Topological Invariant for Chern Insulators under Decoherence

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In this report, we share on progress on defining a topological invariant for Chern insulators under non-thermal decoherence. Three approaches are considered: the parallel transport of density matrices, Thouless pump in an open system, and the double system formalism. It is found that Thouless pump is robust for open-system fermion chains subject to current-conserving noise. To develop a more general theory with different types of decoherence, classification of quantum channels based on $U(1)$ symmetry and the double system formalism are discussed.

1. Introduction

Typically, Chern insulators are a class of 2D insulators with broken time-reversal symmetry [1–3]. They exhibit chiral edge modes at the boundary, with fermions at the edges of the Chern insulators moving in one direction. Geometrically, it is helpful to view the Chern insulator on a cylinder or Corbino disc, e.g. Fig (1), where the chiral edge modes are represented as edge currents at the boundary of the cylinder and Corbino disc. The topological invariant of Chern insulators is given by the number of such chiral edge modes at each edge. As a topological invariant, the number is quantized and robust against any perturbation as long as the energy gap of the system is not closed.

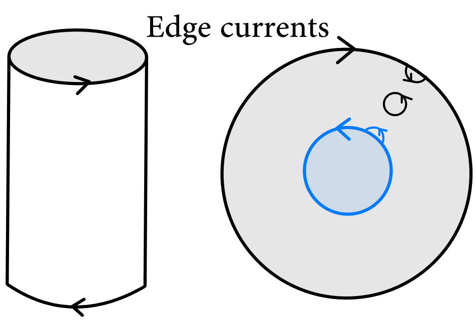

FIG. 1: Schematics of chiral edge modes for a Chern insulator on a torus or Corbino disc.

When coupled to an external electromagnetic field, one can derive the quantized Hall conductance for Chern insulators using the Kubo formula from the linear response theory. This gives the celebrated TKNN formula (at zero temperature) [4],

$$\sigma_H = \frac{e^2}{h}\int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \sum_{n\ \mathrm{occ}} i\left[\left\langle \frac{\partial u_n}{\partial k_x}\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - (x \leftrightarrow y)\right], \tag{1}$$

where $\{|u_n\rangle\}$ are the eigenfunctions of the unperturbed insulator. In terms of the Berry curvature $\mathcal{F}^n_{xy}(k)$, the formula becomes

$$\sigma_H = \frac{e^2}{h}\sum_{n\ \mathrm{occ}}\int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2}\mathcal{F}^n_{xy}(\mathbf{k}), \tag{2}$$

$$\mathcal{F}^n_{xy}(\mathbf{k}) = i\left(\left\langle \frac{\partial u_n}{\partial k_x}\middle|\frac{\partial u_n}{\partial k_y}\right\rangle - \left\langle \frac{\partial u_n}{\partial k_y}\middle|\frac{\partial u_n}{\partial k_x}\right\rangle\right). \tag{3}$$

The integral of the Berry curvature over the full (occupied) 2D Brillouin zone is characterized by a Chern number from the first Chern class [5]. The Chern numbers also correspond to the number of chiral edge modes for the Chern insulator. This further corroborates the topological nature of the invariant.

In this report, we discuss our progress on defining a topological invariant for Chern insulators under non-thermal decoherence. By non-thermal decoherence, we consider the states corrupted by imperfections but do not have time to reach thermal equilibrium. Such states cannot be described as Gibbs states of the form $\rho_G = e^{-\beta\hat{H}}/Z$. Instead, they are better modeled as pure states corrupted by local quantum channels describing finite-time decoherence. We want to characterize the topological order for the Chern insulator states corrupted by such decoherence in the zero temperature limit.

This research is motivated from both theoretical and experimental perspectives. Theoretically, there have been intense interest in studying error-induced

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phase transitions with the development of quantum computation [6, 7]. For example, when a toric code is subject to local noise channels, any existing error correction scheme fails when the error rate exceeds certain threshold [8]. This suggests that there exists some new phase transition induced by noise in the open system that cannot be characterized by topological order in the original system [7]. We want to investigate the effect of decoherence on the topological order and how to redefine a topological invariant in the open system as a measure for recovering quantum information. Experimentally, when the topological order is probed in some system, there always exists impurities, and the system is usually probed before it reaches thermal equilibrium [9]. Thus, it is instructive to understand how the topological order is affected under decoherence also for experimental purposes.

Chern insulators are a class of simplest fermionic systems with topological orders. They provide a reasonable starting ground for us to study the topological order under decoherence. Although we do not yet have a satisfying answer for what the new topological invariant should be, some interesting results have been derived. In the rest of the report, we discuss our current progress on the problem as well as some future directions. In Section (2), we introduce the parallel transport of mixed states based on Uhlmann's construction and why it is difficult to construct a mixed-state topological invariant from it. In Section (3), we derive the charge transport in a 1D gapped fermion chain with current-conserving coupling to the environment, and show that it is a quantized topological invariant robust under such noise. In Section (4), we discuss the double system formalism via Choi–Jamiołkowski isomorphism, classification of quantum channels based on $U(1)$ symmetry, and some future directions of our work. Finally, we conclude the report in Section (5).

2. Parallel Transport of Mixed States

In this section, we introduce the parallel transport of density matrices based on Uhlmann's construction [10] and discuss its connections with our project.

2.1. Motivation and Definition

Recall that in the calculation of the Hall conductance, the Chern number $\mathcal{C}_n$ as a topological invariant is captured by an integral of the Berry curvature over the full (occupied) 2D Brillouin zone:

$$\mathcal{C}_n = \int_{\mathrm{BZ}} \frac{d^2k}{(2\pi)^2} \mathcal{F}^n_{xy}(\mathbf{k}). \tag{4}$$

Some related quantity include the Berry phase and Berry connection [5, 11, 12]. They are defined via the parallel transport of pure states under adiabatic evolution on a closed loop l. For a pure state $|n(\mathbf{R})\rangle$ characterized by time-dependent parameters $\mathbf{R}(t)$, the Berry connection $\mathcal{A}_n$, Berry phase γ_n, and Berry curvature $\mathcal{F}_{ij}$ are defined as follows

$$\mathcal{A}_n = i \langle n(\mathbf{R})| \frac{\partial}{\partial \mathbf{R}} |n(\mathbf{R})\rangle , \tag{5a}$$

$$\gamma_n = \oint_l d\mathbf{R} \cdot \mathcal{A}_n(\mathbf{R}), \tag{5b}$$

$$\mathcal{F}_{ij} = i \left[\langle \nabla_j n(\mathbf{R}) | \nabla_k n(\mathbf{R})\rangle - (j \leftrightarrow k)\right]. \tag{5c}$$

Under decoherence, pure states are corrupted into mixed states. Thus, one naturally wants to extend the notions to mixed states and evaluate whether there exists a topological invariant from the "mixed-state Berry curvature".

Among the existing constructions, the one that comes closest to giving a satisfying answer was proposed by Uhlmann [10, 13, 14]. Given a density matrix ρ, Uhlmann proposed to define the parallel transport and phase difference of mixed states via the amplitude w of $\rho = ww^\dagger$. Note that a pure state $|\psi\rangle$ has a $U(1)$-gauge freedom since $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same physical state. But for the amplitude w of a $N \times N$ density matrix ρ, we have a $U(N)$-gauge freedom. For two different $N \times N$ unitary matrices U_1, U_2, the corresponding amplitudes $w_1 = \sqrt{\rho}U_1, w_2 = \sqrt{\rho}U_2$ represent the same density matrix. More specifically, $w_1 w_1^\dagger = \sqrt{\rho}U_1 U_1^\dagger \sqrt{\rho} = \sqrt{\rho}U_2 U_2^\dagger \sqrt{\rho} = w_2 w_2^\dagger$. Thus, Uhlmann's proposal essentially extends the $U(1)$-gauge freedom for a pure state $|\psi\rangle$ to a $U(N)$-gauge freedom for a density matrix ρ.

The parallel transport condition given by Uhlmann is

$$w_1^\dagger w_2 = w_2^\dagger w_1 > 0, \tag{6}$$

which means that $w_1^\dagger w_2$ and $w_2^\dagger w_1$ are both definite positive and Hermitian. Under this condition, the

Uhlmann connection $\mathcal{A}^U_{\mu,ij}$, Uhlmann phase γ^U, and Uhlmann curvature $\mathcal{F}^U_{\mu\nu}$ for a density matrix $\rho = \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|$ are given by

$$\begin{aligned}\mathcal{A}^U_{\mu,ij} &= (\partial_\mu U U^\dagger)_{ij} \\ &= \frac{(\sqrt{p_i}-\sqrt{p_j})^2}{p_i+p_j} \left|\psi_i\right\rangle\!\left\langle\psi_i\right| \partial_\mu \left|\psi_j\right\rangle\!\left\langle\psi_j\right|, \end{aligned} \tag{7a}$$

$$\gamma^U = \arg \mathrm{Tr}\left[\rho_0 \mathcal{P} \exp\left(\oint_{\mathcal{C}} \mathcal{A}^U_\mu dk^\mu\right)\right], \tag{7b}$$

$$\mathcal{F}^U_{\mu\nu} = \partial_\mu \mathcal{A}^U_\nu - \partial_\nu \mathcal{A}^U_\mu - i[\mathcal{A}^U_\mu, \mathcal{A}^U_\nu]. \tag{7c}$$

Here, $\mu = k_\mu$ in the $\{k_\mu\}$ parameter space. $\mathcal{P}$ is the path ordering operator, and $\mathcal{P}\exp\big(\oint_{\mathcal{C}} \mathcal{A}^U_\mu dk^\mu\big)$ gives the Uhlmann-Wilson loop across the Brillouin zone.

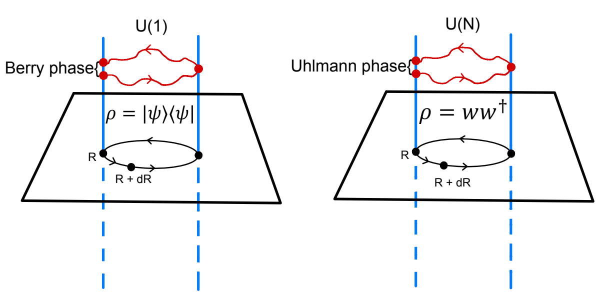

FIG. 2: Schematics of parallel transport for pure states and mixed states. The parallel transport of pure states on a closed loop accumulates a Berry phase while that of mixed states accumulates a Uhlmann phase.

2.2. Problems and Difficulty

However, there are two obstructions with proceeding in this formalism. First, the parallel transport condition requires that the density matrix ρ must be a full-rank matrix, which is not necessarily true in our case. More specifically, consider two amplitudes $w_1 = \sqrt{\rho_1}U_1, w_2 = \sqrt{\rho_2}U_2$ that can be parallel-transported to each other in the Uhlmann scheme. Then by the parallel transport condition, $w_1^\dagger w_2 = \sqrt{w_1^\dagger w_2 w_2^\dagger w_1}$ and consequently $U_1^\dagger \sqrt{\rho_1\rho_2} U_2 = U_1^\dagger \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} U_1$. Thus, the relative phase factor $U_2 U_1^\dagger$ is given by

$$U_2 U_1^\dagger = \sqrt{\rho_2^{-1}}\sqrt{\rho_1^{-1}}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} \tag{8}$$

Since the matrix inverse is involved, the density matrices we consider must be full-rank. However, since we are not working with Gibbs states, there is no guarantee that this condition is met. Thus, Uhlmann's construction does not readily apply to our case.

On a more abstract note, the $U(N)$ bundle that Uhlmann's construction is based on is topologically trivial [15]. Consequently, the Chern numbers extracted from the Uhlmann curvature all yield vanishing results. There have been proposals to modify the Chern-number formula or invent new topological invariants based on Uhlmann's construction [16, 17]. However, due to the topological triviality of the $U(N)$ bundle, these modifications still do not reflect the actual topology of the system. In this vein, the difficulty in satisfying the full-rank constraint for density matrices also illustrates the existence of non-trivial topology in our system of interest.

3. Thouless Pump in Open Systems

In this section, we introduce a more concrete and physically intuitive way to address the problem. In particular, we consider the robustness of the quantized particle transport in an open system with current-conserving coupling. Thouless first proved in 1983 that there exists quantized particle transport in 1D fermionic systems under adiabatic cyclic variation of both time t and the spatial variable x [18]. Such quantized particle transport is often referred to as the Thouless pump [19]. We test and show that the Thouless pump is robust in face of current-conserving coupling that does not close the energy gap.

3.1. Thouless Pump

Consider a 1D fermionic system periodic in time t with period τ and in the spatial variable x with period L. Suppose the changes in the system within the periods are slow such that adiabatic approximations can be applied. Then according to Thouless, in the zero temperature limit, the particle transport for such a system with n filled bands over the time period τ is given by

$$Q = \frac{i}{2\pi}\sum_{n\ \mathrm{occ}} \int_0^\tau dt \int_0^{\frac{2\pi}{L}} dk \left[\left\langle \frac{\partial u_n}{\partial t}\middle|\frac{\partial u_n}{\partial k}\right\rangle - (k \leftrightarrow t)\right], \tag{9}$$

where $|u_n\rangle$ is the eigenfunction of the n-th band. Recall from Section (1) that the integrals of the Berry curvature over the 2D Brillouin zones are characterized by Chern numbers:

$$\begin{aligned}\mathcal{C} &= \sum_{n\ \mathrm{occ}} \mathcal{C}_n \\ &= \frac{i}{2\pi} \sum_{n\ \mathrm{occ}} \int_{\mathrm{BZ}} \frac{d^2k}{2\pi} \left[\left\langle \frac{\partial u_n}{\partial k_x} \middle| \frac{\partial u_n}{\partial k_y} \right\rangle - (x \leftrightarrow y) \right].\end{aligned} \tag{10}$$

Comparing Eqn (9) and Eqn (10), one sees that the particle transport defines the first Chern class on a torus in the tk-space. Thus, the particle transport is quantized and captures a topological invariant of interest that we can study under decoherence.

Alternatively, one can follow the derivation in [20] using momentum operators to write the particle transport in terms of the ground state $|\psi_0\rangle$ of the system. This gives a simpler formula that we will work with

$$Q = \frac{i}{2\pi} \int_0^\tau dt \int_0^{\frac{2\pi}{L}} dk \left[\left\langle \frac{\partial \psi_0}{\partial t} \middle| \frac{\partial \psi_0}{\partial k} \right\rangle - (k \leftrightarrow t) \right]. \tag{11}$$

3.2. Current-conserving Coupling in an Open System

To investigate the robustness of Thouless pump, we test whether it remains unchanged in an open system with no net current exchanges with the environment for each time and spatial period. Suppose instead that there exists some arbitrary current gain or loss in the open system, then its charge transport in general varies accordingly and is less interesting to pinpoint. Moreover, we require that there exists no long-range correlation in the open system. Then the coupling between the system and the environment must be short-ranged and we can always choose some time and spatial period for the combined system and the environment.

More specifically, we consider a gapped free fermion chain S with energy gap Δ_S coupled to an environment E with energy gap Δ_E. For simplicity, set the combined system of S-E to be periodic in time t with period τ and in the spatial variable x with period L. The complete Hamiltonian for S-E can be written as

$$\hat{H} = \hat{H}_S + \hat{H}_E + g\hat{H}_{SE}, \tag{12}$$

where we take $g > 0$ and $g \ll \Delta_S, \Delta_E$ so that S-E are only weakly coupled. In the language of perturbation theory, the unperturbed Hamiltonian for S-E is $\hat{H}_0 = \hat{H}_S + \hat{H}_E$ and the perturbation term $\hat{H}_1 = g\hat{H}_{SE}$ creates some coupling between S-E.

Under adiabatic approximation, to the first-order in the time derivatives, the ground state of S-E can be written as

$$|\psi(t)\rangle \approx |\psi_0(t)\rangle + i \sum_{n \neq 0} \frac{|\psi_n(t)\rangle \langle \psi_n(t)|\partial_t \psi_0(t)\rangle}{\epsilon_n - \epsilon_0}, \tag{13}$$

where $|\psi_n(t)\rangle$ gives the n-th unperturbed eigenstate of $\hat{H}_0$. Since no net current exchanges occur for each time and spatial period, the charge transport in S can be accounted for by operators on its own fermions. Therefore, following the treatment in [18], the charge transport of S in a period τ with lowest-order corrections is

$$Q_s = \frac{1}{L} \int_0^\tau dt \left(\sum_{n \neq 0} \frac{(\hat{P}_s)_{0n} \langle \psi_n(t)|\partial_t \psi_0(t)\rangle + \text{c.c.}}{\epsilon_n - \epsilon_0} \right), \tag{14}$$

where $\hat{P}_s$ is the total momentum operator of S and $(\hat{P}_s)_{ij} = \langle \psi_i(t)| \hat{P}_s |\psi_j(t)\rangle$.

Since we are only interested in the transport of particles in S, define a unitary transformation acting solely on S as

$$\widetilde{|\psi_j\rangle} = e^{-i\alpha \hat{X}_s} |\psi_j\rangle, \tag{15a}$$

$$\widetilde{H} = e^{-i\alpha \hat{X}_s} \hat{H} e^{i\alpha \hat{X}_s}. \tag{15b}$$

The transformation is equivalent to a boost in the momenta of the S fermions by α. After the unitary transformation, $\widetilde{P}_s = \partial_\alpha \widetilde{H}$. As explained in [20], without long-range correlations in the ground state, the charge transport in the bulk should be insensitive to different boundary conditions. Therefore, Q_s should be equal to its average over $0 \leq \alpha < 2\pi/L$, which specifies different boundary conditions. Substituting $\widetilde{P}_s = \partial_\alpha \widetilde{H}$ and averaging over α, we arrive at the desired expression for Q_S:

$$Q_s = \frac{i}{2\pi} \int_0^\tau dt \int_0^{\frac{2\pi}{L}} d\alpha \left[\left\langle \frac{\partial \psi_0}{\partial t} \middle| \frac{\partial \psi_0}{\partial \alpha} \right\rangle - (\alpha \leftrightarrow t) \right]. \tag{16}$$

Obviously, this expression is in the form of Chern numbers. Thus, the particle transport in S is still quantized as an integer when subject to coupling $g\hat{H}_{SE}$ with E.

3.3. Robust Charge Transport

Now, we proceed with perturbation theory to show that the particle transport in the open system S is exactly the same as before coupling to the environment.

To the 0-th order in the perturbation, S and E decouples and $\hat{H}_1 = 0$. Then the ground state for the combined system S-E can be factorized into a product state of the unperturbed ground states of S and E, $\left|\psi_0^{(0)}\right\rangle = \left|\varphi_0^{(S)}\right\rangle \left|\varphi_0^{(E)}\right\rangle$. In this case, the particle transport in S is determined solely by its unperturbed ground state. Thus, to the 0-th order, the transported particle $Q_s^{(0)} = Q_s$, where Q is the original particle transport in the closed-system S.

Taking into account the 1-st order correction from $\hat{H}_1$, the corrected ground state for the combined system S-E takes the form

$$\left|\psi_0^{(1)}\right\rangle = \sqrt{p}\left|\psi_0^{(0)}\right\rangle + \sqrt{1-p}\left|\psi_0^{(c)}\right\rangle. \tag{17}$$

Here, p gives the probability that $\left|\psi_0^{(1)}\right\rangle$ is in the same ground state as before the 1-st order perturbation and $\left|\psi_0^{(c)}\right\rangle$ is the correction orthogonal to $\left|\psi_0^{(0)}\right\rangle$. Substitute $\left|\psi_0^{(1)}\right\rangle$ into the expression for Q_s from Eqn (16), we have

$$\begin{aligned} Q_s^{(1)} &= \frac{i}{2\pi}\int_0^\tau dt \int_0^{\frac{2\pi}{L}} d\alpha \left[\left\langle \frac{\partial\psi_0^{(1)}}{\partial t} \middle| \frac{\partial\psi_0^{(1)}}{\partial\alpha}\right\rangle - (t\leftrightarrow\alpha)\right] \\ &= \frac{i}{2\pi}\int_0^\tau dt \int_0^{\frac{2\pi}{L}} d\alpha \left\{ p\left[\left\langle \frac{\partial\psi_0^{(0)}}{\partial t} \middle| \frac{\partial\psi_0^{(0)}}{\partial k}\right\rangle - (t\leftrightarrow k)\right]\right. \\ &\left. + (1-p)\left[\left\langle \frac{\partial\psi_0^{(c)}}{\partial t} \middle| \frac{\partial\psi_0^{(c)}}{\partial k}\right\rangle - (t\leftrightarrow\alpha)\right]\right\}, \end{aligned} \tag{18}$$

where the cross terms between $\left|\psi_0^{(0)}\right\rangle$ and $\left|\psi_0^{(c)}\right\rangle$ vanish by the orthogonality of the two states and periodicity in α and t. Ignoring the prefactors of p and $(1-p)$, the rest of the two terms in Eqn (18) are both integrals of Berry curvatures for many-body states over the $t\alpha$-space torus. Therefore, two terms both result in integer charges. Up to the 1-st order correction, the charge transport in S is

$$Q_s^{(1)} = pQ_s^{(0)} + (1-p)Q_s^{(c)}, \tag{19}$$

where $Q_s^{(1)}, Q_s^{(0)}$ and $Q_s^{(c)}$ are all integers.

Let $\left|\psi_{n,m}^{(0)}\right\rangle = \left|\varphi_n^{(S)}\right\rangle \left|\varphi_m^{(E)}\right\rangle$ denote the product of the n-th excited state of S and the m-th excited state of E under 0-th order perturbation. The probability $(1-p)$ for $\left|\psi_0^{(c)}\right\rangle$ scales as

$$\begin{aligned} (1-p) &\sim \sum_{m,n\neq 0}\left(\frac{\left\langle \psi_{n,m}^{(0)}\middle| H_1 \middle|\psi_0^{(0)}\right\rangle}{\epsilon_{m,n}^{(0)} - \epsilon_{0,0}^{(0)}}\right)^2 \\ &\sim \mathcal{O}(g^2/\min(\Delta_S,\Delta_E)^2). \end{aligned} \tag{20}$$

In the limit of $g \ll \Delta_S, \Delta_E$, the gap of the combined system S-E is not closed despite the weak coupling. Thus, the combined system S-E does not undergo a phase transition. As a topological invariant, the value of $Q_s^{(1)}$ remains unchanged in this limit for different values of p. For $\lim p \to 1$, $Q_s^{(1)} = Q_s^{(0)} = Q_s$. This holds for all values of p as long as the gap of the combined system S-E is not closed. It can be shown in a similar way that higher-order corrections also do not vary the value of the particle transport. Therefore, we can confirm that Q_s remains invariant for S under open-system coupling of $g\hat{H}_{SE}$.

From the study of Thouless pump, we have found a topological invariant robust against net-current-conserving coupling in an open system that does not close the energy gap. Some physical examples of such coupling include density-density coupling between fermion chains and electron-phonon coupling [21, 22]. Our derivation provides supporting evidence that it is possible to define some topological invariant of interest for a fermionic open system.

4. Double System Formalism

In this section, we investigate the application of the double system formalism and quantum channels to generalize decoherence dynamics of interest. Previously, we have demonstrated that there exists some topological invariant for systems with current-conserving coupling. But it would be more interesting to frame the question in a formalism that admits other types of decoherence and classify them in a more elegant way. We believe that the double system formalism constitutes a promising candidate for such purposes.

4.1. Choi–Jamiołkowski Isomorphism

Via Choi–Jamiołkowski isomorphism [23, 24], a mixed state expressible as a density matrix $\rho = \sum_i \lambda_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|$ in some Hilbert space $\mathcal{H}$ can be purified into a pure state $|\rho\rangle$ in a doubled Hilbert space $\mathcal{H}^* \otimes \mathcal{H}$ as

$$|\rho\rangle = \frac{1}{\sqrt{\dim \rho}} \sum_i \lambda_i \,|\psi_i^*\rangle \otimes |\psi_i\rangle \,. \tag{21}$$

As mentioned in Section (1), non-thermal decoherence on a system can be modeled by local quantum channels acting on ρ. In the original Hilbert space $\mathcal{H}$, let $\mathcal{E}$ be the quantum channel characterizing the decoherence dynamics. Then the original (pure-state) density matrix ρ after decoherence becomes $\mathcal{E}[\rho] = \sum_i K_i \rho K_i^\dagger$, where $\{K_i\}$ are the Kraus operators [25, 26]. In the doubled Hilbert space $\mathcal{H}^* \otimes \mathcal{H}$, $\mathcal{E}$ becomes an operator $\hat{\mathcal{E}}$:

$$\hat{\mathcal{E}} = \sum_i K_{i,(1)}^* \otimes K_{i,(2)}. \tag{22}$$

Here, the subscripts $(1), (2)$ denote the first and second Hilbert space respectively. The decohered double-system state $|\rho\rangle^D$ is then obtained from $|\rho\rangle^D = \hat{\mathcal{E}}\,|\rho\rangle$.

In $\mathcal{H}^* \otimes \mathcal{H}$, the single-system Hamiltonian $\hat{H}$ from $\mathcal{H}$ transforms to the double-system $\widetilde{H}$ as

$$\widetilde{H} = \hat{H}_{(1)}^* \otimes \mathbb{I}_{(2)} + \mathbb{I}_{(1)} \otimes \hat{H}_{(2)}. \tag{23}$$

Taking from the interaction picture, the decohered double-system Hamiltonian then becomes $\widetilde{H}^D = \hat{\mathcal{E}}^\dagger \widetilde{H} \hat{\mathcal{E}}$.

Since the decoherence dynamics is completely incorporated in quantum channels, a general description and classification for quantum channels of interest lead to a general understanding of decohered density matrices and Hamiltonians in the double system. It would be interesting to classify different types of decoherence from quantum channels, and construct some topological invariant based on the decohered double-system states and Hamiltonians.

4.2. Future Directions

Intuitively, since the Chern insulator is protected by $U(1)$ symmetry, any decoherence that conserves $U(1)$ symmetry has the potential of conserving the quantization of Hall conductance and producing some topological invariant even though the system consists of mixed states. On the other hand, if $U(1)$ symmetry is no longer conserved, one can extrapolate to the extreme case where all charges are lost to the environment. Obviously, no quantized Hall conductance or any Hall conductance at all can possibly exist in such a system.

Thus, two classes of quantum channels that we can consider are those that conserve the $U(1)$ symmetry and those that do not. Two example channels of these classes acting on $\mathcal{H}^* \otimes \mathcal{H}$ are given as below

$$\begin{aligned}\hat{\mathcal{E}}_a = \prod_i [&(1-p)\mathbb{I}_{i,(1)} \otimes \mathbb{I}_{i,(2)} \\ &+ p(2\hat{n}_{i,(1)} - 1) \otimes (2\hat{n}_{i,(2)} - 1)],\end{aligned} \tag{24a}$$

$$\begin{aligned}\hat{\mathcal{E}}_b = \prod_i [&(1-p)\mathbb{I}_{i,(1)} \otimes \mathbb{I}_{i,(2)} \\ &+ p\frac{1}{\sqrt{\mathcal{N}}}(\hat{c}_{i,(1)}^\dagger + \hat{c}_{i,(1)})^* \otimes \frac{1}{\sqrt{\mathcal{N}}}(\hat{c}_{i,(2)}^\dagger + \hat{c}_{i,(2)})].\end{aligned} \tag{24b}$$

Here, $\hat{n}_{i,(1)}$ $(\hat{n}_{i,(2)})$ is the number density operator on the i-th site in the first (second) Hilbert space, and $\hat{c}_{i,(1)}^\dagger, \hat{c}_{i,(1)}$ $(\hat{c}_{i,(2)}^\dagger, \hat{c}_{i,(2)})$ are the creation and annihilation operators on the i-th site in the first (second) Hilbert space. p is the probability of an error, and $\mathcal{N} = \dim \mathcal{H}$ is put in for normalization purposes. The channel $\hat{\mathcal{E}}_a$ provided in Eqn (24a) manifestly conserves the $U(1)$ symmetry while the channel $\hat{\mathcal{E}}_b$ in Eqn (24b) breaks it.

A future direction for our work would be to systematically work out the expression for quantum channels that fall into these two classes. Then, we can examine the form of the decohered density matrices and Hamiltonians through the quantum channels, and attempt to coin topological invariants for Chern insulators under non-thermal decoherence based on these objects.

The latter, however, remains a nontrivial task. One obstacle originates from the highly non-Gaussian decohered double-system Hamiltonian $\widetilde{H}^D$. For any fermionic system of interest, we can reasonably assume that $\hat{H}$ is at least quadratic in $c^\dagger$ and c. But a nontrivial quantum channel $\hat{\mathcal{E}}$, as the ones in Eqn (24a) and (24b), should also contains at least terms linear in $c^\dagger$ and c. Thus, the decohered $\widetilde{H}^D = \hat{\mathcal{E}}^\dagger \widetilde{H} \hat{\mathcal{E}}$ is likely to be highly non-Gaussian in $c^\dagger$ and c. Such fermionic systems usually cannot be exactly solved and calculations for physical quantities of such systems can be very difficult.

5. Conclusions

With the rapid advancement of quantum computation and error correction [27, 28], questions arise as to what determines the information capacity of quantum computers. From a physical perspective, quantum computers are hosted on many-body systems [29–31]. Therefore, information-theoretic physical quantities that characterize the error-induced phase transitions in such many-body systems may set the threshold for their capacity [7, 32–34].

In this report, we consider a specific kind of many-body system – Chern insulators – and its possible topological invariant in an open system. Despite our limited understanding of the subject, it has been found that Thouless pump is robustly quantized under current-conserving interactions with the environment. The derivation makes little use of specific forms of Hamiltonians and states, and could potentially generalize to higher dimensions under a more field-theoretic approach [35]. To include other types of decoherence in the discussion, preliminary studies on quantum channel classification based on $U(1)$ symmetry and the double system formalism is carried out. But it so far remains unclear whether there exists easily calculable quantities in the resulting non-Gaussian fermionic double systems.

Contributions

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