The Control and Modified Projective Synchronization of a Class of 2,3,4-Dimensional (Chaotic) Systems with Parameter and Model Uncertainties and External Disturbances Via Adaptive Control

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This paper focuses on the control and modified projective synchronization of a class of chaotic systems. Some robust criteria are proposed based on the adaptive control scheme to ensure perfect tracking and modified projective synchronization in the presence of parameter and model uncertainties and external disturbances. Two numerical simulations are given to demonstrate the robustness and efficiency of the proposed approach.

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I. INTRODUCTION

Chaotic phenomena can be found in many scientific and engineering fields, such as biological systems, electronic circuits, power converters, chemical systems, etc. A chaotic system is a nonlinear deterministic system that displays complex and unpredictable behavior. The sensitive dependence on the initial conditions and on the system's parameter variation is a prominent characteristic of chaotic behavior. Research efforts have investigated chaos control and chaos synchronization problems in many physical chaotic systems. Chaos control means designing a controller to mitigate or eliminate the chaotic behavior of nonlinear systems, while the chaos synchronization problem means making two systems oscillate in a synchronized manner. Since the synchronization of chaotic dynamical systems was observed by Pecora and Carroll [1], many types of synchronization such as complete synchronization [1], phase synchronization [2], anti-synchronization [3], generalized synchronization [4], projective synchronization [5], modified projective synchronization (MPS) [6], and combination synchronization [7] have been discovered in a variety of chaotic systems. Modified projective synchronization (MPS), which has been introduced by Li, is a synchronization phenomenon such that the responses of the synchronized dynamical states synchronize up to a constant scaling matrix. Among all kinds of synchronization phenomena, it is easy to see that complete synchronization, anti-synchronization, and projective synchronization belong to special cases of MPS. Meanwhile, the proportional feature can be used to extend binary digital to variety M-nary digital communications for achieving fast communication. Therefore modified projective synchronization is worth studying, and some research results for modified projective synchronization have been obtained in recent

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years [8–14].

Nowadays, different techniques and methods have been proposed to achieve chaos control and chaos synchronization, such as sliding mode control [15], optimal control [16], adaptive control [17], feedback control [18], backstepping method [7], impulsive control [19], etc. However, most of the methods mentioned above that have been found effective for controlling or synchronizing chaotic system are based on exactly knowing the system structure and parameters. But in practical situations, some or all of the system parameters are unknown and even changing from time to time. Moreover, in practical applications, chaotic systems are unavoidably affected by external disturbances. Therefore, the control and synchronization of chaotic systems in the presence of unknown system parameters, model uncertainty, and external disturbance is an important issue. Also, most of the publications concerned with chaos control and synchronization are only valid for some particular chaotic systems. However, from the point of practical applications, it is desired that the control and synchronization scheme can be used for more chaotic systems.

Motivated by the above discussion, in this paper we consider the control and modified projective synchronization of the following chaotic systems:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \vdots \\ \dot{x}_n = f(x)^T(\theta + \Delta\theta) + g(x) + \Delta g(x) + d(t), \end{cases} \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_n)^T \in R^n$ is the state variable, $f(x) = (f_1, f_2, \ldots, f_m)^T \in R^{m\times 1}$ and $g(x) \in R$ are functions, $\theta = (\theta_1, \ldots, \theta_m)^T$ is the system's parameters, $\triangle\theta$ is the parameter uncertainty, $\triangle f(x) \in R^1$ is the model uncertainty, $d(t)$ is the external disturbance. Since the dimensions of most of the famous chaotic systems are less than 4, so in this paper we investigate the cases of $n = 2, 3, 4$.

Remark 1. There are many famous (chaotic) systems which can be expressed or transformed into the form (1.1). Here, we give three examples.

The first example is the famous Duffing system [20], which can be described as:

$$\ddot{x} + a\dot{x} + bx + cx^3 = d\cos(\omega t),$$

where a, b, c are constant parameters, $d\cos(\omega t)$ is an external excitation. By introducing the space variables $(x_1, x_2) = (x, \dot{x})$, the Duffing equation can be written in the following form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -ax_2 - bx_1 - cx_1^3 + d\cos(\omega t). \end{cases} \tag{1.2}$$

The second example is the Genesio-Tesi system which was proposed by Genesio and Tesi [21]. The Genesio-Tesi system is one of the paradigms of chaos, since it captures many features of chaotic systems. The dynamic equations of the system are as follows:

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = x_3, \\ \dot{x_3} = -ax_1 - bx_2 - cx_3 + x_1^2, \end{cases} \tag{1.3}$$

where a, b, c are positive real constants satisfying $bc < a$. When $a = 6$, $b = 2.92$, $c = 1.2$ the Genesio-Tesi system is chaotic.

The third example is the Arneodo-Coullet system [22], which is one of the paradigms of the three dimensional chaotic models described by the dynamics:

$$\begin{cases} \dot{x_1} = x_2, \\ \dot{x_2} = x_3, \\ \dot{x_3} = ax_1 - bx_2 - cx_3 - x_1^3, \end{cases} \tag{1.4}$$

where a, b, and c are positive constants. When $a = 5.5$, $b = 3.5$, $c = 1.0$ the system (1.4) is chaotic.

The Duffing system, the Genesio-Tesi system, and the Arneodo-Coullet system have been studied thoroughly in the literature [20–31]. Compared with those special chaotic systems studied in [20–31], system (1.1) has more research value. To the best knowledge of the authors, the challenging problem of controlling and synchronizing the chaotic systems (1.1) in spite of the parameter and model uncertainties and external disturbances, has yet to be studied to this date. Therefore, the main purpose of this paper is to design robust controllers to control and synchronize systems (1.1) in the presence of parameter and model uncertainties and external disturbances.

This paper is organized as follows: In Section II, we discuss the control of 2,3,4-dimensional systems, some robust criteria are proposed based on the adaptive schemes. The synchronization based on the control of 2-dimensional system is investigated in Section III. Section IV includes several numerical examples to demonstrate the effectiveness of the proposed approach; finally, some conclusions are shown in Section V.

II. THE CONTROL SCHEMES OF THE 2,3,4-DIMENSIONAL SYSTEMS

The aim of this section is to design a proper controller u based on the reference signal, such that the output of the (chaotic) system follows the given reference signal asymptotically.

II-1. The control scheme of 2-dimensional systems

Let us consider the following the 2-dimensional system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + f(x,t) + \Delta f(x,t) + d + u, \end{cases} \tag{2.1.1}$$

where $x = (x_1, x_2)^T \in R^{2\times1}$ is the state vector of system (2.1.1), a_1, a_2 are the system's parameters, $\triangle a_1$, $\triangle a_2$ are the parameter uncertainties, $f(x,t) \in R^1$ is a function, $\triangle f(x,t) \in R^1$ and $d \in R^1$ are the model uncertainty and external disturbance of the system (2.1.1), respectively, u is the controller to be designed later.

Let $y(t)$ be an arbitrarily given bounded reference signal with well defined first and second derivatives. The goal of this subsection is to design a proper controller u based on

$y(t)$ and its derivatives such that

$$\lim_{t\to\infty}(x_1-y)=\lim_{t\to\infty}(x_2-\dot{y})=0.$$

By defining the error variables as $e_1=x_1-y$, $e_2=x_2-\dot{y}$, we get the error dynamic system:

$$\begin{cases}\dot{e}_1=e_2,\\ \dot{e}_2=(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+f(x,t)+\Delta f(x,t)-\ddot{y}+d+u.\end{cases}\tag{2.1.2}$$

In order to derive our results, we introduce a lemma which will be used in the proof of Theorem 2.1.1.

Lemma 2.1.1. For the system (2.1.2), if

$$\lim_{t\to\infty}(e_1+e_2)=0,$$

then

$$\lim_{t\to\infty}e_1=\lim_{t\to\infty}e_2=0,$$

which means that $\lim\limits_{t\to\infty}(x_1-y)=\lim\limits_{t\to\infty}(x_2-\dot{y})=0.$

Proof. If

$$\lim_{t\to\infty}(e_1+e_2)=0,$$

then

$$\lim_{t\to\infty}(\dot{e}_1+\dot{e}_2)=0,$$

thus

$$\lim_{t\to\infty}(e_2+\dot{e}_2)=0.$$

By the relationship between the infinitesimal and the limit of a function, one gets

$$e_2+\dot{e}_2=\varepsilon,\tag{2.1.3}$$

where ε is an infinitesimal, i.e., $\lim\limits_{t\to\infty}\varepsilon=0.$

It is easy to show that the general solution of system (2.1.3) is

$$e_2=e^{-t}e_2(0)+e^{-t}\int_0^t e^{\tau}\varepsilon d\tau,\tag{2.1.4}$$

where $e_2(0)$ is the initial value of e_2.

From the first item on the right hand side of Eq. (2.1.4), we have

$$\lim_{t\to\infty}e^{-t}e_2(0)=0.$$

Now, we shall show that

$$e^{-t}\int_0^t e^{\tau}\varepsilon d\tau \to 0 \quad (t \to +\infty).$$

Using L' Hospital's rule to calculate the limit of $e^{-t}\int_0^t e^{\tau}\varepsilon d\tau$, one gets

$$\begin{aligned}\lim_{t\to\infty} e^{-t}\int_0^t e^{\tau}\varepsilon d\tau &= \lim_{t\to\infty}\frac{\int_0^t e^{\tau}\varepsilon d\tau}{e^t}\\ &= \lim_{t\to\infty}\frac{e^t\varepsilon}{e^t} = \lim_{t\to\infty}\varepsilon = 0.\end{aligned}$$

In view of Eq. (2.1.4) we obtain

$$\lim_{t\to\infty} e_2 = 0.$$

Using the condition

$$\lim_{t\to\infty}(e_1+e_2) = 0,$$

yields

$$\lim_{t\to\infty} e_1 = 0.$$

Thus, we have

$$\lim_{t\to\infty} e_1 = \lim_{t\to\infty} e_2 = 0,$$

which means that $\lim\limits_{t\to\infty}(x_1-y) = \lim\limits_{t\to\infty}(x_2-\dot{y}) = 0$.

By using Lemma 2.1.1, the control problem of system (2.1.1) is replaced by the equivalent problem of proving

$$\lim_{t\to\infty}(e_1+e_2) = 0.$$

The derivative of e_1+e_2, with respect to t, is

$$\begin{aligned}\dot{e}_1+\dot{e}_2 &= e_2+\dot{e}_2\\ &= e_2+(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+f(x)+\Delta f(x)-\ddot{y}+d+u\\ &= \omega+u,\end{aligned}$$

where $\omega = e_2+(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+f(x,t)+\Delta f(x,t)-\ddot{y}+d$.

Thus we have

$$\dot{e}_1+\dot{e}_2 = \omega+u. \tag{2.1.5}$$

Suppose the controller u is chosen as:

$$u = -k(e_1 + e_2) = -k(x_1 - y + x_2 - \dot{y}), \tag{2.1.6}$$

and the update law is selected as

$$\dot{k} = l > 0. \tag{2.1.7}$$

By plugging Eq. (2.1.6) into Eq. (2.1.5), we get

$$\dot{v} = -kv + \omega, \tag{2.1.8}$$

where $v = e_1 + e_2$.

It can easily be seen that the control problem of system (2.1.1) is equivalent to the problem of stabilizing the system (2.1.8).

In order to get further results, we make the following assumption.

Assumption 1. The parameter uncertainties $\Delta a_1, \Delta a_2$, model uncertainty $\Delta f(x,t)$, external disturbance d, and $y^{(i)}$, $i = 1, 2$ are all bounded. It is well known that the trajectory of chaotic systems is bounded, thus there must exist a constant $\varpi \geq 0$ such that $\|\omega\| \leq \varpi$.

The following Theorem 2.1.1 ensures that

$$\lim_{t\to\infty} v = 0.$$

Theorem 2.1.1. If the controller and the update law are designed as Eq. (2.1.6) and Eq. (2.1.7), respectively, then the origin of system (2.1.8) is asymptotically stable, i.e., $\lim_{t\to\infty} v = 0$, which means that $\lim_{t\to\infty}(x_1 - y) = \lim_{t\to\infty}(x_2 - \dot{y}) = 0$.

Proof: It is not difficult to show that the general solution of system (2.1.8) is

$$v = e^{-kt} e^{\frac{lt^2}{2}} v(0) + e^{-kt} e^{\frac{lt^2}{2}} \int_0^t e^{k\tau} e^{-\frac{l\tau^2}{2}} \omega d\tau, \tag{2.1.9}$$

where $v(0)$ is the initial value of v.

By $\dot{k} = l$, we have $k = lt + c_0$, where c_0 is a constant. It is easy to see that

$$\lim_{t\to\infty} e^{-kt} e^{\frac{lt^2}{2}} v(0) = \lim_{t\to\infty} e^{(-\frac{lt^2}{2} + c_0 t)} v(0) = 0. \tag{2.1.10}$$

Using L' Hospital's rule to calculate the limit of $e^{-kt} e^{\frac{lt^2}{2}} \int_0^t e^{k\tau} e^{-\frac{l\tau^2}{2}} \omega d\tau$, one gets

$$\begin{aligned}
\lim_{t\to\infty} |e^{-kt} e^{\frac{lt^2}{2}} \int_0^t e^{k\tau} e^{-\frac{l\tau^2}{2}} \omega d\tau| &\leq \lim_{t\to\infty} \frac{\varpi \int_0^t e^{k\tau} e^{-\frac{l\tau^2}{2}} d\tau}{e^{kt} e^{-\frac{lt^2}{2}}} \\
&= \lim_{t\to\infty} \frac{\varpi e^{kt} e^{-\frac{lt^2}{2}}}{k e^{kt} e^{-\frac{lt^2}{2}} + e^{kt}(lt) e^{-\frac{lt^2}{2}} + e^{kt} e^{-\frac{lt^2}{2}}(-lt)} \\
&= \lim_{t\to\infty} \frac{\varpi}{k} = \lim_{t\to\infty} \frac{\varpi}{lt + c_0} = 0.
\end{aligned} \tag{2.1.11}$$

Combining using Eqs. (2.1.9)–(2.1.11), yields

$$\lim_{t\to\infty} v = 0.$$

According to lemma 2.1.1, we know that $\lim_{t\to\infty}(x_1 - y) = \lim_{t\to\infty}(x_2 - \dot{y}) = 0$.

If $y = 0$, then $\dot{y} = \ddot{y} = 0$. By **Theorem 2.1.1**, we can easily have the following **Corollary 2.1.1.**

Corollary 2.1.1. If the controller is designed as

$$u = -k(x_1 + x_2), \tag{2.1.12}$$

and the update law is chosen as

$$\dot{k} = l > 0, \tag{2.1.13}$$

then the origin of system (2.1.1) is asymptotically stable.

II-2. The control scheme of the 3-dimensional systems

Let us consider the following 3-dimensional chaotic system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + (a_3 + \Delta a_3)x_3 + f(x,t) + \Delta f(x,t) + d + u, \end{cases} \tag{2.2.1}$$

where $x = (x_1, x_2, x_3)^T \in R^{3\times 1}$ is the state vector of system (2.2.1), a_1, a_2, a_3 are the system's parameters, $\triangle a_1$, $\triangle a_2$, $\triangle a_3$ are the parameter uncertainties, $f(x,t) \in R^1$ is a function, $\triangle f(x,t) \in R^1$ and $d \in R^1$ are model uncertainty and external disturbance of the system (2.2.1), respectively, and u is the controller to be designed later.

Let $y(t)$ be an arbitrarily given bounded reference signal with well defined first, second, and third derivatives. The aim of this subsection is to design a proper controller u based on $y(t)$ and its derivatives such that

$$\lim_{t\to\infty}(x_1 - y) = \lim_{t\to\infty}(x_2 - \dot{y}) = \lim_{t\to\infty}(x_3 - \ddot{y}) = 0.$$

Suppose the error variables are $e_1 = x_1 - y$, $e_2 = x_2 - \dot{y}$, $e_3 = x_3 - \ddot{y}$, then the error dynamic system can be obtained as:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \dot{e}_3 = (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + (a_3 + \Delta a_3)x_3 \\ \qquad + f(x,t) + \Delta f(x,t) - y^{(3)} + d + u, \end{cases} \tag{2.2.2}$$

Before presenting the main results, we introduce two lemmas which will be used in the proof of Theorem 2.2.1.

Lemma 2.2.1. For system (2.2.2), if

$$\lim_{t\to\infty}(e_1+e_2)=0,$$

then

$$\lim_{t\to\infty}e_1=\lim_{t\to\infty}e_2=\lim_{t\to\infty}e_3=0,$$

which means that $\lim_{t\to\infty}(x_1-y)=\lim_{t\to\infty}(x_2-\dot{y})=\lim_{t\to\infty}(x_3-\ddot{y})=0$.

Proof. The proof of **Lemma 2.2.1** is similar to that of **Lemma 2.1.1**, so we omit it here.

Lemma 2.2.2. For system (2.2.2), if

$$\lim_{t\to\infty}(e_1+2e_2+e_3)=0,$$

then

$$\lim_{t\to\infty}(e_1+e_2)=0.$$

Proof. Choose the following Lyapunov function:

$$V=\frac{1}{2}(e_1+e_2)^2.$$

The derivative of V along the trajectory of system (2.2.2) is

$$\begin{aligned}\dot{V} &= (e_1+e_2)(\dot{e}_1+\dot{e}_2)\\ &= (e_1+e_2)(e_2+e_3)\\ &= (e_1+e_2)[-(e_1+e_2)+(e_1+2e_2+e_3)]\\ &= -2V+(e_1+e_2)(e_1+2e_2+e_3).\end{aligned}$$

It is noted that e_1+e_2 is bounded and

$$\lim_{t\to\infty}(e_1+2e_2+e_3)=0,$$

we have

$$\lim_{t\to\infty}(e_1+e_2)(e_1+2e_2+e_3)=0.$$

Let

$$\varepsilon=(e_1+e_2)(e_1+2e_2+e_3),$$

then

$$\lim_{t\to\infty}\varepsilon=0.$$

Similar to the proof of system (2.1.3), we obtain

$$\lim_{t\to\infty} V = 0,$$

which means that $\lim_{t\to\infty}(e_1 + e_2) = 0$.

Therefore the control problem of system (2.2.1) is converted to the equivalent problem of proving

$$\lim_{t\to\infty}(e_1 + 2e_2 + e_3) = 0.$$

The derivative of $e_1 + 2e_2 + e_3$, with respect to t, is

$$\begin{aligned}
\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 &= e_2 + 2e_3 + \dot{e}_3 \\
&= e_2 + 2e_3 + (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + \\
&\quad (a_3 + \Delta a_3)x_3 + f(x) + \Delta f(x) - y^{(3)} + d + u \\
&= \omega + u,
\end{aligned}$$

where $\omega = e_2 + 2e_3 + (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + (a_3 + \Delta a_3)x_3 + f(x,t) + \Delta f(x,t) - y^{(3)} + d$. Thus we have

$$\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 = \omega + u. \tag{2.2.3}$$

Suppose the controller u is chosen as:

$$\begin{aligned}
u &= -k(e_1 + 2e_2 + e_3) \\
&= -k(x_1 - y + 2(x_2 - \dot{y}) + (x_3 - \ddot{y})),
\end{aligned} \tag{2.2.4}$$

and the update law is selected as

$$\dot{k} = l > 0. \tag{2.2.5}$$

By inserting Eq. (2.2.4) into Eq. (2.2.3), we get

$$\dot{v} = -kv + \omega, \tag{2.2.6}$$

where $v = e_1 + 2e_2 + e_3$.

Obviously, the control problem of system (2.2.1) is equivalent to the problem of stabilizing system (2.2.6).

Now, we are in a position to make the following assumption.

Assumption 2.2.1. The parameter uncertainties Δa_1, Δa_2, Δa_3, model uncertainty $\Delta f(x,t)$, external disturbance d, and $y^{(i)}, i = 1, 2, 3$ are all bounded. It is well known that the trajectory of chaotic systems is bounded, thus there must be exist a constant $\varpi \geq 0$ such that $\|\omega\| \leq \varpi$.

The following Theorem 2.2.1 ensures that

$$\lim_{t\to\infty}(e_1 + 2e_2 + e_3) = 0.$$

Theorem 2.2.1. If the controller and the update law are designed as Eq. (2.2.4) and Eq. (2.2.5), respectively, then the origin of system (2.2.6) is asymptotically stable, i.e., $\lim_{t\to\infty}(e_1+2e_2+e_3)=0$, which means that $\lim_{t\to\infty}(x_1-y)=\lim_{t\to\infty}(x_2-\dot{y})=\lim_{t\to\infty}(x_3-\ddot{y})=0$.

Proof: The proof of Theorem 2.2.1 is similar to that of Theorem 2.1.1, and we omit it here.

If $y=0$, then $\dot{y}=\ddot{y}=y^{(3)}=0$. By **Theorem 2.2.1**, we can easily have the following **Corollary 2.2.1.**

Corollary 2.2.1. If the controller is designed as

$$u=-k(x_1+2x_2+x_3), \tag{2.2.7}$$

and the update law is chosen as

$$\dot{k}=l>0, \tag{2.2.8}$$

then the origin of system (2.2.1) is asymptotically stable.

II-3. The control scheme of the 4-dimensional systems

Let us consider the control of the following the 4-dimensional chaotic system:

$$\begin{cases}\dot{x}_1=x_2,\\ \dot{x}_2=x_3,\\ \dot{x}_3=x_4,\\ \dot{x}_4=(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+(a_3+\Delta a_3)x_3\\ \qquad +(a_4+\Delta a_4)x_4+f(x,t)+\Delta f(x,t)+d+u,\end{cases} \tag{2.3.1}$$

where $x=(x_1,x_2,x_3,x_4)^T\in R^{4\times 1}$ is the state vector of system (2.3.1), a_1, a_2, a_3, a_4 are the system's parameters, $\triangle a_1$, $\triangle a_2$, $\triangle a_3$, $\triangle a_4$ are the parameter uncertainties, $f(x,t)\in R^1$ is a function, $\triangle f(x,t)\in R^1$ and $d\in R^1$ are the model uncertainty and external disturbance of the system (2.3.1), respectively, u is the controller to be designed later.

Let $y(t)$ be an arbitrarily given bounded reference signal with well defined first, second, third, and forth derivatives. The objective of this subsection is to design a proper controller u based on $y(t)$ and its derivatives such that

$$\lim_{t\to\infty}(x_1-y)=\lim_{t\to\infty}(x_2-\dot{y})=\lim_{t\to\infty}(x_3-\ddot{y})=\lim_{t\to\infty}(x_4-y^{(3)})=0.$$

Let the error variables be $e_1=x_1-y$, $e_2=x_2-\dot{y}$, $e_3=x_3-\ddot{y}$, $e_4=x_4-y^{(3)}$, then we get the error dynamic system:

$$\begin{cases}\dot{e}_1=e_2,\\ \dot{e}_2=e_3,\\ \dot{e}_3=e_4,\\ \dot{e}_4=(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+(a_3+\Delta a_3)x_3\\ \qquad +(a_4+\Delta a_4)x_4+f(x,t)+\Delta f(x,t)-y^{(4)}+d+u,\end{cases} \tag{2.3.2}$$

Similarly, we introduce two lemmas which will be used in the proof of Theorem 2.3.1.

Lemma 2.3.1. For system (2.3.2), if

$$\lim_{t\to\infty}(e_1+e_2)=0,$$

then

$$\lim_{t\to\infty}e_1=\lim_{t\to\infty}e_2=\lim_{t\to\infty}e_3=\lim_{t\to\infty}e_4=0,$$

which means that $\lim_{t\to\infty}(x_1-y)=\lim_{t\to\infty}(x_2-\dot{y})=\lim_{t\to\infty}(x_3-\ddot{y})=\lim_{t\to\infty}(x_4-y^{(3)})=0$.

Proof. The proof of **Lemma 2.3.1** is similar to that of **Lemma 2.1.1**, and we omit it here.

Lemma 2.3.2. For system (2.3.2), if

$$\lim_{t\to\infty}(2e_1+4e_2+3e_3+e_4)=0,$$

then

$$\lim_{t\to\infty}(e_1+e_2)=0.$$

Proof. Choose the following Lyapunov function:

$$V=\frac{1}{2}(e_1+e_2)^2+\frac{1}{2}(e_1+2e_2+e_3)^2.$$

The derivative of V along the trajectory of system (2.3.2) is

$$\begin{aligned}\dot{V} &= (e_1+e_2)(\dot{e}_1+\dot{e}_2)+(e_1+2e_2+e_3)(\dot{e}_1+2\dot{e}_2+\dot{e}_3)\\ &= (e_1+e_2)(e_2+e_3)+(e_1+2e_2+e_3)(e_2+2e_3+e_4)\\ &= (e_1+e_2)[-(e_1+e_2)+(e_1+2e_2+e_3)]\\ &\quad +(e_1+2e_2+e_3)[-(e_1+2e_2+e_3)+(e_1+3e_2+3e_3+e_4)]\\ &= -2V+(e_1+2e_2+e_3)(2e_1+4e_2+3e_3+e_4).\end{aligned}$$

It is noted that $e_1+2e_2+e_3$ is bounded and

$$\lim_{t\to\infty}(2e_1+4e_2+3e_3+e_4)=0,$$

we have

$$\lim_{t\to\infty}(e_1+2e_2+e_3)(2e_1+4e_2+3e_3+e_4)=0.$$

Let

$$\varepsilon=(e_1+2e_2+e_3)(2e_1+4e_2+3e_3+e_4),$$

then

$$\lim_{t\to\infty} \varepsilon = 0.$$

Similar to the proof of system (2.1.3), we obtain

$$\lim_{t\to\infty} V = 0,$$

which means that $\lim\limits_{t\to\infty}(e_1+e_2)=0$.

Therefore the control problem of system (2.3.1) is equivalent to the problem of proving

$$\lim_{t\to\infty}(2e_1+4e_2+3e_3+e_4)=0.$$

The derivative of $2e_1+4e_2+3e_3+e_4$ with respect to t is

$$\begin{aligned} 2\dot{e}_1+4\dot{e}_2+3\dot{e}_3+\dot{e}_4 &= 2e_2+4e_3+3e_4+\dot{e}_4 \\ &= 2e_2+4e_3+3e_4+(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+(a_3+\Delta a_3)x_3 \\ &\quad +(a_4+\Delta a_4)x_4+f(x)+\Delta f(x)-y^{(4)}+d+u \\ &= \omega+u, \end{aligned}$$

where $\omega = 2e_2+4e_3+3e_4+(a_1+\Delta a_1)x_1+(a_2+\Delta a_2)x_2+(a_3+\Delta a_3)x_3+(a_4+\Delta a_4)x_4+f(x,t)+\Delta f(x,t)-y^{(4)}+d$. Thus we have

$$2\dot{e}_1+4\dot{e}_2+3\dot{e}_3+\dot{e}_4=\omega+u. \tag{2.3.3}$$

Suppose the controller u is chosen as

$$\begin{aligned} u &= -k(2e_1+4e_2+3e_3+e_4) \\ &= -k(2(x_1-y)+4(x_2-\dot{y})+3(x_3-\ddot{y})+(x_4-y^{(3)})), \end{aligned} \tag{2.3.4}$$

and the update law is selected as

$$\dot{k}=l>0. \tag{2.3.5}$$

By substituting Eq. (2.3.4) into Eq. (2.3.3), we get

$$\dot{v}=-kv+\omega, \tag{2.3.6}$$

where $v=2e_1+4e_2+3e_3+e_4$.

Obviously, the control problem of system (2.3.1) is converted to the problem of stabilizing system (2.3.6).

Similarly, we make the following assumption.

Assumption 2.3.1. The parameter uncertainties Δa_1, Δa_2, Δa_3, Δa_4, model uncertainty $\Delta f(x,t)$, external disturbance d, and $y^{(i)}, i=1,2,3,4$ are all bounded. It is well known that the trajectory of chaotic systems is bounded, thus there must be exist a constant $\varpi \geq 0$ such that $\|\omega\| \leq \varpi$.

The following Theorem 2.3.1 ensures that

$$\lim_{t\to\infty}(2e_1 + 4e_2 + 3e_3 + e_4) = 0.$$

Theorem 2.3.1. If the controller and the update law are designed as Eq. (2.3.4) and Eq. (2.3.5), respectively, then the origin of system (2.3.6) is asymptotically stable, i.e., $\lim_{t\to\infty}(2e_1+4e_2+3e_3+e_4) = 0$, which means that $\lim_{t\to\infty}(x_1-y) = \lim_{t\to\infty}(x_2-\dot{y}) = \lim_{t\to\infty}(x_1-\ddot{y}) = \lim_{t\to\infty}(x_4 - y^{(3)}) = 0$.

Proof: The proof of Theorem 2.3.1 is similar to that of Theorem 2.1.1, and we omit it here.

If $y = 0$, then $\dot{y} = \ddot{y} = y^{(3)} = y^{(4)} = 0$. By **Theorem 2.3.1**, we can easily have the following **Corollary 2.3.1.**

Corollary 2.3.1. If the controller is designed as

$$u = -k(2x_1 + 4x_2 + 3x_3 + x_4), \tag{2.3.7}$$

and the update law is chosen as

$$\dot{k} = l > 0, \tag{2.3.8}$$

then the origin of system (2.3.1) is asymptotically stable.

Some remarks are listed as follows:

Remark 2. In practical applications, the exact value of the upper bound ϖ in **Assumption 2.1.1**, **Assumption 2.2.1**, and **Assumption 2.3.1** cannot be determined in advance. Fortunately, in our control scheme we don't need to know the exact value of ϖ, which can be seen as one of the merits of our paper.

Remark 3. Theorem 2.1.1, **Theorem 2.2.1**, and **Theorem 2.3.1** can be extended to control the general (chaotic) systems. Here, we take the 2-dimensional system as an example.

Suppose the controlled 2-dimensional system is

$$\begin{cases} \dot{x}_1 = f_1(x) + u_1, \\ \dot{x}_2 = f_2(x) + u_2, \end{cases} \tag{2.3.9}$$

where $x = (x_1, x_2)^T \in R^{2\times 1}$ is the state variable, $f_i(x) \in R^1, i = 1, 2$ are functions, u_1, u_2 are controllers to be designed later.

Let $y(t)$ be an arbitrarily given bounded reference signal with well defined first derivative. The goal of this subsection is to design a proper controller u based on $y(t)$ and its derivatives such that

$$\lim_{t\to\infty}(x_1 - y) = \lim_{t\to\infty}(x_2 - \dot{y}) = 0.$$

By Theorem 2.1.1, we can easily obtain the following Theorem 2.3.2.

Theorem 2.3.2. If the controller is designed as

$$\begin{cases} u_1 = x_2 - f_1(x), \\ u_2 = -k(e_1 + e_2), \end{cases} \tag{2.3.10}$$

where $e_1 = x_1 - y$, $e_2 = x_2 - \dot{y}$, and the update law is chosen as

$$\dot{k} = l > 0, \tag{2.3.11}$$

then

$$\lim_{t\to\infty}(x_1 - y) = \lim_{t\to\infty}(x_2 - \dot{y}) = 0.$$

Remark 4. The second equation of system (2.3.9) may contain the parameter uncertainty, model uncertainty, and external disturbance.

III. THE MODIFIED PROJECTIVE SYNCHRONIZATION SCHEME BETWEEN TWO 2-DIMENSIONAL SYSTEMS

In this section, based on Theorem 2.1.1 we consider the modified projective synchronization between two different 2-dimensional systems with parameter and model uncertainties and external disturbances via adaptive control.

Suppose the drive chaotic system is

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = (a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + f(x,t) + \Delta f(x,t) + d_1, \end{cases} \tag{3.1}$$

where $x = (x_1, x_2)^T \in R^{2\times 1}$ is the state vector of system (3.1), a_1, a_2 are the system's parameters, $\triangle a_1$, $\triangle a_2$ are the parameter uncertainties, $f(x,t) \in R^1$ is a function, $\triangle f(x,t) \in R^1$ and $d_1 \in R^1$ are the model uncertainty and external disturbance of the system (3.1), respectively.

The response chaotic system is

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = (b_1 + \Delta b_1)y_1 + (b_2 + \Delta b_2)y_2 + g(y,t) + \Delta g(y,t) + d_2 + u, \end{cases} \tag{3.2}$$

where $y = (y_1, y_2)^T \in R^{2\times 1}$ is the state vector of system (3.2), b_1, b_2 are the system's parameters, $\triangle b_1$, $\triangle b_2$ are the parameter uncertainties, $g(y,t) \in R^1$ is a function, $\triangle g(y,t) \in R^1$ and $d_2 \in R^1$ are the model uncertainty and external disturbance of the system (3.2), respectively, u is the controller to be designed.

Suppose the synchronization error between the drive and response systems is defined as $e(t) = y(t) - \alpha x(t)$, where $\alpha = \mathrm{diag}(\alpha_1, \alpha_2)$ is the scaling matrix, then by multiplying both sides of Eq. (3.1) by α and then subtracting it from Eq. (3.2), the synchronization error system is achieved as follows:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = (b_1 + \Delta b_1)y_1 + (b_2 + \Delta b_2)y_2 + g(y,t) + \Delta g(y,t) + d_2 \\ \qquad - \alpha_2[(a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + f(x,t) + \Delta f(x,t) + d_1] + u. \end{cases} \tag{3.3}$$

If the controller u is chosen as:

$$u = -k(e_1 + e_2), \tag{3.4}$$

and the update law is selected as

$$\dot{k} = l > 0, \tag{3.5}$$

then, by inserting Eq. (3.4) into Eq. (3.3), we obtain

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = -k(e_1 + e_2) + \omega, \end{cases} \tag{3.6}$$

where $\omega = (b_1 + \Delta b_1)y_1 + (b_2 + \Delta b_2)y_2 + g(y,t) + \Delta g(y,t) + d_2 - \alpha_2[(a_1 + \Delta a_1)x_1 + (a_2 + \Delta a_2)x_2 + f(x,t) + \Delta f(x,t) + d_1]$.

It is clear that the modified projective synchronization problem between systems (3.1) and (3.2) is replaced by the equivalent problem of stabilizing system (3.6).

In order to obtain further results, we make the following assumption.

Assumption 3.1. The parameter uncertainties $\Delta a_i, \Delta b_i, i = 1, 2$, model uncertainties $\Delta f(x,t), \Delta g(y,t)$, and the external disturbances d_1, d_2 are all bounded. Since the trajectory of chaotic systems is bounded, thus there must exist a constant $\varpi \geq 0$ such that $\|\omega\| \leq \varpi$.

Similar to Theorem 2.1.1, we obtain the following Theorem 3.1 and the proofs are trivial and thus they are omitted here.

Theorem 3.1. If the controller and the update law are designed as Eq. (3.4) and Eq. (3.5), respectively, then the origin of system (3.6) is asymptotically stable, which means that the modified projective synchronization between systems (3.1) and (3.2) is achieved.

In the following we extend Theorem 3.1 to the general systems.

Suppose the drive system is

$$\begin{cases} \dot{x}_1 = f_1(x), \\ \dot{x}_2 = f_2(x), \end{cases} \tag{3.7}$$

where $x = (x_1, x_2)^T \in R^{2\times 1}$ is the state variable, $f_i(x) \in R^1, i = 1, 2$ are functions.

The controlled response system is

$$\begin{cases} \dot{y}_1 = g_1(y) + u_1, \\ \dot{y}_2 = g_2(y) + u_2, \end{cases} \tag{3.8}$$

where $y = (y_1, y_2, y_3)^T \in R^{2\times 1}$ is the state variable, $g_i(y) \in R^1, i = 1, 2$ are functions, u_1, u_2 are controllers to be designed later.

The synchronization error between the drive and response systems is defined as $e(t) = y(t) - \alpha x(t)$, where $\alpha = \mathrm{diag}(\alpha_1, \alpha_2)$ is the scaling matrix. Multiplying both sides of Eq. (3.7) by α and then subtracting it from Eq. (3.8), the synchronization error system is achieved as follows:

$$\begin{cases} \dot{e}_1 = g_1(y) - \alpha_1 f_1(x) + u_1, \\ \dot{e}_2 = g_2(y) - \alpha_2 f_2(x) + u_2, \end{cases} \tag{3.9}$$

It is easy to see that the modified projective synchronization problem between systems (3.7) and (3.8) is equivalent to the problem of stabilizing the system (3.9).

Similar to Theorem 2.3.2, we obtain Theorem 3.2; the proofs are trivial so they are omitted here.

Theorem 3.2. If the controller is designed as

$$\begin{cases} u_1 = e_2 + \alpha_1 f_1(x) - g_1(y), \\ u_2 = -k(e_1 + e_2), \end{cases} \tag{3.10}$$

and the update law is chosen as

$$\dot{k} = l > 0, \tag{3.11}$$

then the origin of system (3.9) is asymptotically stable, which means that the modified projective synchronization between systems (3.7) and (3.8) is achieved.

Remark 5. The second equations of systems (3.7) and (3.8) may contain the parameter uncertainty, model uncertainty, and the external disturbance.

Remark 6. In this section, we only investigate the modified projective synchronization scheme between two 2-dimensional systems. Sufficient conditions for modified projective synchronization between two 3,4-dimensional systems can be easily derived in the same way, and we omit them here .

IV. NUMERICAL SIMULATIONS

This section presents two illustrative examples to verify and demonstrate the effectiveness of the proposed control scheme. The simulation results are carried out using the MATLAB software. The fourth order Runge-Kutta integration algorithm was used to solve the differential equations. A time step size of 0.001 was employed.

Example 1. The control of the two-degrees-of-freedom dissipative gyroscope.

The research of gyroscope dynamics goes back one hundred years. Gyroscope systems are extensively applied in navigation, aeronautics, and space engineering. In 1981, Leipnik and Newton [32] first presented the chaotic motion in a heavy, symmetrical gyroscope. In 2005, Chen and Ge [33] investigated a two-degree-of-freedom system for a dissipative gyroscope mounted on a vibrating base. This dissipative gyroscope containing a mechanical vibration absorber in the interior in the form of a spring-mass-dashpot mounted on a vibrating base is shown in Fig. 1.

The dynamical equations for the symmetric gyroscope can be described by the angle θ, ϕ, and φ. The vibration dynamics of the base can be described by the harmonic motions $\bar{l}\sin\omega t$. Let the state variables be $x_1 = \theta, x_2 = \dot{\theta}, x_3 = z-p$, and $x_4 = \dot{z}$, then the dynamical

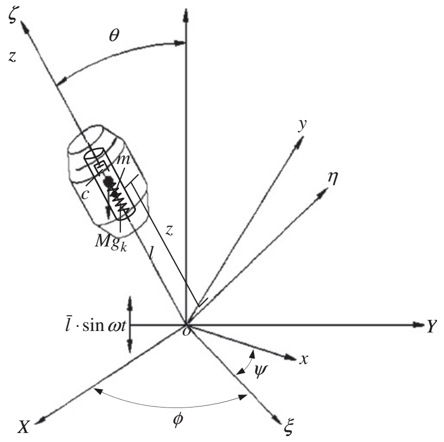

FIG. 1: A schematic diagram of a heavy symmetric dissipative gyroscope.

equations can be expressed as [33]

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)}{\sin^3 x_1} + \frac{[(Mgl+mgp)+mgx_3+(M+m)g\bar{l}]\sin\omega t\sin x_1}{[I_1+m(x_3+p)^2]}, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \left(\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)^2}{\sin^2 x_1}\right)(x_3+p) + g(1-\cos x_1) - \frac{k}{m}x_3 \\ \quad + (x_3+p)x_2^2 - 2cx_4. \end{cases} \tag{4.1}$$

With the specific values set to $I_1 = 1$, $k = 100$, $l = 0.1$, $M = 0.5$, $m = 0.1$, $p = 0.1$, $\beta_\phi^2 = 100$, $\omega = 2$, $2c = 0.5$, and $\bar{l} = 5$ in a numerical simulation, the dynamical behavior of the gyroscope system (4.1) will exhibit an irregular motion, as shown in Fig. 2, with the initial conditions of $(x_1, x_2, x_3, x_4) = (0.1, 0.2, 0.3, -0.1)$.

System (4.1) can be decomposed into two coupled subsystems:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)}{\sin^3 x_1} + \frac{[(Mgl+mgp)+mgx_3+(M+m)g\bar{l}]\sin\omega t\sin x_1}{[I_1+m(x_3+p)^2]}. \end{cases} \tag{4.2a}$$

and

$$\begin{cases} \dot{x}_3 = x_4, \\ \dot{x}_4 = \left(\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)^2}{\sin^2 x_1}\right)(x_3+p) + g(1-\cos x_1) - \frac{k}{m}x_3 \\ \quad + (x_3+p)x_2^2 - 2cx_4. \end{cases} \tag{4.2b}$$

Obviously, systems (4.2a) and (4.2b) are all in the form of system (2.1.1), so we can use Theorem 2.1.1 to control system (4.1). We add two controllers u_1, u_2 on the righthand side

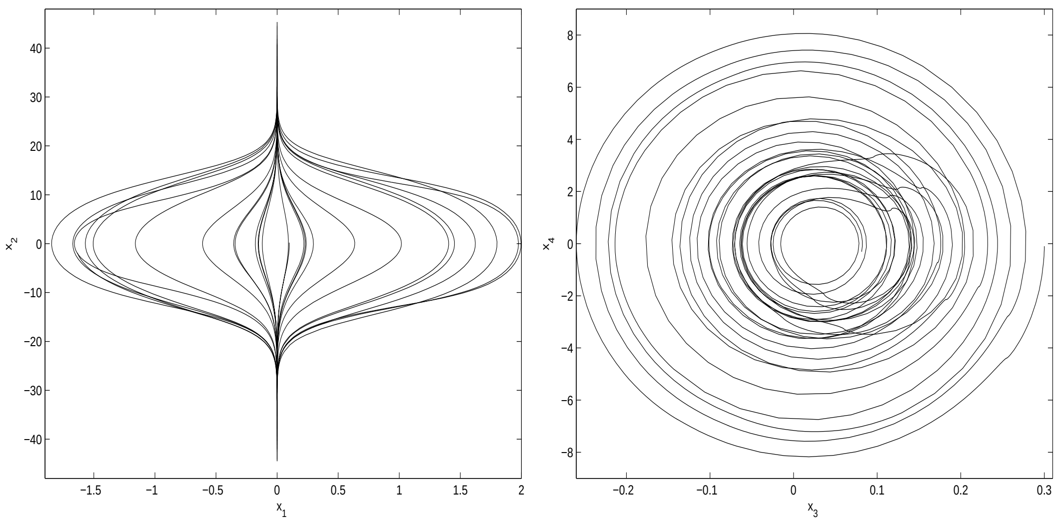

FIG. 2: Phase plane trajectories of a chaotic dissipative gyroscope.

of system (4.1), then the controlled system (4.1) with parameter and model uncertainties and external disturbances can be rewritten as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)}{\sin^3 x_1} + \frac{[(Mgl+mgp)+mgx_3+(M+m)g(\bar{l}+\overbrace{0.5}^{\Delta\bar{l}})]\sin\omega t\sin x_1}{[I_1+m(x_3+p)^2]} \\ \quad + 2\underbrace{\cos(x_1x_2)}_{f(x)} + 3\underbrace{\sin(t)}_{d_1} + u_1, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \left(\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)^2}{\sin^2 x_1}\right)(x_3+p) + g(1-\cos x_1) - \frac{(k-\overbrace{0.1}^{\Delta k})}{m}x_3 \\ \quad + (x_3+p)x_2^2 - 2cx_4 - 3\underbrace{\sin(x_1x_2x_3x_4)}_{g(x)} + \underbrace{\cos(t)}_{d_2} + u_2. \end{cases} \tag{4.3}$$

Suppose the desired signal $y(t) = \sin t$, the two controllers $u_1 = -k(x_1 - \sin t + x_2 - \cos t)$, $u_2 = -k(x_3 + \sin t + x_4 + \cos t)$ and the update law $\dot{k} = l > 0$, then according to Theorem 2.1.1 we have $\lim_{t\to+\infty} |x_1 - \sin t| = 0$, $\lim_{t\to+\infty} |x_2 - \cos t| = 0$, $\lim_{t\to+\infty} |x_3 + \sin t| = 0$, $\lim_{t\to+\infty} |x_4 - \cos t| = 0$. The time response of tracking error states are shown in Figure 3 with $x_1(0) = 8$, $x_2(0) = 12$, $x_3(0) = -5$, $x_4(0) = -11$, and $l = 30$.

Example 2. Modified projective synchronization between two two-degree-of-freedom dissipative gyroscopes.

Suppose the drive system with parameter and model uncertainties and external dis-

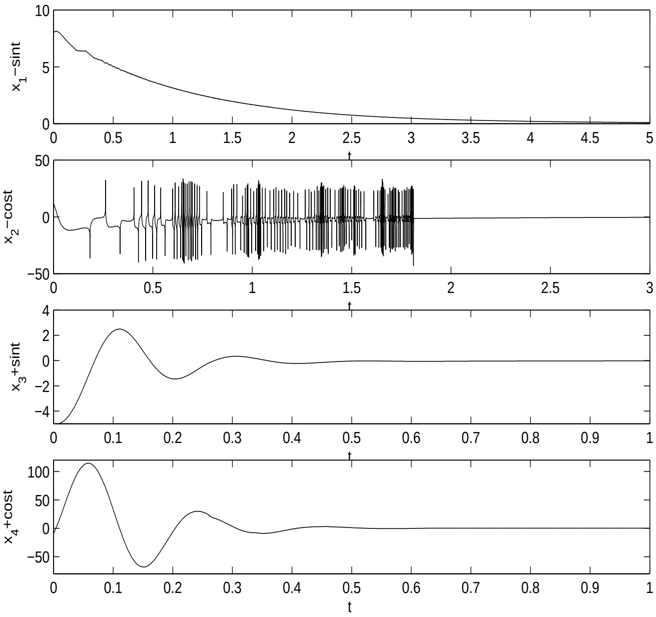

FIG. 3: The time response of tracking error states.

turbances is

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)}{\sin^3 x_1} + \frac{[(Mgl+mgp)+mgx_3+(M+m)g(\bar{l}+\overbrace{0.5}^{\Delta\bar{l}})]\sin\omega t\sin x_1}{[I_1+m(x_3+p)^2]} \\ \qquad + \underbrace{\cos(x_1x_2)}_{f_1(x)} + \underbrace{\sin(t)}_{d_{11}}, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = \left(\frac{\beta_\phi^2}{[I_1+m(x_3+p^2)]^2}\frac{(1-\cos x_1)^2}{\sin^2 x_1}\right)(x_3+p) + g(1-\cos x_1) - \frac{(k-\overbrace{0.1}^{\Delta k})}{m}x_3 \\ \qquad + (x_3+p)x_2^2 - 2cx_4 - \underbrace{\sin(x_1x_2x_3x_4)}_{f_2(x)} + \underbrace{\cos(t)}_{d_{12}}. \end{cases} \tag{4.4}$$

The controlled response system with parameter and model uncertainties and external disturbances is

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = -\frac{\beta_\phi^2}{[I_1+m(y_3+p^2)]^2}\frac{(1-\cos y_1)}{\sin^3 y_1} + \frac{[(Mgl+mgp)+mgy_3+(M+m)g(\bar{l}+\overbrace{0.1}^{\Delta\bar{l}})]\sin\omega t\sin y_1}{[I_1+m(y_3+p)^2]} \\ \quad + \underbrace{2\cos(y_1y_2)}_{g_1(y)} - \underbrace{\sin(t)}_{d_{21}} + u_1, \\ \dot{y}_3 = y_4, \\ \dot{y}_4 = \left(\frac{\beta_\phi^2}{[I_1+m(y_3+p^2)]^2}\frac{(1-\cos y_1)^2}{\sin^2 y_1}\right)(y_3+p) + g(1-\cos y_1) - \frac{(k+\overbrace{0.1}^{\Delta k})}{m}y_3 \\ \quad + (y_3+p)y_2^2 - 2cy_4 - \underbrace{2\sin(y_1y_2y_3y_4)}_{g_2(y)} - \underbrace{\cos(t)}_{d_{22}} + u_2. \end{cases} \tag{4.5}$$

Let the scaling factors be $\alpha_1 = 1$, $\alpha_2 = -2$, $\alpha_3 = -1$, $\alpha_4 = 1$, and $u_1 = -k(e_1 + e_2)$, $u_2 = -k(e_3 + e_4)$, and the update law $\dot{k} = l > 0$, then by **Theorem 3.1.** we know systems (4.4) and (4.5) can achieve modified projective synchronization. The time response of modified projective synchronization error states between systems (4.4) and (4.5) with $(x_1(0), x_2(0), x_3(0), x_4(0)) = (8, 4, -5, -14)$, and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (9, 6, 2, -10)$ is shown in Figure 4.

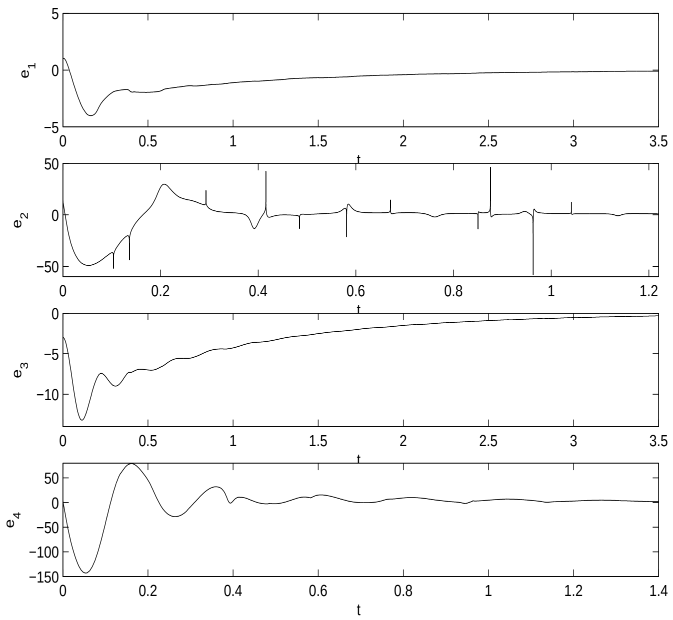

FIG. 4: Time response of modified projective synchronization error states.

Remark 7. Computer simulation shows that we can accelerate the rate of convergence by changing the value of l. The bigger the value, the faster the convergence. In this paper we take $l = 30$ in order to guarantee an appropriate rate of convergence.

Remark 8. To the best of our knowledge, so far, no result on the control of the two-degree-of-freedom dissipative gyroscope are available in the literature, which is still open and remains unsolved. However, modified projective synchronization of the two-degree-of-freedom dissipative gyroscope has been well discussed in papers [12–14]. It is easy to see that the controllers presented in papers [12–14] are more complex than those proposed in this paper. Furthermore, the synchronization schemes proposed in papers [12–14] are under the assumption that $\alpha_1 = \alpha_2, \alpha_3 = \alpha_4$, however, in this paper this limitation has been eliminated.

V. CONCLUSION

This work investigates the control and synchronization of a class chaotic systems by designing an adaptive controller. Some new, simple, and yet easily verified sufficient conditions are established for chaos control and modified projective synchronization, by using the general solution of chaotic system and the adaptive control schemes. Our controllers are simple and we don't need to know the exact value of ϖ, thus our control approach may be easily used in application situations. Finally, numerical simulations show the applicability and feasibility of the proposed scheme.

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