THE HONG KONG POLYTECHNIC UNIVERSITY
香港理工大學

AAE6102 – Satellite Communication and Navigation

Positioning

Dr. Yiping Jiang

Position Determination with TOA

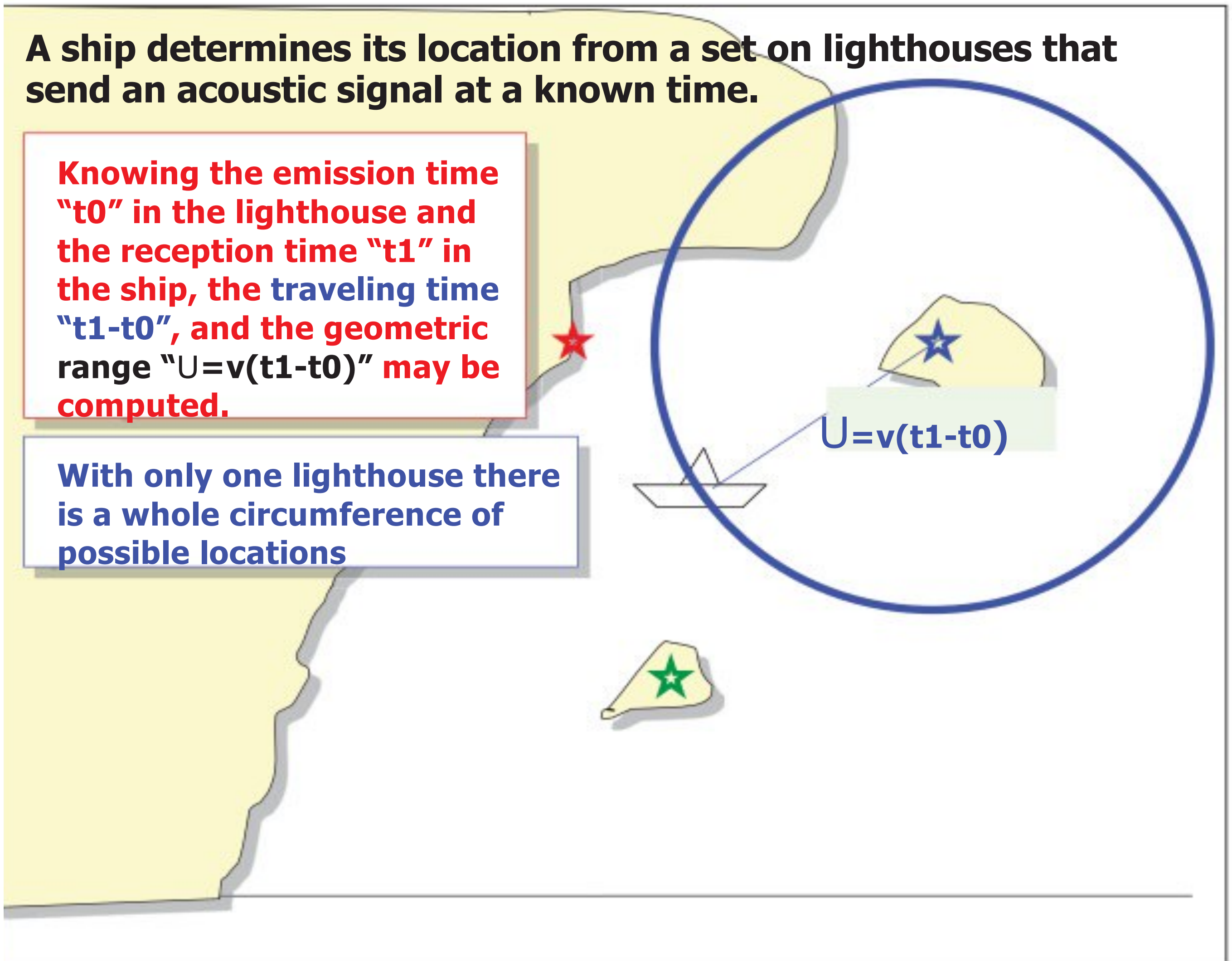
- The system utilizes the concept of one-way **time of arrival (TOA) ranging**. Satellite transmissions are referenced to highly accurate atomic frequency standards onboard the satellites, which are in synchronism with a GPS time base.
- If the receiver clock were synchronized with the satellite clocks, only three range measurements would be required. However, a crystal clock is usually employed in navigation receivers to minimize the cost, complexity, and size of the receiver. Thus, four measurements are required to determine user latitude, longitude, height, and receiver clock offset from internal system time.

Position Determination with TOA

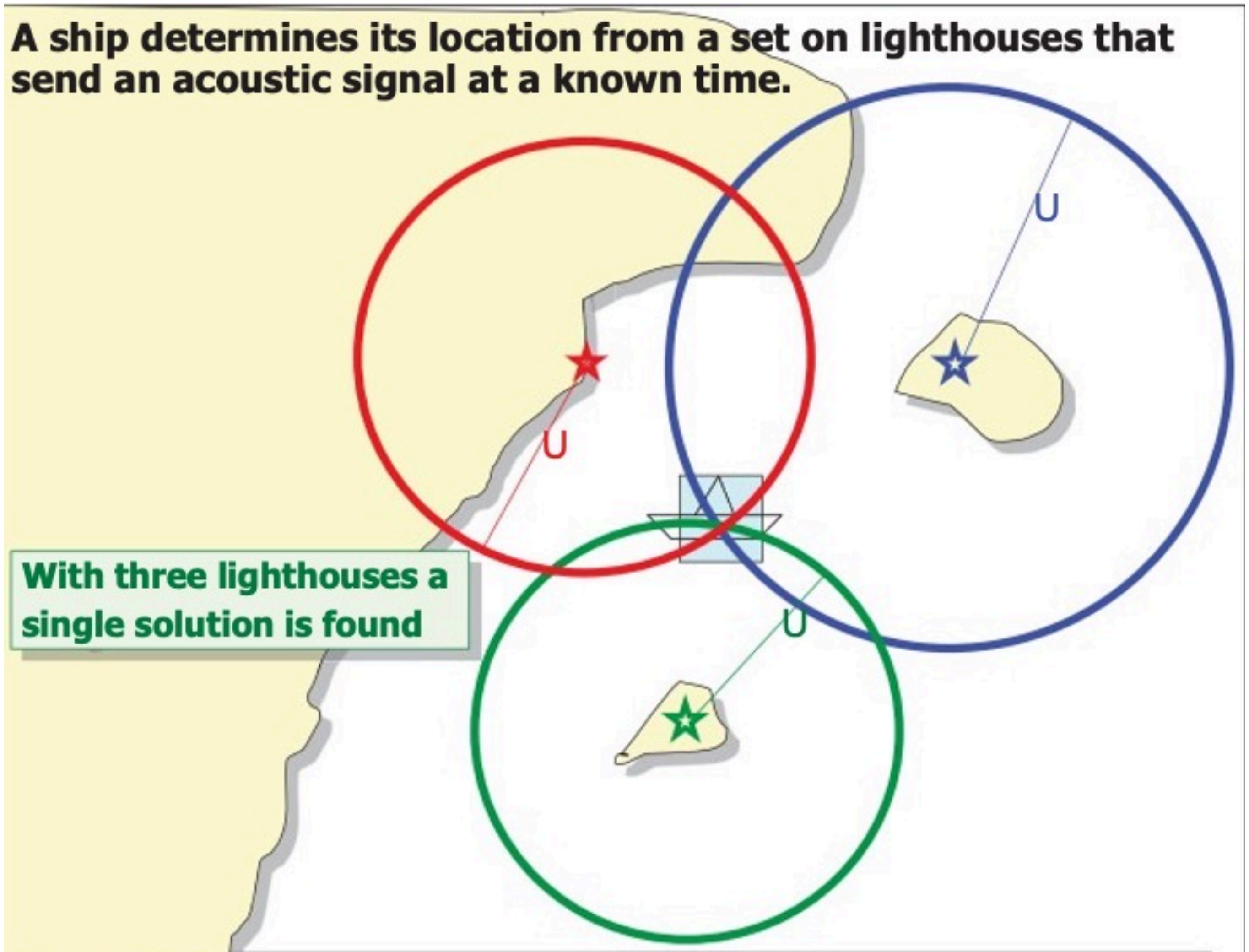
A ship determines its location from a set on lighthouses that send an acoustic signal at a known time.

Knowing the emission time "t0" in the lighthouse and the reception time "t1" in the ship, the traveling time "t1-t0", and the geometric range "U=v(t1-t0)" may be computed.

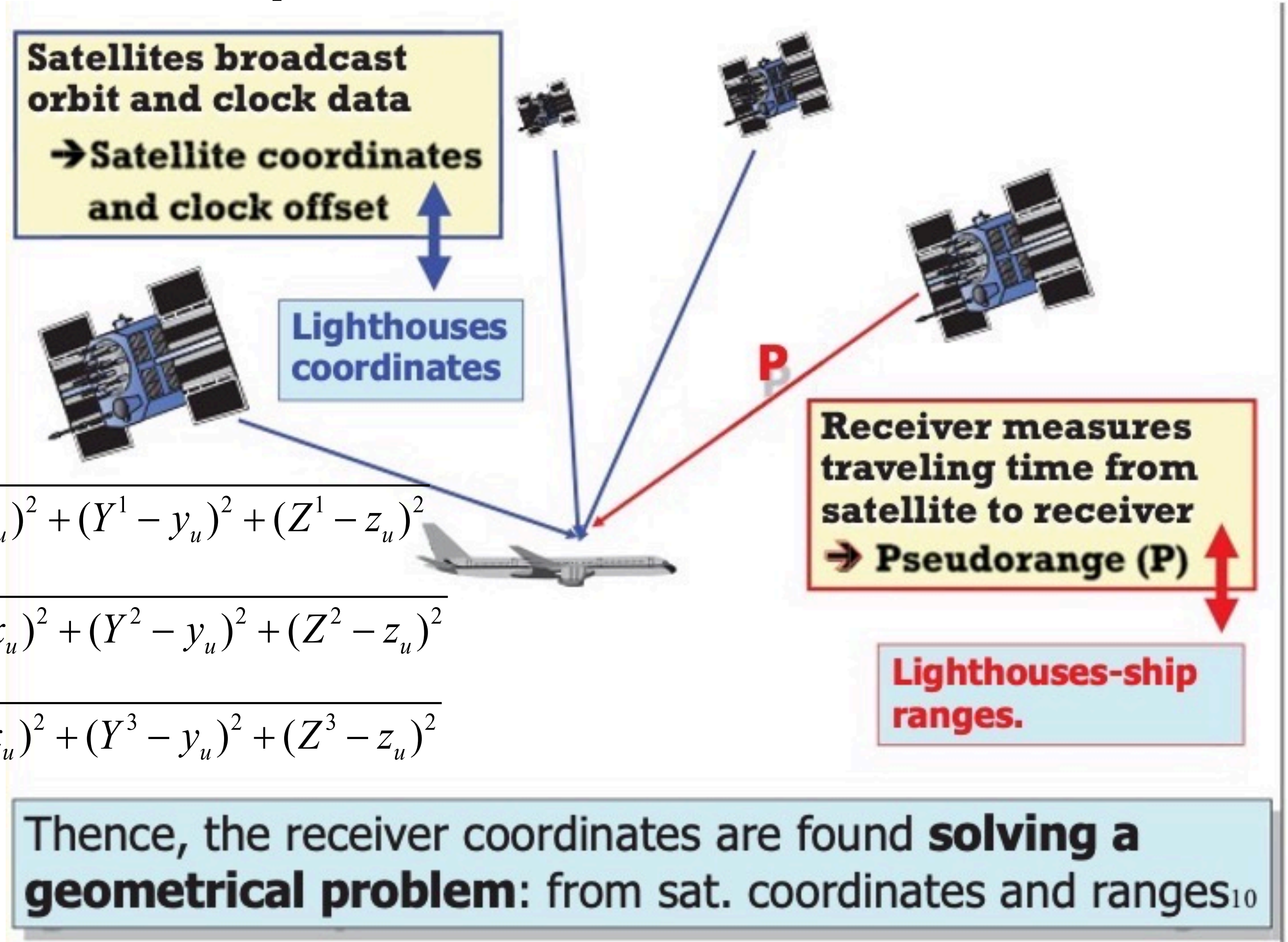
With only one lighthouse there is a whole circumference of possible locations



Position Determination with TOA



GNSS Principles



$$\rho_1 = \sqrt{(X^1 - x_u)^2 + (Y^1 - y_u)^2 + (Z^1 - z_u)^2}$$

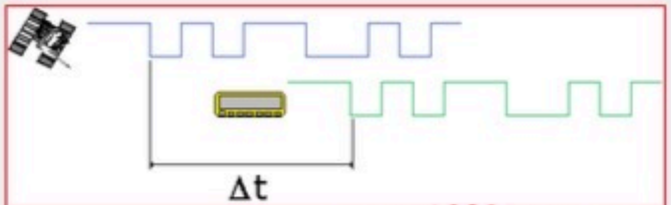
$$\rho_2 = \sqrt{(X^2 - x_u)^2 + (Y^2 - y_u)^2 + (Z^2 - z_u)^2}$$

$$\rho_3 = \sqrt{(X^3 - x_u)^2 + (Y^3 - y_u)^2 + (Z^3 - z_u)^2}$$

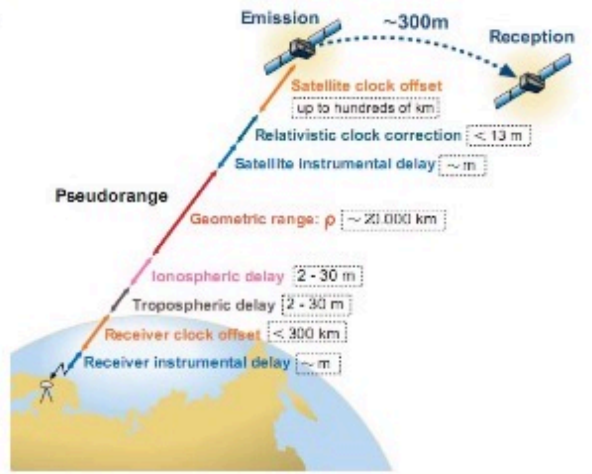
THE HONG KONG POLYTECHNIC UNIVERSITY
香港理工大學

1 nanosecond equivalent to ~30cm distance!

GNSS Measurement



$$P = c\,\Delta T = c\,[t_{rec}(T) - t^{sat}(T-\Delta T)]$$



$$P^{sat}_{rec} = \underbrace{\rho^{sat}_{rec}}_{\text{Geometric range}} + c\cdot(\underbrace{dt_{rec} - dt^{sat}}_{\text{Clock offsets}}) + \sum\delta$$

$$\sum\delta = \underbrace{Trop^{sat}_{rec}}_{\text{Tropospheric delay}} + \underbrace{Ion^{sat}_{rec}}_{\text{Ionospheric delay}} + \underbrace{K_{rec} + K^{sat}}_{\text{Instrumental delays}} + \underbrace{\varepsilon}_{\text{noise}}$$

GNSS Measurement

- Pseudorange
  - Geometric range ρ: true distance between satellite and receiver
  - Range: geometric range plus path delays
  - Pseudo- + range: range plus effects of clock errors

$$\boldsymbol{Pseu = \rho + c * (clock\ errors) + c * (path\ delays)}$$

- Ranging code
  - GNSS signals contain **ranging codes** to allow the users to compute the travel time time ΔT of signal from satellite to receiver
  - ΔT multiply the speed of light c gives us pseudorange

$$\boldsymbol{Pseu = c * \Delta T = c * (\underbrace{t_r(T_2)}_{\text{Time of signal reception}} - \underbrace{t^s(T_1)}_{\text{Time of signal transmission}})}$$

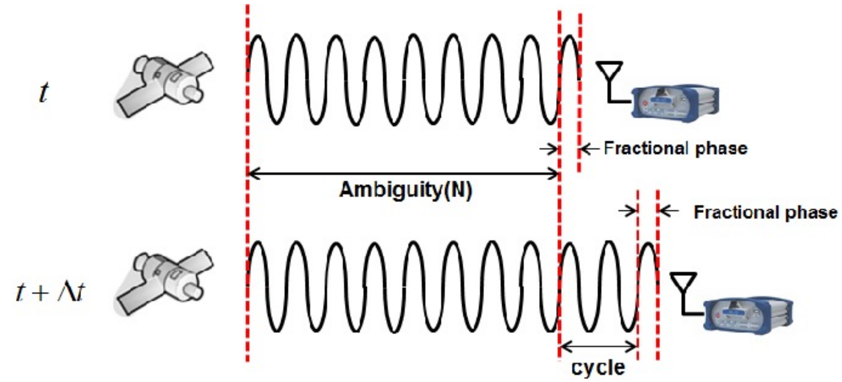
GNSS Measurement

- Carrier phase
  - Another GNSS signal **carrier** is also used to obtain pseudorange
  - Using carrier phase difference $\Delta\varphi$ between satellite and receiver multiply wavelength λ gives pseudorange

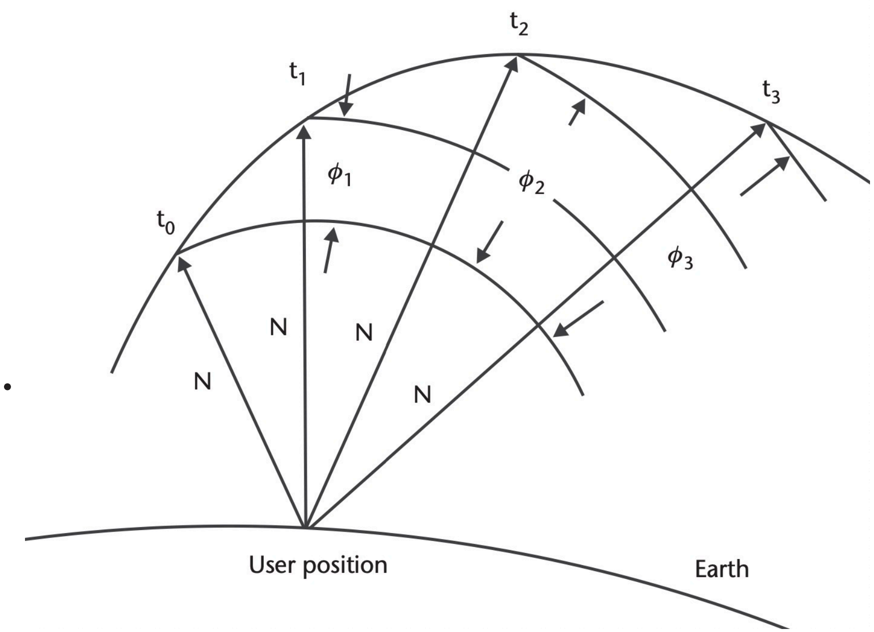
$$\boldsymbol{Pseu} = \boldsymbol{\lambda} * \Delta\boldsymbol{\varphi} = \boldsymbol{\lambda} * (\underbrace{\boldsymbol{\varphi_r}(\boldsymbol{T_2})}_{\text{Phase of signal reception}} - \underbrace{\boldsymbol{\varphi^s}(\boldsymbol{T_1})}_{\text{Phase of signal transmission}})$$

- Comparison between code and carrier measurement
  - Carrier measurement is much more precise than code measurement,
  - Ambiguous by an unknown integer number of wavelengths exists in carrier measurement

Carrier-cycle integer ambiguity



- Even though the receiver carrier-phase measurement can be made with some precision (better than 0.01 cycle for receivers in the marketplace) and any advance in carrier cycles since satellite acquisition by the receiver can be accurately counted, the overall phase measurement contains an unknown number of carrier-cycles. This is called the carrier-cycle integer ambiguity (*N*).
- This ambiguity exists because the receiver merely begins counting carrier cycles from the time a satellite is placed in active track.

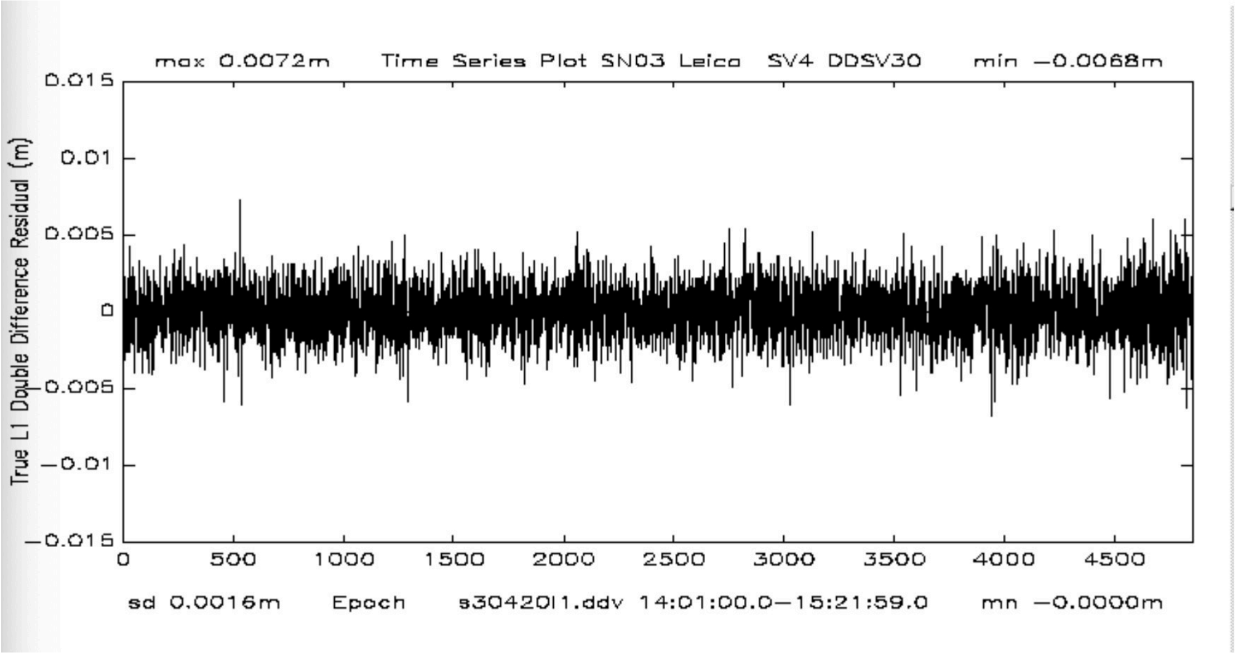


Principle of Least Squares

- Observations → Parameters
- Number of observations ≥ Number of parameters
- 1. The observations are not perfect! How to account for uncertainty? Via the stochastic model
  - Difference between random error (noise), systematic error (bias) and gross error (fault, outlier)
- 2. There must exist a mathematical relationship between observations and parameters, functional model
  - need linearization

Random Error

- Errors are unpredictable -- *they do not necessarily obey a well defined pattern or model*.
- Error behaviour can be studied using probability theory.
- The statistics generated from probability analysis are useful for evaluation.

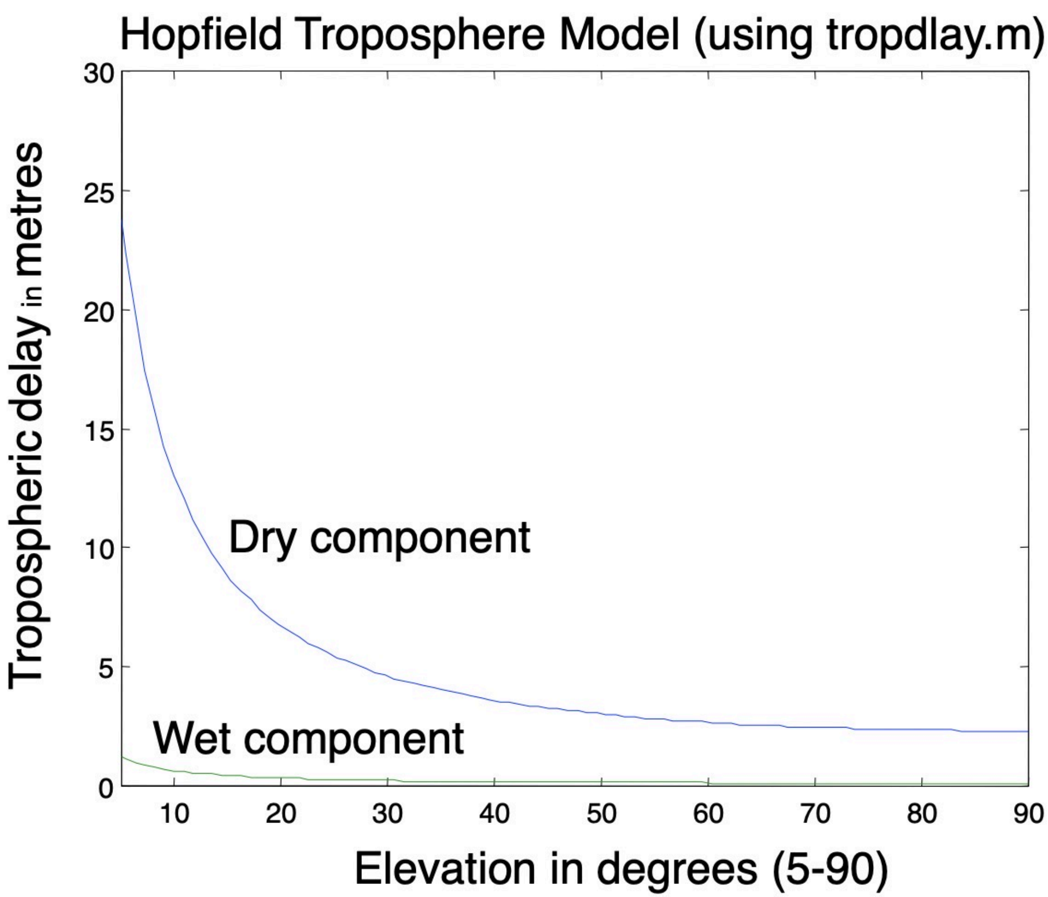


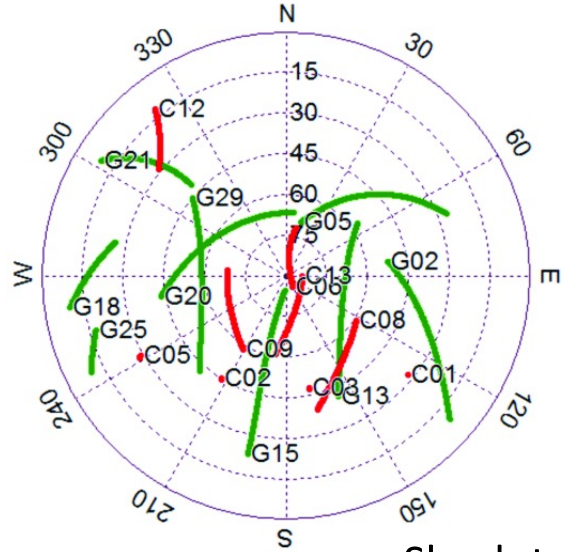
GPS Receiver Noise

Systematic Errors

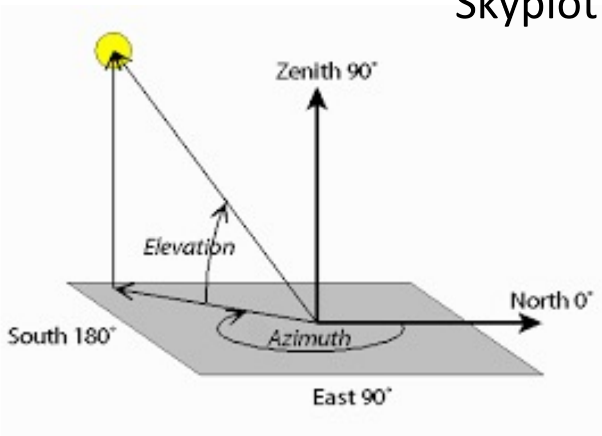
- ***Systematic errors*** or ***biases*** occur according to some pattern which, if know, can be described mathematically.
- They may also be called *constant errors* if their magnitude (and sign) remain the same throughout the measurement process.
- May be induced by the instrument, the observer, physical or environmental conditions.
- Examples:
  - Environmental: tropospheric refraction error
  - Instrumental: GPS antenna phase centre variation
  - Observer: incorrect entry of GPS antenna offset parameters

Systematic Errors: GPS Troposphere Model (not perfect!)





Skyplot



Elevation vs. Azimuth

Gross Errors (*Fault, Outliers*)

- These are the result of ***blunders, mistakes, or unpredicted small probability event***
- If these errors have large magnitude they are usually easy to identify and can be easily removed. *Small gross errors may go unnoticed, requiring careful procedures to detect them.*
- Gross errors can be reduced by careful observation procedures, and redundant observations allow identification within the estimation process (*increase 'reliability'*).

GNSS Fault Examples

- **Ephemeris failure**: between 1999-2007, errors greater than 50 meters occurred on 24 occasions.
- **Clock runoff**: (>1000m, 4 times) SV22 on July 28, 2001; SV27 on May 26, 2003; SV35 on June 11, 2003, and SV23 on January 1, 2004.
- **Signal deformation**: once, in 1993, SV19, caused by the failure of the modulation unit
- **Ionosphere Anomaly**: 40 significant events in the last solar peak period

Functional Model

- Using only user receiver, without dependence on other facilities/receivers/sensor, user estimates its own position with pseudorange or carrier phase measurements
- With satellites' position x_j, y_j, z_j and pseudorange ρ_j known, to determine user's position and receiver clock offset x_u, y_u, z_u, t_u:

$$\begin{aligned}\rho_j &= \left\| \mathbf{s}_j - \mathbf{u} \right\| + ct_u \\ &= \sqrt{\left(x_j - x_u\right)^2 + \left(y_j - y_u\right)^2 + \left(z_j - z_u\right)^2} + ct_u\end{aligned}$$

- How to solve the above nonlinear equations to get x_u, y_u, z_u, t_u?

Linearization

- Linearization: expanding nonlinear equations into a Taylor series about the approximate position:

User position and offset: $(x_u, y_u, z_u)\ t_u$

Single pseudorange:
$$\rho_j = \sqrt{(x_j - x_u)^2 + (y_j - y_u)^2 + (z_j - z_u)^2} + ct_u$$
$$= f(x_u, y_u, z_u, t_u)$$

Approx position and offset: $(\hat{x}_u, \hat{y}_u, \hat{z}_u)\ \hat{t}_u$

Approx pseudorange:
$$\hat{\rho}_j = \sqrt{(x_j - \hat{x}_u)^2 + (y_j - \hat{y}_u)^2 + (z_j - \hat{z}_u)^2} + c\hat{t}_u$$
$$= f(\hat{x}_u, \hat{y}_u, \hat{z}_u, \hat{t}_u)$$

$$\boxed{\begin{aligned} x_u &= \hat{x}_u + \Delta x_u \\ y_u &= \hat{y}_u + \Delta y_u \\ z_u &= \hat{z}_u + \Delta z_u \\ t_u &= \hat{t}_u + \Delta t_u \end{aligned}}$$

Linearization

- Expanding in Taylor series

$$f(x_u,y_u,z_u,t_u) = \overbrace{f\left(\hat{x}_u+\Delta x_u,\hat{y}_u+\Delta y_u,\hat{z}_u+\Delta z_u,\hat{t}_u+\Delta t_u\right)}^{\rho_j} = \overbrace{f\left(\hat{x}_u,\hat{y}_u,\hat{z}_u,\hat{t}_u\right)}^{\hat{\rho}_j}$$

$$+\underbrace{\frac{\partial f\left(\hat{x}_u,\hat{y}_u,\hat{z}_u,\hat{t}_u\right)}{\partial \hat{x}_u}}_{a_{xj}}\Delta x_u + \underbrace{\frac{\partial f\left(\hat{x}_u,\hat{y}_u,\hat{z}_u,\hat{t}_u\right)}{\partial \hat{y}_u}}_{a_{yj}}\Delta y_u$$

$$+\underbrace{\frac{\partial f\left(\hat{x}_u,\hat{y}_u,\hat{z}_u,\hat{t}_u\right)}{\partial \hat{z}_u}}_{a_{zy}}\Delta z_u + \underbrace{\frac{\partial f\left(\hat{x}_u,\hat{y}_u,\hat{z}_u,\hat{t}_u\right)}{\partial \hat{t}_u}}_{c}\Delta t_u + \ldots$$

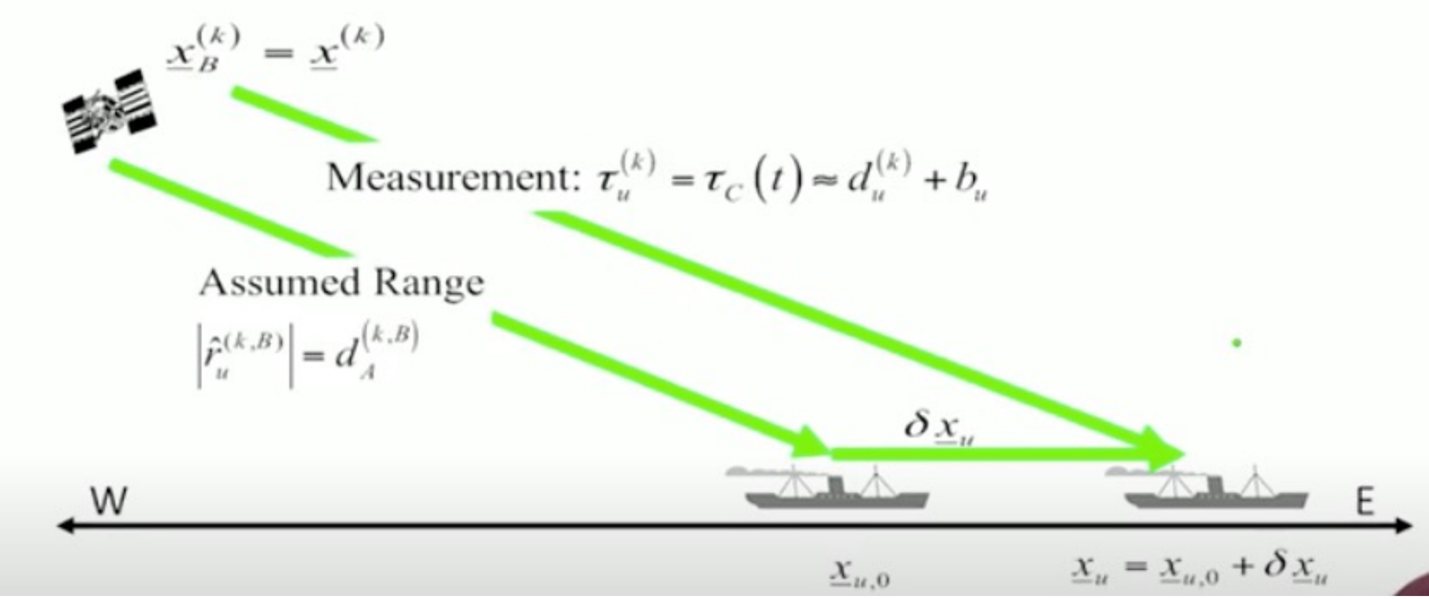
- Rewritten as:

$$\rho_j - \hat{\rho}_j = \Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u + c\Delta t_u$$

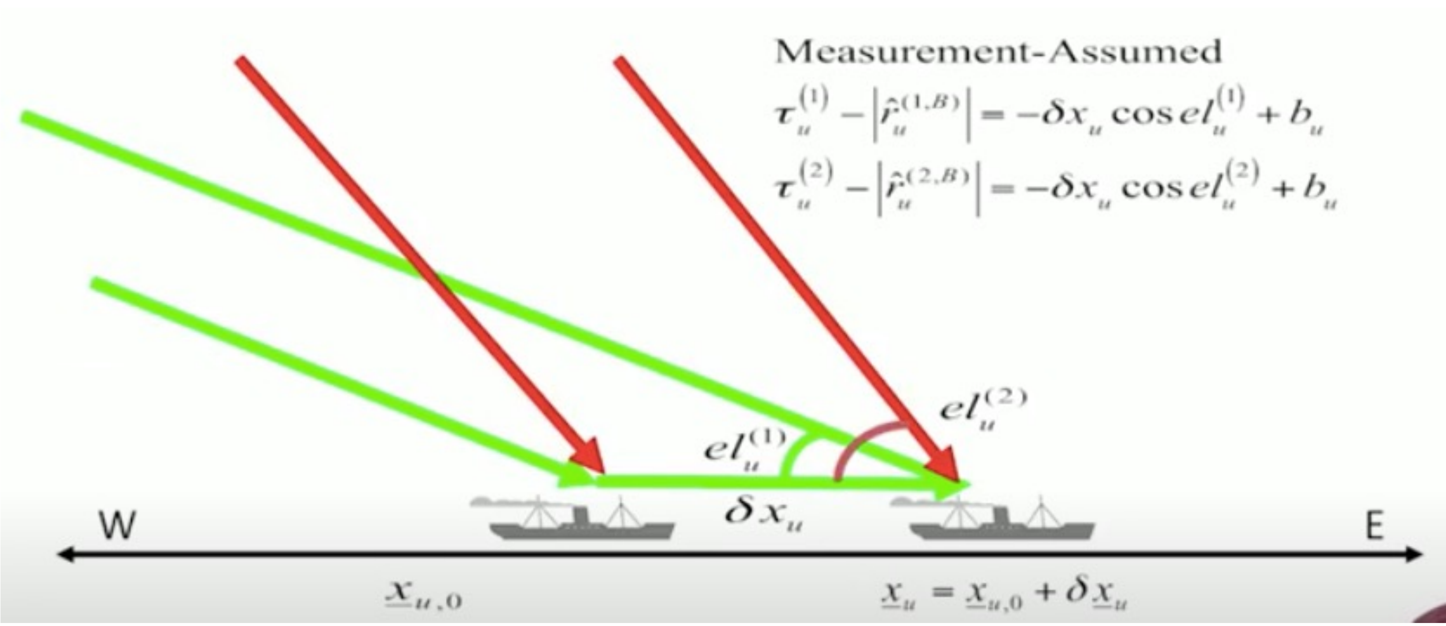
- Once the four unknowns: $\Delta x_u, \Delta y_u, \Delta z_u, \Delta t_u$ are computed
- The user's coordinates and clock offset can be calculated by:

$$x_u = \hat{x}_u + \Delta x_u \quad y_u = \hat{y}_u + \Delta y_u \quad z_u = \hat{z}_u + \Delta z_u \quad t_u = \hat{t}_u + \Delta t_u$$

Linearization



a is a function of the elevation angles.



Least Squares

- How to solve the four unknowns $\Delta x_u, \Delta y_u, \Delta z_u, \Delta t_u$ with the below linear equation?

$$\hat{\rho}_j - \rho_j = \Delta\rho_j = a_{xj}\Delta x_u + a_{yj}\Delta y_u + a_{zj}\Delta z_u - c\Delta t_u$$

- Need at least four ranging measurements (four visible satellites).
- If we have exactly four ranging measurements:

$$\begin{cases} \Delta\rho_1 = a_{x1}\Delta x_u + a_{y1}\Delta y_u + a_{z1}\Delta z_u - c\Delta t_u \\ \Delta\rho_2 = a_{x2}\Delta x_u + a_{y2}\Delta y_u + a_{z2}\Delta z_u - c\Delta t_u \\ \Delta\rho_3 = a_{x3}\Delta x_u + a_{y3}\Delta y_u + a_{z3}\Delta z_u - c\Delta t_u \\ \Delta\rho_4 = a_{x4}\Delta x_u + a_{y4}\Delta y_u + a_{z4}\Delta z_u - c\Delta t_u \end{cases} \Rightarrow \boldsymbol{\Delta\rho} = \mathbf{H}\boldsymbol{\Delta}\mathbf{x} \Rightarrow \boldsymbol{\Delta}\mathbf{x} = \mathbf{H}^{-1}\boldsymbol{\Delta\rho}$$

- If we have more than four ranging measurements?

Least Squares

- More than four ranging measurements: **Least Squares**
- Brief introduction of the least square techniques:

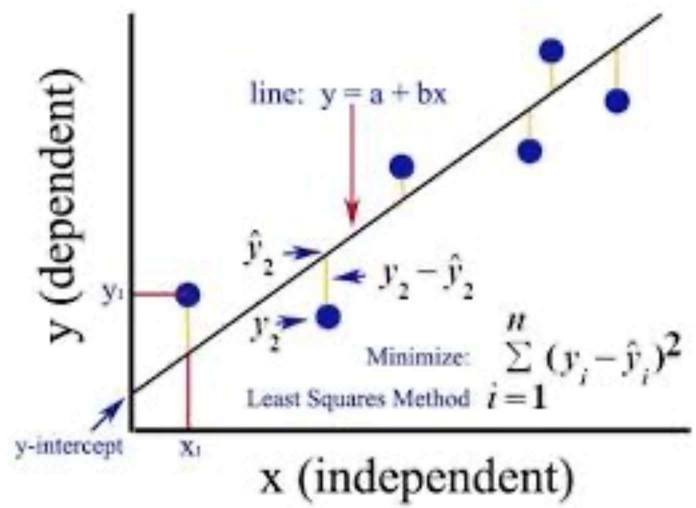
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

$\mathbf{y} = [y_1 y_2 \dots y_N]^T$	$\mathbf{x} = [x_1 x_2 \dots x_M]^T$	$\mathbf{n} = [n_1 n_2 \dots n_N]^T$
Measurements	Unknown parameters	Gaussian distributed errors

- The maximum likelihood estimate of **x**:

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}} p(\mathbf{y}/\mathbf{x}) = \arg\min_{\mathbf{x}} \left\| \mathbf{y} - \mathbf{H}\mathbf{x} \right\|^2$$

p(y/x) is the pdf of the measurement y for a fixed x



THE SUM OF THE SQUARES OF THE WEIGHTED RESIDUALS IS A MINIMUM

Least Squares

- To obtain the solution:

$$\frac{d}{d\hat{\mathbf{x}}}\left\|\mathbf{y}-\mathrm{H}\hat{\mathbf{x}}\right\|^{2} = 2\mathrm{H}^{T}\mathrm{H}\hat{\mathbf{x}} - 2\mathrm{H}^{T}\mathbf{y} = 0$$

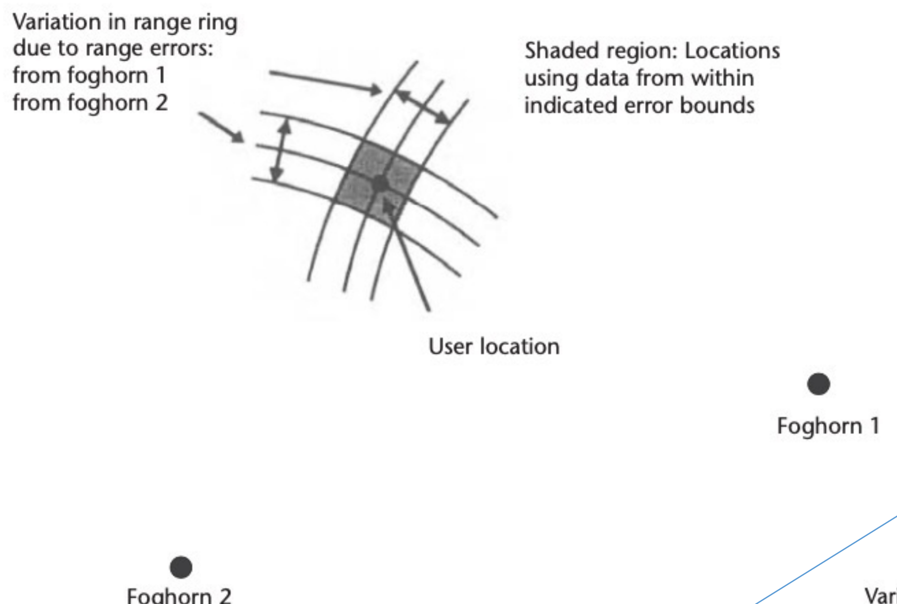
- Estimate result:

$$\hat{\mathbf{x}} = \left(\mathrm{H}^{T}\mathrm{H}\right)^{-1}\mathrm{H}^{T}\mathbf{y}$$

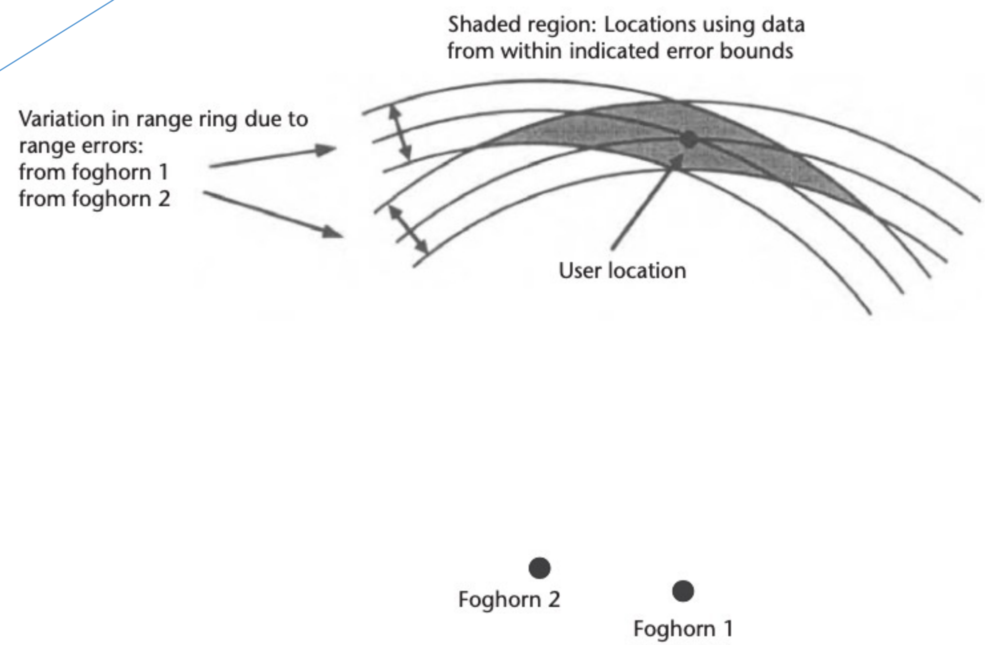
- Applying to positioning:

$$\Delta\boldsymbol{\rho} = \mathrm{H}\Delta\mathbf{x} \quad \Rightarrow \quad \Delta\mathrm{x} = (\boldsymbol{H}^{T}\boldsymbol{H})^{-1}\,\boldsymbol{H}^{T}\Delta\boldsymbol{\rho}$$

Dilution of Precision (DOP)



(a)



DOP

- DOP is used to specify error propagation as a function of a satellite geometry on positioning precision
- Satellite evenly distributed in the sky has lower DOP.
- The lower the DOP, the better the geometry of the available satellites, and more precisely determined are the parameters assumed to be.
- In general, the more satellites used in a solution, the smaller the DOP values
- VDOP values are generally larger than HDOP values

DOP Definition

$$\mathbf{H}\Delta\mathbf{x} = \Delta\boldsymbol{\rho}$$

$$\mathbf{H} = \begin{bmatrix} a_{x1} & a_{y1} & a_{z1} & 1 \\ a_{x2} & a_{y2} & a_{z2} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ a_{xn} & a_{yn} & a_{zn} & 1 \end{bmatrix}$$

$$\Delta\mathbf{x} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\Delta\boldsymbol{\rho}$$

$$\mathrm{cov}(d\mathbf{x}) = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\sigma_{UERE}^2$$

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix}$$

$$\mathrm{PDOP} = \sqrt{D_{11} + D_{22} + D_{33}}$$

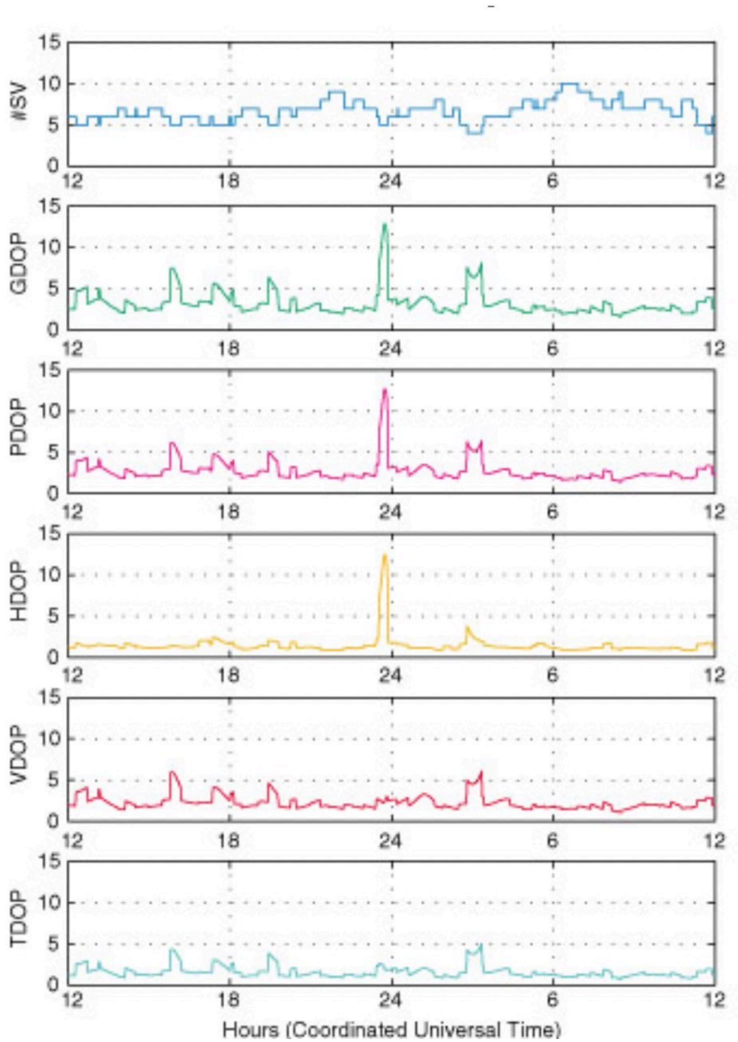
$$\mathrm{HDOP} = \sqrt{D_{11} + D_{22}}$$

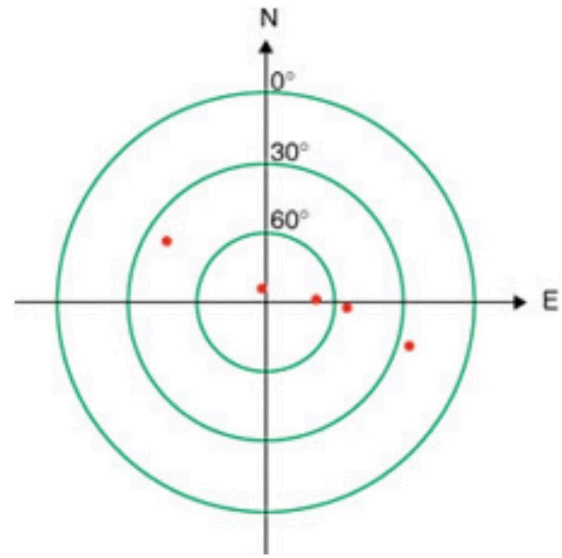
$$\mathrm{VDOP} = \sqrt{D_{33}}$$

$$\mathrm{TDOP} = \sqrt{D_{44}}/c$$

$$\mathrm{GDOP} = \sqrt{D_{11} + D_{22} + D_{33} + D_{44}}$$

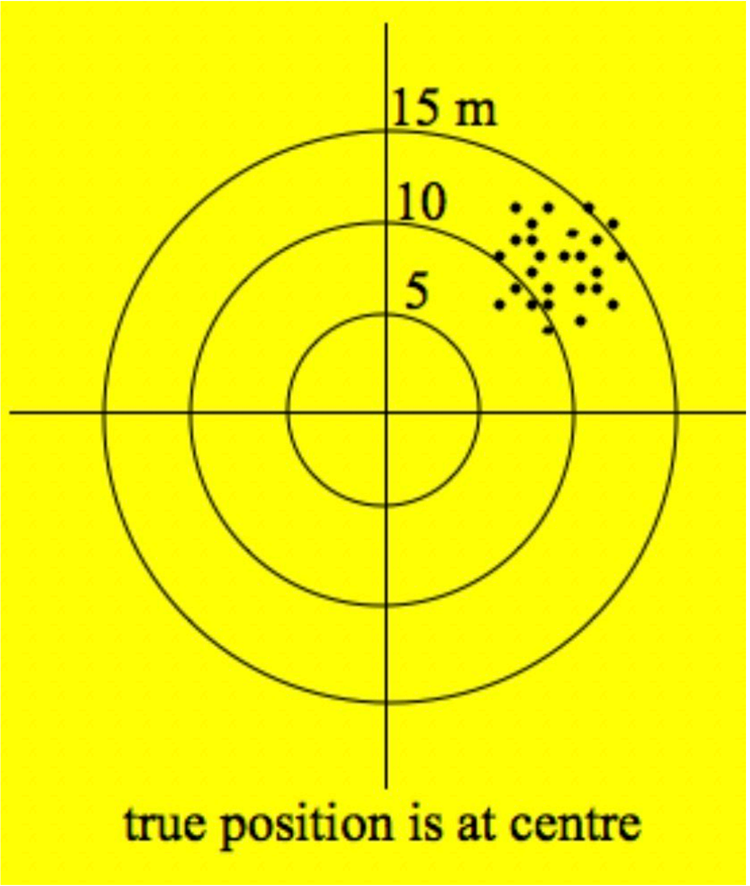
Example DOP values, elevation mask 15

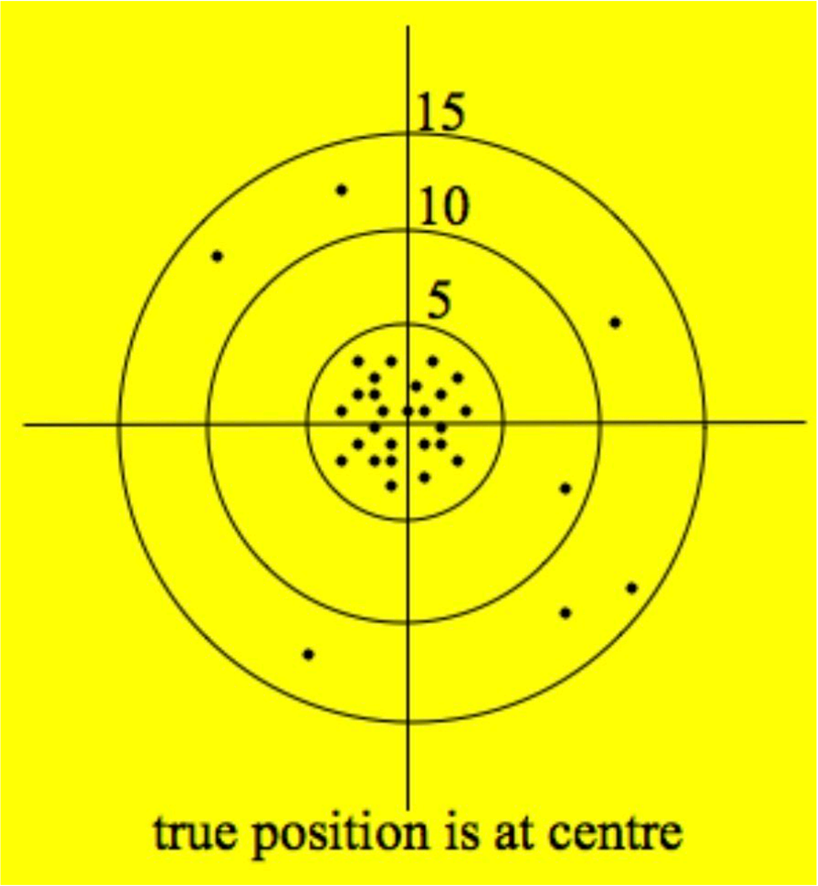




Spike in DOP values is caused by almost perfect alignment of 5 satellites… *implying poor geometry*

What factors are influencing the estimation accuracy and precision?





accuracy vs. precision

Sample RINEX navigation message file

SV No

Epoch date & time (t_{oc})

This is GPS time

a_0 a_1 a_2

```
----|---1|0---|---2|0---|---3|0---|---4|0---|---5|0---|---6|0---|---7|0---|---8|
 2              N: GPS NAV DATA                             RINEX VERSION / TYPE
XXRINEXN V2.0       AIUB                12-SEP-90 15:22     PGM / RUN BY / DATE
EXAMPLE OF VERSION 2 FORMAT                                 COMMENT
     .1676D-07   .2235D-07  -.1192D-06  -.1192D-06          ION ALPHA
     .1208D+06   .1310D+06  -.1310D+06  -.1966D+06          ION BETA
     .133179128170D-06  .107469588780D-12   552960       39 DELTA-UTC: A0,A1,T,W
     6                                                      LEAP SECONDS
                                                            END OF HEADER
 6 90  8  2 17 51 44.0 -.839701388031D-03 -.165982783074D-10  .000000000000D+00
     .910000000000D+02  .934062500000D+02  .116040547840D-08  .162092304801D+00
     .484101474285D-05  .626740418375D-02  .652112066746D-05  .515365489006D+04
     .409904000000D+06 -.242143869400D-07  .329237003460D+00 -.596046447754D-07
     .111541663136D+01  .326593750000D+03  .206958726335D+01 -.638312302555D-08
     .307155651409D-09  .000000000000D+00  .551000000000D+03  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .910000000000D+02
     .406800000000D+06
13 90  8  2 19  0  0.0  .490025617182D-03  .204636307899D-11  .000000000000D+00
     .133000000000D+03 -.963125000000D+02  .146970407622D-08  .292961152146D+01
    -.498816370964D-05  .200239347760D-02  .928156077862D-05  .515328476143D+04
     .414000000000D+06 -.279396772385D-07  .243031939942D+01 -.558793544769D-07
     .110192796930D+01  .271187500000D+03 -.232757915425D+01 -.619632953057D-08
    -.785747015231D-11  .000000000000D+00  .551000000000D+03  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .389000000000D+03
     .410400000000D+06
```

$$\text{Clock correction} = a_0 + a_1(t-t_{oc}) + a_2(t-t_{oc})^2 \text{ (seconds)}$$

SV Clock Correction

- Clock correction model is broadcast in each satellite's Navigation Message.
- a0, a1 and a2 are the transmitted polynomial coefficients for the “satellite clock error” model.
- It is a *prediction* of the satellite clock error in the future (hours to days).
- t is the time of signal transmission. *Clock correction/error varies with time!*
- toc is the time of the *clock error reference time*.

Sample RINEX navigation message file

SV No

Epoch date & time (t_{oc})

```
----|---1|0---|---2|0---|---3|0---|---4|0---|---5|0---|---6|0---|---7|0---|---8|
 2              N: GPS NAV DATA                         RINEX VERSION / TYPE
XXRINEXN V2.0       AIUB                12-SEP-90 15:22     PGM / RUN BY / DATE
EXAMPLE OF VERSION 2 FORMAT                                 COMMENT
     .1676D-07   .2235D-07  -.1192D-06  -.1192D-06          ION ALPHA
     .1208D+06   .1310D+06  -.1310D+06  -.1966D+06          ION BETA
     .133179128170D-06  .107469588780D-12   552960       39 DELTA-UTC: A0,A1,T,W
     6                                                      LEAP SECONDS
                                                            END OF HEADER
 6 90  8  2 17 51 44.0 -.839701388031D-03 -.165982783074D-10  .000000000000D+00
     .910000000000D+02  .934062500000D+02  .116040547840D-08  .162092304801D+00
     .484101474285D-05  .626740418375D-02  .652112066746D-05  .515365489006D+04
     .409904000000D+06 -.242143869400D-07  .329237003460D+00 -.596046447754D-07
     .111541663136D+01  .326593750000D+03  .206958726335D+01 -.638312302555D-08
     .307155651409D-09  .000000000000D+00  .551000000000D+03  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .910000000000D+02
     .406800000000D+06
13 90  8  2 19  0  0.0  .490025617182D-03  .204636307899D-11  .000000000000D+00
     .133000000000D+03 -.963125000000D+02  .146970407622D-08  .292961152146D+01
    -.498816370964D-05  .200239347760D-02  .928156077862D-05  .515328476143D+04
     .414000000000D+06 -.279396772385D-07  .243031939942D+01 -.558793544769D-07
     .110192796930D+01  .271187500000D+03 -.232757915425D+01 -.619632953057D-08
    -.785747015231D-11  .000000000000D+00  .551000000000D+03  .000000000000D+00
     .000000000000D+00  .000000000000D+00  .000000000000D+00  .389000000000D+03
     .410400000000D+06
```

Quasi-Keplerian orbital elements

Satellite coordinates (at a time instant or epoch) are computed using the *quasi-Keplerian elements* contained within the Navigation Message.

find the RINEX format definition from internet.

Sample RINEX observation file

```
     2.10           OBSERVATION DATA    M (MIXED)           RINEX VERSION / TYPE
BLANK OR G = GPS,  R = GLONASS,  E = GALILEO,  M = MIXED    COMMENT
XXRINEXO V9.9       AIUB                24-MAR-01 14:43     PGM / RUN BY / DATE
EXAMPLE OF A MIXED RINEX FILE (NO FEATURES OF V 2.11)       COMMENT
A 9080                                                      MARKER NAME
9080.1.34                                                   MARKER NUMBER
BILL SMITH          ABC INSTITUTE                           OBSERVER / AGENCY
X1234A123           XX                  ZZZ                 REC # / TYPE / VERS
234                 YY                                      ANT # / TYPE
  4375274.       587466.      4589095.                      APPROX POSITION XYZ
         .9030         .0000         .0000                  ANTENNA: DELTA H/E/N
     1     1                                                WAVELENGTH FACT L1/2
     1     2     6   G14   G15   G16   G17   G18   G19      WAVELENGTH FACT L1/2
     0                                                      RCV CLOCK OFFS APPL
     5    P1    L1    L2    P2    L5                        # / TYPES OF OBSERV
    18.000                                                  INTERVAL
  2005     3    24    13    10   36.0000000                 TIME OF FIRST OBS
                                                            END OF HEADER
 05  3 24 13 10 36.0000000  0  4G12G09G06E11                        -.123456789
  23629347.915           .300 8         -.353   23629364.158
  20891534.648          -.120 9         -.358   20891541.292
  20607600.189          -.430 9          .394   20607605.848
                         .324 8                                          .178 7
 05  3 24 13 10 50.0000000  4  4
     1     2     2   G 9   G12                              WAVELENGTH FACT L1/2
  *** WAVELENGTH FACTOR CHANGED FOR 2 SATELLITES ***        COMMENT
      NOW 8 SATELLITES HAVE WL FACT 1 AND 2!                COMMENT
                                                            COMMENT
 05  3 24 13 10 54.0000000  0  6G12G09G06R21R22E11                  -.123456789
  23619095.450      -53875.632 8    -41981.375  23619112.008
  20886075.667      -28688.027 9    -22354.535  20886082.101
  20611072.689       18247.789 9     14219.770  20611078.410
  21345678.576       12345.567 5
  22123456.789       23456.789 5
                     65432.123 5                                    48861.586 7
 05  3 24 13 11  0.0000000  2  1
```

Exercise 1

- Open GNSS planning online: http://www.gnssplanningonline.com/
- Set your location and time.
- Select GPS + BDS constellations
- Plot the DOP of HK during yesterday.

Exercise 2

- Download RINEX file from SATREF

https://www.geodetic.gov.hk/en/satref/RINEX_download.htm

- Set the RINEX version, ground station and time.