

Linear Analysis of the Radial Perturbed Displacement in a Cylindrical Plasma

Yujie Dai* and Xuehui Wang

College of Sciences, Liaoning Shihua University, Fushun 113001, China

(Received February 18, 2013; Revised April 20, 2013)

The effect of different plasma pressures on the radial perturbed displacement with wave numbers k and growth rate γ in a cylindrical plasma column is given. The linear numerical simulation indicates that the radial perturbed displacements almost never change when the plasma wavelength is longer, that is to say, the effect of plasma pressure on a longer plasma wavelength is insignificant. The range of wave numbers k increases with an increase of the plasma pressure. Moreover, radial perturbed displacements oscillate obviously when the plasma pressure is non-uniform, which indicates that non-uniform plasma pressure has great destructiveness on the kink instability.

DOI: 10.6122/CJP.52.205

PACS numbers: 52.65.Kj

I. INTRODUCTION

Magnetohydrodynamics (MHD) instabilities play an important role in laboratory and astrophysical plasmas. For all MHD instabilities, the kink instability in a cylindrical plasma column is a basic one which can be studied by solving the linear MHD equations. Many efforts have been made to study the kink instability. For example, Li *et al.* simulated the large-scale behavior of astrophysical jets in astrophysics by solving the ideal MHD equations in a three-dimensional Cartesian coordinate system [1]. Umansky studied the stability of cylindrical localized ideal pressure-driven interchange plasma modes and got the result that near the marginal stability limit the growth rate γ decays with an exponential law [2]. The hydromagnetic characteristics in plasma physics have been mentioned in many papers [3, 4]. For further study of the kink instability in a cylindrical plasma column, a new method for studying the linear MHD instability in cylindrical geometry was introduced by Evstatiev and Delzanno *et al.* [5–9]. The method can change MHD equations into a second-order ordinary differential equation which only contains one variable (radial perturbed displacement). Then the growth rate and radial perturbed displacement can be obtained by solving the second-order ordinary differential equation. However, the effect of plasma pressure P_0 on the kink instability in Refs. [5–9] is not mentioned at all, though it can be neglected when the plasma beta is small, $\beta \approx 1\%$ [10]. In fact, the effect of plasma pressure on the kink instability cannot be ignored when the plasma beta is large enough.

Many efforts have been made to study the effect of plasma pressure on the kink instability. Svidzinski *et al.* simulated the internal kink instability with a finite difference

*Electronic address: yjdai2006@126.com

method in a line-tied screw pinch when the plasma pressure is uniform [10]. The conclusion is that finite compressibility makes the mode more stable than that of zero plasma pressure. Dai *et al.* [11, 12] studied the kink instability in a cylindrical plasma with line-tied boundary conditions; the growth rate and radial eigenfunctions were obtained for the two cases of $P_0=0$ and $P_0 \neq 0$; the linear analysis of the radial perturbed displacement indicates that the uniform plasma pressure provides a stabilizing effect in comparison with the zero plasma pressure case. However, the study of the effect of plasma pressure on the kink instability was not systematic.

The goal of this paper is to give the linear analysis of the radial perturbed displacement with different plasma pressures. In Section II, the idea of the numerical model is described. Section III gives the linear analysis of the growth rate and radial perturbed displacement. Section IV summarizes the study and gives the main conclusions.

II. NUMERICAL MODEL

In this section, we investigate the linearized MHD equations in a cylindrical (r, θ, z) geometry. The linearized MHD equations in normalized units can be written as

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}_1) = 0, \quad (1)$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} = -\nabla \cdot P_1 + \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_0, \quad (2)$$

$$P_1 = \frac{\Gamma P_0}{\rho_0} \rho_1, \quad (3)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0), \quad (4)$$

$$\nabla \times \mathbf{B}_1 = \mathbf{J}_1, \quad (5)$$

$$\nabla \cdot \mathbf{B}_1 = 0, \quad (6)$$

where Γ is the adiabatic index and $\Gamma = 5/3$, a zero subscript indicates equilibrium quantities, and a one subscript indicates perturbed quantities. In the following calculation, we let the plasma mass density $\rho_0 = 1$, the initial velocity $\mathbf{u}_0 = 0$, and the initial magnetic field $B_{0r}(r) = 0$. The initial equilibrium magnetic field components $B_{0\theta}(r)$, $B_{0z}(r)$ and plasma pressure P_0 in a cylindrical plasma column satisfy the pressure balance equation

$$\frac{d}{dr} \left(P_0 + \frac{B_{0\theta}^2}{2} + \frac{B_{0z}^2}{2} \right) + \frac{B_{0\theta}^2}{r} = 0. \quad (7)$$

For convenience of calculation, we let the plasma displacement $\xi = \mathbf{r} - \mathbf{r}_0$, then

$$\mathbf{u} = \mathbf{u}_1 = \frac{d\mathbf{r}}{dt} = \frac{\partial \xi}{\partial t} + \mathbf{u}_1 \cdot \nabla \xi \approx \frac{\partial \xi}{\partial t}. \quad (8)$$

Then Eq. (2) can be written as

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla P_1 + \mathbf{J}_0 \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B}_0. \quad (9)$$

Combining Ampere's law $\mathbf{J} = \nabla \times \mathbf{B}$ and Eq. (9), one can get the MHD equations with plasma displacement:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} + \nabla P_1 + \mathbf{B}_1 \times (\nabla \times \mathbf{B}_0) + \mathbf{B}_0 \times (\nabla \times \mathbf{B}_1) = 0, \quad (10)$$

$$P_1 + \xi \cdot \nabla P_0 + \Gamma P_0 \nabla \cdot \xi = 0, \quad (11)$$

$$\mathbf{B}_1 + \nabla \times (\mathbf{B}_0 \times \xi) = 0. \quad (12)$$

One can change Equations (10)–(12) into a second-order ordinary differential equation with Fourier transforms to get the linear analysis. The computational process is to take the time and space dependence as $\exp[i(m\theta + kz - \omega t)]$ in cylindrical coordinates, where m , k , and ω are the poloidal mode number, toroidal mode number, and frequency, respectively. Then the vector equations can be transformed into a second-order ordinary differential equation which contains one variable ξ_r [13, 14]. The linear analysis of the radial perturbed displacement and growth rate were given in this paper.

$$\frac{d}{dr} \left[\frac{D}{C_2} \frac{1}{r} \frac{d}{dr} (r \xi_r) \right] + \left[\frac{1}{D} \left(C_3 - \frac{C_1^2}{C_2} \right) - r \frac{d}{dr} \left(\frac{C_1}{r C_2} \right) \right] \xi_r = 0, \quad (13)$$

where ξ_r is the radial plasma displacement, the coefficients F , D , C_1 , C_2 , and C_3 in Eq. (13) are as follows:

$$F = \frac{m}{r} B_{0\theta} + k B_{0z}, \quad (14)$$

$$D = (\rho_0 \omega^2 - F^2) [\rho_0 \omega^2 (\Gamma P_0 + B_0^2) - \Gamma P_0 F^2], \quad (15)$$

$$C_1 = \frac{2B_{0\theta}}{r} \{ \rho_0^2 \omega^4 B_{0\theta} - \frac{mF}{r} [\rho_0 \omega^2 (\Gamma P_0 + B_0^2) - \Gamma P_0 F^2] \}, \quad (16)$$

$$C_2 = \rho_0^2 \omega^4 - \left(k^2 + \frac{m^2}{r^2} \right) [\rho_0 \omega^2 (\Gamma P_0 + B_0^2) - \Gamma P_0 F^2], \quad (17)$$

$$\begin{aligned}
C_3 = & D \left[(\rho_0 \omega^2 - F^2) + 2B_{0\theta} \frac{d}{dr} \left(\frac{B_{0\theta}}{r} \right) \right] + \rho_0 \omega^2 (\rho_0 \omega^2 - F^2) \left(\frac{2B_{0\theta}^2}{r} \right)^2 \\
& - [\Gamma P_0 (\rho_0 \omega^2 - F^2) + \rho_0 \omega^2 B_{0z}^2] \left(\frac{2B_{0\theta} F}{r} \right)^2.
\end{aligned} \tag{18}$$

We have reduced the partial differential equations to a second-order ordinary differential equation by expanding in perturbations of the form $\exp[i(m\theta + kz - \omega t)]$. But it is impossible to find the solution of Eq. (13) analytically for an arbitrary initial equilibrium field. Therefore, we resort to numerical means.

III. RESULTS OF LINEAR ANALYSIS

We now present the numerical results of Eq. (13). The initial magnetic field and initial plasma pressure are considered as Eq. (19), where c is a parameter [5] which can alter the plasma pressure. The formulas in (19) satisfy the pressure balance equation (7), so the expression is reasonable. The growth rate $\gamma = -i\omega$ is defined in the following calculations:

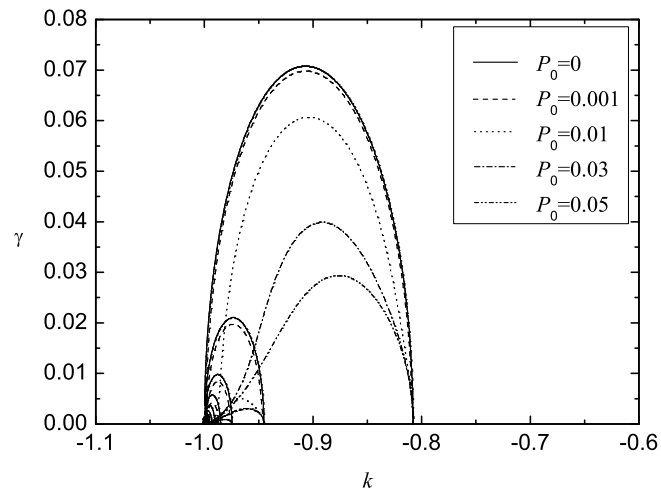
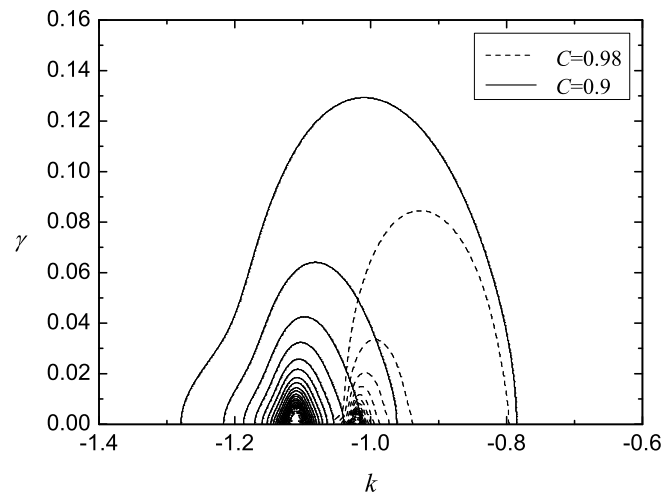
$$B_{0\theta} = \frac{r}{1+r^2}, \quad B_{0z} = \frac{c}{1+r^2}, \quad P_0 = \frac{(1-c^2)}{2(1+r^2)^2}. \tag{19}$$

We first start with the effect of a finite uniform plasma pressure on the growth rate when the poloidal mode number $m = 1$. The relationship between k and γ with different finite uniform plasma pressure P_0 is shown in Fig. 1. The maximum value of the growth rate decreases significantly with an increase of the finite uniform plasma pressure, though the variation of the plasma pressure is only 2%, which indicates that the finite compressibility makes the mode more stable. The mechanism is similar to the compressible stabilization in the configurations with open field lines in Z-pinch. All curves intersect at the common point of $k = -1, \gamma = 0$, which is because $F = \mathbf{k} \cdot \mathbf{B} = (1+k)/(1+r^2) = 0$ when $k = -1$.

Compared with Fig 1, the range of k and γ are increased with a decrease of the parameter c , which indicates the destructive effect of non-uniform plasma pressure on the kink instability. This is because the plasma pressure gradient is not zero when $P_0 = f(r)$. The results can be seen in Fig. 2. The function of Figs. 1–2 is to determine the range of γ and k in the following numerical simulation.

Since the range of the growth rate γ and wave number k with different plasma pressure has been determined in Figs. 1–2, now we can give the linear analysis of the radial perturbed displacements ξ_r with wave number k and growth rate γ . We consider the case of zero plasma pressure ($c = 1$) with the initial magnetic field and plasma pressure that has been given by Eq. (19). In Figs. 3–4, the radial plasma displacements almost never changes when the plasma wavelength is longer.

With an increase of the uniform plasma pressure, the radial perturbed displacement is equal to zero when the range of the wave numbers k is $(-0.94, -0.82)$. This stable effect is caused by plasma compressibility, which can be seen in Figs. 5–6. The stabilization

FIG. 1: The relationship between k and γ with different P_0 .FIG. 2: The relationship between k and γ with different C .

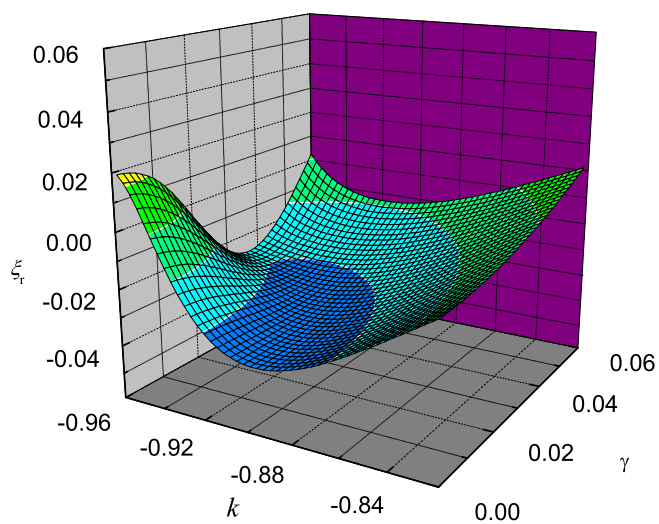


FIG. 3: The evolution of ξ_r with k and γ when $P_0 = 0$.

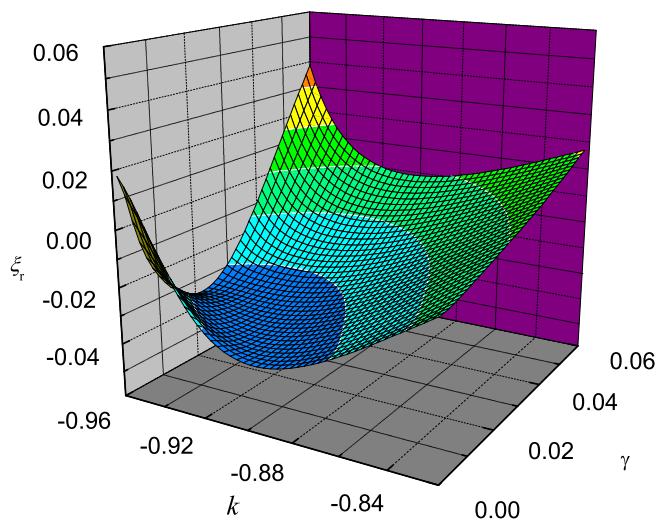


FIG. 4: The evolution of ξ_r with k and γ when $P_0 = 0.01$.

result in the reduction of the growth rate is shown in Fig. 1. Finite uniform plasma pressure provides a stabilizing effect in comparison with the zero plasma pressure case.

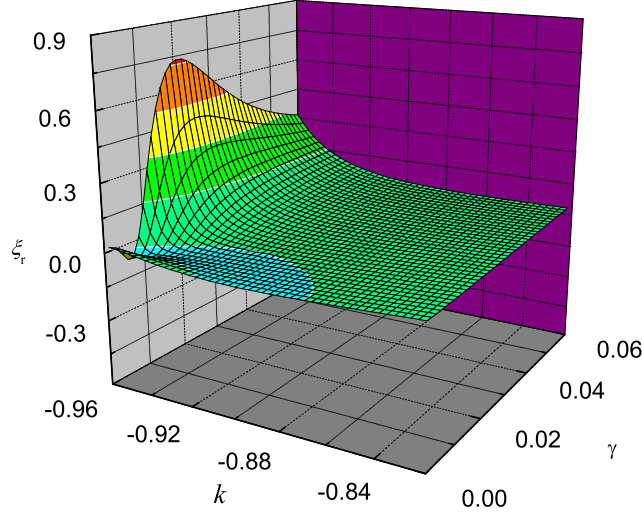


FIG. 5: The evolution of ξ_r with k and γ when $P_0 = 0.03$.

The plasma β can be defined as $\beta = P_0 / (B^2 / 2\mu_0)$, to show the constraint efficiency of a magnetic field for the plasma. The plasma β increases with an increase of the plasma pressure P_0 if the magnetic field \mathbf{B} is fixed. The linear analysis of the radial perturbed displacement with different parameter c in Eq. (19) is shown in Figs. 7–8. The maximum value of the radial perturbed displacement for $c = 0.98$ is smaller than that of $c = 0.9$. The radial perturbed displacement oscillates obviously near $\xi_r = 0$ when the range of k is $(-1.2, -1.0)$. The mechanism can be analyzed via the following formula [15, 16]:

$$\delta W_p = \frac{1}{2} \int_V d\mathbf{r} \left[\frac{|B_{1\perp}|^2}{\mu_0} + \frac{B_0^2}{\mu_0} |\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \mathbf{k}|^2 + \gamma p_0 |\nabla \cdot \xi|^2 - \frac{\mathbf{J}_0 \cdot \mathbf{B}_0}{B_0^2} (\xi_\perp^* \times \mathbf{B}_0) \cdot \mathbf{B}_{1\perp} - 2(\xi \cdot \nabla p_0)(\xi_\perp^* \cdot \mathbf{k}) \right], \quad (20)$$

where $|B_{1\perp}|^2 / \mu_0$ is the magnetic tension, which makes the mode more stable. Here, the terms $B_0^2(|\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \mathbf{k}|^2) / \mu_0$ and $\gamma p_0 |\nabla \cdot \xi|^2$ denote the stabilization caused by compressibility and the magnetic field. $(\mathbf{J}_0 \cdot \mathbf{B}_0)[(\xi_\perp^* \times \mathbf{B}_0) \cdot \mathbf{B}_{1\perp}] / B_0^2$ leads to the kink instability. $2(\xi \cdot \nabla p_0)(\xi_\perp^* \cdot \mathbf{k})$ denotes an instability which arises from the plasma pressure gradient and the curvature of the magnetic line.

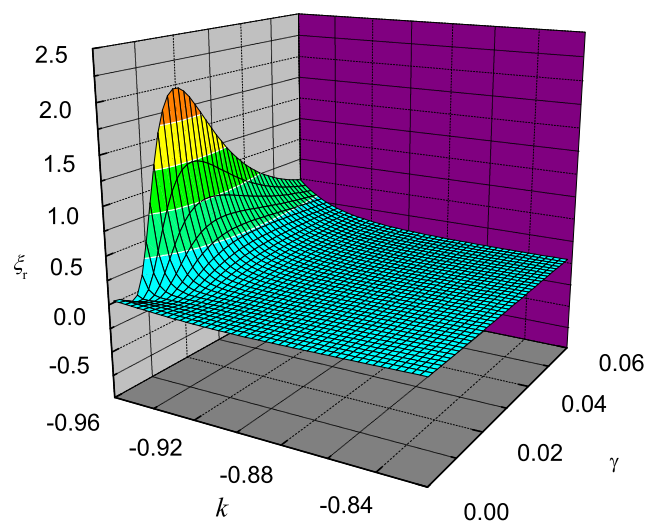


FIG. 6: The evolution of ξ_r with k and γ when $P_0 = 0.05$.

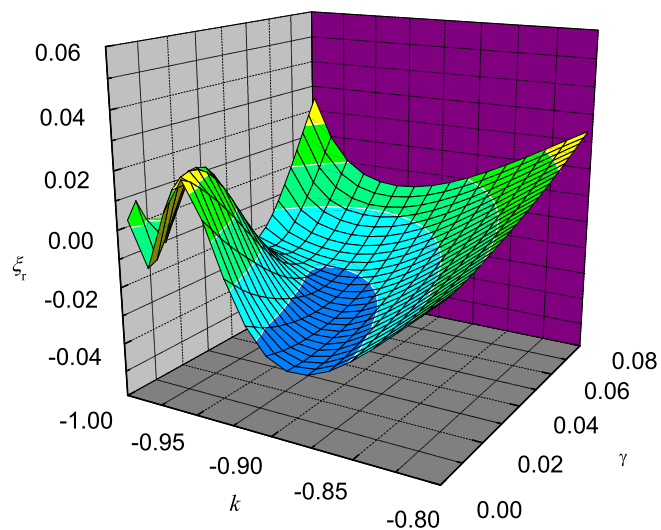


FIG. 7: The evolution of ξ_r with k and γ when $c = 0.98$.

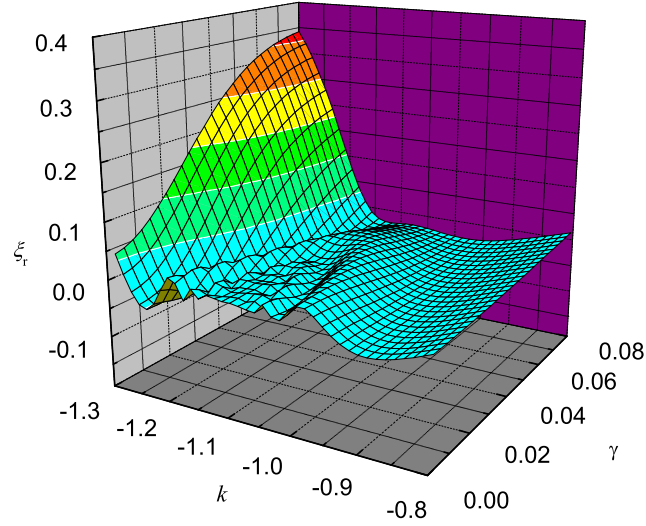


FIG. 8: The evolution of ξ_r with k and γ when $c = 0.9$.

IV. SUMMARY

The kink instability with different plasma pressure ($P_0 = 0$ and $P_0 \neq 0$) is studied. One can transform the MHD equations into a second-order ordinary differential equation which only contains one variable (radial perturbed displacement) by using Fourier transforms. The relationship between the growth rate γ and wave number k is obtained by solving the ordinary differential equation if the parameters $B_{0\theta}$, B_{0z} , and P_0 in the coefficients has been defined. Then the linear analysis of ξ_r with k and γ with uniform and non-uniform plasma pressure is given. The results indicate that the effect of plasma pressure on the kink instability for longer wavelength is unobvious, the non-uniform plasma pressure on the kink instability is destructive.

Acknowledgments

This research is supported by the Foundation of the Education Department of Liaoning province under grant L2013148.

References

- [1] H. Li, G. Lapenta, J. M. Finn, S. Li, and S. A. Colgate, *Astrophys J.* **643**, 92 (2006). doi: 10.1086/501499
- [2] M. V. Umansky, *Contrib. Plasma Phys.* **47**, 447 (2007). doi: 10.1002/ctpp.200710058
- [3] S. H. Chuang, L. N. Hau, and P. K. Shukla, *Chin. J. Phys.* **48**, 460 (2010).
- [4] A. Rahman and A. A. Mamun, *Chin. J. Phys.* **46**, 601 (2008).
- [5] E. G. Evstatiev, G. L. Delzanno, and J. M. Finn, *Phys. Plasmas* **13**, 072902 (2006). doi: 10.1063/1.2219426
- [6] G. L. Delzanno, E. G. Evstatiev, and J. M. Finn, *Phys. Plasmas* **14**, 072902 (2007). doi: 10.1063/1.2749252
- [7] G. L. Delzanno, E. G. Evstatiev, and J. M. Finn, *Phys. Plasmas* **14**, 070702 (2007). doi: 10.1063/1.2746213
- [8] G. L. Delzanno, E. G. Evstatiev, and J. M. Finn, *Phys. Plasmas* **14**, 042701 (2007). doi: 10.1063/1.2711173
- [9] G. L. Delzanno, J. M. Finn, *Phys. Plasmas* **15**, 032904 (2008). doi: 10.1063/1.2876666
- [10] V. A. Svidzinski, V. V. Mirnov, and H. Li, *Phys. Plasmas* **15**, 092106 (2008). doi: 10.1063/1.2917916
- [11] Y. J. Dai, J. Y. Liu, and X. H. Wang, *Plasma Sci. Technol.* **11**, 269 (2009). doi: 10.1088/1009-0630/11/3/04
- [12] Y. J. Dai, J. H. Huang, and X. H. Wang, *Adv. Mat. Res.* **518**, 5627 (2012). doi: 10.4028/www.scientific.net/AMR.518-523.5627
- [13] J. P. Goedbloed and H. J. L. Hagebeuk, *Phys. Fluids* **15**, 1090 (1972). doi: 10.1063/1.1694034
- [14] K. Appert, R. Gruber, and J. Vaclavik, *Phys. Fluids* **17**, 1471 (1974). doi: 10.1063/1.1694918
- [15] D. Li, Y. H. Chen, J. X. Ma, and W. H. Yang, *Plasma Physics* (Higher Education Press, 2006).
- [16] X. W. Hu, *Foundation of Plasma Physics* (Peking University Press, 2006).