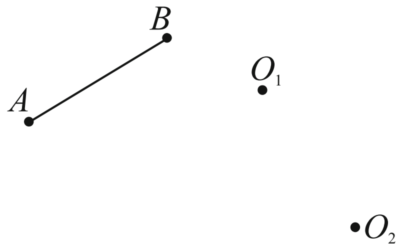


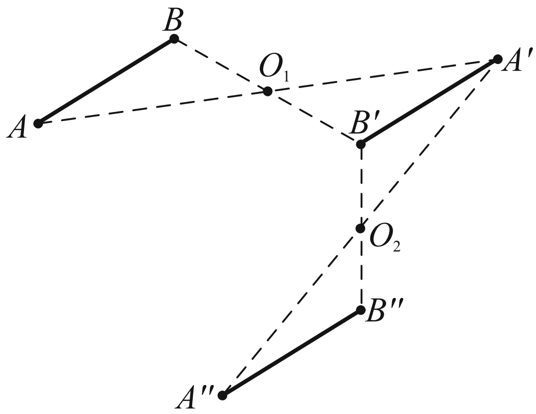
Fig. 2.89

In the problem above, you were actually performing two operations of central symmetry consecutively. In mathematics such an operation is called the *composition* (or *product*) *of two central symmetries*.

More generally, suppose F is a figure. If we apply an operation $\mathscr{A}$ to this figure and obtain the figure $F' = \mathscr{A}(F)$ and then apply the operation $\mathscr{B}$ to the figure F', we say that we have a performed the *composition of two operations* $\mathscr{A}$ *and* $\mathscr{B}$. This is usually written as $\mathscr{B} \cdot \mathscr{A}$ or $\mathscr{B}(\mathscr{A}(F))$. The operation $\mathscr{A}$ was applied first to the figure F, then the operation $\mathscr{B}$ was applied to the figure $F' = \mathscr{A}(F)$.



a)



b)

Fig. 2.90

Let us consider another example.

Exercise 15. Consider a segment AB and two points O_1 and O_2 (see Fig. 2.90). Draw the segment symmetric to AB with respect to point O_1, and then apply the symmetry with respect to point O_2 to the segment you have drawn.

Solution. The segment symmetric to AB with respect to the point O_1 is $A'B'$ and the segment symmetric to the segment $A'B'$ with respect to the point O_2 is $A''B''$ (see Fig. 2.90b).

Exercise 16. Could the segment $A''B''$ in Fig. 2.90b be obtained from the segment AB in "one step"? If yes, how?

Solution. It appears that the segment $A''B''$ is just a parallel translation of segment AB. Let us check if this is true.

For this we need to show that the figure $ABB''A''$ (see Fig. 2.91) is a parallelogram.

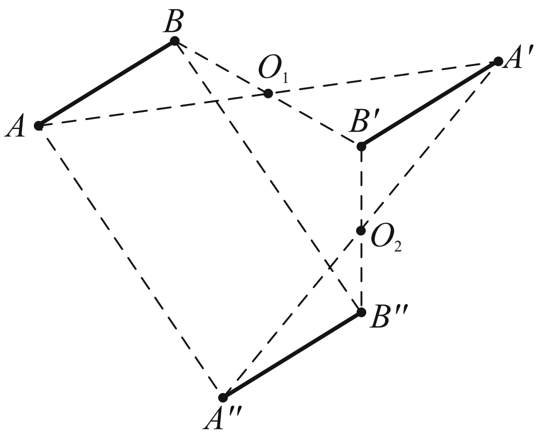


Fig. 2.91

Figure $ABA'B'$ is a parallelogram (see Problem 5 in Section 5.4), because $AB \parallel A'B'$ and $AB = A'B'$. (Note, however, that segment $A'B'$ is not a parallel translation of the segment AB since the endpoints have switched places).

Similarly, figure $A'B''A''B'$ is also a parallelogram (but again $A''B''$ is not a parallel translation of $A'B'$). Due to transitivity of parallelism and equality, we have $AB \parallel A''B''$ and $AB = A''B''$. Therefore, $ABB''A''$ is also a parallelogram.

Is this a coincidence, or is the following statement true?

Proposition 15. Let a figure F and two points O_1 and O_2 be given. To the figure F, we apply the central symmetry with respect to the point O_1 and then apply the central symmetry with respect to the point O_2 to the new figure. The figure obtained is a parallel translation of the initial figure.

This statement is true. We will not prove it in the general case. We have already proved it for a segment in the above exercise. Moreover, the converse statement is also true: if there is a parallel translation of a figure it can also be represented as a product of two central symmetries with respect to certain points. (Note that this representation is not unique.)

PROBLEM 44. Consider a figure F and two points O_1 and O_2 (see Fig. 2.92).

Fig. 2.92

(a) Construct figure F' symmetric to figure F with respect to point O_1.

(b) Construct figure F'' symmetric to figure F' with respect to point O_2.

PROBLEM 45. Consider figure F and the two points O_1 and O_2 in Fig. 2.92.

(a) Construct figure F_1 symmetric to figure F with respect to point O_2.

(b) Construct figure F_2 symmetric to figure F_1 with respect to point O_1.

PROBLEM 46. Compare the figures obtained in Problem 44 and Problem 45. Do they coincide?

We have considered the composition of two symmetries with respect to a point. We already know another important operation: parallel translation of a figure. Let us consider the composition of two parallel translations.

Exercise 17. Fig. 2.93 shows a figure F and three points A_1, A_2, and A_3.

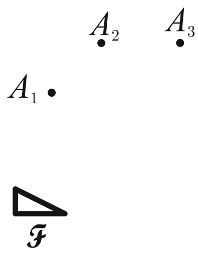


Fig. 2.93

(a) Apply the parallel translation defined by points A_1, A_2 to figure F. Let the new figure be F'.

(b) Apply the parallel translation defined by the points A_2, A_3 to the figure F'. Let the new figure be F''.

(c) Is it possible to translate figure F onto figure F'' by a single parallel translation? If yes, make the necessary construction.

Solution.

(a) See Fig. 2.94a.

(b) See Fig. 2.94b.

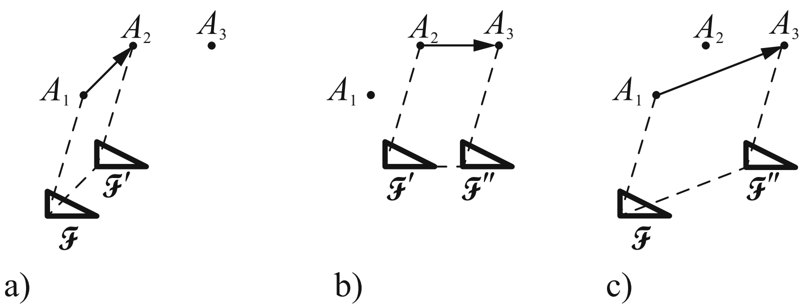


Fig. 2.94

(c) Yes, a single parallel translation that translates figure F onto figure F'' can be defined by the points A_1, A_3; see Fig. 2.94c.

Exercise 18. Consider again Fig. 2.93.

Exercise 18. Consider again Fig. 2.93.

(a) Apply the parallel translation defined by the points A_2, A_3 to figure F. Let the translated figure be F_1.

(b) Apply the parallel translation defined by the points A_1, A_2 to figure F_1. Let the translated figure be F_2.

(c) Is it possible to translate figure F onto figure F_2 by a single parallel translation? If yes, make the necessary construction.

Solution.

(a) See Fig. 2.95a.

(b) See Fig. 2.95b.

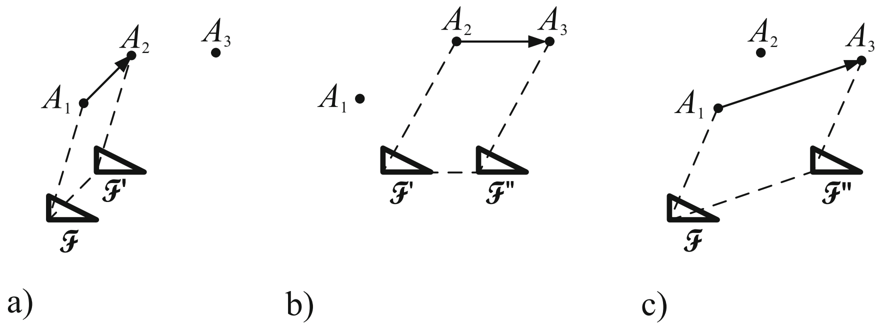


Fig. 2.95

(c) Yes, a single parallel translation that translates figure F onto figure F_2 can be defined by the points A_1, A_3 (see Fig. 2.95c).

Note that figure F'' in Exercise 17 coincides with figure F_2 in Exercise 18.

For those of you who like formulas, we can summarize the main observations of this section in the following way:

Let $\mathscr{S}_1, \mathscr{S}_2$ denote central symmetries and $\mathscr{P}_1, \mathscr{P}_2$ denote parallel translations. Then

$\mathscr{S}_1 \cdot \mathscr{S}_2$ is a parallel translation (Proposition 15);

In general $\mathscr{S}_1 \cdot \mathscr{S}_2 \neq \mathscr{S}_2 \cdot \mathscr{S}_1$ (Problem 46);

$\mathscr{P}_1 \cdot \mathscr{P}_2$ is a parallel translation (Exercises 17c, 18c);

$\mathscr{P}_1 \cdot \mathscr{P}_2 = \mathscr{P}_2 \cdot \mathscr{P}_1$ (Solution of Exercise 18c).

12 Vectors

12.1 Vectors and parallel translations

A *vector* is a directed segment. A vector is usually denoted as $\overrightarrow{a}$ or as $\overrightarrow{AB}$ (see Fig. 2.96a). For a vector $\overrightarrow{AB}$ the point A is called the *beginning* of this vector and the point B is called the *end* of this vector.

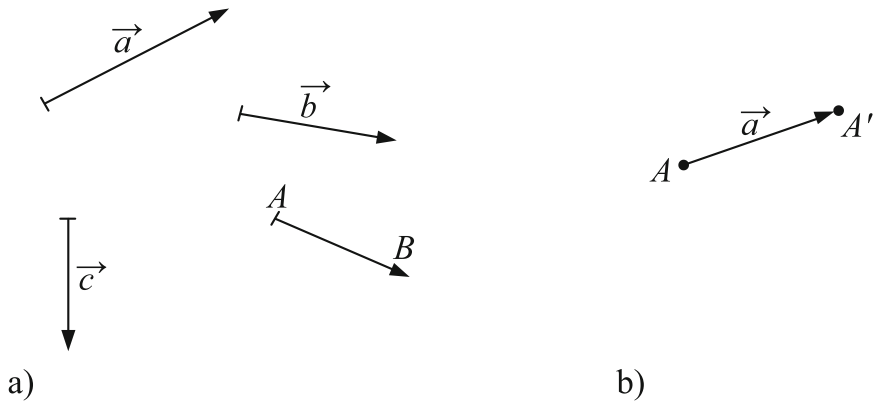


Fig. 2.96

In fact, we were already dealing with vectors in Section 10 (see Fig. 2.60). Indeed, we can now say, that a parallel translation in the plane is defined by a vector (in Fig. 2.96b the point A is translated onto the point A' by the vector $\overrightarrow{a}$).

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{A_1B_1}$ are considered equal if $AB \parallel A_1B_1$, $AB = A_1B_1$, and the direction of the arrows is the same (see Fig. 2.97a).

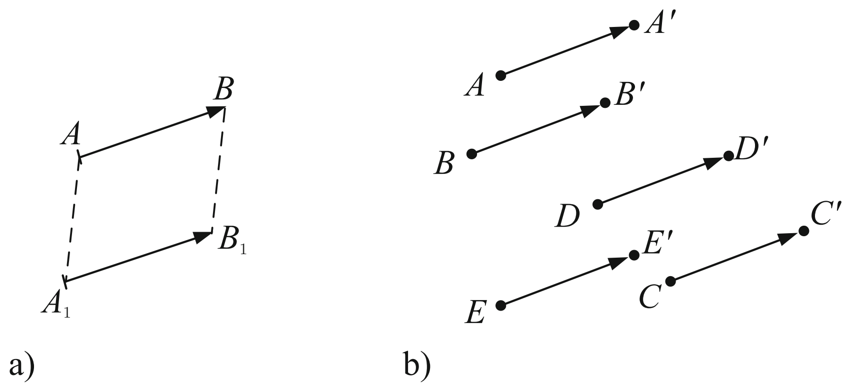


Fig. 2.97

Continuing our comparison with parallel translation, we can conclude (see Proposition 13) that equal vectors define the same parallel translation on the plane (Fig. 2.97b).

PROBLEM 47. Consider a vector $\overrightarrow{AB}$ and a point A_1, Fig. 2.98. Construct the vector equal to $\overrightarrow{AB}$ with the beginning point A_1.

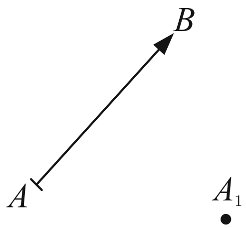


Fig. 2.98

12.2 Addition of vectors

Exercise 19. Find where point A is translated to under the parallel translation defined by the vector below (Fig. 2.99a).

Fig. 2.99

Solution. Point A is translated onto point A' (see Fig. 2.99b).

Exercise 20. Fig. 2.100 shows a point A and two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

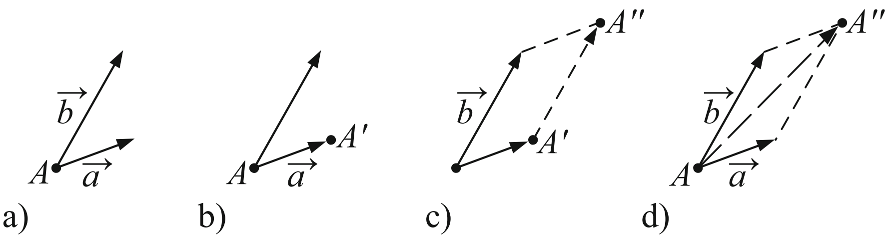


Fig. 2.100

(a) Apply the parallel translation defined by $\overrightarrow{a}$ to point A, and then apply the parallel translation defined by $\overrightarrow{b}$[24] to the point A' obtained. Let the resulting point be A''. Find it. Indicate by the vector $\overrightarrow{AA''}$ the result of the composition of these two parallel translations.

(b) Apply the parallel translation defined by $\overrightarrow{b}$ to point A, and then apply the parallel translation defined by $\overrightarrow{a}$ to the point A' obtained. Find the resulting point A''. Mark the vector $\overrightarrow{AA''}$.

(c) Compare the answers for (a) and (b).

Solution.

(a) In Fig. 2.100b, point A is translated onto point A' by vector $\overrightarrow{a}$. In Fig. 2.100c, point A' is translated onto point A'' by vector $\overrightarrow{b}$.

We can see (Fig. 2.100d) that the vector $\overrightarrow{AA''}$ is the diagonal of a parallelogram with sides $\overrightarrow{a}$ and $\overrightarrow{b}$.

(b) The point A is translated by vector $\overrightarrow{b}$ onto point A' (see Fig. 2.101a). The point A' is translated by vector $\overrightarrow{a}$ onto point A'' (see Fig. 2.101b). The vector $\overrightarrow{AA''}$ is the diagonal of a parallelogram with sides $\overrightarrow{a}$ and $\overrightarrow{b}$.

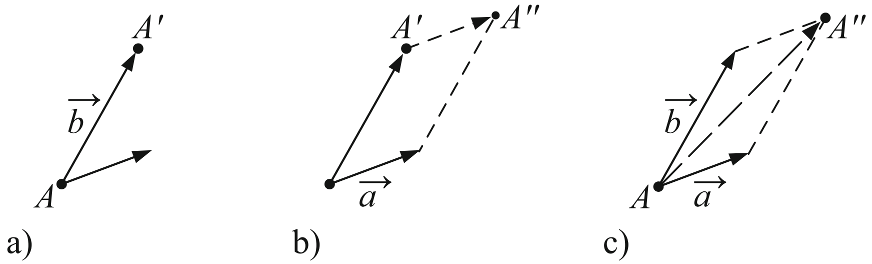


Fig. 2.101

(c) We see that point A'' is the same in the two cases (a) and (b), and so is the vector $\overrightarrow{AA''}$.

[24] As mentioned in Section 11.1 of this chapter, such an operation is called the composition of two parallel translations.

In this exercise we have proved the following theorem:

Theorem 6. Suppose there are two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ that define parallel translations $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively. The parallel translation $\mathscr{P}_2 \cdot \mathscr{P}_1$ (or $\mathscr{P}_1 \cdot \mathscr{P}_2$) translates a point A onto the point A'', where AA'' is the diagonal of the parallelogram constructed on the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ both beginning at the point A.

The vector AA'' in this theorem (Fig. 2.101) is called *the sum of two vectors* $\overrightarrow{a}$ and $\overrightarrow{b}$ and is denoted by $\overrightarrow{a} + \overrightarrow{b}$.

Consider two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ (see Fig. 2.102a). Let us draw the vector $\overrightarrow{a} + \overrightarrow{b}$.

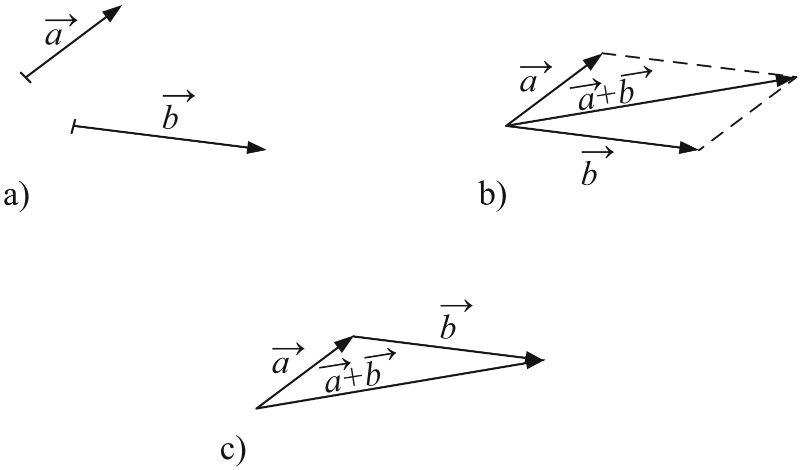


Fig. 2.102

We translate one of the vectors so that the beginning points of both vectors coincide and complete the figure to a parallelogram (Fig. 2.102b). The diagonal in Fig. 2.102b of this parallelogram is the vector $\overrightarrow{a} + \overrightarrow{b}$.

Notice that there is also another way to construct the vector $\overrightarrow{a} + \overrightarrow{b}$. We make a parallel translation of one of the vectors as in Fig. 2.102c, i.e., in such a way that the beginning of vector $\overrightarrow{b}$ coincides with the end of vector $\overrightarrow{a}$. The vector connecting the beginning of vector $\overrightarrow{a}$ with the end of vector $\overrightarrow{b}$ is the vector $\overrightarrow{a} + \overrightarrow{b}$.

The sum of two vectors does not depend on the order of summands, i.e., $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$.

PROBLEM 48. Prove the statement above, i.e., prove that $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{b} + \overrightarrow{a}$. This means that addition of vectors is commutative.

PROBLEM 49. Using the two constructions above, draw the vector $\vec{a} + \vec{b}$ for the vectors $\vec{a}$ and $\vec{b}$ in Fig. 2.103. (Illustrate your two solutions).

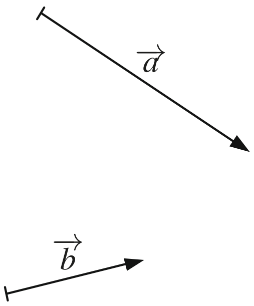


Fig. 2.103

PROBLEM 50. Fig. 2.104a, b, c, and d show different pairs of vectors $\vec{a}$ and $\vec{b}$.

(a) In each figure find the sum of these vectors.

(b) In each figure mark the point A'' onto which point A is translated under the parallel translation defined by the sum of these vectors.

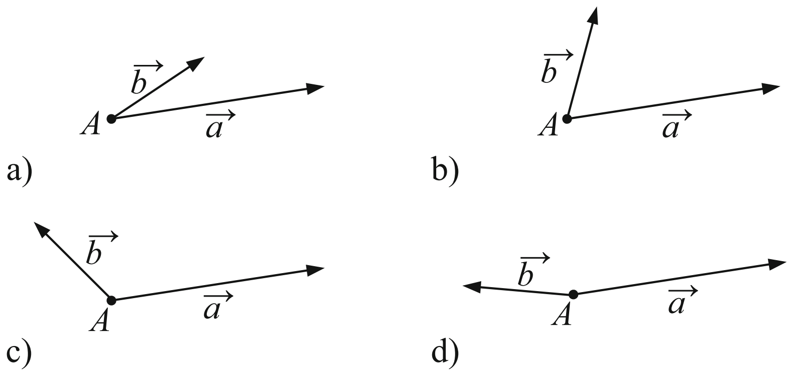


Fig. 2.104

Exercise 21. Find the sum of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in Fig. 2.105a.

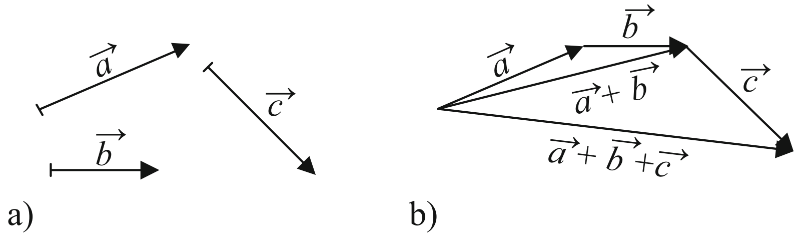


Fig. 2.105

Solution. Let us make a parallel translation of the vectors $\vec{b}$ and $\vec{c}$ as in Fig. 2.105b. We can first find the sum $\vec{a}+\vec{b}$ and then add $\vec{c}$ to this sum. One can see that the vector $\vec{a}+\vec{b}+\vec{c}$ is the vector that connects the beginning of the vector $\vec{a}$ with the end of the vector $\vec{c}$.

Note that what we have found above is the vector $(\vec{a}+\vec{b})+\vec{c}$. In order to omit the brackets, we need to make sure that if we add these vectors in a different way as $\vec{a}+(\vec{b}+\vec{c})$ we will obtain the same result.

PROBLEM 51 (∗) Prove that $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$. This means that addition of vectors is associative.

Exercise 22. Find the vector $\vec{a}+\vec{b}+\vec{c}$ for the vectors in Fig. 2.106.

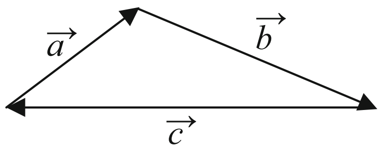


Fig. 2.106

Solution. The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are already positioned properly. Then according to our rule, the vector $\vec{a}+\vec{b}+\vec{c}$ is the vector that has its beginning point at the beginning of the vector $\vec{a}$ and its endpoint at the end of the vector $\vec{c}$. But for the vectors in Fig. 2.106a, these points coincide. Such a vector is called a *zero vector*. Thus, the sum of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in Fig. 2.106a is a zero vector.

A vector whose beginning point coincides with its end point is called a *zero vector*. It is usually denoted as $\vec{0}$.

12.3 Vectors lying on parallel lines

Consider two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ lying on parallel lines (see Fig. 2.107a). How can we find the sum of these vectors?

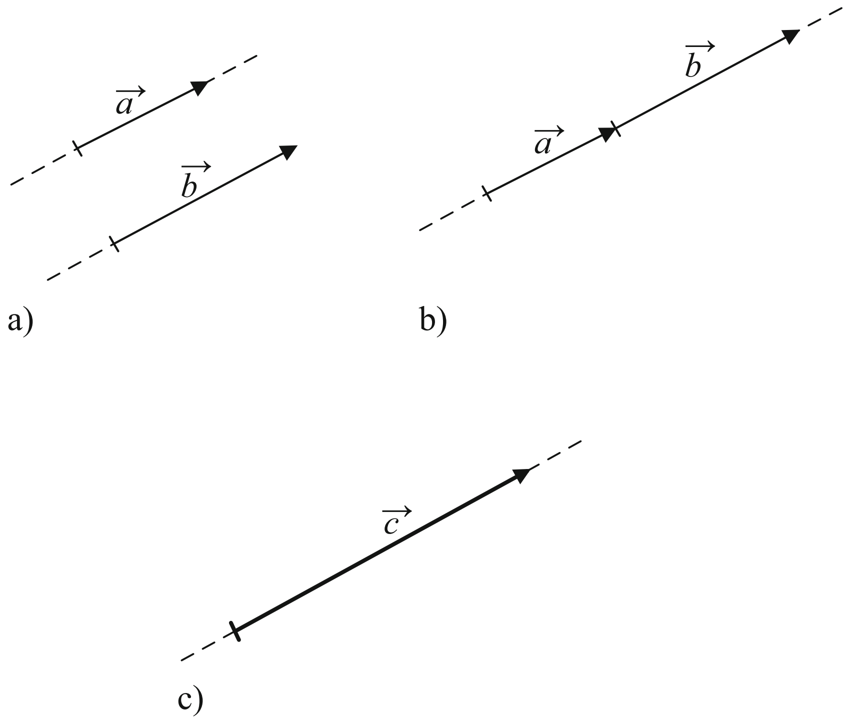


Fig. 2.107

According to the rule[25] illustrated in Fig. 2.102c, we make a parallel translation of the vector $\overrightarrow{b}$ (see Fig. 2.107b) and obtain the vector $\overrightarrow{c} = \overrightarrow{a} + \overrightarrow{b}$ (see Fig. 2.107c).

[25]When vectors lie on parallel lines, the other rule of adding vectors illustrated in Fig. 2.102b does not work because there is no parallelogram in this case.

Exercise 23. In Fig. 2.108a, a vector $\overrightarrow{a}$ is given. Construct the vector $2\overrightarrow{a} = \overrightarrow{a} + \overrightarrow{a}$.

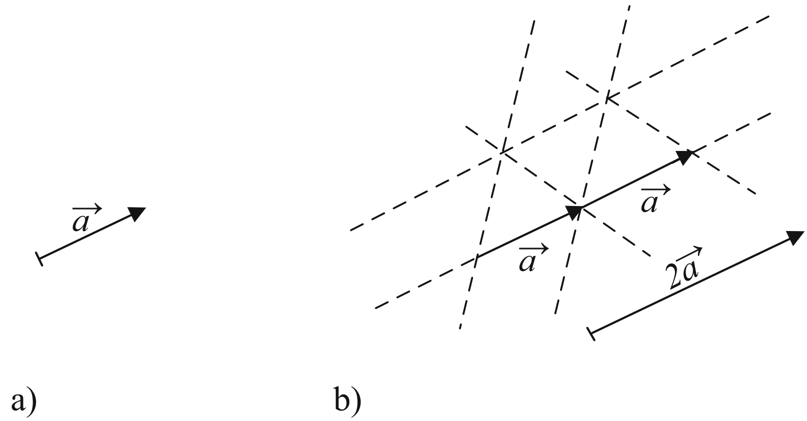


Fig. 2.108

Solution. The vector $2\overrightarrow{a}$ is shown in Fig. 2.108b.

Similarly we can construct a vector $m\overrightarrow{a}$, where m is a natural number, and even a vector $\frac{m}{n}\overrightarrow{a}$, where m, n are natural numbers.

Exercise 24. Find the sum of the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in each of the following figures:

(a) Fig. 2.109a;

(b) Fig. 2.109b.

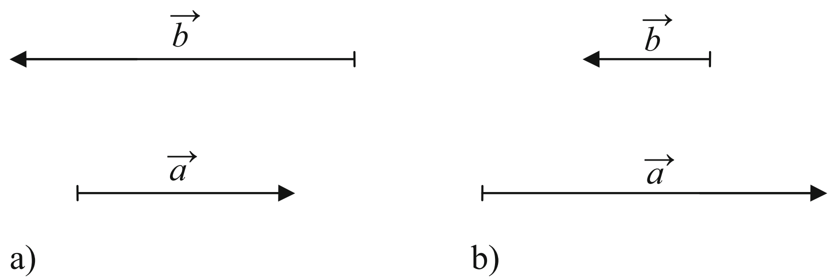


Fig. 2.109

Solution. We make a parallel translation of the vector $\overrightarrow{b}$ so that its beginning coincides with the end of the vector $\overrightarrow{a}$. Then the vector $\overrightarrow{a}+\overrightarrow{b}$ is the vector connecting the beginning of the vector $\overrightarrow{a}$ with the end of the vector $\overrightarrow{b}$.

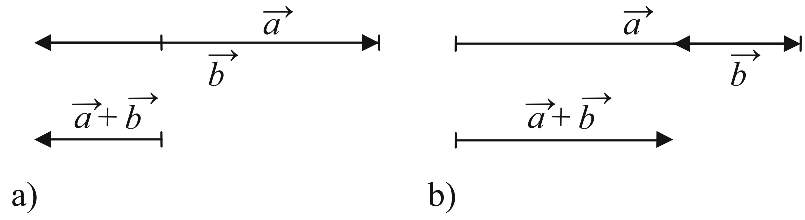


Fig. 2.110

Solutions of (a) and (b) are presented in Fig. 2.110a and Fig. 2.110b respectively.

We say that a vector $-\overrightarrow{a}$ is the vector obtained from the vector $\overrightarrow{a}$ by switching the beginning with the end. Fig. 2.111 shows a vector $\overrightarrow{a}$ and the vector $-\overrightarrow{a}$.

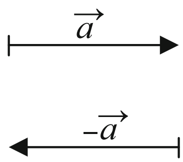


Fig. 2.111

PROBLEM 52.

(a) Prove that $\overrightarrow{a}+(-\overrightarrow{a})=\overrightarrow{0}$.

(b) Prove that $\overrightarrow{a}+\overrightarrow{0}=\overrightarrow{a}$.

12.4 Subtraction of vectors

Consider two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ (see Fig. 2.112a). How shall we define the vector $\overrightarrow{a} - \overrightarrow{b}$? We already know how to draw the vector $-\overrightarrow{b}$ (see Fig. 2.112b). Then we can add these vectors $\overrightarrow{a} + (-\overrightarrow{b})$. For this we translate the vector $-\overrightarrow{b}$ as in Fig. 2.112c. The vector $\overrightarrow{a} - \overrightarrow{b}$ is shown in Fig. 2.112d.

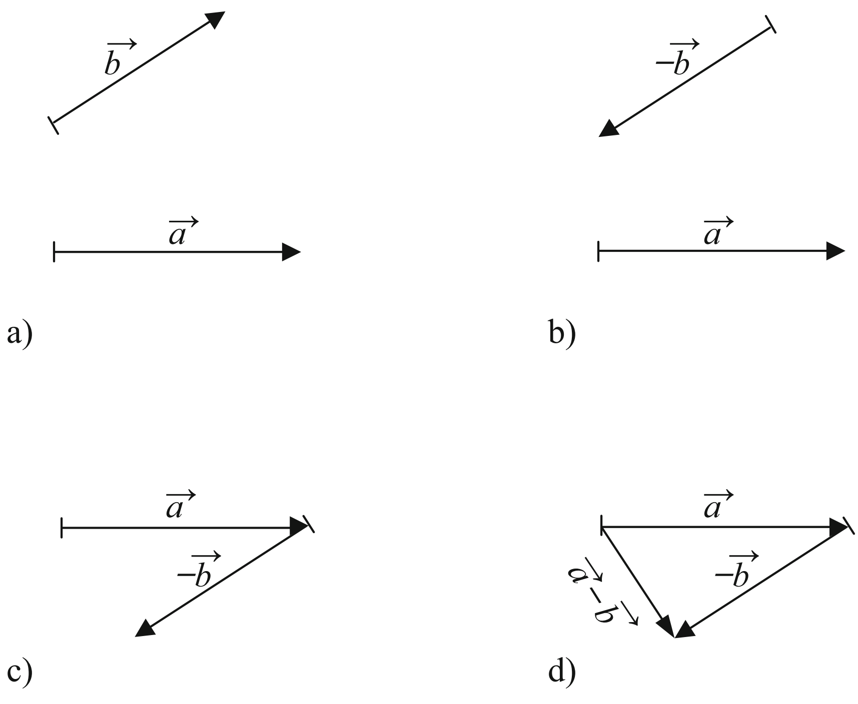


Fig. 2.112

As mentioned above, for two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ there are two ways to draw the vector $\overrightarrow{a}+\overrightarrow{b}$ (see Fig. 2.113a,b,c). Note that there are also two ways to draw the vector $\overrightarrow{a}-\overrightarrow{b}$ (see Fig. 2.113d,e).

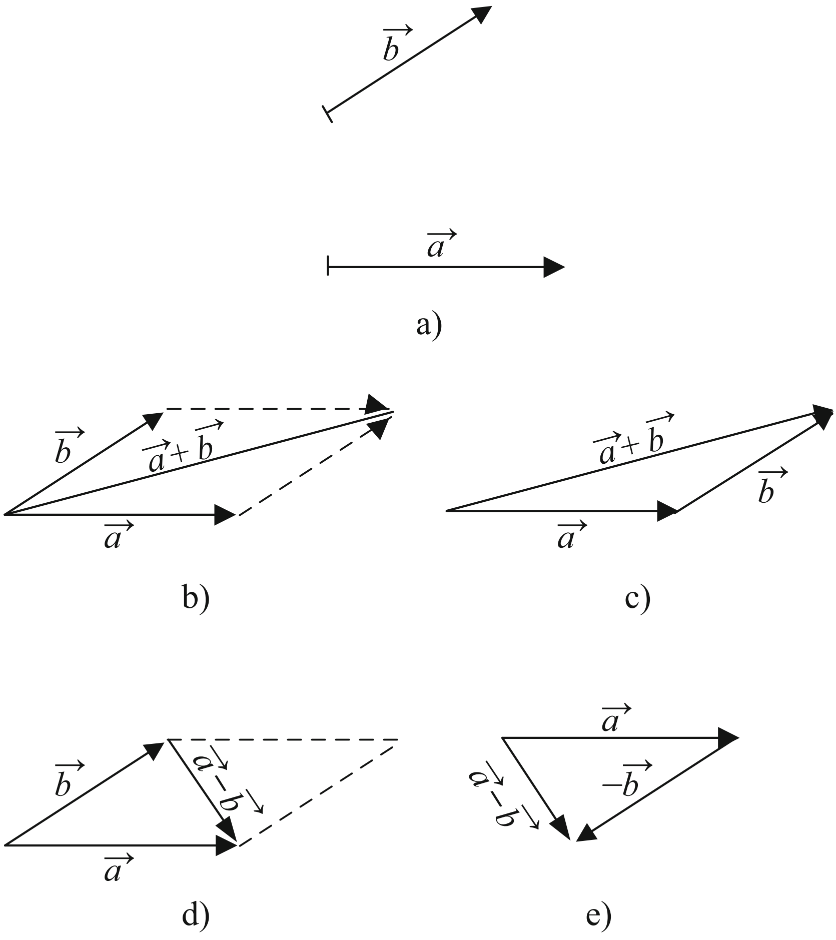


Fig. 2.113

PROBLEM 53. Prove that the diagonal of the parallelogram in Fig. 2.113d is indeed the vector $\overrightarrow{a}-\overrightarrow{b}$.

12.5 More problems on vectors

You may skip this section if you like.

PROBLEM 54. Draw the vector $\vec{a} + \vec{b} + \vec{c}$ for:

(a) the vectors in Fig. 2.114a;

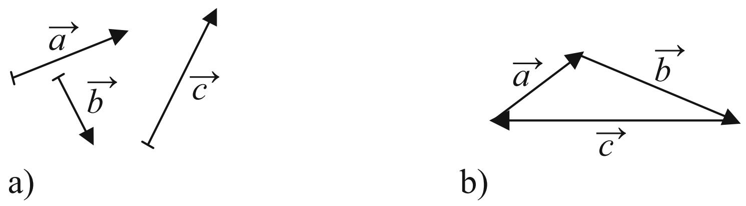


Fig. 2.114

(b) the vectors in Fig. 2.114b.

PROBLEM 55. Consider two vectors $\vec{a}$ and $\vec{b}$ with the same beginning point (see Fig. 2.115a). Complete this figure as in Fig. 2.115b.

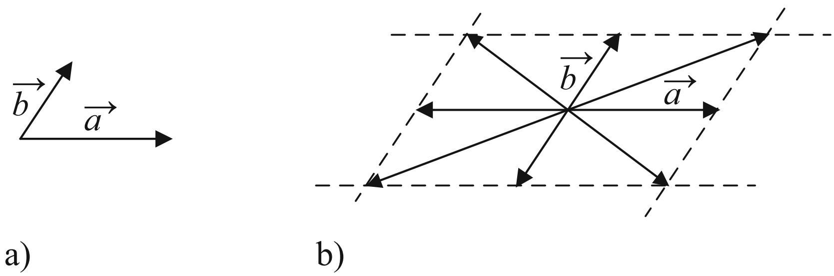


Fig. 2.115

Find an expression for each of the vectors obtained in terms of $\vec{a}$ and $\vec{b}$ and write down each expression above the vector it stands for.

PROBLEM 56. Find the sum of the vectors shown:

(a) in Fig. 2.116a;

(b) in Fig. 2.116b;

(c) in Fig. 2.116c.

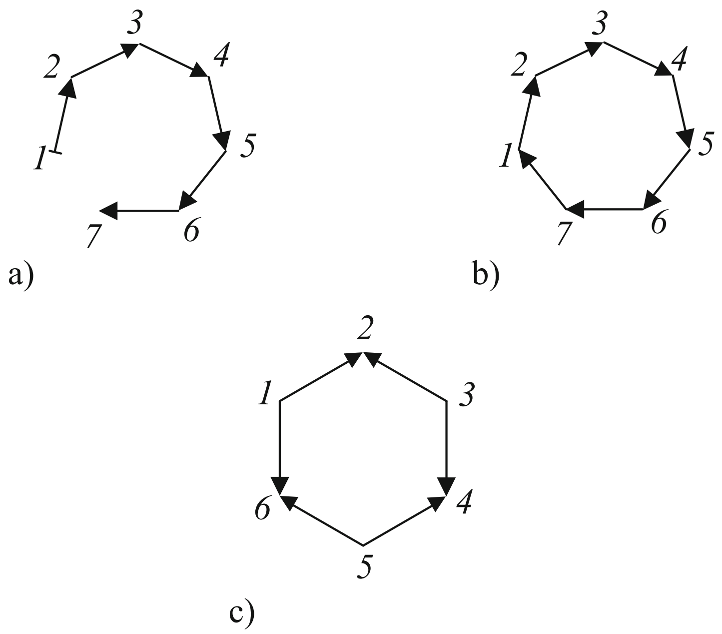


Fig. 2.116

PROBLEM 57.

(a) Fig. 2.117a shows four vectors: $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$. Find their sum.

(b) Fig. 2.117b shows four vectors: $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$. Draw the following vectors: $\overrightarrow{a}+\overrightarrow{b}$, $\overrightarrow{a}+\overrightarrow{c}$, $\overrightarrow{a}+\overrightarrow{d}$.

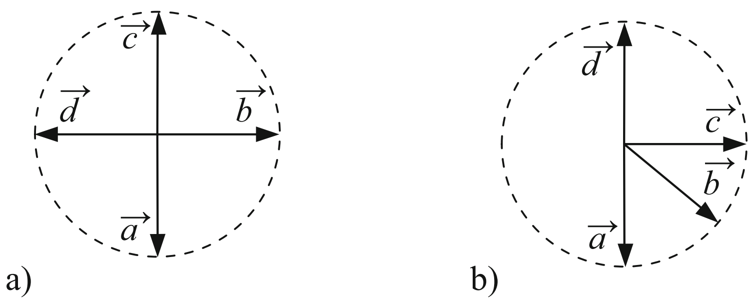


Fig. 2.117

PROBLEM 58.

(a) Fig. 2.118a shows a regular hexagon and vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$, $\overrightarrow{e}$, $\overrightarrow{f}$. Find the sum of these vectors.

(b) Find the sum of all vectors in Fig. 2.118b.

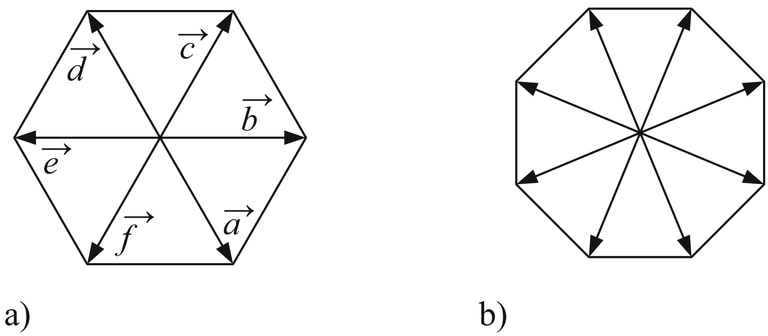


Fig. 2.118

PROBLEM 59. On a straight line there are two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ (see Fig. 2.119).

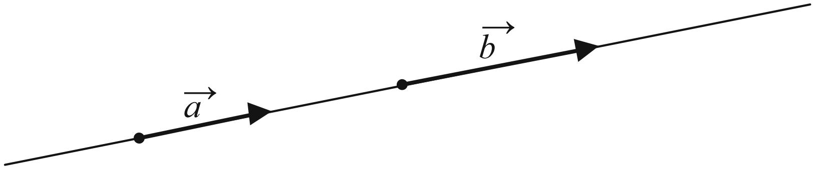


Fig. 2.119

1. Draw the vector $\overrightarrow{a} + \overrightarrow{b}$.

2. Draw the vector $\overrightarrow{b} - \overrightarrow{a}$.

3. Draw the vector $\overrightarrow{a} - \overrightarrow{b}$.

13 Overview of Chapter II

In Chapter II, we defined parallel lines and introduced one more operation, based on Euclid's fifth postulate. We added another tool, a rolling ruler, for drawing parallel lines. We introduced the notion of equal segments on parallel lines and on the same line. We were able to compare segments on these lines and construct a segment of any rational length. These operations and methods are sufficient to allow us to define an affine coordinate system. We consider this in Section 5 of the Appendix that follows this chapter.

We also defined more objects and figures: the midpoint of a segment, the medians and bimedians of a triangle, parallelograms, trapezoids, and the median of a trapezoid.

We considered a particular correspondence, called parallel projection, which established a relation between all the points on one line and all the points on another line. Parallel projection is not a one-to-one correspondence.

We defined some operations with figures: Minkowsky addition of two figures, parallel translation of a figure, and symmetry with respect to a point (central symmetry). We considered applying some of these operations to a given figure consecutively. Parallel translation can be defined with the help of a directed segment, called a vector. Vectors can be added, and this addition corresponds to applying two parallel translations consecutively.

Parallel translation and central symmetry can be applied to any point of the plane. If we apply one of these operations to a given point A, we obtain a new point A' in a unique way. The correspondences established in this way are examples of one-to-one correspondences, and are called transformations of the plane. There will be more about transformations in Chapter IV.

Appendix for Chapter II

1 Why we cannot define equal segments in Chapter II

In Section 4, Chapter II, we defined equal segments lying on parallel lines. However, it is in principle impossible in Chapter II to define equal segments lying on non-parallel lines. Why?

Consider two non-parallel lines a and b. Suppose that we have succeeded in marking equal segments on each of these lines. For simplicity, let us assume that they have one common end at the intersection point of the lines a and b (see segments OA and OB in Fig. 1).

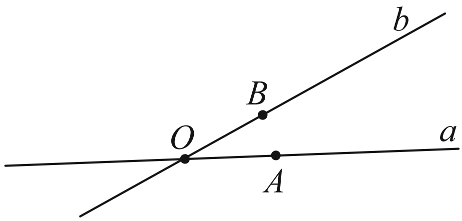


Fig. 1

Imagine that we make a parallel projection (see Section 9, Chapter II) of these segments onto another line x. We denote by O', A', B' the projections of the points O, A, B correspondingly.

As one can see from Fig. 2, depending on the direction of parallel rays, we obtain different cases. Indeed, in Fig. 2a the segment $O'A'$ is shorter than the segment $O'B'$, in Fig. 2b the segment $O'A'$ is equal to the segment $O'B'$, and in Fig. 2c the segment $O'A'$ is longer than the segment $O'B'$. Thus, no matter how we choose "equal" segments on the lines a and b, their projections by parallel rays will no longer be equal and therefore such "equality" is not correctly defined.

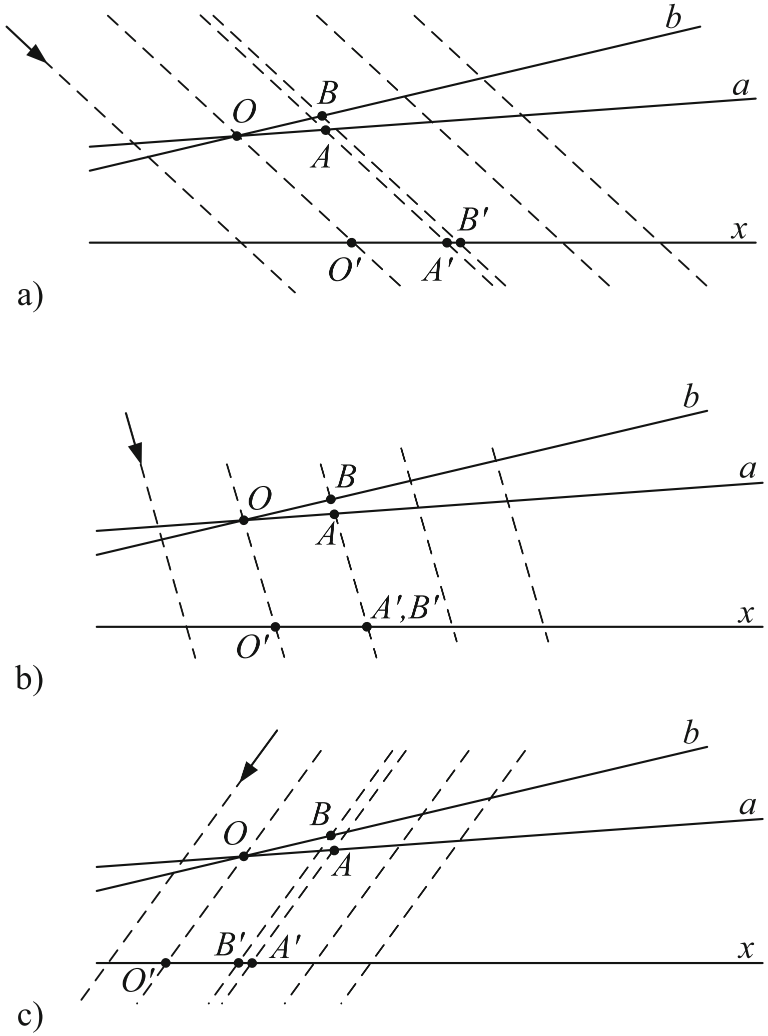


Fig. 2

In this example the parallel rays lie on the same plane as the segments *OA* and *OB*. Even though this is all that we consider in this book, if we allow the rays not to lie on the same plane as the segments, our statement might be easier to illustrate by examples from our daily life.

On a sunny day do the following "experiments" or exercises:

1. Take a pencil, place it over a flat surface, and watch its shadow from the sun rays. We can imagine that a pencil is just a "thick" segment. Its

shadow then is also a segment. Then change the position of the pencil in regard to the sun rays. Watch what happens with its shadow. Can you make the shadow shorter than, equal to, longer than the pencil? Can you make the shadow become a “point”?

2. Stand with your back to the sun and your arms to the sides. Watch your shadow. Now turn so that the sun’s rays are not directly behind you. What happens to your shadow? Did the length of your arm’s shadow change? Do this when the sun is high up around lunch time, and repeat when the sun is close to the horizon. Observe how the length and the shape of your shadow change.

3. Take a round object, a small ball, for example, and place it above a “plane”, such as a firm sheet of paper or a piece of cardboard. What is the shape of the shadow of the ball if the plane is perpendicular to the sun’s rays? For example, if the sun is high, then place the sheet of paper parallel to the ground. Now change the position of the sheet of paper with respect to the sun’s rays, making it angled to the ground. Watch what shapes the shadow of the ball can take. You should be able to see the shadow as a circle, as an ellipse, and if you really try all possibilities, as two other shapes we are not dealing with in this book. We can tell you secretly that they are a *parabola* and a *hyperbola*; these curves are studied in calculus.

   This exercise shows that in Chapter II we cannot introduce a circle. Indeed, a parallel projection will turn it into an ellipse, hyperbola or parabola.

4. Another way to illustrate the examples above is to imagine a square and a circle being drawn on a sheet of some transparent material (or actually do this if you can). Place this sheet under the sun’s rays and watch its shadows on a flat surface (a desk, a floor, or a sheet of paper). By changing the position of the transparent plane, you will be able to make the square into a parallelogram or some other quadrilateral, and the circle into an ellipse.

Note that, unlike equality of segments on non-parallel lines, all the other notions and statements from Chapter II will remain true after applying a parallel projection. We suggest that you check this for the three operations (1), (2), (3) defined in Section 2, Chapter II.

PROBLEM 1. In sunny-day exercise 4 above, check that no matter how you position the transparent sheet, operations (1), (2), (3) will remain true on the projection plane, i.e., on the plane on which you watch the shadow.

More precisely, for operation (1) we need to check that if there is a line that passes through two given points, then the parallel projection of this line will pass through the projections of the two points.

For operation (2) we need to check that if two lines intersect, then their parallel projections will also intersect.

For operation (3) we need to check that if there are two parallel lines, then their parallel projections are also parallel lines.

2 Parallel lines, equal segments, and the Desargues configuration

2.1 Variation of the Desargues configuration in the case of parallel lines

We suggest that you reread Sections 8, 9, Chapter I, about the Desargues configuration and Desargues' theorem.

Let us consider two triangles ABC and $A'B'C'$ such that the lines e, f, g passing through the corresponding vertices of these triangles are parallel (see Fig. 3a).

Will the points of intersection of the corresponding sides of these triangles (points E, F, G in Fig. 3b) lie on one line? The answer is "yes" (see Fig. 3c). Thus, the following statement is true:

> Given two triangles such that the lines passing through the corresponding vertices of these triangles are parallel, the points of intersection of the corresponding sides of these triangles lie on one line.

PROBLEM 2. Draw another pair of triangles such that the lines passing through the corresponding vertices of these triangles are parallel, and check that the points of intersection of the corresponding sides of these triangles lie on one line.

The configuration in Fig. 3c and the statement above are like the Desargues configuration (see Fig. 1.80) and Desargues' theorem. The only difference is that in the configuration above, the lines passing through the corresponding vertices of the two triangles are parallel, while in the Desargues configuration these lines intersect at one point.

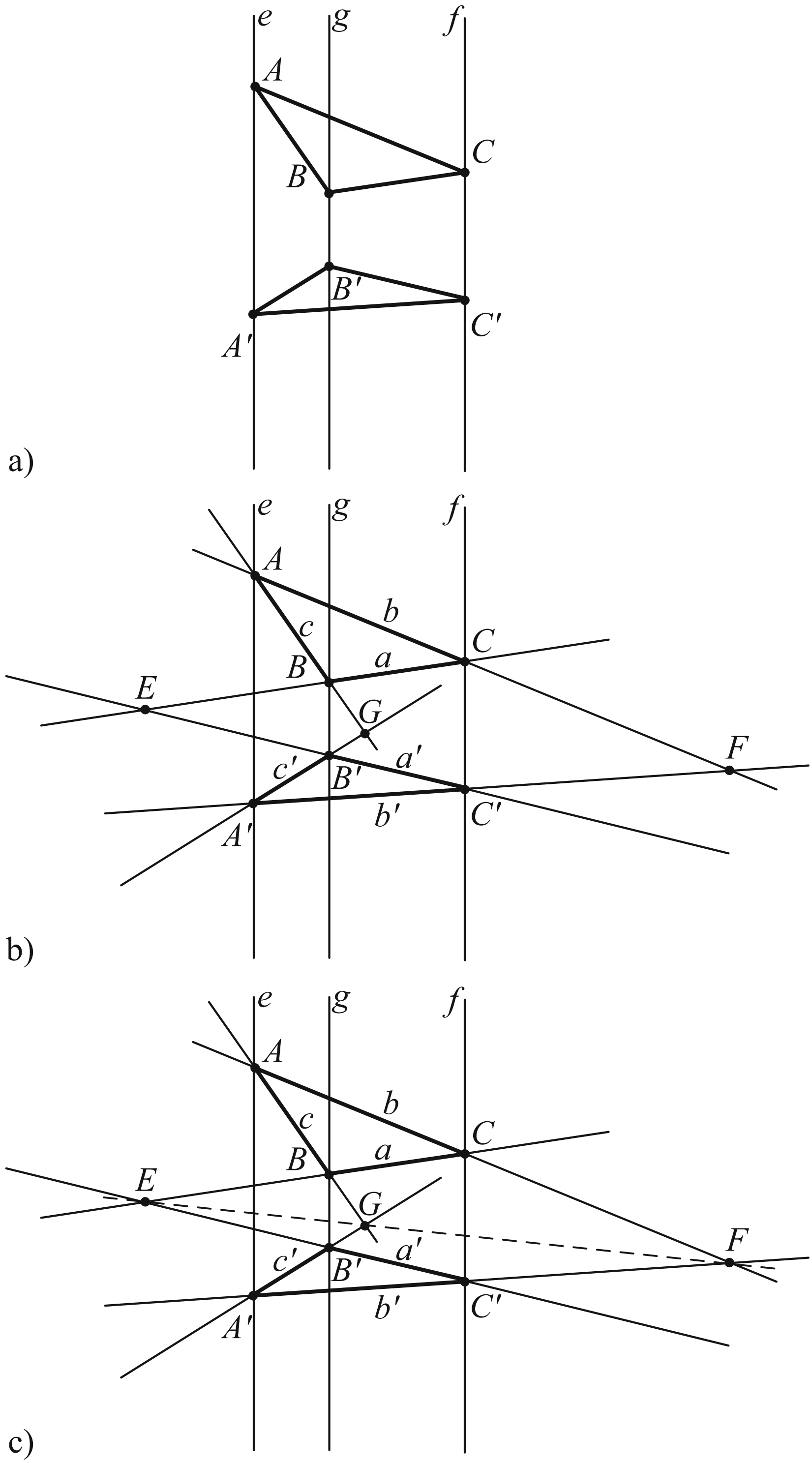


Fig. 3

As you might have heard, mathematicians like to make uniform rules. They would like to imagine that parallel lines intersect somewhere but not at any actual point. This is why they made a convention that parallel lines intersect at a *point at infinity* or at *infinity*, and they denote this point as ∞.

Once this convention is accepted, some interesting things follow.

For example, Desargues' theorem now also includes the statement above.

Footnote 3 from the first page of Chapter II—but do not forget to also look at Fig. 2.1—does not seem strange any more. We can reformulate it as follows: when the point of intersection of the lines a and b moves very far away, at a certain position these lines will intersect at a point at infinity. If you continue to turn the line b, the point of the intersection of a and b moves closer again. There is no more "jump" from "far left" to "far right" as mentioned in footnote 3.

There is another interesting observation that uses the convention above. Let us consider again two triangles and lines e, f, g passing through the corresponding vertices of these triangles. This time let not only lines e, f, g be parallel, but also the sides of these triangles be correspondingly parallel; see Fig. 4.

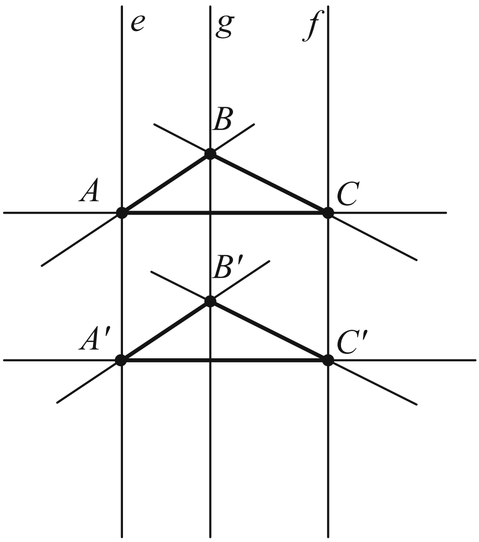


Fig. 4

It turns out that Fig. 4 also presents a Desargues configuration. Indeed, lines e, f, g are parallel, and therefore, as we agreed, they intersect at one point, a point at infinity. At the same time, we have $AB \parallel A'B'$, $BC \parallel B'C'$, and $AC \parallel A'C'$, i.e., the point of intersection of AB and $A'B'$, the point of intersection of BC and $B'C'$, and the point of intersection of AC and $A'C'$ are all points at infinity. Mathematicians go further and accept that all points at infinity lie on one *line at infinity*. Thus, Desargues' theorem is true also for the configuration in Fig. 4.

In projective geometry there are other beautiful observations which we will not discuss in this book.

2.2 Transitivity of equal segments

Consider three parallel lines a, b, c and three segments $\bar{a}$, $\bar{b}$, and $\bar{c}$ on lines a, b, c respectively,[1] as in Fig. 5a.

Transitivity of equal segments means that if $\bar{a} = \bar{b}$ and $\bar{b} = \bar{c}$ then $\bar{a} = \bar{c}$. Let us prove this statement.

Since $\bar{a} = \bar{b}$, the lines connecting their ends are parallel; since $\bar{b} = \bar{c}$, the lines connecting their ends are also parallel.

In order to prove transitivity, we need to show that $\bar{a} = \bar{c}$; therefore, we need to prove that the lines connecting the endpoints of these segments are also parallel, as in Fig. 5b.

Let us compare Fig. 5b with the Desargues configuration presented in Fig. 4. In order to make these pictures look more alike, let us turn the figure Fig. 5b to make the lines a, b, c be more vertical and mark the intersection points as A, B, C and A', B', C' as in Fig. 5c.

It is clear that the triangles ABC and $A'B'C'$ in Fig. 5c form a Desargues configuration (see Fig. 4) because the lines a, b, and c intersect at a point at infinity. Then from Desargues' theorem in the case of parallel lines (see Section 2.1 of this appendix), it follows that lines AC and $A'C'$ are also parallel. This means that $\bar{a} = \bar{c}$, which finishes the proof of transitivity.

2.3 A property of parallel translation

In this section we will prove Proposition 13 (from Chapter II), which we repeat below as Proposition 1.

Proposition 1. Let the points A, A' define a parallel translation of the plane. Two points B, B' define the same parallel translation of the plane as the points A, A' if and only if $AA'B'B$ is a parallelogram.

[1] We can also consider the case where the segments $\bar{a}$, $\bar{b}$, and $\bar{c}$ lie on two parallel lines or even on a single line.

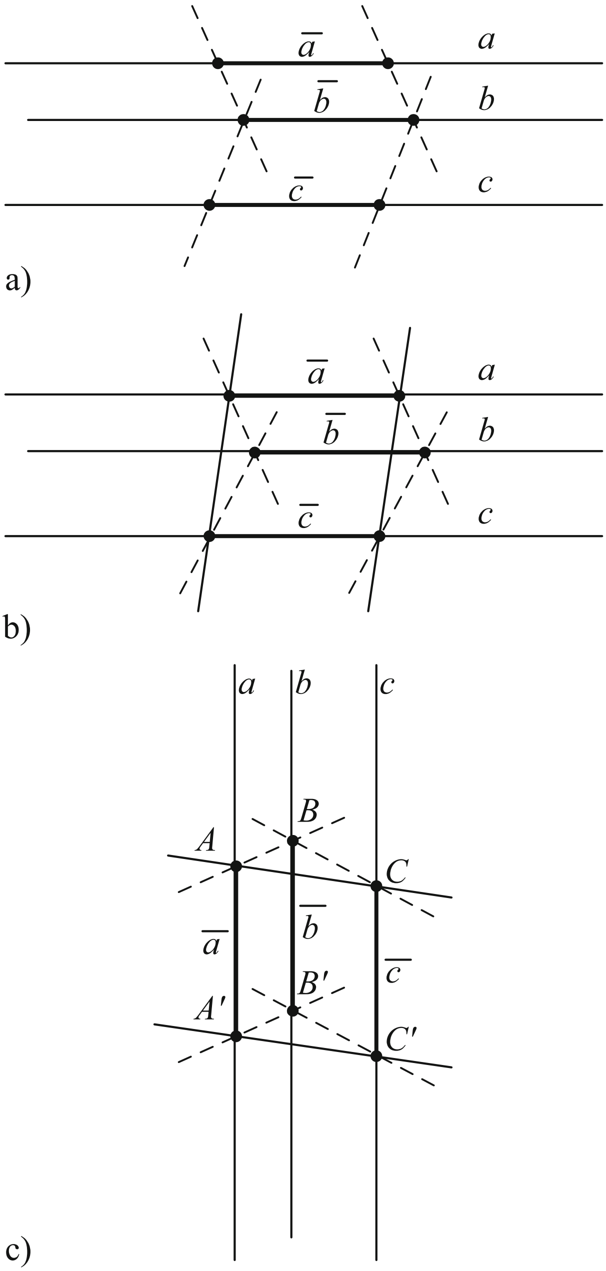


Fig. 5

As was mentioned in Chapter II, the words *if and only if* mean that two statements are true. Before formulating them, let us consider some examples from real life.

What does it mean to say: “One can legally travel by train if and only if s/he has a ticket”? If a person has a ticket s/he can take a train without getting a fine. However, if a person can travel legally in a train this does not mean that she must have a ticket; indeed, any train conductor travels legally without it. Therefore, the expression “if and only if” makes this statement false because one of the statements in the phrase “if and only if” is false.

And what about this example: “One can legally drive a car alone if s/he is at least of a certain required age (16–17 years old depending on the state) and has a valid driver’s license”? If a person is at least of the required age and has a valid driver’s license s/he can legally drive a car alone. And if one can legally drive a car alone then this person is at least of a required age and has a valid driver’s licence. Therefore, the statement above with the expression “if and only if” is true.

In mathematical terms[2] when we say “X is true if and only if Y is true,” this means that Y follows from X and also that X follows from Y.

Let us return to the proposition above.

Proof. We need to prove two statements.

1. If $AA'B'B$ is a parallelogram, then points B, B' define the same parallel translation of the plane as points A, A'.

2. If points A, A' and B, B' define the same parallel translation, then $AA'B'B$ is a parallelogram.

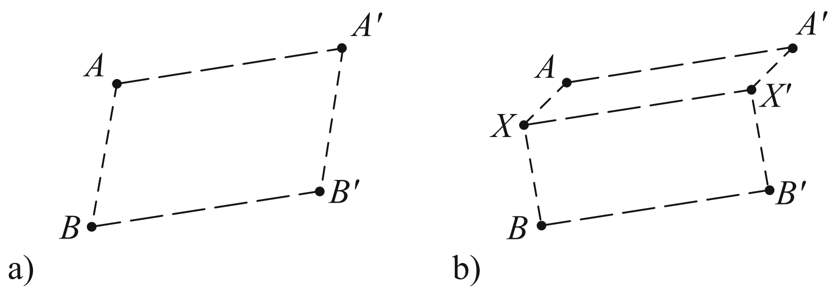


Fig. 6

[2]This topic is a subject of mathematical logic. Those who are interested in logic can compare the proposition above with Theorems 1 and 2 in Section 9, Chapter I. Try to reformulate these two theorems into one.

Let us first prove statement (1). Fig. 6a shows a parallelogram $AA'B'B$. We need to prove that if a point X is moved to a point X' by the translation defined by AA', then the translation BB' will also move the point X onto the point X'. In other words (see Section 10.1 in Chapter II), we need to prove that if $AA'X'X$ is a parallelogram then $BB'X'X$ is also a parallelogram.

In Fig. 6b, $AA'X'X$ is a parallelogram. Then $XX' \parallel AA'$ and $XX' = AA'$. We need to show that $BB'X'X$ is also a parallelogram, which we may do by showing that both (a) $XX' \parallel BB'$ and (b) $XX' = BB'$. The statement (a) follows from the transitivity of parallel lines, and the statement (b) follows from the transitivity of equal segments on parallel lines.

Now let us prove statement (2). We know that AA' and BB' define the same parallel translation, so a point X is translated onto the same point X' by each of them; i.e., $AA'X'X$ and $BB'X'X$ are parallelograms. We need to prove that $AA'B'B$ is also a parallelogram, or that (a) $AA' \parallel BB'$ and (b) $AA' = BB'$. Statement (a) follows from transitivity of parallel lines, and statement (b) follows from transitivity of equal segments. This finishes the proof. □

3 Arithmetic operations with segments

3.1 Addition and subtraction

We can perform arithmetic operations with segments just as we do with numbers. In Section 4.3 of Chapter II, we already constructed a segment $2\bar{a}$, where $\bar{a}$ is a given segment; i.e., we can construct $\bar{a} + \bar{a} = 2\bar{a}$.

Let us construct a segment $\bar{a} + \bar{b}$, where $\bar{a}$ and $\bar{b}$ are two given segments. Let the segments $\bar{a}$ (or AB) and $\bar{b}$ (or CD) lie on line a (see Fig. 7a).

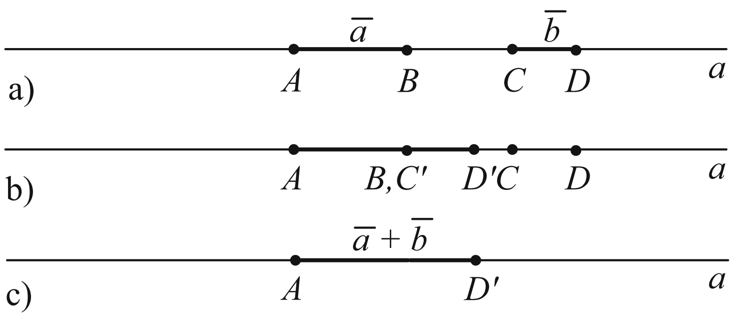


Fig. 7

Let us make a parallel translation of segment CD so that point C' coincides with point B (see Fig. 7b). The segment AD' (see Fig. 7c) is called *the sum of the segments AB* and CD, i.e., $\bar{a} + \bar{b}$.

How can we subtract one segment from another?

Fig. 8a shows two segments $\bar{a}$ or AB and $\bar{b}$ or CD on line a. Let us make a translation of the segment CD so that point C' coincides with point A (see Fig. 8b). The segment $D'B$ (see Fig. 8c) is called *the difference between the segments AB* and CD, i.e., $\bar{a} - \bar{b}$.

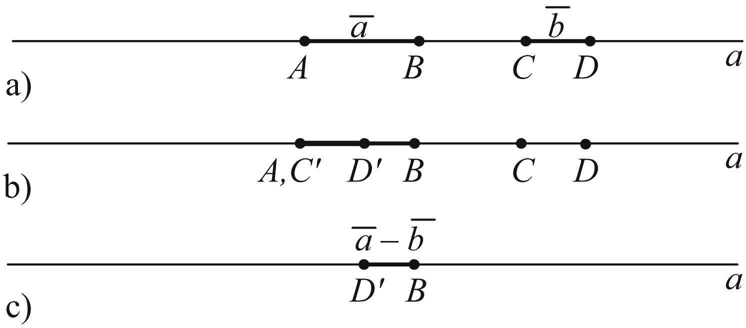


Fig. 8

Exercise 1. Subtract a segment CD from a segment AB if the length of CD is bigger than the length of AB. We will leave the segments of the same lengths as above, but simply change their notations.

Solution. Consider two such segments AB and CD (see Fig. 9a). Let us repeat the construction described above. We translate segment CD so that points A and C' coincide (Fig. 9b). Then according to our definition of subtraction, segment $D'B$ (see Fig. 9c) is the difference, i.e., $\bar{a} - \bar{b}$.

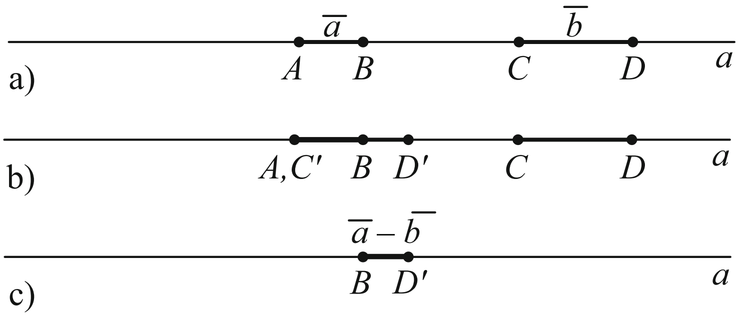


Fig. 9

Note that in this exercise $\bar{a} - \bar{b} = D'B$, and while in the example in Fig. 8 we also obtained $\bar{a} - \bar{b} = D'B$, the roles of the two endpoints are reversed:

endpoint D' in Fig. 9 is in the position occupied by endpoint B in Fig. 8, and vice versa. Thus, the "direction" of the segment[3] has changed. Therefore, when doing arithmetic operations with segments, one should be careful about the direction in which we read the notation of a segment. With this consideration, the segments BD' and $D'B$ are not the same. If their lengths are equal and their directions are opposite, it is natural to assume that one is the negative of the other, i.e., $D'B = -BD'$.

Exercise 2. Fig. 10a shows two segments. Subtract segment AB from segment CD.

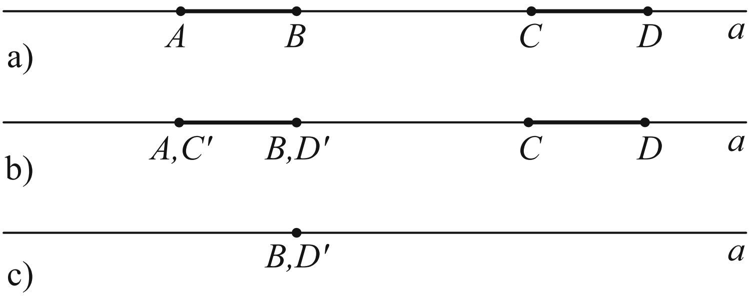


Fig. 10

Solution. Let us repeat the construction above (see Fig. 10b). Since segments AB and CD have equal lengths, points B and D' coincide. A segment such as BD' which has its beginning and endpoint at the same one point is said to have *zero length* (see Fig. 10c).

PROBLEM 3. Suppose there are three segments $\bar{x}$, $\bar{y}$, and $\bar{z}$ on a line a. Check that these segments satisfy the following properties:

(1) $\bar{x}+\bar{y}=\bar{y}+\bar{x}$,

(2) $(\bar{x}+\bar{y})+\bar{z}=\bar{x}+(\bar{y}+\bar{z})$.

Draw different segments $\bar{x}$, $\bar{y}$, $\bar{z}$ on a line and check that these equalities will be satisfied again.

[3]In Chapter II, Section 12 we already considered directed segments called vectors. Thus, these segments BD' and $D'B$ are vectors with opposite directions.

3.2 Multiplication and division

A segment can be multiplied by a positive whole number.

Exercise 3. Given a segment $\bar{a}$, construct a segment whose length is $3\bar{a}$.

Solution. We already know how to construct the segment $2\bar{a}$ (Section 4.3 in Chapter II; see also Fig. 11a, b, c). We can then add the segment $2\bar{a}$ to the segment $\bar{a}$ (see Fig. 11d). The segment in Fig. 11e is $3\bar{a}$.

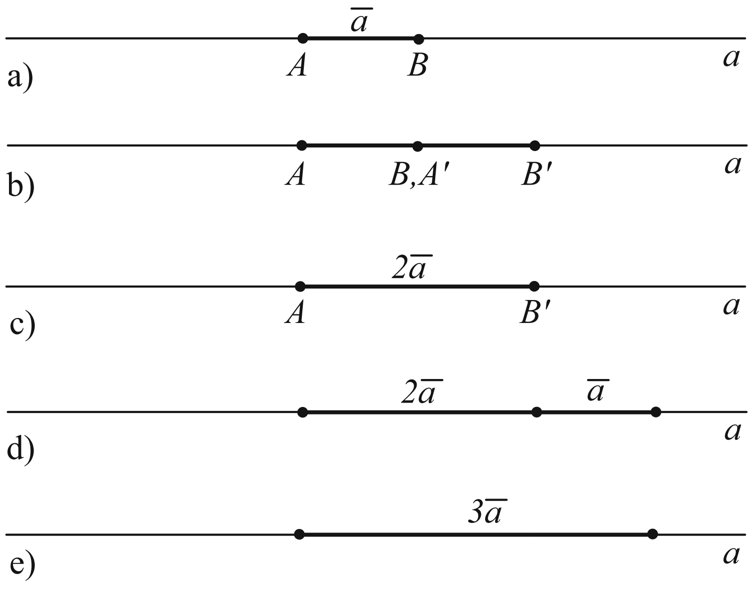


Fig. 11

PROBLEM 4. A segment $\bar{a}$ is given (choose a segment yourself). Construct the segment $5\bar{a}$.

A segment can be divided in a given ratio. This means that we can multiply a segment by a number $\frac{m}{n}$, where m and n are positive whole numbers. We already know how to multiply a segment by a number m. Let us show how to divide a segment into n equal parts.

In Section 4.4, Chapter II, we already divided a segment into two equal parts using the Lemma. We illustrate this in Fig. 12 where, for a given segment $\bar{a} = AB$, we construct two equal segments $AO = OB = \frac{1}{2}\bar{a}$. We present below another solution without a proof.

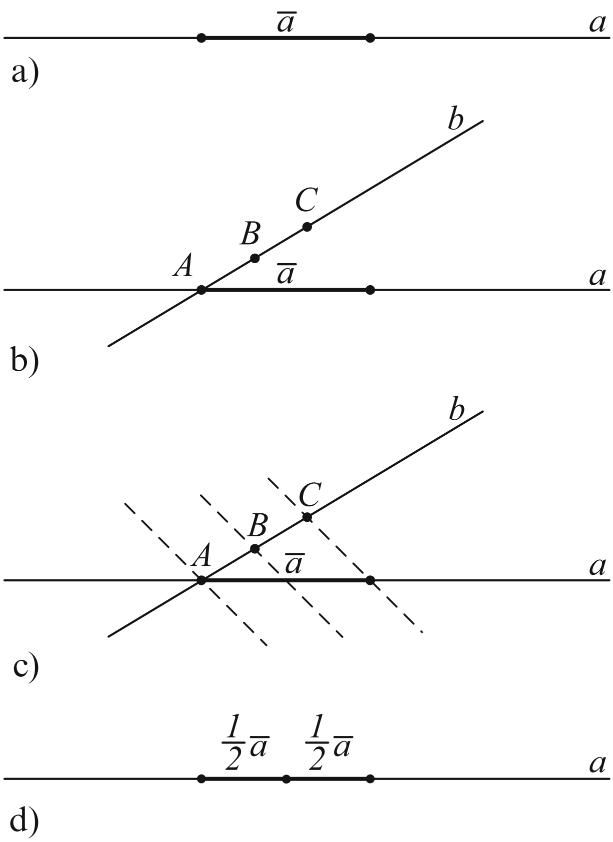


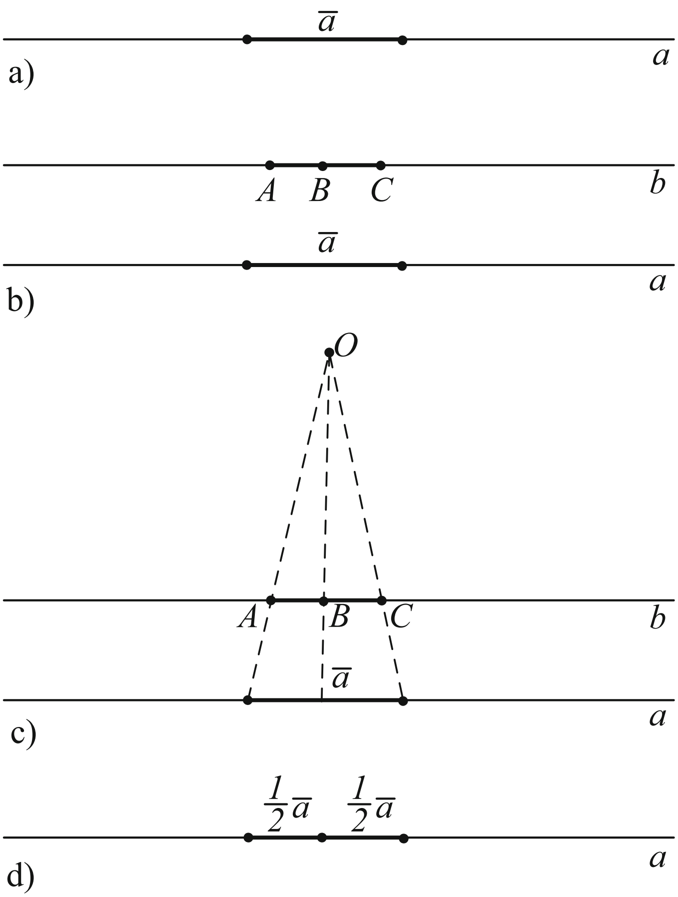
Fig. 12

Exercise 4. Divide a segment $\bar{a}$ into two equal parts, i.e., construct a segment $\frac{1}{2}\bar{a}$.

Solution. Fig. 13a shows a segment $\bar{a}$ on line a. Let us draw line b parallel to line a. On line b let us mark two equal segments AB and BC of any length (Fig. 13b). Through the ends of $\bar{a}$ and the segment AC draw the lines until they intersect,[4] say at a point O, and then draw line OB, extending it until it intersects line a (Fig. 13c).

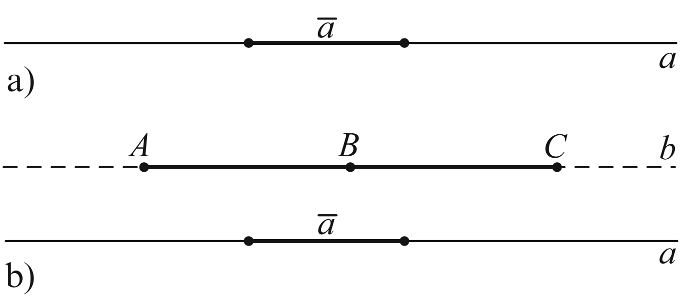
Line OB divides the segment $\bar{a}$ into two equal segments. The two segments obtained (see Fig. 13d) are equal and each has length $\frac{1}{2}\bar{a}$.

[4]If by chance, they are parallel, then the midpoint of $\bar{a}$ lies on the line passing through B and parallel to these lines.



$\overline{a}$
a)
a
A
B
C
b
b)
O
c)
$\frac{1}{2}\overline{a}$
d)

Fig. 13



$\overline{a}$
a)
a
A
B
C
b
b)

Fig. 14

PROBLEM 5. A student Mike decided to divide segment $\bar{a}$ (in Fig. 14a) into two equal parts according to the solution above. He drew a line b parallel to a and marked two equal segments as in Fig. 14b.

Finish this construction and find the midpoint of $\bar{a}$.

Exercise 5. Divide a segment $\bar{a}$ into three equal parts, i.e., construct a segment $\frac{1}{3}\bar{a}$.

Solution. We have described two ways of dividing a segment into equal parts. In Fig. 15a there is a segment $\bar{a}$.

We can use the Lemma. We draw some equal segments on any line b which intersects the line a. Then parallel lines through the ends of these segments mark equal segments on the other line (see Fig. 15b).

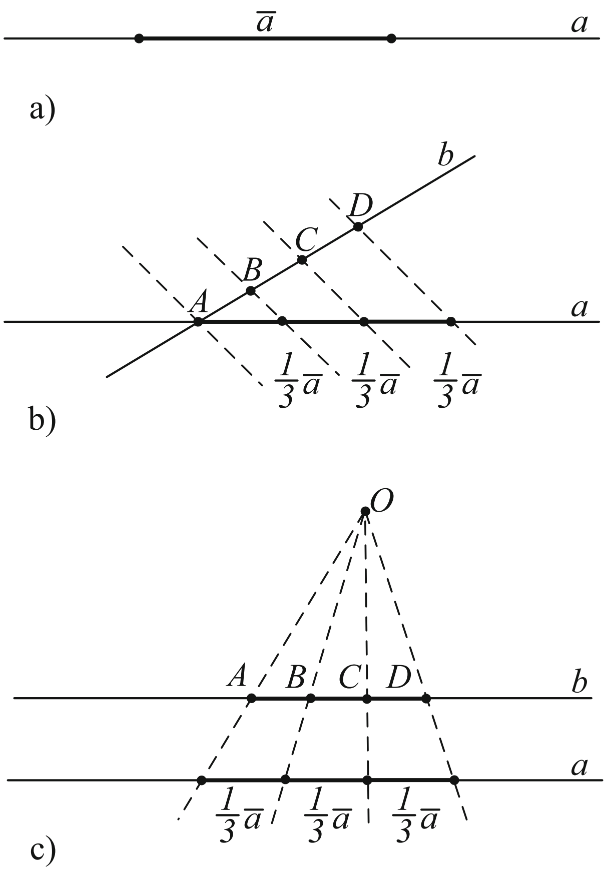


Fig. 15

Or we can draw a line b parallel to line a and mark some equal segments on this line. Then the lines passing through the point O mark equal segments on the line a (see Fig. 15c).

PROBLEM 6. Given a segment $\bar{a}$ construct a segment $\frac{1}{5}\bar{a}$.

PROBLEM 7. Construct a segment with length equal to $\frac{3}{5}$ of the length of a given segment AB.

Hint. Divide the segment AB into 5 parts and then mark a segment which consists of three of them.

To summarize this Section 3.2, we can say that for a given segment $\bar{x}$ on a line a and natural[5] numbers m, n we can construct a segment nx, a segment $\frac{1}{n} \cdot x$, and a segment $\frac{m}{n} \cdot x$.

A number which can be written in the form $\frac{m}{n}$, where m, n are natural numbers, is a *rational number*. Therefore, we showed how to multiply a segment by a rational number.

4 Segments and rational numbers

4.1 Number axis

Arithmetic operations with segments give us the possibility of describing any point on the line with the help of a number. This number is called the *coordinate* of this point.

Consider any straight line, which we will denote by x and call the *axis*. Let us choose a point on it and mark it as 0 or "zero" (see Fig. 16a). Let us take another point[6] on the line x and mark it as 1 (see Fig. 16b). We say that the first point has the *coordinate* 0 and the second point has the *coordinate* 1.

A line with two marked points 0 and 1 on it is called a *number axis* or a *coordinate axis*. The point 0 is called the *origin*.

Besides Fig. 16b, another way to illustrate the number axis is as in Fig. 16c. The arrow indicates that on this number axis the point 1 has to be to the right of the point 0.

On the number axis, the two points 0 and 1 define a *unit segment* or a segment of unit length. This segment is denoted by $[0,1]$.

[5] A natural number is a positive integer.

[6] Usually this point is chosen to the right of the point 0, but no one forbids us from choosing it to the left. If a teacher does not like this, you can say that you have chosen a *number axis with the opposite orientation*.

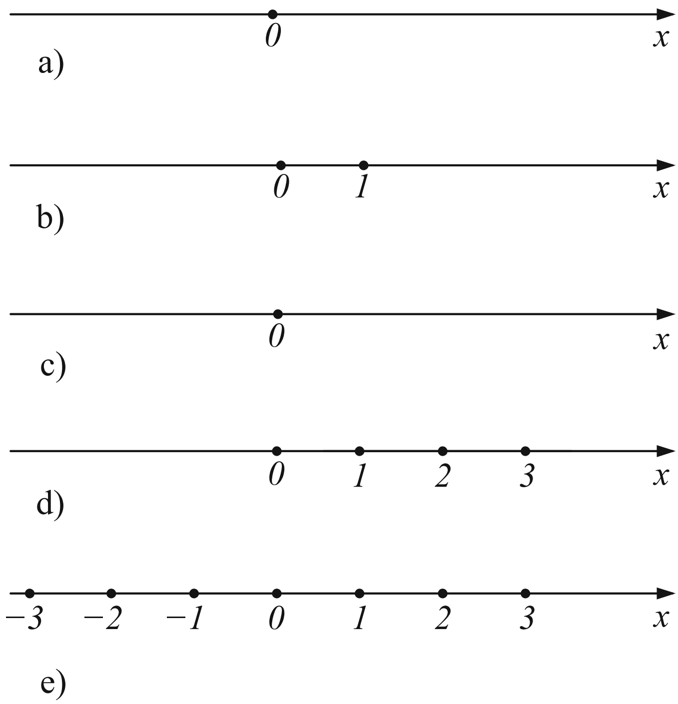


Fig. 16

Note that unlike a segment marked by letters, e.g., AB, a segment marked by numbers has a different notation. By the way, the brackets $[,]$ have the precise meaning that the endpoints are included in the segment. In calculus books you might also meet the notation $(0,1)$, which indicates that the endpoints are not included in that segment.

Given a unit segment, we can construct a segment of any rational length (see Section 3.2 of this appendix) and thus mark on the number axis a point with any rational coordinate $\frac{m}{n}$.

Indeed, we mark segments equal to $[0,1]$ to the right of the point 1 and mark the endpoints as 2, 3 and so on (see Fig. 16d).

Similarly, we mark points to the left of the origin and mark them as $-1,-2,-3,\ldots$ respectively (see Fig. 16e).

We can also divide the segment $[0,1]$ into equal parts. For example, if we divide it into 3 equal parts, we can mark the points $\frac{1}{3}$ and $\frac{2}{3}$. If we divide the segment $[0,1]$ into n parts and mark m of them to the right of the point 0, we obtain the point with the coordinate $\frac{m}{n}$.

Remark 1. We remind you that in Chapter II, in order to draw a segment equal to a given one on the same line, we have to do an auxiliary construction (see Fig. 17a, b and also Section 4, Chapter II). Thus, in order to turn a line into a number axis, we need to use the whole plane and cannot remain on the line. We also need the plane when dividing a segment into a given ratio (Section 3.2 of this appendix).

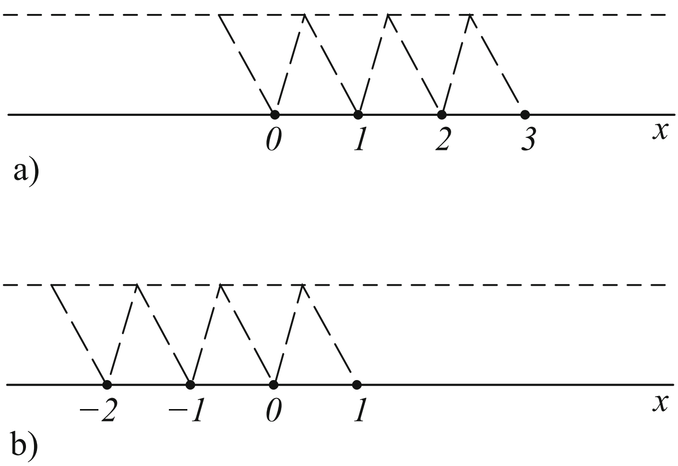


Fig. 17

Remark 2. On the number axis defined above we can mark any point with a rational coordinate. However, as mathematicians say, such a number axis has some "holes." This was discovered already by ancient Greeks, who studied the problem very thoroughly. There are segments with a length that is not a rational number and cannot be marked on the number axis above. For example, if we have a square with side equal to 1, then its diagonal is a segment whose length cannot be expressed by any rational number (see Section 10.1, Chapter IV). There are also other segments whose lengths are not rational numbers. The points on the number axis which do not have rational coordinates are called *irrational points*. Do not think that there are just a few of them. In fact, there are many more irrational points than there are rational points. We will leave this interesting topic for calculus.

PROBLEM 8. On a number axis mark the points with the coordinates $\frac{1}{2}$, $\frac{4}{3}$, $-\frac{4}{5}$.

4.2 Finding the coordinate of a point and length of a segment

We have shown above that on a number axis we can mark a point with a rational coordinate $\frac{m}{n}$ and construct a segment of length $\frac{m}{n}$.

Can we also find the coordinate of a given point on the number axis? Or, given a segment on the number axis, can we determine its length?

The answer to both questions is "yes." In principle this can be done if the coordinate of this point is a rational number or the segment has a rational length. We will show how this can be done.

Consider a number axis x and a point A on it (see Fig. 18a). Let us measure the length of the segment OA, where O denotes the origin.

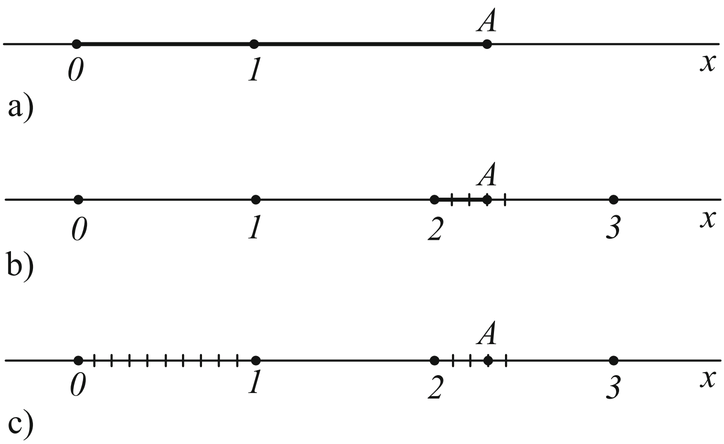


Fig. 18

Note that when defining the number axis, we have chosen the segment $[0,1]$ arbitrarily. Thus, we could choose the unit segment $[0,1]$ to coincide with the segment OA, and then its length[7] would be equal to 1. However, in real life when we measure segments, we usually have to use a unit segment of a certain length, for example, 1 inch, 1 cm, 1 mm, or 1 yard or even 1 mile. For mathematical reasoning, the units of length do not matter because the measuring procedure remains the same.

First, we count how many unit segments can fit completely inside the segment OA. For our example, there are 2 unit segments contained in the segment OA (see Fig. 18b). This means that the length of segment OA is more than 2 but less than 3. Now we only need to measure the length of segment $[2,A]$.

[7] In this case we would have solved the problem of measuring the length of OA, but what about another segment OB on the same number axis?

For this we divide the unit segment into 10 equal parts and count how many such parts will fit completely inside the segment $[2,A]$. In Fig. 18c, there are 3 such parts, so the segment OA has a length bigger than 2.3 but less than 2.4.

We continue the measuring process by dividing the unit segment into 100 parts and counting how many of them fit completely in the remaining part of the segment OA. If the length is rational we will eventually be able to measure it. However, this involves the notion of limit, which is beyond the subject of this book.

If this measuring process never ends, we say that the length of the segment OA is an irrational number (and so is the coordinate of the point A).

5 Affine coordinate systems on the plane

In the section above we have seen that a point A can be represented by a number, its coordinate. To make this representation, we had to choose a line x which passes through point A and two points 0 and 1 on this line.

We can also represent a point A by a pair of numbers, or a pair of coordinates. For this we choose two intersecting lines on the plane and two points. The lines are usually denoted by x and y, and are called the *x-axis* and the *y-axis*. One point is chosen at the intersection of these axes (see Fig. 19a) and is called the origin. We denote it by O (this is not "zero" but the letter "O") and agree upon its coordinates both being equal to 0, a designation which is written as $(0,0)$.

The second point is chosen arbitrarily and its coordinates are considered to be $(1,1)$. This second point $(1,1)$ is usually chosen in the right upper corner of the axes as in Fig. 19b.

These two intersecting lines and the two points define an *affine coordinate system* on the plane.[8]

Besides Fig. 19b the affine coordinate system can be illustrated as in Fig. 19c. The arrows indicate that the point $(1,1)$ is to be chosen in the right upper corner of the axes but has not yet been chosen.

On the affine coordinate plane, the point $(1,1)$ defines a *unit parallelogram*. Indeed, through this point let us draw the lines parallel to the axes (see Fig. 19d). We obtain a point on each axis, and we mark each of these points as 1. The coordinate of the point on the x-axis is called the x-coordinate

[8] Note that in Chapter II, we cannot draw the Cartesian coordinate system that is usually studied at school, because we can neither draw the two perpendicular axes that it requires nor mark equal segments on these axes.

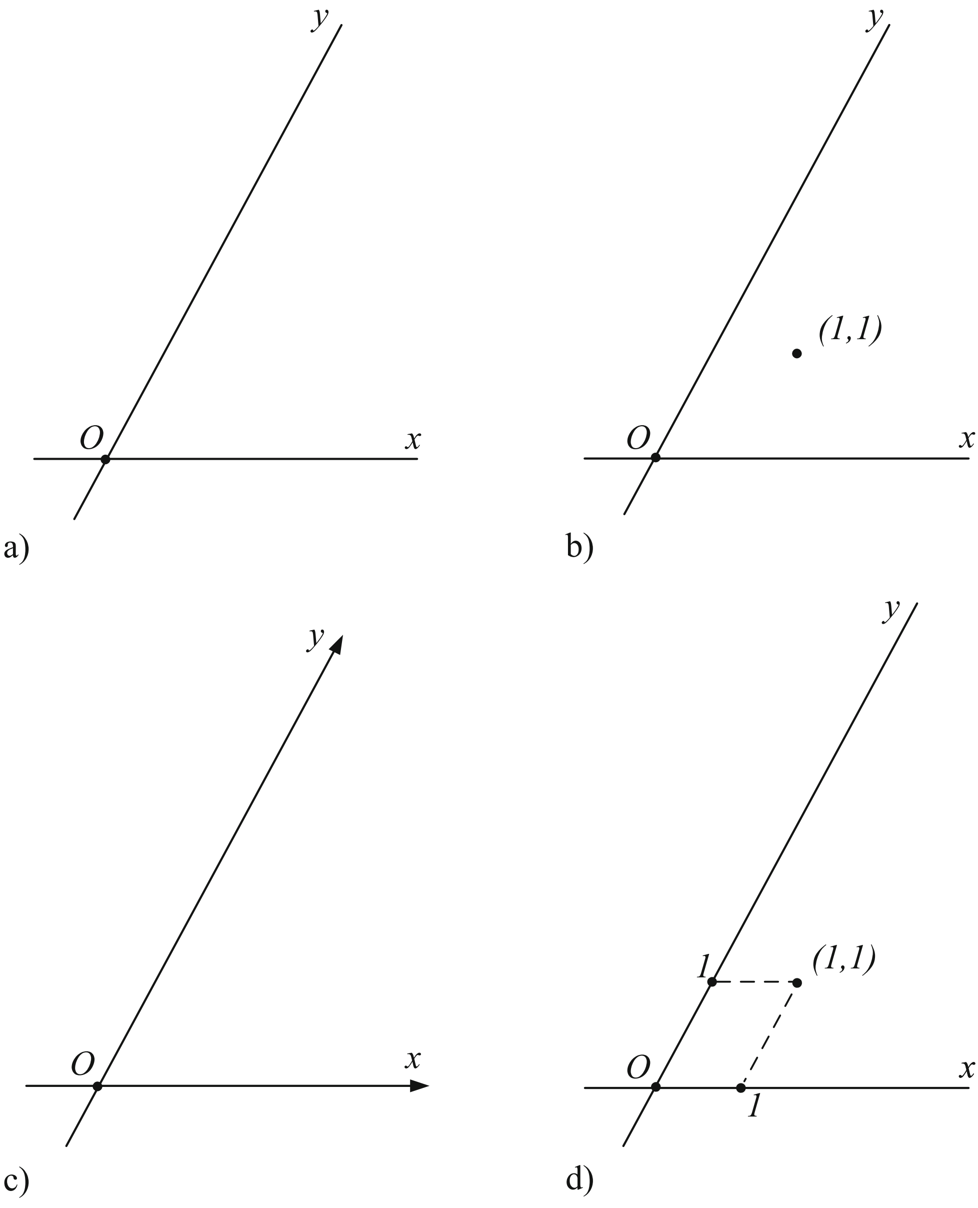


Fig. 19

of the point $(1,1)$. The coordinate of the point on the y-axis is called its y-coordinate. Note that the order of coordinates of a point is very important. First we write its x-coordinate and then its y-coordinate.

If we have chosen a unit parallelogram (see Fig. 19d) on the plane, we

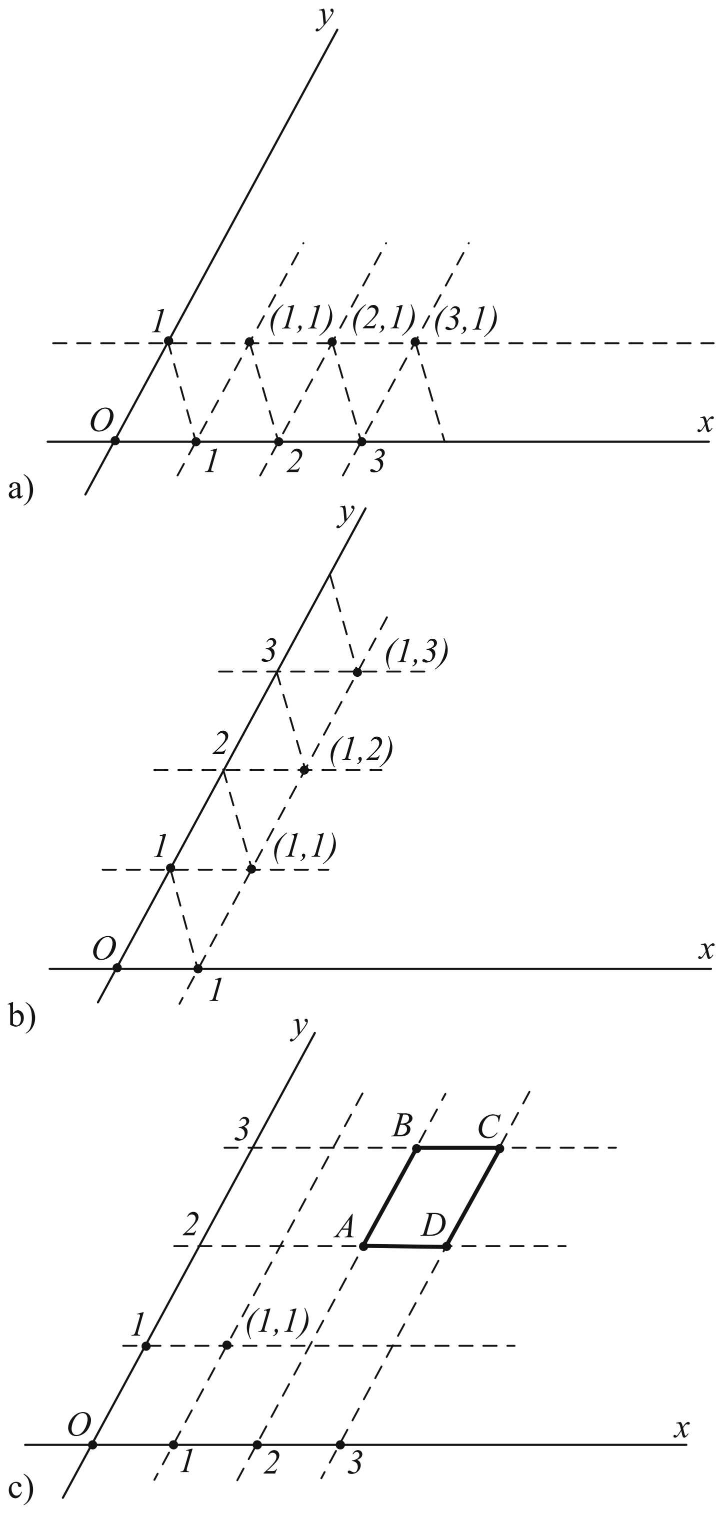


Fig. 20

can now mark points on the plane with any rational coordinates (x, y) similarly to the way we marked points with rational coordinates x on the number axis. For this we just need to draw parallel lines and construct equal segments on them.

For example, Fig. 20a shows how to mark points with x-coordinate equal to 2, 3, or any other whole number and y-coordinate equal to 1. Similarly, Fig. 20b shows points with y-coordinate equal to 2, 3, or any other whole number and x-coordinate equal to 1.

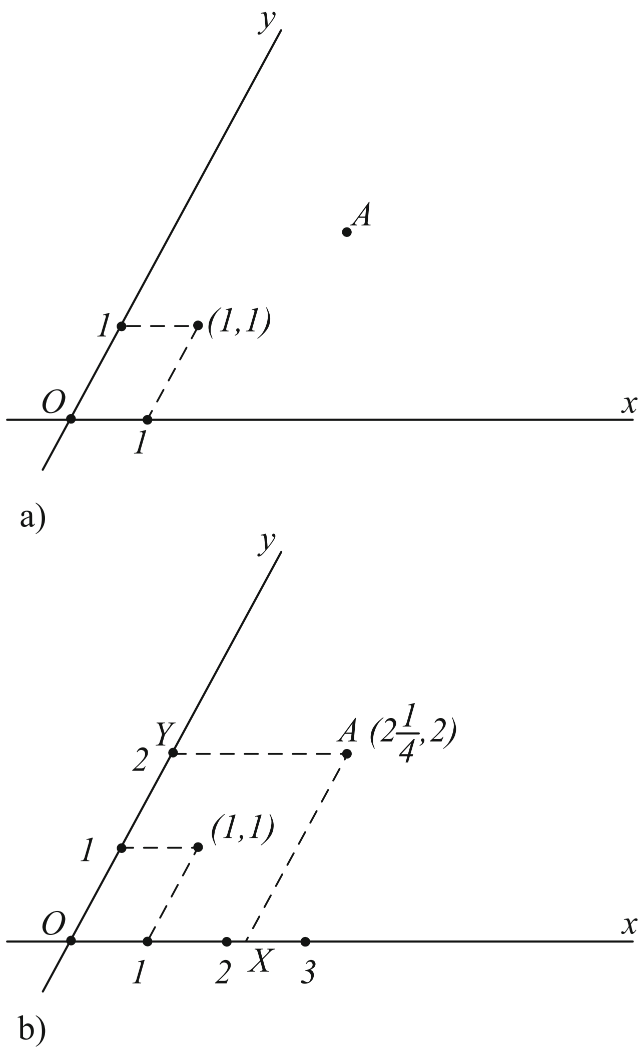


Fig. 21

PROBLEM 9. On a fragment of "graph paper" in the affine coordinate system (Fig. 20c), mark the coordinates of vertices A, B, C, D of a parallelogram.

Exercise 6. A point A with rational coordinates is marked on the affine coordinate plane (see Fig. 21a). Determine the coordinates of this point A.

Solution. Through point A we draw lines parallel to each of the two axes (see Fig. 21b) and extend each of them until it intersects the other axis. We obtain two points, X and Y, on the axes. Each of these axes is a number axis, and we already know how to find the coordinate of a point on it. We find the coordinates of points X and Y, and these two numbers (x, y) are the coordinates of the point A. For our example, point A has coordinates $(2\frac{1}{4}, 2)$.

PROBLEM 10. Draw affine coordinate axes and mark points with the following coordinates:

(a) $(2,1)$, $(1,2)$;

(b) $(2,3)$, $(-2,3)$, $(2,-3)$, $(-2,-3)$;

(c) $(2,5)$, $(-2,-5)$, $(5,2)$, $(-5,-2)$.

PROBLEM 11. What is the convex hull[9] of the following points: $(2,5)$, $(5,2)$, $(-2,5)$, $(5,-2)$, $(-2,-5)$, $(-5,-2)$, $(-5,2)$, $(2,-5)$?

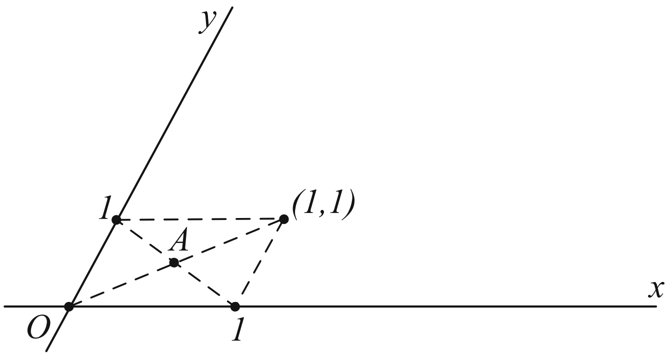


Fig. 22

PROBLEM 12. Fig. 22 shows a unit parallelogram on the affine coordinate plane. Find the coordinates of the point A, which is the center of this parallelogram.

[9]The convex hull of points was defined in Section 12, Chapter I.

PROBLEM 13. Consider a unit parallelogram on the affine coordinate plane (see Fig. 23a).

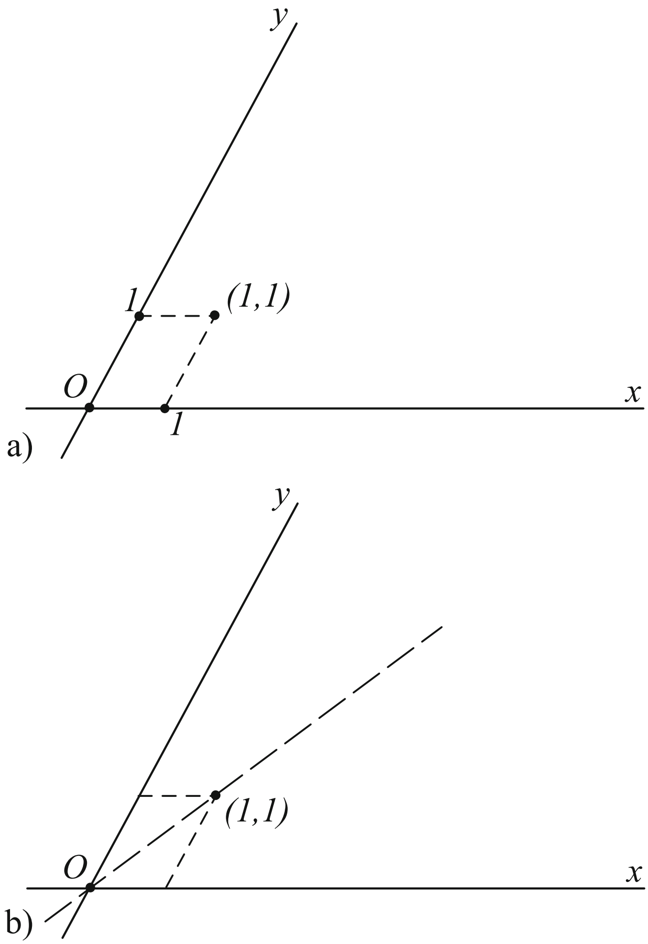


Fig. 23

(a) Mark the points with coordinates $(2,2)$, $(3,3)$, $(4,4)$, $(-1,-1)$, $(-2,-2)$, $\left(\frac{1}{2},\frac{1}{2}\right)$.

(b) In the unit parallelogram, draw the diagonal passing through the origin and extend it in both directions (see Fig. 23b). What can you say about the coordinates (x,y) of a point lying on this line?

PROBLEM 14. What can you say about all the points with affine coordinates (x,y) for which $y=-x$? Is it possible to draw all such points? If yes, do this.

PROBLEM 15. A student John drew a unit parallelogram on the affine coordinate plane (see Fig. 24a). Then he changed his mind and made the parallelogram twice as wide (see Fig. 24b).

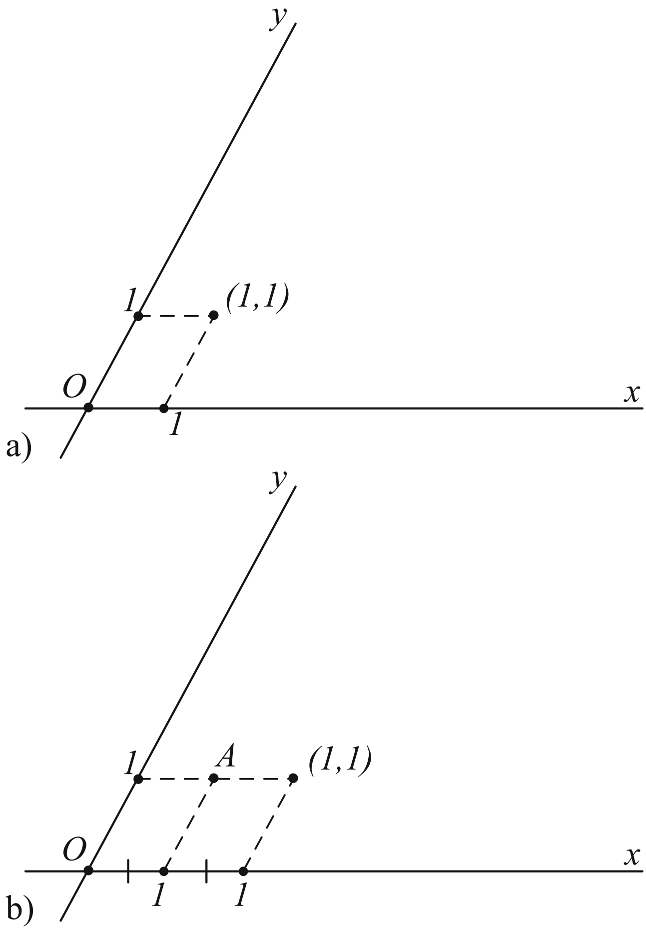


Fig. 24

What are the coordinates of the original point $(1,1)$, i.e., the coordinates of the point A in Fig. 24b?

PROBLEM 16 ($*$) A student John drew a unit parallelogram on the affine coordinate plane (see Fig. 25a) and marked a point A with coordinates (x, y).

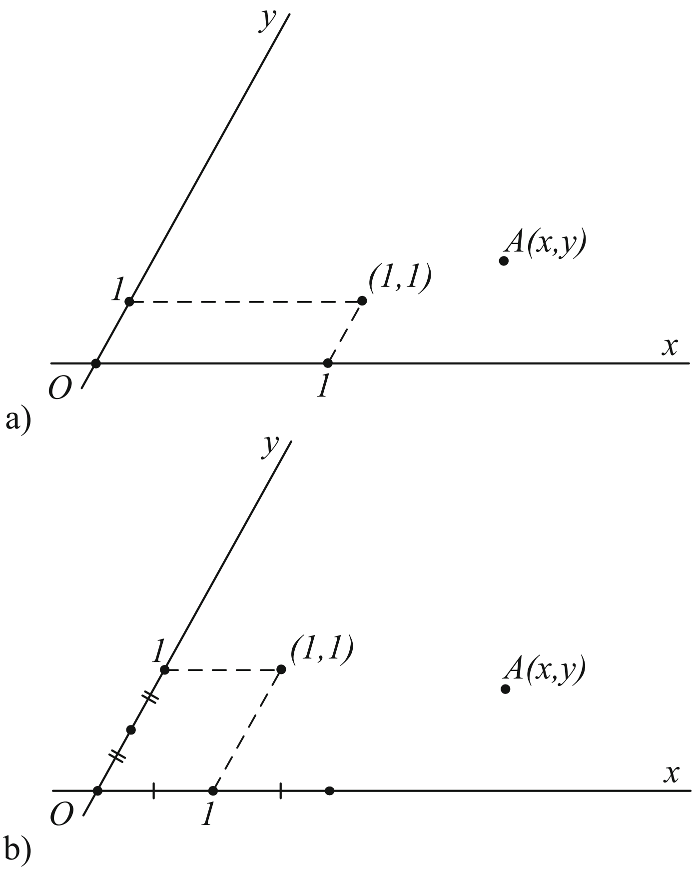


Fig. 25

Then he changed his mind and made the unit parallelogram twice as high and twice as narrow (Fig. 25b shows what John drew). What are the coordinates of the point A in Fig. 25b?

Chapter III

Area: A Look at Symplectic Geometry

1 The area of a figure

Chapter III is optional reading. It does not contain the formulas for calculating area that you may have studied at school. Calculation of areas of different figures based on formulas will be presented in Chapter IV. In this chapter we want to show that it is possible to introduce the area of a figure using only the operations from Chapters I and II, i.e., without introducing a measure of length.[1] We can measure the area of figures geometrically, by comparing these figures with a *unit parallelogram* (see below).

You probably remember from school what the area of a figure is. Let us consider a few questions that require the notion of area for the answer.

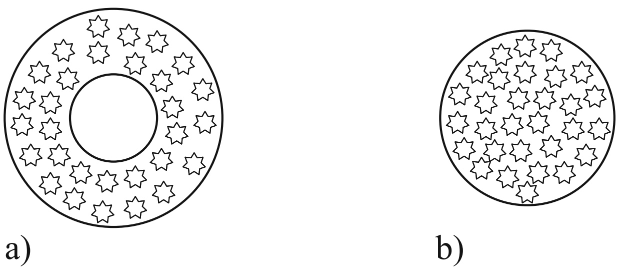


Fig. 3.1

[1] Such geometry is called *symplectic geometry*.

I. M. Gelfand, T. Alekseyevskaya (Gelfand), *Geometry*,
https://doi.org/10.1007/978-1-0716-0299-7_3

Question 1. Fig. 3.1 shows two pieces of land prepared for planting flowers. Suppose that a flower must be one foot away from the fence or from any other flower. In which piece of land will more flowers be planted?

For this problem we need to be able to compare these two pieces of land in the sense above. We can say that the larger the yard, the more flowers we can plant in it. That is, in this problem area may be characterized by the number of flowers planted.

Question 2. Suppose one has to buy paint to paint an apartment. How much paint is necessary?

In this problem the area of the walls of an apartment is described by the amount of paint required to cover them. Note that when buying paint, we do not need to know the shape of the apartment; we just need to know one number—a number characterizing the area of the walls of this apartment.

A similar example is provided by buying fabric for a dress. We do not need to describe exactly what the shape of a dress will be. We just need to know a number describing how much fabric is needed for sewing the dress. Of course, the amount of fabric depends on the design (pattern) and the size of the person for whom the dress is being made.

Question 3. Suppose that we need to cover a big hole in the roof and we have two sheets of the same material (Fig. 3.2). Which of them can cover a larger part of the roof?

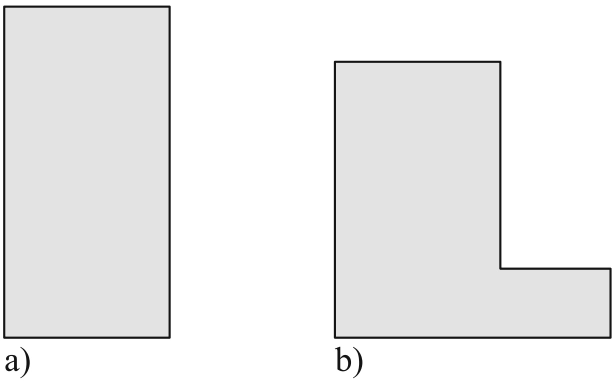


Fig. 3.2

Again, we are interested in the area of these pieces. How can it be measured?

This problem has a good solution: weigh each of these pieces. We will get a number (weight) describing each piece. Clearly, a heavier piece has a larger area, assuming uniform thickness. However, this method cannot always be used to determine the area (see Question 1).

In Fig. 3.3 there are more examples. Which of these figures has the larger area?

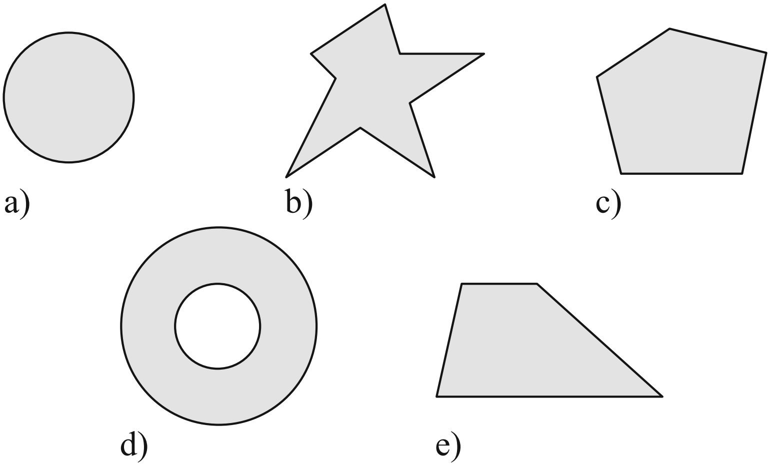


Fig. 3.3

Summarizing the comments above, we can say that *area* is an important characteristic of a figure in the plane. To *measure the area* of a figure is to assign a positive number to this figure. Note that figures with different shapes can have the same area, and figures with the same shape can have different area.

What are the requirements that area has to satisfy? First of all, in order to define area we choose a *unit area*. For our unit area let us choose the area of a parallelogram—the simplest figure that we have considered so far.[2] In Fig. 3.4 there is a parallelogram whose area will be considered equal to 1. Such a parallelogram is called a *unit parallelogram*.

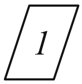


Fig. 3.4

[2]Note that since we have not introduced the length of a segment, we cannot define what a square is.

Remark 1. The unit parallelogram is not chosen once and for all. For example, we can think of a unit parallelogram with area equal to 1 square inch or with area equal to 1 square foot or 1 square mile. Therefore, in each problem we will indicate the corresponding unit parallelogram.

Area satisfies the following conditions (axioms):[3]

(1) If a figure is composed of several parts, then the area of this figure is the sum of the areas of these parts. (See Fig. 3.5.)

(2) If a figure has the same area as some part of a second figure, then the area of the first figure is smaller than the area of the second. (This axiom follows from Axiom 1.) (See Fig. 3.6.)

(3) If two figures are obtained from one another by a parallel translation, then these figures have equal areas. (See Fig. 3.7.)

(4) If two figures are obtained from one another by symmetry with respect to a point, then these figures have equal areas. (See Fig. 3.8.)

The area of a figure is usually denoted by the letter S.

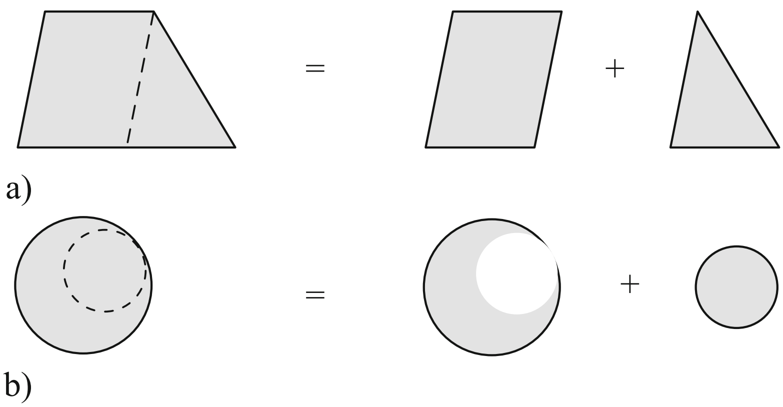


Fig. 3.5

[3]The area of a figure is indicated in the following four figures (Fig. 3.5 through Fig. 3.8) by shading the figure.

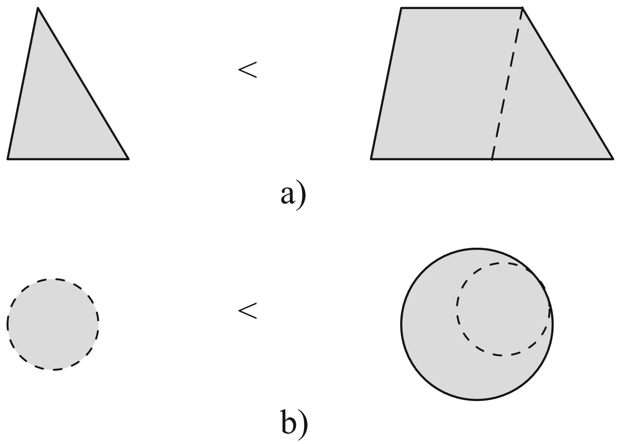


Fig. 3.6

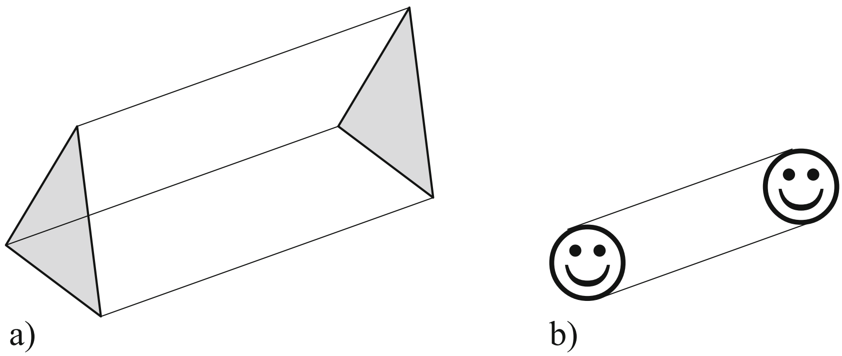


Fig. 3.7

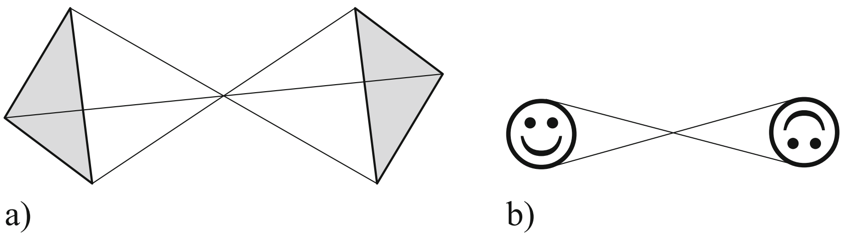


Fig. 3.8

2 Area of a parallelogram

2.1 Constructing parallelograms with rational area

We have already chosen a unit parallelogram (Fig. 3.4). Let us show that we can construct parallelograms whose area is equal to any rational number.

First of all, it is clear that the area of the parallelograms in Fig. 3.9 is equal to 2. Indeed, we made a parallel translation of the unit parallelogram and composed the new parallelogram by putting two unit parallelograms together (see Axioms 1 and 3).

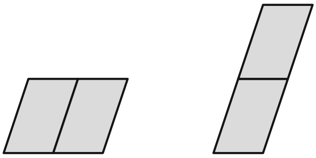


Fig. 3.9

Similarly, the area of each parallelogram in Fig. 3.10a is equal to three. In Fig. 3.10b there are figures (non-parallelograms) whose area is also equal to 3.

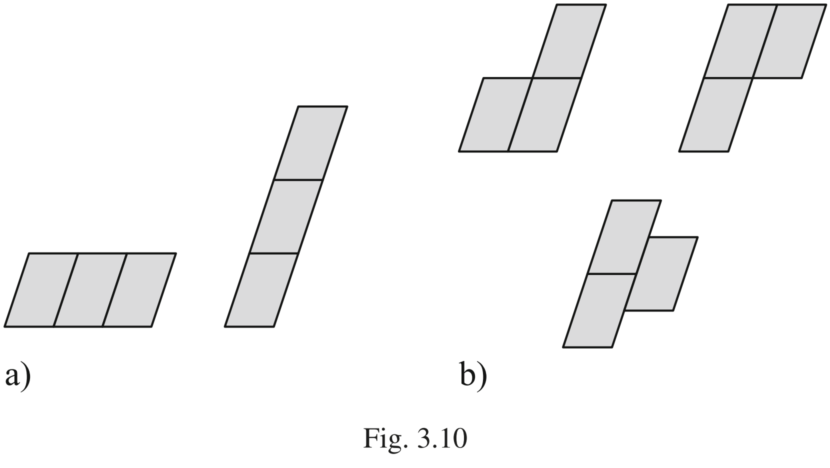


Fig. 3.10

We can compose parallelograms (and some other figures) by combining more than two or three unit parallelograms. Thus, we can construct a parallelogram with area equal to any integer.

Exercise 1. Construct a parallelogram with area equal to 6 and with integer sides.

Solution. Fig. 3.11 shows all four possible solutions.

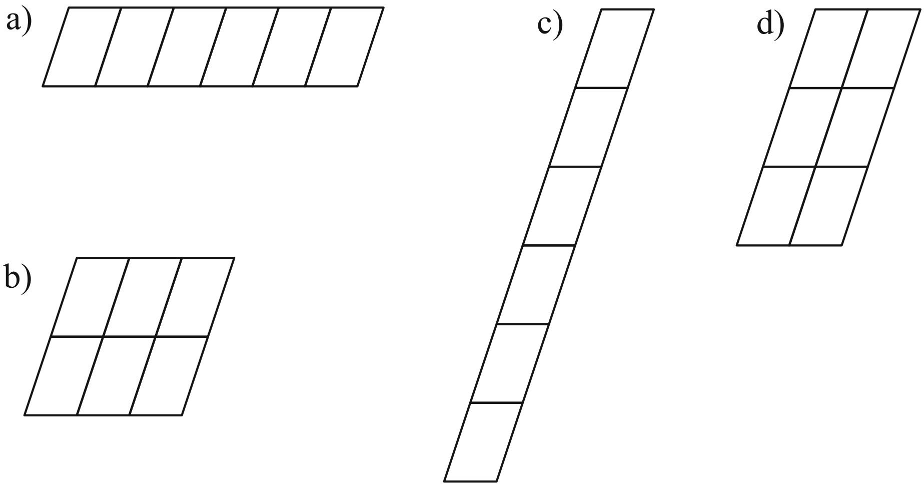


Fig. 3.11

Let us construct a parallelogram with area equal to $\frac{1}{2}$ (see Fig. 3.12).

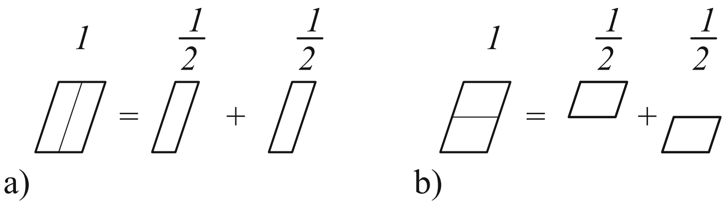


Fig. 3.12

For this we draw a line through the midpoint of one side of the unit parallelogram parallel to the neighboring sides of this parallelogram. Since the unit parallelogram is divided by this line into two equal parts, the area of each of these parts is equal to $\frac{1}{2}$.

Clearly, if we divide[4] a side of the unit parallelogram into three equal segments and draw lines through their endpoints parallel to the neighboring sides, then the unit parallelogram will be divided into three parts with equal area. That is, we can draw a parallelogram with area equal to $\frac{1}{3}$ (see Fig. 3.13).

[4]We remind you that in Chapter II and Section 3.2 of the Appendix, we described how to divide a segment into *n* equal parts.

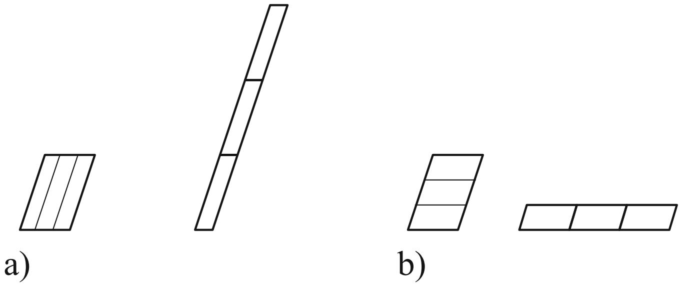


Fig. 3.13

Similarly, by dividing a side of a parallelogram into k equal parts we can draw a parallelogram whose area is equal to $\frac{1}{k}$. (For example, Fig. 3.14a shows a parallelogram with area $\frac{1}{5}$.) A new parallelogram composed of m such parts will have area $\frac{m}{k}$ (e.g., the parallelogram in Fig. 3.14b has area $\frac{3}{5}$).

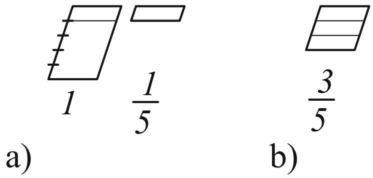


Fig. 3.14

Thus, just as in Chapter II we could construct a segment with any rational length, in Chapter III we can construct a parallelogram with any rational area.

Proposition 1. Consider a parallelogram *ABCD* whose sides are correspondingly parallel to the sides of the unit parallelogram. If one of the sides of *ABCD* is a times larger than the corresponding side of the unit parallelogram, and another side of *ABCD* (not parallel to the first side) is b times larger than the corresponding side of the unit parallelogram, then[5] $S_{ABCD} = ab$.

Proof. First, let a and b be integers. From Fig. 3.15a (in which $a = 6$ and $b = 2$), it is easy to see that we can divide parallelogram *ABCD* into ab parallelograms which are parallel translations of the unit parallelogram. Thus, indeed, $S_{ABCD} = ab$.

[5]Read the proposition carefully. We do *not* claim that the area of any parallelogram whatsoever with sides a and b is equal to ab.

Now let a and b be rational numbers. For example, in Fig. 3.15b $a = \frac{1}{2}$ and $b = \frac{1}{3}$. In this figure we have shaded parallelogram $ABCD$. What is its area? As we can see, six such parallelograms fit into the unit parallelogram. Therefore, the area of the shaded parallelogram is equal to $\frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$.

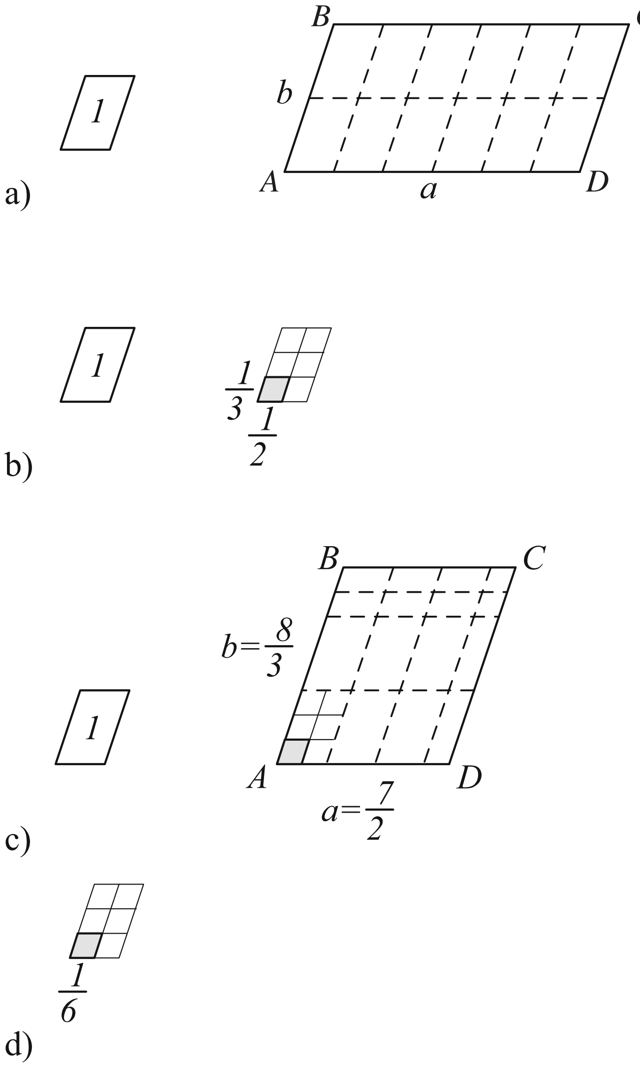


Fig. 3.15

Consider the general case, $a = \frac{m_1}{n_1}$ and $b = \frac{m_2}{n_2}$. For example, in Fig. 3.15c we have $a = \frac{7}{2}$ and $b = \frac{8}{3}$. In this case we divide the unit parallelogram into smaller parts by dividing side a into n_1 equal parts and side b into n_2 equal parts (see Fig. 3.15d, where $n_1 = 2$ and $n_2 = 3$). The unit parallelogram consists of $n_1 \cdot n_2$ such parts, so the area of each of these small parts is $\frac{1}{n_1 \cdot n_2}$.

Now we count how many such parts fit into our parallelogram. Along side a there are m_1 such parts, and along side b there are m_2 such parts. (In Fig. 3.15c there are 7 and 8 parts along the corresponding sides.) Therefore, all together there are $m_1 \cdot m_2$ parts, each of which has the area $\frac{1}{n_1 \cdot n_2}$. (In Fig. 3.15c we have $7 \cdot 8 = 56$ parts, so the area of $ABCD$ is $7 \cdot 8 \cdot \frac{1}{2 \cdot 3} = \frac{56}{6} = \frac{7}{2} \cdot \frac{8}{3}$.)

Finally, taking all parts together, the area of a parallelogram with sides $a = \frac{m_1}{n_1}$ and $b = \frac{m_2}{n_2}$, provided that its sides are parallel to the sides of the unit parallelogram, is $S_{ABCD} = m_1 \cdot m_2 \cdot \frac{1}{n_1 \cdot n_2} = ab$. □

PROBLEM 1.

(a) Fig. 3.16a shows a unit parallelogram and a shaded parallelogram one of whose sides is two times larger than the longer side of the unit parallelogram and whose other side is 5 times larger than the shorter side of the unit parallelogram. What is the area of the shaded parallelogram in Fig. 3.16a?

(b) What is the area of the parallelogram in Fig. 3.16b?

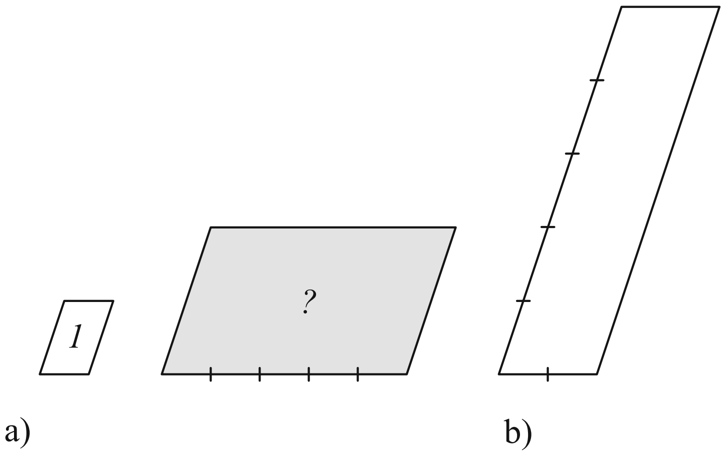


Fig. 3.16

PROBLEM 2.

(a) Suppose the big parallelogram in Fig. 3.17 has unit area. What is the area of the shaded parallelogram?

(b) Choose a unit parallelogram and construct a parallelogram with area equal to $\frac{1}{4}$. Present several solutions.

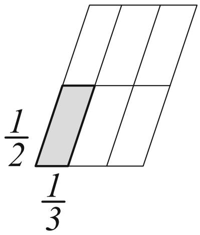


Fig. 3.17

PROBLEM 3. Let a parallelogram (as in Fig. 3.18) be a unit parallelogram. Divide one side into 5 equal parts and a neighboring side into 4 equal parts. Through these points draw lines parallel to the sides of the parallelogram.

(a) Into how many parts has the unit parallelogram been divided?

(b) What is the area of each of these parts?

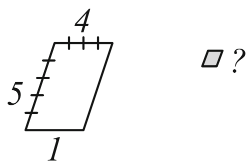


Fig. 3.18

2.2 Different unit parallelograms

We have chosen a unit parallelogram and, using parallel translations of this unit parallelogram, constructed parallelograms with rational area. It is also useful to have different parallelograms with unit area.

Changing the length of the sides of a unit parallelogram

Given a unit parallelogram (Fig. 3.19a), is it possible to construct a parallelogram one of whose sides is two times smaller than the side of the unit parallelogram and such that it has area equal to 1? The answer is "yes."

In order to show this let us divide the unit parallelogram into two parts and then put these parts together. There are two possibilities for this (see Fig. 3.19b and Fig. 3.19c). We obtain a new parallelogram whose area is still

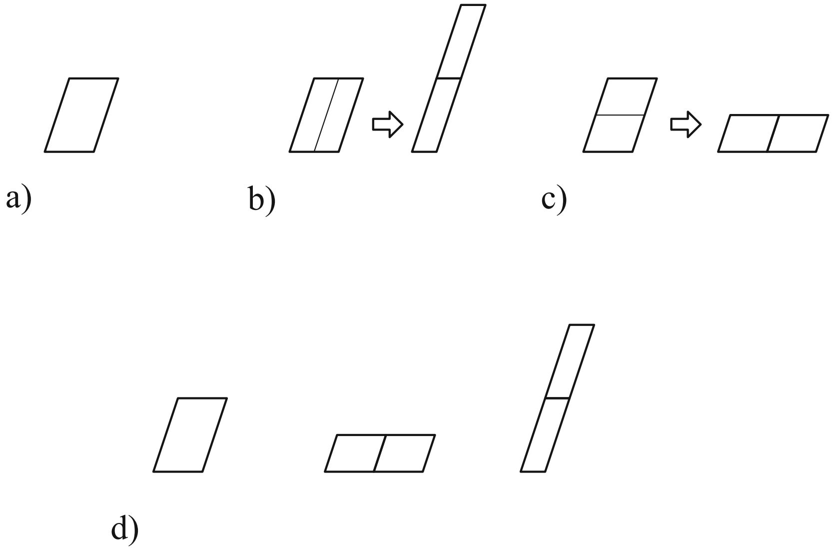


Fig. 3.19

equal to 1 and one of whose sides is two times smaller than the corresponding side of the given unit parallelogram.

Thus, we now have three unit parallelograms (Fig. 3.19d).

Similarly, we can construct a parallelogram one of whose sides is three times smaller than the corresponding side of the unit parallelogram, and construct it in such a way that its area is equal to 1 (see Fig. 3.20a, b, c).

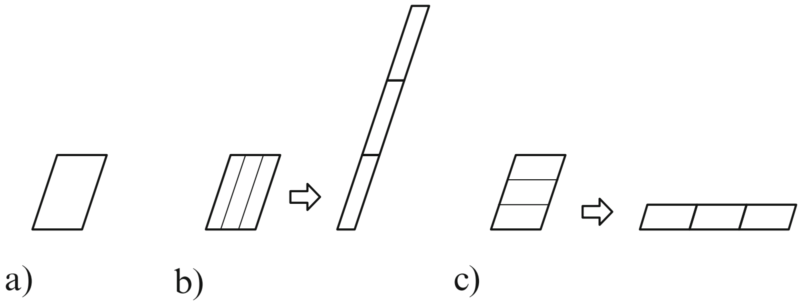


Fig. 3.20

PROBLEM 4. Fig. 3.21 shows a parallelogram. Construct a parallelogram with equal area that has a side equal to one fourth of one side of the original parallelogram (see Fig. 3.21).

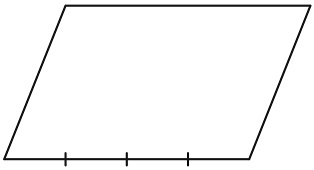


Fig. 3.21

Changing the direction of the sides of a unit parallelogram

Suppose we have chosen a unit parallelogram, but then we would like its sides to lie on certain specific lines a and b (see Fig. 3.22a). Can we replace it with a new parallelogram which does this but whose area will remain the same? The answer is "yes." We will first show how to do this, and then prove that our construction gives us what we want.

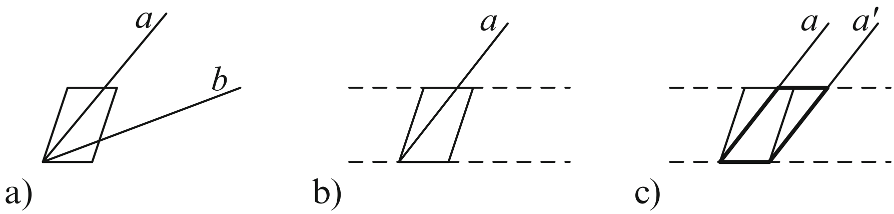


Fig. 3.22

Let us consider a unit parallelogram and line a. Draw line $a' \parallel a$ as in Fig. 3.22b). As we will prove in Proposition 2, the parallelogram formed by the bold lines in Fig. 3.22c has area equal to 1.

Now consider this new parallelogram in Fig. 3.22c. It has two sides parallel to line a. We need to replace it with another parallelogram whose other pair of sides is parallel to line b, a parallelogram which also has area equal to 1. We can repeat the procedure above (see Fig. 3.23b).

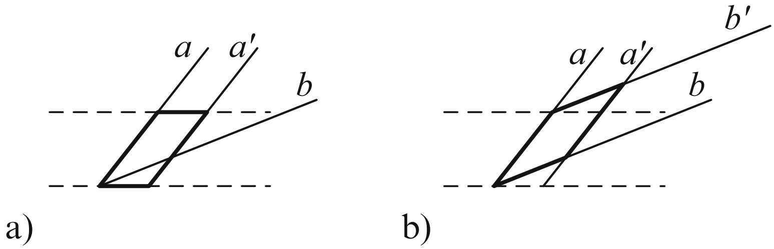


Fig. 3.23

The new parallelogram (formed by the bold lines in Fig. 3.23b) has its sides parallel to lines a and b and area equal to 1.

Proposition 2. Suppose there are two parallel lines a and a'. Consider two parallelograms which have a common side lying on line a. Suppose that the opposite side of each of these parallelograms lies on line a'. Then these parallelograms have equal areas.

Proof. Let us draw such parallelograms (see Fig. 3.24a). We need to prove that the area of $ABCD$ is equal to the area of $ABC'D'$, i.e., $S_{ABCD} = S_{ABC'D'}$.

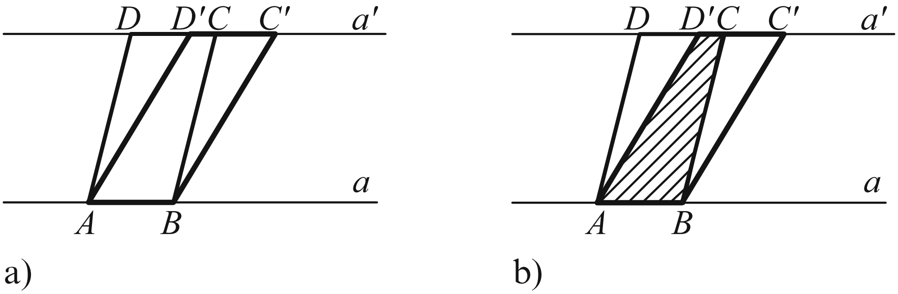


Fig. 3.24

We see that the part $ABCD'$ is common to these parallelograms (it is shaded in Fig. 3.24b). Triangle ADD' is a parallel translation of triangle BCC', and from Axiom 3, $S_{ADD'} = S_{BCC'}$. Each parallelogram is composed of the common area and one of the translated triangles. Therefore, the areas of the two parallelograms are equal: $S_{ABCD} = S_{ABC'D'}$.

However, the proof is not finished. Indeed, there is another possible case (see Fig. 3.25), where the sides DC and $D'C'$ of the parallelograms do not overlap.

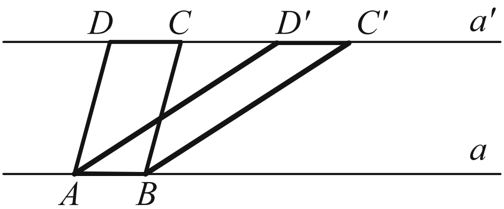


Fig. 3.25

The statement can be also proved for this case, but it is more complicated and we will not do it here. □

Remark 2. From Proposition 2 it follows that, when measuring area, we can substitute for all such parallelograms a rectangle with the same area (see

Fig. 3.26). However, since in Chapter III we cannot define a rectangle, we will do this only in Chapter IV.

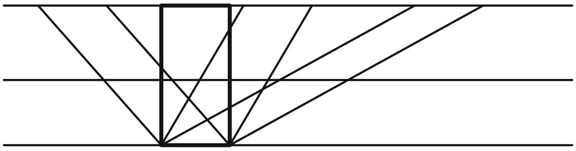


Fig. 3.26

PROBLEM 5. In Fig. 3.27 lines a and b are parallel to each other. Prove that $S_{ABCD} = S_{ABC'D'}$.

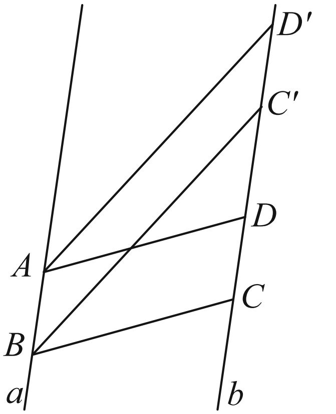


Fig. 3.27

PROBLEM 6.

(a) Fig. 3.28a shows a parallelogram and two rays a and b. Construct a parallelogram whose area is the same as the area of the given parallelogram and whose sides are parallel to a and b.

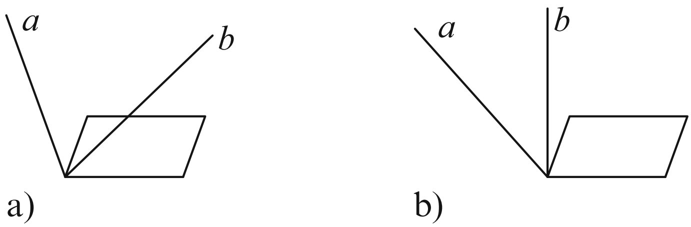


Fig. 3.28

(b) Solve this problem for Fig. 3.28b.

2.3 How to measure the area of a parallelogram

Fig. 3.29a shows an arbitrary parallelogram. How can we measure its area?

First of all we need a unit parallelogram. If we have already chosen one (see Fig. 3.29b), we can replace it (according to Proposition 2) by a parallelogram whose sides are parallel to the sides of the given parallelogram and whose area is equal to 1. (See Fig. 3.30a.) If we have not yet chosen a unit parallelogram, we may choose it initially with the sides parallel to the sides of the given parallelogram.

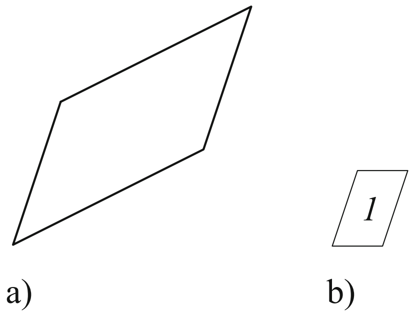


Fig. 3.29

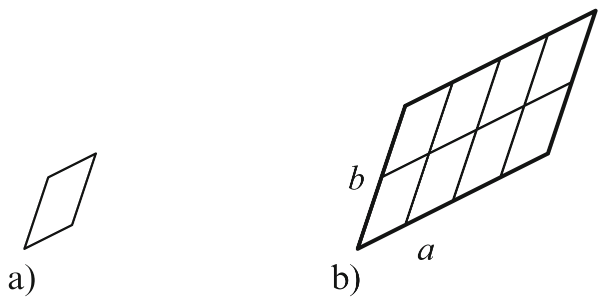


Fig. 3.30

Then we compare the corresponding sides of the given parallelogram and the unit parallelograms. If these ratios are a and b, then according to Proposition 1, the area of the given parallelogram is ab. For example, in Fig. 3.30b we have $a = 4$ and $b = 2$. Thus, the area that we measure is $2 \cdot 4 = 8$.

2.4 How a diagonal of a parallelogram divides its area

We have shown above how to measure the area of an arbitrary parallelogram. In the following sections we will show how to reduce measuring the area of a geometric figure to measuring the area of a parallelogram.

Proposition 3. A parallelogram is divided by its diagonal into two parts with equal area.

Proof. Consider parallelogram $ABCD$ with diagonal BD (see Fig. 3.31).

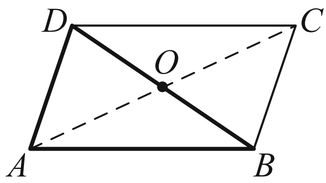


Fig. 3.31

Let us mark the center O of this parallelogram. Triangles ABD and BCD are symmetric to each other with respect to point O (this follows from Proposition 14 in Chapter II). Therefore, according to Axiom 4, $S_{ABD} = S_{BCD}$. □

It follows from Proposition 3 above that in Fig. 3.31 $S_{ABD} = \frac{1}{2}S_{ABCD}$.

We can also reformulate Proposition 3 as follows:

Any triangle can be completed to form a parallelogram whose area is two times larger than the area of this triangle.

PROBLEM 7.

(a) Draw a triangle. Complete this triangle to form a parallelogram with area two times larger than the area of the triangle. Find three different ways to do this. (Draw three separate pictures).

(b) Prove that all three parallelograms obtained have the same area.

3 Area of a triangle

Consider an arbitrary triangle. We have already seen that its area is half the area of a certain parallelogram. This parallelogram has two sides in common with the triangle, and one of its diagonals coincides with the third side of the triangle. We can also compare the area of a triangle with the area of another parallelogram.

Consider a triangle (see Fig. 3.32a). Let us construct a parallelogram which has one common side with the triangle and another side coincident with half of another side of the triangle (see Fig. 3.32b).

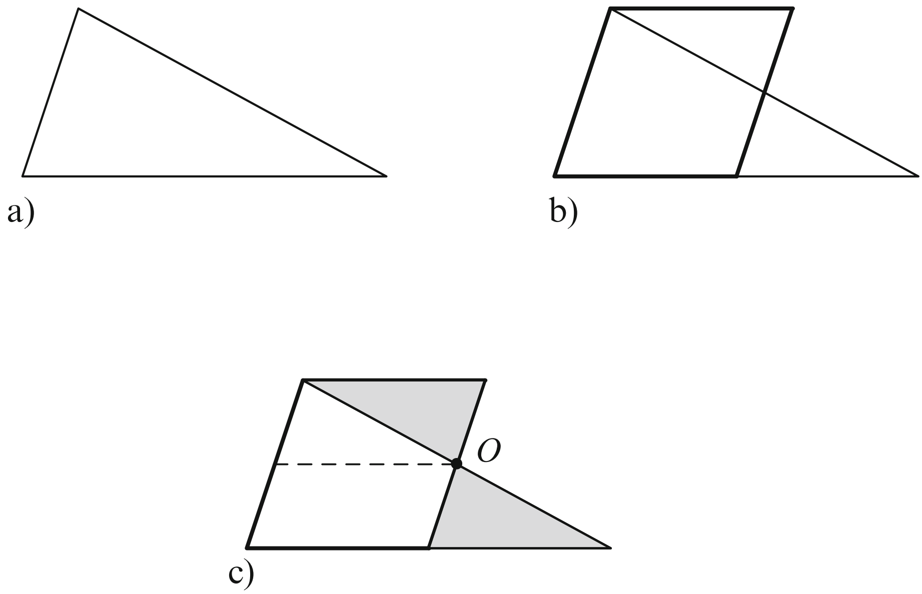


Fig. 3.32

Theorem 1. The area of a triangle is equal to the area of a parallelogram, one of whose sides is a side of the triangle, while another of its sides is half of another side of the triangle.

Proof. We need to prove that the area of the triangle (see Fig. 3.32a) and the area of the parallelogram (see Fig. 3.32b) are equal.

Notice that the parallelogram can be obtained from the triangle by replacing one of the shaded triangles in Fig. 3.32c by the other. By Axiom 4, these two shaded triangles have equal areas, since they are symmetric to each other with respect to point O. This finishes the proof. □

Theorem 1 means that if we want to measure the area of a triangle, we can substitute for it a parallelogram of equal area and then measure the area of this parallelogram.

Remark 3. In Theorem 1 there was no indication which side of the triangle is also a side of the parallelogram. Different choices are possible. Let us denote the triangle in Fig. 3.32a as ABC (see Fig. 3.33a). If we choose the midpoint of side AC and draw a line parallel to AB we obtain the parallelogram in Fig. 3.33b (the same as in Fig. 3.32b). There is another possibility for the same midpoint: we can draw a line parallel to BC and obtain another parallelogram (see Fig. 3.33c).

There are three sides in a triangle, so we can choose the midpoint on any of them. Thus, there are six possibilities for drawing a parallelogram with its area equal to the area of a triangle.

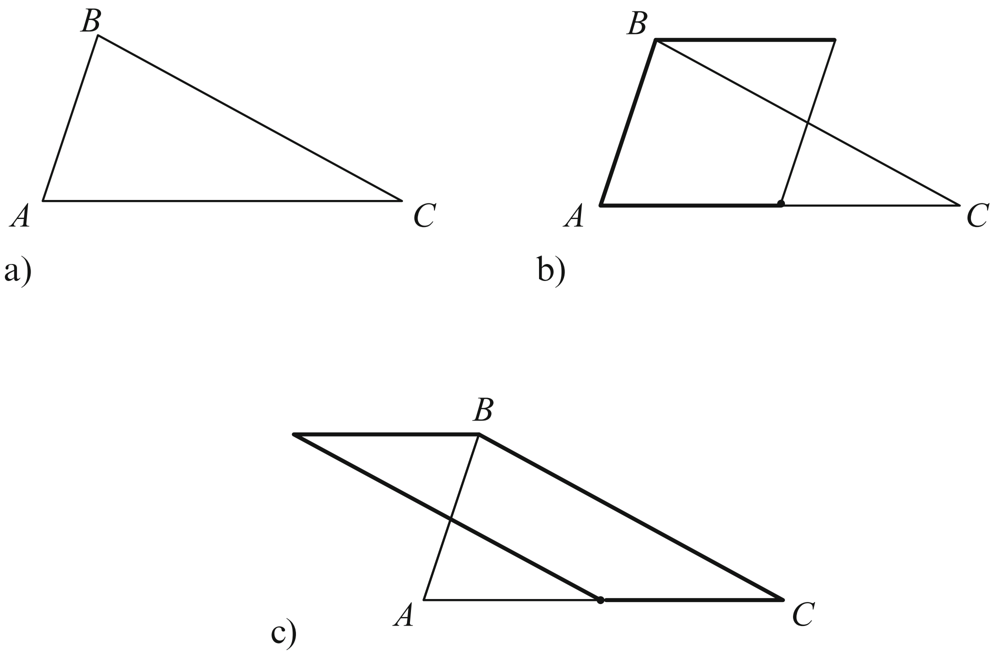


Fig. 3.33

PROBLEM 8. Draw a triangle. For this triangle draw all six possible parallelograms mentioned in Theorem 1 separately.

Proposition 4. Let ABC and $A'B'C'$ be two triangles such that sides AC and $A'C'$ lie on the same line a and points B, B' lie on a line b parallel to line a. Let $AB \parallel A'B'$. Let also $A'C' = r \cdot AC$, where r is a rational number. Then $S_{A'B'C'} = r \cdot S_{ABC}$.

Proof. In Fig. 3.34a we have two triangles that satisfy the proposition. Let us complete each of them to form a parallelogram as in Fig. 3.34b.

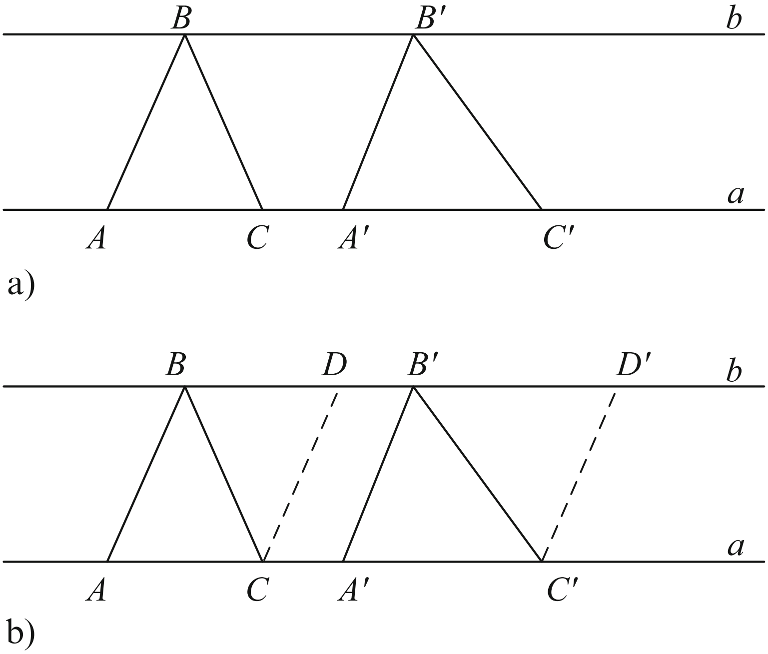


Fig. 3.34

These parallelograms have sides correspondingly parallel to each other. Let us choose a unit parallelogram with sides parallel to the sides of these parallelograms. According to Proposition 1, the area of parallelogram $ABDC$ is equal to the length of AC multiplied by the length of AB, and the area of parallelogram $A'B'D'C'$ is equal to the length of $A'C'$ multiplied by the length of $A'B'$. But $AB = A'B'$ and $A'C' = r \cdot AC$, and therefore $S_{A'B'D'C'} = r \cdot S_{ABDC}$.

Since the area of each of the triangles ABC and $A'B'C'$ equals half of the area of the corresponding parallelogram, we obtain $S_{A'B'C'} = r \cdot S_{ABC}$. □

There are some useful statements about area related to a triangle, its bimedian, and its median.

Proposition 5. A bimedian of a triangle cuts off a triangle whose area is $1/4$ of the area of the initial triangle.

Proof. Consider a triangle with a bimedian in it (see Fig. 3.35).

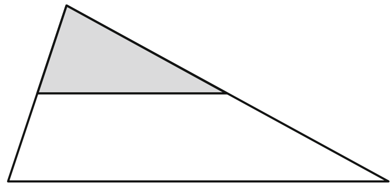


Fig. 3.35

We need to prove that the area of the shaded triangle is $\frac{1}{4}S_{\triangle}$. Let us draw the other two bimedians (Fig. 3.36). Triangles 1, 2, and 3 can be obtained from each other by parallel translation. Therefore, their areas are equal according to Axiom 3.

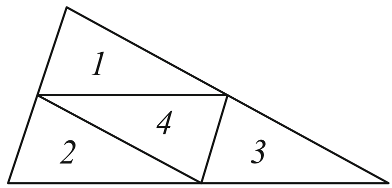


Fig. 3.36

Note that the quadrilateral formed by the two triangles 1 and 4 put together is a parallelogram, since bimedians are parallel to the corresponding sides of the triangle. As we know from Proposition 3, a diagonal divides a parallelogram into two triangles with equal area. Thus, the areas of triangles 1 and 4 are equal. We obtain the result that a given triangle is divided by its bimedians into 4 parts with equal area. Therefore, the area of the shaded triangle is indeed $\frac{1}{4}S_{\triangle}$. □

Proposition 6. A median of a triangle divides it into parts with equal area.

Proof. Consider a triangle with a median in it (see Fig. 3.37a). Let us complete it to form a parallelogram *ABCD* as shown in Fig. 3.37b.

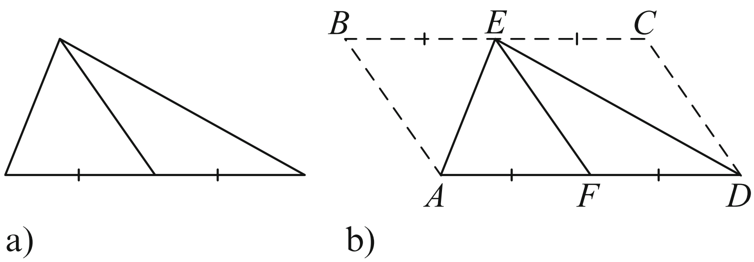


Fig. 3.37

Median EF divides this parallelogram into two parallelograms $ABEF$ and $FECD$. One can easily check that each of these parallelograms is a parallel translation of the other. Therefore, their areas are equal. But AE is a diagonal in $ABEF$ and ED is a diagonal in $FECD$. We obtain $S_{AEF} = \frac{1}{2}S_{ABEF}$ and $S_{FED} = \frac{1}{2}S_{FECD} = \frac{1}{2}S_{ABEF}$. Thus, $S_{AEF} = S_{FED}$. □

PROBLEM 9. Suppose we have a triangular piece of chocolate. Divide it into four parts of equal area (or we can say, divide this chocolate fairly into four parts).

(a) Find solutions based on drawing only bimedians in a triangle.

(b) Find solutions based on drawing only medians in a triangle.

PROBLEM 10. Suppose you have a piece of chocolate in the shape of a parallelogram. Divide it into 4 parts with equal areas (or divide the chocolate fairly into four parts).

(a) Find solutions based on drawing parallel lines (or finding midpoints of segments).

(b) Find a solution which does not require any such construction.

PROBLEM 11. In Fig. 3.38 there is a triangle with a midpoint A on one of its sides. Through point A we draw two lines parallel to the two sides of the triangle. Prove that the area of the parallelogram obtained (shaded in Fig. 3.38) is half of the area of the triangle.

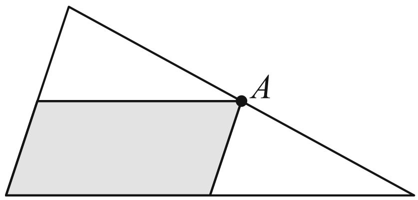


Fig. 3.38

PROBLEM 12. In triangle ABC in Fig. 3.39, points K, L, and M are the midpoints of the sides. Prove that the area of the shaded triangle KLM is four times as small as the area of $\triangle ABC$.

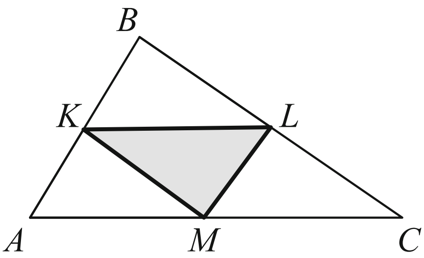


Fig. 3.39

Exercise 2. Fig. 3.40 shows triangle ABC with point D on side BC. Segment MN is the bimedian parallel to side BC, and points K and L are the midpoints of segments BD and DC, respectively.

Which area is larger: the area of quadrilateral[6] $KLMN$ or the area of the rest of triangle ABC? That is, we need to compare S_{KLMN} and $S_{AMN} + S_{BKN} + S_{CLM}$.

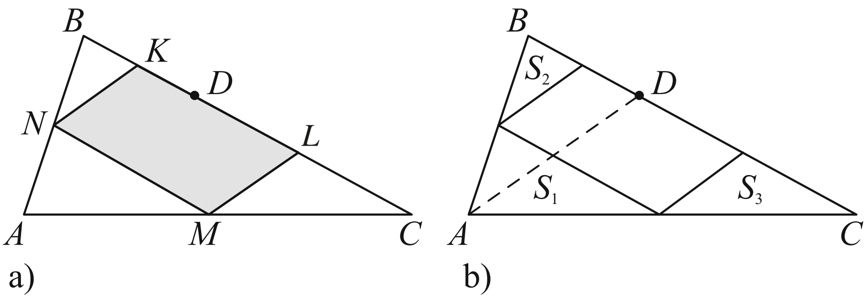


Fig. 3.40

Solution. Let us imagine that we cut out parallelogram $KLMN$ and let us find the area of the "leftovers." For simplicity denote the area of these pieces (triangles) as S_1, S_2, and S_3 (see Fig. 3.40b). From Proposition 5 we have $S_1 = \frac{1}{4}S_{ABC}$.

What can we say about the area of the other pieces? If we connect the points A and D, we see that $S_2 = \frac{1}{4}S_{ABD}$ and $S_3 = \frac{1}{4}S_{ADC}$. Therefore, $S_1 + S_2 + S_3 = \frac{1}{4}S_{ABC} + \frac{1}{4}S_{ABD} + \frac{1}{4}S_{ADC} = \frac{1}{4}S_{ABC} + \frac{1}{4}S_{ABC} = \frac{1}{2}S_{ABC}$. Thus, the area of the shaded quadrilateral $KLMN$ is equal to the area of the rest of $\triangle ABC$.

[6] The quadrilateral $KLMN$ is a parallelogram (see Problem 12 in Chapter II).

PROBLEM 13. Fig. 3.41 shows two parallel lines a and b and several triangles that have a common side. Prove that these triangles have the same area.

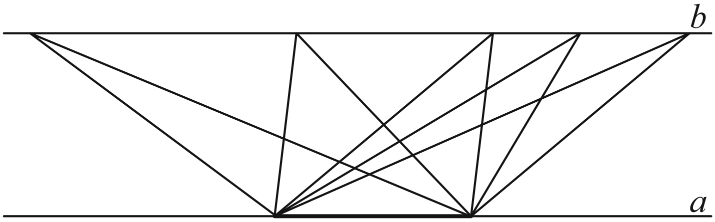


Fig. 3.41

Hint. Compare with Fig. 3.26 and Remark 2.

PROBLEM 14. Consider a point O and a straight line a. On line a let us mark equal segments AB, BC, CD, ... and connect them with the point O (see Fig. 3.42).

Prove that $S_{OAB} = S_{OBC} = S_{OCD} = \cdots$.

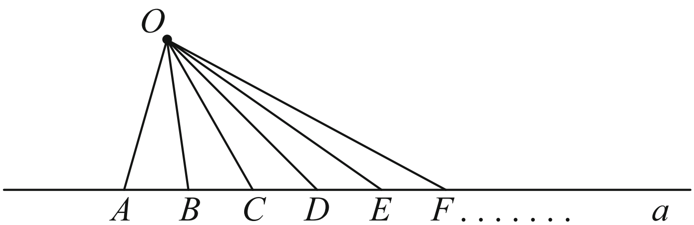


Fig. 3.42

PROBLEM 15. Consider a triangle. Divide one of its sides into three equal parts. Through the first mark draw the straight line parallel to another side of the triangle as in Fig. 3.43.

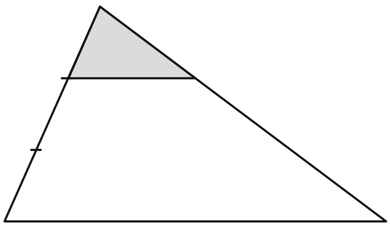


Fig. 3.43

Prove that the area of the shaded triangle in Fig. 3.43 is equal to $1/9$ of the area of the given triangle.

PROBLEM 16. Consider a triangle. Divide one of its sides into ten equal parts. Through the first mark draw the straight line parallel to another side of the triangle as in Fig. 3.44. If the area of the initial triangle is $S_\triangle$, what can you say about the area of the shaded triangle in Fig. 3.44?

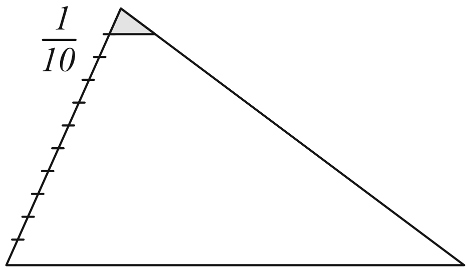


Fig. 3.44

PROBLEM 17 (∗) In Fig. 3.45a and Fig. 3.45b there are two triangles. One of them is a parallel translation of the other. On the first triangle, we divide one side into three equal parts, and through one mark draw the line parallel to another side of the triangle (see Fig. 3.45a). On the second triangle we further divide the corresponding side into 9 equal segments, and through their endpoints we draw lines parallel to the other two sides of the triangle as shown in Fig. 3.45b.

(a) Compare the area of the shaded triangle in Fig. 3.45a with the area of all shaded triangles together in Fig. 3.45b.

(b) In the triangle in Fig. 3.45a, let us draw the line parallel to the divided side of the triangle (see Fig. 3.45c). In what ratio does this line divide the area of the given triangle?

(c) We divide each of two sides of the triangle into three equal segments, and through their endpoints, draw lines parallel to other sides of the triangle as in Fig. 3.45d. Compare the areas S_1, S_2, S_3, and S_4.

(d) In the triangle in Fig. 3.45b, do all the vertices of the shaded triangles lie on the same line as it is drawn in Fig. 3.45e? If the answer is "yes," what can you say about this line?

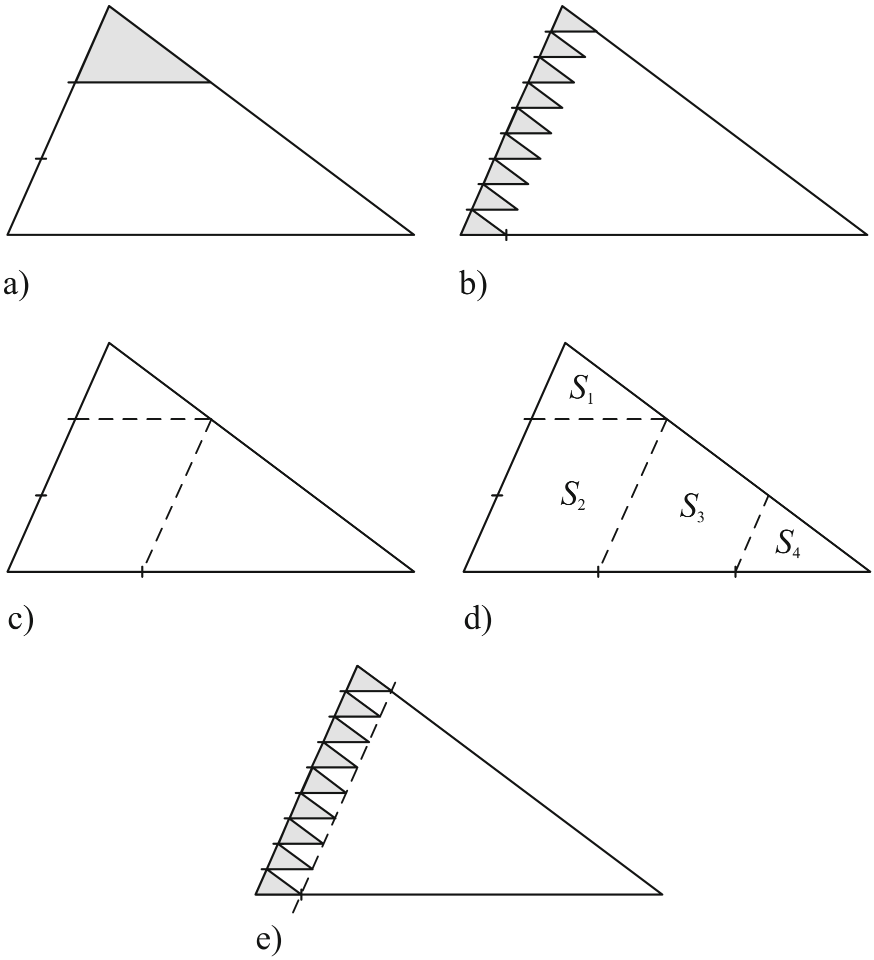


Fig. 3.45

4 Area of a trapezoid

When we considered the area of a triangle, we substituted for it a parallelogram with equal area. Let us do the same for a trapezoid.

Consider a trapezoid with median EF (see Fig. 3.46a). Through point F let us draw a line parallel to another side of the trapezoid (see Fig. 3.46b). We obtain a parallelogram. The theorem below states that the area of the trapezoid and the area of this parallelogram are equal.

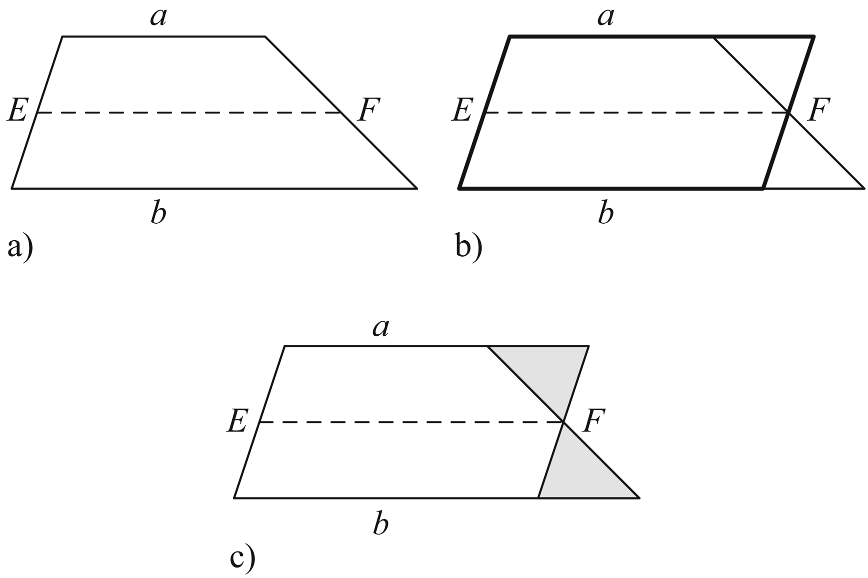


Fig. 3.46

Theorem 2. The area of a trapezoid is equal to the area of a parallelogram that has two of its sides lying on the parallel sides of the trapezoid with lengths equal to that of the trapezoid's median, and its other two sides parallel to one of the other sides of the trapezoid.

Proof. We need to prove that the area of the trapezoid in Fig. 3.46a is equal to the area of the parallelogram in Fig. 3.46b.

We notice that the parallelogram can be obtained from the trapezoid by replacing one of the shaded triangles in Fig. 3.46c with the other. It is easy to check that these two shaded triangles are symmetric to each other with respect to point F. Thus, their areas are equal. Therefore, the area of the trapezoid is equal to the area of the parallelogram. □

PROBLEM 18. Construct a parallelogram equal in area to the trapezoid in Fig. 3.46a.

Hint. Use point E.

Consider a trapezoid with its diagonal (see Fig. 3.47a). The diagonal divides the trapezoid into two triangles (see Fig. 3.47b and Fig. 3.47c). Clearly, the area of the trapezoid is equal to the sum of the areas of these two triangles.

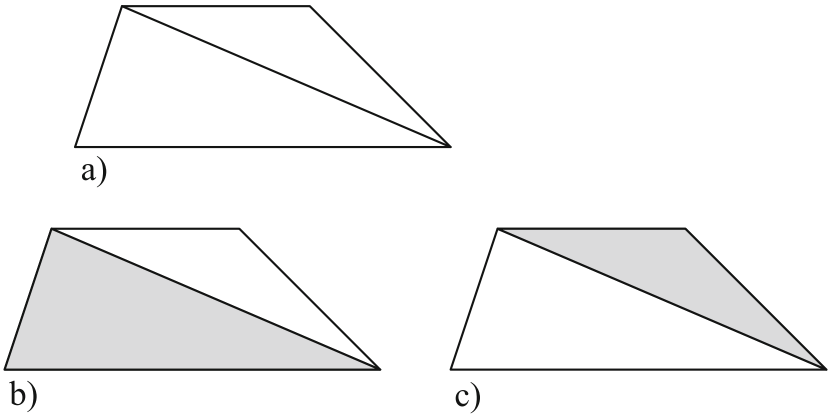


Fig. 3.47

PROBLEM 19. Fig. 3.48 shows two identical trapezoids divided by a diagonal into triangles S_1, S_2, S_3, S_4. Are there triangles with equal areas? If so, which ones?

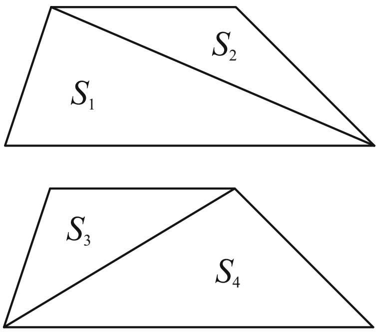


Fig. 3.48

5 Area of a polygon

Consider the polygon in Fig. 3.49a. How can we measure its area?

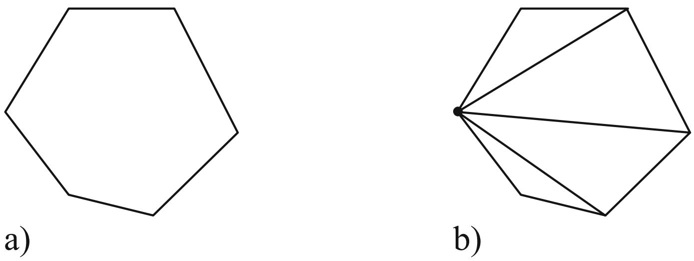


Fig. 3.49

Let us recall that we measure the area of a figure by determining how many unit parallelograms fit exactly into this figure. We have already shown how to measure the area of an arbitrary parallelogram. Then, for an arbitrary triangle and trapezoid, we showed how to substitute a parallelogram with the same area as these figures. For an arbitrary polygon, we substitute several triangles that, taken together, have the same area as this polygon. Then we can find the areas of these triangles and thus know the area of the polygon.

Let us choose a vertex and draw all the diagonals of the polygon from this vertex (see Fig. 3.49b). The polygon has been divided into triangles which do not overlap.[7] It is clear (see Axiom 1 in Section 1) that the area of the polygon is equal to the sum of the areas of these triangles. The following proposition can be proved, but we will not do this here.

Proposition 7. Consider a polygon. Let us divide it in two different ways into non-overlapping triangles by drawing some of the diagonals in this polygon.[8] The sum of the areas of these triangles is the same in each case.

Thus, the sum of the areas of all triangles in a triangulation of a polygon is the same for any triangulation of this polygon. This number is assigned to the polygon as its area.

Exercise 3. Suppose we have a piece of chocolate that has the shape of a parallelogram. Mark the midpoints of its sides and connect them (see Fig. 3.50).

[7] The operation of dividing a polygon into non-overlapping triangles is called *triangulation of a polygon*. There are many ways to triangulate a polygon.

[8] That is, we consider two different triangulations of a given polygon.

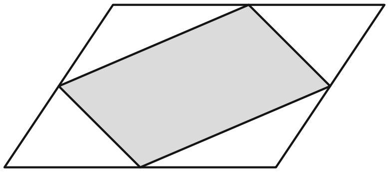


Fig. 3.50

(a) Prove that the shaded domain in Fig. 3.50 is a parallelogram.

(b) Where is there more chocolate: inside this parallelogram or outside it?

Solution. (a) Solve the problem and compare your solution with Problem 13 in Chapter II.

(b) Let us find the area outside of the parallelogram. For this we introduce some notations (see Fig. 3.51a) and draw diagonal AC. Then according to Proposition 5, $S_1 = \frac{1}{4}S_{ABC}$ and $S_3 = \frac{1}{4}S_{ACD}$. Therefore, $S_1 + S_3 = \frac{1}{4}S_{ABC} + \frac{1}{4}S_{ACD} = \frac{1}{4}S_{ABCD}$.

Now consider diagonal BD, Fig. 3.51b. Similarly, we obtain that $S_2 + S_4 = \frac{1}{4}S_{BCD} + \frac{1}{4}S_{ABD} = \frac{1}{4}S_{ABCD}$. By adding these two equalities we get

$$S_1 + S_2 + S_3 + S_4 = \frac{1}{4}S_{ABCD} + \frac{1}{4}S_{ABCD}$$

or

$$S_1 + S_2 + S_3 + S_4 = \frac{1}{2}S_{ABCD}.$$

We conclude that the shaded area equals half of the area of the parallelogram. Thus, the chocolate is divided “fairly” into two parts.

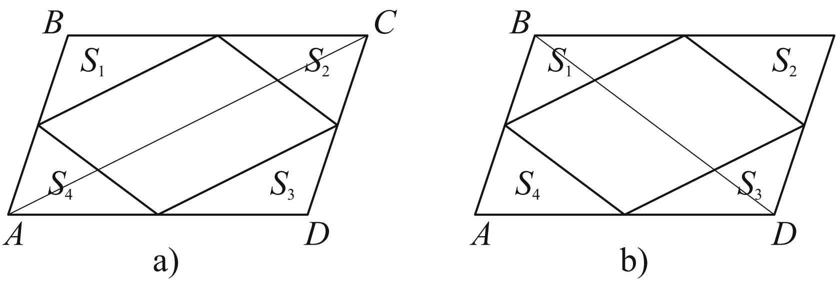


Fig. 3.51

PROBLEM 20. Suppose that a piece of chocolate has the shape of a quadrilateral. Let us again mark the midpoints of all sides and connect them in succession to form another quadrilateral (see Fig. 3.52).

(a) Do we obtain a parallelogram?

(b) Where is there more chocolate: inside the shaded quadrilateral or outside it?

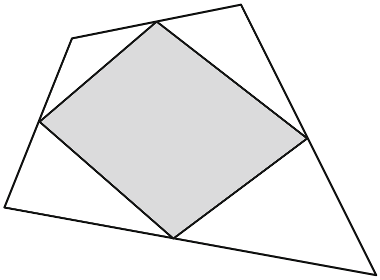


Fig. 3.52

The following problem is more difficult. You can try to solve it yourself or see our solution below.

PROBLEM 21 (∗) Consider a nonconvex quadrilateral $ABCD$ (see an example in Fig. 3.53a). (This case might be more difficult to check on a piece of chocolate.) Connect the midpoints K, L, M, N of its sides, as in Fig. 3.53b.

(a) Is quadrilateral $KLMN$ a parallelogram?

(b) Which area is larger: the shaded area or the non-shaded part of the quadrilateral $ABCD$?

(c) What equality holds between the areas of triangles and quadrilaterals in Fig. 3.53b? (Compare your answer with the answer of Exercise 3.)

Solution. (a) Let us connect point B with point D and point A with point C (see Fig. 3.54a).

In triangle ABD, segment MN is a bimedian and is parallel to BD. In triangle BCD, segment KL is a bimedian and is also parallel to BD. Therefore, $MN \parallel KL$. Similarly, in triangle ABC, segment NK is a bimedian and is parallel to AC; in triangle ACD, the bimedian LM is also parallel to AC. Thus, $NK \parallel ML$. As we know, a quadrilateral whose opposite sides are parallel is a parallelogram. Therefore $KLMN$ is a parallelogram.

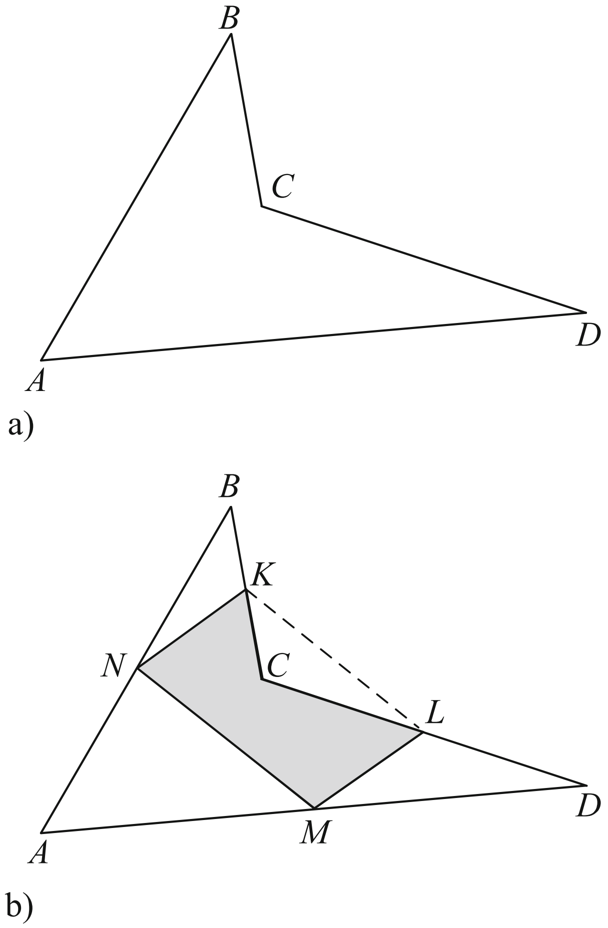


Fig. 3.53

(b) We need to compare the shaded area S_{sh} and the non-shaded area of $ABCD$. Let us denote the non-shaded pieces of the original quadrilateral by S_1, S_2, and S_3, and denote triangle CKL by S_4 (see Fig. 3.54b).

From Proposition 5 we have the following equalities:

$$\begin{aligned} S_1 &= \tfrac{1}{4}S_{ABD}, \\ S_2 &= \tfrac{1}{4}S_{ABC}, \\ S_3 &= \tfrac{1}{4}S_{ACD}, \\ S_4 &= \tfrac{1}{4}S_{BCD}. \end{aligned}$$

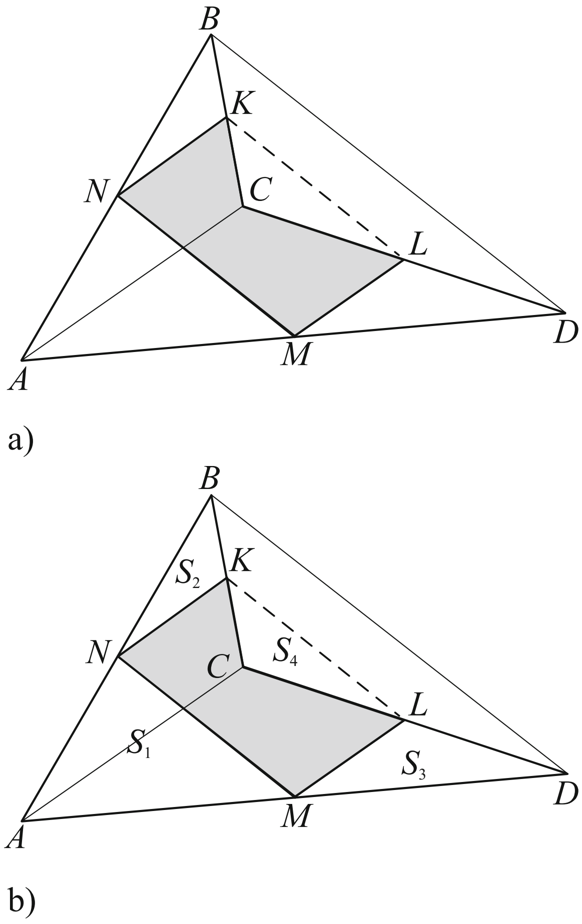


Fig. 3.54

From Fig. 3.54 we also have

$$S_{ABCD} = S_1 + S_2 + S_3 + S_{\text{sh}},$$
$$S_{ABCD} = S_{ABC} + S_{ACD},$$
$$S_{ABCD} = S_{ABD} - S_{BCD}.$$

Comparing all these equalities we notice that

$$S_2 + S_3 = \tfrac{1}{4}S_{ABC} + \tfrac{1}{4}S_{ACD} = \tfrac{1}{4}S_{ABCD} \quad \text{and}$$
$$S_1 - S_4 = \tfrac{1}{4}S_{ABD} - \tfrac{1}{4}S_{BCD} = \tfrac{1}{4}S_{ABCD}.$$

Therefore,

$$S_1+S_2+S_3-S_4 \;=\; \tfrac{1}{4}S_{ABCD}+\tfrac{1}{4}S_{ABCD} \;=\; \tfrac{1}{2}S_{ABCD},$$

or

$$S_1+S_2+S_3 \;=\; \tfrac{1}{2}S_{ABCD}+S_4.$$

Then from the equality for S_{sh} we obtain

$$S_{\mathrm{sh}} \;=\; S_{ABCD}-(S_1+S_2+S_3) \;=\; S_{ABCD}-\left(\tfrac{1}{2}S_{ABCD}+S_4\right),$$

or

$$S_{\mathrm{sh}} \;=\; \tfrac{1}{2}S_{ABCD}-S_4.$$

Thus, the non-shaded part of $ABCD$, i.e., $S_1+S_2+S_3$, is larger than $\frac{1}{2}S_{ABCD}$, and the shaded area S_{sh} of $ABCD$ is smaller than $\frac{1}{2}S_{ABCD}$. The difference in each case is precisely S_4. This also means that the non-shaded area of $ABCD$ is larger than the shaded area of $ABCD$ by $2S_4$. (We leave it to you to make these calculations.)

(c) For the convex quadrilateral in Exercise 3, we obtained the equality

$$S_1+S_2+S_3+S_4 \;=\; \tfrac{1}{2}S_{ABCD}.$$

In part (b) of the current problem, for a nonconvex quadrilateral we obtained

$$S_1+S_2+S_3-S_4 \;=\; \tfrac{1}{2}S_{ABCD}.$$

Consider again a convex quadrilateral $ABCD$, and connect the midpoints of its sides (see Fig. 3.55a).

This figure has already been considered in Problem 20. Let us start moving point C a little bit towards the segment BD (see Fig. 3.55b). The shape of the quadrilateral $ABCD$ changes, but it still remains a convex quadrilateral.

Continue moving point C towards segment BD. At a certain moment point C will end up lying on segment BD (see Fig. 3.55c). We obtain a triangle with a point marked on its side. This figure has already been considered in Exercise 2.

If we continue moving point C farther in the same direction, we will obtain a nonconvex quadrilateral (see Fig. 3.55d) like the one we have been considering here in Problem 21.

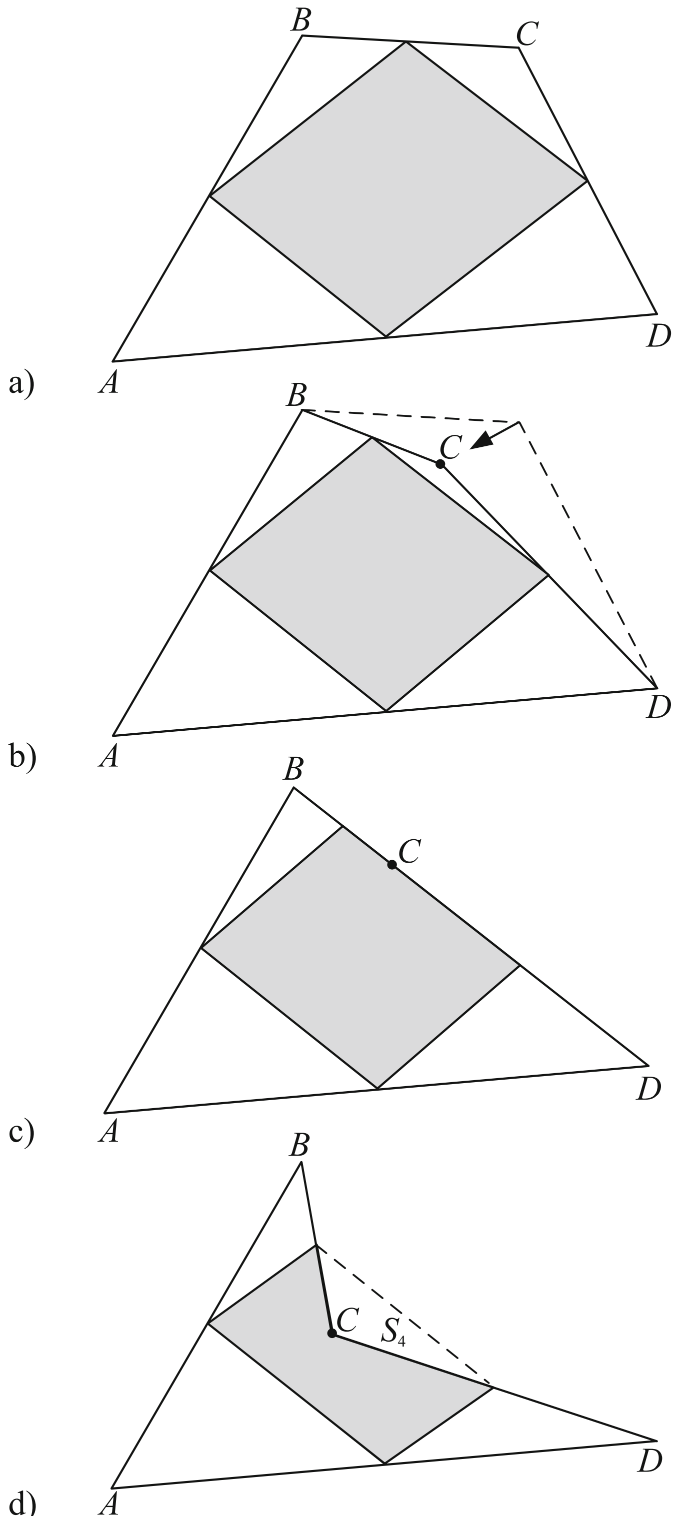


Fig. 3.55

Thus, Problem 20, Exercise 2, and Problem 21 are related. Let us see what happens with the solutions, i.e., what the area of the shaded quadrilateral is in each case.

Recall that the answer for Problem 20(b) is

$$S_{\text{sh}} = \tfrac{1}{2}S_{ABCD}.$$

In Exercise 2 we also obtained the result that the area of the shaded quadrilateral (a parallelogram) is half of the area of $ABCD$, i.e.,

$$S_{\text{sh}} = \tfrac{1}{2}S_{ABCD}.$$

However, in Problem 21 we found that the area of the shaded figure (which is not a parallelogram) is less than half of the area of $ABCD$. For this case we have:

$$S_{\text{sh}} = \tfrac{1}{2}S_{ABCD} - S_4,$$

where S_4 is the area of the triangle which is missing from the shaded figure in order to make it a parallelogram (see Fig. 3.55d). This summary shows that it is often useful not only to solve problems but also to see the relations between them.

6 More problems on areas

PROBLEM 22. Consider a parallelogram $ABCD$, and let E and F be two points on opposite sides such that $AE = ED$ and $BF = FC$ (see Fig. 3.56). Let the area of $ABCD$ be equal to $S_{ABCD} = S$. Find the ratio of areas of the three parts, i.e., $S_{DEC} : S_{AECF} : S_{ABF}$.

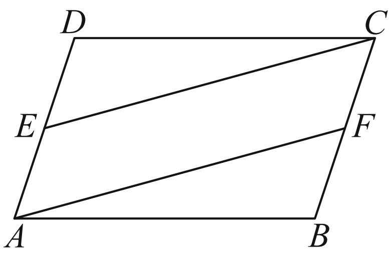


Fig. 3.56

Hint. Draw diagonal AC. Then EC is a median in the triangle ACD.

PROBLEM 23. Consider a parallelogram $ABCD$, and let E and F be two points on opposite sides such that $AE : ED = 2 : 1$ and $BF : FC = 1 : 2$ (see Fig. 3.57). Let the area of $ABCD$ be equal to S. Find the ratio of areas of the three parts, i.e., $S_{DEC} : S_{AECF} : S_{ABF}$.

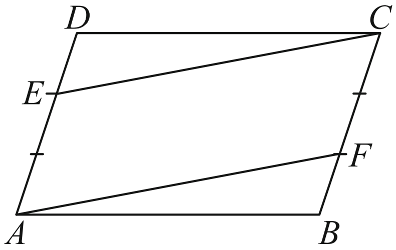


Fig. 3.57

PROBLEM 24. Consider a parallelogram $ABCD$ with area equal to S. Let E and F be two points on adjacent sides such that $ED : AD = 1 : 3$ and $FB : AB = 1 : 4$ (see Fig. 3.58). Find the area of triangle AEF, i.e., S_{AEF}.

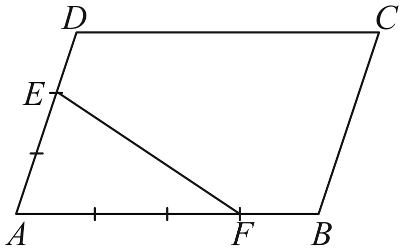


Fig. 3.58

PROBLEM 25. Consider a parallelogram $ABCD$ and draw diagonal BD. Divide diagonal BD into three equal segments, and connect points E and C, then F and A (see Fig. 3.59).

Find the ratio of areas of the parts (i.e., $S_{DEC} : S_{ECB} : S_{ABF} : S_{AFD}$).

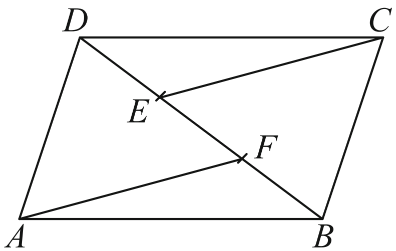


Fig. 3.59

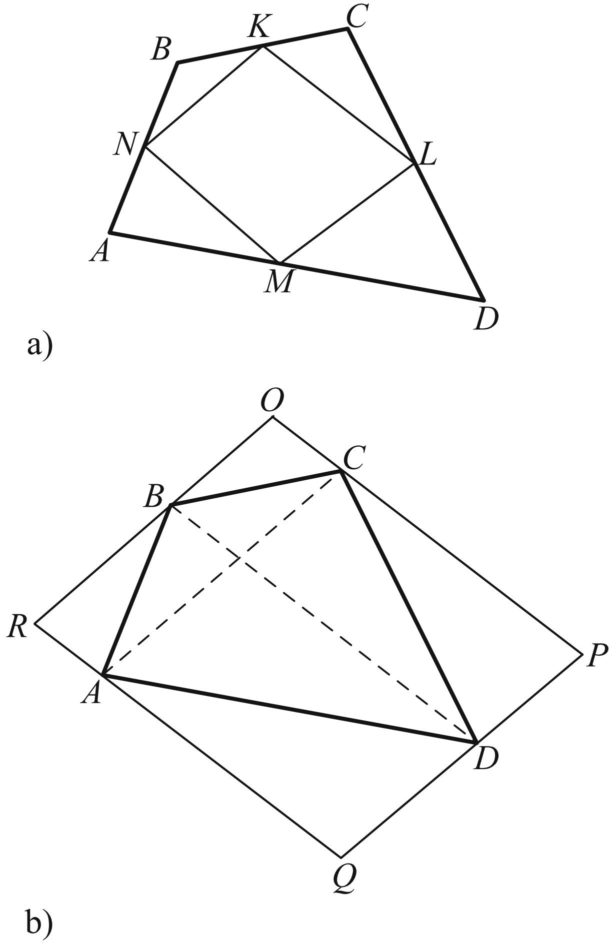


Fig. 3.60

In Problem 20 we considered a quadrilateral, connected the midpoints of its sides, and obtained a new quadrilateral (shaded in Fig. 3.52) inside the first one. We repeat this construction again in Fig. 3.60a. We call the quadrilateral *KLMN* a quadrilateral "inscribed" in the quadrilateral *ABCD*.

We can also construct a quadrilateral "circumscribed" around the same quadrilateral *ABCD* by drawing, through points *A*, *B*, *C*, and *D*, lines parallel to the diagonals of *ABCD*. We obtain quadrilateral *OPQR*, as in Fig. 3.60b.

PROBLEM 26.

a. Is quadrilateral *OPQR* in Fig. 3.60b a parallelogram?

b. Find the ratio $S_{KLMN} : S_{OPQR}$.

PROBLEM 27. Fig. 3.61 shows two parallel lines a and b. On line a there are four equal consecutive segments $A_1B_1 = B_1C_1 = C_1\,D_1 = D_1E_1$. On line b there are also four equal consecutive segments $A_2B_2 = B_2C_2 = C_2D_2 = D_2E_2$, but these are not equal to the segments on line a. The points are connected as shown in Fig. 3.61. We obtain two sets of triangles marked in the figure as white and shaded triangles.

Find the ratio of the sum of the areas of all shaded triangles to the sum of the areas of all white triangles.

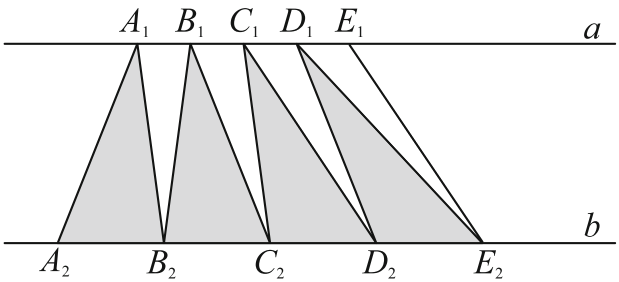
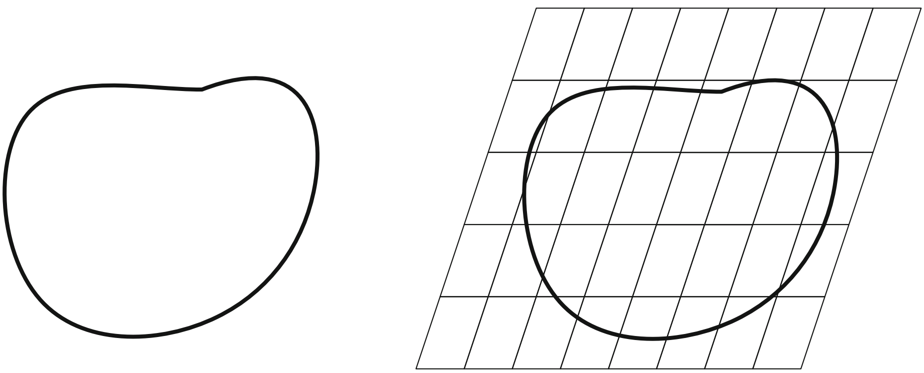


Fig. 3.61

7 How to measure the area of a figure

Consider a figure in the plane, for example, Fig. 3.62a. We need to find its area. We choose a unit parallelogram, then count how many unit parallelograms fit into this figure. In order to do this let us make parallel translations of the unit parallelogram across the figure (see Fig. 3.62b).

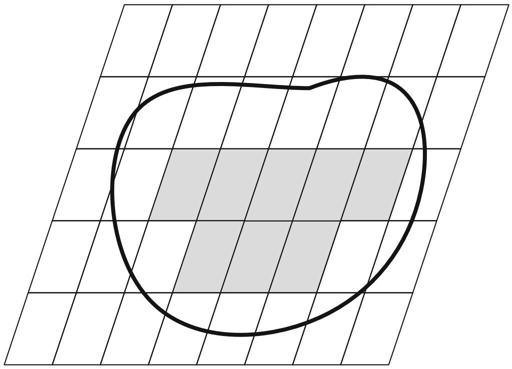


a) b)

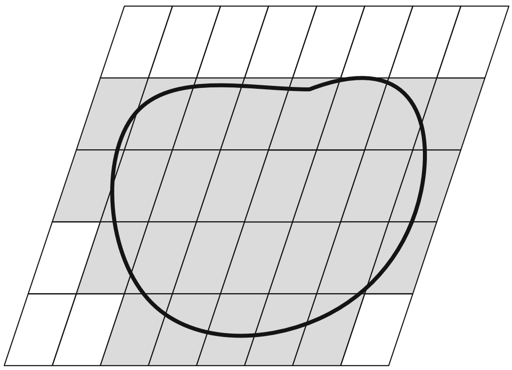
Fig. 3.62

Now let us count the number of whole unit parallelograms (shaded in Fig. 3.63a) which lie inside the figure.

We see that inside the figure there are some parts which are not unit parallelograms. Therefore, it is clear that by counting only whole unit parallelograms we obtain a value which is smaller than the actual area of the figure: we thus obtain only some approximation to its actual area.



a)



b)

Fig. 3.63

On the other hand, some of the parts which we did not count are "almost" unit parallelograms; thus, we would have a better estimate of the area of the figure if we counted them as well. This suggests another idea: we can count those unit parallelograms (shaded in Fig. 3.63b) which contain at least some part of the figure. In this case the value obtained will be larger than the actual area of the figure. These two ways of measuring the area of a figure by counting unit parallelograms are called a "lower estimate" and an "upper estimate," or "lower and upper bound" of the figure's area.

The actual area of the figure is between the two values that we have obtained from Fig. 3.63a and Fig. 3.63b. If the unit parallelogram were smaller, then these values would be closer to each other and to the area of the figure.[9] It is not always convenient, however, to choose a very small unit parallelogram because it is more time-consuming to draw a lot of lines and count so many parallelograms.

In order to measure the area more accurately, we can do the following. First, we count unit parallelograms which lie inside the figure. Then for each of those unit parallelograms which lie partially inside the figure, we divide one of its sides into 10 equal parts and draw a set of parallel lines (see Fig. 3.64).

Then we count all the whole parts which lie inside the figure. The area of each such part is equal to $\frac{1}{10}$.

If we want to measure the area even more accurately, we divide the rest of these parts again into smaller parts, and continue the process.

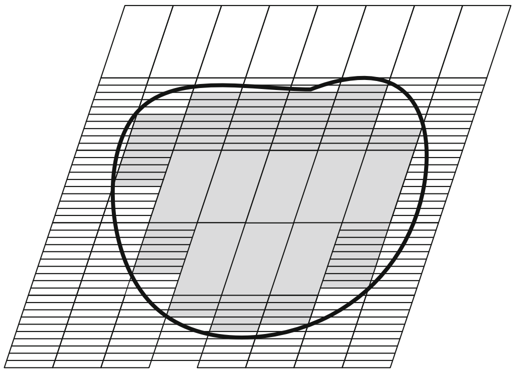


Fig. 3.64

[9]If you study calculus you will learn this in more detail.

8 Overview of Chapter III

In Chapter III we did not introduce new instruments and operations and did not define new figures. Instead we defined the area of a figure. Area is a characteristic of a figure, like length is a characteristic of a segment. The figures that we consider in this book, e.g., triangle, parallelogram, trapezoid, lie on the plane but do not lie on a line. These are *two-dimensional* objects. Segments lie on a line and they are *one-dimensional*[10] objects. To each figure on the plane we assign a positive number, called its area. In order to determine the area of a figure, we first introduce a unit area and then determine how many unit areas fit into the given figure. The geometry considered in Chapter III corresponds to symplectic geometry. In symplectic geometry the length of a segment is not always defined.

In Chapter IV we will define length of a segment and finally “build” Euclidean geometry. We will obtain more figures, such as squares, circles, sectors. We will then measure their areas as well as the areas of some figures we already considered in Chapter III. It is often easier to determine the areas of figures in Euclidean geometry than in symplectic geometry.

[10]We do not define here the notion of a dimension, but rely on your intuitive understanding of it.

Chapter IV

Circles: A Look at Euclidean Geometry

PART I. Introduction to the circle

1 Operations available in Chapter IV

First, let us review the operations which we defined in Chapters I–III and what we can do using these operations.

In Chapter I we had only two operations:

(1) Draw the unique line through any two points.

(2) Mark the point of intersection of two lines if these lines intersect.

Using these operations we can draw different configurations of points and lines; we defined angles, segments, and polygons. However, we can distinguish different polygons only by the number of their vertices, since we have no way to compare segments and angles.

In Chapter II, we had three operations:

(1) Draw the unique line through any two points.

(2) Mark the point of intersection of two lines if these lines intersect.

(3) Draw a straight line that passes through a given point A and is parallel to a given line a.

In Chapter II, we drew parallel lines and considered certain types of polygons. We compared segments lying on parallel lines or on the same line. If

I. M. Gelfand, T. Alekseyevskaya (Gelfand), *Geometry*,
https://doi.org/10.1007/978-1-0716-0299-7_4

we chose a unit segment, we could construct other segments, either on the same line or on a parallel line, that are a times larger or smaller than the unit segment, where the number a is a rational number. Thus, we could construct a number axis. We could also measure a segment lying on the number axis or on a line parallel to it. In Chapter II, however, we could not compare segments lying on non-parallel lines.

In Chapter III, we did not add any more operations, but we introduced the area of a figure.

In Chapters I–III, we had a straightedge, which enabled us to draw lines, and an instrument to draw parallel lines.

In Chapter IV, we will add a new instrument: a *compass*. We assume that with the help of a compass one can draw a special figure on the plane called a circle. A *circle* is a set of all points such that the segment connecting any point on it with a fixed point O has the same length. The point O is called the *center* of the circle. A segment connecting a point on the circle with the center O is called a *radius*[1] of the circle. Thus, by definition, any radius of a particular circle has the same length.

Fig. 4.1a shows a circle with center O. Segments AO and BO are different radii of the circle. A radius of a circle is usually denoted by r (see Fig. 4.1b).

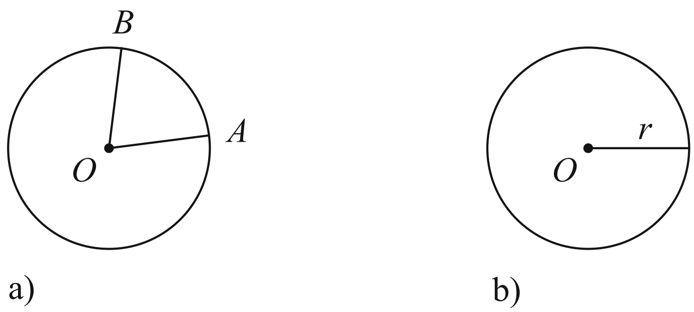


Fig. 4.1

We assume the following axioms:

(1) Two circles either do not intersect, intersect at one point, or intersect at two points.

(2) A circle and a straight line either do not intersect, or intersect at one point, or intersect at two points.

[1] The term radius is also used to indicate the length of this segment.

For a more detailed description of possible intersections between two circles or between a circle and a line, see Sections 17 and 18.

Thus, in Chapter IV, we assume that we have the following operations in addition to the three operations of Chapter II:

(4) Given any two points A and B, we can draw a circle with its center at one of these points and radius AB. Thus, we can draw a circle with any center and any radius.

(5) Mark the intersection points of two circles if they intersect.

(6) Mark the intersection points of a line and a circle if they intersect.

With the help of these operations we will be able to compare and measure any segments in the plane, not just those lying on parallel lines. We will be able to measure angles and enrich our knowledge about the figures and notions which we have already introduced. The geometry constructed in Chapter IV corresponds to *Euclidean geometry*.

1.1 Properties of a circle. Some related definitions

With the help of a compass we can draw infinitely many circles. These circles differ by their centers and radii.

The system of all circles in the plane has the following properties:

(1) Given two points A and O, there is one and only one circle such that O is its center and AO is its radius.

(2) A parallel translation of a circle is a circle.

(3) If we draw a figure similar[2] to a circle with any center of similarity, we obtain a circle.

A circle also satisfies the following important property:

Suppose a circle is given. If we draw a set of parallel lines intersecting this circle, the midpoints of all the segments lying inside the circle lie on the same straight line (see Fig. 4.2). This line passes through the center of the circle.

[2]Similarity of figures is defined in Section 15.

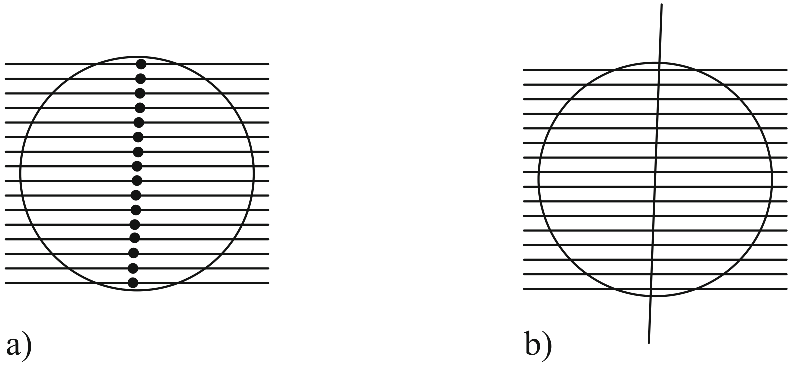


Fig. 4.2

A circle divides the plane into two domains: the bounded domain inside the circle, usually called a *disk* (see Fig. 4.3a), and the unbounded domain outside the circle (see Fig. 4.3b). Note that a disk is a convex[3] domain while the outside domain is not.

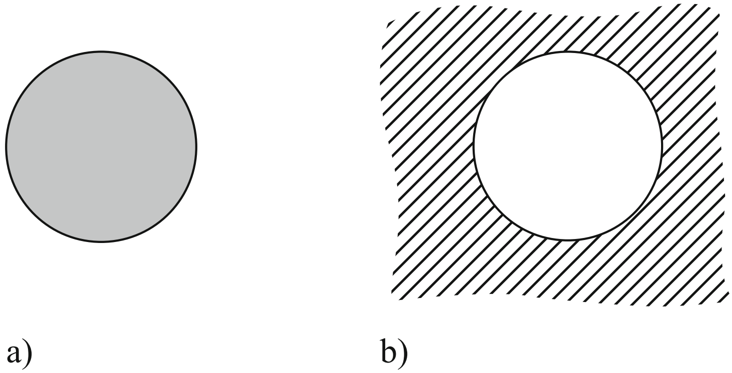


Fig. 4.3

The boundary of a disk is a circle. The distance around the circle, or the length of the circle, is called its *circumference*.[4]

Figure 4.4 illustrates some notions related to the circle. The part of a circle between[5] two points on this circle is called an *arc*. A segment connecting two points on the circle is called a *chord*. A chord passing through the center of the circle is called a *diameter* of this circle. A *sector* is a part of a disk bounded by two radii and the arc between them.

[3]Convex figures are defined in Section 2.1, Chapter I.

[4]Sometimes the term circumference is used to indicate the boundary itself, while the term circle is used to indicate the whole figure with the interior.

[5]See footnote 6 in Section 1.3, Chapter I.

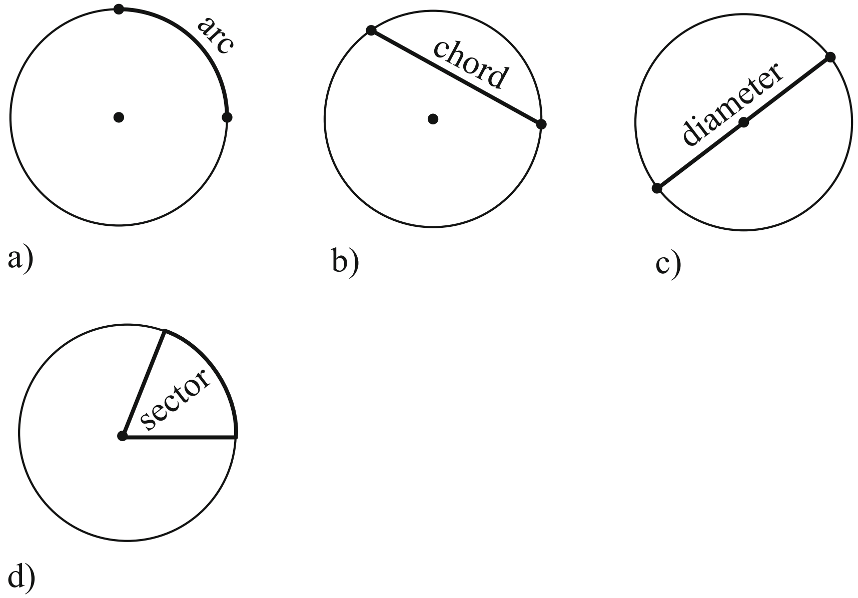


Fig. 4.4

2 Comparing segments

In Chapter II the only segments we could compare were segments lying on parallel lines or on the same line. In this chapter we compare any two segments and construct a segment equal to another one on any straight line.

Consider two segments AB and CD on non-parallel lines a and b (see Fig. 4.5a). If we need to compare these segments, we can mark one of them (e.g., AB) with the help of a compass set to the length of that first segment, and then move the compass to the other segment. If we make selected endpoints (e.g., A and C) of the two segments coincide, then we can see whether the other endpoints coincide or not. If they also coincide, the segments are equal. For the segments in Fig. 4.5a, point B will lie outside of segment CD (see Fig. 4.5b), so $AB > CD$. If point B lies inside CD (as is the case for the segments in Fig. 4.5c), then $AB < CD$ (see Fig. 4.5d).

Sometimes in order to illustrate the use of a compass in constructions better, the whole circle is drawn, as for example in Fig. 4.5e (which illustrates the same situation as Fig. 4.5b).

Remark 1. If two lines a and b happen to be parallel to each other, then we have two ways to compare these segments. One is described above. We can also compare segments on parallel lines by making a parallel translation as

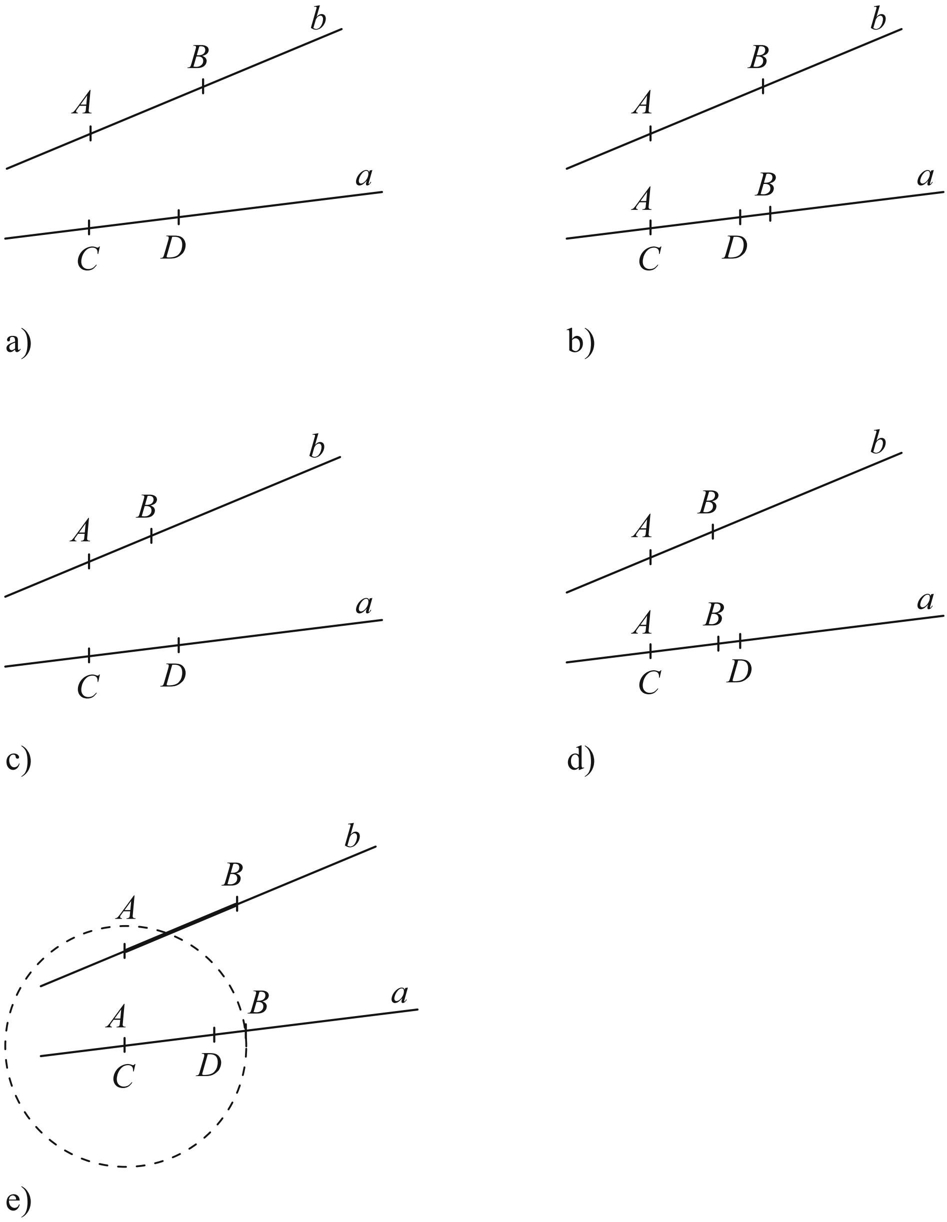


Fig. 4.5

we did in Chapter II. One can prove that these two ways do not contradict each other and will give the same result. The proof uses the fact that a parallel translation of a circle is also a circle.

3 Angles

In Chapter I we defined an angle as a domain bounded by two rays emanating from a single point. In this chapter we will describe how to compare and measure angles.[6]

3.1 Comparing angles. Degree measure

First consider a convex angle. A convex angle is an angle whose sides are conceived as bounding a convex domain (see Chapter I). Let us denote the vertex of this angle by O. Draw a circle with unit radius[7] and center O (see Fig. 4.6).

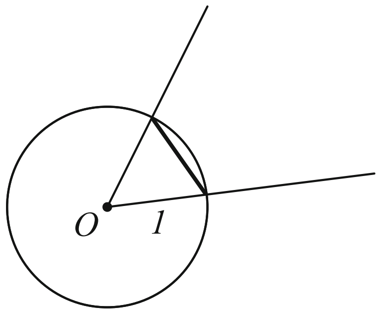


Fig. 4.6

Definition 1. We say that two convex angles with vertices at point O are equal if the chords corresponding to them on the unit circle are equal.

Thus, we can compare convex angles by comparing the lengths of their chords on a unit circle: the bigger the angle, the longer the corresponding chord is, and vice versa.

However, we cannot compare nonconvex angles by their chords. Indeed, a nonconvex angle (see the angle β in Fig. 4.7a) might have the same chord as a convex angle (see the angle α in Fig. 4.7b).

[6]One instrument used to measure angles is called a *protractor*. It has been known since ancient times. Measuring of angles was used in a "sundial," which consisted of a pole on the ground casting a shadow. As the sun moved during the day, the shadow would move. The angle between one position of the shadow and another showed how much time had passed.

[7]We could draw a circle of any radius r, but this radius has to be the same for all the angles we compare.

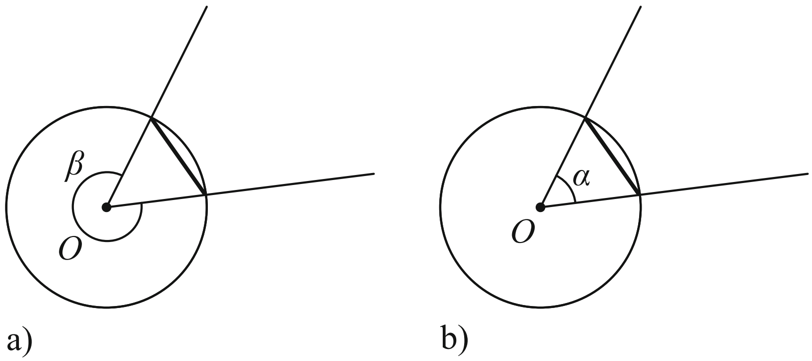


Fig. 4.7

In order to compare convex and nonconvex angles, the following is usually done: given an angle with vertex O, we draw a circle with center O and divide it by rays from the center into 360 equal parts. An angle corresponding to one such part is called an *angle of one degree* and is denoted by 1°. Clearly, 360 angles of one degree put together cover the whole circle and the whole plane. The more degrees an angle has, the larger it is. Any part of the plane bounded by two rays emanating from one point represents an angle whose measure ranges from 0° to 360°. The number of degrees assigned in this way to each angle is called the *degree measure* of this angle.

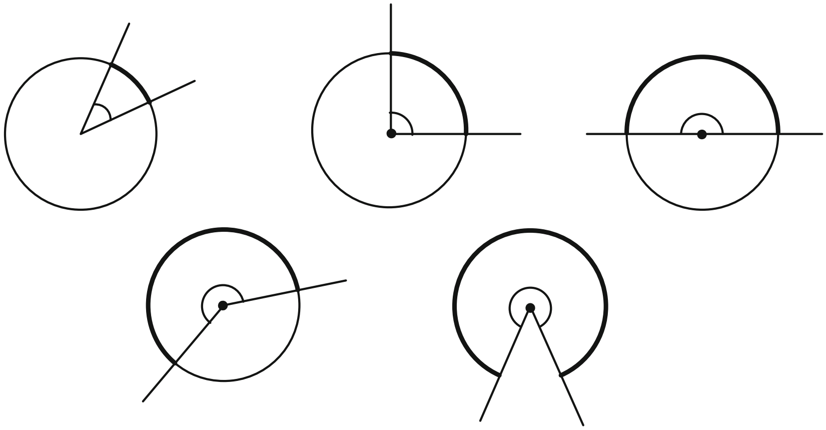
Arc degree measure

To every angle with its vertex at the center O of a circle there corresponds one arc on this circle. Fig. 4.8a shows some examples where the arcs corresponding to the angles are marked by bold lines. Conversely, given an arc on a circle, one can indicate only one angle which contains this arc. Therefore, there is a one-to-one correspondence between arcs on a given circle and angles with their vertices at the center of this circle.

Thus, we can naturally define the *degree measure of an arc* on a given circle. An arc of one degree (1°) is the arc corresponding to $\frac{1}{360}$ of the circle.

It is important to keep in mind that the arc degree measure is not the length of the arc. Calculating the length of an arc is not so easy (see Section 22.3) because an arc is not a segment and we cannot use a ruler. A circle is different from a straight line. Therefore, the degree measure of an arc is often more convenient to use than the length of the arc.

The arc degree measure indicates what part of the circle this arc represents. Two arcs with the same degree measure can have different lengths (see Fig. 4.8b). This means that, given an arc degree measure, we can indicate


a)

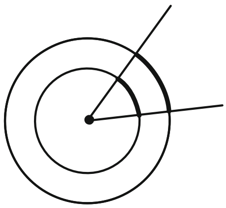

b)

Fig. 4.8

more than one arc with this degree measure. This is like the area of a figure. For a given area measurement we can indicate many figures which have this area. Arc degree measure and area are examples of correspondences that are not one-to-one.

Note that neither angle degree measure nor arc degree measure depends on the radius of the circle that is divided into 360 parts.

Some angles are of special interest and have names. An angle containing 180° is called a *straight angle* (Fig. 4.9a).

An angle containing 90° is called a *right angle*. It is often denoted as in Fig. 4.9b.

The lines determined by rays forming the sides of a right angle are called *perpendicular lines*. The fact that lines a and b are perpendicular to each other is often written as $a \perp b$.

An angle smaller than 90° is called an *acute angle*, and an angle bigger than 90° and smaller than 180° is called an *obtuse angle*.

An angle smaller than 180° is a *convex angle*, and an angle bigger than 180° is *nonconvex*. For example, an angle of 270° (see Fig. 4.9c) is nonconvex.

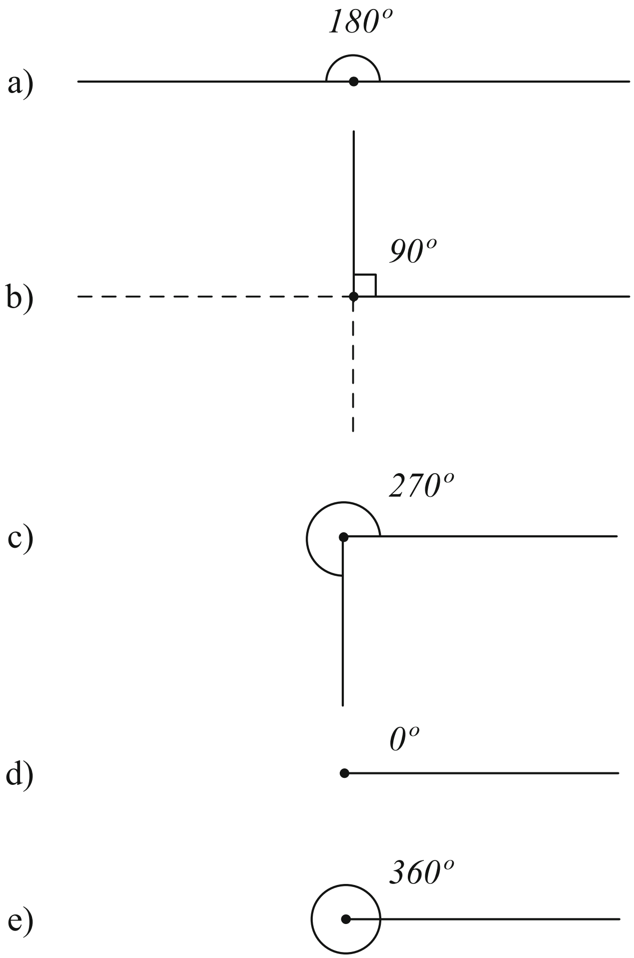


Fig. 4.9

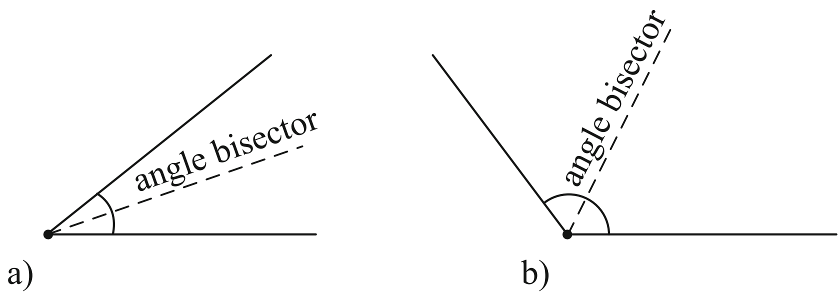


Fig. 4.10

What if two rays with the same vertex O coincide? In this case there are two possibilities. If we consider the convex domain bounded by these rays (see Fig. 4.9d), we say that this angle has $0°$. If we consider the nonconvex angle bounded by these rays, we obtain an angle containing $360°$ (see Fig. 4.9e). This angle is called a *complete angle*.

As we will see in Section 3.3, we can add angles. Therefore, we can get an angle bigger than $360°$.

An important ray associated with an angle is its *angle bisector*. The bisector of an angle is a ray passing through its vertex that is positioned in such a way that it divides the angle into two equal angles (see Fig. 4.10).

3.2 Construction of equal angles

Exercise 1. Consider a convex[8] angle α with vertex O. From point O' let us draw a ray a (see Fig. 4.11a). Construct an angle equal to angle α with vertex O' such that one of its sides is the ray a.

Solution. Let us draw a circle with radius r and center O. Angle α is measured by its chord AB (see Fig. 4.11b). Now draw the circle with radius r and center O' and mark its point of intersection with ray a as A' (see Fig. 4.11c). From point A' we draw segment $A'B' = AB$ such that point B' lies on the circle with center O'. We can do this by drawing a circle with radius AB and center A' (see Fig. 4.11d). Then angle $\angle B'O'A'$ (see Fig. 4.11e) is equal to angle $\angle BOA$, since their chords are equal.

Notice that there is another possibility for drawing a chord $A'B' = AB$, and an angle $\angle B'O'A' = \angle BOA$ (see Fig. 4.11f).

3.3 Addition of angles

In this chapter, we can add any two angles.

For simplicity of notation, angles are sometimes denoted by one letter, for example angle α or angle β.[9]

We will show how to construct an angle called the *sum of two angles*.

Exercise 2. Fig. 4.12a shows two angles α and β. Construct the angle $\alpha + \beta$.

---

[8] As we know, two rays from a point form two angles. Therefore, if you want to construct an angle equal to a nonconvex angle α, you can still use the construction for convex angles and then choose the nonconvex angle formed by the same rays.

[9] Angle α can also mean that the measure of this angle is equal to the number α. This rarely causes confusion. Compare with footnote 1 in this chapter.

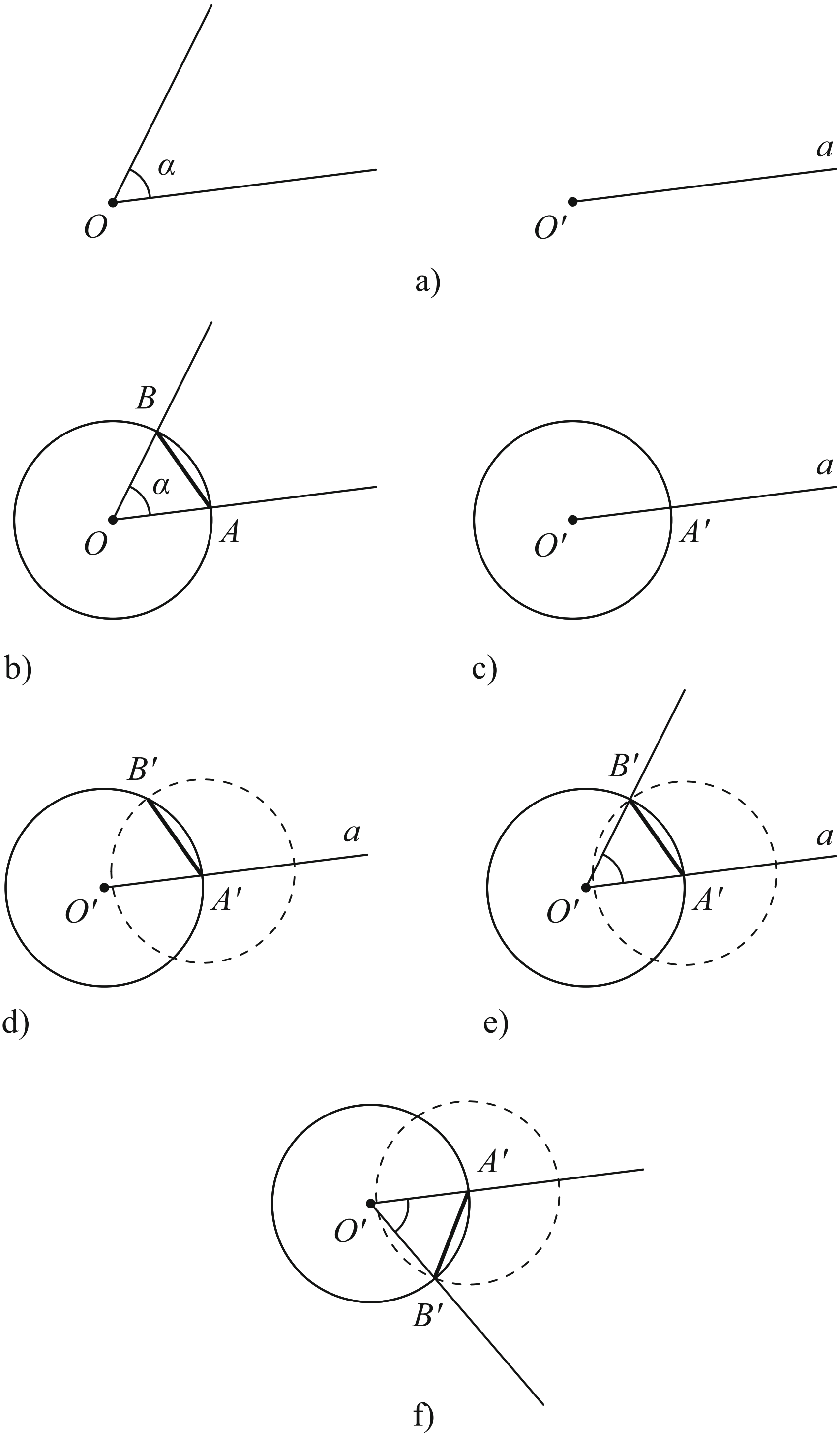


Fig. 4.11

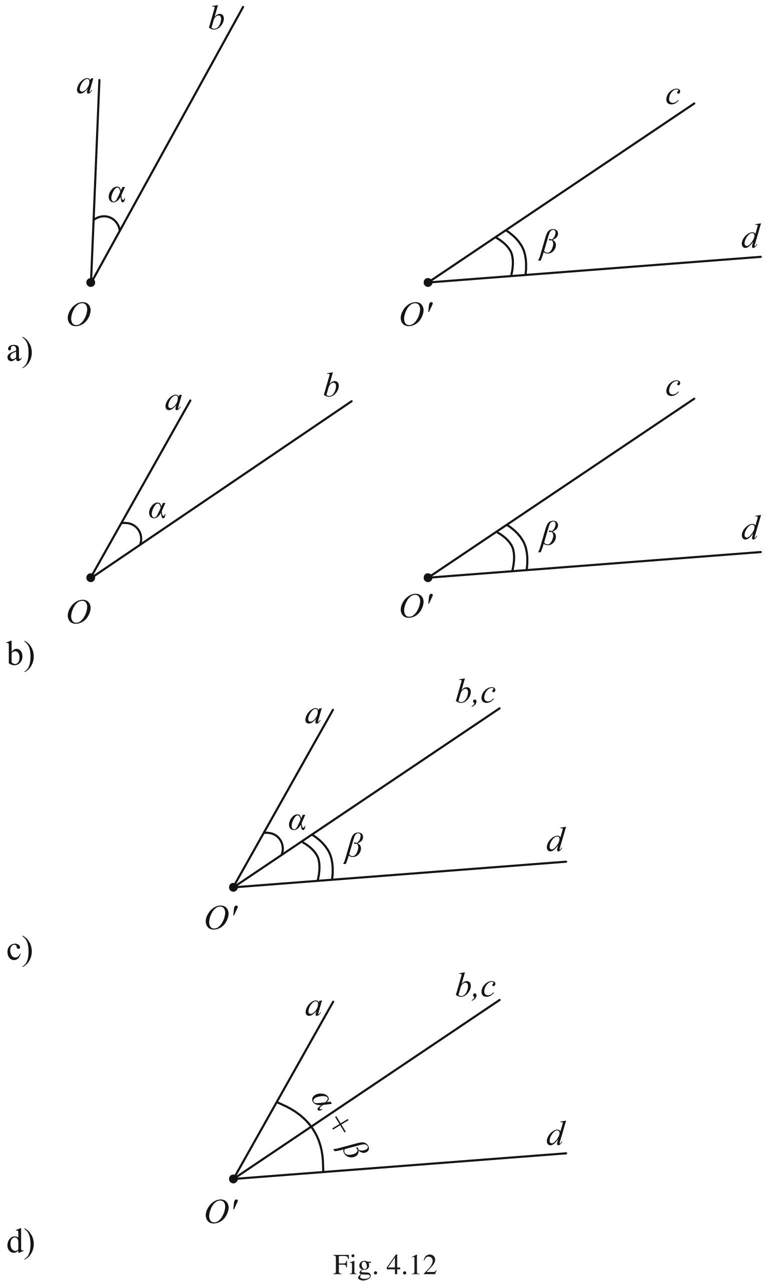


Fig. 4.12

Solution. Let us construct an angle equal to angle $\angle\alpha$ such that one of its rays (ray b in Fig. 4.12b) is parallel to ray c of angle $\angle\beta$ (use Exercise 1). Now we need to add the two angles, $\angle\alpha$ and $\angle\beta$ (see Fig. 4.12b), which have a pair of sides parallel to each other. Using a parallel translation, we can move angle α so that both angles have the same vertex and a common ray.[10] The angle formed by rays a and d (see Fig. 4.12d) is called the *sum of angles* α and β and is denoted as $\angle(\alpha+\beta)$.

Clearly, if α and β are degree measures of these angles, the angle $\angle(\alpha+\beta)$ has the degree measure $(\alpha+\beta)^\circ$.

PROBLEM 1. Find the sum of angles α and β in Fig. 4.13a, b and the sum of angles α, β, and γ in Fig. 4.13c.

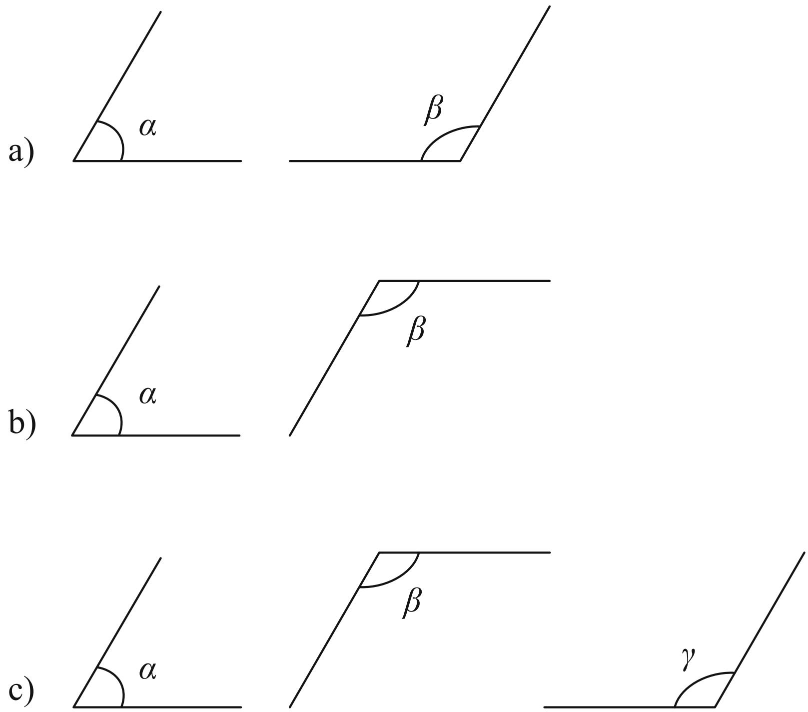


Fig. 4.13

Using the addition of angles, we can construct an angle of a given number of degrees. For example, we can add two angles of 1° and obtain an angle of 2°.

PROBLEM 2. Construct the angles below. Since it is not easy to mark an angle of 1° accurately, do this approximately.

(a) 181°; (b) 182°; (c) 179°; (d) 271°; (e) 269°; (f) 359°.

[10] Such angles (e.g., angles α and β in Fig. 4.12c) are called *adjacent angles*.

PROBLEM 3. Imagine that we placed a mark on one of the spokes of a bicycle wheel.[11] Let us look at the wheel with the marked spoke in such a way that when the bicycle starts moving, this spoke spins counterclockwise. If we notice and remember the initial position of the spoke (just like a policeman does), we can measure the angle through which the spoke turned. Answer the following questions:

(a) How many degrees is the angle that the spoke turns when the wheel makes half of the full rotation?

(b) How many degrees is the angle that the spoke turns when the wheel makes three quarters of the full rotation?

(c) How many degrees is the angle that the spoke turns when the wheel makes a full rotation?

(d) How many degrees is the angle that the spoke turns when the wheel makes a full and a quarter rotation?

As we know, an angle of 360° covers the whole plane (see Fig. 4.14a). Can we still add another angle to it? Based on common sense and the desire to make notions universal, mathematicians agree that there can be an angle of any number of degrees.

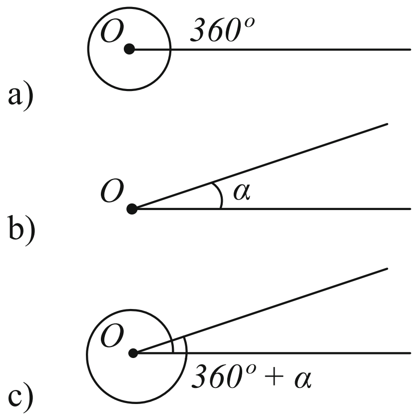


Fig. 4.14

[11]This can happen, for example, when someone leaves a bicycle on a lot with a time limit for parking. A policeman often comes and places a chalk mark on a vehicle's wheel in order to know later how long this vehicle had been there.

In order to imagine an angle larger than 360°, we can think of a wheel with a marked spoke (see Problem 3 above) which makes more than a full rotation.

We can also think of an angle larger than 360° in the following way: if an angle α is added to the 360° angle, we imagine that, as the angle becomes larger than 360°, it starts "covering" the plane for a second time. We sketch this in Fig. 4.14a, b, and c.

PROBLEM 4. Construct the angles below (mark an angle of 1° approximately):

(a) 361°; (b) 359°; (c) 450°; (d) 720°; (e) 540°.

Let us look again at the wheel with a marked spoke. Imagine that the wheel makes a quarter of a full rotation, but then the bicyclist backs up, and the wheel makes a quarter of a full rotation in reverse. How can we describe this in terms of angles? First, the spoke turned through a 90° angle, then it turned again; the result of these two turns together is the same as if the spoke had remained in place. This means that the wheel turned by a 0° angle. It is natural to suggest that the angle "in reverse" should be −90°. This is exactly the way negative angles are introduced. We do need to agree upon how to look at the "wheel." On the plane we fix a ray which we call the *x-axis*; the angles counted from the *x*-axis in the counterclockwise direction are considered positive and the angles counted in the clockwise direction are considered negative (see Fig. 4.15a, b).

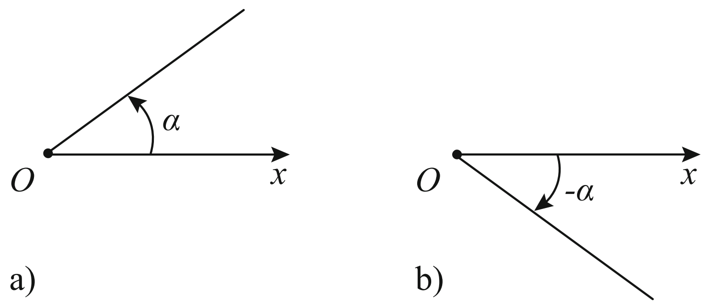


Fig. 4.15

PROBLEM 5. For each angle in Problem 2 find the corresponding negative angle and the number of degrees it contains.

Negative angles and angles more than 360° are usually studied in trigonometry. We introduce them here to show that there is a one-to-one correspondence between angles on the plane and numbers. This correspondence is used to measure angles. In Section 3.1 we introduced the degree measure of angles less than 360°. In the present section we have shown that degree measure of angles can be extended to angles of any number of degrees.

There is another way to measure angles which is widely used in trigonometry: it is called *radian measure of angles*. In radian measure, an angle is measured by the length of the corresponding arc on the unit circle (see Section 22.4).

Remark 2. As we mentioned in Section 3.1, we can compare convex angles by comparing their chords on a unit circle. Is it possible to measure convex angles with the help of their chords?

The answer is "no," at least not in a way that allows us to add angle measures together. Let us look at Fig. 4.16. Angle α has the chord a and angle β has the chord b. But the angle $\alpha+\beta$ has the chord $c \neq a+b$.

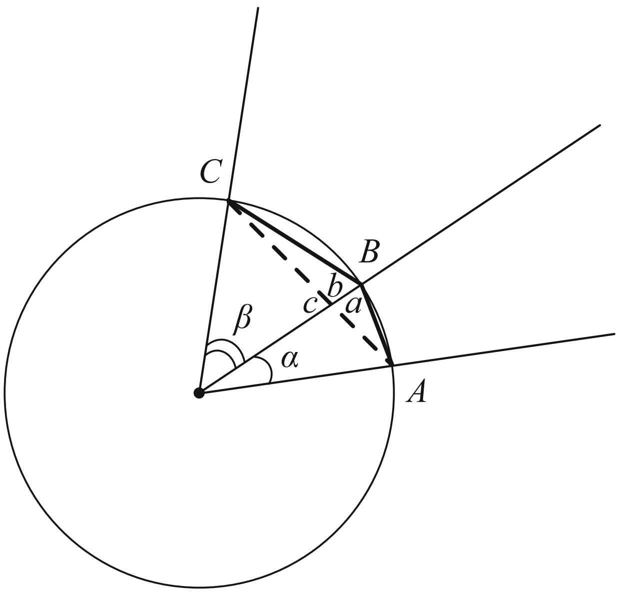


Fig. 4.16

Indeed, these three chords form a triangle ABC. In a triangle one side is always smaller than the sum of the other two sides (see Proposition 2 in Section 7).

3.4 Vertical angles and angles with respectively parallel sides

Theorem 1. Vertical angles are equal.

Proof. Two intersecting lines a and b form two pairs of *vertical angles* (Chapter I, Section 2.2). Let us denote one of the pairs of vertical angles formed by these lines by α and β, and denote by x one of the other angles lying between them (see Fig. 4.17).

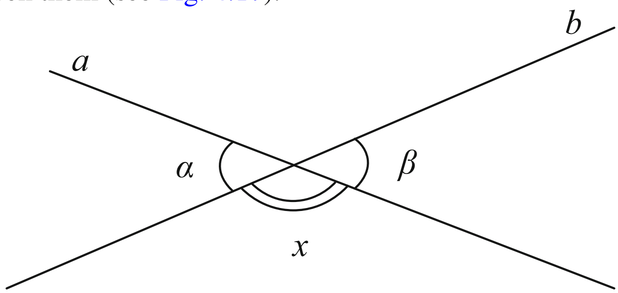


Fig. 4.17

It is clear from this figure that $\angle\alpha + \angle x = 180^\circ$ and $\angle\beta + \angle x = 180^\circ$. Therefore, $\angle\alpha = \angle\beta$. □

A pair of angles whose sum is a straight angle are called *supplementary angles*. In Fig. 4.17, angles α and x are supplementary angles.

Consider two parallel lines a and b and a line c intersecting them (see Fig. 4.18a). There are several angles formed by these lines. Some pairs of these angles are of particular interest and have special names.

The pair of angles marked in Fig. 4.18b are called *corresponding angles*. The pair of angles marked in Fig. 4.18c are called *alternate interior angles*. The pair of angles marked in Fig. 4.18d are called *alternate exterior angles*.

PROBLEM 6.

(a) Find other pairs of corresponding angles in Fig. 4.18b. (All together there are four such pairs of angles).

(b) Find other pairs of alternate interior angles in Fig. 4.18c. (All together there are two such pairs of angles).

(c) Find other pairs of alternate exterior angles in Fig. 4.18d. (All together there are two such pairs of angles).

(d) Find pairs of supplementary angles other than angles α and x in Fig. 4.17.

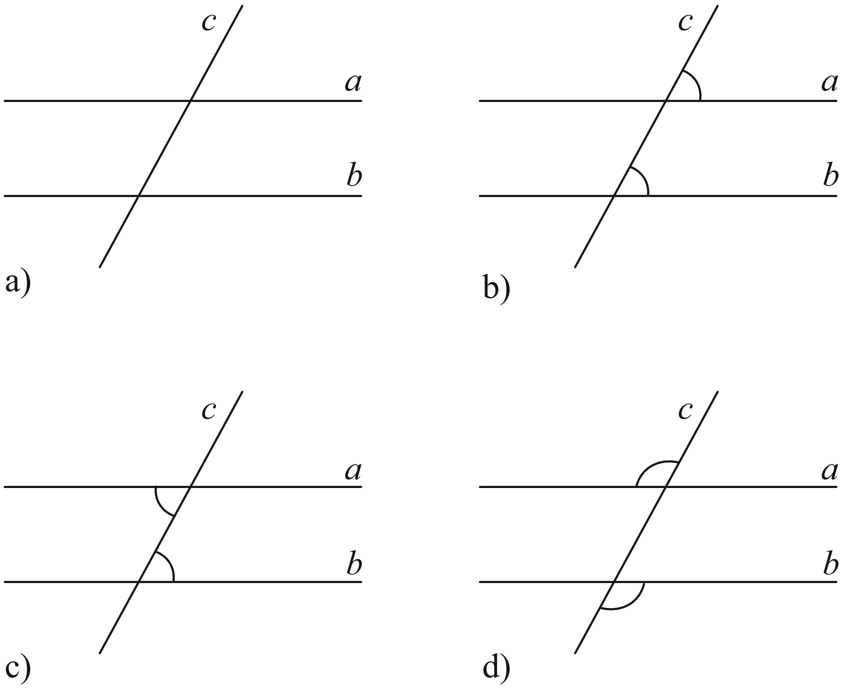


Fig. 4.18

Theorem 2. Suppose two parallel lines are intersected by a third line.

(1) A pair of corresponding angles are equal to each other.

(2) A pair of alternate interior angles are equal to each other.

(3) A pair of alternate exterior angles are equal to each other.

Proof.

(1) Consider the corresponding angles α and β in Fig. 4.19a. If we make a parallel translation of angle α to the vertex of angle β, these angles will coincide, since the lines a and b are parallel. Therefore, $\angle\alpha = \angle\beta$.

(2) Consider the alternate interior angles α and β in Fig. 4.19b. We can see that $\angle\alpha = \angle\alpha'$ because they are vertical angles. But $\angle\alpha' = \angle\beta$ since they are corresponding angles. Therefore, from transitivity of equality we obtain $\angle\alpha = \angle\beta$.

(3) Consider the alternate exterior angles α and β (see Fig. 4.19c). Angle α is equal to angle α' since they are vertical angles. And $\angle\alpha' = \angle\beta$

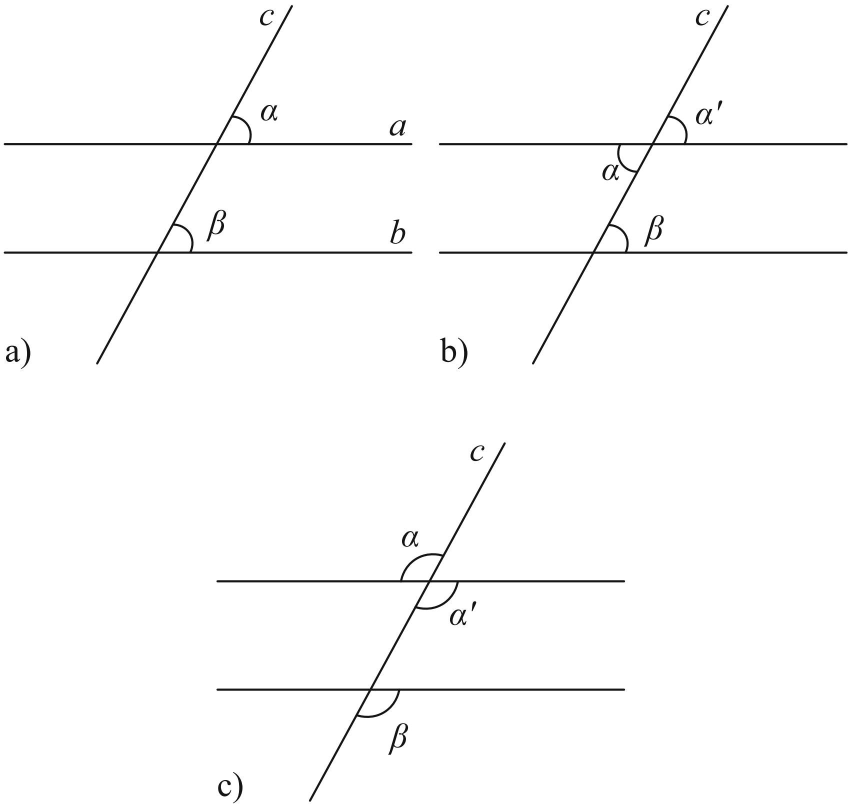


Fig. 4.19

since they are corresponding angles. Therefore, again due to transitivity of equality[12] we obtain $\angle\alpha = \angle\beta$. □

Remark 3. One can prove that the converses of statements (1), (2), and (3) above are also true. That is, if two lines are intersected by a third line and a pair of corresponding angles are equal to each other, then the two lines are parallel. The converse theorem is also true for a pair of alternate interior angles, or a pair of alternate exterior angles.

Remark 4. When two parallel lines a and b are intersected by a line c, we can notice that, in any pair of angles formed by these lines, either the two angles are equal or their sum is equal to a straight angle (180°).

[12]In Chapter II, Section 3 we proved that parallel lines satisfy the property of transitivity and symmetry. It can also be proved that equality of angles or equality of other objects satisfies the properties of transitivity and symmetry.

Now consider two pairs of intersecting lines such that the lines in one pair are correspondingly parallel to the lines in the second pair.

Proposition 1. If the sides of two angles are correspondingly parallel, then either these angles are equal or their sum is equal to 180°.

Proof. Consider an angle α formed by rays a and b. It is proposed that there is a second angle whose sides are parallel to these rays, so let us draw two lines a' and b' with a' parallel to a and b' parallel to b (see Fig. 4.20a).

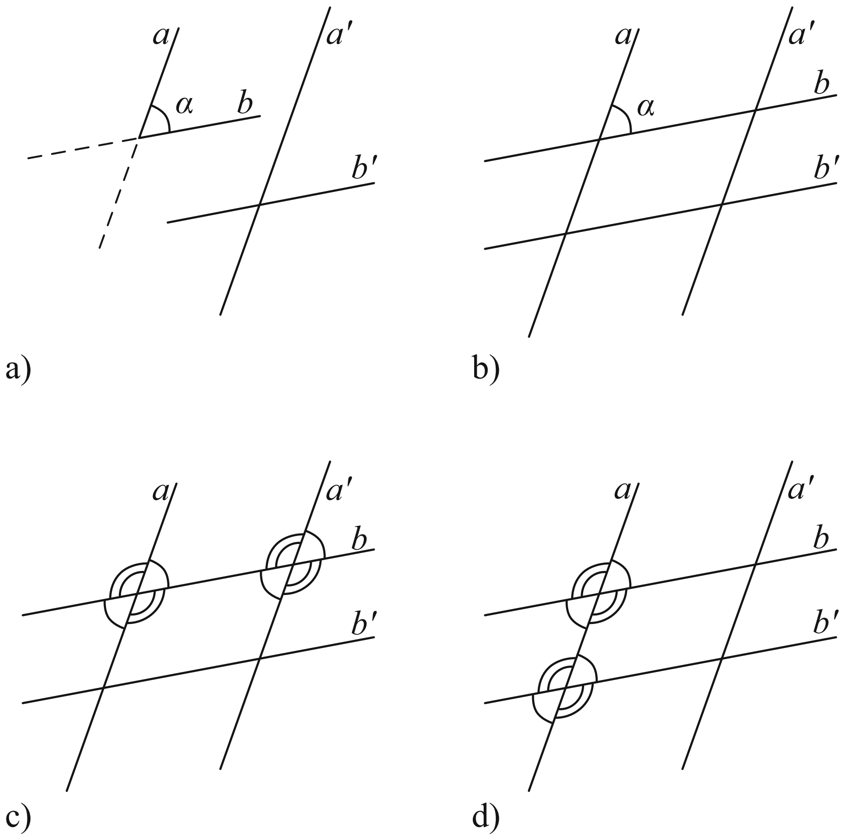


Fig. 4.20

By extending rays a and b, we obtain lines that we may refer to also as line a and line b. Let us continue to extend these lines a, b, a', and b' until all those that are not parallel intersect as in Fig. 4.20b.

From Remark 4, since $a \parallel a'$, in any pair of these angles with a side on line b, either the two angles are equal to each other or their sum is 180° (see

Fig. 4.20c). On the other hand, since $b \parallel b'$, in any pair of these angles with a side on line a (see Fig. 4.20d), either the two angles are equal to each other or their sum is $180°$. Therefore, from transitivity of equality we conclude that any two angles marked in Fig. 4.20c or d either are equal to each other or are such that their sum is $180°$. But for the given lines a and b, these angles cover all the possibilities of angles with correspondingly parallel sides. $\square$

4 Operations with figures

4.1 Turns and reflections

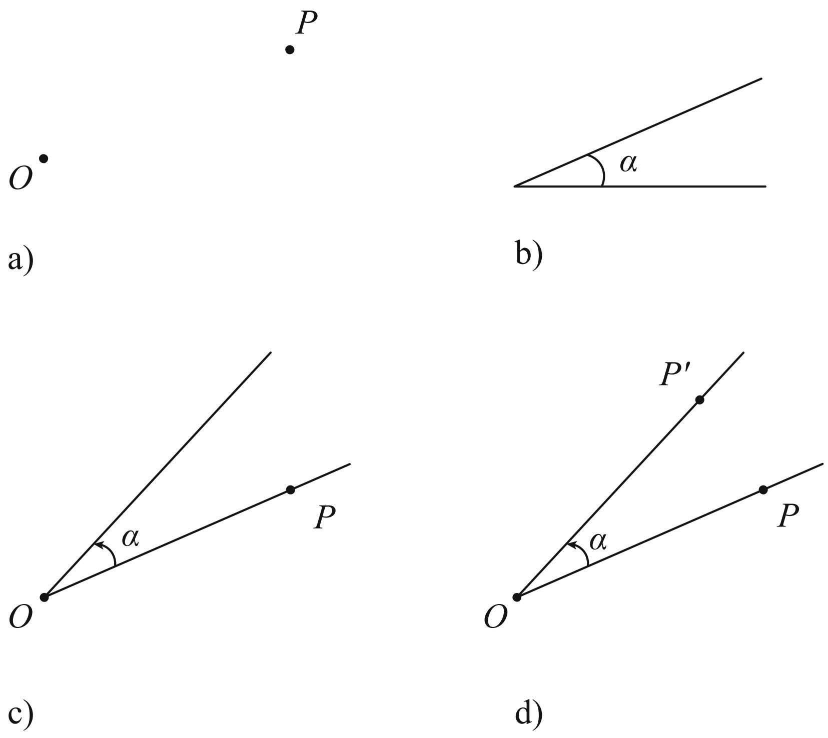


Fig. 4.21

In the previous section we defined the degree measure of an angle. We now define a *turn (rotation) through a certain angle*.

Let us choose a point O, which we will call the *center of a turn*. Let P be any point on the plane (see Fig. 4.21a). We may draw a ray from point O passing through point P. Suppose we want to turn this ray through an angle α (for example, the angle shown in Fig. 4.21b). To do this, we construct angle

α with its vertex on point O such that one of its rays is OP and the other ray lies in the counterclockwise direction from the ray OP (see Fig. 4.21c).

On this other ray we mark point P' such that $OP = OP'$ (see Fig. 4.21d). We say that point P' is obtained from point P by a *turn through the angle* α, or that we *turned point P through the angle* α.

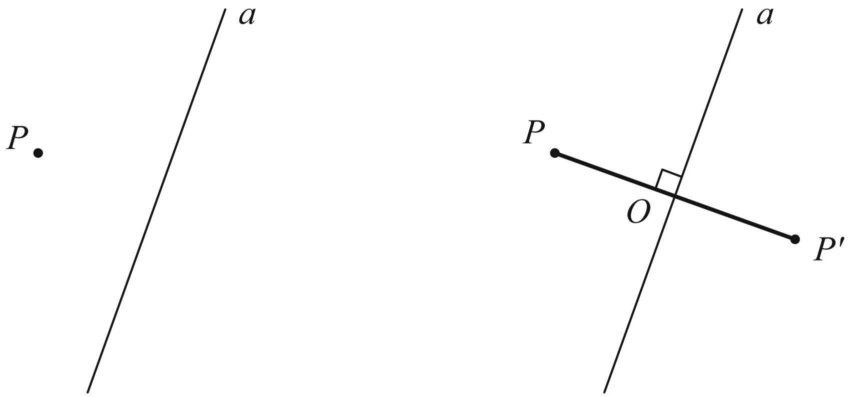
We can turn a point through an angle of a given degree measure.

PROBLEM 7. Choose a center O and three points A, B, and C. Connect these points to form a triangle. Turn triangle ABC through the following angle (that is, we have to turn each of its points through this angle):

(a) 90°; (b) 180°; (c) 270°.

We can turn any figure on the plane through an angle with respect to any center O. We can also reflect any figure on the plane with respect to a line.

Consider a point P and a line a (see Fig. 4.22a). In order to reflect this point in line a, we draw a line perpendicular to line a passing through point P. If we denote by O the intersection of these lines, then the point P' such that $OP' = OP$ is called the *reflection of the point P with respect to line a*.



a) b)

Fig. 4.22

PROBLEM 8.

(a) Choose a line a and three points A, B, and C all lying on the same side of line a. Connect these points to form a triangle. Reflect triangle ABC in line a.

(b) Repeat the same construction for the case where the points A, B, and C do not all lie on the same side of line a.

Consider a figure on the plane and a line a intersecting this figure. If we reflect one part of the figure with respect to line a and it coincides with the part on the other side of the line a, we say that this figure is *symmetric* with respect to line a. We call line a a *line of symmetry* of the figure. A figure might have more than one line of symmetry. Each of the figures in Fig. 4.23 has one line of symmetry. Note, that in nature, living things are rarely, if ever, strictly symmetric. Only the images of them that we create are perfectly symmetric.

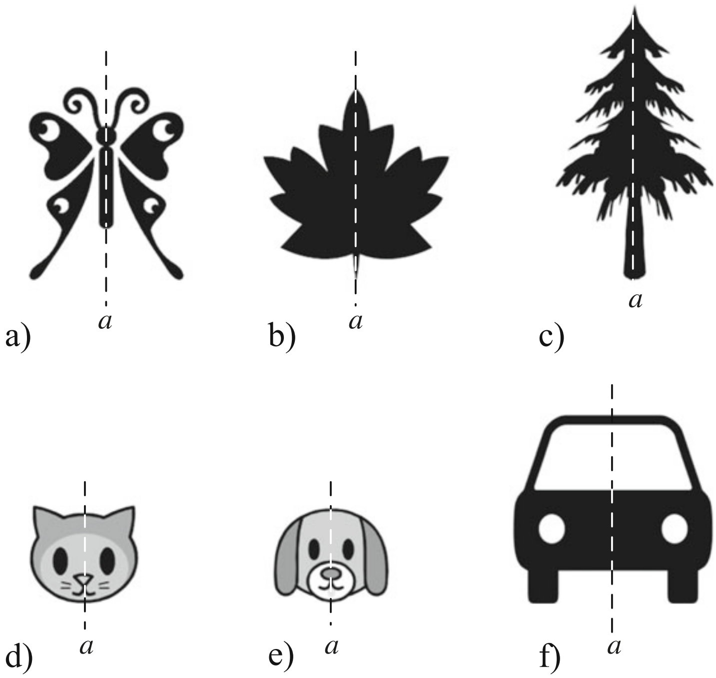


Fig. 4.23

4.2 Consecutive operations with a figure. Congruent figures

We already know the following four operations with figures on the plane: parallel translation, central symmetry with respect to a point (see Chapter II), rotation with respect to a point, and reflection in a line (defined in this chapter). We can apply any one of these four operations to any figure on the plane. We assume that they do not change a figure. We will use these operations to define *congruent figures*.

Given a figure on the plane, let us apply to it one of the four operations above. We obtain another figure. Now we can again apply one of these operations to this "new" or "resulting" figure. We might also say that "the two operations were applied consecutively to a given figure." It is important to understand the concept and not get confused about the terminology.

For example, we can make a parallel translation of a given figure and then reflect the resulting figure in a line. Of course, we can apply these operations to a given figure in the opposite order. Will the result be the same?

PROBLEM 9.

(a) Two points O and O' and a figure (a triangle) are given (see Fig. 4.24). Turn this figure about point O by 30°. Then turn the resulting figure about point O' by 330° (or by -30°).

(b) Turn the figure in Fig. 4.24 around point O' by 30°. Then turn the resulting figure around point O by 330° (or by -30°).

Do the results in (a) and (b) coincide?

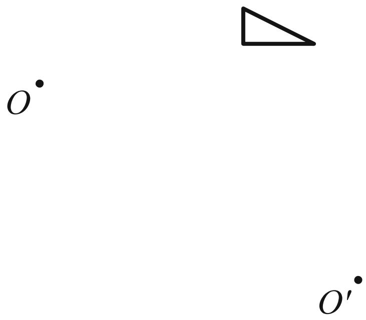


Fig. 4.24

PROBLEM 10. Consider the points O and O' again and the figure in Fig. 4.24.

(a) Turn the figure about point O by 30°. Then apply the parallel translation defined by points O and O' to the resulting figure.

(b) Turn the figure about point O' by 30°. Then apply the parallel translation defined by points O and O' to the resulting figure.

Compare the results in (a) and (b). Do they coincide?

PROBLEM 11.

(a) Draw a simple figure and a point O. Draw a figure that is centrally symmetric to your figure with respect to the point O.

(b) Consider the same figure and turn it about point O by 180°. Do the results in (a) and (b) coincide?

(c) Prove that a central symmetry with respect to a given point is the same as a turn through 180° about the same point.

Remark 5. From the problem above it follows that we do not need to consider central symmetry as a separate operation because we can always substitute for it a turn through 180°. Note, however, that there is no turn that can substitute for reflection in a line.

We recommend that you draw a simple figure and play with it by applying any two of the operations in a different order: parallel translation, rotation through an angle, and reflection. Observe and compare the results. You can also consider one operation but vary it. For example, you can choose two different centers of rotation or two different lines of reflection.

Parallel translation and rotation about a point are also called *motions* on the plane. Thus, we can move[13] or reflect a figure on the plane. We can see intuitively that these operations do not change the figure itself.

Definition 2. Two figures on the plane are considered *congruent figures* if one is obtained from the other with the help of one or more of the following operations: parallel translation, rotation about a point, or reflection in a line.

Motion of a figure is a one-to-one correspondence and is an example of a *transformation*.

In Section 15 we will consider one more operation with a figure, called *similarity*. Similarity is also a transformation. However, similarity changes a figure. Therefore, similar figures are not congruent figures.

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[13] In the Introduction to the book we said that we will "move figures." We see now that this phrase has a precise meaning.

PART II. The geometry of the triangle and other figures

5 Elements of a triangle. Congruent triangles

Consider a triangle (Fig. 4.25).

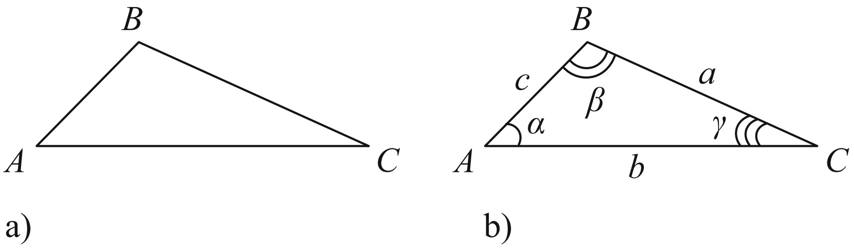


Fig. 4.25

It has three sides, e.g., a, b, and c, and three angles,[14] e.g., α, β, and γ. There is one equality relation among these six elements of a triangle, i.e., $\angle\alpha + \angle\beta + \angle\gamma = 180°$ (see Theorem 3).

What triangles are considered congruent?[15]

Definition 3. *Two triangles are congruent* if one of the following three conditions is satisfied:

(1) Three sides of one triangle are equal to the three sides of the other triangle.

(2) Two sides and the angle between those two sides in one triangle are correspondingly equal to two sides and the angle between them in the other triangle.

(3) A side and two adjacent angles of one triangle are correspondingly equal to a side and two adjacent angles of the other triangle.

[14]By angles of a triangle we usually mean its interior angles. There is no need to consider the exterior angles since an exterior angle of a triangle is supplementary to one of its interior angles (their sum is 180°).

[15]Strictly speaking, in Section 4 we have already defined congruent figures on the plane. Here we define congruent triangles in a more useful and practical way. Of course, one has to prove that these definitions do not contradict each other. Such a proof is far beyond what can be explained in this book.

We have formulated three definitions of congruent triangles. To be precise, one needs to prove that they are equivalent to each other. This can be done using Hilbert's axiomatization of geometry, which is far beyond the level of this book.

It follows from these definitions that if two triangles are congruent, all six elements of one triangle are correspondingly[16] equal to the six elements of the other.

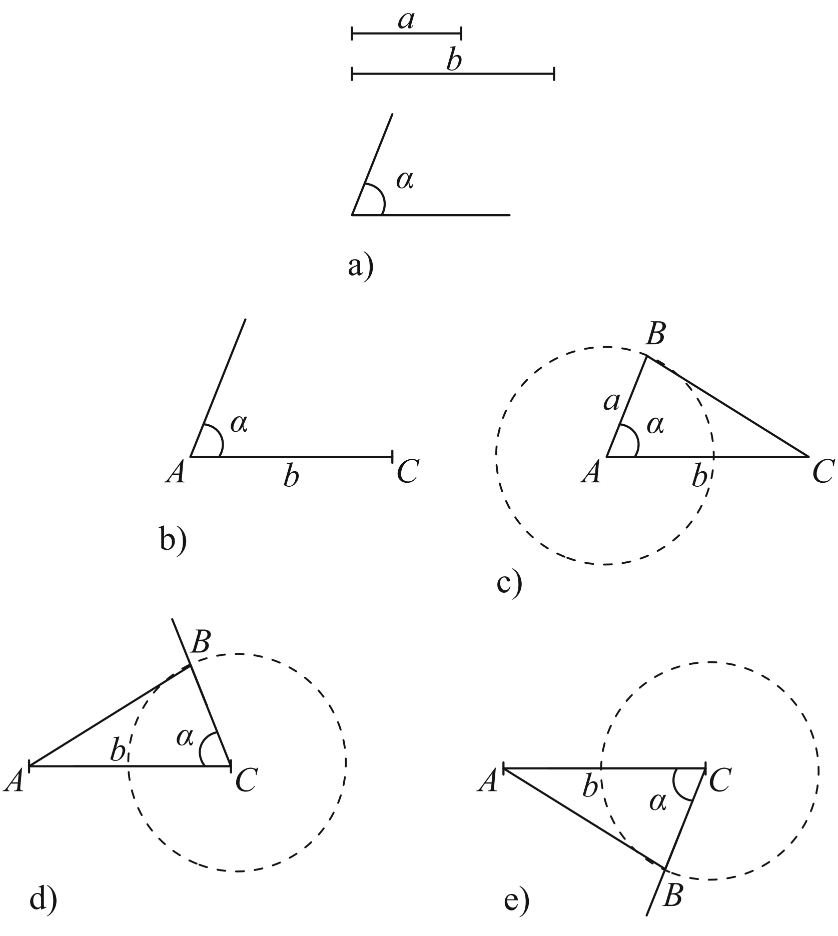


Fig. 4.26

[16]By correspondingly equal we mean that if, for example, angle α has sides a and b in one triangle, then the angle with sides a and b in the other triangle has to be equal to α.

6 Construction of a triangle from its elements

Exercise 3. Two segments a and b and an angle α are given (see Fig. 4.26a). Construct a triangle with the sides a and b and the angle α between them.

Solution. We start with a segment b (or AC). We know that the vertex of the angle α is either at point A or at point C. For example, let it be at point A (see Fig. 4.26b). It is given that another side of angle α in the triangle has to be a. In order to mark its end point B we use a compass and draw a circle with center A and radius a (see Fig. 4.26c). By connecting points B and C, we obtain a triangle which has the sides a, b and the angle α between them.

There are different possibilities for this construction. We could draw angle α from point C (see Fig. 4.26d, e). Thus, we can construct several different triangles with sides a and b and the angle α between them. By our definition of congruent triangles, all these constructed triangles are congruent.

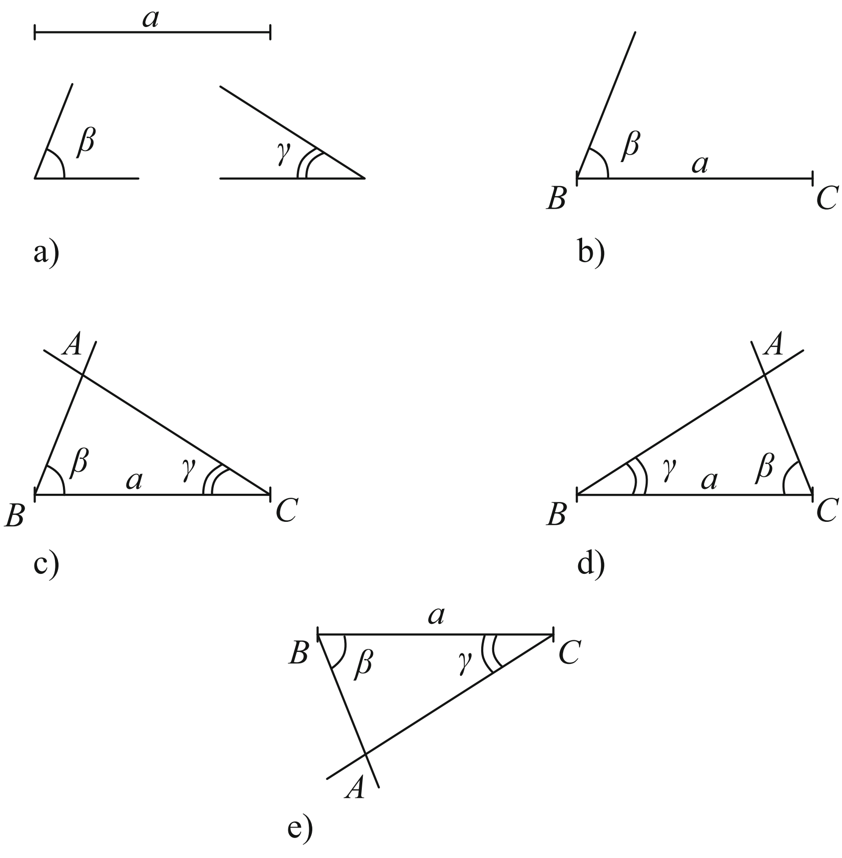


Fig. 4.27

Exercise 4. A segment a and two angles β and γ are given (see Fig. 4.27a on the previous page). Construct a triangle with side a and two angles β and γ adjacent to this side.

Solution. First we draw a segment equal to a with endpoints B and C. This is our side a. Then we draw an angle equal to β with vertex B such that one of its sides coincides with a (see Fig. 4.27b). Now we draw angle γ with vertex C such that one of its sides coincides with a (see Fig. 4.27c). We obtain triangle ABC, which has a side a and two adjacent angles β and γ.

Of course, we could have drawn angle β with vertex C and angle γ with vertex B instead, as in Fig. 4.27d or Fig. 4.27e. All these triangles are congruent according to our definition.

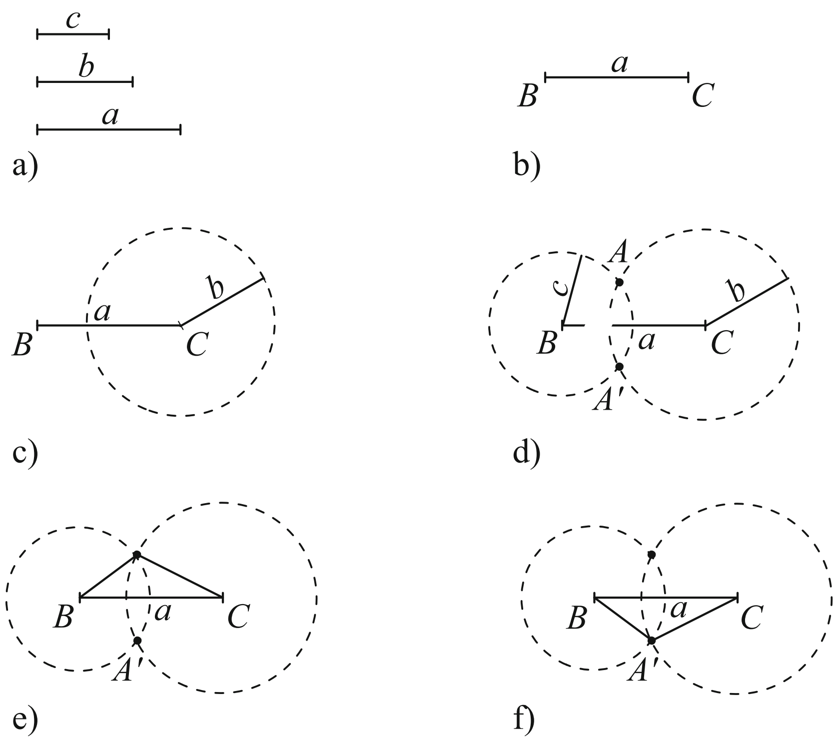


Fig. 4.28

Exercise 5. Consider three segments a, b, and c (Fig. 4.28a). Construct a triangle with sides a, b, and c.

Solution. First, draw a segment a, or BC (see Fig. 4.28b).

The other sides b and c will meet at point A. Thus, point A will lie at a distance b from one of the points B or C and at a distance c from the other. Let us work with side b intersecting side a at point C. Then point A must lie on the circle with center C and radius b (see Fig. 4.28c). On the other hand, point A must also lie at a distance c from point B, i.e., on the circle with center B and radius c (see Fig. 4.28d). Point A has to lie on these two circles simultaneously. Therefore, it lies at an intersection of these two circles.

There are two points A and A' at which these circles intersect.[17] Either of the points A or A' can serve as the third vertex of a triangle with sides a, b, and c (see Fig. 4.28e, f).

We have constructed two triangles with the given sides a, b, and c. According to our definition these triangles are congruent.

PROBLEM 12. Using the segments a, b, c in Fig. 4.28a, draw a triangle with sides a, b, c. Start drawing with side a and use the following information.

(a) In clockwise order the sides of the triangle are a, b, c;

(b) In clockwise order the sides of the triangle are a, c, b.

(c) Will the triangles in (a) and (b) coincide?

Additional constructions of a triangle from its elements

You can omit this section unless you are interested in why we have not considered all the possibilities of choosing three elements of a triangle in order to construct this triangle.

Exercises 3, 4, and 5 correspond to the alternative conditions (1)–(3) for triangles to be congruent given in Definition 3. Thus, in each of these exercises, all the triangles that we can construct are congruent.

Let us try to construct a triangle using other choices of three elements of a triangle.

[17] Note that not every pair of circles intersect at two points. They can also intersect at one point or not intersect at all. In these cases we will not be able to construct a triangle (see Section 7.1 and Fig. 4.31c, d).

Exercise 6. Construct a triangle with sides a and b and angle β, which is not the angle between a and b.

Solution. Let the segments a and b and angle β be given (see Fig. 4.29a).

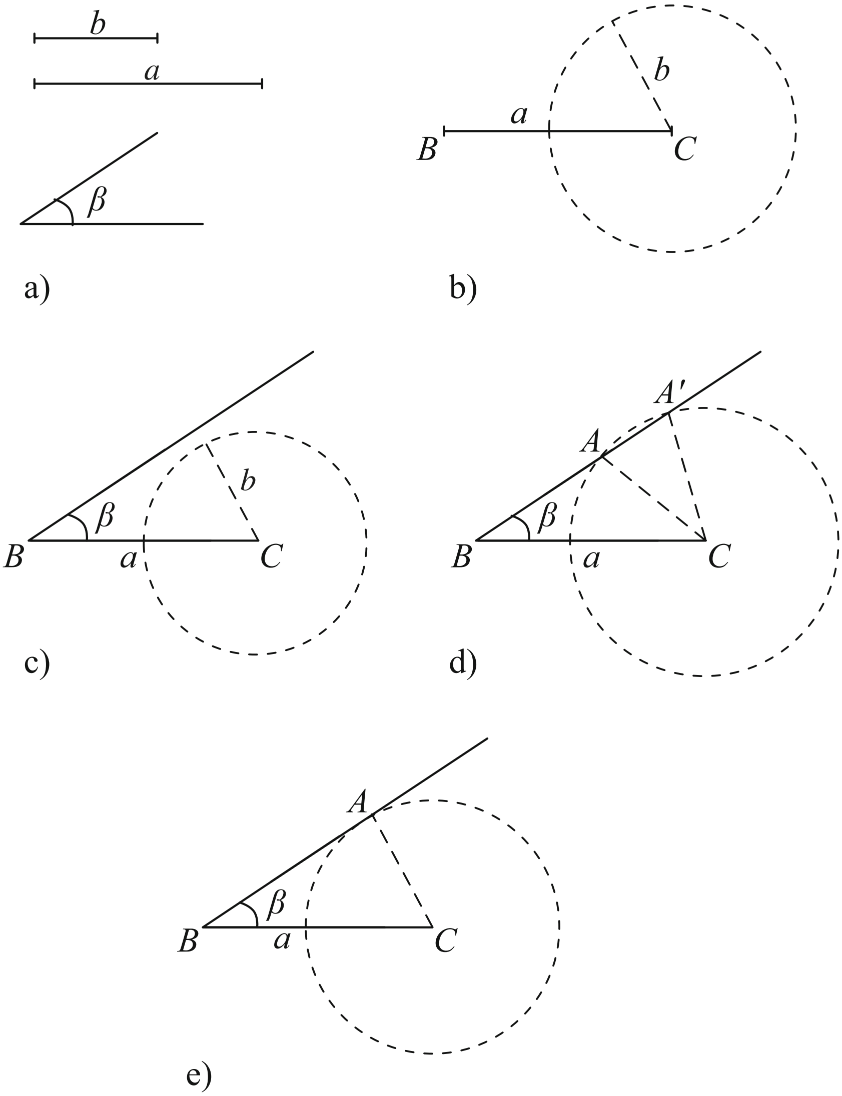


Fig. 4.29

Let us draw a segment a (or BC). We know that side b either starts at point B or at point C. Let it start at point C. Then the other end of b (i.e., point A) has to lie on the circle with center C and radius b (see Fig. 4.29b).

Since angle β is not the angle between a and b, we cannot draw it from vertex C. It has to have its vertex at B.

As we can see from Fig. 4.29c, the circle does not intersect the side of the angle. Therefore, there cannot be any triangle with the given elements a, b, and $\angle\beta$.

However, depending on the given parameters, there might be other possibilities. If segment b had been longer, then the picture would look like Fig. 4.29d. In this case the circle intersects the ray at two points, and we can construct two triangles: ABC and $A'BC$. But one can easily see that these triangles are not congruent. Which one do we choose then?

There is also the case where, by chance, segment b is such that the circle will intersect the ray at exactly one point (as in Fig. 4.29e). Only in this case do we obtain one triangle with three given elements.

We can conclude that the three elements of a triangle chosen in this exercise are not reliable and cannot define a triangle.

Exercise 7. Construct a triangle with a given side a and two angles such that one angle is adjacent to the side a and the other angle is not.

Solution. Due to Theorem 3 (see below), we can find the third angle of the triangle. Thus this case does not differ much from Exercise 4.

Exercise 8. Construct a triangle if its three angles are given.

Solution. According to Theorem 3, these angles cannot be arbitrary. Their sum must be equal to 180°. However, even in such a case there is no unique solution. We need to choose one side of a triangle. Then we can use Exercise 4 for the construction of the triangle. Since we can choose a side arbitrarily, we can construct infinitely many triangles with the given three angles. (See also Section 15 on similarity.)

7 Relations between elements of a triangle

7.1 Relations between the sides of a triangle

An important question arises: given three segments, is it always possible to construct a triangle such that its sides will have lengths a, b, and c? In other words, what segments a, b, and c can serve as sides of a triangle?

PROBLEM 13. Can the sides of a triangle have the following lengths:

(a) 1, 1, 1; (b) 1, 1, 1.5; (c) 1, 1, 1.9; (d) 1, 1, 2;
(e) 1, 1, 2.1; (f) 3, 1, 2; (g) 3, 2, 2?

Proposition 2. If a, b, and c are the sides of a triangle, then the sum of any two of them is bigger than the third side, i.e.,

$$a+b>c, \quad a+c>b, \quad b+c>a.$$

Proof. Consider a triangle. Let us denote its biggest side as a and the other sides as b and c (see Fig. 4.30a).

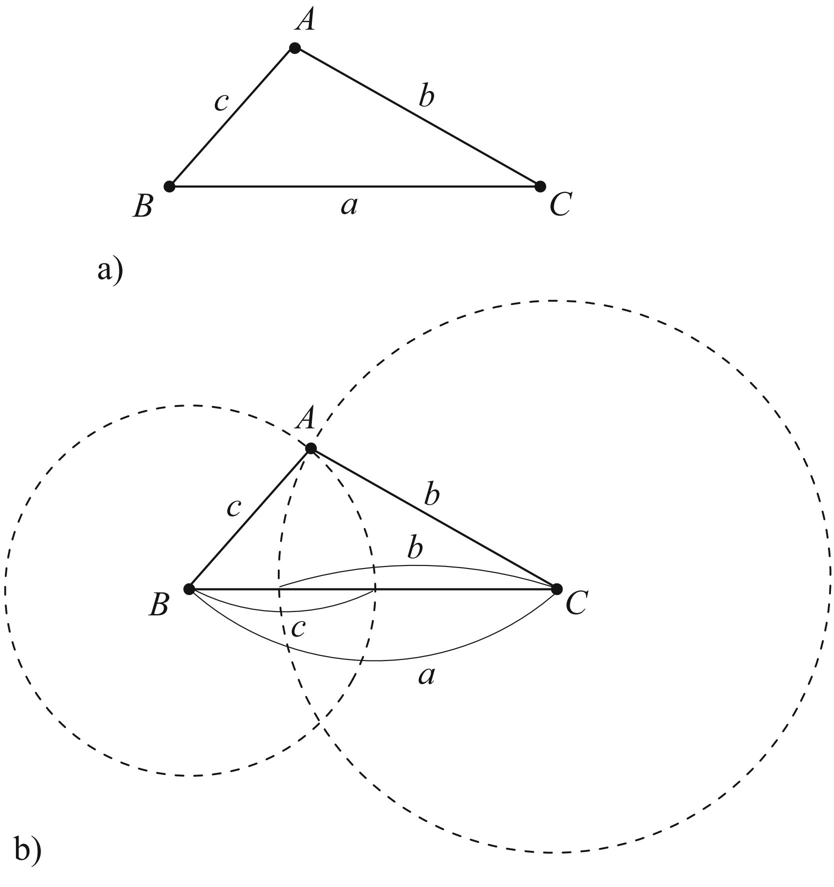


Fig. 4.30

Let us draw a circle with the center C and radius b and another circle with the center B and radius c, Fig. 4.30b. These circles intersect. Therefore, segment a is covered by overlapping segments b and c. Therefore, $b+c>a$. Since $a>b$ and $a>c$, we also obtain $a+b>c$ and $a+c>b$. □

The converse statement is also true.

Proposition 3. If a, b, and c are three numbers such that $a+b>c$, $a+c>b$, and $b+c>a$, then it is always possible to draw a triangle with sides a, b, and c.

Proof. Let a, b, and c be three segments such that $a+b>c$, $a+c>b$, and $b+c>a$ (Fig. 4.31a).

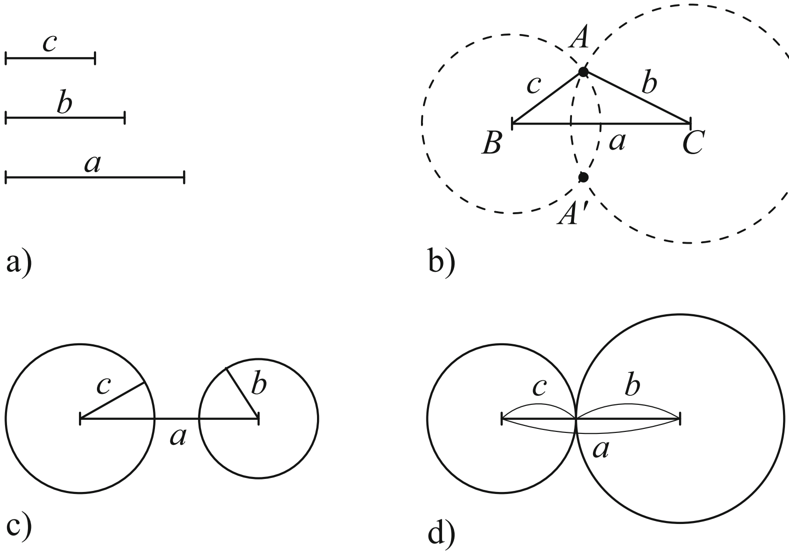


Fig. 4.31

Let us repeat the construction of a triangle with the sides a, b, and c from Exercise 5 starting from the biggest side (in our case, the side a). We draw side BC equal to a. Then we draw a circle with radius b and center C. Finally, we draw a second circle with radius c and center B. An intersection point of these two circles would give the triangle's third vertex. Since $b+c>a$, these circles do have to intersect at precisely two points (see Remark 6 below), e.g., A and A' as in Fig. 4.31b. This means that there is a triangle ABC (and also a triangle $A'BC$) which has the three given sides a, b, and c. □

Remark 6. The two circles in Proposition 3 have to intersect at precisely two points. Indeed, two circles either do not intersect or, if they intersect, can intersect at one or two points (see Section 18.1). If these circles do not intersect (see Fig. 4.31c), then we obtain $b+c<a$; if the circles intersect at one point (see Fig. 4.31d), we obtain $b+c=a$. In both cases, these results contradict the condition that $b+c>a$. Therefore, the only remaining possibility for the circles in Proposition 3 is that they intersect at two points as in Fig. 4.31b.

7.2 Relations between the angles of a triangle

Let us find the restrictions on angles of a triangle. For example, is there a triangle which has a right angle? Yes: see Fig. 4.42.

PROBLEM 14.

(a) Is there a triangle which has two right angles?

(b) Is there a triangle which has two angles each equal to 60° and a third angle equal to 65°?

Explain your answers.

The restriction on angles of a triangle is expressed in the following theorem.

Theorem 3. The sum of interior angles of a triangle is equal to 180°.

Proof. Consider a triangle ABC with angles α_1, α_2, and α_3 (see Fig. 4.32a).

Let us extend side AC through D (Fig. 4.32b) and draw a ray a, or CE, from point C parallel to side AB (Fig. 4.32c). We obtain two angles: $\angle BCE$ and $\angle ECD$. We have $\angle ECD = \alpha_1$ since they are corresponding angles; we also have $\angle BCE = \alpha_2$ since they are alternating angles (see Fig. 4.32d).

With vertex C we have three adjacent angles α_1, α_2, α_3 whose sum is a straight angle; therefore, $\alpha_1 + \alpha_2 + \alpha_3 = 180^\circ$. But α_1, α_2, α_3 are the interior angles of $\triangle ABC$. Thus, we have proved that their sum is equal to 180°. $\square$

We can see that the following statement is also true.

Proposition 4. For any three given angles such that their sum is 180°, it is always possible to construct a triangle with its interior angles equal to the given angles.

Proof. See the solution to Exercise 8. $\square$

Another theorem follows from Theorem 3:

Theorem 4. An exterior angle of a triangle is equal to the sum of the two interior angles not adjacent to this exterior angle.

Proof. Indeed, let us look at an exterior angle of a triangle such as $\angle BCD$ in Fig. 4.32b. We can see in Fig. 4.32d that $\angle BCD = \angle\alpha_1 + \angle\alpha_2$. Thus, indeed, the exterior angle $\angle BCD$ is equal to the sum of the interior angles not adjacent to $\angle BCD$. $\square$

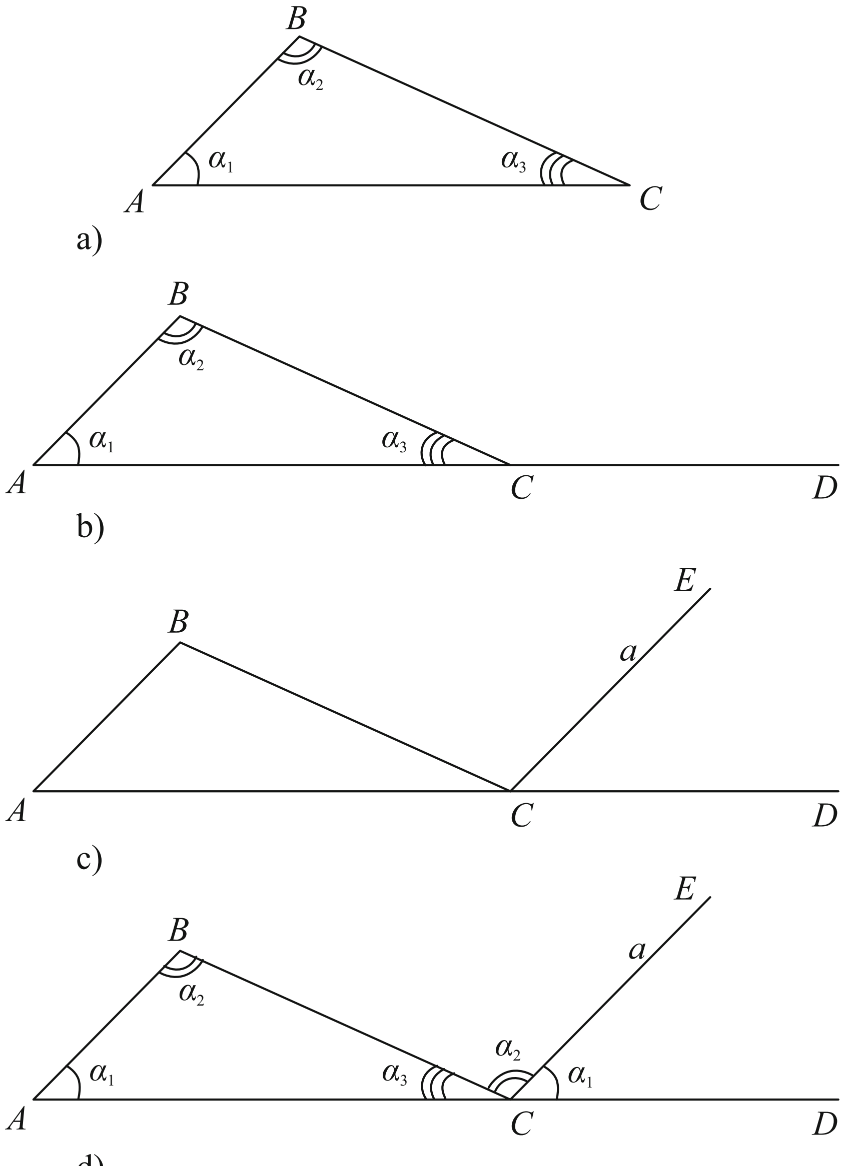


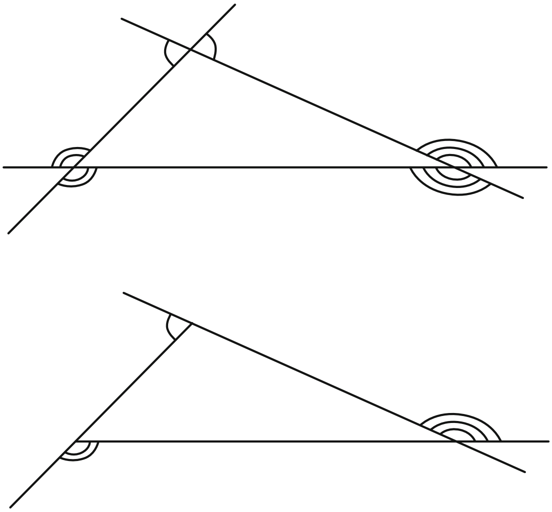
Fig. 4.32

A useful corollary follows from this theorem:

Corollary 1. An exterior angle of a triangle is larger than either of the non-adjacent interior angles of this triangle.

We have considered a relation between the sides of a triangle and a relation between the angles of a triangle. There is also a relation between the sides and the angles of a triangle (see Theorem 6 in Section 8.1).

PROBLEM 15. Consider a triangle. Let us extend all its sides. Around each vertex of the triangle there are four angles. For each interior angle of the triangle there are two exterior angles which are equal to each other, since they are vertical angles (see Fig. 4.33a).



a)

b)

Fig. 4.33

Let us choose only one exterior angle for each vertex as in Fig. 4.33b. Find the sum of these exterior angles of the triangle.

7.3 More about angles in a triangle

Consider a triangle ABC with angles $\angle\alpha$, $\angle\beta$, $\angle\gamma$ (see Fig. 4.34a).

Through vertices A, B, C let us draw three arbitrary lines such that they intersect at a single point O (see Fig. 4.34b).

Each of these lines divides an angle of the triangle into two angles: angle α into $\angle\alpha_1$ and $\angle\alpha_2$, angle β into $\angle\beta_1$ and $\angle\beta_2$, angle γ into $\angle\gamma_1$ and $\angle\gamma_2$. Thus, we have

$$\alpha = \alpha_1 + \alpha_2, \quad \beta = \beta_1 + \beta_2, \quad \gamma = \gamma_1 + \gamma_2.$$

These lines form several new angles in triangle ABC: six angles with vertex O (see Fig. 4.34c) and six angles at the points of intersection between these lines and the sides of the triangle (see Fig. 4.35a).

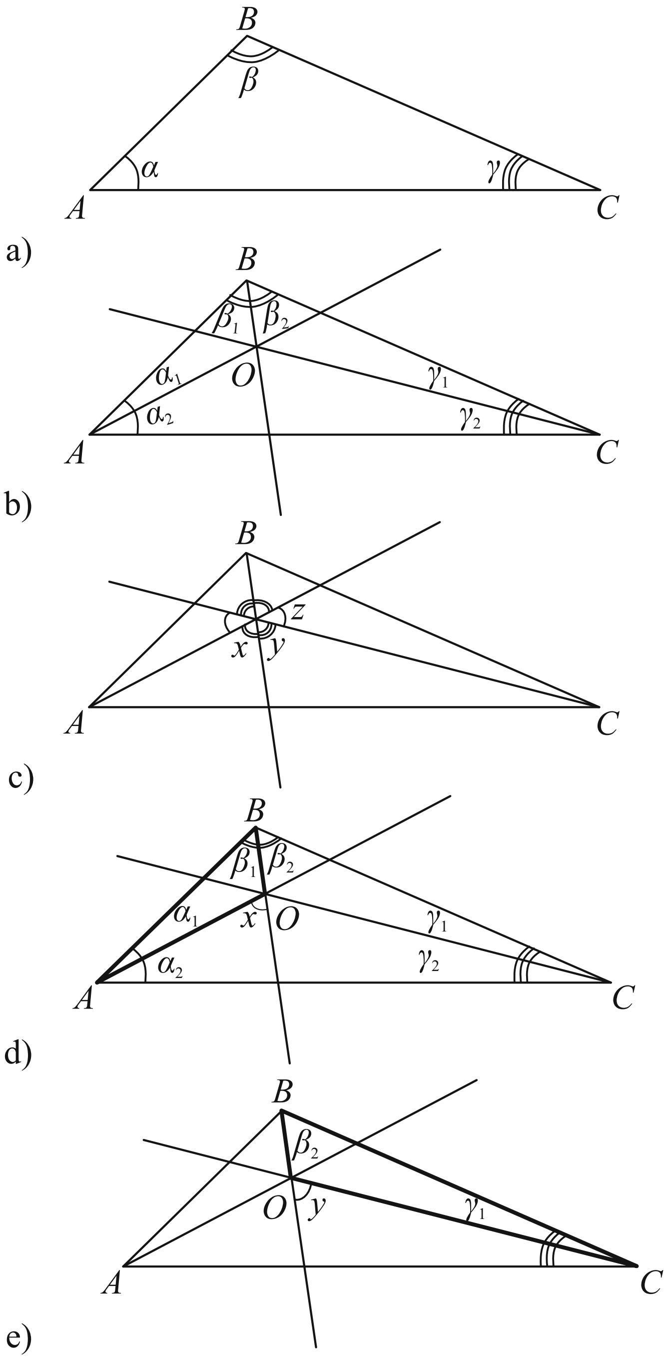

a)
B
β
α
γ
A
C
b)
B
β1 β2
α1
O
γ1
α2
γ2
A
C
c)
B
z
x y
A
C
d)
B
β1 β2
α1
x O
γ1
α2
γ2
A
C
e)
B
β2
O y
γ1
A
C

Fig. 4.34

Exercise 9. Suppose that the angles α_1, α_2, β_1, β_2, γ_1, γ_2 in the triangle ABC (see Fig. 4.34b) are known. Find all the angles with their vertices at O in Fig. 4.34c.

Solution. First, note that there are only three different angles with vertex O, i.e., x, y and z. The other three angles are correspondingly equal to these angles since they are vertical angles.

Let us find angle x. Consider triangle AOB (see Fig. 4.34d). Angle x is its exterior angle and from Theorem 4 we have

$$x = \alpha_1 + \beta_1.$$

Similarly let us find angle y by considering triangle BOC (see Fig. 4.34e). We obtain

$$y = \beta_2 + \gamma_1.$$

We leave it for you to find angle z.

Let us focus now on finding the angles in Fig. 4.35a. For example, in order to find angle δ in Fig. 4.35b we consider the triangle marked with bold lines in this figure. We notice from Theorem 4 that

$$\delta = \beta_1 + \beta_2 + \gamma_1 = \beta + \gamma_1.$$

Note that we could have also marked the smaller triangle shown with bold lines in Fig. 4.35c. Then, again making use of Theorem 4, we obtain $\delta = \beta_1 + y$, i.e., $\delta = \beta_1 + \beta_2 + \gamma_1$. Thus, both solutions give the same answer.

PROBLEM 16.

(a) Find angle z in Fig. 4.34c.

(b) Find all the angles marked in Fig. 4.35a.

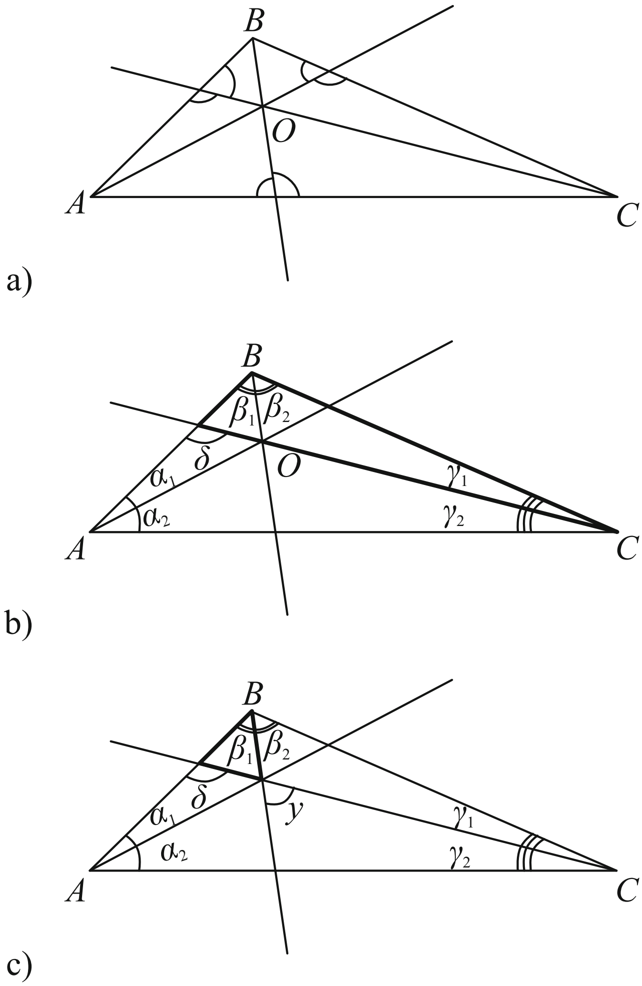


Fig. 4.35

8 Properties of a triangle. Particular kinds of triangles

There are some special lines related to a triangle: *median*, *angle bisector* and *altitude*.

A *median* is a line passing through a vertex of a triangle and dividing the opposite side into two equal parts.

An *angle bisector* is a line passing through a vertex of a triangle and dividing the angle at that vertex into two equal parts.

An *altitude* is a line that passes through a vertex of a triangle and is perpendicular to the opposite side of the triangle.

In a triangle there are 3 medians, 3 angle bisectors and 3 altitudes.[18] Fig. 4.36a shows a triangle and its 3 medians; Fig. 4.36b shows the same triangle and its 3 angle bisectors; Fig. 4.36c shows the same triangle and its 3 altitudes.

Note that in some instances, a triangle's altitude might not lie inside the triangle, as in Fig. 4.36d. This happens if the triangle has an obtuse angle. Such a triangle is sometimes called an *obtuse triangle*.

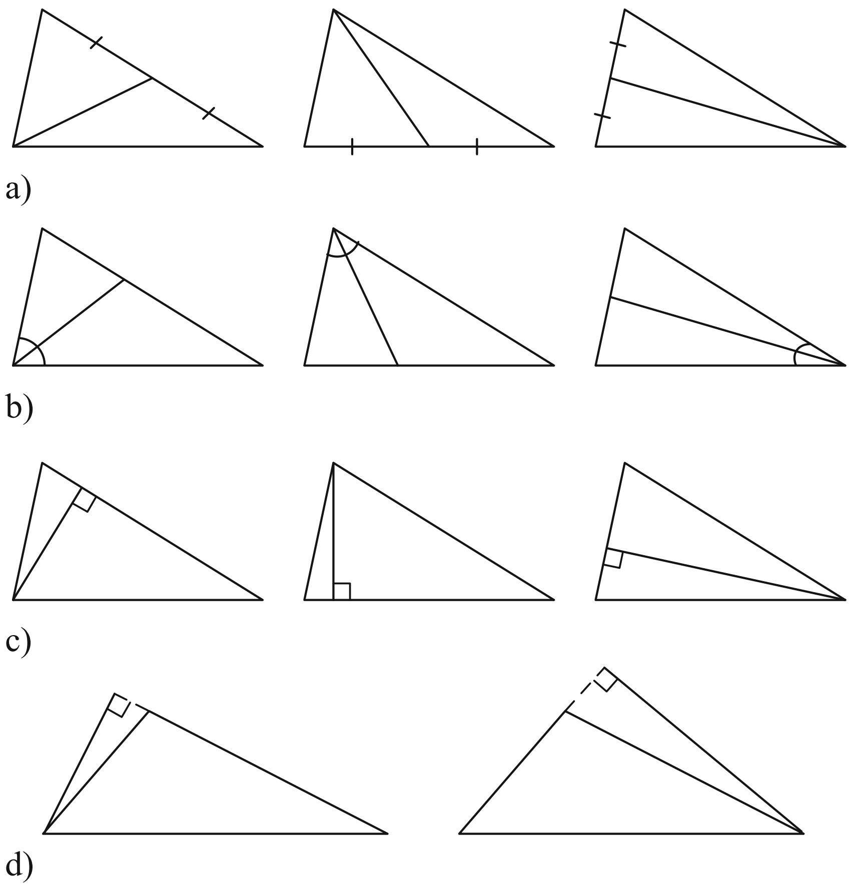


Fig. 4.36

[18] Sometimes instead of a line we will draw only a ray or a segment lying on this line and inside the triangle.

8.1 The isosceles triangle

An important particular case of a triangle is a triangle with two equal sides.

A triangle with two sides equal to each other is called an *isosceles triangle*. The third side of an isosceles triangle is called its *base*.

An isosceles triangle has the following interesting property.

Theorem 5. In an isosceles triangle the median to the base coincides with the altitude to the base and with the bisector of the angle opposite the base.

Proof. Consider an isosceles triangle ABC with $AB = BC$. Draw the median BD to the base AC (see Fig. 4.37a). Let us prove that BD is an angle bisector.

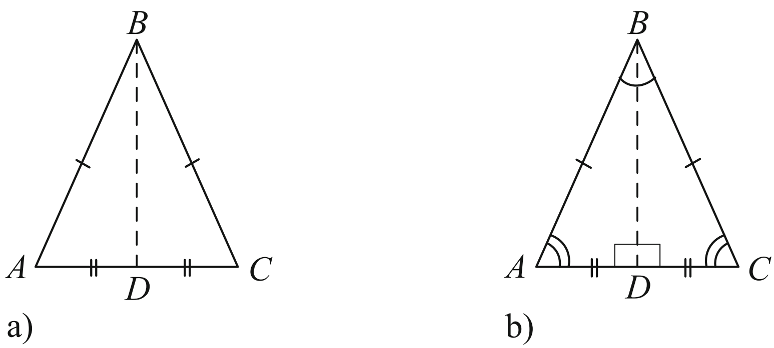


Fig. 4.37

Note that the two triangles ABD and BCD are congruent to each other. Indeed, they have a common side BD and two other sides which are correspondingly equal to each other (see (1) in Definition 3). Therefore, the angles in these triangles are correspondingly equal, i.e., $\angle BAD = \angle BCD$, $\angle ABD = \angle CBD$, and $\angle BDA = \angle BDC$ (see Fig. 4.37b).

Since $\angle ABD = \angle CBD$, segment BD is the bisector of the angle ABC.

Since $\angle BDA = \angle BDC$ and points A, D, C form a straight angle ($180°$), we also conclude that $\angle BDA = \angle BDC = 90°$. Thus, line BD is the altitude of triangle ABC to its base AC. □

Remark 7. Theorem 5 contains, in fact, two statements:

(1) In an isosceles triangle, the median to the base coincides with the altitude to the base.

(2) In an isosceles triangle, the median to the base coincides with the bisector of the angle opposite the base.

It is true that in an isosceles triangle all three lines—the median, altitude, and angle bisector to the base—coincide. This means that in an isosceles triangle this statement about all pairs of these lines can be proved. For example, we can say that in an isosceles triangle the altitude to the base is at the same time the bisector of the angle opposite the base.

The converse statements are also true. For example, if in a triangle a median coincides with the altitude, then this triangle is isosceles. Indeed, in triangle ABC let the altitude BD also be the median (see Fig. 4.37b). Then the two triangles ABD and BCD are congruent by (2) in Definition 3. Therefore, the sides AB and BC are equal and the triangle ABC is isosceles.

All the other converse statements can be proved as well. It is a useful exercise to prove all the statements of Theorem 5 and all the converse statements.

We have also already proved the following important corollary.

Corollary 2. In an isosceles triangle the angles adjacent to the base are equal.

One can also prove the converse statement.

Proposition 5. If two angles in a triangle are equal to each other, then this triangle is isosceles and its base is the side to which these equal angles are adjacent.

Proof. Consider triangle ABC with angles α, β, and γ. Let $\angle\alpha = \angle\gamma$ (see Fig. 4.38a).

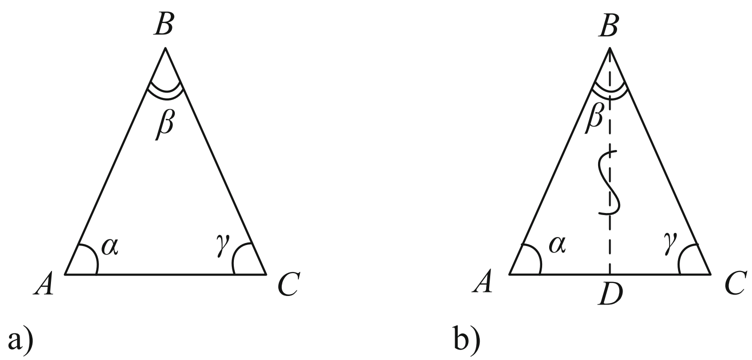


Fig. 4.38

Let us draw the bisector BD of angle $\angle ABC$, or $\angle\beta$ (see Fig. 4.38b). Then in triangles ABD and BDC we have $\angle\alpha = \angle\gamma$ and $\angle ABD = \angle CBD$. From Theorem 3 we also have $\angle BDA = \angle BDC$. Since BD is a common side of triangles ABD and BDC, then from (3) in Definition 3, we conclude that $\triangle ABD = \triangle BDC$. Thus $AB = BC$. □

From Fig. 4.37b we can also notice that a triangle with two equal angles has two equal sides which lie opposite to these equal angles. An even more general fact holds.

Theorem 6. The biggest side of a triangle lies opposite its biggest angle.

Proof. Consider triangle ABC with angles α, β, γ. Let angle β be the biggest angle in $\triangle ABC$, as in Fig. 4.39a. We need to prove that side AC is bigger than AB and bigger than BC.

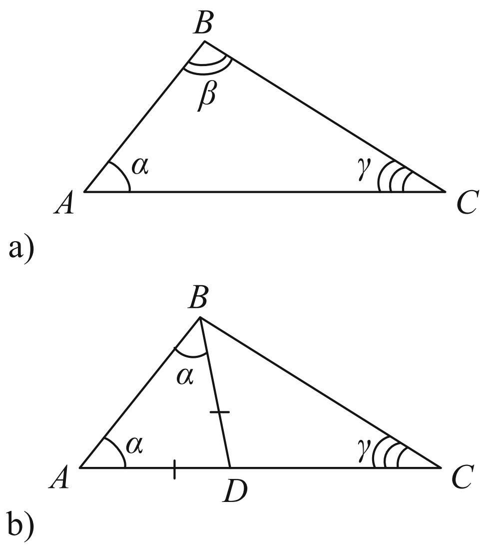


Fig. 4.39

Let us draw an angle equal to $\angle\alpha$ from vertex B (see Fig. 4.39b). Then according to Proposition 5, triangle ABD is isosceles and $BD = AD$.

Now consider triangle BCD. By Proposition 2, $BC < BD + DC$. But $BD + DC = AD + DC = AC$. Therefore, we have proved that $BC < AC$. Similarly, we can prove that $AB < AC$. □

One can also prove the converse theorem: the biggest angle of a triangle lies opposite its biggest side.

8.2 Equilateral triangle

There is another important particular case of a triangle.

A triangle whose sides are all equal is called an *equilateral triangle*.

It follows from Corollary 2 that all angles of an equilateral triangle are equal to each other. Since the sum of all the angles of a triangle is 180° we find that an equilateral triangle has three angles of 60° (see Fig. 4.40).

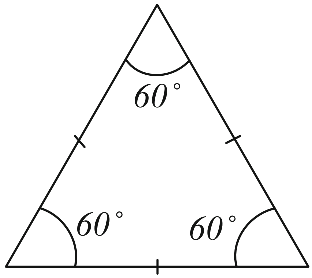


Fig. 4.40

The next theorem follows easily from Theorem 5.

Theorem 7. In an equilateral triangle all the medians coincide with the angle bisectors and with the altitudes.

PROBLEM 17. Prove that the medians, angle bisectors, or altitudes in an equilateral triangle also serve as lines of symmetry of this triangle (see Section 4.1).

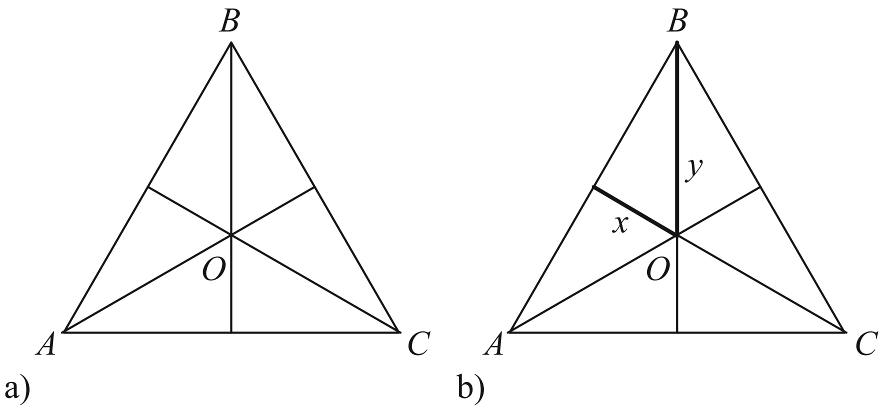


Fig. 4.41

PROBLEM 18. Repeat the construction from Section 7.3 (see Fig. 4.34c) for an equilateral triangle. Find angles x, y, z in Fig. 4.34c and the angles marked in Fig. 4.35a.

PROBLEM 19. Consider an equilateral triangle. Draw three medians in it which, as we know, intersect at a single point (see Fig. 4.41a).

If x and y are the two segments shown in Fig. 4.41b, then $x = \frac{1}{2}y$. Prove this.

8.3 Right triangle

Another important particular case of a triangle is a right triangle.

A triangle is called a *right triangle* if one of its interior angles is a right angle. The two sides of the right angle are called *legs* and the third side is called the *hypotenuse*.

In Fig. 4.42 there are examples of right triangles.

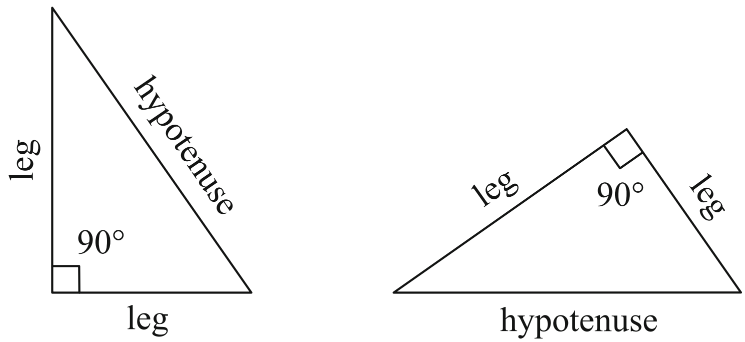


Fig. 4.42

Proposition 6. The hypotenuse of a right triangle is longer than either of its legs.

Proof. Indeed, the 90° angle is the biggest angle in a right triangle (since the sum of the angles is 180°). It follows from Theorem 6 that the side opposite this angle, which is the hypotenuse, is the biggest. □

Proposition 7. In a right triangle with a 30° angle, the leg opposite this angle is equal to one-half of the hypotenuse.

PROBLEM 20. Prove Proposition 7.

Hint. Complete the right triangle to form an equilateral triangle by reflecting it in the leg that is adjacent to the 30° angle. Then apply Theorem 7.

In a right triangle we already know one angle, an angle of 90°. Therefore, from Definition 3 one can obtain the following definitions of congruent right triangles:

(1) Two right triangles are congruent if the two legs of one triangle are correspondingly equal to the two legs of the other triangle.

(2) Two right triangles are congruent if one leg and the hypotenuse of one triangle are correspondingly equal to one leg and the hypotenuse of the other triangle.

(3) Two right triangles are congruent if the hypotenuse and an angle adjacent to it in one triangle are correspondingly equal to the hypotenuse and an angle adjacent to it in the other triangle.

(4) Two right triangles are congruent if a leg and the angle adjacent to it (of course, not the right angle) are correspondingly equal to the leg and the angle adjacent to it in the other triangle.

Proof.

(1) Since we know the angle between the legs of the right triangle, this statement follows from (2) of Definition 3.

(2) Consider two right triangles ABC and $A'B'C'$ which satisfy definition (2) above. Then either we can move triangle $A'B'C'$ so that leg $B'C'$ will coincide with leg BC (as in Fig. 4.43a) or we first need to reflect triangle $A'B'C'$ in line $A'C'$ in order to do this (as in Fig. 4.43b).

In either case, when sides BC and $B'C'$ coincide, leg $A'C'$ will lie on the same ray as AC since both angles $\angle ACB$ and $\angle A'C'B'$ are right angles.

We need to prove that points A and A' lying on the same ray coincide. Let us draw a circle with the radius AB from point B, (see Fig. 4.43c). Note that the endpoint C of segment AC lies inside this circle because $CB < AB$ (since a leg is smaller than the hypotenuse).

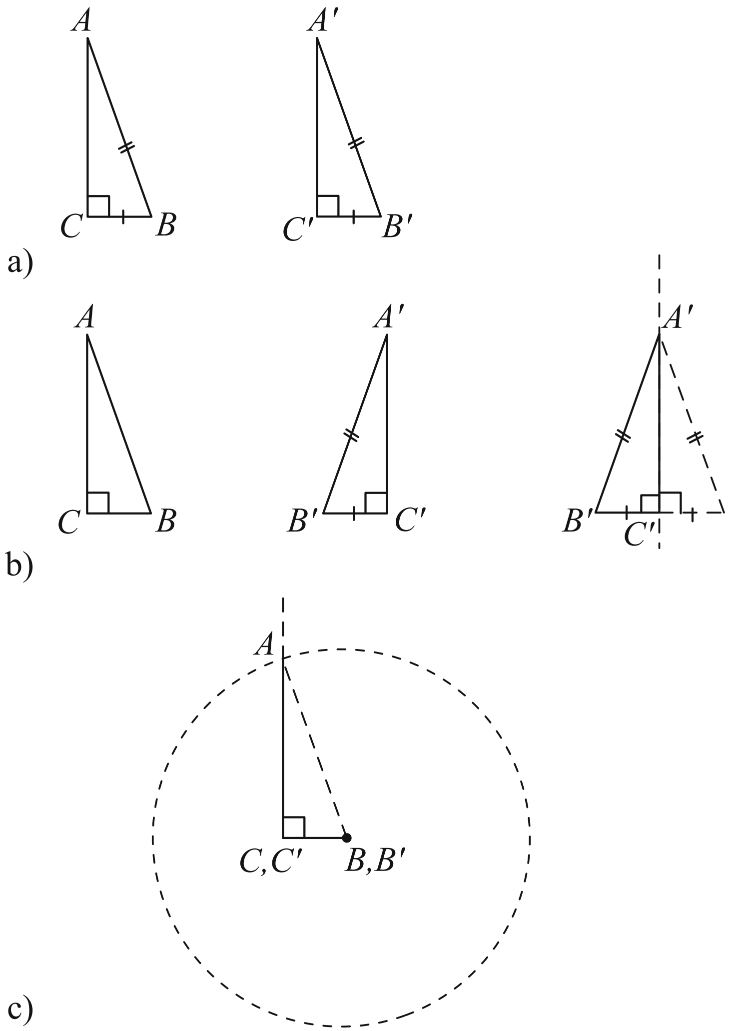


Fig. 4.43

Now consider ray $C'A'$ with vertex C'. It can intersect this circle in only one point. Since $AB = A'B'$ and points B and B' coincide, point A' must also lie on this circle. Therefore, point A' coincides with point A. Thus, both triangles ABC and $A'B'C'$ coincide, and all their sides and all their angles are correspondingly equal to one another. According to Definition 3 they are congruent triangles.

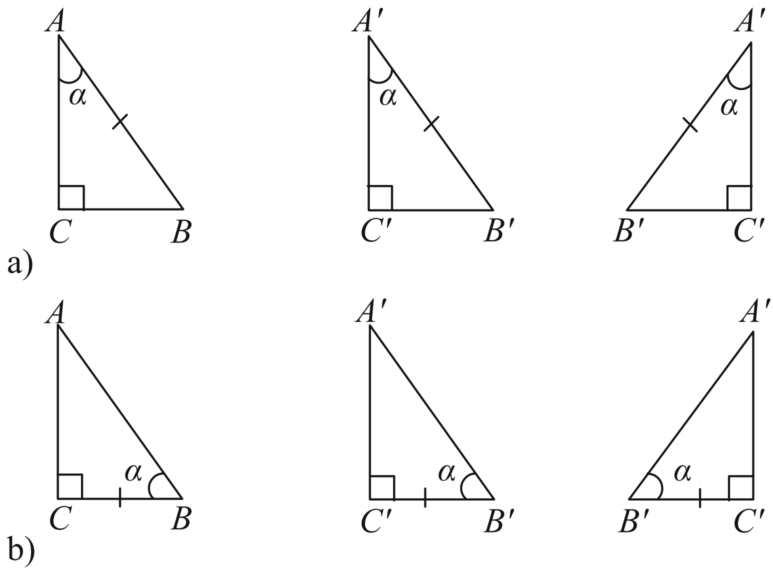


Fig. 4.44

(3) Consider two right triangles ABC and $A'B'C'$ which satisfy the condition in definition (3) (see Fig. 4.44a). Then $AB = A'B'$ and, for example, angles $\angle CAB$ and $\angle C'A'B'$ are equal. But $\angle ACB = \angle A'C'B' = 90°$. Since the sum of all the angles of a triangle is equal to $180°$, we have $\angle ABC = 180° - 90° - \angle CAB = 180° - 90° - \angle C'A'B' = \angle A'B'C'$. Therefore, triangles ABC and $A'B'C'$ are congruent according to (3) in Definition 3.

(4) Let two right triangles ABC and $A'B'C'$ satisfy the condition in definition (4) (see Fig. 4.44b). Then for example, $CB = C'B'$ and $\angle ABC = \angle A'B'C'$. According to (3) in Definition 3, we have $\triangle ABC = \triangle A'B'C'$. □

PROBLEM 21. Construct right triangles given the data in each definition above, i.e.,

(a) given two legs;

(b) given one leg and a hypotenuse;

(c) given a hypotenuse and an angle;

(d) given a leg and an angle.

There will be more about right triangles in Section 10.

9 Area in Euclidean geometry

9.1 Measurement of area. Area of a rectangle

In Chapter III we defined the area of a figure and used a unit parallelogram to measure areas.[19]

In this section we present formulas for the area of a rectangle and that of a triangle. Areas of some other figures will be presented in Section 14.

First of all we need to choose a unit area. As is usually done in Euclidean geometry, we will choose a unit square as the unit area.

A *rectangle* is a quadrilateral which has four right angles. A *square* is a rectangle with all sides equal to each other.[20] A *unit square* has side length equal to 1.

Proposition 8. The area of a rectangle with sides a and b is equal to the product of its side lengths, i.e., $S = a \cdot b$.

Proof. Consider a rectangle and let us choose a unit square (see Fig. 4.45a). Just as we measured area in Chapter III, we can measure the area of a rectangle by counting the number of unit squares that fit in it.

In the case where sides a and b are integers (see the example in Fig. 4.45b, where $a = 5$ and $b = 4$), there are ab unit squares in the rectangle, i.e., $S = ab$.

In the case where the numbers a and b are rational numbers, we can repeat the reasoning from Chapter III and still obtain the same formula.

One can also prove that this formula is true for any real numbers a and b. $\square$

Remark 8. Note that in Chapter III we found the same formula $S = a \cdot b$ for the area of a parallelogram. This does not mean that the area of a rectangle with sides a, b and the area of a parallelogram with sides a, b are the same. It is very important to remember that in Chapter III the unit area was defined as a unit parallelogram and not a unit square. For the area of a parallelogram in this chapter, see Section 14.

The following fact is a consequence of Proposition 8.

Proposition 9. The area of a square with side a is equal to a^2.

---

[19] However, in Chapter III we could not obtain formulas for the area of some figures since we could not measure length on non-parallel lines and had no angle measure. We could not even define some figures, e.g., a rectangle or a square.

[20] We can also say that a square is a rhombus with one right angle.

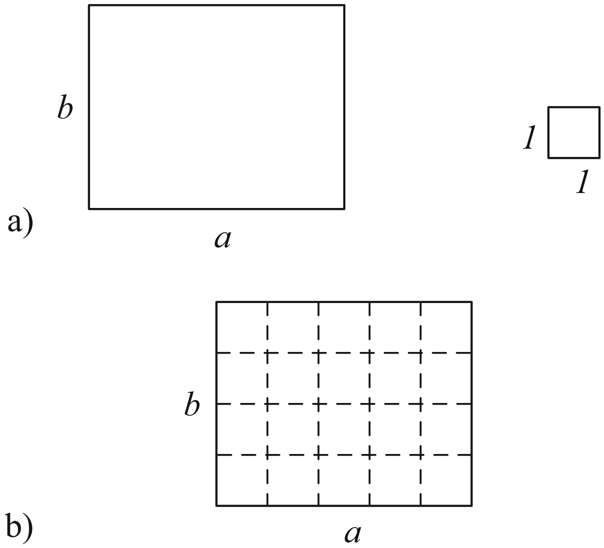


Fig. 4.45

9.2 Area of a triangle

Proposition 10. The area of a right triangle with the legs a and b is equal to

$$\frac{1}{2}a \cdot b.$$

Proof. Let us complete a right triangle (see Fig. 4.46a) to form a rectangle (see Fig. 4.46b).

Fig. 4.46

We can see that there are two congruent triangles in this rectangle—indeed, both triangles have sides equal to a and b and angle 90° between them. Therefore, the area of the original right triangle is $\frac{1}{2}a \cdot b$. $\square$

Proposition 11. The area of a triangle is equal to half of a side multiplied by the altitude drawn to this side, i.e.,

$$S_\triangle = \frac{1}{2}a \cdot h.$$

Proof. We have proved the formula for a right triangle. In an arbitrary triangle any angle is either acute or obtuse.

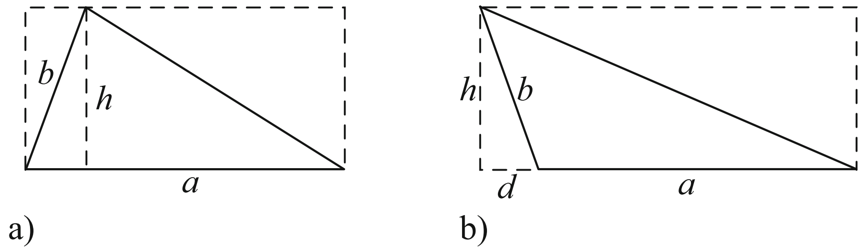


Fig. 4.47

First, consider an acute triangle. Let us complete this triangle to form a rectangle (see Fig. 4.47a). We also draw the altitude h of the triangle to side a. The altitude divides the rectangle into two rectangles with two congruent right triangles in each of them. The area of each right triangle is half of the area of the corresponding rectangle. Therefore, the area of the original triangle is half of the area of the whole rectangle, which is equal to $a \cdot h$. Thus, $S_\triangle = \frac{1}{2}a \cdot h$.

Consider an obtuse triangle. Again let us complete it to form a rectangle and draw the altitude h of the triangle to the side a (see Fig. 4.47b). The altitude intersects the extension of the side a. Let us denote the segment between the altitude h and side a by d. The area of the rectangle is then $(a+d) \cdot h$. The diagonal of the rectangle splits it into two congruent right triangles with area $\frac{1}{2}(a+d) \cdot h$. The area of the original triangle is smaller than one of them by the area of the right triangle with base d and altitude h. Thus,

$$S_\triangle = \frac{1}{2}(a+d) \cdot h - \frac{1}{2}d \cdot h = \frac{1}{2}a \cdot h.$$

As we can see, the area for either an acute or an obtuse triangle is given by the same formula:

$$S_\triangle = \frac{1}{2}a \cdot h,$$

where a is a side of the triangle and h is the altitude drawn to this side. $\square$

Here is another proof of Proposition 11. Consider a triangle (Fig. 4.47a). Let us draw a bimedian in this triangle and draw a rectangle as in Fig. 4.48a.

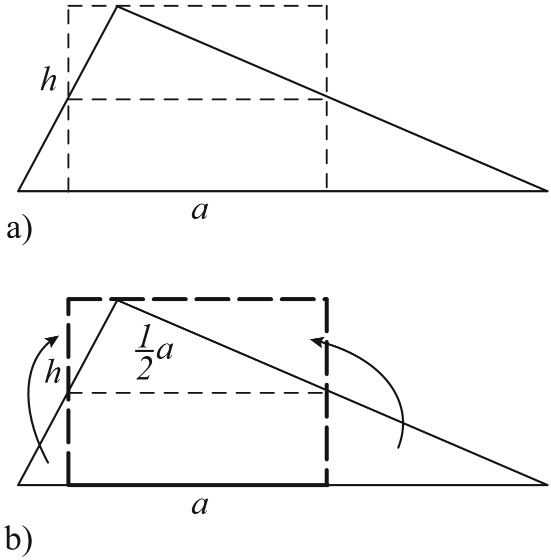


Fig. 4.48

This rectangle cuts two triangles from the original triangle. If we move these triangles as shown in Fig. 4.48b, we can see that the area of the original triangle is the same as the area of the rectangle. But the area of this rectangle is equal to $\frac{1}{2}a \cdot h$ (due to Theorem 2 in Chapter II). Thus, we have obtained the same formula for the area of a triangle.

We offer the case in Fig. 4.47b to you as a problem. □

PROBLEM 22. Complete the second proof of Proposition 11 by considering the case in Fig. 4.47b.

The following statement, illustrated in Fig. 4.49, follows from Proposition 11.

Corollary 3. In a triangle with sides a_1, a_2, a_3, let h_1 be the altitude to the side a_1 and h_2 the altitude to the side a_2. Then $a_1h_1 = a_2h_2$ or

$$\frac{a_1}{a_2} = \frac{h_2}{h_1}.$$

In other words, the altitudes to the sides of a triangle are inversely proportional to the corresponding sides.

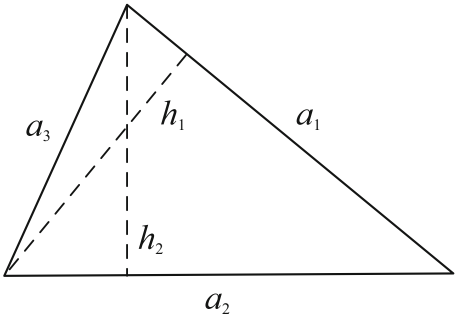


Fig. 4.49

The area of a triangle can also be expressed by a formula in terms of the three sides of this triangle (see Section 10).

10 The Pythagorean theorem and its applications

10.1 The Pythagorean theorem

The Pythagorean theorem is over 2,500 years old.[21] However, its significance was increased when geometries other than Euclidean geometry appeared. In fact, the Pythagorean theorem is what distinguishes Euclidean geometry from, for example, Lobachevskian geometry.

Consider a right triangle with sides a, b, and hypotenuse c (see Fig. 4.50a). There is a very important relation among the sides a, b, and c, known as the Pythagorean theorem.

Theorem 8 (Pythagorean theorem) In a right triangle with legs a and b and hypotenuse c, the following relation holds:

$$a^2 + b^2 = c^2.$$

[21] The Greek philosopher and mathematician Pythagoras lived about 580–500 BC. Like many other ancient philosophers, Pythagoras had interests ranging from mathematics, astronomy, and music to ethics. For example, he had writings about beans and cabbage which set the base for vegetarianism in his days. People who followed his ethical vegetarian style of living were called Pythagoreans. This term was used until the middle of the 19th century. Pythagoras wrote: "As long as man continues to be the ruthless destroyer of lower living beings, he will never know health or peace. For as long as men massacre animals, they will kill each other. Indeed, he who sows the seeds of murder and pain cannot reap joy and love."

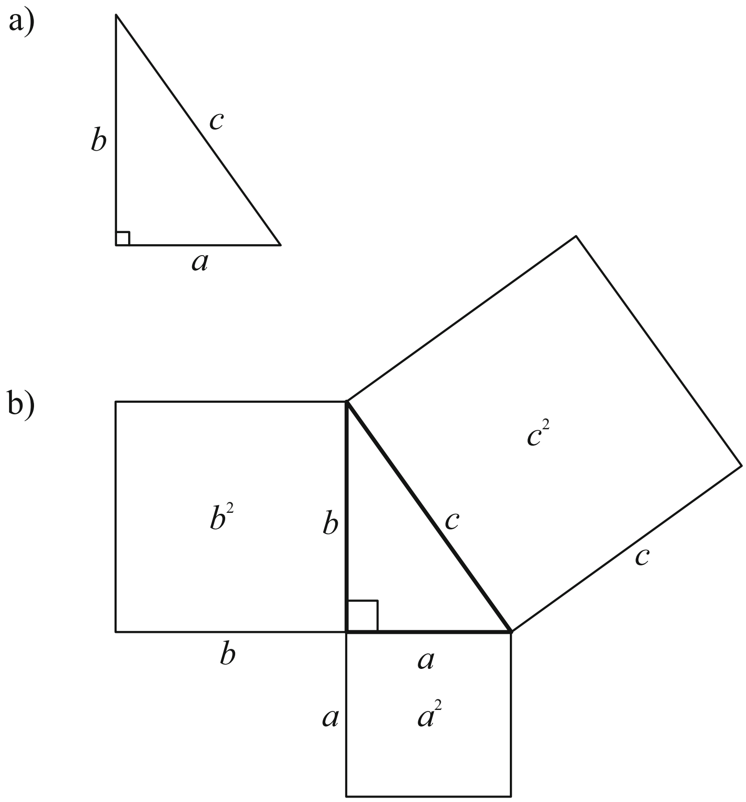


Fig. 4.50

Geometrically, this theorem means that the sum of the areas of squares constructed on the legs of a right triangle is equal to the area of a square constructed on the hypotenuse of this triangle (see Fig. 4.50b).

There are many proofs of this theorem. In this section we give two proofs, both using the notion of area. In Section 15 we will use the notion of similarity to present a third proof of the Pythagorean theorem.

Proof. I. We need to prove that the area of the two squares a^2 and b^2 together is equal to the area of the square c^2, where a and b are the legs and c is the hypotenuse of a right triangle (see Fig. 4.50b; see also Fig. 4.51a).

Consider the squares on the left of the equality. Let us mark segment $a = CB$ on the bottom side of square b^2 and connect point B with point A, as in Fig. 4.51b. We obtain right triangle ABC with hypotenuse AB equal to c.

If we connect point B with point D (see Fig. 4.51c), we obtain another right triangle BDE with hypotenuse BD equal to c and legs a and b. Indeed,

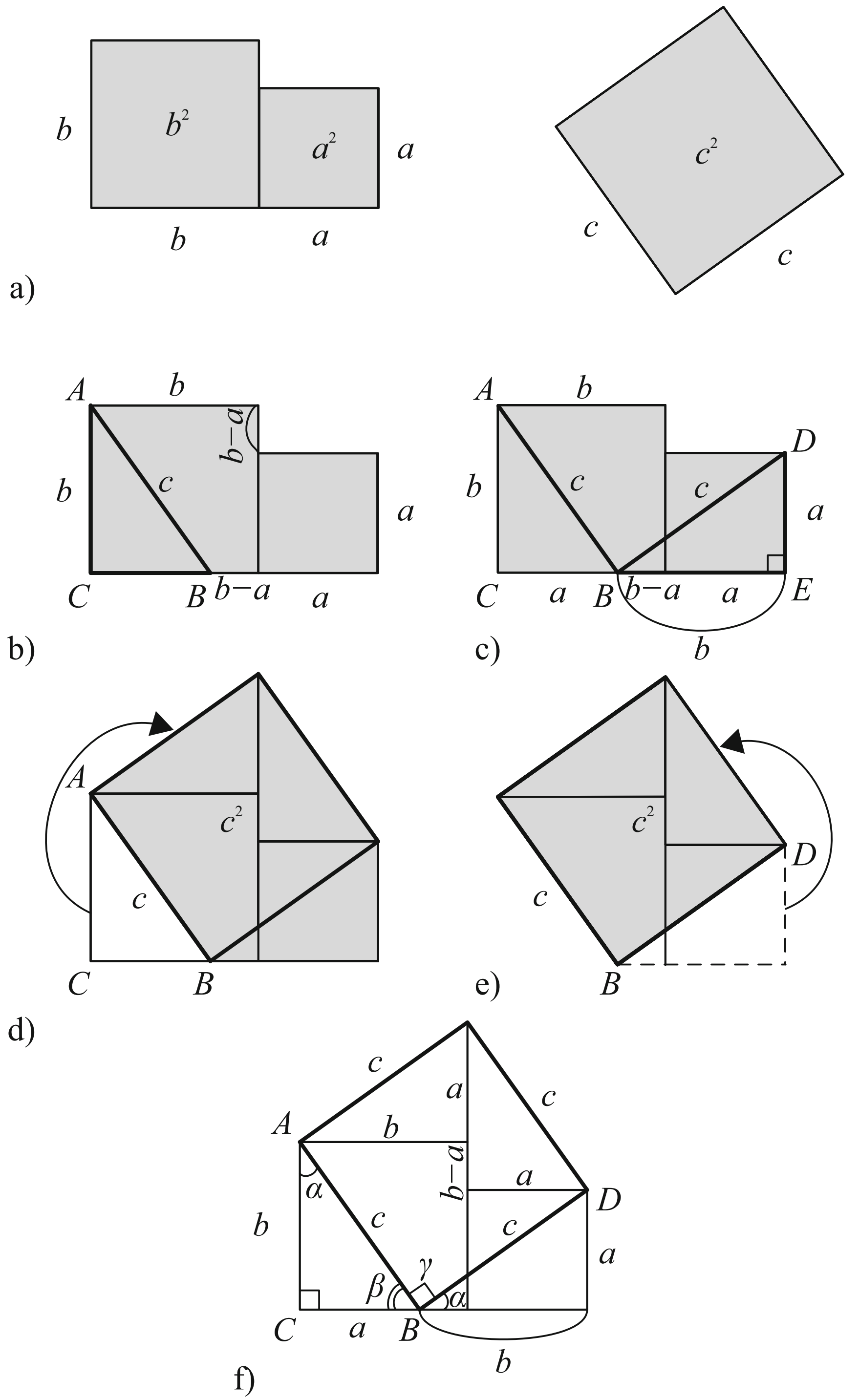


Fig. 4.51

$BE = b - a + a = b$, and $BD = c$ since this right triangle has legs a and b, and is, therefore, congruent to the initial right triangle.

Let us imagine that squares a^2 and b^2 are made, for example, from cardboard. Now let us cut this figure along segment AB. This will cut out triangle ABC, which we then place as in Fig. 4.51d. If we now cut the figure along BD, we cut out triangle BDE, which we then place as in Fig. 4.51e. The figure obtained is a square with side c.

Indeed, as you can see from Fig. 4.51f, angle γ of the quadrilateral obtained (marked by bold lines) is equal to $\angle\gamma = 180^\circ - (\angle\alpha + \angle\beta)$, but $(\angle\alpha + \angle\beta) = 90^\circ$ (as the angles of the original right triangle), so $\angle\gamma = 90^\circ$. Since all the sides of this quadrilateral are equal, it is a square with side c. The area of this square is c^2, and it is composed of two other squares with areas a^2 and b^2. Therefore, $a^2 + b^2 = c^2$. $\square$

Proof. II. Consider a right triangle ABC (see Fig. 4.52a). Let us extend the side a by a length b and the side b by a length a, and complete the figure to form a square with side $a + b$ (see Fig. 4.52b).

On each side of this square we mark segments a and b, and then we connect these marks to obtain a quadrilateral (see Fig. 4.52c). Each side of this quadrilateral is equal to c, since all the triangles around it are right triangles with legs a and b.

Let us find the angles of this quadrilateral. It is easy to see that each of them is equal to 90°. Indeed, all the right triangles around this quadrilateral are congruent and have angles α and β, where $\alpha + \beta = 90^\circ$. Then, as you can see from Fig. 4.52d, each angle of the quadrilateral is equal to $180^\circ - (\alpha + \beta) = 180^\circ - 90^\circ = 90^\circ$. Therefore, this quadrilateral is a square.

We can write the following equality for the area. If we denote the area of the outside square by S_o, the area of the inside square by S_i, and the area of the right triangle by $S_\triangle$, we have

$$\begin{aligned} S_\mathrm{o} &= (a+b)^2, \\ S_\mathrm{i} &= c^2, \\ S_\mathrm{o} &= S_\mathrm{i} + 4S_\triangle, \quad \text{or} \\ S_\mathrm{o} &= c^2 + 4\tfrac{ab}{2} = c^2 + 2ab. \end{aligned}$$

Therefore,

$$\begin{aligned} (a+b)^2 &= c^2 + 2ab, \quad \text{or} \\ a^2 + 2ab + b^2 &= c^2 + 2ab. \end{aligned}$$

Finally, $a^2 + b^2 = c^2$. $\square$

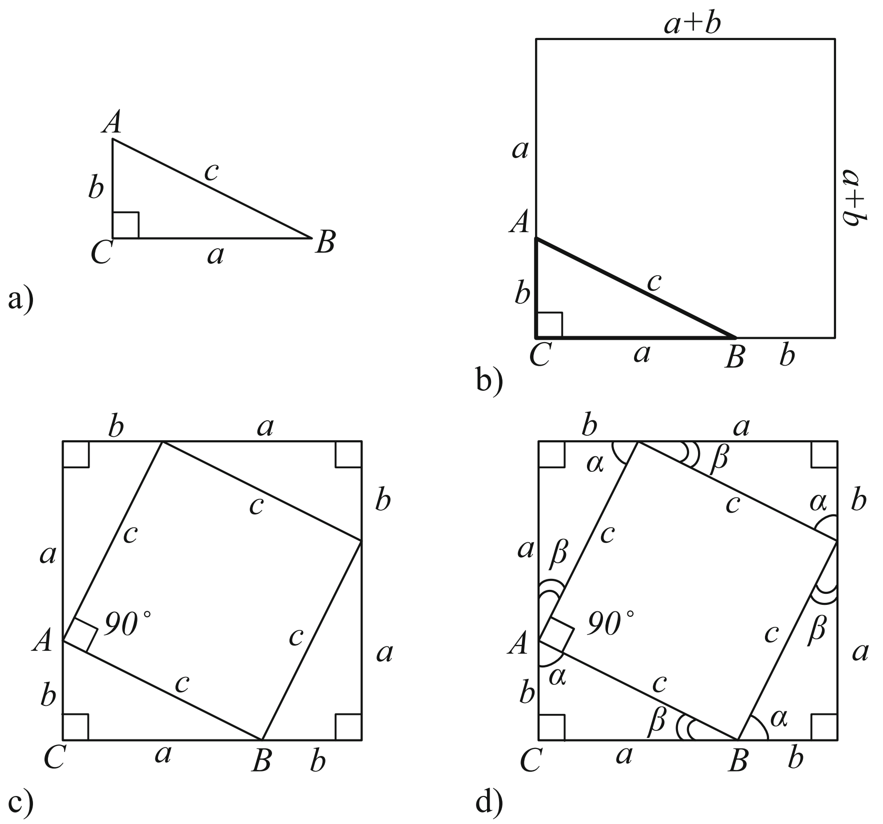


Fig. 4.52

Exercise 10.

(a) A right triangle has two legs which are 3 and 4 cm long. What is the length of the hypotenuse?

(b) A right triangle has two legs each of which is 1 cm long. What is the length of the hypotenuse?

Solution. According to the Pythagorean theorem,

(a) the hypotenuse is $c = \sqrt{3^2+4^2} = \sqrt{25} = 5$;

(b) the hypotenuse is $c = \sqrt{1^2+1^2} = \sqrt{2}$.

As we can see from this exercise, some segments have lengths which cannot be expressed as rational numbers. The number $\sqrt{2}$ is an example of a number that cannot be so expressed, and so it is called an *irrational number*.

Remark 9. Rational and irrational numbers are usually defined in calculus. This is also the subject of an interesting theory (number theory). For the purposes of this book we accept the definition that a rational number is a number that can be represented as a rational fraction, i.e., as a ratio of two integers, either positive or negative. Rational numbers can be added, subtracted and multiplied by one another. The result of these operations is a rational number. One must define equality for rational numbers. For example, $\frac{17}{34} = \frac{1}{2}$; or $\frac{34}{17} = 2$. Rational numbers can be represented as decimal fractions, either as a finite decimal or as an infinite repeating decimal. For example, 0.25, $\frac{1}{3} = 0.333\ldots$, $\frac{2}{3} = 0.666\ldots$, $\frac{1}{7} = 0.142857142857142857\ldots$

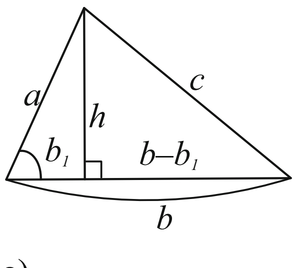
However, the number $\sqrt{2}$, which is the length of the hypotenuse of a right triangle with legs equal to 1, is not a rational number and cannot be represented as a rational fraction.

10.2 The use of the Pythagorean theorem in arbitrary triangles

The Pythagorean relationship $c^2 = a^2 + b^2$ is a very important characteristic of a right triangle with the legs a and b and hypotenuse c. Moreover, we can also use this relationship to obtain some other interesting relationships (see Propositions 12 and 13) in an arbitrary triangle. Consider first an acute triangle, and draw an altitude in it (see Fig. 4.53a).

Proposition 12. In an acute triangle with sides a, b, and c, let h be the altitude to the side b. If b_1 is the segment of side b included between the side a and the altitude,[22] then the following relation holds:

$$c^2 = a^2 + b^2 - 2bb_1.$$



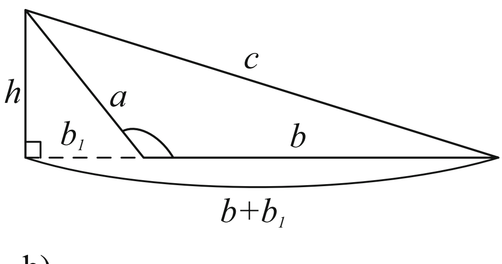


Fig. 4.53

[22] The segment b_1 is sometimes called the *projection* of side a onto side b.

Proof. Consider an acute triangle with sides a, b, and c. Draw altitude h to side b (see Fig. 4.53a). This altitude divides side b into two segments b_1 and $b - b_1$. We want to express c in terms of a, b, and b_1.

Applying the Pythagorean equality to each of the two right triangles in Fig. 4.53a, we obtain

$$\begin{aligned} c^2 &= h^2 + (b - b_1)^2 \quad \text{and} \\ a^2 &= h^2 + b_1^2. \end{aligned}$$

If we subtract the second equality from the first one, we obtain

$$\begin{aligned} c^2 - a^2 &= (b - b_1)^2 - (b_1)^2, \quad \text{or} \\ c^2 - a^2 &= b^2 - 2bb_1 + (b_1)^2 - (b_1)^2, \quad \text{or} \\ c^2 - a^2 &= b^2 - 2bb_1. \end{aligned}$$

Thus,

$$c^2 = a^2 + b^2 - 2bb_1. \qquad \square$$

We have considered an acute triangle. For an obtuse triangle the following proposition holds.

Proposition 13. In an obtuse triangle with sides a, b, and c, where c is the largest side, draw altitude h to side b. If b_1 is the segment on the extension of side b included between side a and the altitude,[23] then the following relation holds:

$$c^2 = a^2 + b^2 + 2bb_1.$$

Proof. In an obtuse triangle with sides a, b, and c, draw altitude h to side b (see Fig. 4.53b). The altitude intersects the extension of side b.

Applying the Pythagorean equalities, we obtain

$$\begin{aligned} a^2 &= h^2 + b_1^2 \quad \text{and} \\ c^2 &= h^2 + (b + b_1)^2. \end{aligned}$$

From these equalities we obtain the following equality for c^2:

$$c^2 = h^2 + b^2 + 2bb_1 + b_1^2 = a^2 + b^2 + 2bb_1,$$

i.e.,

$$c^2 = a^2 + b^2 + 2bb_1. \qquad \square$$

[23] The segment b_1 in this case is also the *projection* of side a onto side b.

Summarizing Theorem 8 and Propositions 12 and 13, we can say that in a triangle, one of three relations among its sides holds. In a right triangle with legs a, b and hypotenuse c we have the equality

$$c^2 = a^2 + b^2;$$

in an acute triangle with sides a, b, c we have an inequality,

$$c^2 < a^2 + b^2;$$

in an obtuse triangle with sides a, b, c, where the side c lies opposite the obtuse angle, we have the inequality

$$c^2 > a^2 + b^2.$$

PROBLEM 23. Three triangles are given with sides:
(1) 3, 4, 4.5 (2) 3, 4, 5 (3) 3, 4, 5.5
Which of these triangles is acute, which is a right triangle, and which is an obtuse triangle?

A statement about a parallelogram follows from the above summary.

Proposition 14. In a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of all its sides.

Proof. Consider a parallelogram $ABCD$ with sides a and b and diagonals c_1 and c_2 (Fig. 4.54a). From the points B and C let us draw the perpendiculars to side b as in Fig. 4.54b. Let b_1 denote the segment on side b included between side a and the altitude (see Fig. 4.54b).

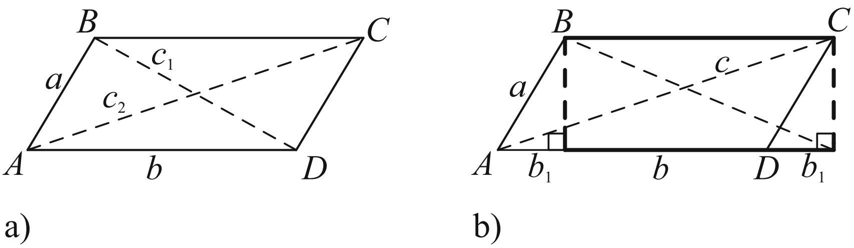


Fig. 4.54

Then in acute triangle ABD we have

$$c_1^2 = a^2 + b^2 - 2bb_1.$$

In obtuse triangle ACD we have

$$c_2^2 = a^2 + b^2 + 2bb_1.$$

Therefore, $c_1^2 + c_2^2 = 2a^2 + 2b^2$. □

Note that if the parallelogram is a rectangle, we obtain $2c^2 = 2a^2 + 2b^2$ or $c^2 = a^2 + b^2$, i.e., the Pythagorean theorem.[24]

Exercise 11. Consider a triangle with all acute angles, sides a, b, and c and a median m to side c (see Fig. 4.55a). Find a relation between the sides a, b, c and the median m.

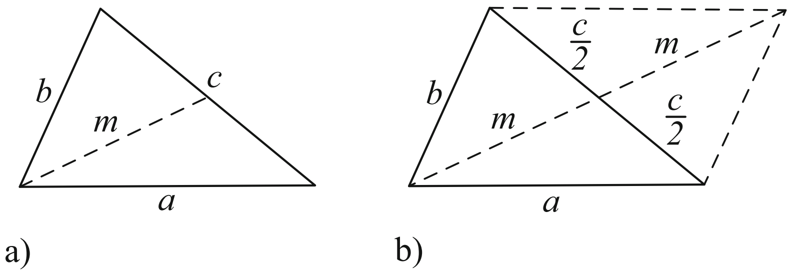


Fig. 4.55

Solution. Let us complete the triangle to form a parallelogram as in Fig. 4.55b. Using Proposition 14 we obtain $2a^2 + 2b^2 = c^2 + 4m^2$. So $m^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2$.

PROBLEM 24. Do the exercise above for a triangle with an obtuse angle in the following two cases:

(a) Draw the median from the obtuse angle.

(b) Draw the median from an acute angle.

PROBLEM 25. Prove that in an equilateral triangle with side a, the altitude is equal to $\frac{\sqrt{3}}{2}a$.

[24] Note that this is not a proof of the Pythagorean theorem since we have used the Pythagorean theorem to derive this fact.

10.3 Heron's formula for the area of a triangle

There is a formula for the area of a triangle proved by the ancient mathematician Heron.[25] This formula expresses the area in terms of the sides of the triangle.

Consider a triangle ABC with sides a, b, and c (see Fig. 4.56a). Let us denote by s the *semiperimeter*[26] of this triangle, i.e., $s = \frac{1}{2}(a+b+c)$.

Then Heron's formula for the area of the triangle ABC is:

$$S_{\triangle} = \sqrt{s(s-a)(s-b)(s-c)}.$$

We will omit the proof of this formula because it is quite long, but we will show how one could have guessed it.

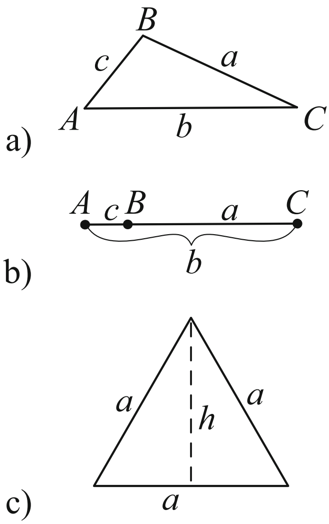


Fig. 4.56

We know that the formula has to contain a, b, and c, which are measured in units of length—for example, in centimeters. The area of a triangle must be expressed in cm^2. This means that the expression for the area should be proportional to the square of a side.

We also know that the area of a triangle must be a non-negative number. From Proposition 2 we know that in a triangle the sum of two sides is larger

[25]Heron, who lived in Alexandria supposedly around 150 BC, was well known for his works in geometry and mechanics.

[26]The *semiperimeter* of a polygon is half its perimeter. The *perimeter* of a polygon is the sum of the lengths of all its sides.

than the third side; that is, $a+b-c>0$, $b+c-a>0$, $c+a-b>0$. If, for example, $c+a=b$, then the triangle "degenerates" into a segment (as in Fig. 4.56b). In this case the area is equal to 0. This means that the formula for the area has to contain the factor $(a+b-c)$.

Since the expression for the area should not discriminate against any side of the triangle, it is natural to assume that it has to contain all three factors:

$$(a+b-c)(b+c-a)(c+a-b).$$

Unfortunately, this cannot be the final answer because this expression is proportional not to cm^2, but to cm^3. In order to obtain cm^2, we can assume that the expression for the area has another factor proportional to a side; this will give us cm^4. Then by taking the square root $\sqrt{\mathrm{cm}^4}$, we will obtain the correct measure for area, i.e., cm^2.

Let us look for such an expression proportional to a side which will not discriminate against any of the sides a, b, or c. More precisely, this means that this expression will not change if sides a, b, c change places in this expression.[27]

There is only one linear expression in a, b, and c that is symmetric in this way: it is $k(a+b+c)$, where k is a number[28] that does not depend on a, b, c.

Thus, our hypothetical expression for the area of a triangle looks like

$$S_{\triangle} = \sqrt{k(a+b+c)(a+b-c)(b+c-a)(c+a-b)}.$$

We now need only to find the coefficient k. In order to do this, let us consider a particular triangle for which we can calculate its area without this formula. For example, consider an equilateral triangle with side a (see Fig. 4.56c). Then $b=a$ and $c=a$.

It is easy to check[29] that its area is equal to

$$S_{\triangle} = \frac{a}{2}h = \frac{a}{2}\cdot\frac{\sqrt{3}a}{2} = \frac{\sqrt{3}a^2}{4}.$$

On the other hand, according to our hypothetical formula we have

$$S_{\triangle} = \sqrt{k(a+a+a)(a+a-a)(a+a-a)(a+a-a)} = \sqrt{3k}a^2.$$

From comparing the two expressions for the area, we can find k. We have $\frac{\sqrt{3}a^2}{4} = \sqrt{3k}a^2$, or $\sqrt{k}=\frac{1}{4}$, or $k=\left(\frac{1}{4}\right)^2=\frac{1}{16}$.

---

[27] An expression which does not change if the variables in it change their places is called *symmetric* with respect to these variables.

[28] In an expression, a letter which substitutes for a number is often called a *coefficient*.

[29] Apply Proposition 11, Theorem 7, and the Pythagorean theorem.

Finally, the expression we are guessing takes the form

$$S_\triangle = \sqrt{\frac{1}{16}(a+b+c)(a+b-c)(b+c-a)(c+a-b)},$$

or

$$S_\triangle = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}.$$

PROBLEM 26. Prove that the expression $S_\triangle$ obtained above is the same as Heron's formula. That is, prove that

$$\sqrt{s(s-a)(s-b)(s-c)} \\ = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)},$$

where $s = \frac{a+b+c}{2}$.

11 Relations between lines and points

11.1 Perpendicular from a point to a line

As we know, a line is perpendicular to another line when the angle between these lines measures 90°. A *perpendicular from a point A onto a line a* is a segment connecting point A with a point on the line chosen in such a way that the angle between this segment and line a is equal to 90°.

Exercise 12. Given a point A and a straight line a, drop a perpendicular from point A onto line a.

Solution. Consider a point A and a line a (see Fig. 4.57a).

Let us draw a circle with center A big enough that it intersects line a (see Fig. 4.57b). Denote the part of line a inside the circle by BC. We find the midpoint O of chord BC and connect points A and O (see Fig. 4.57c). Let us prove that the segment AO is a perpendicular to the line a.

Note that triangle ABC (see Fig. 4.57d) is isosceles since $AB = AC$ since they are two radii of the circle. Segment AO is the median of $\triangle ABC$, and from Theorem 5 is also its altitude. Therefore, segment AO is perpendicular to BC. □

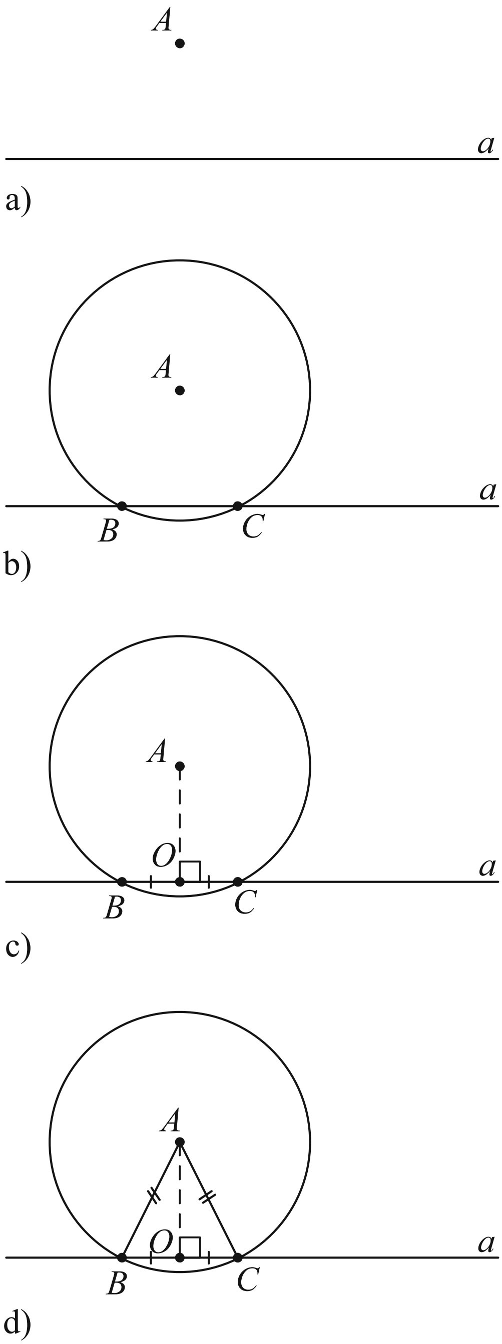


Fig. 4.57

11.2 Distance from a point to a line

Given a point A and a line a, the length of the segment AO perpendicular to line a is called the *distance from the point A to the line a*.

Proposition 15. The distance from a point A to a line a is the shortest length of any segment connecting the point A with a point on the line a.

Proof. Consider a point A and a line a. Let AO be the perpendicular from point A onto line a. Let O' be a point on line a different from point O (see Fig. 4.58). From Proposition 6, $AO < AO'$. □

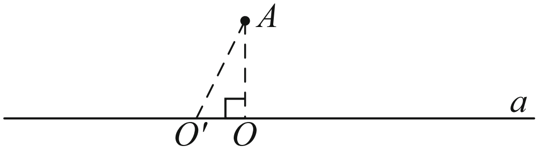


Fig. 4.58

Exercise 13. Suppose there is a straight road passing close to a University building. Where should one build a bus stop on this road so that the distance from the building to the bus stop will be minimal?

Solution. One must draw a perpendicular from the building to the road. The bus stop should be where the perpendicular intersects the road.

Now consider two parallel lines a and b and two points A and A' on line a (see Fig. 4.59).

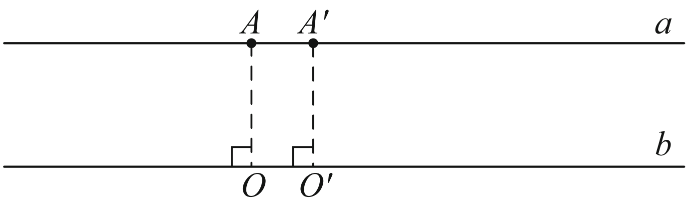


Fig. 4.59

Proposition 16. For two parallel lines a and b, the distance from any point lying on line a to line b is the same.

Proof. Consider any two points A and A' on the line a. Let us draw a perpendicular AO from point A onto line b and a perpendicular $A'O'$ from point A' onto line b (see Fig. 4.59). Figure $AA'O'O$ is a parallelogram (Theorem 2, Remark 3—and it is, in fact, a rectangle because by Theorem 2 all interior angles of this quadrilateral are equal to 90°). Therefore, $AO = A'O'$. □

11.3 The locus of points lying at equal distance from two given points

Exercise 14. Consider two points A and B. Find all the points lying at an equal distance from point A and from point B, i.e., find all points X such that $AX = BX$.

This exercise can be interpreted as follows. Suppose there are two administrative buildings on a campus, and we need to make a path for pedestrians such that the distance from anywhere on this path to each building will

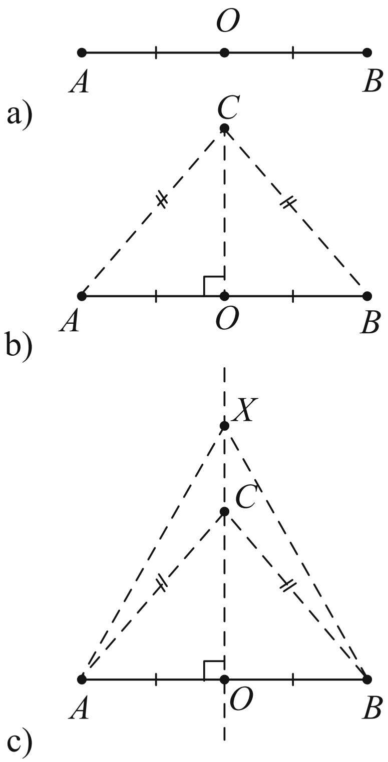


Fig. 4.60

be the same (so that neither of the buildings will have an advantage). Draw such a path.

Solution. Consider two points A and B. We are looking for points X such $AX = BX$. Note that we already know one such point—the midpoint of segment AB. Let us denote it by O (see Fig. 4.60a).

Let C be a point such that $AC = CB$ (see Fig. 4.60b). Then triangle ACB is isosceles. Segment OC is perpendicular to segment AB. Indeed, since OC is a median of $\triangle ABC$ it is also its altitude (due to Theorem 5). Thus, any point C such that $AC = BC$ lies on a perpendicular to the midpoint of AB.

The converse statement is also true, i.e., any point X lying on the perpendicular OC is such that $AX = BX$. Indeed, if X lies on the perpendicular OC (see Fig. 4.60c) then $\triangle AOX = \triangle BOX$ since OX is a common side and $AO = OB$. Therefore, $AX = BX$.

We have found that all points that lie at an equal distance from points A and B must lie on the perpendicular at the midpoint of AB.

Mathematicians often use the word *locus* to describe the set of all points which have a certain property or satisfy a certain condition—a set which, conversely, is such that no other point (not in this set) has this property.

Thus we have proved the following proposition.

Proposition 17. The locus of points lying at equal distances from two given points A and B is the perpendicular at the midpoint of segment AB.

A perpendicular at the midpoint of a segment is called its *perpendicular bisector*.

Exercise 15. Construct a triangle with sides a, b, and c and angle α adjacent to side b if the following elements are given: side b, angle α, and a segment s such that $s = a + c$ (see Fig. 4.61a).

Solution. On one ray of angle α, let us mark off segment b (or AC); on the other ray, let us mark off segment s (or AD) (see Fig. 4.61b). We need to find a point B on segment AD such that in the triangle ABC we have $AB + BC = s$.

For this let us connect points C and D, then draw the perpendicular bisector to CD to where it intersects the side AD at some point B (see Fig. 4.61c). It is easy to check that $BD = BC$. Therefore, $AB + BC = AB + BD = s$. Thus, triangle ABC satisfies the conditions of the exercise.

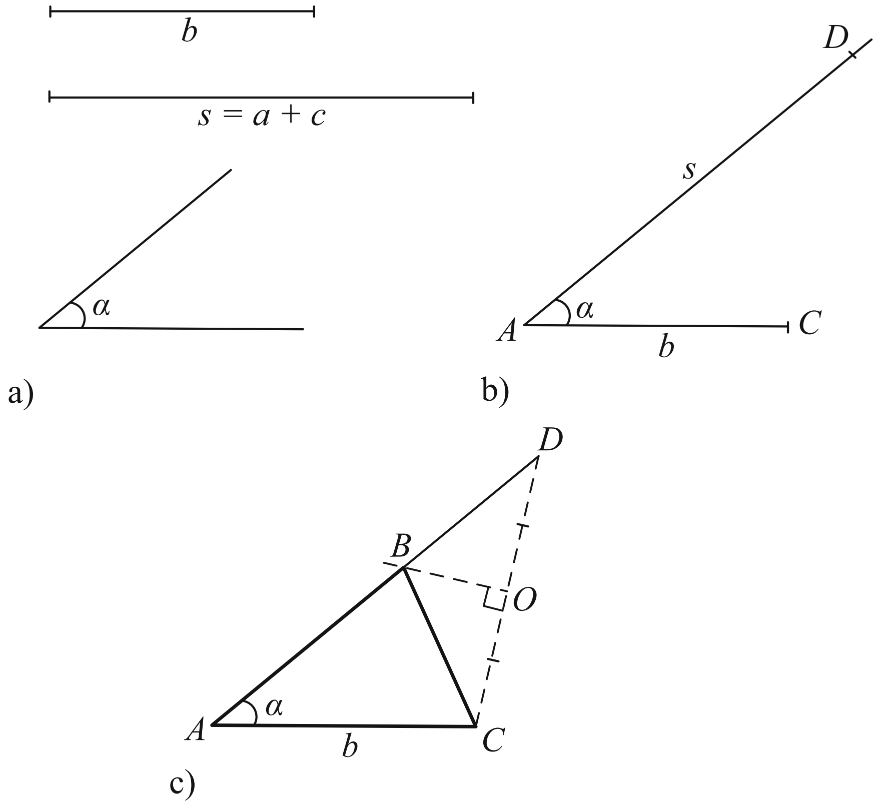


Fig. 4.61

11.4 The locus of points lying at equal distance from two given lines. Two definitions of an angle bisector

We have considered the locus of points lying at an equal distance from two given points. Now let us find all points that lie at an equal distance from two given lines.

Exercise 16. Consider two intersecting lines a and b. Find all the points lying at an equal distance from line a and from line b.

This exercise can be interpreted, for example, as follows. Suppose there are two intersecting straight roads and we need to build several buildings such that the distance from any of them to each road will be the same (so that neither of the roads will have an advantage). Find possible locations for these buildings.

Solution. We are looking for points lying at equal distance from both lines a and b. We already know one such point: it is the intersection point between a and b. Let us denote it by O (see Fig. 4.62a).

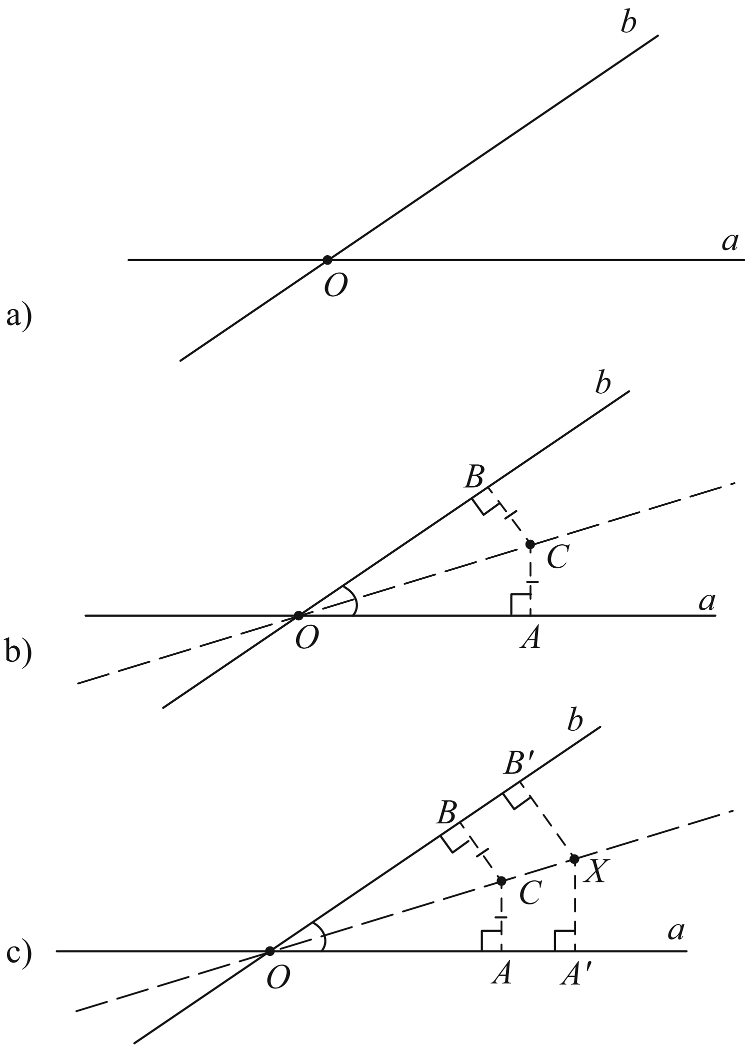


Fig. 4.62

Let C be a point such that $AC = CB$, where AC is perpendicular to a and BC is perpendicular to b (see Fig. 4.62b). Then $\triangle AOC = \triangle BOC$ (since they have a common hypotenuse and $BC = CA$). Therefore, $\angle AOC = \angle BOC$. This means that point C lies on the bisector of angle AOB. Thus, any point lying within angle AOB at equal distance from lines a and b lies on the bisector of the angle formed by these lines.

The converse statement is also true, i.e., any point X lying on the angle bisector OC is such that $XA' = XB'$, where XA' and XB' are perpendiculars from X to the two given lines. Indeed, let point X lie on angle bisector OC (see Fig. 4.62c). Then $\triangle XOA' = \triangle XOB'$ (since they have a common hypotenuse and equal angles adjacent to it). Therefore, $XA' = XB'$.

We have found that all points that lie at an equal distance from lines a and b must lie on the bisector of the angle between a and b.

Thus we have proved the following proposition.

Proposition 18. The locus of points lying at equal distance from two given lines a and b is the bisector of the angle formed by these lines.

This proposition gives us a new definition of the bisector of an angle.

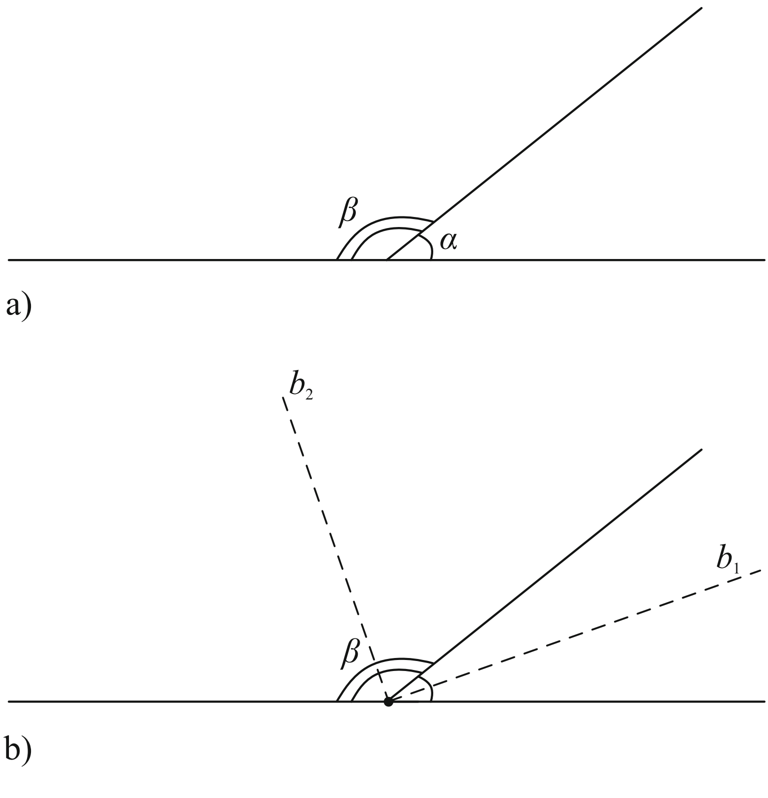


Fig. 4.63

One can prove (which we will not do here) that both definitions are equivalent. Thus, either of them can be used:

(1) The bisector of an angle is a ray passing through the vertex of the angle and dividing it into two equal angles.

(2) The bisector of an angle is the locus of points lying at equal distances from the two lines forming the angle.

PROBLEM 27. Consider two supplementary angles α and β placed as in Fig. 4.63a.

Draw the bisector of each of these angles. Prove that these angle bisectors b_1 and b_2 (see Fig. 4.63b) are perpendicular to each other.

PROBLEM 28. Suppose lines a and b are parallel. Find the locus of points lying at equal distance from each of these lines.

Exercise 17. Two rays emanating from point O form an angle (Fig. 4.64a). Draw the bisector of the angle formed by these rays.

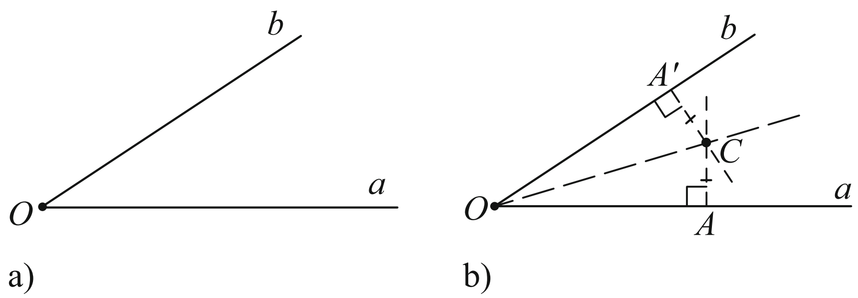


Fig. 4.64

Solution. We choose an arbitrary segment and mark it on each ray from point O, so that $OA = OA'$ (see Fig. 4.64b). Then we draw a perpendicular to line a from point A and a perpendicular to line b from point A'. These perpendiculars intersect at some point C. The line passing through points O and C is the bisector of the angle formed by a and b. Indeed, right triangles AOC and $A'OC$ are equal by definition (2) of congruent right triangles (i.e., hypotenuse and one leg are equal). Therefore, $\angle AOC = \angle A'OC$.

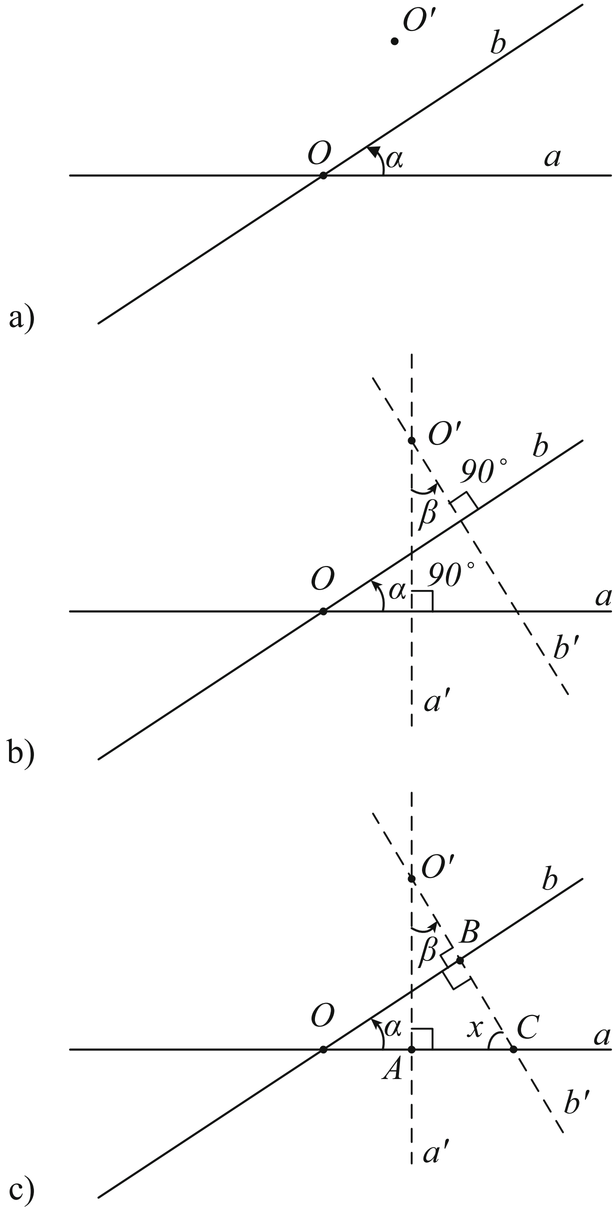


Fig. 4.65

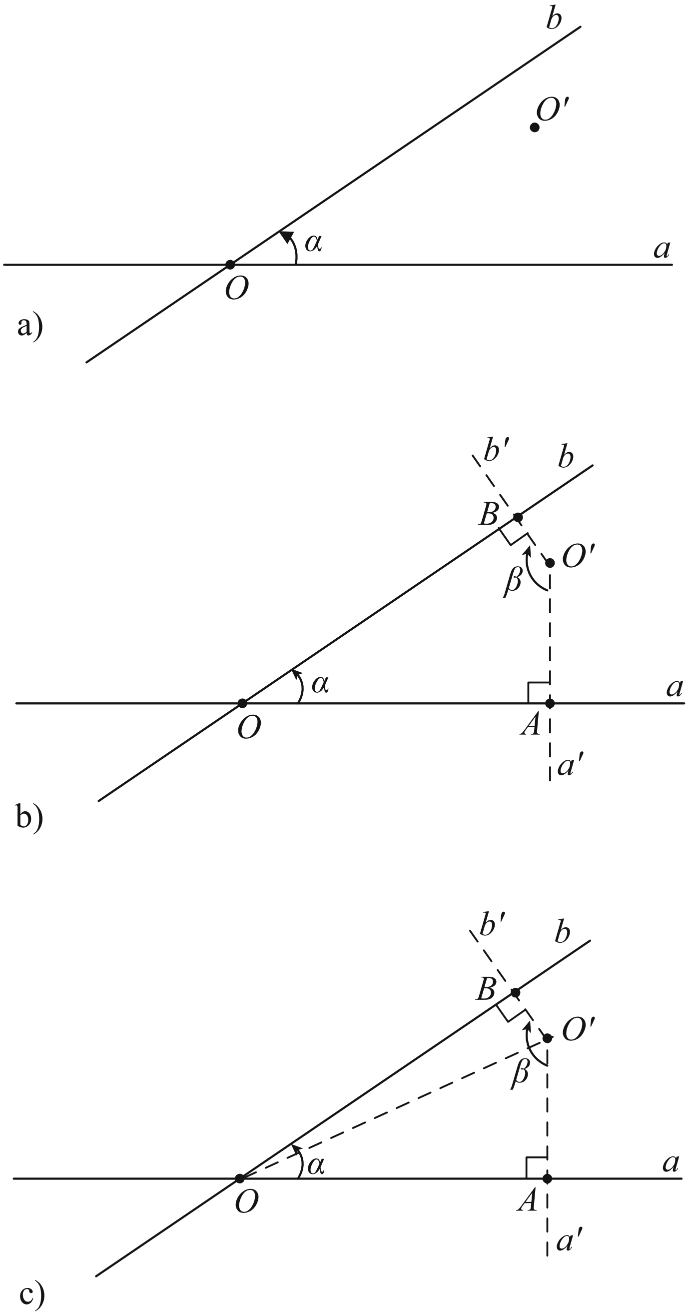


Fig. 4.66

11.5 Angles with respectively perpendicular sides

Consider an angle α between[30] two lines a and b and a point O'. The point O' can lie inside this angle or outside it (see Figs. 4.65a and 4.66a).

Through point O' let us draw lines a' and b' perpendicular to the sides of angle α (see Figs. 4.65b, 4.66b). Let β be the angle between lines a' and b'. The angles α and β are called *angles with respectively perpendicular sides*.

Proposition 19. Two angles with respectively perpendicular sides are either equal or supplementary.

Proof. Let us draw two angles α and β with respectively perpendicular sides. First, consider the case where the vertex O' of angle β lies outside angle α (as in Fig. 4.65a, b).

Let us mark points A, B, and C and angle x as in Fig. 4.65c. Triangles OBC and $O'AC$ are right triangles. Then $\angle\alpha + \angle x = 90^\circ$ and $\angle\beta + \angle x = 90^\circ$. Therefore, $\angle\alpha = \angle\beta$.

Now consider the case where the vertex O' of angle β lies inside angle α (as in Fig. 4.66a, b). Let us connect points O and O' and mark points A and B as in Fig. 4.66c. In quadrilateral $AOBO'$ there are two right angles. Since the sum of the interior angles of a quadrilateral is equal to 360° (see Section 13), we have $\angle\alpha + \angle\beta = 360^\circ - 180^\circ = 180^\circ$. This finishes the proof. □

PROBLEM 29. Consider a right triangle ABC and draw the altitude CD from the right angle (see Fig. 4.67).

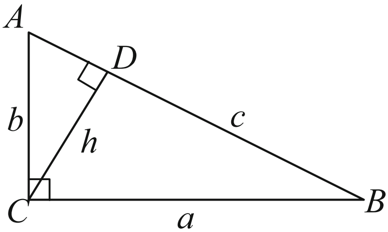


Fig. 4.67

Draw all the triangles from Fig. 4.67 separately (there are three of them). Indicate all angles that are equal to each other, and explain why they are equal.

Compare this problem with Problem 45.

[30] In Section 3 of this chapter we mentioned that mathematicians agree to measure an angle between two lines in the counterclockwise direction from the first line to the second.

12 Special lines and special points in a triangle

We have already mentioned in Section 8 that there are three special lines in a triangle: median, altitude, and angle bisector. There is one more special line in a triangle: the perpendicular bisector. In this section we collect the definitions and facts about these special lines and about some special points of a triangle. Some of the facts have already been proved, some of them we prove here, and some of them will be proved later in other sections.

12.1 The median

Let us repeat the facts we already know about the medians of a triangle.

(1) A median is a line passing through a vertex of a triangle and the midpoint of the opposite side (see Chapter II, Section 6.2).

(2) The three medians of a triangle intersect at a single point (see Ch. II, Section 6.2. See also Fig. 4.73a).

(3) Any two medians of a triangle are divided by their point of intersection in the ratio 2 : 1 (see Ch. II, Section 6.2).

(4) A median divides a triangle into two triangles with equal area (see Ch. III, Section 3).

PROBLEM 30. Prove that the length of a median to the hypotenuse of a right triangle is equal to one-half of the length of the hypotenuse.

12.2 The angle bisector

Below are some facts about the angle bisectors of a triangle.

(1) The bisector of an angle of a triangle is the line passing through a vertex of the triangle and dividing the angle at that vertex into two equal angles (see Ch. IV, Section 3.1).

(2) The bisector of an angle is a locus of points lying at an equal distance from the two lines forming this angle (see Ch. IV, Section 11.4).

(3) The three angle bisectors of a triangle intersect at a single point (see this section below).

(4) The intersection point of the three angle bisectors of a triangle is the center of its inscribed circle (see Ch. IV, Section 20.1).

(5) In a triangle with sides a and b, the bisector of the angle between sides a and b divides the third side of the triangle into segments proportional to the sides a and b (see this section below).

Let us prove the following theorem.

Theorem 9. The three angle bisectors of a triangle intersect at a single point.

Proof. Consider a triangle ABC and draw the bisectors of angles BAC and ABC; see Fig. 4.68a. Denote their point of intersection by O.

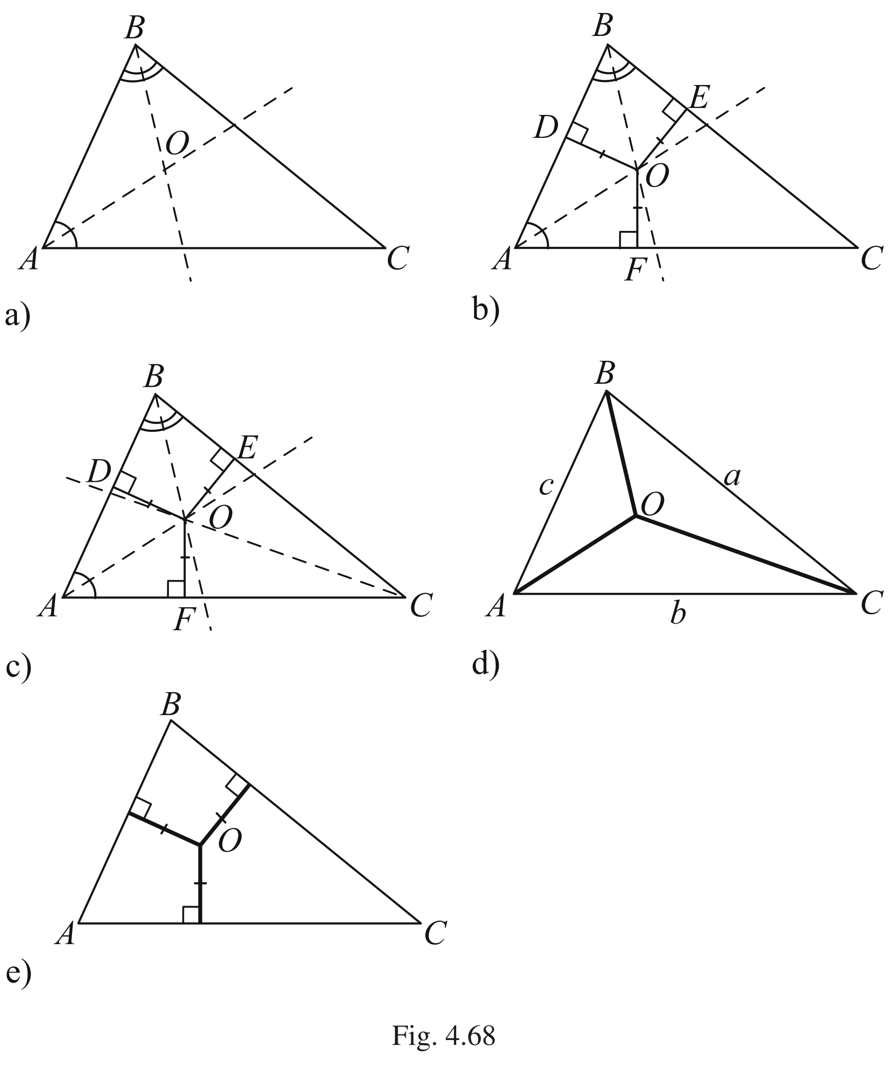


Fig. 4.68

As we know, point O lies at equal distances from all the sides of these angles. This means that the perpendiculars OD, OE, and OF are equal to each other (see Fig. 4.68b). From Proposition 18, this means that point O lies on the bisector of angle BCA (see Fig. 4.68c). Therefore, all three angle bisectors of triangle ABC intersect at a single point (see also Fig. 4.73b). □

Corollary 4. The point of intersection of the angle bisectors in a triangle is equidistant from each side of the triangle.

From the statements above we can also obtain an additional formula for the area of a triangle.

Proposition 20. The area S of a triangle with sides a, b, and c is equal to

$$S = \frac{1}{2}r(a+b+c),$$

where r is the radius of the inscribed circle.[31] This formula means that the area of a triangle with sides a, b, and c is equal to half of the product of its *perimeter*[32] and the radius of the inscribed circle.

Proof. In a triangle with sides a, b, and c, let us connect the intersection point of the three angle bisectors (point O) to each vertex of the triangle (see Fig. 4.68d). We obtain three triangles AOB, BOC, and AOC. Since point O is equidistant from the sides (Corollary 4), it is the center of the inscribed circle. The altitudes from this point onto the sides (see Fig. 4.68e) are radii of this circle, and thus all have the same length r. If we denote area by S, we have $S_{\triangle ABC} = S_{\triangle AOB} + S_{\triangle BOC} + S_{\triangle AOC} = \frac{1}{2}rc + \frac{1}{2}ra + \frac{1}{2}rb = \frac{1}{2}r(a+b+c)$. □

Exercise 18. Suppose there are three intersecting straight roads (see Fig. 4.69a). Find a place for a gas station so that the distance from it to each road will be the same.

Solution. A solution follows from Corollary 4. We draw the bisectors of the angles formed by these three straight lines. Point O (see Fig. 4.69b), where they intersect, lies at equal distance from each of the three given lines.

PROBLEM 31. Suppose that two of the roads in Exercise 18 are parallel and the third one intersects them. Find the place for a gas station in this case.

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[31] See the definition of an inscribed circle in Section 20.1.

[32] The perimeter of a triangle is the sum of the sides of this triangle.

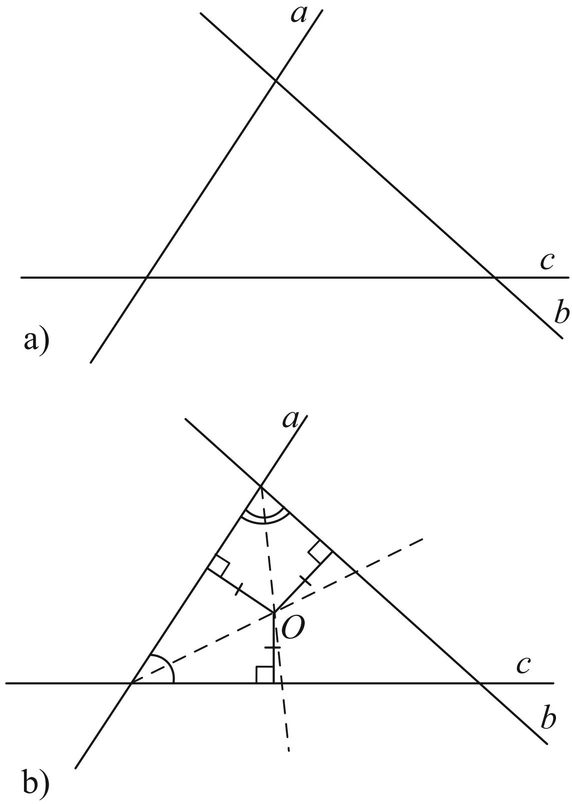


Fig. 4.69

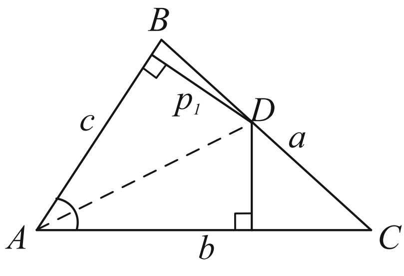
Proposition 21. In a triangle with sides a, b, and c, the angle bisector of the angle between sides b and c divides the side a of the triangle into segments a_1 and a_2 proportional to the sides b and c. That is,

$$\frac{b}{c} = \frac{a_1}{a_2},$$

where a_1 is the segment adjacent to side b, and a_2 is the segment adjacent to side c.

Proof. Consider a triangle ABC with sides a, b, and c. Draw the angle bisector AD in it. From point D we draw the perpendicular p_1 to side AB and the perpendicular p_2 to side AC (see Fig. 4.70a). We have $p_1 = p_2$ (see Section 11.4).

Let us compare the areas $S_{\triangle ABD}$ and $S_{\triangle ADC}$. We can find the areas in two different ways. On one hand, we have $S_{\triangle ABD} = \frac{1}{2}cp_1$ and $S_{\triangle ADC} = \frac{1}{2}bp_2$.



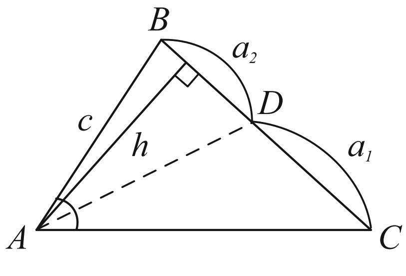


Fig. 4.70

Since $p_1 = p_2$ we have

$$\frac{S_{\triangle ABD}}{S_{\triangle ADC}} = \frac{c}{b}.$$

On the other hand, we can draw the altitude h from vertex A (see Fig. 4.70b). Then $S_{\triangle ABD} = \frac{1}{2}ha_2$ and $S_{\triangle ADC} = \frac{1}{2}ha_1$ (where a_1 and a_2 are the parts into which side a is divided by the angle bisector). We have

$$\frac{S_{\triangle ABD}}{S_{\triangle ADC}} = \frac{a_2}{a_1}.$$

Therefore,

$$\frac{c}{b} = \frac{a_2}{a_1}. \qquad \square$$

12.3 The perpendicular bisector

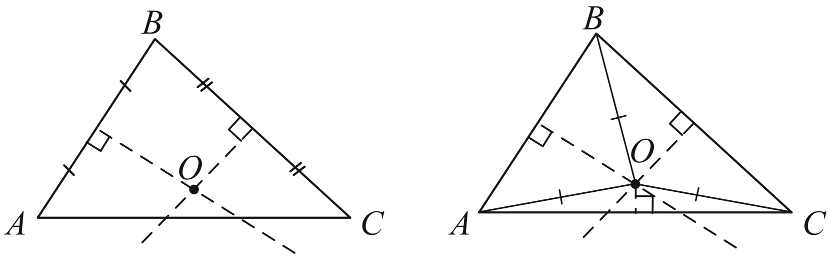
In Section 11.3 we introduced a special line that we called the perpendicular bisector.

(1) A perpendicular bisector in a triangle is a perpendicular at the midpoint of a side of this triangle.

(2) The three perpendicular bisectors of a triangle intersect at a single point that is the center of its circumscribed circle.

Theorem 10. The three perpendicular bisectors of a triangle intersect at a single point.

Proof. Consider a triangle ABC. Through the midpoint of AB draw the perpendicular to AB and through the midpoint of BC draw the perpendicular to BC (see Fig. 4.71). Let O be their point of intersection.



a) b)

Fig. 4.71

From Section 11.3, we have $AO = BO$ and also $BO = OC$. Therefore $AO = BO = OC$; i.e., point O is equidistant from each vertex of triangle ABC. This means that the perpendicular bisector of side AC also passes through point O (see Fig. 4.71b). □

12.4 The altitudes

Below are some facts about altitudes of a triangle.

(1) An altitude of a triangle is a line that passes through a vertex of this triangle and is perpendicular to the opposite side of this triangle (see Ch. IV, Section 8).

(2) The three altitudes of a triangle intersect at a single point (see below) called its *orthocenter.*

Theorem 11. The three altitudes of a triangle intersect at a single point.

Proof. We will use Theorem 10. Consider a triangle ABC. We will construct another triangle whose altitudes are the perpendicular bisectors of triangle ABC. To do this, through each vertex of triangle ABC, draw a line parallel to the opposite side of this triangle, and extend these lines (denoted a, b, c) until they intersect (see Fig. 4.72a). We obtain a new triangle KLM.

Let us draw the altitudes of triangle ABC and prove that each altitude in triangle ABC is also a perpendicular bisector of triangle KLM.

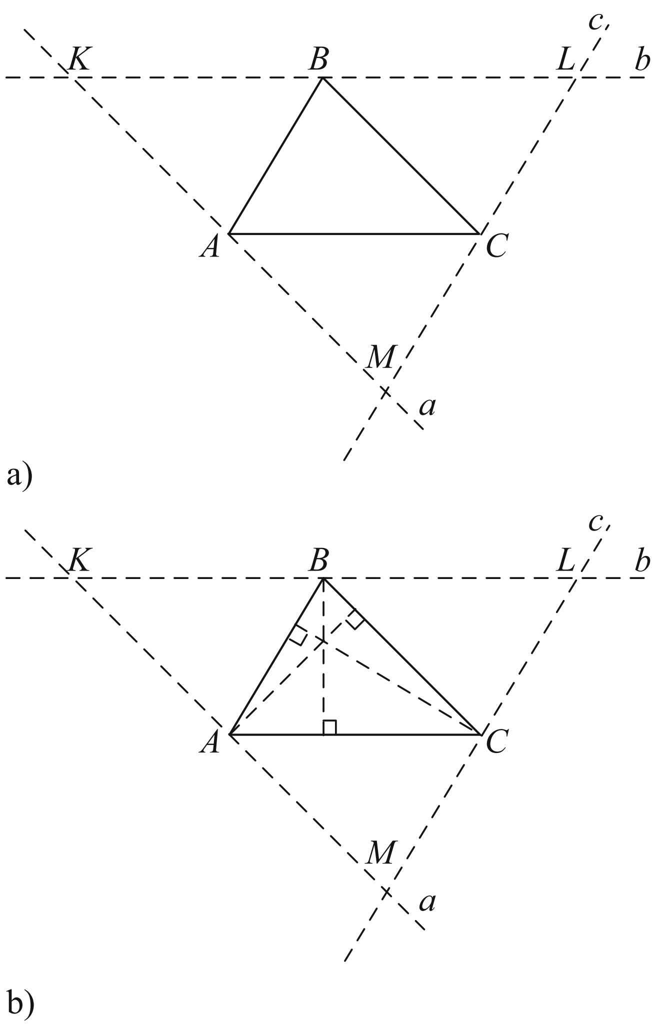


Fig. 4.72

Indeed, from our construction we have $BK \parallel AC$, $AB \parallel MC$, $BC \parallel AM$. So figures $AKBC$ and $ABCM$ are parallelograms, and therefore $\triangle ABK = \triangle ACM$ and $AK = AM$. Thus, A is the midpoint of the side KM. Similarly we can prove that $KB = BL$ and $LC = CM$.

As we know, the three perpendicular bisectors of a triangle ($\triangle KLM$) intersect at a single point. Since they coincide with the altitudes of the triangle ABC, this means that the three altitudes of $\triangle ABC$ also intersect at a single point (see Fig. 4.73c). $\square$

12.5 Special lines of a triangle at a glance

The figure below shows a triangle with its three medians (see Fig. 4.73a), its three angle bisectors (see Fig. 4.73b), its three perpendicular bisectors (see Fig. 4.73c), and its three altitudes (see Fig. 4.73d).

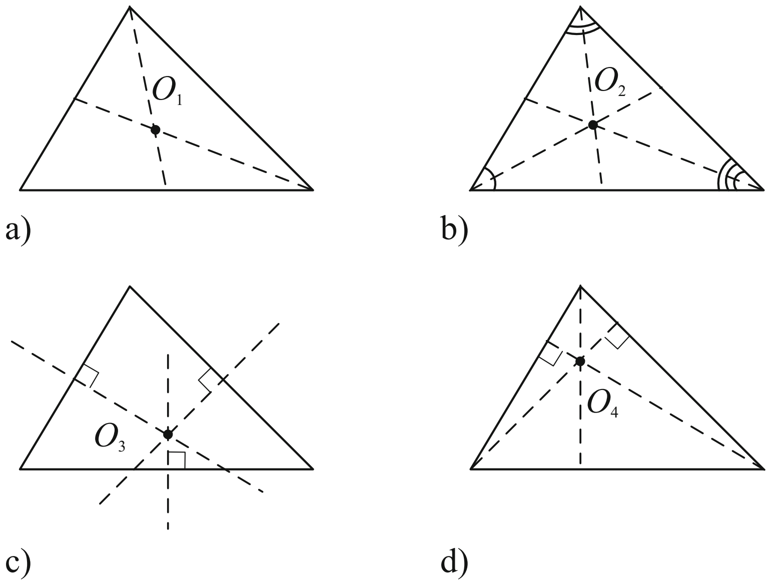


Fig. 4.73

PROBLEM 32. Fig. 4.74 shows an obtuse triangle.

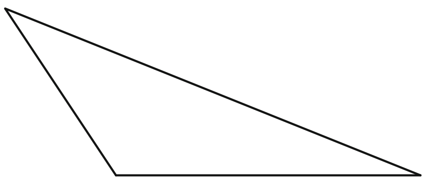


Fig. 4.74

On separate pictures of this triangle draw all its medians, all its angle bisectors, all its altitudes, and all its perpendicular bisectors.

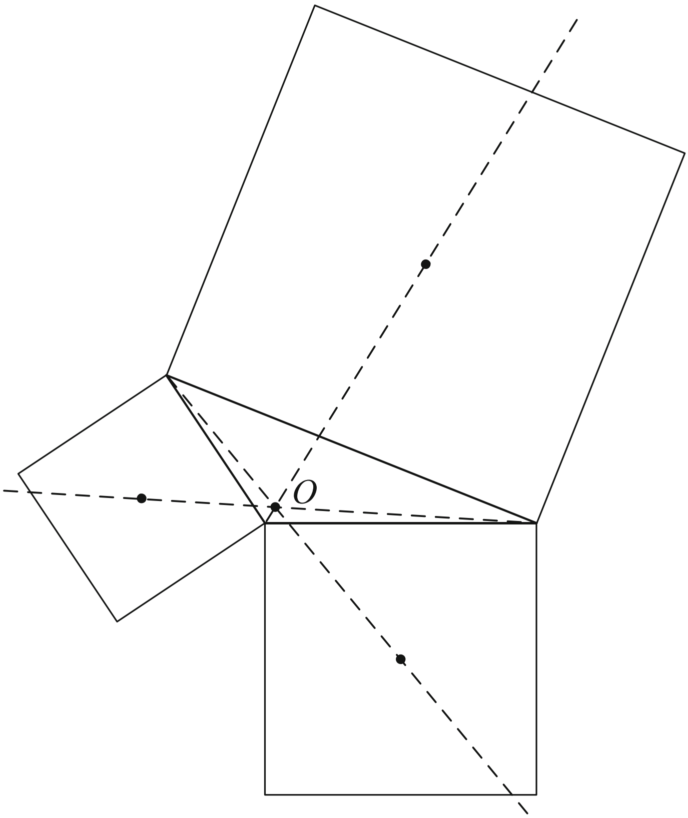
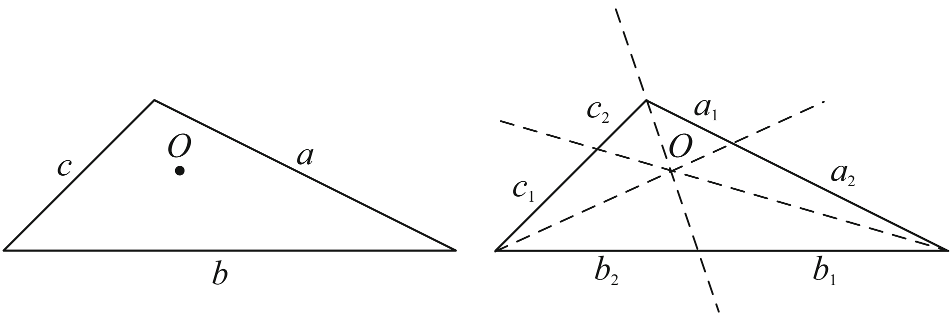


Fig. 4.75



a) b)

Fig. 4.76

12.6 Special points in a triangle

Special lines also create special points in a triangle: the point of intersection of the medians, the point of intersection of the angle bisectors, the point of intersection of the perpendicular bisectors, and the point of intersection of the altitudes.

Here is a brief summary of the special points in a triangle.

- The three medians of a triangle intersect at a single point. This point divides each median in the ratio 1 : 2 counting from the side to which the median is drawn. The point of intersection of the medians of a triangle is called the *centroid* and is the center of mass of this triangle. (This will not be defined in the book.)
- The three angle bisectors of a triangle intersect at a single point. This point is the center of the inscribed circle.
- The three perpendicular bisectors of a triangle intersect at a single point. This point is the center of the circumscribed circle.
- The three altitudes of a triangle intersect at a single point, called its orthocenter.

In an equilateral triangle these four points coincide.

PROBLEM 33. Draw an equilateral triangle and mark the four special points in it.

In a triangle there are many other interesting points and relations. We have no space in the book to consider them in detail.

We want to mention one more special point in a triangle and only show how to find it. If we construct squares on the sides of a triangle, then the lines connecting each vertex of the triangle with the center of the opposite square intersect at a single point O (see Fig. 4.75).

There is also an interesting relation in a triangle with an arbitrary point O in it. Consider a triangle with sides a, b, and c and a point O inside (see Fig. 4.76a). Let us draw lines through point O and each of the vertices of the triangle. Denote the segments on the sides as in Fig. 4.76b.

The following relation holds:

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1.$$

We will not prove this here. Let us just check this relation for medians and for angle bisectors.

If a point O is the point of intersection of the medians, we have $a_1 = a_2$, $b_1 = b_2$, and $c_1 = c_2$. Then we have $1 \cdot 1 \cdot 1 = 1$. which is true.

If a point O is the point of intersection of the angle bisectors, we have (from Proposition 21)

$$\frac{a_1}{a_2} = \frac{c}{b}, \quad \frac{b_1}{b_2} = \frac{a}{c}, \quad \frac{c_1}{c_2} = \frac{b}{a}.$$

Therefore,

$$\frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = \frac{c}{b} \cdot \frac{a}{c} \cdot \frac{b}{a} = 1. \qquad \square$$

13 Polygons

13.1 Definitions of special quadrilaterals

In Chapters I and II, we defined polygons, both convex and nonconvex. We considered the simplest examples of a polygon: triangles and quadrilaterals. In Chapter II we considered some special cases of convex quadrilaterals: trapezoids and parallelograms. In this chapter, we will define other special cases of quadrilaterals: *rhombus*, *rectangle*, and *square*. Below are several definitions of all these figures, some of which are equivalent.[33]

A quadrilateral that has two opposite sides parallel to each other is a *trapezoid*.

A quadrilateral in which any pair of opposite sides are parallel to each other is a *parallelogram*.

A quadrilateral with two opposite sides that are parallel and equal to each other is a *parallelogram*.

A quadrilateral in which any two opposite sides are correspondingly equal to each other is a *parallelogram*.

A parallelogram that has all sides equal to each other is a *rhombus*.

A parallelogram that has one angle equal to 90° is a *rectangle*. (In Section 14.3 we prove that all angles of a rectangle are equal and each of them is 90°.)

A rectangle that has all sides equal is a *square*. A rhombus that has one angle equal to 90° is a *square*.

[33] We prove in Section 14.2 that the three definitions of a parallelogram are equivalent to each other.

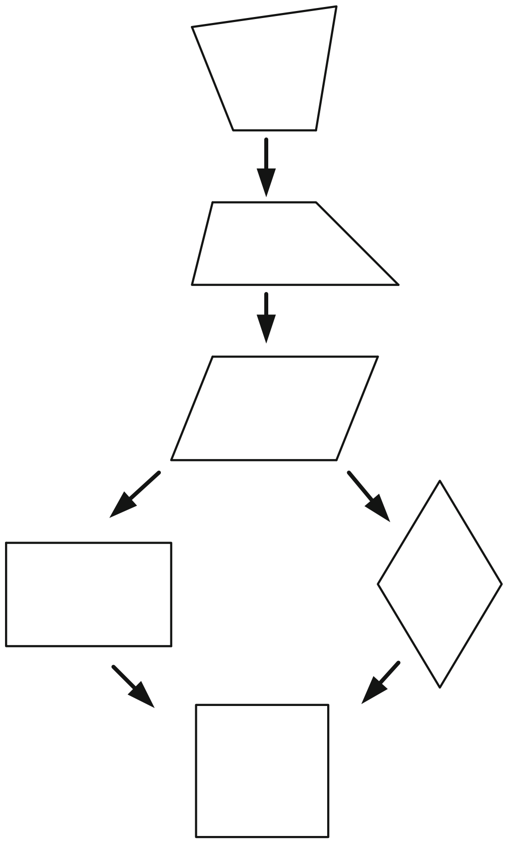


Fig. 4.77

PROBLEM 34.

(a) The above two definitions of a square are equivalent. Explain why.

(b) Answer the following questions:

Is a parallelogram a trapezoid?

Is a rectangle a parallelogram?

Is a rhombus a parallelogram?

Is a square a rhombus?

Is a square a rectangle?

Explain your answers.

(c) Label the types of all quadrilaterals in Fig. 4.77.

In the schema shown in Fig. 4.77, one can see the relations among all quadrilaterals.

13.2 Regular polygons

A convex polygon whose sides are all equal and whose interior angles are all equal is called a *regular polygon*. Examples of such polygons are: an equilateral triangle, a square, a regular pentagon, a regular hexagon, a regular heptagon, etc. (see Fig. 4.78). A regular polygon with n vertices is sometimes called a regular n-gon.

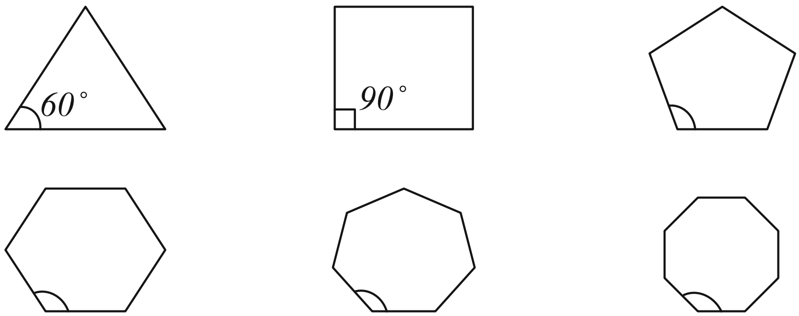


Fig. 4.78

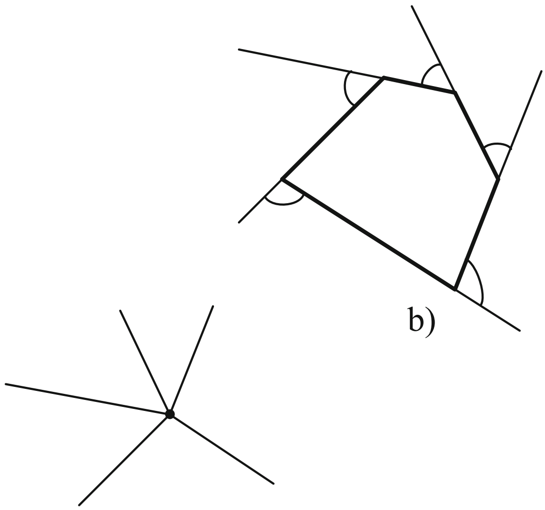


Fig. 4.79

13.3 The sum of the angles of a polygon

Consider a polygon (see Fig. 4.79a). In Chapter II (see Section 10.2), we proved that the sum of the exterior angles of a polygon (see Fig. 4.79b) is equal to 360°. Indeed, if we make a parallel translation of each of these angles so that all their vertices coincide, these angles become adjacent to each other and fill in the whole plane (see Fig. 4.79c).

This means that the following proposition holds.

Proposition 22. The sum of the exterior angles of a polygon is equal to 360°.

Exercise 19. Find the sum of the interior angles of a convex pentagon.

Solution 1. Consider a pentagon (see Fig. 4.80a). Let us denote its interior angles by $\angle\alpha_1, \angle\alpha_2, \ldots, \angle\alpha_5$ and its exterior angles by $\angle\beta_1, \angle\beta_2, \ldots, \angle\beta_5$.

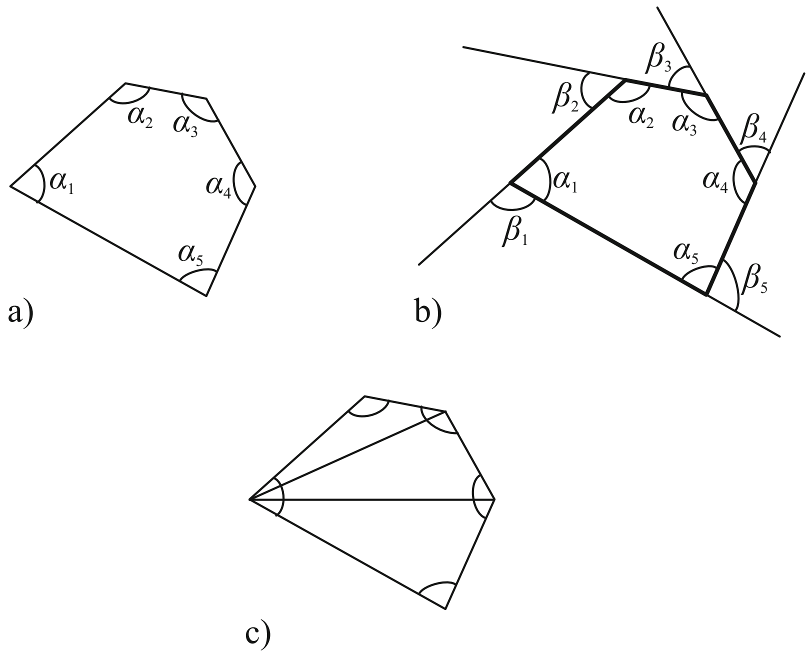


Fig. 4.80

We need to find $\angle\alpha_1 + \angle\alpha_2 + \cdots + \angle\alpha_5$. We know that $\angle\beta_1 + \angle\beta_2 + \cdots + \angle\beta_5 = 360°$. For each vertex of the pentagon we have one of the following equalities: $\angle\alpha_1 + \angle\beta_1 = 180°$, $\angle\alpha_2 + \angle\beta_2 = 180°$, etc. Since there are 5 vertices we see that the sum of all the interior angles ($\angle\alpha_1 + \angle\alpha_2 + \cdots + \angle\alpha_5$) plus the sum of all the exterior angles ($\angle\beta_1 + \angle\beta_2 + \cdots + \angle\beta_5$) is equal to $5 \cdot 180^\circ$. Therefore, $\angle\alpha_1 + \angle\alpha_2 + \cdots + \angle\alpha_5 = 5 \cdot 180^\circ - 360^\circ = 5 \cdot 180^\circ - 2 \cdot 180^\circ = 3 \cdot 180^\circ = 540^\circ$.

Solution 2. We can decompose[34] a pentagon into triangles as in Fig. 4.80c. We obtain three triangles. That is, the number of these triangles is two less than the number of sides of the polygon.

As we can see, the sum of all the interior angles of the pentagon is the sum of all the interior angles of all these triangles. Since the sum of the interior angles of a triangle is 180°, we have $\angle\alpha_1 + \angle\alpha_2 + \cdots + \angle\alpha_5 = 180^\circ \cdot 3 = 540^\circ$.

PROBLEM 35. Find the sum of the interior angles of a convex quadrilateral.

PROBLEM 36. Find the sum of the interior angles of a convex heptagon (convex 7-gon).

PROBLEM 37 (*) Find the sum of the interior angles of a convex polygon with 102 vertices.

If you have solved the problems above you can now prove the following proposition.

Proposition 23. The sum of the interior angles of a convex polygon with n vertices is equal to $180^\circ \cdot (n-2)$.

Since all the angles of a regular polygon are equal to each other, we obtain from the above proposition that an interior angle of a regular polygon with n vertices is equal to $\frac{180^\circ(n-2)}{n}$.

PROBLEM 38. Find the interior angles of all the polygons in Fig. 4.78.

---

[34] This procedure is also called *triangulation* from one vertex.

It is also possible to describe how to find the sum of the exterior angles and the sum of the interior angles of a nonconvex polygon. We will not do this here. Instead, we consider an example.

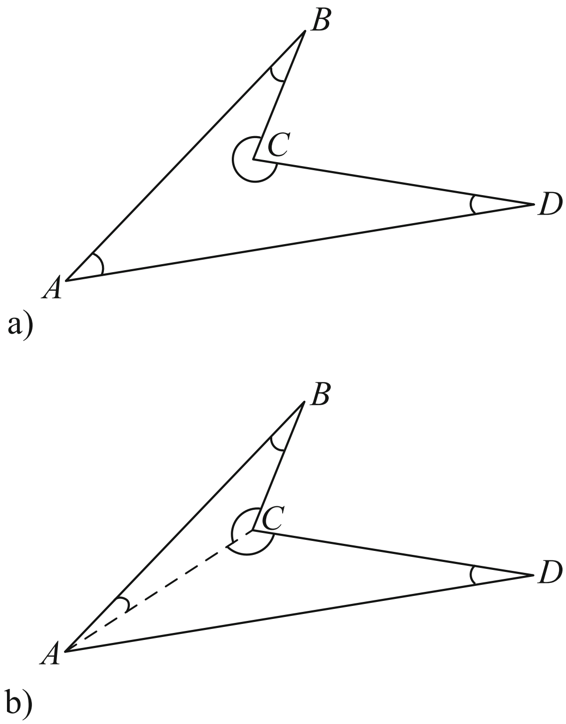


Fig. 4.81

Exercise 20. Consider the nonconvex quadrilateral *ABCD* in Fig. 4.81a. Find the sum of its interior angles.

Solution. Let us connect vertices *A* and *C* (see Fig. 4.81b). The quadrilateral is divided into two triangles *ABC* and *ACD*. We can see that the sum of all interior angles of *ABCD* equals the sum of the interior angles of the two triangles. But in any triangle the sum of its interior angles is 180°. Therefore, the sum of the interior angles of the nonconvex quadrilateral *ABCD* is equal to $180^\circ + 180^\circ = 360^\circ$.

14 Summary of facts about different quadrilaterals

In this section we put together some important facts about each quadrilateral. Some facts we proved earlier, and some will be presented and proved here.

14.1 Trapezoid

We repeat what we already studied in Chapter II about trapezoids.

- The diagonals of a trapezoid are divided in half by the median of the trapezoid (see Ch. II, Section 7).
- The length of the median of a trapezoid is equal to half of the sum of the lengths of its bases (see Ch. II, Section 7).
- The sum of the angles adjacent to a leg of a trapezoid is equal to 180° (see Ch. IV, Section 3.4).
- The altitude of a trapezoid to its base is defined as a perpendicular from a vertex of a trapezoid to its opposite base. Since there are four vertices, one can draw four altitudes onto bases of a trapezoid; all these four altitudes have the same length. (Note that the four altitudes to the other sides—the legs—of the trapezoid are not necessarily equal.)

Fig. 4.82 shows a trapezoid $ABCD$. We denote the angles adjacent to its side AB by α_1 and β_1 and the angles adjacent to its side CD by α_2 and β_2. We have $\alpha_1 + \beta_1 = 180^\circ$ and $\alpha_2 + \beta_2 = 180^\circ$ (see Proposition 1).

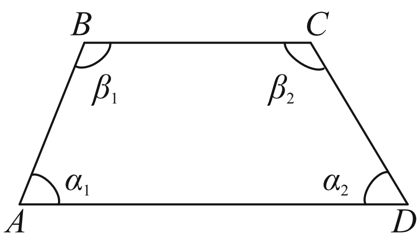


Fig. 4.82

Area of a trapezoid

Proposition 24. The area of a trapezoid is equal to half the sum of its bases multiplied by the altitude to the base, i.e.,

$$S = \frac{1}{2}(a+b)h.$$

Proof 1. Consider a trapezoid with bases a and b (Fig. 4.83a).

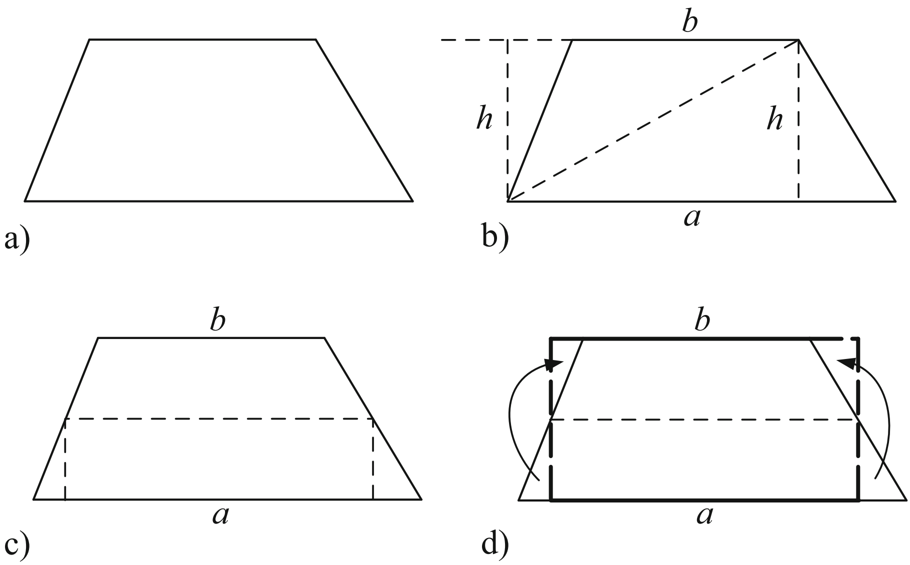


Fig. 4.83

Draw a diagonal in it (Fig. 4.83b). We obtain two triangles. The area of one of them is equal to $S_1 = \frac{1}{2}ah$, and the area of the other one is equal to $S_2 = \frac{1}{2}bh$. Therefore, the area of the trapezoid is

$$S = S_1 + S_2 = \frac{1}{2}(a+b)h. \qquad \square$$

Proof 2. Given a trapezoid with bases a and b (Fig. 4.83a), draw its median and drop perpendiculars from its endpoints onto side a (Fig. 4.83c).

This cuts two little triangles out of the trapezoid. If we move them as in Fig. 4.83d, we obtain a rectangle. The area of this rectangle is equal to the altitude of the trapezoid multiplied by its median, i.e., $\frac{1}{2}(a+b)h$. $\square$

14.2 Parallelogram

As we mentioned in Section 13.1, there are three definitions of a parallelogram.[35]

Proposition 25. All three definitions of a parallelogram are equivalent to each other.

Proof. We have already proved (in Chapter II, Section 5.1) that two of the definitions of a parallelogram are equivalent. Let us prove that the third definition is equivalent to the second.

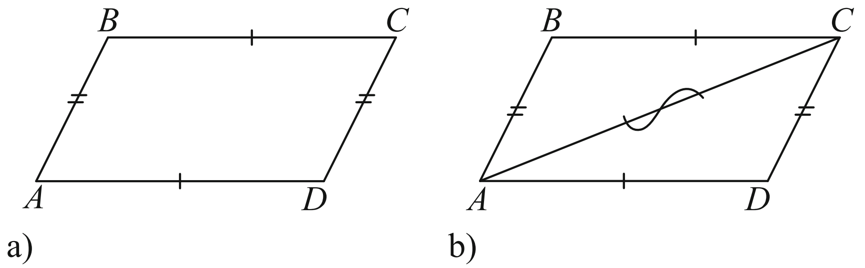


Fig. 4.84

Consider a quadrilateral $ABCD$ (see Fig. 4.84) with $AB = CD$ and $AD = BC$. According to the third definition, $ABCD$ is a parallelogram. We need to verify that it also satisfies the second definition, i.e., that a pair of opposite sides (e.g., BC and AD) are parallel to each other as well as equal.

Let us draw diagonal AC (see Fig. 4.84b). We have $\triangle ABC = \triangle ACD$, since their three sides are correspondingly equal. Therefore, $\angle BCA = \angle CAD$. But these equal angles are alternate interior angles (see Theorem 2 and Remark 3) and, therefore, $BC \parallel AD$.

Let us prove the converse statement, i.e., that if two opposite sides of a quadrilateral are equal and parallel, then it has two pairs of equal opposite sides. Indeed, let $BC = AD$ and $BC \parallel AD$. Draw diagonal AC (see Fig. 4.84b). We have $\angle BCA = \angle CAD$ as alternate interior angles. Therefore, due to (2) in Definition 3, we have $\triangle ABC = \triangle ACD$. Then $AB = CD$. $\square$

We review some facts about parallelograms:

- In a parallelogram each diagonal passes through its center and is divided in half by this center (see Ch. II, Section 5.2).

[35] Note that in Chapter II we could not compare segments lying on non-parallel lines and therefore could not give the third definition.

- Any straight line that passes through the center O of a parallelogram is divided by point O and the sides of the parallelogram into two equal segments (see Ch. II, Section 5.4).

- In a parallelogram the sum of the squares of its diagonals is equal to the sum of the squares of all its sides. (See Ch. IV, Section 10.2)

Area of a parallelogram

Proposition 26. The area of a parallelogram is equal to one of its sides multiplied by the altitude of the parallelogram drawn to this side, i.e.,

$$S = a \cdot h_a,$$

where a is a side of the parallelogram and h_a is the altitude to this side.

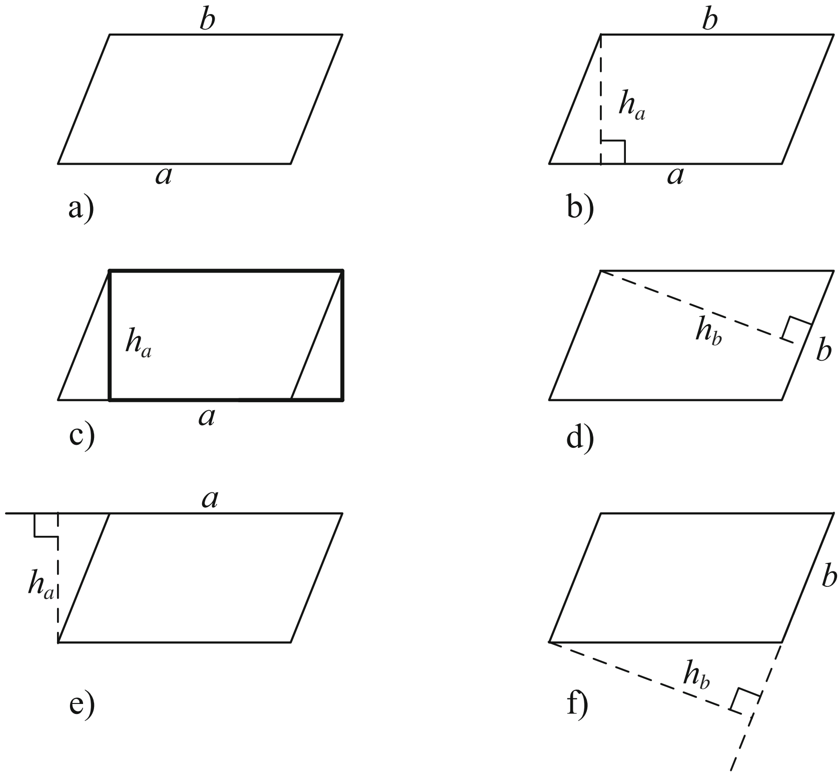


Fig. 4.85

Proof. Consider a parallelogram with sides a and b (see Fig. 4.85a). Let us draw altitude h_a to side a (see Fig. 4.85b).

Altitude h_a cuts a triangle out of the parallelogram. If we make a parallel translation of this triangle as in Fig. 4.85c, we obtain a rectangle with the same area as the parallelogram. This rectangle has sides a and h_a. Therefore, the area of both the rectangle and the parallelogram is equal to $a \cdot h_a$.

Note that from two vertices of the parallelogram we can draw four different altitudes (see Fig. 4.85b, d, e, f). □

PROBLEM 39. Prove that the area of the parallelogram in Fig. 4.85f is equal to $b \cdot h_b$.

PROBLEM 40. Two sides of a parallelogram are equal to 4 cm and 6 cm. The altitude to the side that measures 6 cm in length is equal to 2 cm. Find the length of the perpendicular to the other side of this parallelogram.

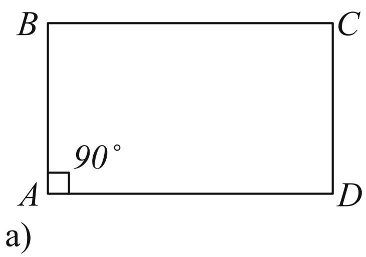
PROBLEM 41. Consider a parallelogram with sides that measure 4 cm and 6 cm in length, and the angle between them is equal to 30°. Find the area of this parallelogram.

14.3 Rectangle

We defined a rectangle as a parallelogram with one 90° angle. The following proposition holds.

Proposition 27. All angles in a rectangle are equal to 90°.

Proof. Consider a rectangle $ABCD$ (see Fig. 4.86a).



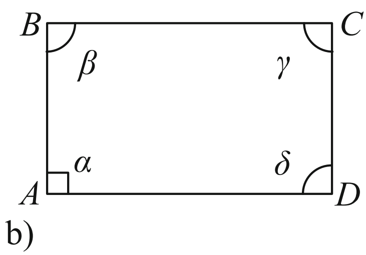


Fig. 4.86

Denote its angles by α, β, γ, δ. Since $ABCD$ is also a parallelogram, we have $\alpha+\beta=180°$, $\gamma+\delta=180°$, and also $\alpha+\delta=180°$, $\beta+\gamma=180°$. But angle α is equal to 90°, so $\beta=180°-90°=90°$. Similarly, $\gamma=\delta=90°$. □

Area of a rectangle

The area S of a rectangle with sides a and b is

$$S = a \cdot b$$

We have already proved this in Section 9.1.

14.4 Rhombus

The following theorem can serve as a second definition of a rhombus.

Theorem 12. A parallelogram is a rhombus if and only if its diagonals are perpendicular to each other.

Proof. Let us first prove that in a rhombus the diagonals are perpendicular to each other. In Fig. 4.87a there is a rhombus $ABCD$ whose diagonals intersect at point O. Since $ABCD$ is also a parallelogram, in the triangle ABC segment BO is a median (see Theorem 1, Ch. II). Since triangle ABC is isosceles, BO is also its altitude, i.e., BD is perpendicular to AC.

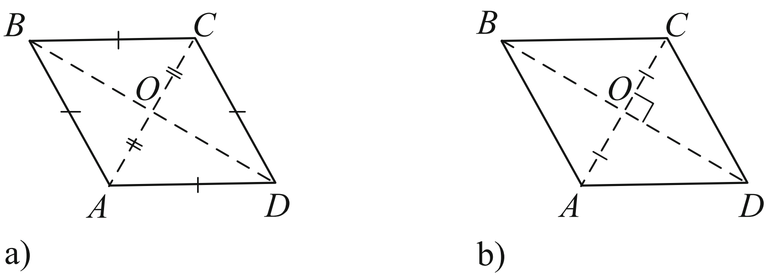


Fig. 4.87

Let us prove the converse statement: let $ABCD$ (Fig. 4.87b) be a parallelogram whose diagonals are perpendicular, i.e., BD is perpendicular to AC. We need to prove that $AB = BC$. Since in this parallelogram $AO = OC$, segment BO is a median of $\triangle ABC$. It is also given that BO is an altitude. Therefore, triangle ABC is isosceles (see Section 8.1), and $AB = BC$. $\square$

Area of a rhombus

Since a rhombus is a parallelogram, we can use the formula for the area of a parallelogram:

$$S = a \cdot h_a.$$

There is also another formula for the area of a rhombus.

Proposition 28. The area of a rhombus is equal to half of the product of its diagonals d_1, d_2, i.e.,

$$S = \frac{1}{2}d_1 d_2.$$

First let us do the following exercise.

Exercise 21. Draw a rhombus and cut it along its diagonals. Rearrange the four pieces obtained into a rectangle.

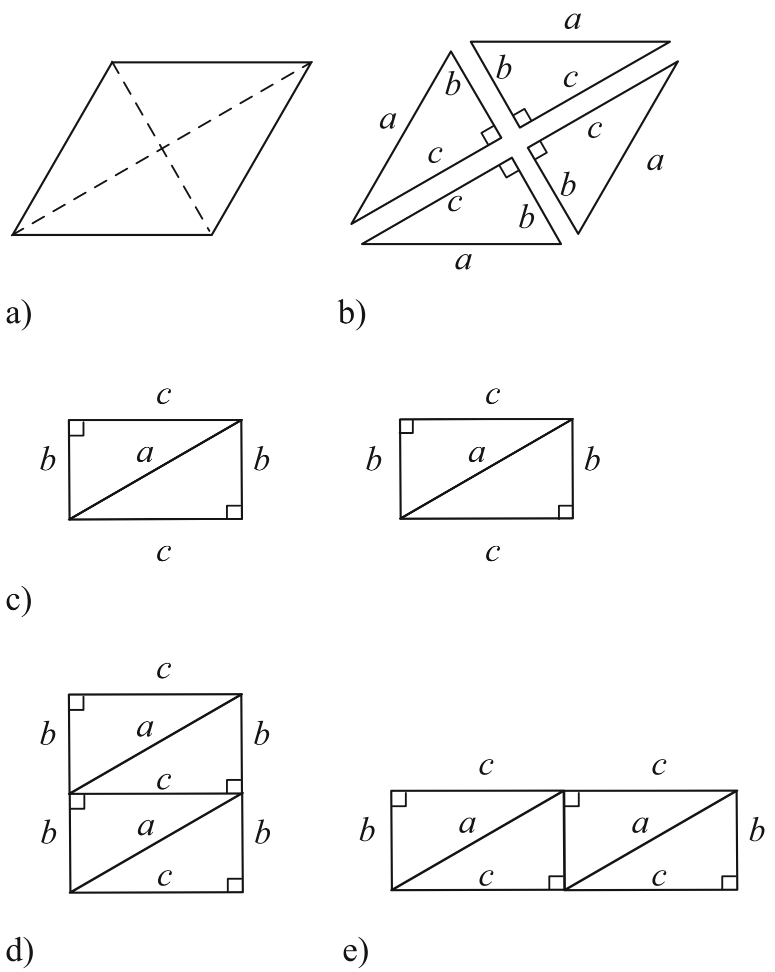


Fig. 4.88

Solution. Fig. 4.88a shows a rhombus with its diagonals. If we cut along the diagonals we obtain four pieces as in Fig. 4.88b. We have also labeled all their sides.

We can put these pieces together to form two rectangles as in Fig. 4.88c, or to form one rectangle as in Fig. 4.88d or in Fig. 4.88e.

Now we will prove Proposition 28.

Proof 1. Let $ABCD$ be a rhombus with side a and diagonals $AC = d_1$ and $BD = d_2$ (Fig. 4.89a).

According to Exercise 21, we can cut this rhombus along its diagonals and rearrange the pieces into a rectangle (see Fig. 4.88d,e). For the rhombus in Fig. 4.89 $c = \frac{1}{2}AC = \frac{1}{2}d_1$ and $b = \frac{1}{2}BD = \frac{1}{2}d_2$. Thus the area of the rhombus $ABCD$ is $S = 2b \cdot c = 2c \cdot b = \frac{1}{2}d_1 d_2$. □

Proof 2. A rhombus $ABCD$ is divided into two congruent triangles ABC and ACD by its diagonal AC (see Fig. 4.89b). From Theorem 12, BO is an altitude in $\triangle ABC$, therefore, we have

$$S = 2S_{ABC} = 2 \cdot \frac{1}{2}BO \cdot AC = \frac{1}{2}BD \cdot AC = \frac{1}{2}d_1 \cdot d_2. \qquad \square$$

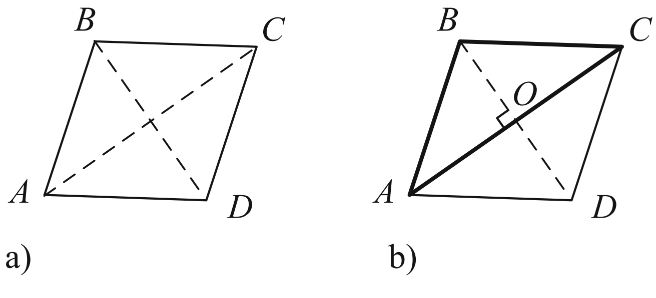


Fig. 4.89

The following statement is a consequence of Exercise 21:

Proposition 29. The diagonals of a rhombus divide it into four congruent right triangles.

14.5 Square

Consider a square with side a and diagonal d (see Fig. 4.90).

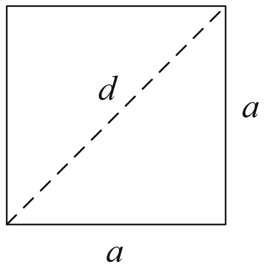


Fig. 4.90

From the Pythagorean theorem, it follows that $d^2 = 2a^2$, or $d = \sqrt{2}a$, or $d \approx 1.41a$.

PROBLEM 42. In England, the students of Oxford University must walk only on paved passages from one building to another. Professors are allowed to make a short-cut by walking on the lawn.

In Fig. 4.91, the points A and B represent two buildings. The segments are paved passages, and the dashed line is a short-cut across the lawn.

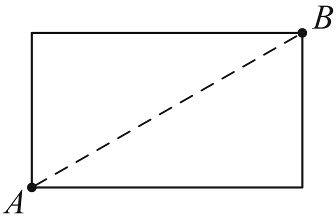


Fig. 4.91

(a) If the lawn is a rectangle with sides 300 and 400 ft in length, what is the percentage of the path that a professor saves by walking across the lawn?

(b) Answer the same question in the case where the lawn is a square with a side 300 ft long.

Area of a square

The area of a square with side a is

$$S = a^2$$

15 Similarity

Similarity is a transformation of a figure. We have already considered the following transformations with figures: parallel translation, rotation through an angle, and reflection with respect to a line. We assumed that these transformations did not change the figure itself, and we used them to define congruent figures (see Section 4). Similarity changes a figure.

Let us define *similar figures*.[36] First, we choose a point O that we call the *center of similarity*.

Definition 4. A *similarity* (or *similarity transformation*) with the center of similarity O is a one-to-one correspondence between points on the plane according to which to any point A there corresponds a point A' such that $A'O = k \cdot AO$, where k is a constant factor that is a rational or real number.

The point A' is called the *image* of point A under similarity with center O and *coefficient* k.

As you will see in this section, similarity does not change the shape of a figure (for example, a triangle will remain a triangle), but it changes the size of the figure. One can think of the relation between similar figures as one figure being rescaled from the other by the coefficient k.

Exercise 22. Consider two points: the center of similarity O and a point A (see Fig. 4.92). Find the point A' that is similar to point A with a similarity coefficient of 2.

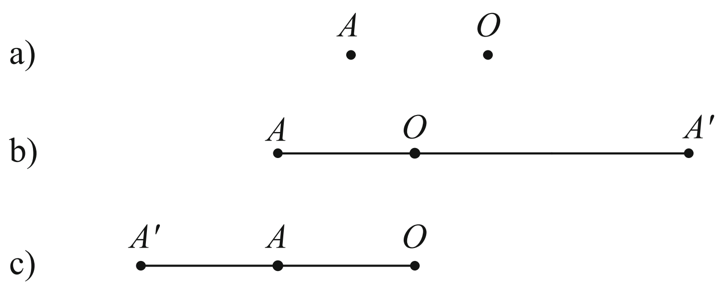


Fig. 4.92

Solution. We connect point A with point O and mark point A' on line AO in such a position that $A'O = 2AO$.

As we can see from Fig. 4.92b and c, there are two ways to do this.

[36]Note that it would have been possible to define similar figures in Chapter II and then study them more in this chapter.

In one case, the points A and A' lie on different sides of point O; in the other case, these two points lie on the same side of point O.

Given a figure, if we need to construct a similar figure, we have to find the image of each point of the given figure. In such a case, we must choose all the points of the figure in only one of these two different ways, i.e., either lying on the same side of the center of similarity or not.

PROBLEM 43. Fig. 4.93 shows a point O and a segment AB. Find the segment $A'B'$ similar to AB with similarity coefficient 2 and center of similarity O.

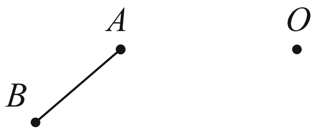


Fig. 4.93

15.1 Similar triangles

According to Definition 4, for a given triangle and a point O as center of similarity, we can construct a similar triangle with a coefficient k. How can we determine whether two triangles are similar or not? It is not easy to answer these questions using Definition 4.

The following definitions present a more useful and practical way to describe similar triangles. Of course, one has to prove that the definitions below are equivalent to Definition 4; however, such a proof cannot be explained in this book.[37]

Definition 5.

(1) If three sides of one triangle are correspondingly proportional to the three sides of another triangle, then these triangles are similar.

(2) If an angle of one triangle is equal to an angle of another triangle, and the two sides of this angle are correspondingly proportional to the two sides of the equal angle in the other triangle, then these triangles are similar.

(3) If two angles of one triangle are correspondingly equal to two angles of another triangle, then these triangles are similar.

[37] The main difficulty arises from determining the center of similarity for two given similar triangles.

These definitions of similar triangles[38] are illustrated in Fig. 4.94a, b, and c.

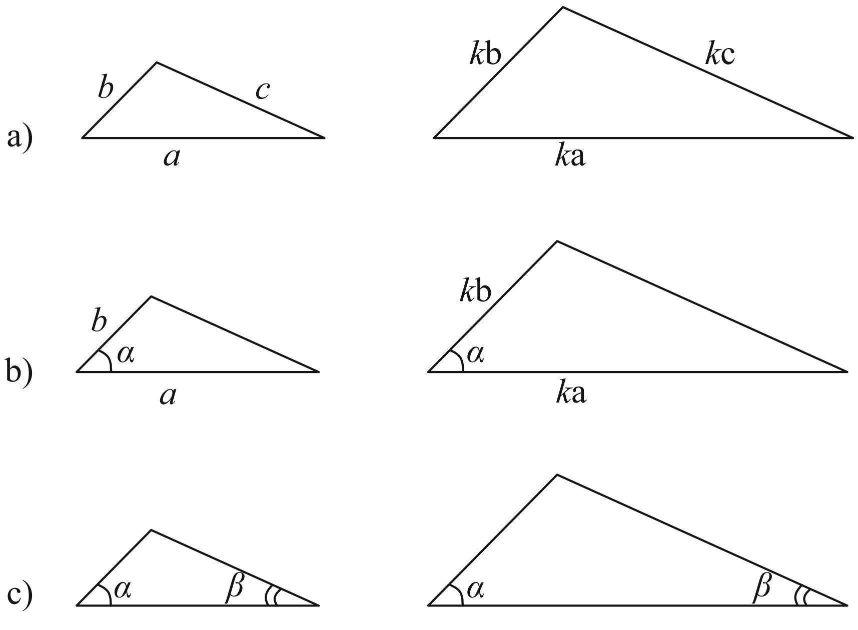


Fig. 4.94

The following statements are also presented without a proof.

Lemma 1.

(a) If the sides of a triangle are correspondingly parallel to the sides of another triangle, these triangles are similar.

(b) If an angle is intersected by two parallel lines, then the two triangles formed are similar.

This lemma is illustrated in Fig. 4.95a, where $a \parallel a'$, $b \parallel b'$, and $c \parallel c'$, and in Fig. 4.95b, where $a \parallel b$.

PROBLEM 44. In an equilateral triangle all angles are equal to $60°$. Are any two equilateral triangles similar?

PROBLEM 45. In a right triangle ABC draw perpendicular CD from the right angle to the hypotenuse (see Fig. 4.96a). Find all pairs of similar triangles among $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$ (see Fig. 4.96b).

[38]Compare this definition with Definition 3 in Section 5.

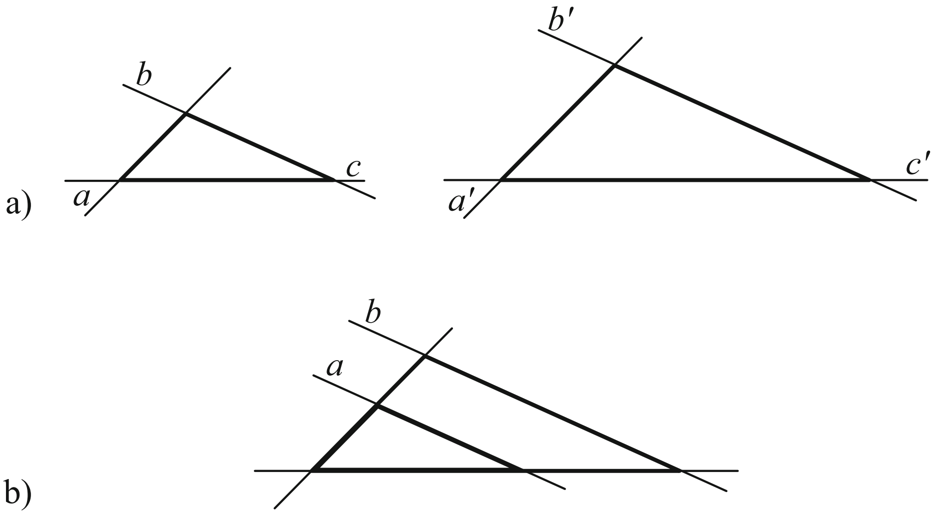


Fig. 4.95

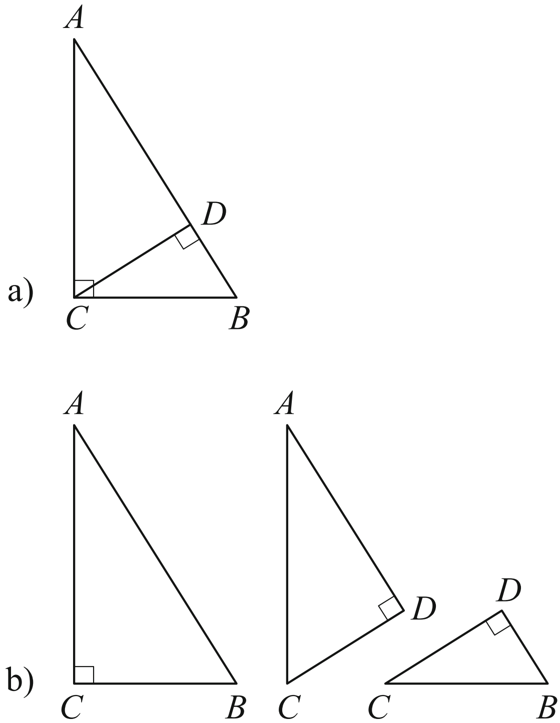


Fig. 4.96

15.2 Similarity of polygons and area of similar polygons

Consider a polygon and a point O. In order to construct a similar polygon with similarity coefficient k and center of similarity O, we need to find the image of each vertex of the given polygon and then connect these images.

PROBLEM 46.

(a) Choose a point O. Draw a triangle ABC. Construct a triangle $A'B'C'$ similar to $\triangle ABC$ with center O and with coefficient of similarity equal to 2.

(b) Solve the same problem for a square $ABCD$.

(c) Solve the same problem for a parallelogram $ABCD$.

PROBLEM 47.

(a) In a square, every angle is equal to 90°. Are any two squares similar?

(b) In any rectangle, every angle is equal to 90°. Are any two rectangles similar?

Proposition 30. Suppose we have two similar polygons with coefficient of similarity k. If the area of the first polygon is S, then the area of the second polygon is k^2S.

We will not prove this proposition here, but instead ask you to check some examples.

PROBLEM 48.

(a) Check this Proposition for two similar triangles for $k = 2$, and for $k = 3$.

(b) Check this Proposition for two squares when $k = 2$, and when $k = 3$.

15.3 A third proof of the Pythagorean theorem

Consider a right triangle ABC (see Fig. 4.97a). Let us denote the angles of ABC by α and β. Let CD be the perpendicular dropped from the right angle onto the hypotenuse. Let $AD = x$ and $BD = y$ (see Fig. 4.97b).

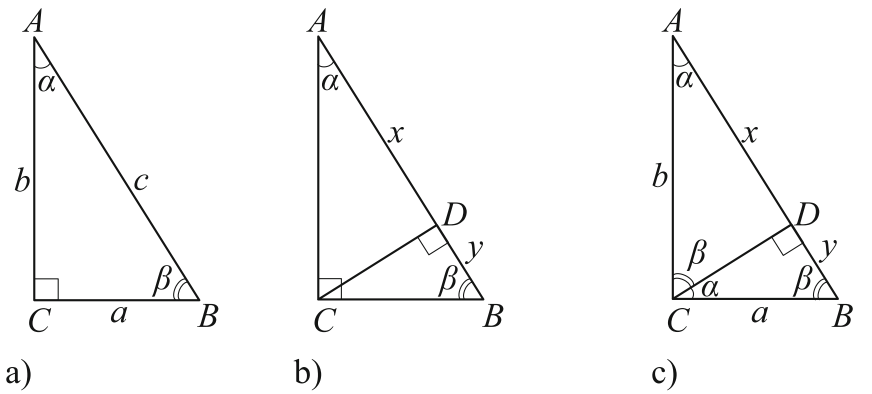


Fig. 4.97

Note that the angles in triangles BCD and ABC are correspondingly equal to each other. Indeed, in triangle ABC the angles are α, β, and 90°, where $\alpha + \beta = 90^\circ$ (since the sum of the angles in a triangle is equal to 180°). In triangle BCD, one angle is β and another angle is 90°. Therefore, the third angle of triangle BCD is equal to $180^\circ - (90^\circ + \beta) = \alpha$ (see Fig. 4.97c). Thus, triangles BCD and ABC are similar to each other and

$$\frac{c}{a} = \frac{a}{y}.$$

Note that triangle ACD also has angles α, β, and 90°. Thus, triangles ABC and ACD are similar and

$$\frac{c}{b} = \frac{b}{x}.$$

From these ratios we can find x and y:

$$x = \frac{b^2}{c} \text{ and } y = \frac{a^2}{c}.$$

But $x + y = c$. Therefore,

$$\frac{b^2}{c} + \frac{a^2}{c} = c,$$

or finally, $a^2 + b^2 = c^2$. □

PART III. Circles

In Chapter IV we introduced a circle. We repeat the definition that a circle is a closed curve such that all the segments connecting a point on this curve with a certain point O, called the center of the circle, are equal to each other.

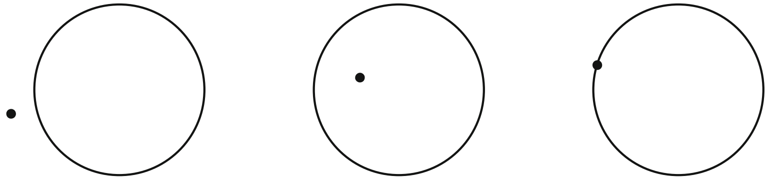
A circle divides the plane into two regions (see Fig. 4.3a, b) such that one cannot move on the plane from one region into the other without crossing the boundary.

We have also considered different polygons, such as triangles, parallelograms, and trapezoids, and studied their properties. For a circle the situation becomes more complicated: a circle can have different relationships with points, lines, and figures. We will consider some of these relationships.

16 Circles and points

What are the possible positions of a point A relative to a circle?

Since a circle divides the plane into two regions, the point A can lie outside the circle (see Fig. 4.98a), inside the circle (see Fig. 4.98b), or on the circle (see Fig. 4.98c).



a) b) c)

Fig. 4.98

16.1 Circles passing through a point

Given a point A, how many circles can we draw passing through A? Where can the centers of these circles lie?

In order to draw a circle, we need to choose its center O. If we draw a circle with radius AO, this circle passes through point A (see Fig. 4.99a). However, we can choose another center for the circle, a point O' (see Fig. 4.99b) or any other point on the plane (see Fig. 4.99c). Since the center can be anywhere on the plane, one can draw infinitely many circles passing through the point A.

This fact is similar to the fact that through a point one can draw infinitely many lines.

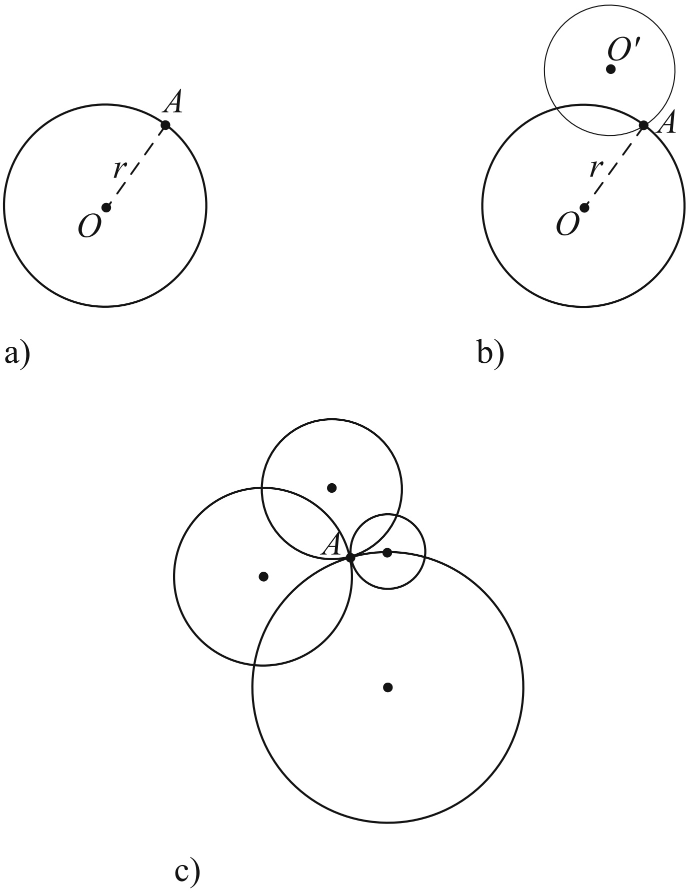


Fig. 4.99

16.2 Circles passing through two points

Consider two points A and B. How many circles can we draw passing through these points? Where can the centers of these circles lie?

In order to draw a circle, we must choose its center O. However, this time we cannot choose any point as the center O. Indeed, since both OA and OB are radii of the circle, the point O has to be equidistant from A and B.

As we know from Section 11.3, all points equidistant from two given points A and B lie on the perpendicular a to the midpoint of segment AB (see Fig. 4.100a).

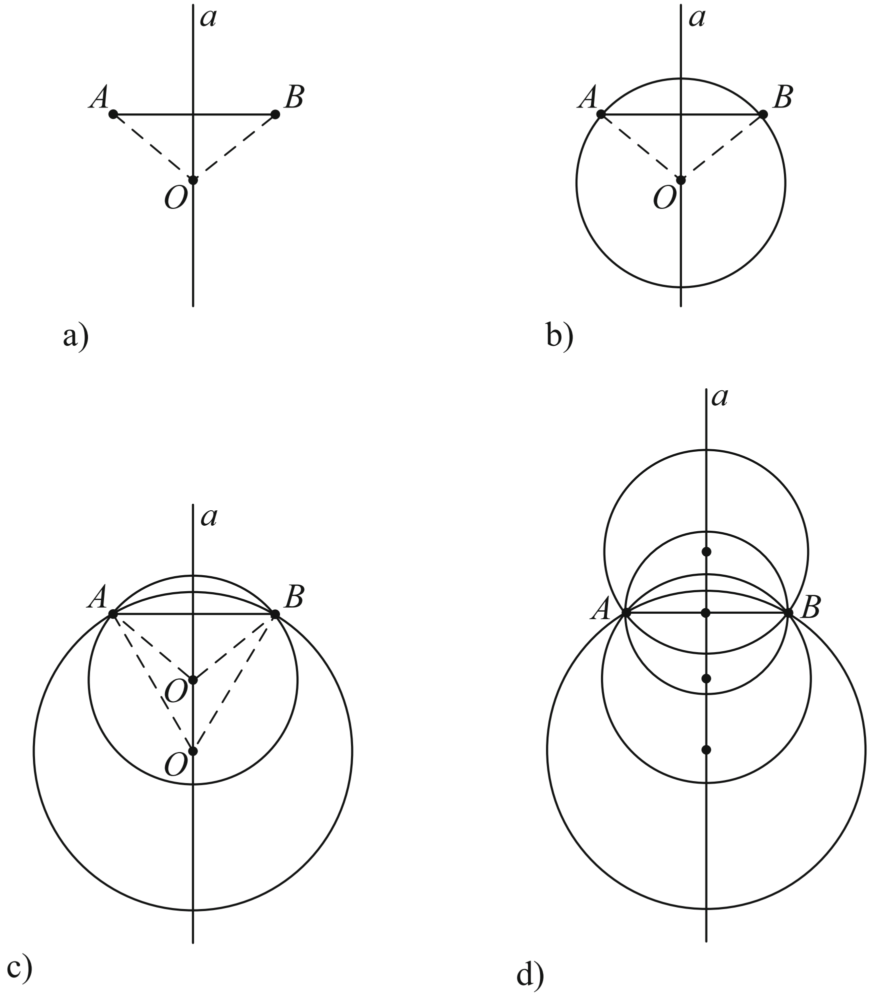


Fig. 4.100

With center O and radius OA we can now draw a circle (see Fig. 4.100b). This circle passes through both points A and B.

We can also choose another point O' on the perpendicular a as the center of the circle (see Fig. 4.100c), or any other point on this perpendicular (see Fig. 4.100d). Thus, we can draw infinitely many circles passing through points A and B. The center of each one of them must lie on the perpendicular bisector of segment AB.

This fact differs from a corresponding fact about lines: there is only one straight line passing through two points.

PROBLEM 49. Describe what happens to the circle as its center O moves closer to the segment AB (or even lies on AB), and what happens when the point O moves far away from segment AB.

16.3 Circles passing through three points

Consider three points A, B, and C. How many circles can we draw passing through these points? Where can the centers of these circles lie?

Since the circle is to pass through points A and B, its center must lie on the perpendicular a to the midpoint of AB. This circle must also pass through points B and C. Therefore, the center must lie also on the perpendicular b to the midpoint of BC (see Fig. 4.101a). If the perpendiculars a and b intersect at a point O, then O is the center of the circle passing through points A, B, and C (see Fig. 4.101b). Indeed, by construction, $OA = OB$ and $OB = OC$.

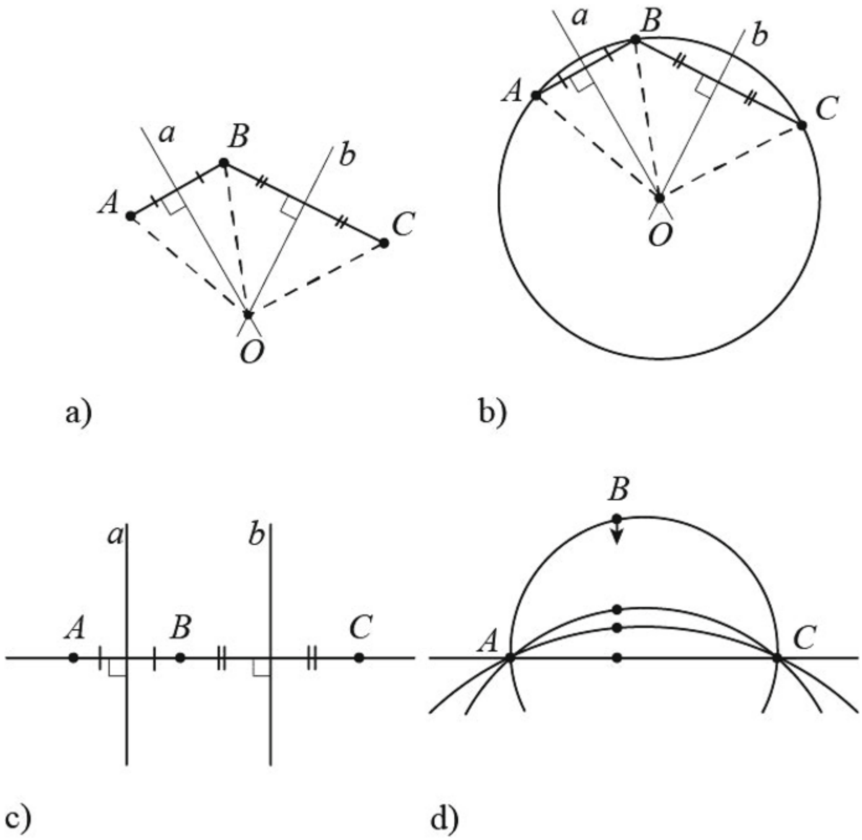


Fig. 4.101

Therefore, $OA = OC$, i.e., the points A, B, and C are equidistant from point O (see also Theorem 10 in Section 12.3).

What if the perpendiculars a and b do not intersect? This means that lines a and b are parallel. This is possible only if points A, B, and C lie on the same straight line[39] (see Fig. 4.101c). In this case, points A, B and C are not in general position.

Thus, for any three points A, B, and C in general position one can draw a unique circle passing through these points, but one cannot draw a line through them. If three points lie on the same line, one cannot draw a circle passing through these three points.

Notice (see Fig. 4.101d) that when point B is closer to line AC, the radius of the circle passing through A, B, and C becomes larger while the arc of the circle between A and C becomes "flatter" and closer to segment AC. When the point B lies on AC, we obtain a straight line. Therefore, sometimes a straight line is called a *circle with infinite radius*. We can obtain the following theorem.

Theorem 13. Given three points A, B, and C, one can always draw either a unique circle or a unique line passing through these three points.

17 Circles and lines

17.1 The relative positions of a circle and a line

In Section 7 of Chapter I, we described the *duality* between points and straight lines. We have already considered relative positions of a circle and a point. Let us now consider the relative positions of a circle and a line.

A circle and a line can be in one of the following three positions (see Fig. 4.102 and compare it with Fig. 4.98).

In Fig. 4.102a there are no points of intersection between the circle and the line; in Fig. 4.102b there are two points of intersection; and in Fig. 4.102c there is only one common point between the circle and the line.

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[39] This can be proved by contradiction. Let us assume that points A, B, C do not lie on one line (see Fig. 4.101a). If we extend line AB until it intersects line b at some point D, then (since $a \parallel b$ and $a \perp AD$) we have $b \perp AD$. But then in the triangle formed by lines AD, b, and BC there are two right angles, which is impossible. Therefore, A, B, C must lie on one straight line.

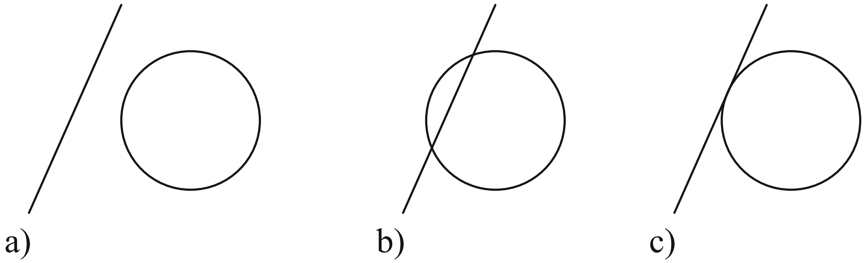


Fig. 4.102

A line which has only one common point with a circle is called *tangent* to this circle.

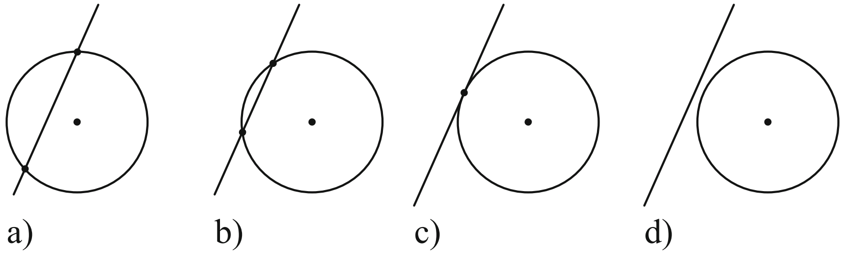


Fig. 4.103

In this case (Fig. 4.102c), mathematicians say that the two points of intersection coincide. To see how these points can come to coincide, let us look again at the case in Fig. 4.102b and start making a parallel translation of this line away from the center (see Fig. 4.103a, b, c). We can see that the two points of intersection come closer to each other until at a certain position (Fig. 4.103c) they coincide. If we continue moving the line there will be no more common points between the line and the circle (see Fig. 4.103d).

17.2 Circles tangent to one, two and three straight lines

In Section 16, we considered all the circles passing through one, two, or three points. Let us consider all the circles tangent to one, two, or three lines.

Circles tangent to one straight line

There are infinitely many circles tangent to a given line.[40] First, a circle can be tangent to any point on the straight line (see Fig. 4.104a) and second, a circle can have any radius (see Fig. 4.104b).

[40]Compare this with all circles passing through one point.

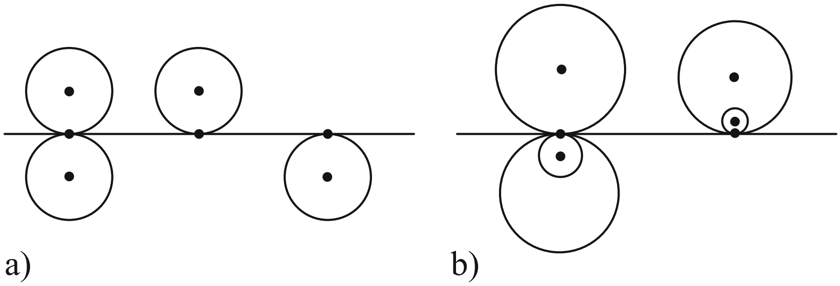


Fig. 4.104

Circles tangent to two straight lines

Let us find all the circles tangent to two given lines a and b in general position, i.e., when a and b intersect. If a circle is tangent to two lines, then its center has to lie at the same distance from each of the lines, or (see Section 11.4) on the bisector of the angle formed by these lines (see Fig. 4.105a). Therefore, all the circles tangent to two given lines have their centers on the angle bisector of the angle between the lines a and b (see Fig. 4.105b). It is clear that there are infinitely many such circles.[41]

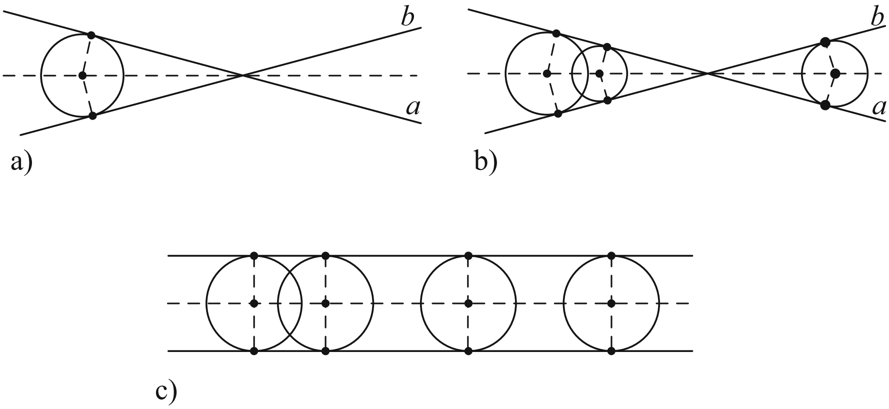


Fig. 4.105

Note that if the lines a and b are parallel, then all the circles tangent to both a and b have the same radius (see Fig. 4.105c).

[41] Compare this with all circles passing through two points.

Exercise 23. Fig. 4.105b shows where circles tangent to both a and b might lie. Are there other possibilities?

Solution. Two intersecting lines form two pairs of vertical angles. Thus there are two angle bisectors. Therefore, one can draw circles tangent to both lines a and b as in Fig. 4.106.

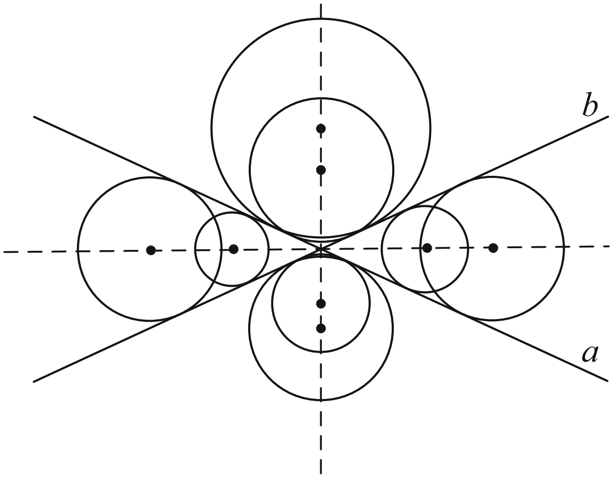


Fig. 4.106

Circles tangent to three straight lines

Consider three lines a, b, and c in general position (see Fig. 4.107a). We need to find all the circles tangent simultaneously to these lines. The center O of such a circle has to lie at the same distance from the lines a, b, and c. This means that the center O lies on the bisectors of the angles between a and b, between b and c, and between a and c (see Fig. 4.107b). Since these angle bisectors intersect at a single point (see Section 12.2), we obtain one circle tangent to the lines a, b, and c.

Exercise 24. In Fig. 4.107b there is a single circle tangent to the lines a, b, and c. Are there other possibilities?

Solution. For any two intersecting lines one can draw two angle bisectors. For three lines a, b, c we obtain six angle bisectors. There are four circles[42]

[42] Compare this with the circle passing through three given points.

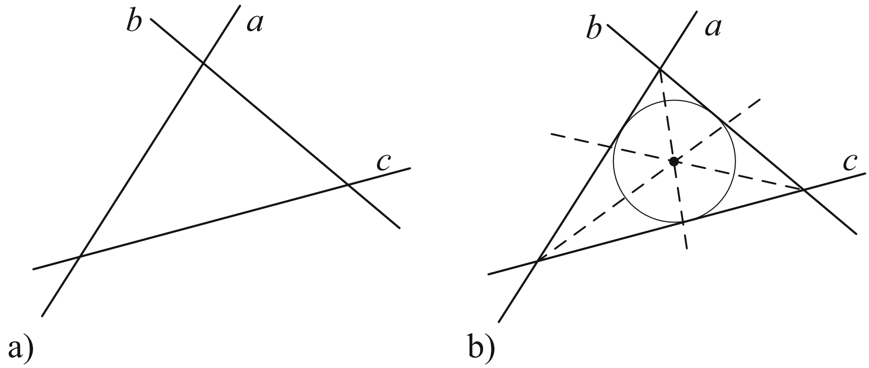


Fig. 4.107

tangent to lines a, b, and c (see Fig. 4.108[43]). Their centers are the intersections of all the bisectors of the angles between the lines a and b, b and c, and a and c.

Note that since the bounded domain formed by three intersecting lines a, b, c is a triangle, we have constructed a circle tangent to the sides of this triangle. Read more about this in Section 20.1.

18 Two or more circles

18.1 The relative positions of two circles

What are the possible relative positions of two circles?

Note that relative positions of two or more circles are examples of what we call configurations. In Chapter I, Section 3.1 we have defined a configuration of lines and points. We can extend this definition to a configuration of circles.

We say that two collections of two or more points, lines, or circles are examples of the same configuration if one can gradually move the objects from one collection into the objects of the other collection on the plane while obeying the following rule:

- at no moment are we allowed to obtain a new intersection point or lose an existing one.

[43]In order to fit this figure on the paper, the triangle formed by the lines a, b, and c has been reduced. Actually, we have applied a similarity transformation, which is what is used when scaling maps of a territory.

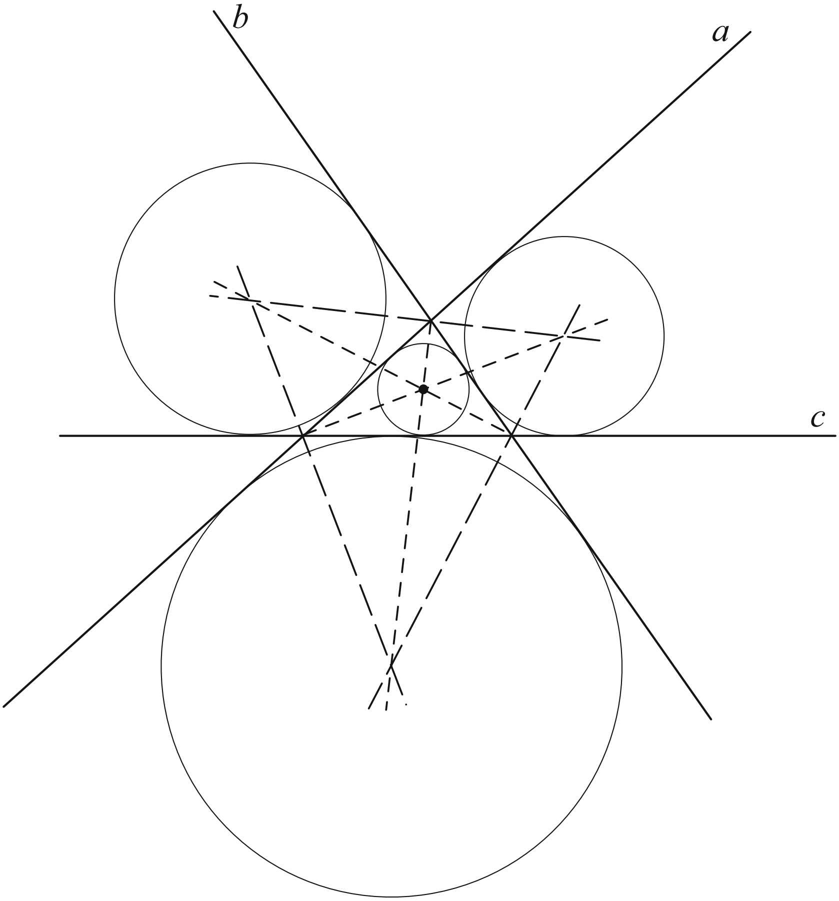


Fig. 4.108

Note that in this chapter, the term “move a figure on the plane” has a precise meaning: a move is a parallel translation or a rotation.

In Chapter I (Section 3.1 and Section 7), we considered a notion about the configuration of a general position of lines and a general position of points. This can also be extended to circles.

We say that two or more circles on the plane are *in general position* if no two of them have only one intersection point, and no two of them have the same center.

There are three general positions of two circles (see Fig. 4.109a, b, c). In Fig. 4.109a, b, two circles do not intersect (they either lie outside each other as in Fig. 4.109a or lie inside each other as in Fig. 4.109b). The two circles in Fig. 4.109c intersect and have two points of intersection.

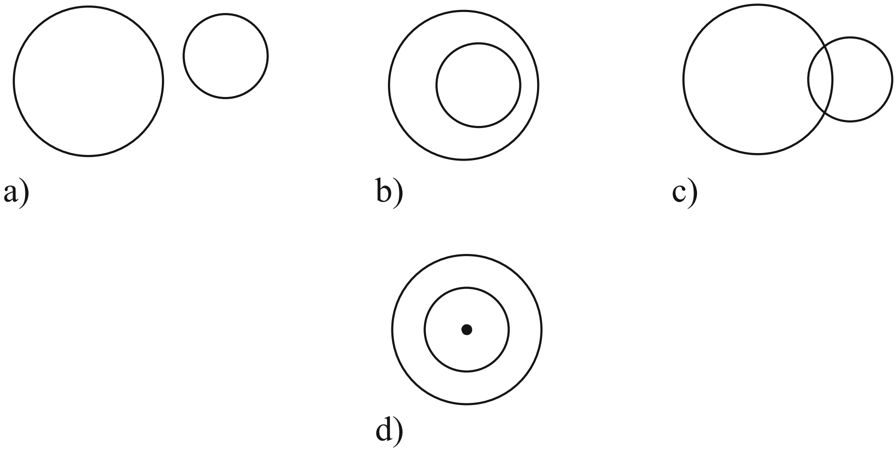


Fig. 4.109

The position of circles in Fig. 4.109b includes a particular case where the centers of these two circles coincide, which is, of course, not a general position. We remind the reader that for a non-general position of figures, any move of these figures, no matter how small, changes this position. In this particular case (see Fig. 4.109d) the circles are called *concentric circles*.

There are also two non-general positions of two circles when the two circles have only one common point (see Fig. 4.110a, b). If two circles have only one common point, they are called tangent to each other. The circles in Fig. 4.110a are tangent to each other externally, and the circles in Fig. 4.110b are tangent to each other internally.

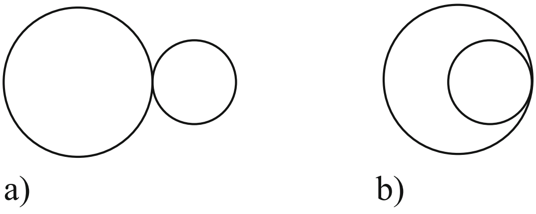


Fig. 4.110

Exercise 25. Sketch all the possible positions of two circles with marked centers. Do not consider any tangent positions of the two circles. To make it easier not to skip a case, imagine, for example, gradually moving one circle towards the other.

Solution. See Fig. 4.111.

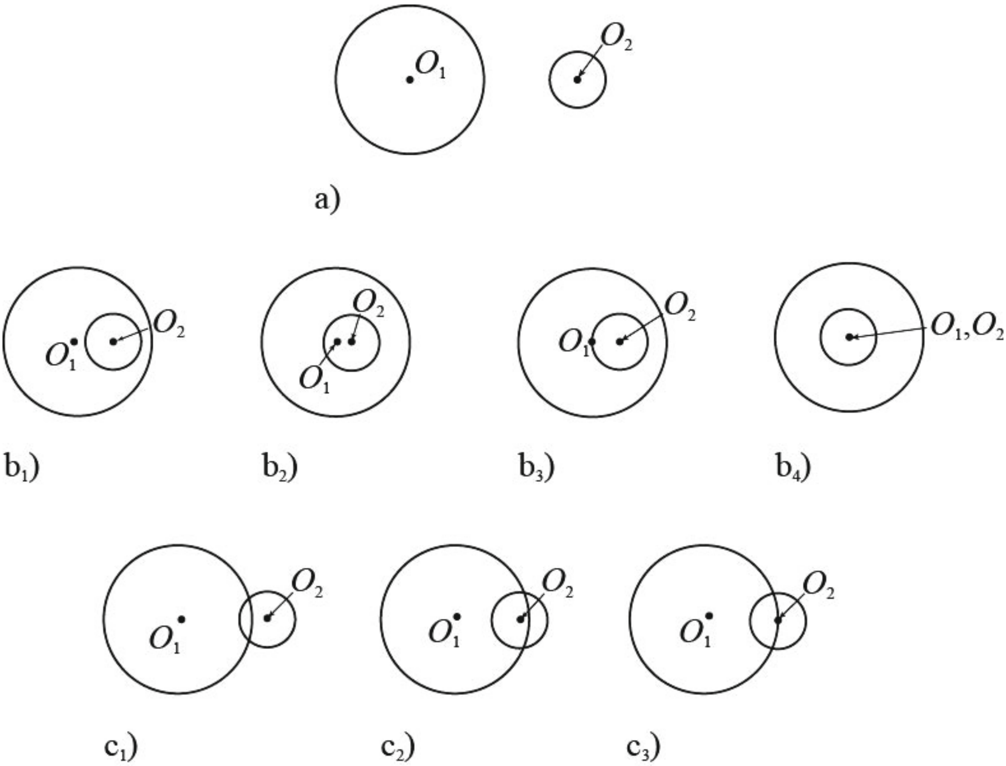


Fig. 4.111

Each position of these two circles can be described by the following three numbers: the radii r_1 and r_2 of the two circles and the distance d between the centers O_1 and O_2. Note that in the particular case in Fig. 4.111b$_4$, the circles are concentric.

PROBLEM 50. The numbers r_1, r_2, and d are given. For each case below, determine the relative position of the circles, draw them, and find this position in Fig. 4.111 (if it is there).

(a) $r_1 = 1, r_2 = 2, d = 5$.　(b) $r_1 = 1, r_2 = 2, d = 4$.

(c) $r_1 = 1, r_2 = 2, d = 3$.　(d) $r_1 = 1, r_2 = 2, d = 2$.

(e) $r_1 = 1, r_2 = 2, d = 1$.　(f) $r_1 = 1, r_2 = 2, d = 0$.

PROBLEM 51 (*) Given three numbers r_1, r_2, and d, how can one determine without drawing the circles which relative position these two circles have?

PROBLEM 52. Write down all the relations that you can find among the numbers r_1, r_2, and d for each of the figures in Fig. 4.111.

For example, in each figure we have $r_1 > r_2$. Additionally, for the position shown in Fig. 4.111a, we have $r_1 + r_2 < d$.

PROBLEM 53.

(a) Suppose the numbers r_1, r_2, and d describing two circles are such that $r_1 + r_2 > d$. Do the circles intersect?

(b) Describe the relations among r_1, r_2, d when the two circles intersect.

18.2 The relative positions of three circles

What are the possible relative positions of three circles if we consider only their general positions?

We already know that two circles can either intersect or not intersect, and if they do not intersect they may lie either inside or outside of each other. Let us use the same reasoning for each pair of three circles. After several attempts, we can find the cases presented in Fig. 4.112. Check whether there are any other cases, and if you find one, please, let us know.

PROBLEM 54. For every pair of circles in each of the cases in Fig. 4.112, determine what relation they have according to the cases in Fig. 4.109.

For example, in Fig. 4.112a_2, we have two pairs of non-intersecting circles lying outside each other (case Fig. 4.109a) and a third pair of non-intersecting circles lying inside each other (case Fig. 4.109b). We can denote this as the sequence N_o, N_o, N_i for short.

For the case in Fig. 4.112b_3, we have I, N_i, N_i, where I means that this pair of circles intersect. Determine the relations for all other cases.

Remark 10. In Sections 16, 17, and 18, we analyzed the relative positions of a circle with respect to points, lines, and other circles. We have not distinguished their other characteristics or measurements, such as size, area, or distance between them. The famous mathematician Leonhard Euler[44] called such a field of mathematics *analysis situs*. This is the study of figures and their relative positions without taking into account any of their quantitative characteristics. It served as an origin for the development of modern topology, which at the beginning of the 20th century was still called analysis situs.

[44] Leonhard Euler (1707–1783) was born in Switzerland and worked mostly in St. Petersburg and Berlin. He made enormous contributions to many fields of mathematics, and is considered one the greatest mathematicians of all time.

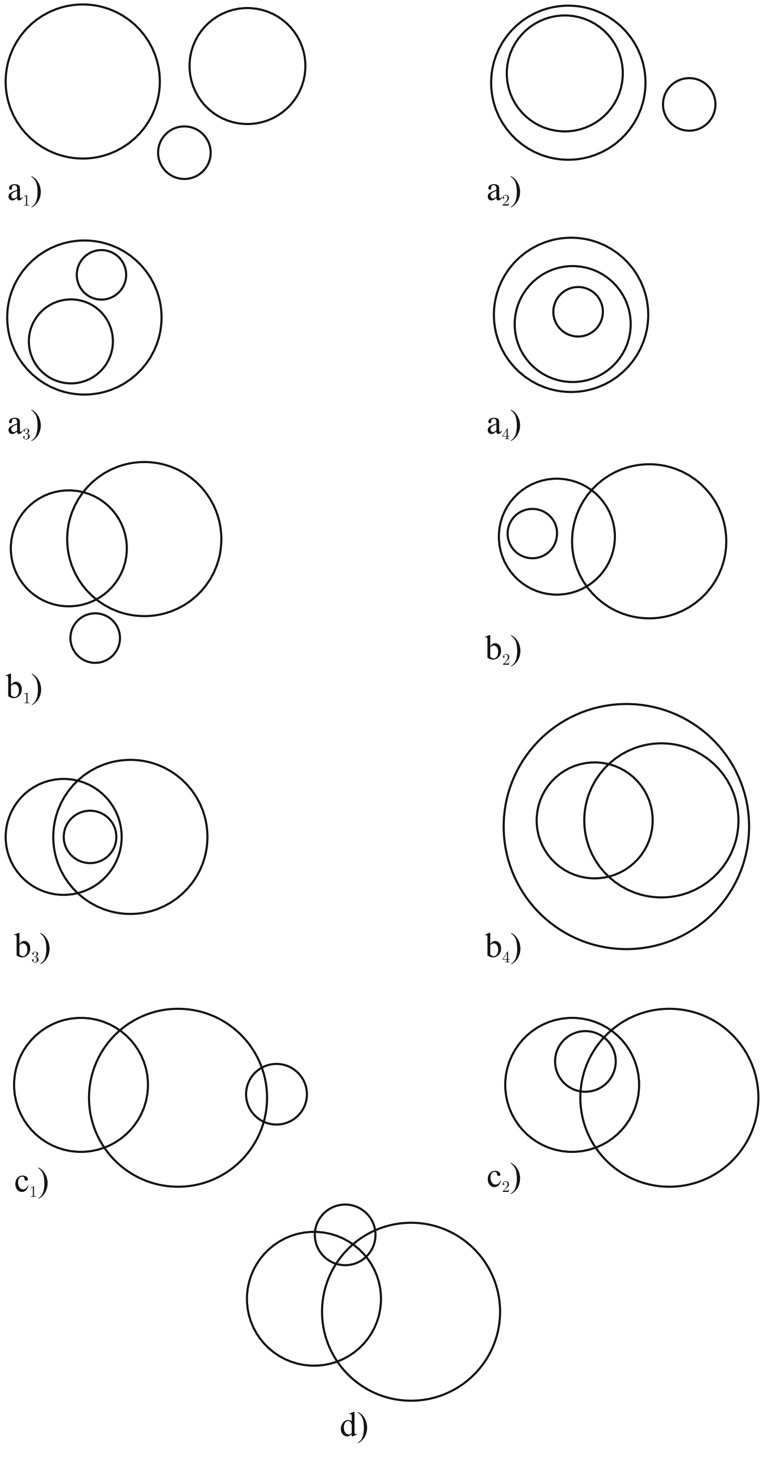


Fig. 4.112

19 Circles and angles

In this section, we consider relations between a circle and an angle. We are interested in angles formed by two intersecting lines which also intersect a circle. These lines intercept one or two arcs on this circle. What is the relative measure of the angle and these arcs?

In general a circle and an angle formed by two intersecting lines can be in one of the following positions (shown in Fig. 4.113a, b, c), where

(a) the vertex of the angle lies on the circle (Fig. 4.113a);

(b) the vertex of the angle lies inside the circle (Fig. 4.113b);

(c) the vertex of the angle lies outside the circle (Fig. 4.113c).

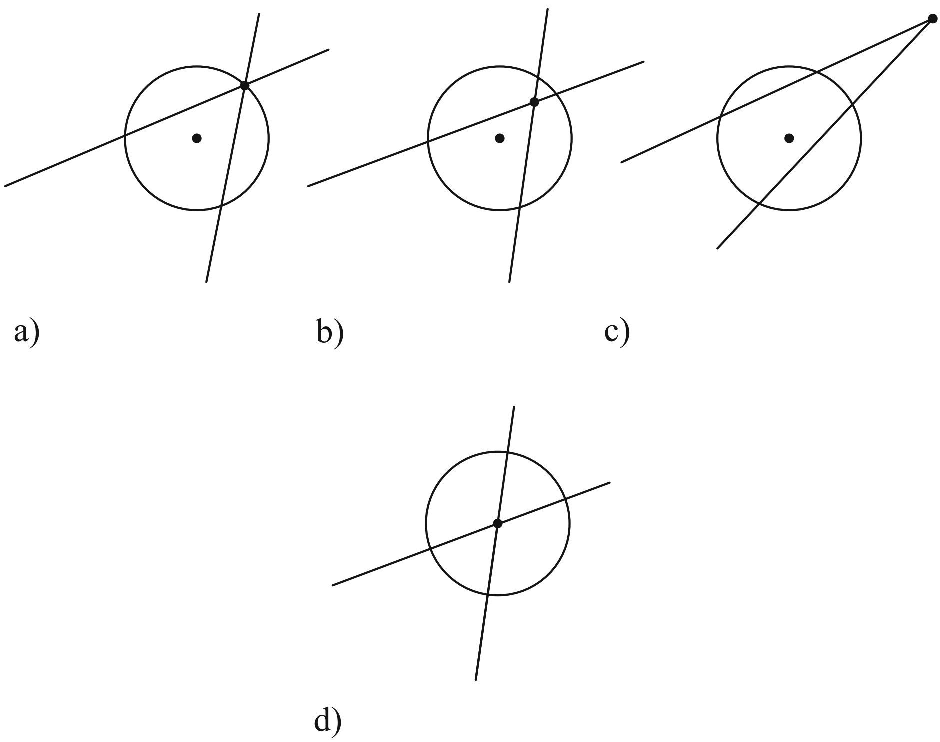


Fig. 4.113

In Section 3, we already studied one relation between an angle and a circle, which is a particular instance of case (b) (see Fig. 4.113d), namely the central angle. A *central angle* is an angle with its vertex at the center of a circle.

Case (a) will be considered in Section 19.1. In Sections 19.2 and 19.3 we reduce cases (b) and (c) to case (a). Theorem 14 summarizes all the cases. Some interesting particular cases will also be considered at the end of Section 19.3.

We remind the reader (see Section 3) that a central angle can be measured in two different ways:

(a) By comparing it with a fixed central angle of 1°. A central angle of 1° is one 360th of the whole circle. Because of the one-to-one correspondence between central angles and their intercepted arcs on a given circle, we can also introduce arc degree measure. Angle degree measure and arc degree measure do not depend on the radius of the circle that is divided into 360 equal parts.

(b) By measuring the length of the arc which this central angle intercepts on a unit circle. This is radian angle measure (see Section 22.4).

From the above, we get the following:

Proposition 31. A central angle is measured by the arc which it intercepts on the circle.

What can we say about measuring non-central angles?

19.1 Inscribed angles

An angle is called *inscribed* in a circle if its vertex lies on the circle. An example of an inscribed angle is shown in Fig. 4.113a.

Proposition 32. An angle inscribed in a circle is measured by half of the arc intercepted on the circle.

Proof. First, we consider the case where an inscribed angle α contains the center O of the circle (as in Fig. 4.114a). Let A, B, and C be the intersection points of the angle α with the circle.

We need to prove that $\angle\alpha = \frac{1}{2}\overset{\frown}{BC}$, where $\overset{\frown}{BC}$ is a notation for arc BC. How can we measure $\overset{\frown}{BC}$, which is marked by a bold line in Fig. 4.114a? Right now, all we know is the relation between a central angle and its arc. Note that the angle α defines a central angle (see Fig. 4.114b)—the angle BOC or $\angle\beta$, which is measured by $\overset{\frown}{BC}$ due to Proposition 31. Therefore, we need to prove that $\angle\alpha = \frac{1}{2}\angle\beta$.

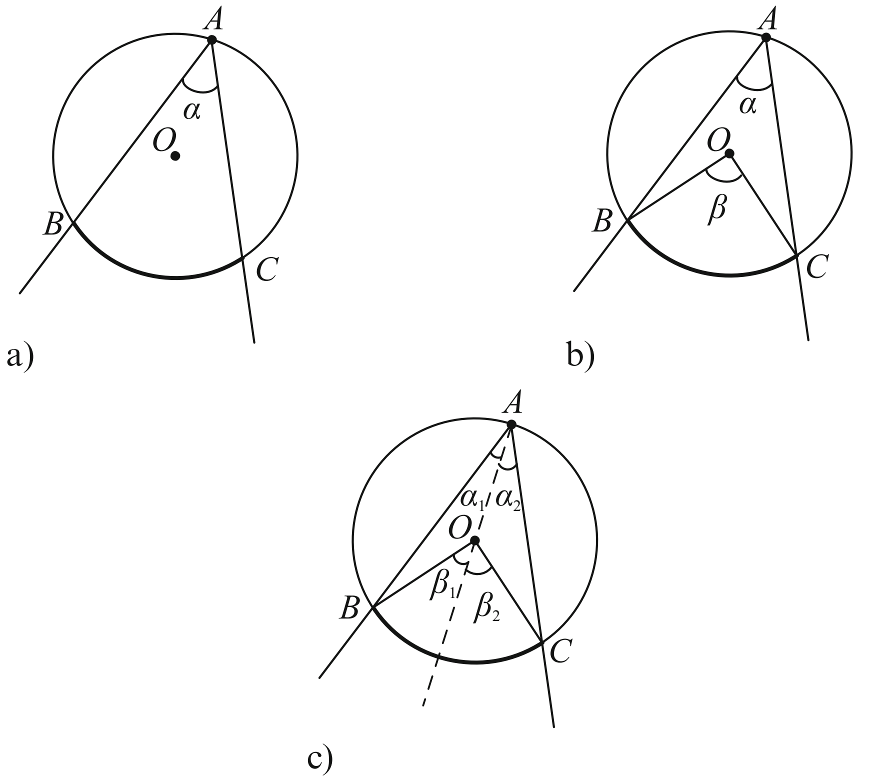


Fig. 4.114

Let us draw a ray passing through points A and O, and denote the angles as in Fig. 4.114c. The triangle AOB is isosceles, since AO and BO are radii. Then we have $\angle ABO = \angle BAO = \alpha_1$. Angle β_1 is an exterior angle of $\triangle ABO$. Therefore, $\beta_1 = \alpha_1 + \alpha_1 = 2\alpha_1$.

Similarly, from isosceles triangle AOC we obtain $\beta_2 = \alpha_2 + \alpha_2 = 2\alpha_2$. Since $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ we obtain $\beta = 2\alpha$ or

$$\alpha = \frac{1}{2}\beta.$$

Thus, for the case in Fig. 4.114 we have proved that the inscribed angle α is measured by half of the arc BC that it intercepts. This proof can also be applied to an inscribed angle one of whose rays passes through the center O of the circle (as in Fig. 4.115a).

The only other possible case is when the domain between the rays of an inscribed angle does not contain the center O (such as the angle α in

Fig. 4.115b). However, as we can easily see from Fig. 4.115c, $\alpha+\beta=\frac{1}{2}\overset{\frown}{BD}$ and $\beta=\frac{1}{2}\overset{\frown}{CD}$. Therefore, $\alpha=\frac{1}{2}\overset{\frown}{BD}-\frac{1}{2}\overset{\frown}{CD}=\frac{1}{2}\overset{\frown}{BC}$. □

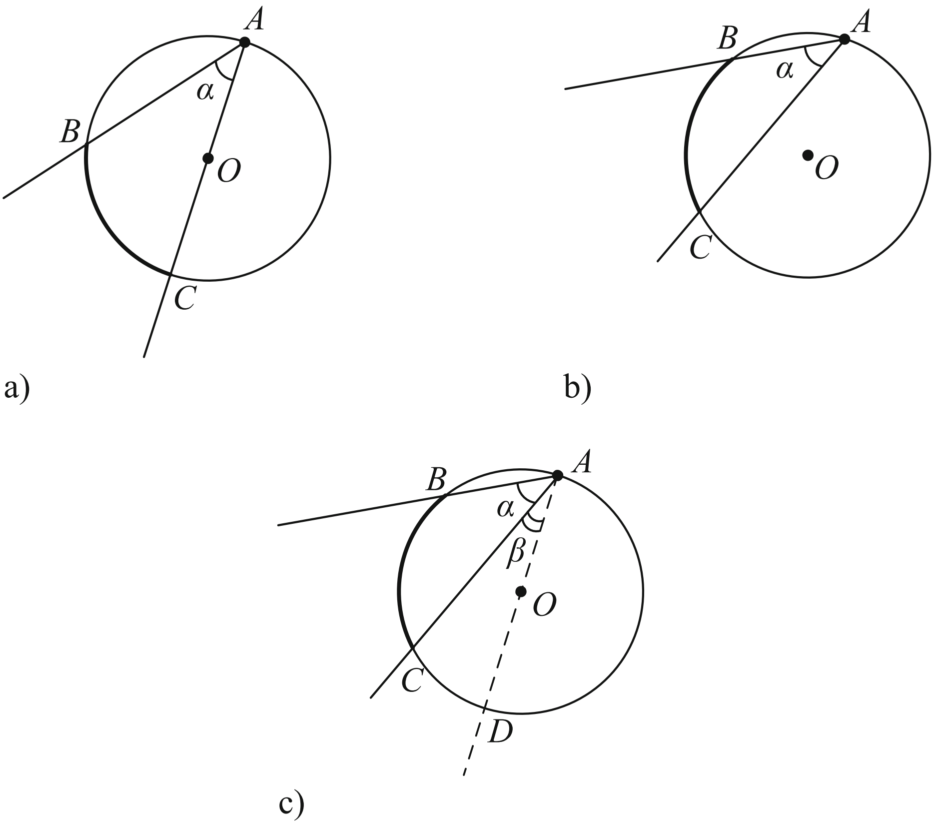


Fig. 4.115

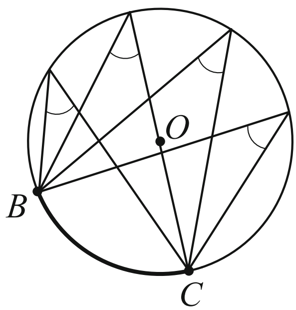


Fig. 4.116

Corollary 5. In a circle all inscribed angles which intercept the same arc on the circle are equal to each other.

This statement is illustrated in Fig. 4.116.

Consider again an inscribed angle α with the vertex A. Let the rays of the angle intersect the circle at some points B and C (see Fig. 4.117a). Segment BC is a chord of this circle. We will refer to this chord as the chord corresponding to the inscribed angle α.

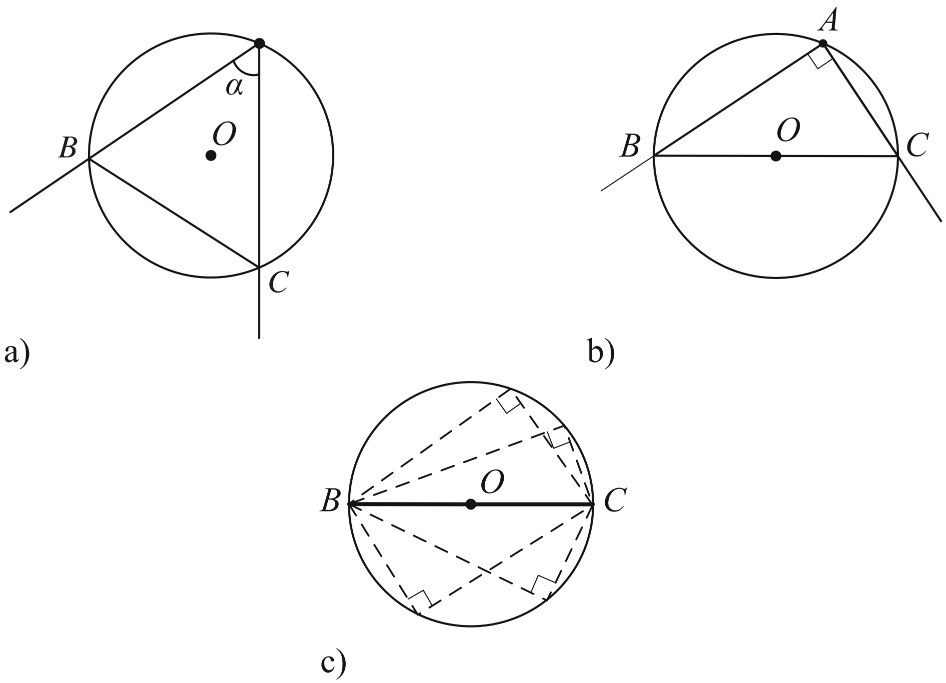


Fig. 4.117

Proposition 33. If an angle α is inscribed in a circle and the chord corresponding to this angle is a diameter of the circle, then $\angle\alpha$ is a right angle.

Proof. In Fig. 4.117b, angle $\angle BAC$ is an inscribed angle whose corresponding chord is a diameter. As we know, $\angle BAC = \frac{1}{2}\overset{\frown}{BC}$. Since BC is a diameter, we have $\overset{\frown}{BC} = 180^\circ$. Therefore, $\alpha = 90^\circ$. $\square$

As a result of this proposition, all the angles in Fig. 4.117c are right angles.

Exercise 26. Is this statement true: "All inscribed angles whose corresponding chords coincide are equal to each other"? Compare with Corollary 5.

Solution. At first glance, it seems that the answer must be "yes." Indeed, in Fig. 4.116 all the inscribed angles have the same corresponding chord BC (see also Fig. 4.118a). All these angles are equal.

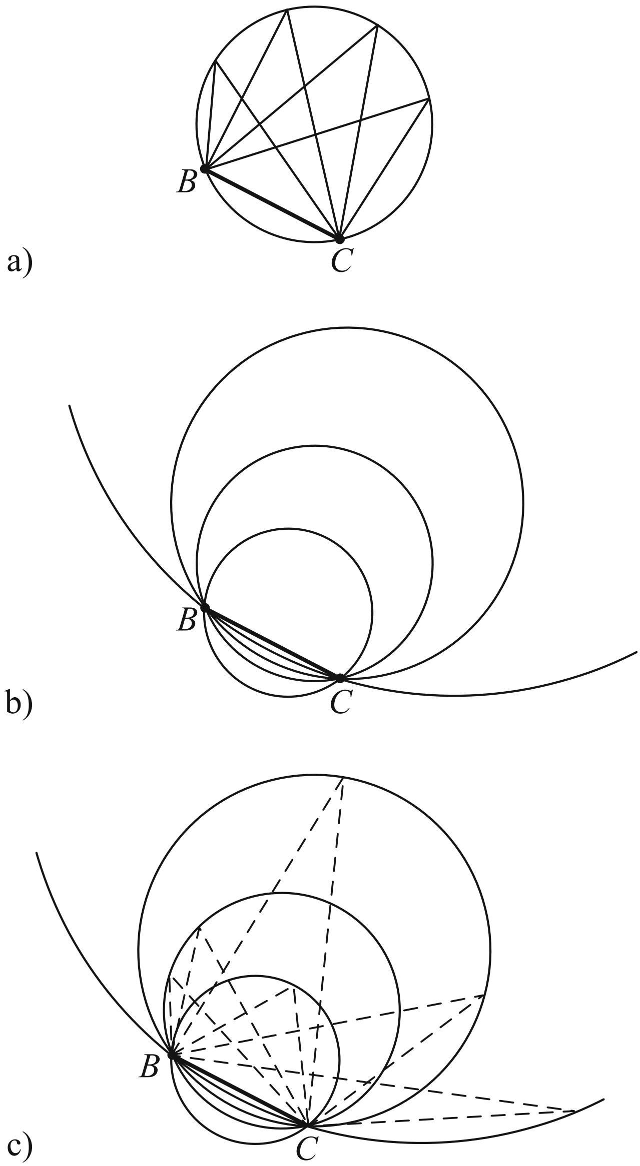


Fig. 4.118

However, as we can see from Fig. 4.118b, there are an infinite number of circles which have the same chord BC. Compare this figure with Fig. 4.100 in Section 16. Therefore, we can draw inscribed angles with the corresponding chord BC in all these circles (see Fig. 4.118c). Clearly, not all of these inscribed angles will be equal to each other. Only the angles inscribed in each particular circle will be equal to each other.

The "trick" is that in Proposition 33 and Corollary 5, there is no need to specify a circle since it is determined uniquely either by a diameter or by an arc. A segment that is a chord in a circle, however, does not uniquely define this circle.

PROBLEM 55. In Fig. 4.119, prove that $\angle\alpha = 2\angle\beta$.

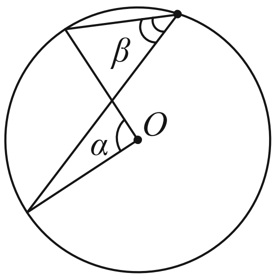
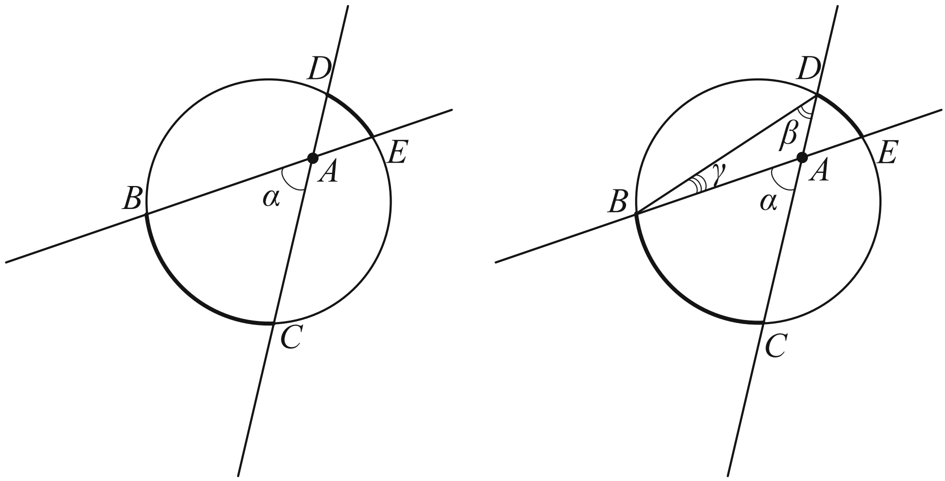


Fig. 4.119



a) b)

Fig. 4.120

19.2 An angle with its vertex inside a circle

Let us consider the case in Fig. 4.113b (see also Fig. 4.120a).

Proposition 34. An angle with its vertex inside a circle is measured by half of the sum of the arcs that this angle intercepts on the circle.

For the angle α in Fig. 4.120a, this means that $\angle\alpha = \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DE})$.

Proof. Let us connect points B and D (see Fig. 4.120b) and denote angles ADB and ABD by β and γ respectively. Then $\angle\beta = \frac{1}{2}\,\overset{\frown}{BC}$ and $\gamma = \frac{1}{2}\,\overset{\frown}{DE}$. The angle α is an exterior angle of triangle ABD, and therefore $\alpha = \beta + \gamma$. Thus, $\angle\alpha = \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DE})$. $\square$

PROBLEM 56. Apply Proposition 34 to the particular case where the vertex A of angle α coincides with the center of the circle. Check whether your answer agrees with Proposition 31.

19.3 An angle with its vertex outside a circle

Consider the case in Fig. 4.113c (see also Fig. 4.121a).

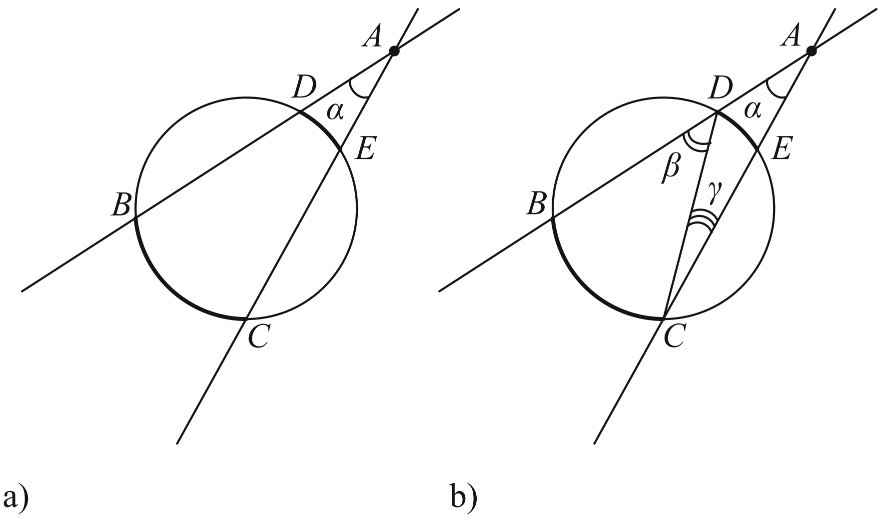


Fig. 4.121

Proposition 35. An angle with its vertex outside a circle is measured by half of the difference between the arcs that this angle intercepts on the circle.

For the angle α in Fig. 4.121a, this means that $\alpha = \frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{DE})$.

Proof. Let us connect points D and C (see Fig. 4.121b) and denote angles BDC and DCE respectively by β and γ. As we already know, $\angle\beta = \frac{1}{2}\,\overset{\frown}{BC}$ and $\angle\gamma = \frac{1}{2}\,\overset{\frown}{DE}$. Note that angle β is an exterior angle of triangle ADC. Therefore, $\beta = \alpha + \gamma$, or $\alpha = \beta - \gamma$. Thus, $\angle\alpha = \frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{DE})$. $\square$

The following theorem summarizes the three cases, together with the particular instance of a central angle, illustrated in Fig. 4.113a, b, c, d.

Theorem 14. An angle whose two rays intersect a circle is measured in terms of the arcs that it intercepts on the circle.

(a) If an angle has its vertex on a circle (inscribed angle), it is measured by half of the intercepted arc.

(b) If an angle has its vertex inside a circle, it is measured by half of the sum of the arcs it intercepts on the circle. (A central angle, however, may equivalently be measured by the single arc it intercepts.)

(c) If an angle has its vertex outside a circle, it is measured by half of the difference between the arcs that it intercepts on the circle.

(d) A central angle is measured by the arc it intercepts on the circle.

PROBLEM 57. Fig. 4.122 shows two points B and C on a circle. We connect them in several different ways by a two-segment polygonal line. Each of the obtained angles intercepts the same arc BC on the circle.

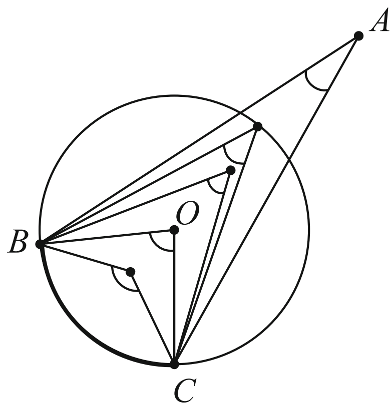


Fig. 4.122

How do the angles in Fig. 4.122 change as the vertex of these angles moves from inside to outside of the circle?

Theorem 14 allows us to prove the following statement (compare it with Proposition 33).

Proposition 36. Consider a right triangle inscribed[45] in a circle. The center of this circle must lie at the midpoint of the hypotenuse of this triangle.

Proof. Consider a right triangle ABC. Let us draw a circle with its center O at the midpoint of hypotenuse BC and with radius $BO = OC$ (see Fig. 4.123a). If point A happens to lie on the circumference (as in Fig. 4.123a), then Proposition 36 is true.

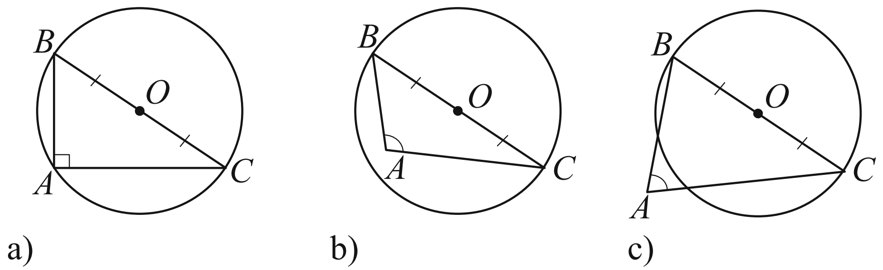


Fig. 4.123

However, suppose that the point A happens to lie inside the circle as in Fig. 4.123b. Then, from Theorem 14, angle BAC would be bigger than $90°$.

Similarly, if point A lies outside the circle (as in Fig. 4.123c), then angle BAC is smaller than $90°$. Therefore, point A must lie on the circumference (as in Fig. 4.123a). □

PROBLEM 58. A circle is given but its center is not marked. Find the center of this circle.

Theorem 14 also has important applications in particular cases which are sometimes called the *extreme positions of a circle and an angle*.

Extreme positions of a circle and an angle

We have considered the three general positions of a circle and an angle, presented in Fig. 4.113a, b, c. Let us consider some interesting particular cases.

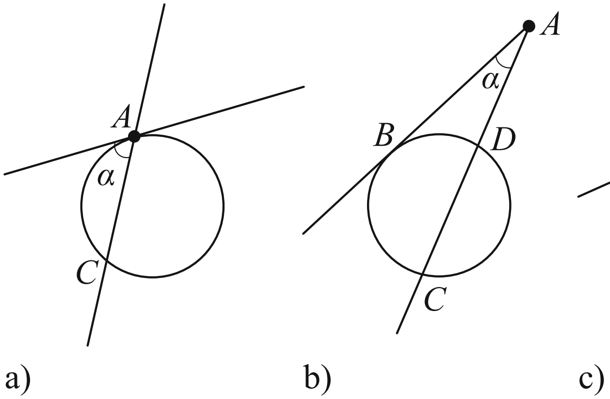


Fig. 4.124

Let one or two rays of the angle α become tangent to the circle (see Fig. 4.124a, b, c).

Proposition 37.

(a) When the vertex of an angle α lies on the circle (i.e., $\angle\alpha$ is an inscribed angle) and one of its rays is tangent to the circle, the angle α is measured by half of the arc which it intercepts. In Fig. 4.124a, $\angle\alpha = \frac{1}{2}\overset{\frown}{AC}$.

(b) When the vertex of an angle α lies outside the circle and one of its rays is tangent to the circle, the angle α is measured by half of the difference of the arcs it intercepts. In Fig. 4.124b, $\angle\alpha = \frac{1}{2}(\overset{\frown}{BC} - \overset{\frown}{BD})$.

(c) When the vertex of an angle α lies outside the circle and both rays of this angle are tangent to the circle, the angle α is measured by half of the difference between the arcs it intercepts. In Fig. 4.124c, $\angle\alpha = \frac{1}{2}(\overset{\frown}{\mathbf{BC}} - \overset{\frown}{BC})$, where $\overset{\frown}{\mathbf{BC}}$ is the arc on the circle marked by a bold line while $\overset{\frown}{BC}$ is the other arc between the points B and C.

Proof.

(a) First suppose that ray a is not tangent but is "almost" tangent as in Fig. 4.125a.

[45] See Definition 7 in Section 20.1.

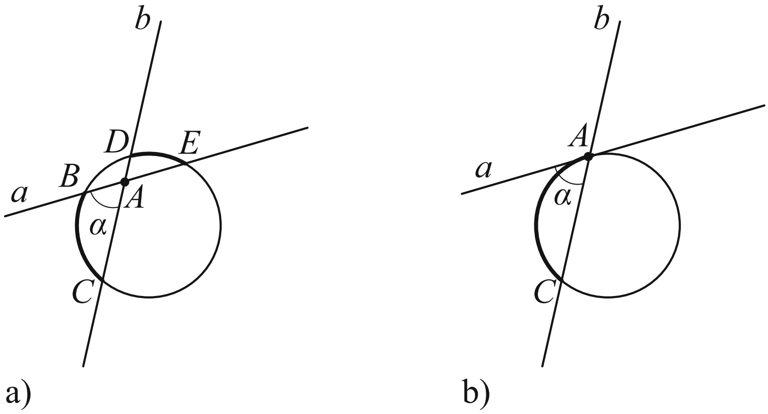


Fig. 4.125

In this case we know that $\alpha = \frac{1}{2}(\overset{\frown}{BC} + \overset{\frown}{DE})$. If point A moves towards the circumference, points B, D, and E move closer together and finally all coincide with the point A. Therefore, for the extreme position when the ray a becomes tangent, only $\overset{\frown}{AC}$ remains (see Fig. 4.125b). We obtain $\angle\alpha = \frac{1}{2}\overset{\frown}{AC}$.

Cases (b) and (c) are considered similarly. Prove them yourself (see Problem 64). □

For Proposition 37 in case (a), one ray of the angle is tangent to the circle and the second ray is a chord. If this chord passes through the center of the circle and becomes its diameter we obtain an interesting particular case (see Fig. 4.126).

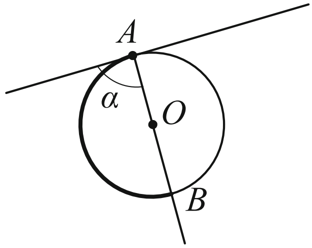


Fig. 4.126

Proposition 38. A straight line tangent to a circle at a point A is perpendicular to the diameter which passes through point A.

Proof. Indeed, from Proposition 37 (case (a)), we have $\angle\alpha = \frac{1}{2}\overset{\frown}{AB}$ (see Fig. 4.126). Since AB is a diameter, $\overset{\frown}{AB} = 180^\circ$. Therefore, $\alpha = 90^\circ$. □

19.4 An angle which a segment subtends

There is an important application of the previous results of this section.

Definition 6. Suppose a point A and a segment BC are given. We say that the *segment BC subtends angle* α at point A if the measure of angle $\angle BAC = \alpha$ (see Fig. 4.127a, b).

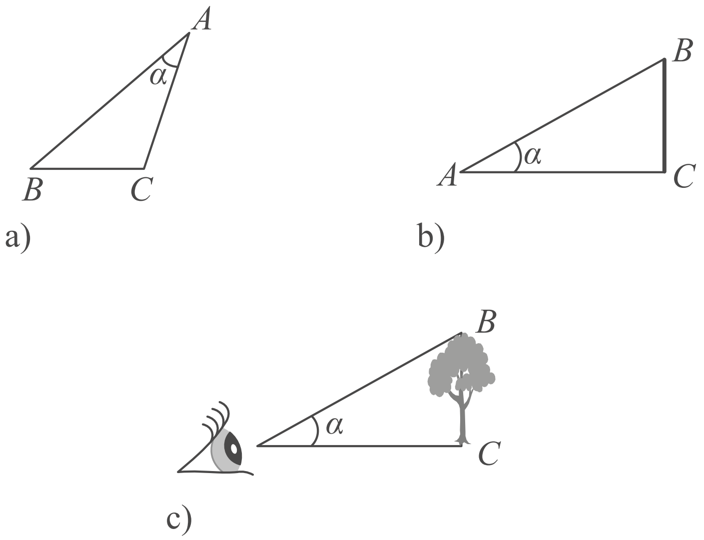


Fig. 4.127

This definition becomes clearer if one imagines looking at an object from the point A (see Fig. 4.127c).

PROBLEM 59.

(a) What angle does segment BC subtend at point A in Fig. 4.128a?

(b) What angle does segment AC subtend at point B in Fig. 4.128a?

PROBLEM 60.

(a) What angle does segment BC subtend at point A in Fig. 4.128b?

(b) What angle does segment AC subtend at point B in Fig. 4.128b?

(c) What angle does segment AB subtend at point C in Fig. 4.128b?

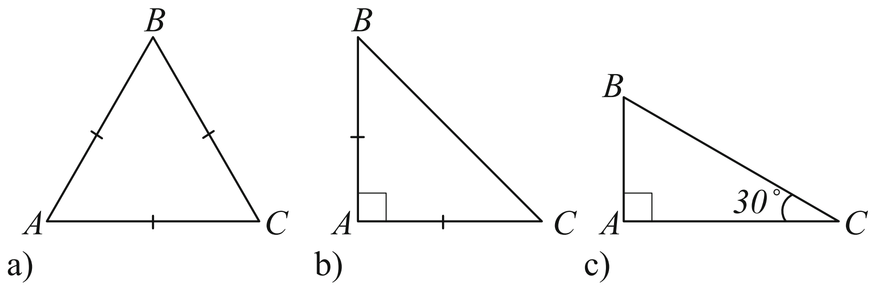


Fig. 4.128

PROBLEM 61.

(a) What angle does segment BC subtend at point A in Fig. 4.128c?

(b) What angle does segment AC subtend at point B in Fig. 4.128c?

(c) What angle does segment AB subtend at point C in Fig. 4.128c?

The following statement follows from Proposition 36.

Corollary 6. Let BC be a diameter of a circle. At a point A lying inside the circle, segment BC subtends an angle larger than $90°$. At a point A lying outside the circle, the segment AB subtends an angle smaller than $90°$. The segment BC subtends an angle $90°$ only at a point A lying on the circumference.

Exercise 27. Find all the points A at which segment BC subtends an angle of $90°$.

Solution. Fig. 4.129a shows a segment BC. From Proposition 36, any point A lying on the circle with diameter BC and center O at the midpoint of BC is such a point. On the other hand, only at a point lying on this circumference

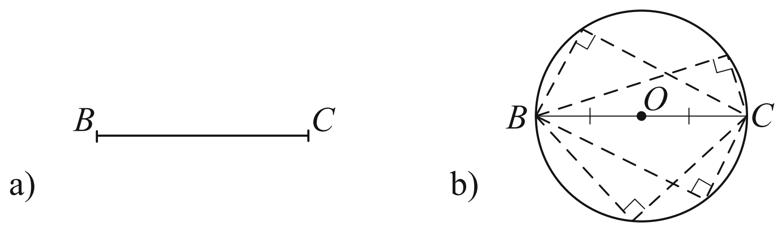


Fig. 4.129

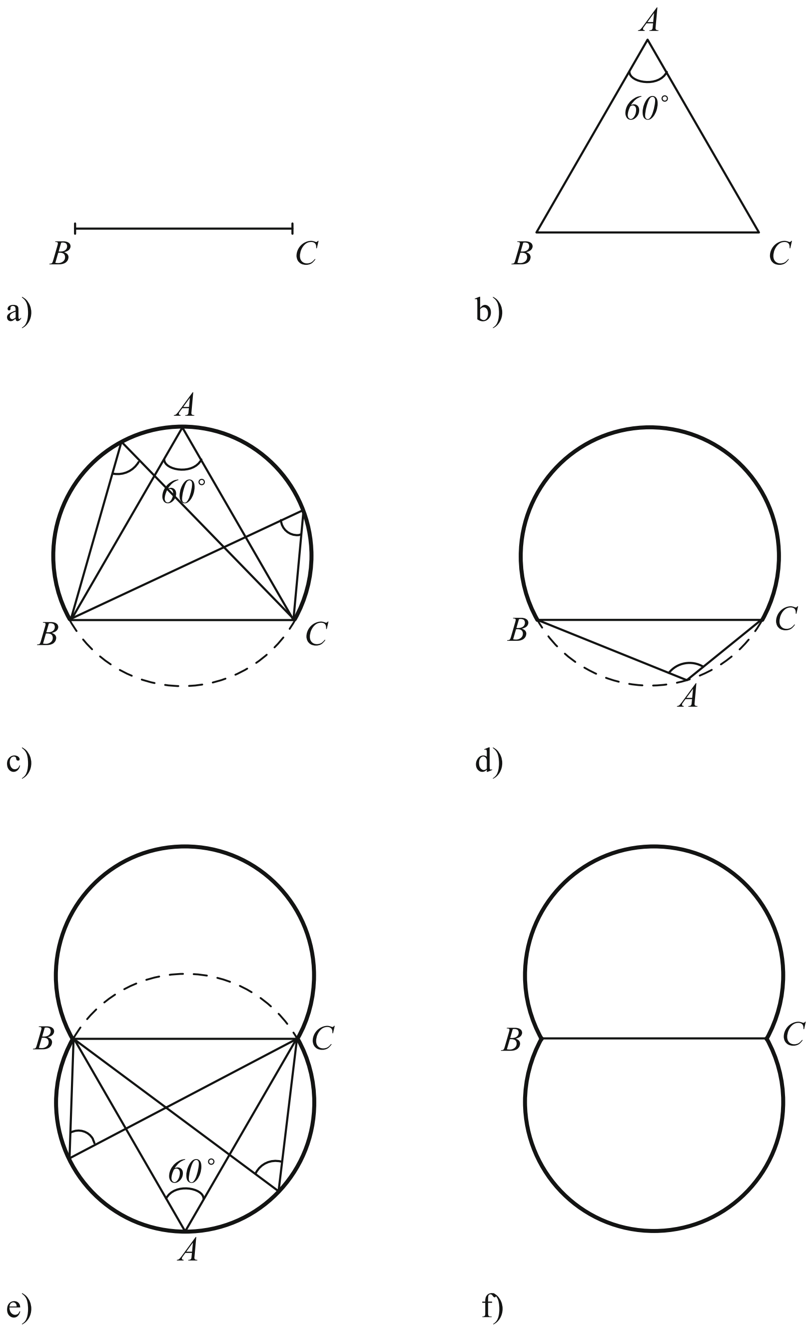


Fig. 4.130

does the segment BC subtend an angle 90°. Thus, the set of all points (or the locus of points) at which a segment BC subtends an angle 90° is a circle (see Fig. 4.129b).

Exercise 28. Find all points at which segment AB subtends an angle of 60°.

Solution. Consider a segment BC (see Fig. 4.130a). We need to find all the points A such that the angle $\angle BAC = 60^\circ$. One such point is easy to guess. Indeed, in an equilateral triangle ABC we have $\angle BAC = 60^\circ$ (see Fig. 4.130b).

Let us use Corollary 5. We need to draw a circle such that the angle BAC will be inscribed in this circle. This means that we need to draw a circle passing through points A, B, and C, as we have done in Section 16.3. Any angle inscribed in this circle and having the same intercepted $\overset{\frown}{BC}$ is equal to 60° (see Fig. 4.130c). We marked in bold all points[46] on this circle at which segment BC subtends an angle of 60°.

Have we obtained all the points at which segment BC subtends 60°?

Let us recall that we chose a point A above the segment BC. But we could have as well chosen it below BC. In this case we will obtain in a similar way the points marked in bold in Fig. 4.130e. Thus, the locus of points, i.e., all the points, at which segment BC subtends a 60° angle is presented in Fig. 4.130f.

Exercise 29.

(a) Find all the points at which a segment BC subtends an angle larger than 60°.

(b) Find all the points at which a segment BC subtends an angle smaller than 60°.

Solution. We already know the locus of points at which a segment BC subtends an angle of 60° (see Fig. 4.130f). Consider a point A lying inside this domain (see Fig. 4.131a).

Let us extend segment BA until it intersects the circle at a point A'. As we know, angle $BA'C = 60^\circ$. The angle BAC is an exterior angle of triangle $AA'C$

[46] Note that even though any point on the lower (i.e., dashed) part of the same circle has the same corresponding chord, an inscribed angle with its vertex at this point will intercept a different arc on the circle. Thus, the angle which BC subtends at such a point A as in Fig. 4.130d is not 60°.

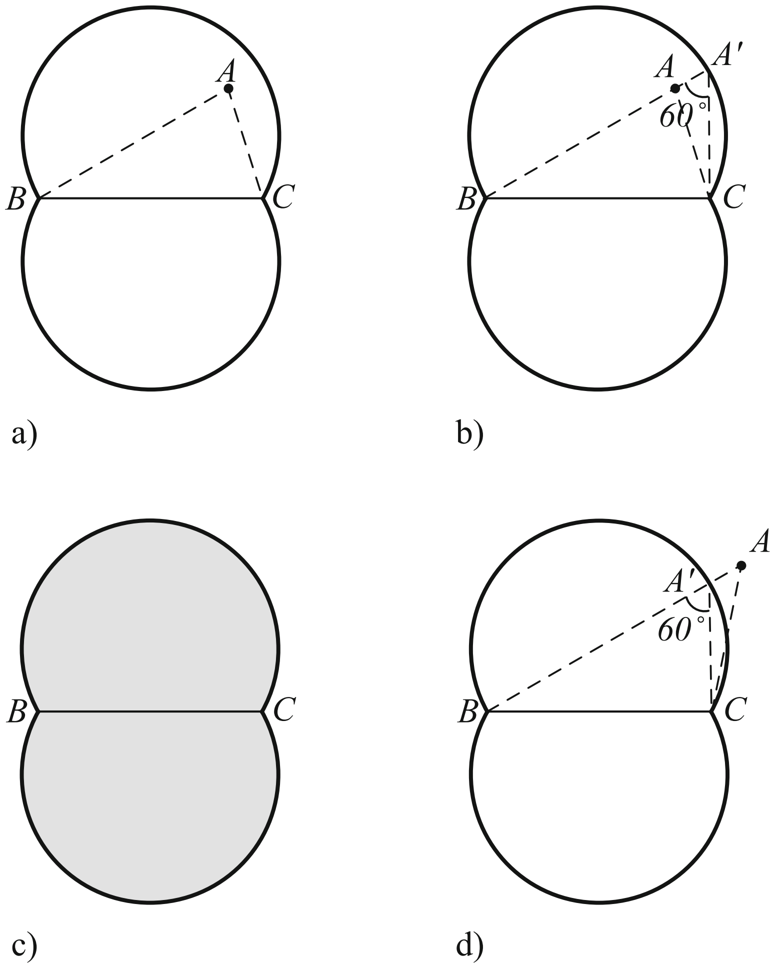


Fig. 4.131

(see Fig. 4.131b). Therefore, it is larger than angle $BA'C$ (see Corollary 1 in Section 7.2). This means that $\angle BAC > 60^\circ$.

Thus, at any point A lying inside the shaded domain in Fig. 4.131c, segment BC subtends an angle larger than 60°.

Similarly, if a point A lies outside the shaded domain as in Fig. 4.131d, angle $BAC < 60^\circ$.

Indeed, let A' be the intersection of the segment AB with the circle again. Since angle $BA'C$ is an exterior angle of triangle $A'AC$, angle BAC is smaller than angle $BA'C = 60^\circ$.

PROBLEM 62. Find all the points at which a segment AB subtends an angle of $30°$.

PROBLEM 63. Find all the points at which a segment AB subtends an angle of $120°$.

PROBLEM 64. Prove statements (b) and (c) of Proposition 37.

Remark 11. In Fig. 4.132 there are two inscribed angles α and β that are supplementary, i.e., $\angle\alpha + \angle\beta = 180°$. Note that the arcs intercepted by these angles form the whole circle, i.e., *arc* $\alpha +$ *arc* $\beta = 360°$.

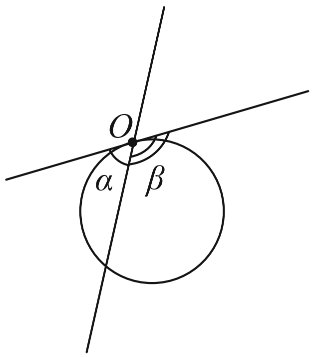


Fig. 4.132

This agrees with statement (a) of Proposition 37.

PROBLEM 65. Find the ratio of arcs α and β in Fig. 4.133.

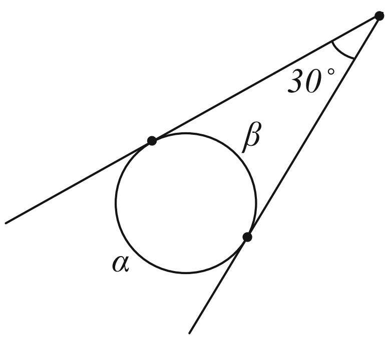
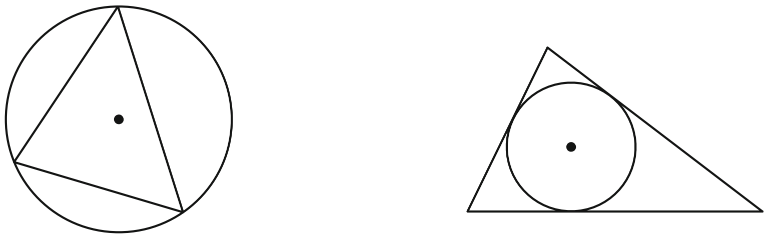


Fig. 4.133

20 A circle and a triangle

20.1 Inscribed and circumscribed triangles

Definition 7. We say that a *triangle is inscribed in a circle* if all the vertices of the triangle lie on this circle. The center of the circle in which a triangle is inscribed is called the *circumcenter* of this triangle.



a) b)

Fig. 4.134

Fig. 4.134a shows a triangle inscribed in a circle. Note that the sides of an inscribed triangle are chords of the circle.

Definition 8. We say that a *triangle is circumscribed around a circle* if each side of the triangle is tangent to this circle.

Fig. 4.134b shows a circumscribed triangle. The sides of a circumscribed triangle are tangent to the circle.

Remark 12. Given a triangle inscribed in a circle, we can also say that this circle is circumscribed around the triangle. Likewise, given a triangle circumscribed around a circle, one can say that the circle is inscribed in the triangle. This is why it is important to understand how these geometric figures look in every case and what their geometric properties are.

Theorem 15. Given a triangle *ABC*, there exists a unique circle in which this triangle can be inscribed. The center of this circle lies at the intersection of the three perpendicular bisectors of the triangle *ABC*.

Let us first prove the following lemma.

Lemma 2. The perpendicular bisector to a chord of a circle passes through the center of this circle.

Proof. Let AB be a chord in a circle. As we know (see Section 11.3), any point equidistant from points A and B must lie on the perpendicular bisector of segment AB.

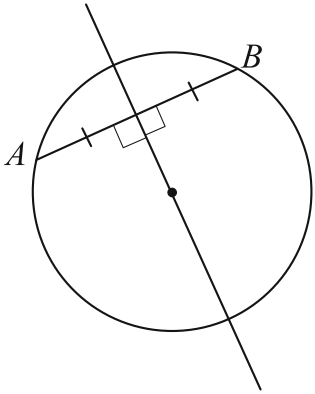


Fig. 4.135

Since the center O of the circle is equidistant from points A and B, it must lie on the perpendicular bisector of AB (see Fig. 4.135). □

Proof of Theorem 15. From Lemma 2, the center of the circle in which a triangle is inscribed must lie on the perpendicular bisectors to the sides of this triangle. According to Theorem 10 (see Fig. 4.71), such a point exists and is unique. □

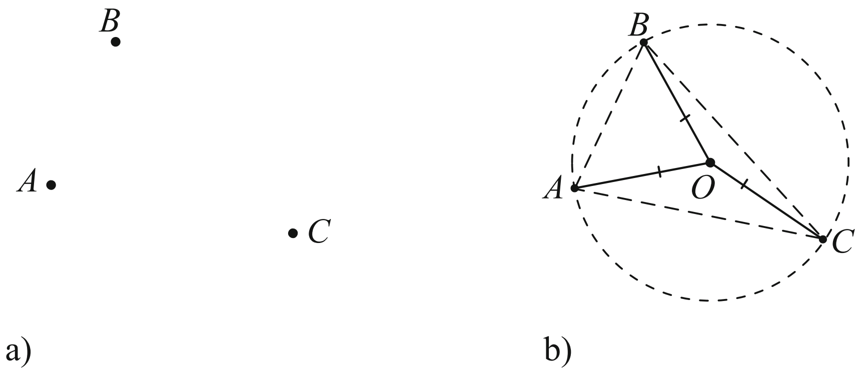


Fig. 4.136

Theorem 15 may be interpreted as in the following exercise.

Exercise 30. Imagine that there are three houses (see points A, B, and C in Fig. 4.136a). Where should one build a shop so that the distance from each house to the shop will be the same?

Solution. The shop should be located at the center O of the circle in which the triangle ABC is inscribed (see Fig. 4.136b). In this case the segments AO, BO, CO are radii and, therefore, are equal.

Theorem 16. Given a triangle ABC, there exists a unique circle around which this triangle can be circumscribed. The center of this circle lies at the intersection of the three angle bisectors of the triangle ABC.

Proof. Each side of the triangle must be tangent to the circle inscribed in this triangle. Therefore, the center of this circle must be equidistant from each side of the triangle. From Theorem 9 and Corollary 4, such a point exists and is unique. This point is the intersection point of the angle bisectors of the triangle. □

Theorem 16 may be interpreted as in the following exercise.

Exercise 31. Imagine that there are three roads (see lines a, b, and c in Fig. 4.137a). Where should one build a gasoline station so that the distance from each road to the gasoline station will be the same?

Solution. Three intersecting lines form a triangle. The gasoline station should be located at the center O of the circle inscribed in this triangle (see Fig. 4.137b). In this case the distances from point O to each line (see Proposition 38) are radii of this circle and are, therefore, equal.

Note that there are also other places where a gasoline station would be equally far from each road. These are the three points O_1, O_2, O_3 in Fig. 4.137c.[47] The circles with centers O_1, O_2, O_3 are sometimes called *escribed circles* of a triangle.

Thus, formally speaking, the solution consists of the four points presented in Fig. 4.137b and Fig. 4.137c. However, point O has the shortest distance from each road, and using common sense, we should choose it as the answer for this exercise.

[47] We had to shrink the triangle formed by the lines a, b, and c to fit this figure on the paper.

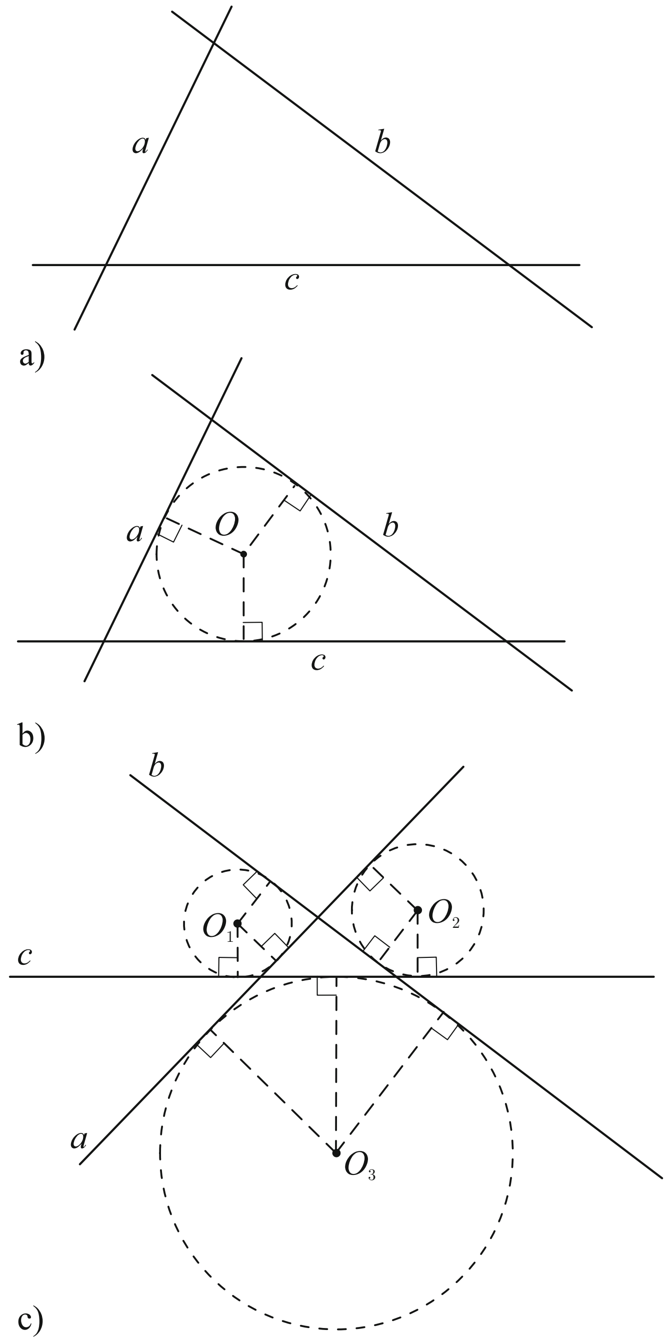


Fig. 4.137

The following figure summarizes this section:

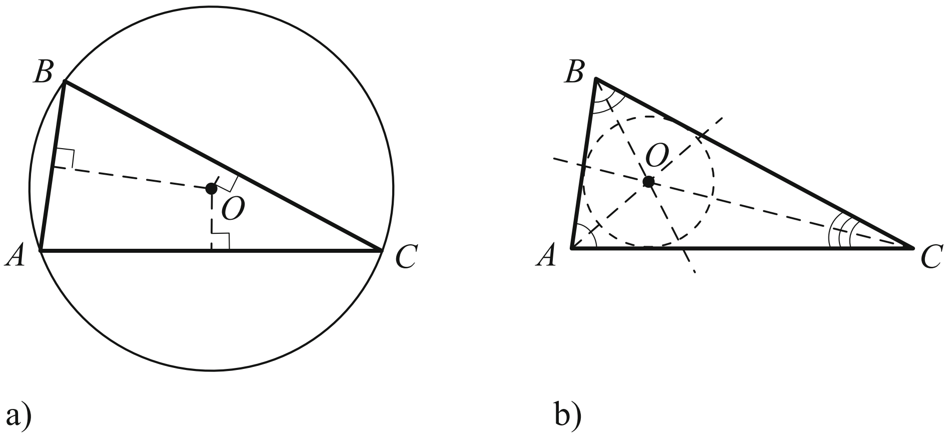


Fig. 4.138

In Fig. 4.138a triangle *ABC* is inscribed in a circle. Its vertices lie on the circle. The center of the circle lies at the intersection of the three perpendicular bisectors of the triangle *ABC*.

In Fig. 4.138b triangle *ABC* is circumscribed around a circle. Its sides are tangent to this circle. The center of the circle lies at the intersection of the three angle bisectors of the triangle *ABC*.

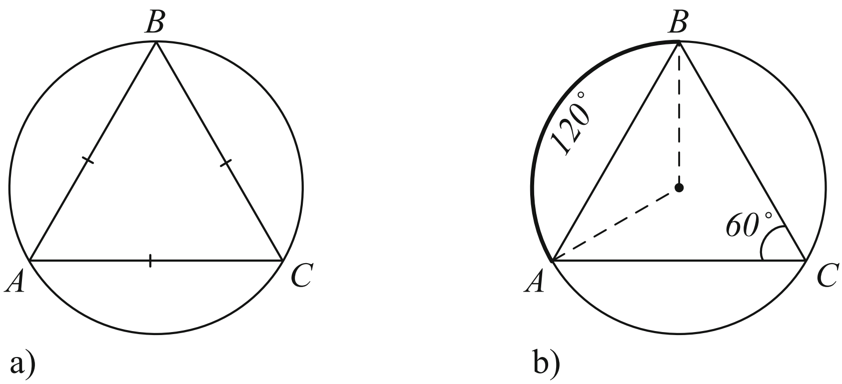


Fig. 4.139

20.2 Some exercises on inscribed and circumscribed triangles

Exercise 32. Consider an equilateral triangle ABC inscribed in a circle (see Fig. 4.139a). What is the degree measure of the arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{AC}$?

Solution. Since chords AB, BC, and AC are equal to each other, their corresponding arcs are equal. All three arcs together form a circle, so the sum of these arcs is 360°. So each arc is 120° (see Fig. 4.139b). (We recall that each angle of $\triangle ABC$ is 60°.)

Exercise 33.

(a) Find the radius x of a circle inscribed in an equilateral triangle with side a (see Fig. 4.140a).

(b) Fig. 4.140b shows an equilateral triangle with a side a. A circle is inscribed in this triangle. After that, three smaller circles are inscribed in such a way that each of them is tangent to the big circle and to two sides of the triangle. Find the radius x_1 of these smaller circles.

Solution.

(a) Let us connect the center O of the circle with vertex A of the triangle. Let P be the point of tangency of the triangle and the circle (see Fig. 4.140c). Radius OP is perpendicular to AP by Proposition 38.

From Theorem 7 it follows that $AP = \frac{a}{2}$ and that $\angle OAP = 30^\circ$. Then, from Proposition 7, in right triangle AOP we have $AO = 2OP$, i.e., $AO = 2x$.

By applying the Pythagorean theorem to this triangle, we obtain the equation: $x^2 + (\frac{a}{2})^2 = (2x)^2$, or $3x^2 = \frac{a^2}{4}$. Finally,

$$x = \sqrt{\frac{a^2}{3 \cdot 4}} = \frac{a}{2\sqrt{3}}.$$

(b) We need to find the radius x_1 of a little circle. Let us draw a line tangent to both circles as in Fig. 4.140d, where Q is the point of tangency. Triangle ABC is equilateral by symmetry of the figure with respect to line OA, and since angle $BAC = 60^\circ$. Let us denote the side of $\triangle ABC$ by a_1.

Notice that if we knew side a_1, we would be solving the same problem as in (a): find the radius of a circle inscribed in an equilateral triangle with side a_1.

Let us find a_1. From (a) we have $OA = 2x$ and $OQ = x$. Therefore, $AQ = x = \frac{a}{2\sqrt{3}}$.

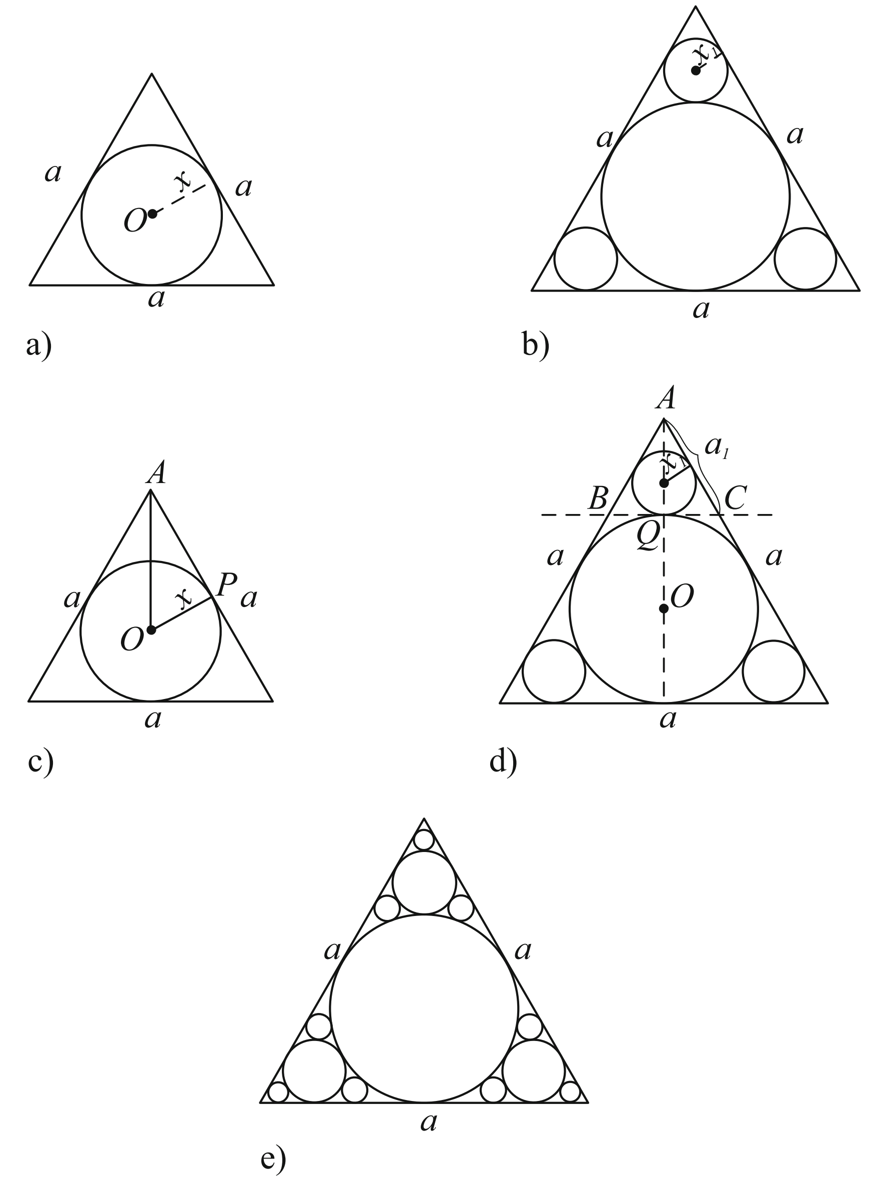


Fig. 4.140

Since AQ is an altitude in $\triangle ABC$, we can use the Pythagorean theorem. We obtain ${a_1}^2 = x^2 + \frac{a_1}{2}^2$, or $\frac{3}{4}{a_1}^2 = x^2$, i.e.,

$$a_1 = \frac{2}{\sqrt{3}}x.$$

We already know x from (a): $x = \frac{a}{2\sqrt{3}}$. Then

$$a_1 = \frac{2}{\sqrt{3}} \cdot \frac{a}{2\sqrt{3}} = \frac{a}{3}.$$

Now in order to find x_1, we can use the solution from (a) for the triangle with side a_1 instead of a. We obtain

$$x_1 = \frac{a_1}{2\sqrt{3}} = \frac{a}{6\sqrt{3}}.$$

We can also see that $x_1 = \frac{1}{3}x$, which makes sense because triangle ABC is similar to the initial triangle with a coefficient of similarity $1:3$.

By repeating this procedure, it is possible to find the radii of all the circles in Fig. 4.140e. This process of inscribing smaller and smaller circles and finding their radii can be repeated infinitely.

20.3 The area of a circumscribed triangle. The area of an inscribed triangle

Consider a triangle ABC with a circle inscribed in it. Let the sides of the triangle a, b, and c and the radius r of the circle be given. What is the area of triangle ABC?

Of course, by Heron's formula we can express the area of triangle ABC even without the radius r. There is a simpler expression, however, if we use the radius r.

We know that the center O of this circle lies at the intersection point of the three angle bisectors of triangle ABC (see Fig. 4.141a). Triangle ABC is divided into three triangles AOB, AOC, and BOC. The area $S_{ABC} = S_{AOB} + S_{AOC} + S_{BOC}$.

In order to find the areas of these triangles, let us draw the altitude from point O in each of them (see Fig. 4.141b). Then $S_{AOB} = \frac{1}{2}cr$, where r is the radius of the circle. Similarly, we find $S_{AOC} = \frac{1}{2}br$ and $S_{BOC} = \frac{1}{2}ar$. Therefore, $S_{ABC} = \frac{1}{2}(ra + rb + rc)$.

Thus, the area of a circumscribed triangle with sides a, b, c and radius r of the circle is

$$S_{\triangle} = \frac{1}{2}(a+b+c)r.$$

This formula may also be written as

$$S_{\triangle} = \frac{1}{2}pr,$$

where $p = a + b + c$ is the perimeter of triangle ABC.

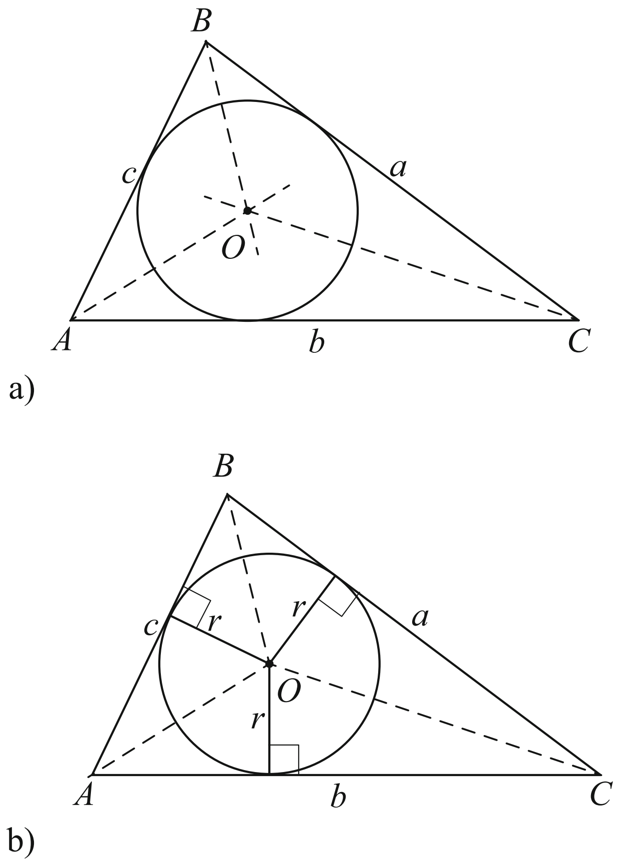


Fig. 4.141

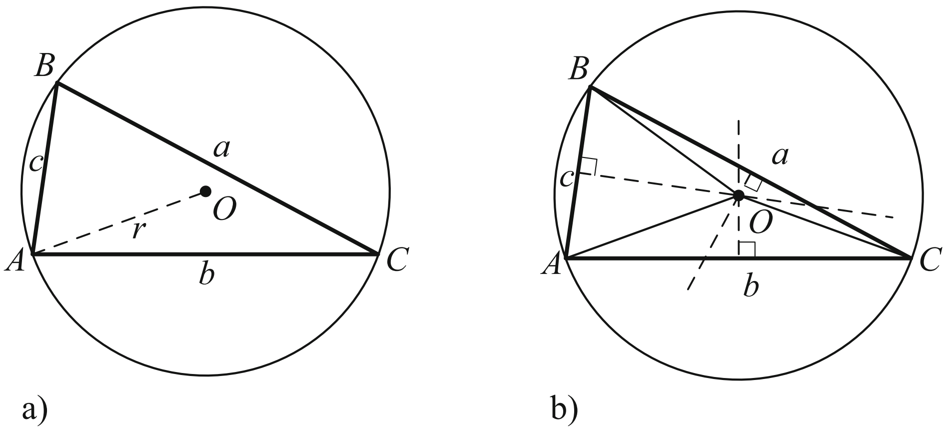


Fig. 4.142

Now consider a triangle ABC inscribed in a circle. Let the sides of the triangle be a, b, c and let the radius r of the circle be given (see Fig. 4.142a). What is the area of triangle ABC?

We know that the center O of this circle lies at the intersection point of the three perpendicular bisectors of triangle ABC. The segments AO, BO, and CO are radii of the circle. They also divide triangle ABC into three triangles (see Fig. 4.142b).

The area of the inscribed triangle ABC can be expressed by the following formula, which we will not prove here:

$$S_{\triangle} = \frac{abc}{4r}.$$

Remark 13. Note that in the formula for the area of a circumscribed triangle, the radius r of the circle is a factor, while in the formula for the area of an inscribed triangle, the radius r of the circle is a denominator. This can be illustrated by the following figures. In the first case, if the vertex B of the triangle ABC is moved closer to segment AC, then the radius becomes smaller

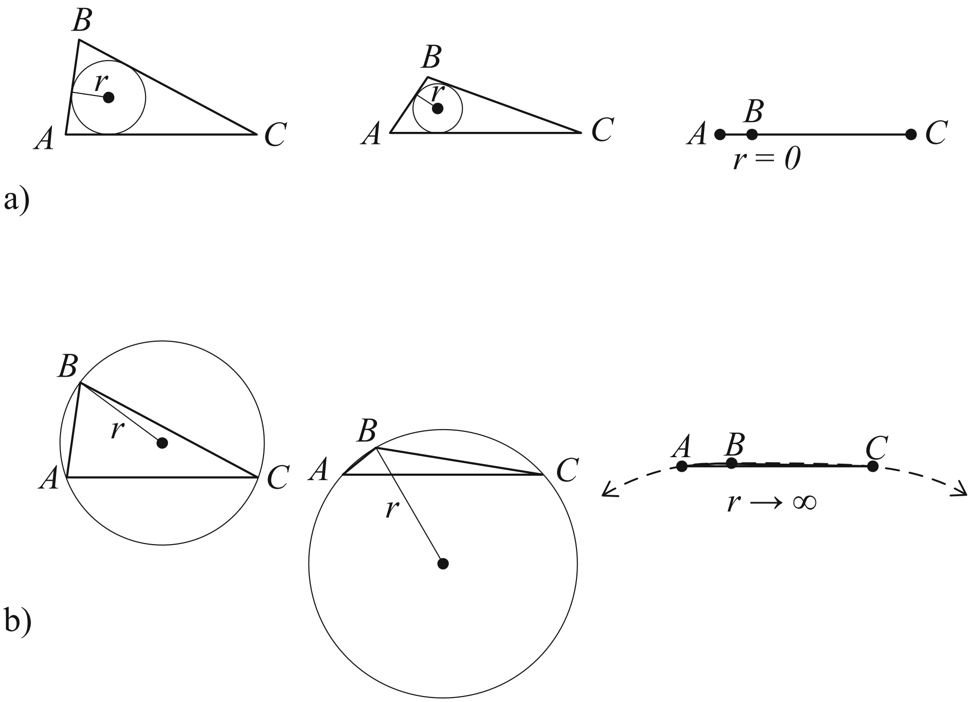


Fig. 4.143

and finally becomes zero (see Fig. 4.143a). This suggests that the area of the circumscribed triangle ABC is proportional to the radius r.

In the second case, if the vertex B is moved closer to the segment AC, the radius becomes bigger and finally becomes infinitely big (see Fig. 4.143b). This means that the area of the inscribed $\triangle ABC$ is inversely proportional to the radius r.

21 Circles and polygons

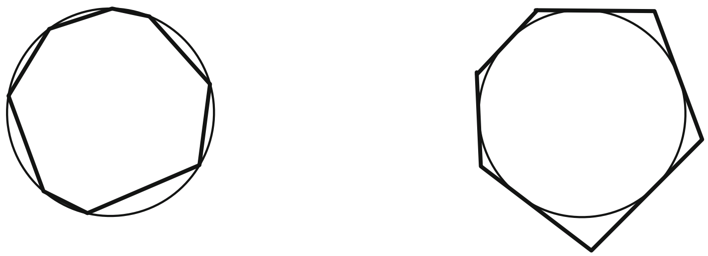
21.1 Inscribed polygons

We can reformulate Definitions 7 and 8 for a convex polygon.

We say that a convex *polygon is inscribed in a circle* if each vertex of this polygon lies on the circle.

We say that a convex *polygon is circumscribed around a circle* if each side of this polygon is tangent to the circle.

Fig. 4.144a shows an irregular octagon inscribed in a circle, and Fig. 4.144b shows an irregular hexagon circumscribed around a circle.



a) b)

Fig. 4.144

Let us consider a quadrilateral. Since three points already define a circle uniquely, it is clear that not every quadrilateral can be inscribed in a circle. Which quadrilaterals can be inscribed in a circle?

Proposition 39. If a quadrilateral is inscribed in a circle, then the sum of its opposite angles is $180°$. And conversely, if the sum of the opposite angles of a quadrilateral is $180°$, then it can be inscribed in a circle.

Proof. Fig. 4.145a shows a quadrilateral $ABCD$ inscribed in a circle. Let us prove that the sum of its opposite angles is $180°$.

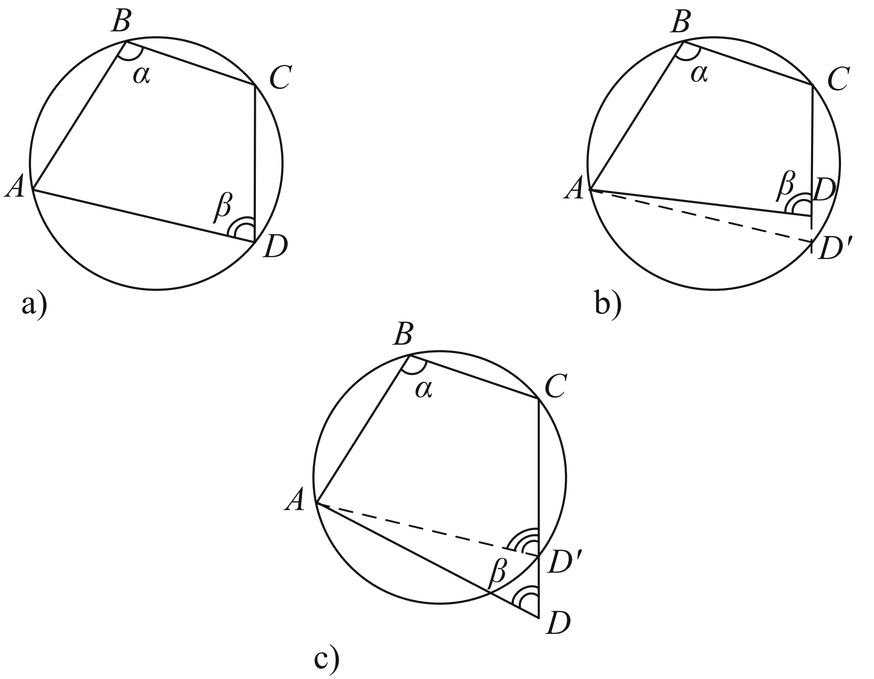


Fig. 4.145

We have $\angle\alpha = \frac{1}{2}\ \overset{\frown}{ADC}$ and $\angle\beta = \frac{1}{2}\ \overset{\frown}{ABC}$.[48] Since the sum of these two arcs is 360°, we obtain $\angle\alpha + \angle\beta = 180^\circ$.

Let us prove the converse statement. Suppose in quadrilateral $ABCD$ the sum of opposite angles α and β is 180°. Let us draw a circle through three of the vertices A, B and C. Then the fourth vertex (point D) can happen to lie inside the circle, outside it, or on the circle.

Suppose that the point D lies inside the circle (as in Fig. 4.145b). Let us extend line CD and denote by D' its intersection point with the circle (see Fig. 4.145b). Angle β is an exterior angle for triangle ADD'. Therefore, $\angle\beta > \angle AD'D$ (see Corollary 1), or $\angle\beta > \angle\gamma$.

Quadrilateral $ABCD'$ is inscribed in the circle and, as we have proved above, $\angle\alpha + \angle\gamma = 180^\circ$.

This means that $\angle\alpha + \angle\beta \neq 180^\circ$, which contradicts what is given. Thus the point D must coincide with the point D' and must lie on the circle.

Similarly, let us assume that point D happens to lie outside the circle, as in Fig. 4.145c. Let us denote by D' the intersection of side CD with the

[48] Here we have indicated arcs by three letters to show what part of the circle we mean.

circle. Then angle γ is an exterior angle of the triangle ADD', and therefore, $\angle\gamma > \angle ADD'$.

But as we have proved above, $\alpha+\gamma = 180°$. Therefore, $\alpha+\beta \neq 180°$, which contradicts what is given. Thus point D must coincide with point D' and must lie on the circle. □

21.2 Inscribed quadrilaterals. Ptolemy's theorem

What else can we say about a quadrilateral inscribed in a circle?

The answer is given by Ptolemy's theorem,[49] which establishes a relation between the lengths of the sides a, b, c, d of an inscribed quadrilateral and its two diagonals d_1, d_2.

Theorem 17 (Ptolemy's theorem) If a quadrilateral is inscribed in a circle, then the product of its diagonals is equal to the sum of the products of its opposite sides, i.e., $d_1 \cdot d_2 = ab + cd$.

Proof. Fig. 4.146a shows an inscribed quadrilateral $ABCD$. We need to prove that $ab+cd = xy$, where $x = AC$ and $y = BD$.

Notice that in this figure there are pairs of equal angles because they correspond to the same arcs. In Fig. 4.146b we denote equal angles and their corresponding arcs by the same numbers 1, 2, 3, or 4.

Note also that if we draw segment AE connecting point A with BD and such that $\angle DAE = \angle BAC$ (see Fig. 4.146c), we will obtain some similar[50] triangles. Indeed, triangles AED and ABC are similar, because they have two correspondingly equal angles marked 1 and 4.

Therefore, $\frac{AD}{AC} = \frac{DE}{BC}$. Using our notations for the sides and diagonals we obtain

$$\frac{d}{x} = \frac{v}{c},$$

where we denote by v the segment DE.

There is another pair of similar triangles: $\triangle ABE$ is similar to $\triangle ACD$ (see Fig. 4.146d). We have $\angle ABE = \angle ACD$ and is the angle marked 2. Also $\angle BAE = \angle CAD$ and is the angle marked 3. Indeed, angle BAD is equal to $\angle(4+3)$ and $\angle CAD = \angle 3$. Thus, $\angle BAE = \angle(4+3) - \angle 4 = \angle 3$.

[49] Ptolemy was an ancient Greek mathematician who lived in Alexandria. He is also known for his work in astronomy and geography. Of course, he knew that the earth is a sphere, and made the first known projection of a sphere onto a plane. In his study of geometry, he derived what are known now as the law of sines and the law of cosines.

[50] See Section 15.

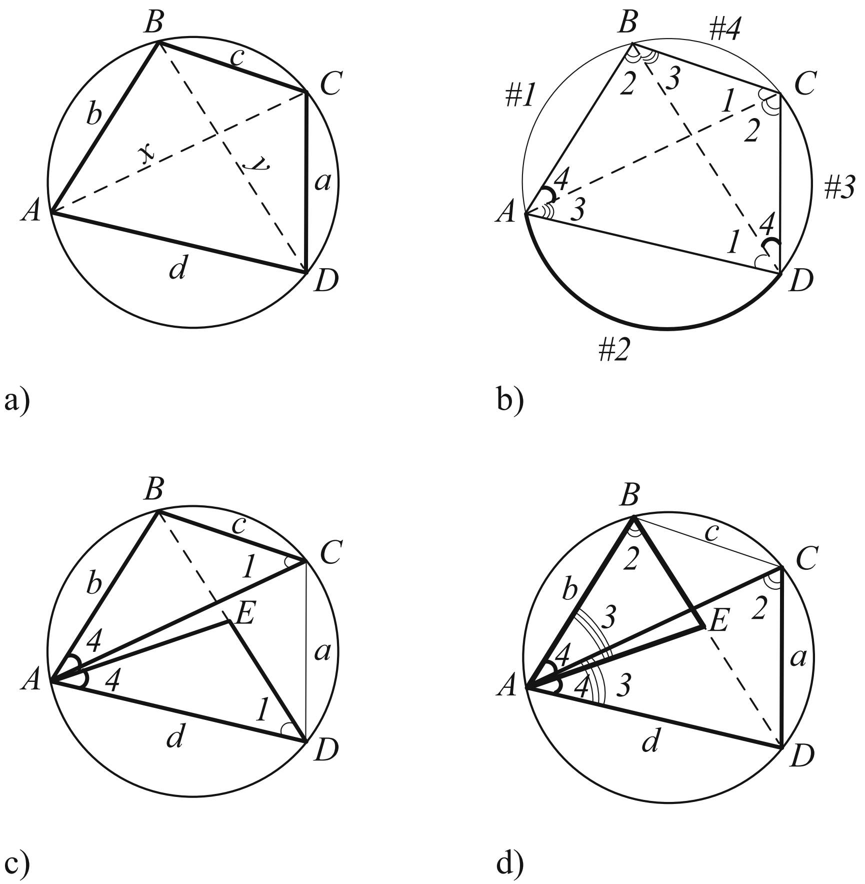


Fig. 4.146

From the similarity of $\triangle ABE$ and $\triangle ACD$, we obtain $\frac{BE}{CD} = \frac{AB}{AC}$, or, using our notations,

$$\frac{y-v}{a} = \frac{b}{x}.$$

By eliminating the fractions from each of the ratios we obtain

$$cd = xv, \text{ and } x(y-v) = ab.$$

Let us add these identities:

$$ab + cd = xv + xy - xv.$$

We obtain Ptolemy's theorem:

$$ab + cd = xy. \qquad \square$$

Remark 14. The converse of Ptolemy's theorem is also true: if a quadrilateral is such that the product of its diagonals is equal to the sum of the products of its opposite sides (i.e., $d_1 \cdot d_2 = ab + cd$), then this quadrilateral can be inscribed in a circle. This statement is not difficult to prove, but we will not do it here.

Remark 15. Consider a particular case of Ptolemy's theorem where the inscribed quadrilateral is a rectangle (see Fig. 4.147a).

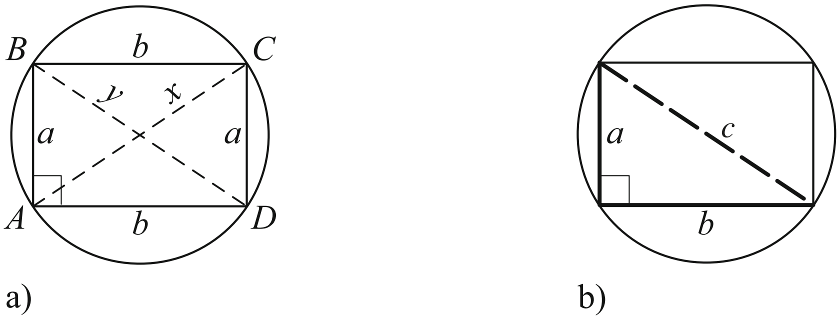


Fig. 4.147

In this case the sides are equal to a and b, and the diagonal is equal to x. The formula takes the form:

$$a^2 + b^2 = x^2.$$

If we change the notation of x to c as in Fig. 4.147b, we obtain the Pythagorean formula for a right triangle with legs a, b and hypotenuse c:

$$a^2 + b^2 = c^2.$$

Thus, the Pythagorean theorem is a particular case of Ptolemy's theorem. In addition, we have obtained one more proof of the Pythagorean theorem that does not use areas but uses instead the notion of similarity.

21.3 Some problems on inscribed quadrilaterals

PROBLEM 66. Fig. 4.148a, b shows inscribed quadrilateral $ABCD$ divided into triangles by its diagonals BD and AC.

Divide the triangle BCD into two triangles such that one of them will be similar to triangle ABC and the other one will be similar to triangle ACD. These three triangles are shown separately in Fig. 4.148c, d.

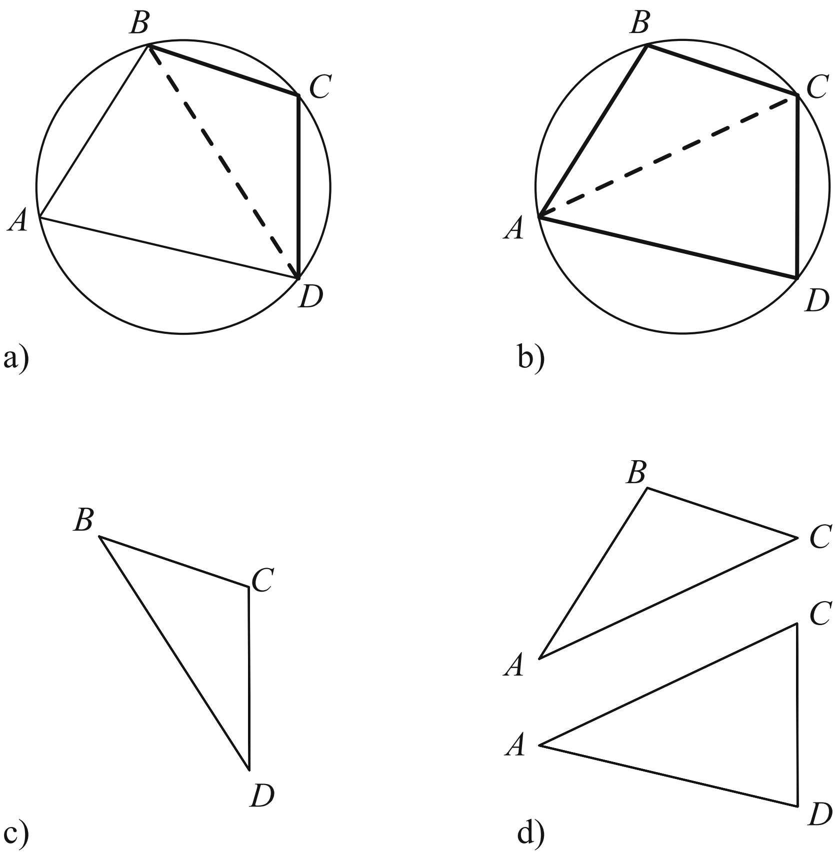


Fig. 4.148

PROBLEM 67. A quadrilateral is given and we know its four angles. Can we say whether it can be inscribed in a circle? If we draw a circle through three vertices of this quadrilateral, can we determine from the four given angles where the fourth vertex will lie: inside or outside this circle?

PROBLEM 68. Consider two right triangles that have hypotenuses of the same length. If we put them together along their hypotenuse, we obtain a quadrilateral as in Fig. 4.149.

(a) Prove that this quadrilateral can be inscribed in a circle.

(b) Find the center of this circle.

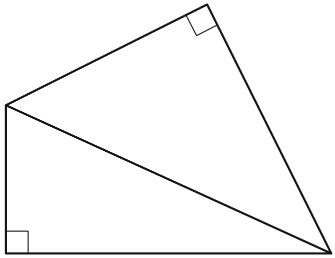


Fig. 4.149

One of the simplest quadrilaterals is a parallelogram. Is it always possible to inscribe one in a circle?

PROBLEM 69. Draw a parallelogram. Choose three of its vertices and guess whether the fourth vertex will be inside or outside the circle passing through these three vertices. Draw a circle and check your guess. Try this on different parallelograms.

Exercise 34. Prove that a parallelogram can be inscribed in a circle if and only if it is a rectangle.

Solution. First, let us show that any rectangle can be inscribed in a circle.

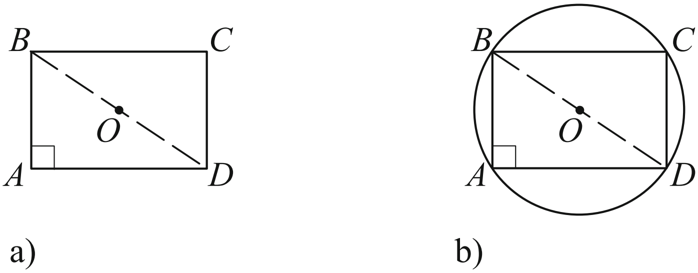


Fig. 4.150

Let us draw a rectangle $ABCD$ and its diagonal BD (see Fig. 4.150a). We can inscribe triangle ABD in a circle (see Fig. 4.150b). The center O of this circle lies at the midpoint of hypotenuse BD.

We can also inscribe triangle BCD in a circle. Its center also lies at the midpoint of BD. Therefore, all four points lie on the same circle.

We now need to prove that if a parallelogram is inscribed in a circle, it is a rectangle. Indeed, from Proposition 39, in an inscribed parallelogram the sum of the opposite angles is equal to 180°. But the opposite angles of a parallelogram are equal. Therefore, each of these angles is equal to 90°, and this parallelogram is a rectangle. □

21.4 The relation between a circle and a regular polygon with n vertices

Consider a regular polygon, such as a pentagon (see Fig. 4.151a). We state the following theorem without a proof.

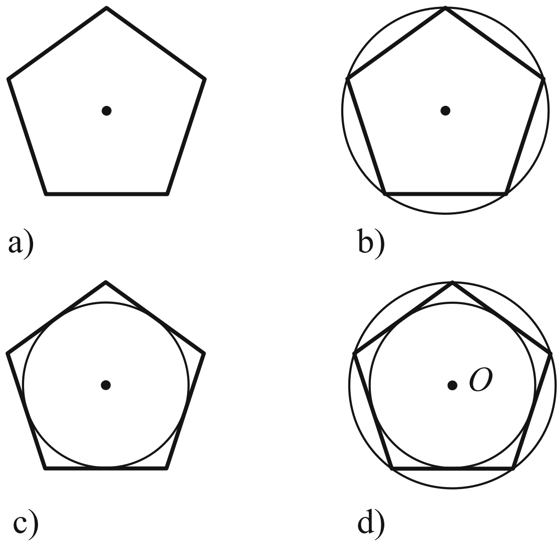


Fig. 4.151

Theorem 18.

(a) Any regular polygon can be inscribed in a circle.

(b) Any regular polygon can be circumscribed around a circle.

(c) The center of the circle circumscribed around a regular polygon coincides with the center of the circle inscribed in this polygon.

Theorem 18 is illustrated in Fig. 4.151a, b, c, d.

The center of the circle in which a regular polygon can be inscribed (or around which it can be circumscribed) is called the *center* of the regular polygon.

This theorem and Proposition 40 are helpful in constructions. If we want to inscribe a circle in a given regular polygon (or circumscribe a circle around it) we can use a compass but we need to know the center of this circle and its radius. In order to find the center of the polygon we can use the following statements, which are easy to check:

Proposition 40.

(a) The center O of a regular polygon lies at the intersection point of the perpendicular bisectors to the sides of this polygon.

(b) The center O of a regular polygon lies at the intersection point of the angle bisectors of the interior angles of the regular polygon.

Statement (a) gives us a way to inscribe a circle in a regular polygon. This circle has its center at the center O of the polygon. Its radius is a segment OA, where A is a midpoint of a side of this polygon.

Statement (b) gives us a way to circumscribe a circle around a regular polygon. This circle has its center at the center O of the polygon. Its radius is a segment OA, where A is a vertex of this polygon.

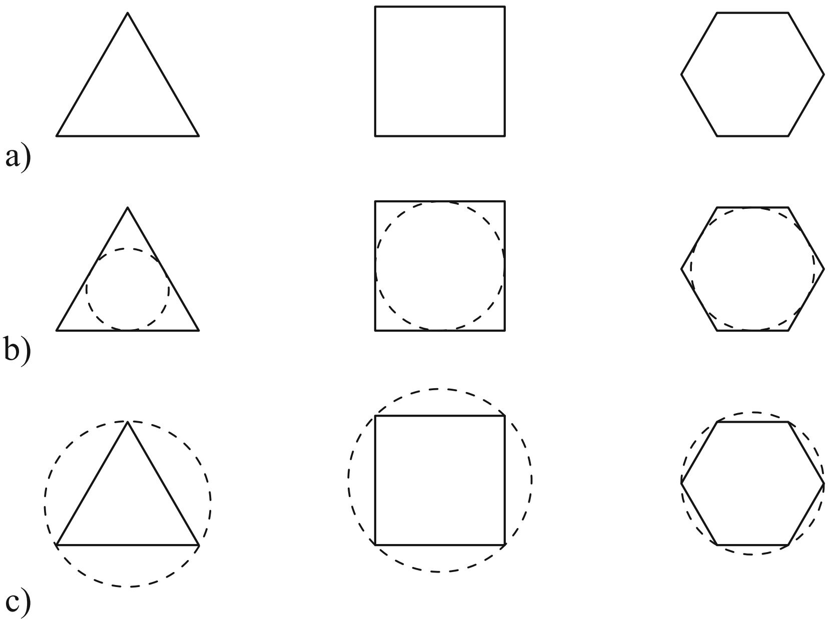


Fig. 4.152

PROBLEM 70. Fig. 4.152a shows a regular triangle, a regular quadrilateral, and a regular hexagon.

(a) Inscribe circles in these figures (see Fig. 4.152b).

(b) Circumscribe circles around these figures (see Fig. 4.152c).

The remark below reminds us that we have to be very clear about what figures are given and what figures we need to construct.

Remark 16. Note that if we are given a circle (see Fig. 4.153a) and asked to inscribe a regular polygon in it (or asked to circumscribe one around it), the

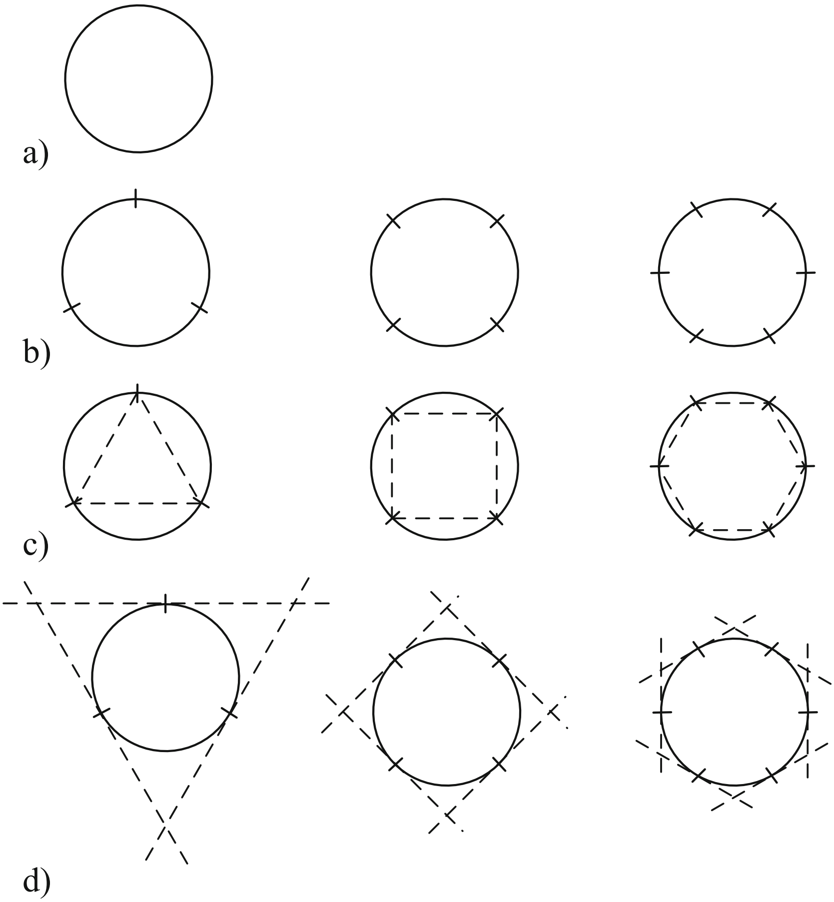


Fig. 4.153

construction will be very different. We will need to divide the circumference into equal arcs, which might not always be an easy task. In the examples in Fig. 4.153b these equal arcs are marked. Think how you would obtain these points yourself.

Then in order to construct inscribed polygons in this circle, we simply connect the points obtained on the circumference (see examples in Fig. 4.153c). However, in order to construct circumscribed polygons around this circle, we need to draw tangent lines through the points obtained on the circumference and extend these lines until they intersect (Fig. 4.153d).

22 Circumference and arc

22.1 Circumference

Consider a circle of radius r (Fig. 4.154a). How can we calculate the circumference?[51]

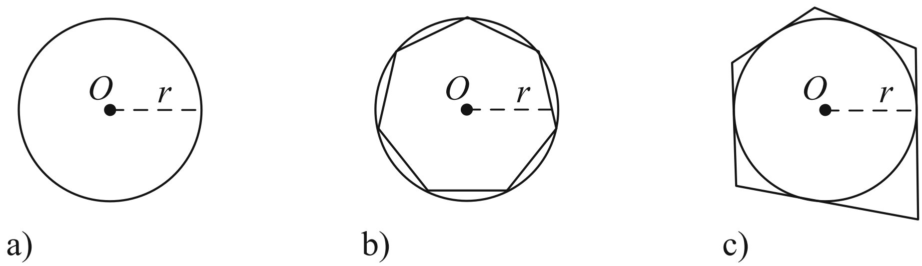


Fig. 4.154

First of all, we need to define what it means to measure the circumference, since until now we could measure only the length of a segment on a straight line.

Let us inscribe a convex polygon in the circle (see Fig. 4.154b), and then also circumscribe a convex polygon around an equal circle (see Fig. 4.154c).

We state the following Propositions 41 and 42 without proofs.

Proposition 41.

(a) Given a circle, the perimeter of any polygon inscribed in this circle is smaller than the perimeter of any polygon circumscribed around this circle.

[51]In this section, by circumference we will mean the distance around the circle.

(b) By choosing different polygons, the difference between the perimeter of an inscribed polygon and that of a circumscribed polygon can be made smaller than any given fixed number, no matter how small.

Definition 9. Given a circle, the length of its circumference is a number C that is larger than the perimeter of any polygon inscribed in this circle and is smaller than the perimeter of any polygon circumscribed around this circle.

Remark 17. The fact that for any circle such a number C exists and is unique is studied in calculus. The existence of such a number follows from Proposition 41(a), and the uniqueness follows from Proposition 41(b). More detailed explanations as well as a proof of Proposition 41 require some techniques from advanced calculus and are not appropriate here.

Proposition 42. Consider a circle with a diameter d and circumference C. The ratio $\frac{C}{d}$ does not depend on the circle. This number, always denoted by π, is the same for any circle, i.e.,

$$C = \pi d,$$

or

$$C = 2\pi r.$$

Remark 18. From Theorem 18, we know that if we are given a circle, it is always possible to inscribe a regular polygon in it and to circumscribe a regular polygon around it. However, it is not always possible to do this using only a straightedge and a compass, i.e., the instruments available for us in this chapter. Such constructions are possible for a square, an octagon, and regular polygons with 16, 32, 64, and other even numbers of vertices. The famous mathematician Gauss[52] proved that it is also possible to construct a regular polygon with 17 vertices using only a straightedge and a compass. This is not an easy problem to solve. Gauss was so proud of his solution that he asked for a regular polygon with 17 vertices to be engraved on his tomb.

[52] Gauss (1977–1855) was a German mathematician who made essential contributions to many fields of mathematics, such as number theory, analysis, and differential geometry, as well as astronomy and optics. He is considered one of the greatest mathematicians of all times.

22.2 The number π

How can the number π be evaluated?

Since $\pi = \frac{C}{d}$ we need to know the length C of the circumference with a given diameter d. According to Proposition 41, C can be approximated by the perimeters l_1 and l_2 of the polygons inscribed in this circle and circumscribed around it.

Let us choose the diameter of a circle to be equal to 1, and calculate the corresponding perimeters l_1, l_2 for some polygons, for example: (a) an equilateral triangle, (b) a square, and (c) a regular hexagon (see Fig. 4.155).

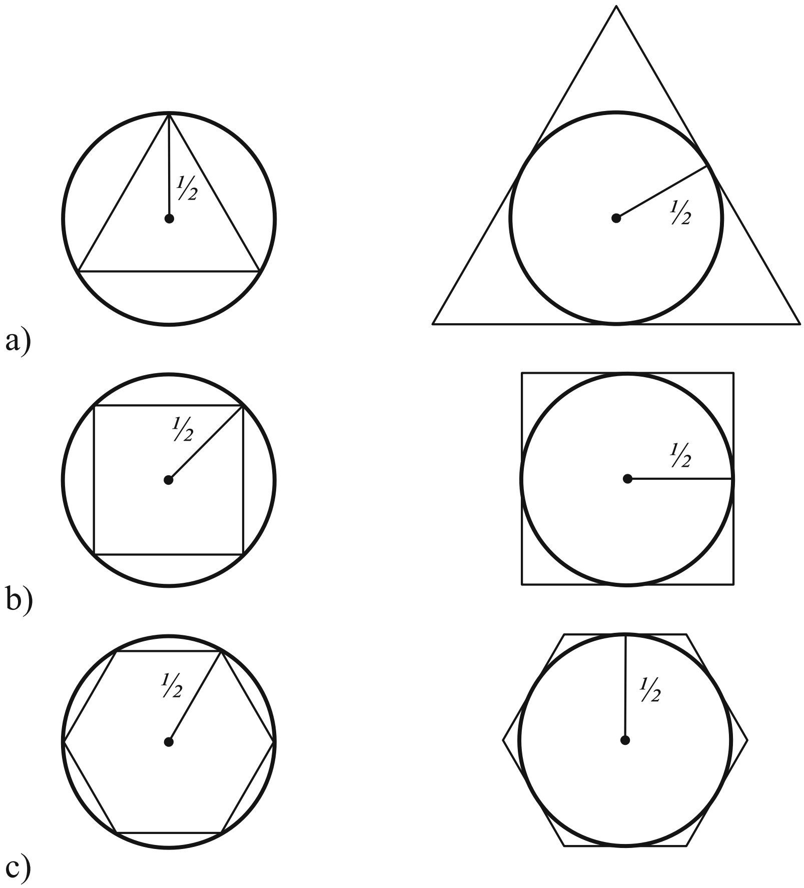


Fig. 4.155

We invite you to do the detailed calculations yourself as an exercise. You can use Fig. 4.156 as guidance for additional helpful constructions.

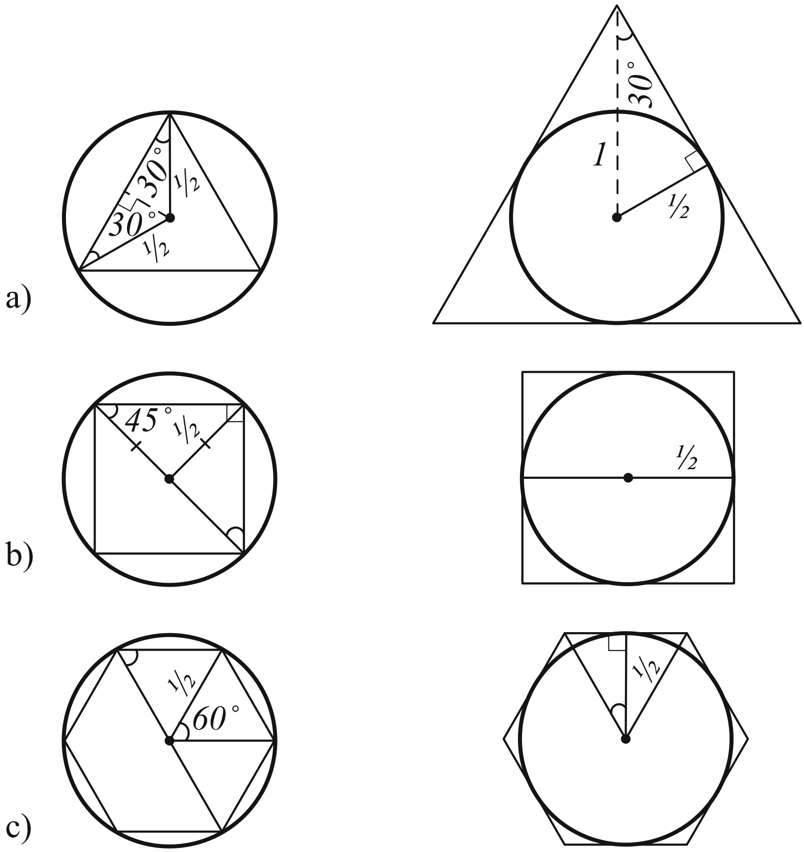


Fig. 4.156

We obtain:
In case (a) $l_1 = \frac{3}{2}\sqrt{3} \approx 2.595$, $l_2 = 3\sqrt{3} \approx 5.19$.
In case (b) $l_1 = 2\sqrt{2} \approx 2.82$, $l_2 = 4$.
In case (c) $l_1 = 3$, $l_2 = 2\sqrt{3} \approx 3.46$.
Thus, for the number π we obtained the inequalities:

$$2.595 < \pi < 5.19,$$

$$2.82 < \pi < 4,$$

$$3 < \pi < 3.46.$$

If we continue the calculations for polygons with a larger number of sides, we will obtain more and more precise approximations for the number π. By approximating the circumference with an inscribed regular polygon with 96 vertices, we will obtain the approximation $\pi \approx 3.14$ with the exactness of 0.01. For many practical calculations, such an approximation is sufficient.

It has been proved that the number π is a non-repeating decimal fraction and cannot be represented as any rational fraction. This is a very important number, and is an example of an *irrational number*.[53]

It is amazing that the famous Greek mathematician Archimedes,[54] who lived two thousand years ago, and who did not know decimal fractions or computers, obtained a good approximation for the number π with a rational (non-decimal) expression: $\pi \approx \frac{22}{7} = 3\frac{1}{7}$. It can even be proved that this expression is the best approximation of the number π by a rational fraction with the numerator and denominator being smaller than 100.

If we start the estimate of π with a square and then double the number of sides at each step (i.e., we consider polygons with 8, 16, 32, ... sides), we obtain an expression for π as

$$\pi \approx \sqrt{2} + \sqrt{\sqrt{2 + \sqrt{\ldots}}}.$$

Nowadays we can calculate more than 1,000 decimal digits of the number π, which is an exactness exceeding that needed for any practical calculations. Even in the 3rd century BC it was possible to obtain π with sufficient exactness to calculate the circumference of Earth and its orbit around the Sun.

In French there is a short phrase that helps to memorize several decimal digits of the number π: *"Que j'aime a faire apprendre un nombre utile aux hommes!"* (The number of letters in each word gives the digits of π, i.e., $\pi \approx 3.1415926536$, where the last digit is rounded up.) The English phrase is: "May I have a large container of coffee?"

PROBLEM 71. Find the percent error between Archimedes' value of π and the actual value of π.

Hint. Represent the rational fraction found by Archimedes as a decimal fraction. Find the decimals in which it will differ from the actual value of π and calculate the error.

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[53] We have already discussed the irrational number $\sqrt{2}$ in Section 10.1.

[54] Archimedes (III century BC), who studied in Alexandria, Egypt, and then lived in Syracuse, Greece, made important contributions to geometry and is known as the father of geometry. Many of his studies and discoveries anticipated the development of science by centuries. He set the base for the integral calculus studied by Newton and Leibnitz 2,000 years later.

22.3 Length of an arc

Consider a central angle of α degrees in a circle of radius r (see Fig. 4.157a).

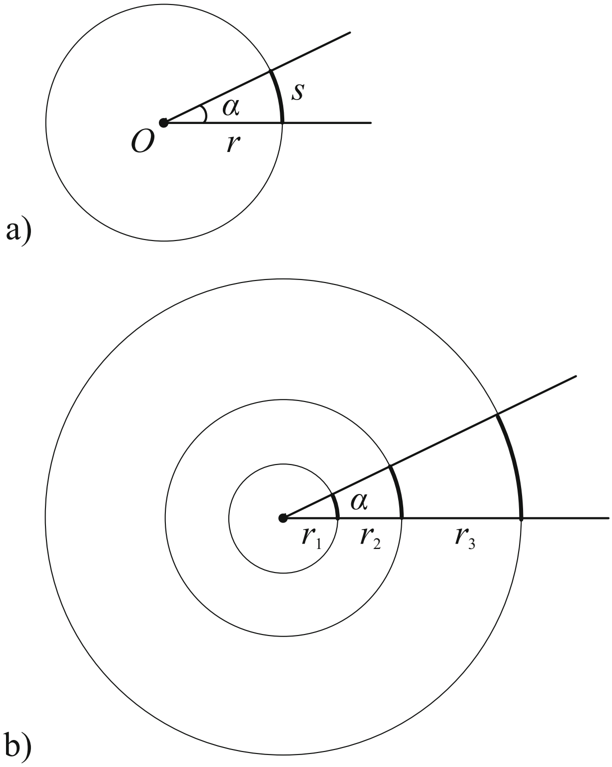

Fig. 4.157

As we know, the circumference is $2\pi r$, and it contains 360°. The angle of α degrees contains an arc which is $\frac{\alpha}{360}$ of the circumference. Let s be the length of the arc. Then $s = \frac{\alpha}{360} \cdot 2\pi r$, or

$$s = \frac{\pi r \alpha}{180}.$$

Clearly, the larger the radius of the circle, the longer the arc corresponding to the angle α (see Fig. 4.157b). However, the ratio of the arc length to the radius is the same.

Exercise 35.

(1) Find the arc length of a 30° central angle if the radius is

(a) $r_1 = 1$; (b) r_2; (c) r_3.

(2) Find the ratio of the arc length s to the corresponding radius r.

Solution.

(1) According to the formula for the arc length, $s = \frac{\pi r 30}{180} = \frac{\pi}{6} r$. We obtain (a) $s = \frac{\pi}{6} r_1 = \frac{\pi}{6}$; (b) $s = \frac{\pi}{6} r_2$; (c) $s = \frac{\pi}{6} r_3$.

(2) We can easily see that in each case the ratio s/r remains the same and is equal to $\frac{s}{r} = \frac{\pi}{6}$.

22.4 Radian measure of an angle

We saw above that given a central angle, the ratio of its arc length to the radius does not depend on the radius. This leads to a new way of measuring angles called *radian measure*.

Consider a central angle of α degrees. Its radian measure q is the ratio of its arc length to the corresponding radius, i.e.,

$$q = \frac{s}{r} = \frac{\pi \alpha}{180}.$$

We can also say that the radian measure of a central angle is the length of its arc on a unit circle.

What is an angle of 1 radian? If $q = 1$, then $s = r$. Thus, the arc length of an angle of 1 radian is equal to the corresponding radius.

Sometimes this important property of a radian is chosen as its definition:

An angle of 1 radian is an angle such that the length of its arc is equal to the radius.

Exercise 36. Find the number of degrees in an angle of 1 radian.

Solution. We have $s = \frac{\pi r \alpha}{180}$. If $s = r$, then $\frac{\pi \alpha}{180} = 1$. Since $\pi \approx 3.14$, we have $\alpha = \frac{180}{\pi} \approx 57.296°$.

Exercise 37. Find the radian measure of the following angles:

(a) $90°$; (b) $30°$.

Solution. As we know, the radian measure $q = \frac{\pi \alpha}{180}$. If $\alpha = 90°$ we obtain $q = \frac{\pi 90}{180} = \frac{\pi}{2}$. If $\alpha = 30°$ we obtain $q = \frac{\pi 30}{180} = \frac{\pi}{6}$.

PROBLEM 72. Find the radian measure of the following angles:

(a) $60°$; (b) $45°$; (c) $180°$; (d) $360°$.

For all these angles and the angles from the exercise above, make a table of their values in degrees and in radians.

PROBLEM 73. Find the radian measure of an angle of 1°.

PROBLEM 74. Find the degree measure of the following angles: $\frac{\pi}{5}$; $\frac{3\pi}{2}$.

We recall (see Section 3.3) that we can measure central angles larger than 360° or 2π radians.

23 Disks and sectors

23.1 Area of a regular polygon

Consider a regular polygon with n vertices (e.g., a heptagon in Fig. 4.158a). Let us connect its center O with the vertices. This divides the polygon into triangles. The area of the polygon is the sum of the areas of these triangles. Therefore, $S = \frac{1}{2}p \cdot h$, where p is the perimeter of the polygon and h is the altitude from the center O.

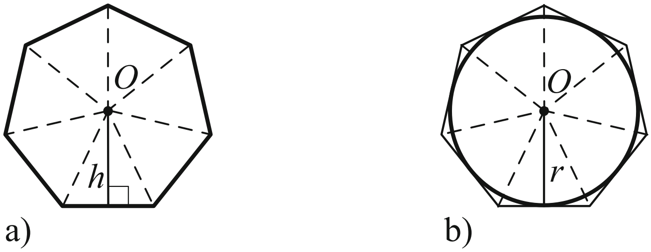

Fig. 4.158

Note that since h is the radius of a circle inscribed in the polygon (see Fig. 4.158b), the following statement is also true.

If a circle is inscribed in a regular polygon, then the area of this polygon is $\frac{1}{2}pr$, where p is the perimeter of the polygon and r is the radius of the circle.

PROBLEM 75. Find the area of a regular polygon with side equal to 1 in the following cases:

(a) a triangle;

(b) a quadrilateral;

(c) a hexagon;

(d) an octagon. Compare these areas with the area of a unit disk, which you can calculate using the theorem below.

23.2 Area of a disk

It is natural to define the area of a disk bounded by a circle similarly to the way in which we have defined the circumference. This means that we can inscribe and circumscribe different regular polygons around the circle and calculate their areas. The area of the disk has to be between the values for the areas of all inscribed and all circumscribed polygons. It is clear that as the number of vertices of a polygon increases, the exactness of such approximations will be higher.

We will not do this process and the calculations here. We present the following statements without proofs.

Theorem 19. The area of a disk with radius r is equal to πr^2.

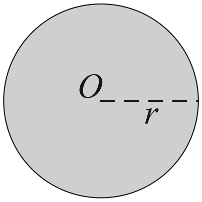

Fig. 4.159

23.3 Area of a sector

A *sector* corresponding to an angle α is part of a disk bounded by the central angle α. In Fig. 4.160, a sector corresponding to an angle α is shown as a shaded domain.

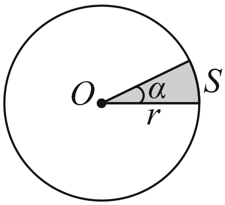

Fig. 4.160

Proposition 43. The area of a sector with radius r is equal to

$$S = \frac{1}{2} s \cdot r,$$

where s is the arc length of this sector.

We already know the length of the arc of a central angle. If the angle α contains α degrees, then the area of the sector is

$$S = \frac{\pi r^2 \alpha}{360}.$$

If the angle α contains q radians, then the area of the sector is

$$S = \frac{1}{2} q r^2.$$

Exercise 38. Consider a unit circle.

(a) Find the area of a unit disk.

(b) Find the area of a sector of 90°.

(c) Find the area of a sector of 60°.

Solution.

(a) According to Theorem 19 we obtain $S = \pi r^2 = \pi$.

(b) Let us use the radian measure of this angle, i.e., $\frac{\pi}{2}$. We have

$$S = \tfrac{1}{2}\tfrac{\pi}{2} r^2 = \tfrac{\pi}{4} \approx \tfrac{3.14}{4} \approx 0.7\ldots.$$

(c) Similarly, we have $S = \frac{1}{2}\frac{\pi}{3} r^2 = \frac{\pi}{6} \approx \frac{3.14}{6} \approx 0.52\ldots$

24 Overview of Chapter IV

In this last chapter of the book we introduced a new figure, the circle. This allowed us to define the length of a segment, thus obtaining the "usual" Euclidean geometry. We can now compare any two or more segments and measure their lengths.

We defined the degree measure of angles, showed how to add angles, and discussed some important types of angles and their properties.

We also defined two important lines: the bisector of an angle and a line perpendicular to another. We were able to expand the set of notions and figures to include perpendicular bisectors, altitudes, rectangles, squares, and rhombi.

Considering relations between figures, we obtained the notions of an arc, sector, tangent, and inscribed and circumscribed polygons. This in turn helped us to discover the circumference of a circle, the area of a disk, and other important facts.

We defined the following transformations of figures: rotation through an angle, reflection in a line, and similarity. Rotation and reflection do not change figures; similarity changes figures but does not change their shape. All these transformations are one-to-one correspondences.

Note that two correspondences, central perspective (introduced in Chapter I) and parallel projection (introduced in Chapter II), are not one-to-one correspondences and are not transformations.

Glossary

Affine coordinate system – a one-to-one correspondence between points on the plane and pairs of numbers. It is defined by two intersecting straight lines and a point on each of these lines, different from their intersection point. The intersection point is chosen as the origin, i.e., the point (0,0). The point on each axis marks the unit coordinate on that axis. If A is an arbitrary point on the plane, its coordinates are determined by drawing lines through this point parallel to the axes. The line intersecting the x-axis gives the x-coordinate, and the line intersecting the y-axis gives the y-coordinate. These numbers are called the *coordinates* of the point A, and the pair of coordinates is denoted as (x, y).

Affine geometry – in our book, geometric constructions performed based only on the following three axioms: 1. Through any two points one can draw a unique line. 2. Any two lines either have a unique point of intersection or are parallel. 3. Given a point A and a line a there exists only one line passing through the point A and parallel to line a.

Altitude – a segment passing through a vertex of a triangle and perpendicular to the opposite side of the triangle.

Analysis situs – a name used by Leonhard Euler for a field in mathematics which studies figures and their relative positions without taking into account any of their quantitative characteristics. This field served as an origin for the modern field of study called *topology*.

Angle – a domain that is bounded by two rays with the same endpoint. Different kinds of angles are described below.

Acute angle – an angle smaller than 90°.

Adjacent angles – two non-overlapping angles with the same vertex that share a ray.

I. M. Gelfand, T. Alekseyevskaya (Gelfand), *Geometry*,
https://doi.org/10.1007/978-1-0716-0299-7

Alternate angles – see *Angles formed by two parallel lines intersected by a third line.*

Angles formed by two parallel lines intersected by a third line – when two parallel lines a and b are intersected by a third line c, some pairs of angles formed by these lines have particular names. One ray of every angle in such a pair lies on the line c. The second ray of one angle of the pair lies on the line a and the second ray of the other angle lies on the line b.

Corresponding angles are a pair of angles both lying on the same side of the line c in such a way that one angle contains the other (i.e., they both are "facing" in the same direction).

Alternate interior angles are a pair of angles lying on different sides of the line c, in such a way that each of these angles intersects the domain between the lines a and b. We can say that each of these angles "faces" into the domain between the lines a and b.

Alternate exterior angles are a pair of angles lying on different sides of the line c, in such a way that each of these angles lies outside the domain between the lines a and b.

Central angle – an angle with its vertex at the center of a circle.

Circumscribed angle – an angle whose sides are tangent to a circle.

Complete angle, *or* **perigon** – an angle formed by two coinciding rays with the same endpoint if we consider that these rays bound the whole plane. A complete angle's measure is $360°$.

Convex angle – an angle which is a convex domain. A convex angle is smaller than or equal to $180°$. **Nonconvex angle** – an angle which is not a convex domain.

Corresponding angles – see *Angles formed by two parallel lines intersected by a third line.*

Equal angles – two convex angles such that if we draw unit circles with their centers at the vertices of these angles, the chords corresponding to these angles will be equal.

Inscribed angle – an angle whose vertex lies on a circle and whose rays intersect the circle.

Obtuse angle – an angle bigger than $90°$ and less than $180°$.

Angles with respectively parallel sides – two angles such that each side of one angle is parallel to the corresponding side of the other angle.

Angles with respectively perpendicular sides – two angles such that each side of one angle is perpendicular to the corresponding side of the other angle.

Right angle – an angle containing 90°.

Straight angle – an angle bounded by two rays with the same endpoint and such that one ray is the extension of the other.

Supplementary angles – two convex angles are supplementary angles if their measures add up to 180°.

Vertical angles – a pair of convex angles such that the sides of one angle are the extensions of the sides of the other angle.

Zero angle – an angle formed by two coinciding rays with the same endpoint if we consider that these rays do not bound any domain. The measure of a zero angle is 0°.

Angle bisector – a ray passing through the vertex of an angle in such a way that it divides the angle into two equal angles.

Angle of one degree – see *Degree measure.*

Arc – part of a circle between two points on the circle.

Arc degree measure – see *Degree measure.*

Area – an important characteristic of a figure in the plane, which satisfies certain axioms:

(a) If a figure is composed of several parts, then the area of this figure is the sum of areas of these parts.

(b) If two figures are obtained from each other by parallel translation, rotation with respect to a point, or line reflection, then these figures have equal areas. Alternatively, (b) states that the area of a figure does not change as a result of a motion or reflection applied to this figure. (Note that in Chapter III, we did not include rotation and line reflection among the notions associated with equality of area because these notions could not have been defined there.)

Area establishes a correspondence between figures and positive numbers. This correspondence is not one-to-one, which means that different figures can have the same area.

Area measure – a positive number assigned to a figure in accordance with the chosen postulates and unit area.

Axiom – A statement in geometry that we agree to accept, because we believe that it is true. Other statements (theorems, lemmas, propositions) are then proved on the basis of axioms.

Axis, *also* **number axis, coordinate axis** – a line with two marked points on it. One point denoted by O is chosen as 0 (zero) and is called the *origin.* The other point A defines a *unit segment* OA. The vector OA defines the positive direction of the number axis. The coordinate of a point B on the axis is equal to the length of the segment OB in terms of the defined unit segment, with the sign $+$ or $-$ depending on the direction in which the point B lies from the point O.

Bimedian of a triangle – the part lying inside the triangle of a line passing through the midpoint of one of the sides of the triangle parallel to one of its other sides.

Boundary of a domain – a collection of lines or a continuous curve which separates this domain from the rest of the plane.

Bounded polygonal domain – see *Polygonal domain.*

Broken line – a path consisting of a finite number of segments connected to each other consecutively. If the first and last points of a broken line coincide, it is called a *closed broken line.* A broken line is said to be *non-self-intersecting* if it does not have intersection points other than the ends of its segments.

Center of a circle – the point O equidistant from all points of a circle.

Center of a regular polygon – the center of the circle in which a regular polygon can be inscribed or around which it can be circumscribed.

Center of a parallelogram – the intersection of diagonals of a parallelogram. This point is also the intersection of two lines each of which passes through the midpoint of one side of a parallelogram parallel to another non-opposite side.

Center of similarity – the only point on the plane which stays in place under a similarity transformation.

Center of symmetry of a figure – see *Centrally symmetric figure.*

Central perspective, *also called* **central projection,** *or sometimes* **projection by a source of light** – a correspondence between the points on the plane and the points on a certain line a established by lines passing through a certain point O. For a point A on the plane, the corresponding point A', which lies on the line a where the line OA intersects it, is called the *image* or *projection* of the point A. Note that points lying on the line passing through the point O and parallel to the line a do not have an image.

Central symmetry – see *Symmetry with respect to a point.*

Centrally symmetric figure, *or* **self-symmetric figure** – a figure symmetric to itself with respect to a certain point inside this figure. This point is called the *center of symmetry* of the figure.

Chamber – a bounded domain in the plane formed by a set of intersecting lines and such that it does not contain within itself any segments of these lines.

Chord – a segment connecting two points on a circle.

Circle – a set of all points such that the segment connecting any point of the circle with a fixed point O has the same length. The point O is called the *center* of the circle.

Circumcenter of a triangle – the center of the circle in which this triangle is inscribed.

Circumference – the distance around a circle, or the length of the circle. Note that sometimes the term circumference is used to indicate the boundary itself, while the term circle is used to indicate the whole figure with the interior.

Circumscribed triangle – see *Polygon circumscribed around a circle.*

Closed polygonal line – see *Polygonal line.*

Compass – an instrument used to draw a circle.

Complete triangle – a configuration consisting of three points in general position and three lines passing through each pair of these points. This configuration is dual to itself.

Complete quadrangle – a configuration consisting of four points in general position and six lines passing through each pair of these points. This configuration is dual to the complete quadrilateral.

Complete quadrilateral – a configuration consisting of four lines in general position and six points of intersection. This configuration is dual to the complete quadrangle.

Composition, *or* **product of two operations** – the result of applying these two operations consecutively.

Concentric circles – circles which have the same center.

Configuration – on the intuitive level, this is a collection of two or more geometric objects (for example, points, lines and circles). Two collections of two or more objects are examples of the same configuration if one can gradually move the objects on the plane from one collection into the objects of the other collection while obeying the following rule: at no moment we are allowed to obtain a new intersection point or lose an existing one.

Configuration dual to itself – a configuration which coincides with its dual.

Congruent figures – two figures on the plane such that one is obtained from the other with the help of one or more of the following operations: parallel translation, rotation about a point, and reflection in a line.

Converse Desargues' theorem – Given two triangles such that the points of intersection of their corresponding sides lie on one line, the three lines that pass through the corresponding vertices of these triangles intersect at one point.

Converse theorems – two theorems such that what is given in one theorem is the conclusion in the other and vice versa.

Convex domain – a domain such that the segment connecting any two points of this domain lies completely in the domain.

Convex hull of a set of points – a minimal convex domain that contains all these points.

Convex hull of a polygon – convex hull of vertices of this polygon.

Convex polygon – a polygon which is a convex domain.

Coordinate of a point – see *Axis,* also *Affine coordinate system.*

Degree measure – is used to measure angles or arcs. For this, a unit circle is divided into 360 equal parts by rays from its center. An angle corresponding to one such part is called an angle of one degree, 1°. Any part of the plane bounded by two rays emanating from one point represents an angle whose measure ranges from 0° to 360°. There is a one-to-one correspondence between arcs on a given circle and the angles with their vertices at the center of this circle. Therefore, we can define the degree measure of an arc on a given circle. An arc of one degree (1°) is the arc corresponding to $\frac{1}{360}$ of this circle.

Desargues configuration – a configuration of two triangles ABC and $A'B'C'$ such that the lines AA', BB', CC' intersect in a single point. In a Desargues configuration the points of intersection of the lines $a = BC$ and $a' = B'C'$, of the lines $b = AC$ and $b' = A'C'$, and of the lines $c = AB$ and $c' = A'B'$ themselves lie on the same straight line.

Desargues' theorem – given two triangles such that the lines passing through the corresponding vertices of these triangles intersect in a single point, the points of intersection of the corresponding sides of these triangles lie on one line.

Desargues triangles – a pair of triangles which form a Desargues configuration.

Diagonal of a polygon – a segment connecting two vertices of a polygon that are not connected by one of its sides. Or: a diagonal of a polygon is a segment connecting two vertices of the polygon that are not adjacent.

Diagonal of a quadrilateral – see *Diagonal of a polygon.*

Diameter – a chord passing through the center of the circle.

Disk – a bounded domain formed by a circle. The boundary of a disk is a circle.

Distance between the points A **and** B – the length of the segment AB.

Distance from a point to a line – the length of the perpendicular from the point onto the line.

Domain, *or* **region** – part of the plane bounded by one or more lines or curves and such that any two points of the domain can be connected by a continuous curve (not necessarily a straight line) that will not cross the boundary of the domain.

Dual configuration – given a configuration of lines and points, its dual configuration is obtained by replacing points with lines and lines with points. The objects (lines and points) are replaced in the initial configuration according to the following rules: a point is replaced by a line; a line is replaced by a point; a point that lies on a line is replaced by a line passing through the point; a line that passes through a point is replaced by a point lying on the line.

Dual Desargues configuration – the dual of a Desargues configuration is that same configuration. See *Desargues configuration.*

Dual to itself, *or* **self-dual** – property of a configuration whereby it coincides with its dual configuration.

Duality between points and straight lines – a correspondence in projective geometry between a configuration and its dual configuration.

Endpoint of a ray – see *Ray.*

Equal segments in affine geometry – two segments $\bar{a}$ and $\bar{b}$ lying on two parallel lines are equal if it is possible to draw straight lines through corresponding ends of the segments $\bar{a}$ and $\bar{b}$ in such a way that these new lines are parallel.

Equal segments in Euclidean geometry – two segments of equal lengths. Two segments are equal if one segment can be superimposed on the other using a line reflection, translation, rotation, or some combination of those operations.

Equivalent definitions or statements – two definitions or statements A and B such that from A follows B and from B follows A.

Euclid's fifth postulate – an assumption that given a line and a point not on this line, there exists one and only one line that passes through this point and is parallel to the given line.

This statement is taken as an axiom in Euclidean geometry.

Euclidean geometry – in our book, geometric constructions performed on the basis of the three axioms of affine geometry and the following axioms: 1. Given two circles, they either do not intersect, or intersect at one point, or intersect at two points. 2. A circle and a straight line either do not intersect, or intersect at one point, or intersect at two points.

Exterior angle of a polygon – an angle supplementary to an interior angle of the polygon.

Exterior angle of a triangle – see *Exterior angle of a polygon.*

Exterior tangent circles – two circles which have only one common point and lie outside each other.

Extreme positions of a circle and an angle – non-general positions in the relative placement of a circle and an angle. Interesting examples arise when one or two rays of the angle become tangent to the circle.

General position of straight lines – two or more straight lines in the plane such that no two of them are parallel and no three of them intersect in one point.

General position of points – points in the plane such that no three of them lie on a straight line.

Half-line – see *Ray.*

Half-plane – a part of the plane bounded by a straight line. This line belongs to the half-plane.

Heptagon – a polygon with seven sides. It also has seven vertices.

Hexagon – a polygon with six sides. It also has six vertices.

Ideal line – see *Line.*

Ideal point – see *Point.*

If and only if – a mathematical expression often used to formulate propositions and theorems; it contains two statements. For example, “A is true if and only if B is true” means: (1) If A is true then B is true, and (2) If B is true then A is also true.

Image of a point A under a transformation – the point A' into which point A is transformed by the transformation.

Image of a figure under a transformation – a collection of images of all points of this figure under this transformation.

Infinity *or* **point at infinity** – by a convention made in mathematics, this is an imaginary point at which parallel lines intersect. This makes rules more uniform. This "point" is denoted as ∞.

Line at infinity – mathematicians make an agreement that all points at infinity lie on one line at infinity.

Inscribed triangle – see *Polygon inscribed in a circle.*

Interior angle of a polygon – an angle formed by two rays whose common endpoint is a vertex of the polygon and which contain two adjacent (neighboring) sides of the polygon.

Interior angle of a triangle – see *Interior angle of a polygon.*

Interior tangent circles – two circles that have only one common point and are positioned so that one circle lies inside the other.

Intersection of two or more lines – a point that belongs to each of the intersecting lines.

Irrational number – a number on a number axis that cannot be represented as a rational number. An irrational number is an infinite non-periodic decimal fraction. Examples used in this book are the numbers π and $\sqrt{2}$.

Line, *or* **straight line** – in this book, a line, or straight line, refers to an "ideal line" in the plane that has no width and is infinite in both directions.

Locus – a set that is made up of all points which have a certain property or satisfy a certain condition and is such that no other point, i.e., not from this set, has this property.

Median of a trapezoid – a segment connecting the midpoints of the legs of the trapezoid.

Median of a triangle – a segment that connects a vertex of the triangle to the midpoint of the opposite side.

Midline of two parallel lines – a straight line parallel to two lines and passing through the midpoint of any segment whose endpoints lie on each of these lines.

Midpoint of a segment – a point on a segment that lies at equal distance from the endpoints of the segment.

Minkowsky addition, *or* **the Minkowsky sum of two figures (or two sets of points)** – a figure that consists of the midpoints of all the segments connecting each point of one figure with each point of the other figure.

Motion – any of a category of one-to-one correspondences consisting of parallel translations and rotations. A motion is an example of a transformation. A motion does not change the figure. (We rely here on your intuitive understanding of what "does not change" means. Compare *Motion* with *Similarity*.)

n-**gon** – a polygon with n vertices. Such a polygon also has n sides.

Non-stable position of lines (or points) – a configuration of lines (or points) such that any move, no matter how small, of any of these lines (or points) will change the configuration.

Number axis – see *Axis.*

Octagon – a polygon with eight sides. It also has eight vertices.

One-to-one correspondence – a correspondence between two sets such that: (a) each element of the first set corresponds to one and only one element of the second set; (b) each element of the second set corresponds to one and only one element of the first set.

Opposite vertices of a parallelogram – two vertices of a parallelogram that do not lie on the same side of the parallelogram.

Origin – see *Axis.*

Orthocenter of a triangle – the intersection of the three altitudes of the triangle.

Parallelogram – a quadrilateral whose opposite sides are parallel.

Parallel lines – two (or more) lines that do not intersect, no matter how far or in which direction we extend them.

Parallel projection – a projection from one line to another established by parallel rays.

Parallel translation on the plane – a one-to-one correspondence defined by two points A, A' (or by a vector $\overrightarrow{AA'}$) that moves any point B on the plane onto a point B' according to the following rule: quadrilateral $AA'B'B$ is a parallelogram. The point B' is called the image of the point B under the parallel translation.

Parallel translation on a line, *or* **shift** – a one-to-one correspondence defined by two points A, A' on the line that moves any point B on the line into the point B' on the line according to the following rule: segment AB is equal to segment $A'B'$. It is important that both segments AB and $A'B'$ be directed in the same way.

Pentagon – a polygon with five sides. It also has five vertices.

Perigon – see *Angle – Complete angle.*

Perimeter of a polygon – the sum of the lengths of all its sides.

Perimeter of a triangle – see *Perimeter of a polygon.*

Perpendicular bisector – a perpendicular at the midpoint of a segment.

Perpendicular from a point to a line – a segment connecting the point with a point on the line in such a way that the angle between this segment and the line is a right angle.

Perpendicular lines – lines determined by rays forming sides of a right angle.

Point – in our geometry, we assume that there exist in the plane "ideal points" that have no length and no width.

Polygon – a domain bounded by a closed polygonal line. Or you can say, a polygon is a domain that is bounded by a set of segments such that no two of them intersect except at their endpoints. These segments are the sides of the polygon.

Polygon circumscribed around a circle – a convex polygon whose sides are tangent to the circle.

Polygon inscribed in a circle – a convex polygon whose vertices lie on the circle.

Polygonal domain – a domain in the plane whose boundary consists of a finite number of rays and/or segments. The boundary of a polygonal domain may contain either two rays or no rays. A polygonal domain can be bounded or unbounded.

Bounded polygonal domain – a polygonal domain whose boundary consists only of a finite number of segments. It is not possible for a ray to lie completely within a bounded polygonal domain. The boundary of a bounded polygonal domain is a closed non-self-intersecting broken line, or a closed polygonal line.

Unbounded polygonal domain – a polygonal domain which is not a bounded polygonal domain. The boundary of an unbounded polygonal domain has exactly two rays and may also contain segments.

Polygonal line – a non-self-intersecting broken line. A closed polygonal line is a closed non-self-intersecting broken line.

Protractor – an instrument used to measure angles.

Projection from a point – see *Central perspective.*

Projection of point A onto line b – the intersection with line b of a perpendicular from point A to line b.

Projection of segment a onto line b – the set of projections of all points of segment a onto line b.

Projective geometry – in our book, geometric constructions that are performed based only on the following two axioms: 1. Through any two points one can draw a unique line. 2. Any two lines either have a unique point of intersection or are parallel.

Ptolemy's theorem – If a quadrilateral is inscribed in a circle then the product of its diagonals is equal to the sum of the products of its opposite sides.

Pythagorean theorem – a very important theorem which establishes a relation between all the sides in a right triangle: $a^2+b^2=c^2$, where a, b are the legs and c is the hypotenuse of the right triangle.

Quadrilateral – a domain bounded by four non-intersecting segments. Or: a polygon with four sides. A quadrilateral has four vertices.

Radian – a central angle such that its arc length is equal to its radius.

Radian measure of a central angle – the ratio of its arc length to the corresponding radius.

Radius – a segment connecting a point on the circle with the center of the circle. Note that sometimes the term radius is also used to indicate the length of such a segment. By the definition of a circle, any of its radii has the same length.

Rational number – a number that can be represented by an ordinary fraction of the form $\frac{m}{n}$, where m and n are whole numbers.

Ray, *or* **half-line** – a part of a straight line that is bounded by a point on one side. This boundary point, called the endpoint, is part of the ray. A ray extends infinitely in only one direction.

Rectangle – a quadrilateral that has four right angles. One can also say that it is a parallelogram which has one right angle. This implies that all angles in a rectangle are equal and are right angles.

Reflection – Reflection of a point with respect to line a is a one-to-one correspondence whereby to any point P in the plane there corresponds a point P' in the following way: we draw a line through the point P perpendicular to line a and choose the point P' so that $OP' = OP$, where O is the intersection of these lines. We can also say that point P' lies at the same distance from line a as point P but on the other side of line a. Point P' is the image of point P under reflection.

Reflexivity of equal segments – the statement that a segment is considered equal to itself.

Reflexivity of parallel lines – the statement that a line is considered parallel to itself.

Regular polygon – a convex polygon such that all its sides are equal and all its interior angles are equal. Examples are: an equilateral triangle, a square, a regular pentagon, a regular hexagon, a regular heptagon.

Rhombus – a parallelogram that has all sides equal to each other.

Rotation through an angle α – a one-to-one correspondence established as follows: to any point P, the corresponding point P' is obtained by drawing a circle with center O and constructing the angle α with one ray lying on OP and its other ray lying in the counterclockwise direction from the ray OP. The point P' is then chosen on the second ray so that $OP = OP'$. (The point O is the center of this rotation.)

Sector – part of a disk bounded by two radii and the arc between them.

Segment – part of a line between two points on this line. A segment includes these two points, which are called endpoints of the segment.

Segment of zero length – a segment, both of whose endpoints are at the same point.

Self-symmetric figure – see *Centrally symmetric figure.*

Semiperimeter of a polygon – half of its perimeter.

Shift – see *Parallel translation on a line.*

Side of an angle – a ray (or a segment on it) that, with another ray or segment, forms this angle.

Side of a polygon – a segment between two neighboring vertices of the polygon.

Side of a triangle – see *Side of a polygon.*

Similarity *or* **similarity transformation with center of similarity** O – a one-to-one correspondence between points on the plane that assigns to each point A a point A' such that $A'O = k \cdot AO$, where k is a constant factor and is a rational or real number. The number k is called the similarity coefficient. Under a similarity, the "shape" of a figure does not change, but lengths change. Note that similarity preserves ratios of distances.

Similar figures – figures which are obtained from one another by a similarity transformation.

Special lines of a triangle – some lines in a triangle are of special interest. They are: median, angle bisector, altitude, bimedian.

Special points of a triangle – some points in a triangle are of special interest. They are: the centroid, or point of intersection of the medians; the incenter, or point of intersection of the angle bisectors; the circumcenter, or point of intersection of the perpendicular bisectors of the sides; and the orthocenter, or point of intersection of the altitudes.

Square – a rectangle that has all sides equal to each other. One can also say that a square is a rhombus that has one right angle.

Stable position of lines (or points) – a configuration of lines (or points) such that no matter how we move them it is always possible to make this move small enough that the lines (or points) still remain in this configuration.

Straight line – we make an assumption that there is an "ideal straight line" that has no width and goes infinitely in both directions.

Subtend – we say that segment BC subtends angle α at point A when the position of segment BC in relation to point A is such that $\angle BAC = \alpha$.

Sum of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ – a vector equal to the diagonal of the parallelogram constructed on the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ both beginning at the point A with the point A being the beginning of the resulting vector $\overrightarrow{a} + \overrightarrow{b}$.

Symmetric figure with respect to line *a* intersecting this figure – a figure such that the reflection in line a of the part of the figure on one side of line a coincides with the part of the figure on the other side of line a. Line a is called a line of symmetry of the figure.

Symmetry of equal segments – the statement that if line a is equal to line b, then line b is also equal to line a.

Symmetry of parallel lines – the statement that if line a is parallel to line b, then line b is parallel to line a.

Symmetry with respect to a line, *or* **line symmetry** – see *Reflection.*

Symmetry with respect to a point, *or* **central symmetry about a point *O*** – a one-to-one correspondence that assigns to every point A on the plane a point A' according to the following rule: point A' lies on the line AO on the other side of the point O from point A, and $OA' = OA$. Point A' is the image of point A under central symmetry. Point O is called the center of symmetry.

Symplectic geometry – in our book, constructions which can be performed in affine geometry in which area measure is introduced.

Tangent – a straight line which has only one common point with a circle.

Tangent circles – two circles which have only one common point.

Transformation on the plane – a term used to indicate a one-to-one correspondence of the plane onto itself. In this book we considered the following transformations: parallel translation, rotation through an angle (these two are called motions), reflection in a line (or symmetry with respect to a line), central symmetry (or rotation through $180°$), and similarity. Note that projection from a point and parallel projection are not transformations on the plane because they are not one-to-one correspondences.

Transitivity of equal segments – the statement that if two segments a and b are equal and the segment b is equal to a segment c, then the segments a and c are also equal.

Transitivity of parallel lines – the statement that if two lines a and b are parallel and the line b is parallel to a line c, then the lines a and c are also parallel.

Trapezoid – a quadrilateral which has two opposite sides parallel.

Bases of a trapezoid – parallel sides of the trapezoid.

Leg of a trapezoid – a side of a trapezoid which is not parallel to any other.

Triangle – a bounded domain formed by three lines in general position. Or: a triangle is a polygon with three sides. Note that a triangle has three vertices. Below are some types of triangles.

Right triangle – a triangle with one right angle. The two sides of this right angle are called legs and the third side is called the hypotenuse.

Isosceles triangle – a triangle two of whose sides are equal to each other. The third side of such a triangle is called its base.

Equilateral triangle – a triangle which has all sides equal.

Triangle circumscribed around a circle – see *Polygon circumscribed around a circle.*

Triangle inscribed in a circle – see *Polygon inscribed in a circle.*

Triangulation of a polygon – a decomposition of a polygon into triangles whose vertices coincide with the vertices of the polygon in such a way that the triangular domains do not mutually intersect except at their boundaries and the union of all these triangles gives the polygon.

Turn through an angle – see *Rotation through an angle* α.

Unbounded polygonal domain – see *Polygonal domain.*

Unit area – a figure whose area is chosen to be equal to one. The area of any other figure is then measured in comparison to this unit area.

Unit circle – a circle whose radius is accepted to be of one unit of length.

Unit parallelogram – a parallelogram chosen to have a unit area.

Unit segment – a segment chosen to have a unit length; see also *Axis.*

Unit square – a square whose side length is equal to 1.

Vector – a directed segment. It is usually denoted as $\vec{a}$ or as $\overrightarrow{AB}$, where the point A is the beginning of the vector and B is its end.

Vertex of an angle – the endpoint of the rays forming the angle.

Vertex of a polygon – an intersection point of two neighboring segments of its boundary.

Vertex of a triangle – see *Vertex of a polygon.*

Vertical angles – a pair of angles such that the sides of one angle are the extensions of the sides of the other angle. We can also say that vertical angles are a pair of opposite angles formed by two intersecting straight lines.

Zero angle – an angle whose bounding rays coincide with each other.

Zero vector – a vector whose beginning coincides with its end.