Ryu-Takayanagi formula for holographic entanglement entropy

Reference: 0605073 || 1609.01287, the pictures used in the notes are taken from the two references.

Introduction

The RT's proposal is a simple yet very elegant proposal based on AdS/CFT correspondence. Essentially, they proposed that the entanglement entropy associated with a spatial region in a holographic QFT is given by the area of a particular minimal area surface in the dual geometry.

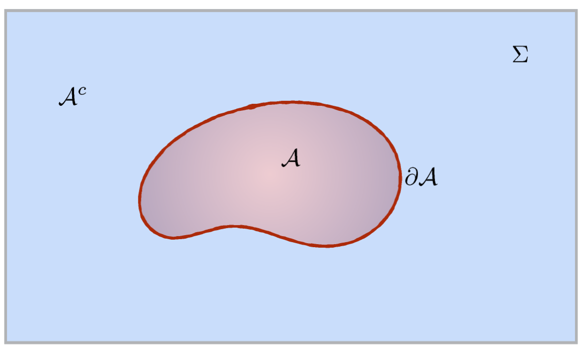
Why would they propose this? What are the implications of the proposal?

The authors were originally motivated by the black hole entropy formula by Hawking and Bekenstein

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N} \tag{1}$$

The black hole entropy is a measure of the information lost to the observers due to the presence of the horizon. The horizon separates spacetime into the interior of the black hole and the exterior of the black hole. The observer sits in the exterior and cannot receive signals from the interior. And the BH formula says that the entropy from this screening is proportional to the area of the horizon.

This is extremely reminiscent of the construction of entanglement entropy in quantum systems. In general, let's take a d-dimensional QFT on some flat Minkowski spacetime, say $\mathbb{R}^{1,d-1}$. Since the entanglement entropy we discuss is a spatial concept, we can pick a spacelike Cauchy slice Σ_{d-1} at some time t_0, and bipartition it into a region A and its complement A^c with ∂A as the boundary.



Let $|\Psi\rangle$ be the ground state of the theory at t_0 and ρ_Σ be the pure-state density matrix defined on the entire slice. Then we can find the entanglement entropy defined on region A by tracing out ρ_Σ on region A^c to have

$$S_A = -\operatorname{Tr}_A(\rho_A \log \rho_A), \rho_A = \operatorname{Tr}_{A^c}(\rho_\Sigma) \tag{2}$$

Since we have traced out region A^c when defining S_A, we can think of S_A as the entropy for an observer only accessible to region A but not A^c. In this sense, region A^c is analogous to the interior of the black hole that an observer in A does not have access to. And the boundary ∂A acts as the horizon for screening. So one sees that there is some flavor of similarity in entanglement entropy to the black hole entropy.

The authors were motivated by this similarity, and further had a Eureka when thinking about the AdS/CFT correspondence. Since the CFT$_d$ vacuum that preserves the $S(d,2)$ conformal symmetry is dual to the vacuum AdS$_{d+1}$ spacetime, maybe it is possible to calculate the entanglement entropy S_A for a subsystem in CFT$_d$ from the area of some surface in its dual AdS$_{d+1}$ spacetime. This led them to propose the following formula

$$S_A = \frac{\text{Area of } \gamma_A}{4G_N^{(d+1)}} \tag{3}$$

where γ_A is some minimal surface in AdS$_{d+1}$ whose boundary is given by ∂A, $G_N^{(d+1)}$ is the Newton constant in $d+1$ dimensions.

This proposal has great implications. It is the first successful attempt that gave a relatively comprehensive gravitational interpretation of the entanglement entropy. Moreover, it established a firm connection between the entanglement structure of quantum states in QFT with the geometry of gravitational spacetime. People working with gravity were inspired to further look into how geometry can be built from entanglement. This has led to great works such as the ER = EPR conjecture.

Note: ER = EPR is a conjecture that two entangled particles (or an EPR pair) are connected by a wormhole (ER bridge).

It also provided an alternative to calculating entanglement entropy in QFT. Before, people relied on the path integral approach; used the replica trick and twisted operators to calculate the entanglement entropy. But the calculations become extremely difficult with higher dimensions. The holographic interpretation gives a nice alternative to this. In principle, if dual geometry for A is not too complicated and we can solve for the minimal surface, one should be able to calculate S_A in any dimensions.

Preliminaries

Before going into any specifics, we clarify that in this talk, we will only consider time-independent CFT and the entanglement entropy for vacuum states of such CFT. The theory can be generalized to the time-dependent case and/or thermal states. But we will skip it for the interest of time. Since we are considering continuum QFT, we will introduce a UV regulator ϵ for it to represent UV divergences. One can regard it as lattice constant in the limit $\epsilon \to 0$.

Under this setting, the particular minimal surface is the RT formula is actually a static minimal surface at some time t_0.

Note: In the time-dependent case, γ_A is some extremal surface that takes time into account.

For the vacuum states in a d-dimensional QFT, people generally believe that the S_A should follow an area law,

$$S_A = \alpha \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \ldots, \alpha \text{ is constant} \tag{4}$$

where the leading order of divergence in S_A is proportional to the area of the boundary of A. This is intuitive since vacuum states should be more local then thermal or excited states. Then the entanglement is strongest at the boundary of A. In general, the UV behavior of S_A should follow the following expansion

$$S_A = \begin{cases} a_{d-2}(\frac{L}{\epsilon})^{d-2} + a_{d-4}(\frac{L}{\epsilon})^{d-4} + \cdots + a_1 \frac{L}{\epsilon} + (-1)^{\frac{d-1}{2}} a^* + O(\epsilon), d \text{ odd} \\ a_{d-2}(\frac{L}{\epsilon})^{d-2} + a_{d-4}(\frac{L}{\epsilon})^{d-4} + \cdots + (-1)^{\frac{d-2}{2}} a^* \log(\frac{L}{\epsilon}) + O(\epsilon^0), d \text{ even} \end{cases} \tag{5}$$

Here $L^{d-2} \propto \text{Area}(\partial A)$. We can compare our calculations to this to check whether they agree at least to the order of magnitudes.

Note: the expansion is originally found numerically and checked in many later arguments. 2D CFT has a logarithmic divergence and fails to follow an area law. May be heuristically understood as the special case of boundary consisting of disconnected points.

Holographic derivation via path integral

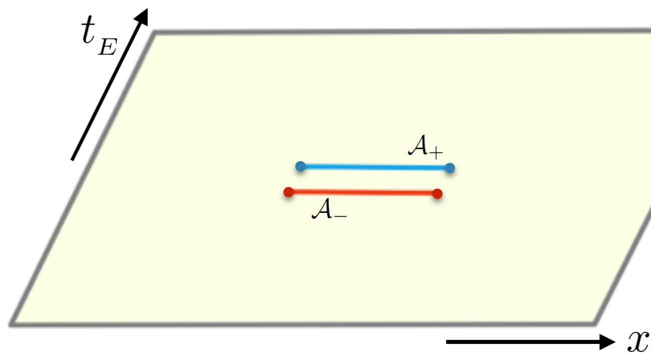
Why RT proposed that the entanglement entropy should be proportional to the static minimal surface?

Intuitively, since the minimal surface locally minimizes its area, when subject to the boundary condition given by ∂A, it should provide the severest entropy bound for the lost information. A rigorous derivation of the formula is very involved. But we may gain some intuition by looking at an example from AdS$_3$/CFT$_2$.

Note: Minimal surface is a surface that locally minimizes its area. Alternatively, it can be defined as a critical point of the area functional for all compactly supported variations. The later definition is analogous to geodesics.

We will derive S_A using the path integral and replica trick. Then we will show that if we take A to be a single line interval, then in the AdS$_3$ dual of the CFT$_2$, we obtain an entanglement entropy proportional to the geodesic distance of two boundary points. Hopefully, the derivation with path integral and the replica trick will also make one appreciate the elegance of holographic interpretation of entropy.

For simplicity, we will work in Euclidean signature. Let's take a Cauchy slice at $t = 0$, and define the region A to be a line interval $A = \{x \in \mathbb{R} | x \in (-a, a)\}$ around the original.



We can find the ground state wavefunction Ψ at $t = 0$ by path-integrating the field $\phi(t_E, x)$ in the 2D CFT from $t_E = -\infty$ to 0,

$$\Psi(t_E = 0, x) = \int_{t_E=-\infty}^{t_E=0} D\phi e^{-S(\phi)} \tag{6}$$

And the complex conjugate of Ψ is given by path-integrating backwards ϕ from ∞ to 0. Then we can compose the density matrix on the slice as $\rho_\Sigma = |\Psi\rangle\langle\Psi|$. To calculate S_A, we want to obtain the reduced density matrix on A. To do so, let's define two fields ϕ_-, ϕ_+ s.t.

$$\phi_A|_{t=0^-} = \phi_-, \phi_A|_{t=0^+} = \phi_+ \tag{7}$$

Then we can project the density matrix onto region A by imposing the boundary conditions that the field ϕ takes value of either ϕ_- or ϕ_+ at time $t = 0^-$ or $t = 0^+$. The projected reduced density matrix is

$$(\rho_A)_{-+} = (Z_1)^{-1} \int_{t_E=-\infty}^{t_E=\infty} D\phi e^{-S(\phi)} \prod_{x \in A} \delta(\phi(0^+, x) - \phi_+(x)) \cdot \delta(\phi(0^-, x) - \phi_-(x)) \tag{8}$$

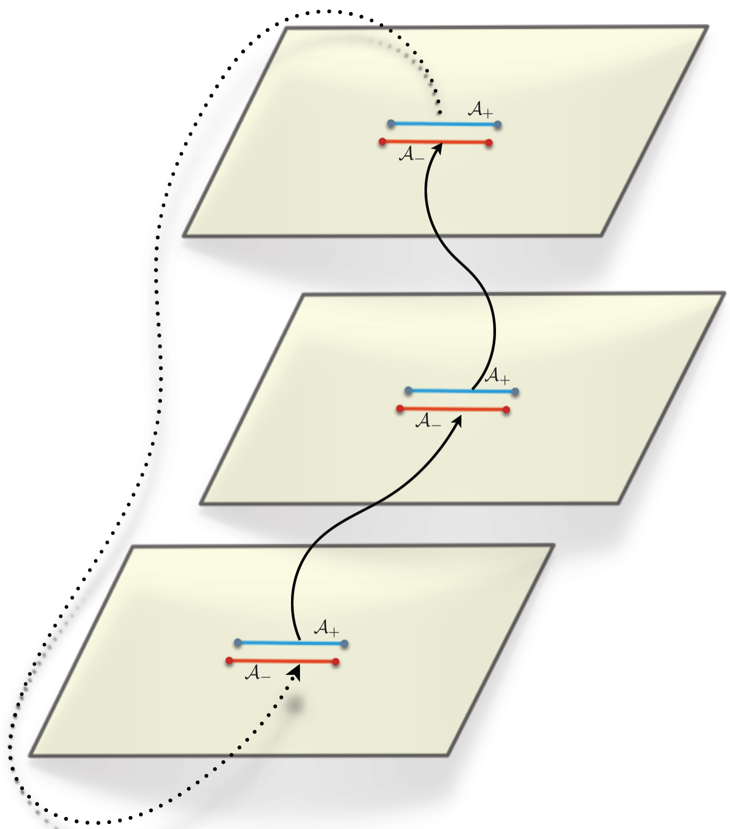
Here, Z_1 is the vacuum partition function used for normalization.

Note: By making it just equal to a ϕ_A value, we essentially say ϕ is ignorant of A^c.

There is a nice entity called the Renyi entropy that converges to the Von Neumann entropy in the limit $q \to 1$. One can think of it as coming from performing the replica trick on the Von Neumann entropy.

$$S_A^{(q)} = \frac{1}{1-q} \log \mathrm{Tr}_A(\rho_A^q) \tag{9}$$

The only quantity that we need to evaluate for the Renyi entropy is $\mathrm{Tr}_A(\rho_A^q)$. To do so, we will take q copies of the path integral, and glue them together.



The extra boundary condition for gluing is $\phi_+^{(k)} = \phi_-^{(k+1)}$, which can be seen from the matrix multiplication $(\rho_A)^q_{-+}$. Observe that after gluing together q copies of ρ_A, their $\mathrm{Tr}_A(\rho_A^q)$ now has a cyclic $\mathbb{Z}_q$ permutation symmetry (or replica symmetry). This also introduces q branch points in the A interval due to the q Riemann sheets.

There is a clever trick in evaluating the seemingly complicated trace. We want to conformally map the q genus-0 Riemann sheets to a complex plane so that we can just calculate CFT$_2$ on $\mathbb{C}$. This can be done by redefining a composite field φ on the complex plane with twisted boundary conditions.

Note: Twisted boundary conditions can be thought of as implementing the $\mathbb{Z}_q$ symmetry with an extra phase factor in defining the boundary conditions.

By orbifold techniques, we can define a set of twist operators $\{\mathcal{T}_q^{(k)}\}$ that can acts on the boundary of A for each of the q sheets. When inserted on the k-th sheet, we want $\mathcal{T}_q^{(k)}$ to implement the twisted boundary conditions and induce a branch-cut of order q at the point of insertion. Thus, we can write the trace as a product of the two-point functions for the twist operators

$$\mathrm{Tr}_A(\rho_A^q) = \prod_{k=0}^{q-1} \langle \mathcal{T}_q^{(k)-}(-a) \mathcal{T}_q^{(k)+}(a) \rangle \tag{10}$$

The hope is that the two-point functions can give an easier way to evaluate the trace. But this is not the case in higher dimensions. In CFT$_2$, properties of the twist operators are easier to calculate since they consist of local fields. They are be thought of holes locally punctured in space. In higher dimensions, this is not necessarily true. The complexity with twist operators also motivates why RT proposal is a nice alternative.

The last use we have for this formula is to see how $S_A \propto \mathrm{Area}(\gamma_A)$ in AdS$_3$/CFT$_2$. As mentioned, we can conformally map the q sheets to a complex plane. A CFT on the q disconnected sheets with central charge c is equivalent to a CFT on a single complex plane with central charge qc and two twisted operators $\mathcal{T}_q^+, \mathcal{T}_q^-$ inserted. When evaluated in the AdS dual of the CFT, it turns out that the two-point function

$$\langle \mathcal{T}_q^-(-a)\mathcal{T}_q^+(a)\rangle \sim \exp\left(-\frac{2q\Delta q \cdot L_{2a}}{R}\right) \tag{11}$$

where L_{2a} is the geodesic distance between the two boundary points in AdS, Δ_q is the conformal scaling dimensions for the operators. Then if we evaluate the S_A via the Renyi entropy, we have

$$S_A = S_A^{(q)}\Big|_{q=1} = -\frac{\partial}{\partial q}\log \mathrm{Tr}_A(\rho_A^q)|_{q=1} = \frac{2L_{2a}}{R}\left(\frac{\partial(q\Delta_q)}{\partial q}\Big|_{q=1}\right) \propto L_{2a} \tag{12}$$

Examples of holographic entanglement entropy

In the following, we give a few examples on how to calculate holographic entanglement entropy. In particular, we will consider the flat Minkowski spacetime $\mathbb{R}^{1,d-1}$ for both the special case of $d=2$, i.e. CFT$_2$ and a general dimension CFT$_d$. Thus, we will resort to the Poincare coordinates in AdS$_{d+1}$, since the boundary of the Poincare patch gives a conformal Minkowski space. As a reminder, the Poincare coordinates are given by:

$$ds^2 = \frac{R^2}{z^2}(-dx_0^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2) \tag{13}$$

The general procedure of evaluating the holographic entanglement entropy works as following.

For a d-dimensional CFT$_d$, in its $d+1$ dimensional AdS spacetime, we first find the induced metric on a bulk surface with boundary ∂A. If we parametrize the surface with a set of intrinsic parameters $\{\xi^i\}$, we can write the induced metric for the surface as

$$ds_A^2 = \frac{1}{z^2}\left(\frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j} + g_{\mu\nu}(x,z)\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\right)d\xi^i d\xi^j \tag{14}$$

where $g_{\mu\nu}$ is a metric function defined on the boundary geometry.

Note: The metric is written in Fefferman-Graham gauge.

Then, we can write down the geodesic action based on this induced metric as

$$S = 4\pi c_{\text{eff}} \int d^{d-1}\xi \frac{1}{z^{d-1}}\sqrt{\det\left(\frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j} + g_{\mu\nu}(x,z)\frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\right)} \tag{15}$$

where we have defined the effective central charge as

$$c_{\text{eff}} = \frac{R^{d-1}}{16\pi G_N^{(d+1)}} \tag{16}$$

The area given by this action is in units of the AdS plank length. We have scaled it s.t. the Newton constant in the RT proposal is incorporated in it. In the following calculations, the minimum of this action directly give the holographic entanglement entropy. We do not need to scale it with extra factors such as $4G_N^{(d+1)}$.

Note: These formulae are generalized s.t. they also apply to the time-dependent holographic entanglement entropy.

Finally, we can minimize the geodesic action S to find the minimal surface from parametrized by the EOMs.

Vacuum state of CFT$_2$ on $\mathbb{R}^{1,1}$:

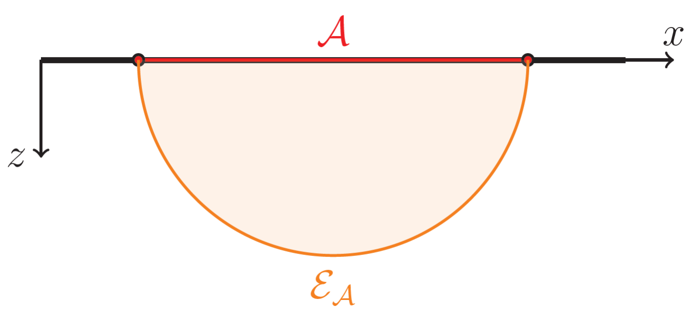
Take the interval to be a line of length $2a$ centered around the origin: $A = \{x \in \mathbb{R} | x \in (-a, a)\}$. WLOG, we will take $t = 0$ since the theory is static. In this case, the geodesic action is given by

$$S = 4\pi c_{\text{eff}} \int d\xi \, \frac{\sqrt{x'(\xi)^2 + z'(\xi)^2}}{z} \tag{17}$$

Under the constraint $\delta S = 0$, we find the EOMs as

$$x(\xi) = a\cos\xi, z(\xi) = a\sin\xi, (\epsilon \le \xi \le \pi - \epsilon) \tag{18}$$

Here, ϵ is the UV cutoff for z. Since $z > 0$, these EOMs parametrize a semi-circle in the xz-plane.



We can integrate over the geodesic action to obtain the entanglement entropy as

$$S_A = 4\pi c_{\text{eff}} 2 \int_{\frac{\epsilon}{a}}^{\frac{\pi}{2}} \frac{d\xi}{\sin\xi} = 8\pi c_{\text{eff}} \log\left(\frac{2a}{\epsilon}\right) = \frac{c}{3} \log\left(\frac{2a}{\epsilon}\right) \tag{19}$$

This S_A has logarithmic divergence with ϵ, as predicted by the aforementioned general expansion formula.

Vacuum state of CFT$_d$ on $\mathbb{R}^{d-1,1}$:

We have more choices of the interval A in the higher dimensional case. We will consider two obvious regions of interest: a strip and a disk.

(a) For the strip, we pick one spatial coordinate x_1 to be in a line interval $(-a, a)$, but puts no interval constraint on the rest of the x^i. More concretely, the region of the strip is characterized by $A_1 = \{\mathbf{x}_{d-1} \in \mathbb{R}^{d-1} | x_1 \in (-a, a), x_i \in \mathbb{R} \text{ with } i = 2, 3, \ldots, d-1\}$. Obviously, the strip preserves the $(d-2)$-dimensional translational invariance.

Given this symmetry, it is convenient for us to pick the parameter $\xi^a = x^a$. Under this choice, the geodesic action is

$$S = 4\pi c_{\text{eff}} \int d^{d-2}x \, dx_1 \frac{\sqrt{1 + z'(x_1)^2}}{z^{d-1}} \tag{20}$$

This gives the EOM

$$\frac{dz}{dx_1} = \frac{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}}{z^{d-1}} \tag{21}$$

where z_* is some constant. To determine the constant, observe that $dz/dx_1 = 0$ when $z = z_*$. This suggests that z_* should be a turning point on the minimal surface. So we can integrate on both sides to obtain

$$\int_0^a dx_1 = \int_0^{z_*} dz \frac{z^{d-1}}{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}} \tag{22}$$

$$\Rightarrow z_* = a \frac{\Gamma(\frac{1}{2(d-1)})}{\sqrt{\pi}\Gamma(\frac{d}{2(d-1)})} \tag{23}$$

The exact form of the surface is not particularly illuminating, but we can compute the area of the minimal surface. Introduce an IR regulator L for the $d-2$ translationally invariant directions, we have

$$S_{A_1} = \frac{4\pi c_{\text{eff}}}{d-2} L^{d-2} \left[\frac{2}{\epsilon^{d-2}} - \left(\frac{2}{z_*} \right)^{d-1} \frac{1}{a^{d-2}} \right] \tag{24}$$

Again, the leading order of the divergence checks our with the previous formula. But note that there exists no subleading divergent terms in this case. This is a special case due to the fact that we have chosen an entangling surface that is intrinsically flat and has no extrinsic curvature.

(b) Another region with a nice symmetry is a spatial region inside a $\mathbf{S}^{d-2}$ of radius l. More specifically, $A_o = \{\mathbf{x}_{d-1} \in \mathbb{R}^{d-1} | \sum_i^{d-1} x_i^2 \leq l^2\}$. Thus, this region has $SO(d-2)$ spherical symmetry.

Given this symmetry, we will take ξ to be the radial coordinate of $\mathbb{R}^{d-1}$ and the rest of the parameters can parametrize the angular coordinates. Then geodesic action in this case is

$$S = 4\pi c_{\text{eff}} w_{d-2} \int d\xi \frac{\xi^{d-2}}{z^{d-1}} \sqrt{1 + z'(\xi)^2} \tag{25}$$

Here w_{d-2} gives the area of a unit $\mathbf{S}^{d-2}$, $w_{d-2} = 2\pi^{(d-1)/2}/\Gamma(\frac{d-1}{2})$. The EOM for this action is

$$z^2 + \xi^2 = l^2 \tag{26}$$

Since $z > 0$, the minimal surface in this case is given by a hemisphere. Parametrize

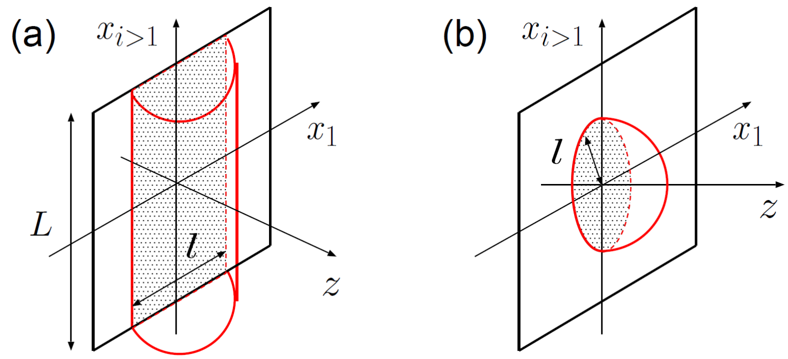
$$z = l\cos\theta, \xi = l\sin\theta \tag{27}$$

The entanglement entropy can be calculated as

$$S_{A_o} = 4\pi c_{\text{eff}} w_{d-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{(\sin\theta)^{d-2}}{(\cos\theta)^{d-1}} \tag{28}$$

$$= \frac{4\pi c_{\text{eff}}}{d-2} \frac{\text{Area}(\partial A)}{\epsilon^{d-2}} + \cdots + \begin{cases} 4(-1)^{\frac{d}{2}-1} a_d \log(\frac{2l}{\epsilon}), d \text{ is even} \\ (-1)^{\frac{d-1}{2}} 2\pi a_d, d \text{ is odd} \end{cases} \tag{29}$$

where $a_d = 2\pi^{d/2} c_{\text{eff}}/\Gamma(\frac{d}{2})$. This entropy fits the previous field theory formula to all orders.



Properties of holographic entanglement entropy

Finally, I wish to discuss how the RT-proposed holographic entanglement entropy follows some general inequalities one expect the entanglement entropy to follow.

What we have done so far is to provide intuition and perform case study to affirm the validity of the holographic entanglement entropy. We can enumerate many more cases and show that they fit our field-theory based formula for entanglement entropy, but that does not touch upon the core of the question.

Holographic entanglement entropy does have a rigorous derivation but is too involved that we have skipped it. But the entropy inequalities for the entanglement entropy are derived based on their definition from reduced density matrices, and are general properties that people believe should be followed by the entanglement entropy. Thus, it's worthwhile to crosscheck these inequalities on the holographic entanglement entropy.

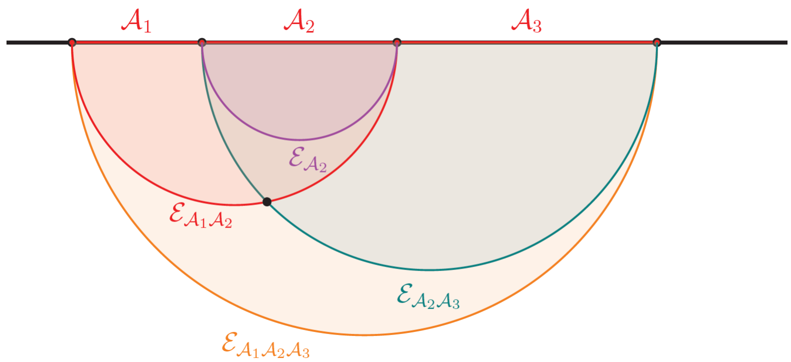
First of all, the entanglement entropy should be positive. By definition, the holographic entanglement entropy is proportional to the area of a spacelike surface, which should give a positive quantity.

An important constraint on the entanglement entropy is the strong subadditivity, which is given by

$$S_{A_1A_2} + S_{A_2A_3} \geq S_{A_1A_2A_3} + S_{A_2} \tag{30}$$

is we partition the system into three regions A_1, A_2, A_3. We will sketch why this is true for the holographic entanglement entropy. Consider three regions A_1, A_2, A_3 and the minimal surfaces $\mathcal{E}_{A_1A_2}, \mathcal{E}_{A_2A_3}$. We did not specify this but in the rigorous derivation of the holographic entanglement entropy, we require the minimal surface $\mathcal{E}_A$ in the bulk for some region A to be homologous to A on the boundary. Now we claim that: given $\mathcal{E}_{A_1A_2}, \mathcal{E}_{A_2A_3}$, we can perform some local surgery to find some new surfaces $\mathcal{F}_{A_1A_2A_3}, \mathcal{F}_{A_2}$. By virtue of the local surgeries, the total area of $\mathcal{F}_{A_1A_2A_3}, \mathcal{F}_{A_2}$ should be no larger than the total area of $\mathcal{E}_{A_1A_2}, \mathcal{E}_{A_2A_3}$. But in general, the new surfaces need not be the minimal surfaces of $A_1A_2A_3$ and A_2. Thus, by transitivity, we have

$$\begin{aligned} \mathrm{Area}(\mathcal{E}_{A_1A_2}) + \mathrm{Area}(\mathcal{E}_{A_2A_3}) &\geq \mathrm{Area}(\mathcal{F}_{A_1A_2A_3}) + \mathrm{Area}(\mathcal{F}_{A_2}) &(31)\\ &\geq \mathrm{Area}(\mathcal{E}_{A_1A_2A_3}) + \mathrm{Area}(\mathcal{E}_{A_2}) &(32) \end{aligned}$$



Then by the definition of the holographic entanglement entropy, we can easily see that the strong subadditivity is satisfied.

Note: Another implicit assumption that we have made is that all the surfaces of interest can be found in the same Cauchy slice.

If we eliminate A_2 as a region, we can obtain from the strong subadditivity the usual subadditivity. Another interesting inequality on the entropies is the Araki-Lieb inequality. It is derived from subadditivity using purification. Since subadditivity holds, the A-L inequality should also hold. Taken together, those two inequalities bound the entropy as

$$|S_{A_1} - S_{A_2}| \leq S_{A_1A_2} \leq S_{A_1} + S_{A_2} \tag{33}$$

The holographic entanglement entropy also has some special properties not followed by entanglement entropies in general. There is an interesting inequality defined on n disjoint regions of a Cauchy slice, the cyclic inequality.

$$\sum_{i=1}^{n} S(A_i \dots A_{i+l-1} : A_{i+l} \dots A_{i+k+l-1}) \geq S_{A_1A_2\dots A_n}, n \geq 2k + l \tag{34}$$

where $S(A_I : A_J)$ is the conditional entropy. Many entropy inequalities turns out to be the special case of this cyclic inequality. For example, with a choice of $(n, k, l) = (2, 0, 1)$, we recover the strong subadditivity.

Note: Actual proof of this is done by mapping to a graph theory problem.

Acknowledgement

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