Simulation & Numerical Analysis of a Simple Harmonic Oscillator (SHO) and the Coulomb Force between a point charge and a uniformly charged bar

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Abstract

This report presents the simulation and analysis of two fundamental problems in physics: the Coulomb force between a point charge and a uniformly charged bar, and the Simple Harmonic Oscillator (SHO). The Coulomb force is computed using numerical integration methods, including Riemann sum, trapezoidal rule, and Simpson's rule, and is compared with the SciPy library's ODE solver. The SHO problem is solved using numerical methods such as Euler's method and the 4th order Runge-Kutta method, with results compared to the exact solution. The accuracy and error behavior of each method are evaluated, and the effects of different parameters and step sizes on the results are discussed.

1 Introduction

In this report, we address two classical physics problems:

- **Coulomb Force**: The interaction between a point charge and a uniformly charged bar. The Coulomb force is calculated using numerical integration methods.
- **Simple Harmonic Oscillator (SHO)**: The motion of an object attached to a spring, described by a second-order differential equation. The SHO is solved numerically and compared to the exact solution.

2 Coulomb Force Calculation

The Coulomb force between a point charge and a uniformly charged bar is described by the following integral:

$$F = \int_{x_{\text{bar}}-L/2}^{x_{\text{bar}}+L/2} \frac{k_e q \lambda}{(x_0 - x)^2} dx$$

where k_e is Coulomb's constant, q is the point charge, $\lambda = Q/L$ is the charge density of the bar, and x_0 is the position of the point charge. The integration is carried out using numerical methods: Riemann sum, trapezoidal rule, and Simpson's rule.

2.1 Implementation

The Coulomb force is computed by discretizing the bar and summing up the contributions from each infinitesimal charge element. The following numerical methods were used:

- Riemann sum
- Trapezoidal rule
- Simpson's rule

We also use SciPy's `quad` function for comparison with the analytic solution.

2.2 Results

Figures 1 and **??** show the Coulomb force as a function of distance from the point charge to the center of the bar, computed using different numerical methods. The results were validated by comparing them with SciPy's solution.

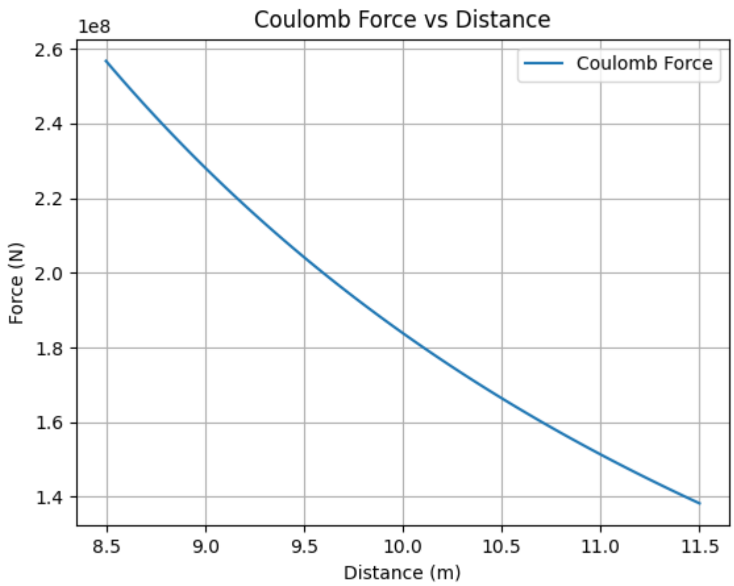

Figure 1: Coulomb force as a function of distance from the point charge to the center of the bar. In this example, the point charge has a charge of 1 Coulomb at x = 0 (m), and the charged bar has total charge of 2 Coulomb at x = 8.5 to 11.5 (m). The method implied was Simpson's Rule.

The comparison of using Simpson's Rule amd Scipy for Figure 1 are:

```
Calculated Coulomb Force using riemann: 114130610.90930647 N
Calculated Coulomb Force using scipy: 114126984.12698413 N
```

3 Simple Harmonic Oscillator (SHO)

The Simple Harmonic Oscillator describes the motion of a mass attached to a spring, governed by the second-order differential equation:

$$m\frac{d^2x}{dt^2} = -kx$$

The solution is periodic and given by:

$$x(t) = A\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency, A is the amplitude, and ϕ is the phase. This equation is solved using numerical methods.

3.1 Methods

The SHO problem was solved using the following methods:

3.1.1 Euler's Method

Euler's method approximates the solution by updating the position and velocity at each time step:

$$x_{n+1} = x_n + \Delta t v_n$$
$$v_{n+1} = v_n - \Delta t\frac{k}{m}x_n$$

3.1.2 Runge-Kutta 4th Order Method

The 4th order Runge-Kutta method provides a more accurate solution by using intermediate slopes:

$$k_1 = \Delta t \cdot f(t_n, x_n)$$
$$k_2 = \Delta t \cdot f\left(t_n + \frac{\Delta t}{2}, x_n + \frac{k_1}{2}\right)$$
$$k_3 = \Delta t \cdot f\left(t_n + \frac{\Delta t}{2}, x_n + \frac{k_2}{2}\right)$$
$$k_4 = \Delta t \cdot f(t_n + \Delta t, x_n + k_3)$$

3.1.3 Comparison with SciPy's ODE Solver

We compared the results from Euler's method and the Runge-Kutta method with SciPy's ODE solver, which provides an efficient and accurate solution.

3.2 Results

Figures 2 and 3 show the position of the object over time using Euler's method and the Runge-Kutta method, respectively, with the exact solution plotted for comparison.

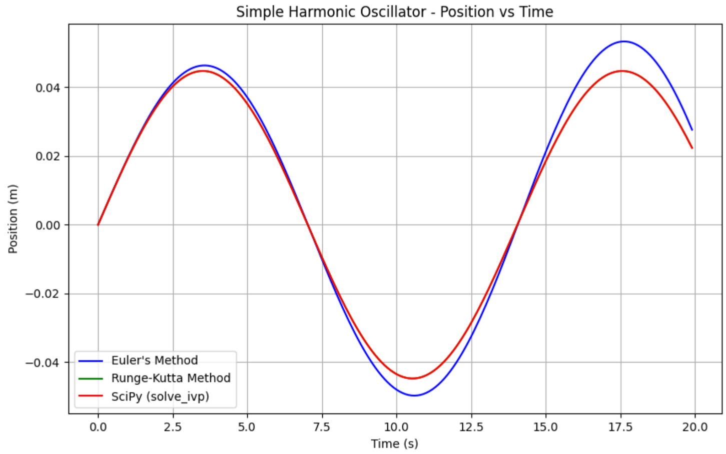

Figure 2: Position of the object over time (20 s): Euler's method vs. exact solution (Scipy-solution). The parameters are: spring mass 0.05 kg with a spring constant of 0.01 N/m, starting at $x = 0\,\text{m}$ with an initial velocity of $0.02\,\text{m/s}^2$ to the right. The time step Δt is 0.1 s, and the total time allowed for this oscillation is 20 s.

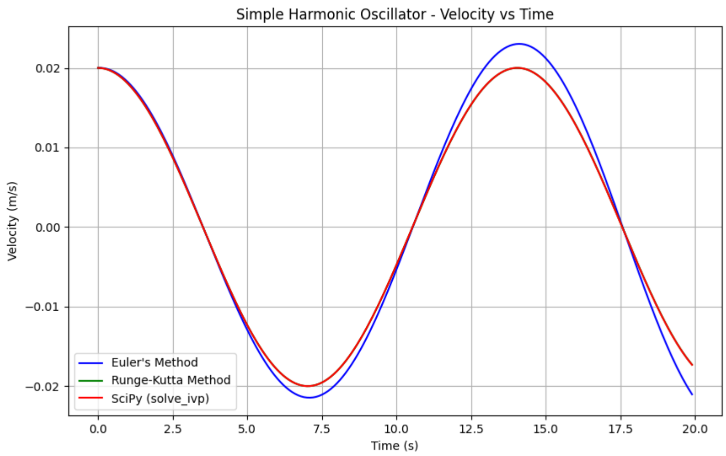

Figure 3: Velocity over time using the same parameters as Figure 2.

3.3 Error Analysis

We analyzed the truncation and global errors for both Euler's method and the 4th order Runge-Kutta method. The errors were observed to scale with the step size, with Euler's method exhibiting linear error scaling, and the Runge-Kutta method exhibiting Δt^4 error scaling.

4 Conclusion

This report presented the simulation and analysis of two fundamental problems in classical mechanics: the Coulomb force between a point charge and a uniformly charged bar, and the Simple Harmonic Oscillator. The Coulomb force was calculated using various numerical integration methods and validated against SciPy's solution. The SHO problem was solved using Euler's method and the 4th order Runge-Kutta method, with results compared to the exact solution. The numerical methods provided good approximations, with the 4th order Runge-Kutta method yielding more accurate results than Euler's method, especially for larger step sizes.