- s is a strategy profile which is an assignment of strategies for each player. E.g. $s = (N, S)$ is a strategy profile for 2 players where player 1 goes north and player 2 goes south.
- s_i is the chosen strategy (chosen action) for player i given profile s. E.g. for the strategy profile $s = (N, S)$, $s_1 = N$, aka player 1 chooses the north strategy.
- S_i is the set of strategies available to player i. E.g. $S_1 = \{N, S\}$, aka player 1 can choose either north or south. For a symmetrical game, $S_1 = S_2$.
- $u_i(s)$ is the utility (payoff) for player i given strategy profile s. E.g. for the strategy profile $s = (N, S)$, $u_1(N, S) = 1$, aka player 1 receives a payoff of 1 when they choose north and player 2 chooses south.
- σ is a mixed strategy profile. E.g. $\sigma = ((0.5, 0.5), (0.25, 0.75))$ is a mixed strategy profile for 2 players where player 1 chooses north with probability 0.5 and south with probability 0.5, and player 2 chooses north with probability 0.25 and south with probability 0.75.
- σ_i is the mixed strategy for player i given mixed profile σ. E.g. for the mixed strategy profile above, $\sigma_1 = (0.5, 0.5)$, aka player 1 chooses north with probability 0.5 and south with probability 0.5.
- $\sigma_i(s_i)$ is the probability that player i chooses strategy s_i given mixed strategy σ_i. E.g. for the mixed strategy $\sigma_1 = (0.5, 0.5)$, $\sigma_1(N) = 0.5$, aka player 1 chooses north with probability 0.5.
- Σ_i is the set of mixed strategies available to player i. E.g. $\Sigma_1 = \{(p, 1-p) \mid 0 \leq p \leq 1\}$, aka player 1 can choose any probability distribution over their strategies.
- $u_i(\sigma)$ is the expected utility (expected payoff) for player i given mixed strategy profile σ.

  $$u_i(\sigma) = \sum_{s \in S} \left(\prod_{j=1}^{I} \sigma_j(s_j) \right) u_i(s)$$

  Steps to find $u_i(\sigma)$:

  - For each strategy profile s in the set of all strategy profiles S:
  - Calculate the probability of that strategy profile occurring by multiplying the probabilities of each player choosing their respective strategies in that profile.
  - Multiply that probability by the utility (payoff) for player i given that strategy profile.
  - Sum these values over all strategy profiles to get the expected utility for player i.

  For the mixed strategy profile $\sigma = ((0.5, 0.5), (0.25, 0.75))$, the expected utility for player i is:

  $$\begin{aligned} u_i(\sigma) &= \sigma_1(N)\sigma_2(N)u_i(N, N) + \sigma_1(N)\sigma_2(S)u_i(N, S) \\ &\quad + \sigma_1(S)\sigma_2(N)u_i(S, N) + \sigma_1(S)\sigma_2(S)u_i(S, S) \end{aligned}$$

- A pure strategy is discrete: either north or south for example
- A mixed Nash equilibrium is the best outcome for both players when they're using mixed strategies (where the strategy is a probability of going north or south rather than a discrete north or south choice).
- A simplex is when each component adds to 1, e.g. $x + y = 1 \leftarrow 2$ strategy game.
- σ is a mixed profile
- $x = f(x)$ is a fixpoint
- Brouwer fixpoint theorem
- Kakutani fixpoint theorem
- Every n-player game has at least one Nash equilibrium.

- Sperner's Lemma:
  - Pick an arbitrary set of $n-1$ colors (let's say red and blue)
  - Count red-blue edges after slicing simplex along cutlines
  - Count exterior edges (call it a)
  - Count interior edges twice (call it b)
  - Total count is $a+2b$

Lecture 3

- A symmetric 2 player game happens when $B = A^T$, where A and B are the matrices of the payoff for each player.
- Let probability vector x be P1's mixed strategy. Let y be player 2's mixed strategy. E.g. $y = (0.5, 0.5)$.
- The vector of payoff for P1's pure strategies is Ay, where A is a matrix of payoffs and y is a vector of probabilities.
- P1 gets $x^t Ay$. This ends up being a single number. P2 gets $x^t By$.
- Ay is a column vector of payoffs for each of p1's pure strategies.
- P1's best response correspondence is

$$r_1(y) = \underset{x \geq 0 \quad \text{and} \quad ||x||_1 = 1}{\arg\max} x^t Ay$$

- $||x||_1 = 1$ means all components of x add to 1.
- Best response can have a mix only when there are ties in the vector Ay.
- P2 best response correspondence is

$$r_2(x) = \arg\max_y x^t By$$

- A nash equilibrium is a pair (x^*, y^*) such that $x^* \in r_1(y^*)$ and $y^* \in r_2(x^*)$.
- For any 2 player game given by (A, B) we can define a related symmetric game $(\tilde{A}, \tilde{B})$ where

$$\tilde{A} = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}$$

  and

$$\tilde{B} = \tilde{A}^T = \begin{bmatrix} 0 & B \\ A^T & 0 \end{bmatrix}$$

- Claim: If (x^*, y^*) is a nash equilibrium for $(\tilde{A}, \tilde{B})$, then $(\frac{x^*}{||x^*||_1}, \frac{y^*}{||y^*||_1})$ is a nash equilibrium for (A, B).
- Ax^* is a column vector of payoffs for each of p2's pure strategies.
- $$A = \begin{bmatrix} 0 & -0.8 & 1.2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

  Guess: Rock, Paper, Scissors all in support of equilibrium symmetric strategy. Call R = x_1, P = x_2, S = x_3.

$$-0.8x_2 + 1.2x_3 = x_1 - x_3 = x_2 - x_1$$

  x_3 is equal to $1 - x_2 - x_1$ since they all must add to 1. Substituting in we get

$$1.2 - 2x_2 - 1.2x_1 = 2x_1 + x_2 - 1 = x_2 - x_1$$

  Solving this we get $x_1 = 1/3, x_2 = 0.377778, x_3 = 1 - 1/3 - 0.377778$.

- "In support" means it's part of the mixed strategy.
- You can never have a dominated strategy in a nash equilibrium.
- Dominated means that you could pick some other strategy such that no matter what the other player does, you always do better.

Lecture 4

- Definition 1 of a Nash Equilibrium: A strategy profile such that each player is playing a best response.
- Definition 2: A strategy profile and beliefs for each player about how each of the other players will play.
  - Each player is maximizing payoff given those beliefs.
  - Beliefs are correct.
- Rational means "I won't do things that I know will give me an inferior payoff.". I know my opponent(s) is rational. I know my opponent knows that I'm rational. Etc etc. This is called common knowledge of rationality.
- Rationalizable profiles: Profile + Beliefs for each player.
  - Each player maximizes payoff with respect to their beliefs.
  - Beliefs are not inconsistent with rationality being common knowledge.
- Example: Sealed first price auction. Each player submits a bid without knowing the other player's bid (sealed). The highest bid wins and pays their bid (first price).

	Valuation
Player 1	3.5
Player 2	1

  Suppose they know the valuations of each other

	Payoff
win	valuation - bid
lose	0
tie	1/2(valuation - bid)

P2

	1	2	3
P1 1	5/4, 0	0, -1,	0, -2
P1 2	3/2, 0	3/4, -1/2	0, -2
P1 3	1/2, 0	1/2, 0	1/4, -1

  - For P2, strategy 1 strictly dominates strategy 3. Strategy 1 weakly dominates strategy 2 because there is a tie in the last row.
  - You can always eliminate strictly dominated strategies. After revising the game by eliminating a dominated strategy, the other player might now have a strictly dominated strategy since it no longer considers the column/row that was eliminated.
  - In the above example, now that we remove column 3, row 1 is strictly dominated by row 2. Furthermore, row 3 is strictly dominated by row 2.
  - This process is called iterated deletion of strictly dominated strategies (IDSDS).
- Another example:

	P2 N	P2 S
P1 N	0, 0	10, -10
P1 S	5, -5	-1, 1

- If you find that a mixed nash equilibrium has a negative number, then the strategies you chose to be in support of the equilibrium were wrong.
- There are always an odd number of Nash equilibria.