

# The PR Wave Equation - a Primary and Realistic Arterial Pressure Wave Equation for the Quantitative and Collective Study of the Cardiovascular System

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We aim to acquire the main dynamic equations in large arteries which are realistic and can be utilized to solve hemodynamic problems. The four momentum equations of the elastic vessel system and the enclosed blood system, both in the axial and in the radial directions, are derived directly from Newton's law, with the contact forces between the two systems being counted explicitly. The resultant generalized axial momentum equations for large arteries in vivo can be used to replace the Navier-Stokes equation for the blood axial motion. We also analyze that the inertial and the longitudinal extending stress terms in the radial equation of motion for the vessel system are two important governing terms. By retaining them simultaneously, an ordinary pressure-radius (PR) wave equation is directly obtained from the two resultant radial momentum equations. According to some physiological facts, we deduce that the ventricular output mainly induces the arterial radial motion. We conclude that the PR wave equation, with low dissipation character, is the primary realistic wave equation that can be utilized as a basis to develop quantitative methods to study cardiovascular diseases in a collective manner, such as the diagnosis and treatment in traditional Chinese medicine.

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## I. INTRODUCTION

Seeking better methods for preventing and curing human diseases, many physicists, mathematicians and engineers have been dedicated to the investigation and modeling of the mechanism in large arteries for centuries. In 1755, the Euler equation [1] was first constructed to describe blood flow in a tube. Since then, almost all of the physical and mathematical modeling of pulse wave propagation have been constructed based on general fluid dynamical principles, by considering the conservation of mass and the Navier-Stokes (NS) equations for the fluid system [2–5]. The elastic vessel was taken into account as the boundary conditions for the enclosed fluid.

The NS equations came up from applying Newton's second law to a viscous fluid system that is driven mainly by the gradient of the hydraulic pressure forces. The equations are mostly used to solve the problem for hydraulic systems enclosed by boundaries which

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are stationary or with known movement. However, these are not the situations for a blood system enclosed by elastic arterial vessels. Due to the arterial pressure wave, the vessel is performing a distributed radial oscillatory motion which can be known only after the pressure wave problem has been solved. Hence, utilizing the NS equations or their modified forms becomes a complicated method for studying the blood motion in arterial systems. Therefore, in order to describe the blood flow and the pressure wave propagation in large arteries realistically, launching a different approach is needed.

Going back to Newton's second law is the fundamental way to bring the hemodynamic studies to the right track. To avoid dealing with the interacting forces on the blood system by the elastic vessel, previously we have applied Newton's law by taking the vessel and the fluid together as one system, and some results that can explain many important physiological phenomena have been found [6–10].

The aim of this study is to develop rigorous and general dynamic equations in large arteries which can be widely accepted and utilized as a bases for solving the hemodynamic problems. In order to elucidate the distinction of the results from the established dynamic equations in the literature, we follow the common approaches by taking the fluid and the wall as two separate systems. However, the interaction of the two systems is not treated merely as the mutual boundary conditions but will be counted directly with the mutual contact forces. Furthermore, the momentum equations of the two systems will be tackled in both the axial and the radial directions. The interacting forces between the two systems in both directions are then revealed explicitly, and a realistic pressure wave equation automatically follows.

## II. METHOD

To study the dynamic equations in large arteries, we first considered the general momentum equations for a rotationally symmetric fluid-filled elastic tube. Figure 1 illustrates a segment of an artery embedded in a surrounding with external pressure  $P_e$ . We assumed that the internal fluid pressure  $P_i$  and the inner radius  $R$ , induced by the pulsatile blood input from the left ventricle, are functions of the axial position  $z$  and the time  $t$ . An element of azimuth angle  $d\theta$  was taken from the segment of axial length  $dz$  (Figure 1), and the motions of the vessel element (see Figure 2a), as well as the wedge-shaped fluid element (see Figure 2b), were considered separately.

### II-1. Momentum equations for the vessel element

The vessel element (see Figure 2a) is in contact with the four neighboring elastic walls (via surfaces I, II, III, and IV), the external surrounding (via surface V), and the enclosed wedge-shaped blood element (via surface BW). Newton's momentum equations of this vessel element are determined by the forces acting on these six surfaces and the damping force associated with its movement.

The vessel is assumed to be isotropic and incompressible, with thickness  $h_w$  and density  $\rho_w$ . Following the conservation law for the mass of the wall,  $Rh_w$  is constant [11].  $E_{\theta\theta}$ ,

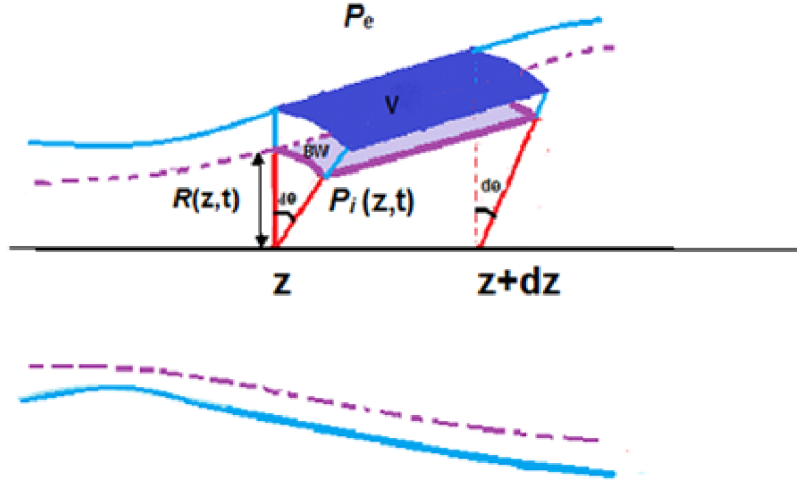


FIG. 1: A segment of the fluid-filled artery embedded in a surrounding with external pressure  $P_e$ . The internal fluid pressure  $P_i$  and inner radius  $R$  are functions of axial position  $z$  and time  $t$ .

$E_{zz}$ , and  $E_{rz}$  are its circumferential, longitudinal, and shearing Young's moduli, respectively.

The areas of surfaces I (at  $z$ ) and II (at  $z+dz$ ) can be expressed as  $A_I = A_{II} = h_w R d\theta$ . The shearing stress in the radial direction is associated with the shearing strain  $\frac{\partial R}{\partial z}$  as  $\sigma_{rz} = E_{rz} \frac{\partial R}{\partial z}$ . The net radial shearing force  $F_S$  acting on the surfaces I and II of the vessel element by the neighboring adjacent vessels would then be

$$F_S = \left[ h_w R d\theta E_{rz} \frac{\partial R}{\partial z} \right]_{II} - \left[ h_w R d\theta E_{rz} \frac{\partial R}{\partial z} \right]_I = E_{rz} h_w R d\theta \frac{\partial^2 R}{\partial z^2} dz. \quad (1)$$

In vivo, the longitudinal tension along the large arteries is high [12–14]. The radial component of the longitudinal stress  $T$  is given by  $T_r = T(\partial R/\partial z)$ . The axial component of the tension  $T$  is given by  $T_z = T \left[ 1 - \left( \frac{\partial R}{\partial z} \right)^2 \right]^{1/2} \cong T \left[ 1 - \frac{1}{2} \left( \frac{\partial R}{\partial z} \right)^2 \right]$ .

The net longitudinal stress force in the radial direction  $(F_T)_r$  and in the axial direction  $(F_T)_z$  acting on the surfaces I and II of the vessel element by the neighboring adjacent vessels would then be

$$(F_T)_r = \left[ T h_w R d\theta \frac{\partial R}{\partial z} \right]_{II} - \left[ T h_w R d\theta \frac{\partial R}{\partial z} \right]_I = T h_w R d\theta \frac{\partial^2 R}{\partial z^2} dz, \quad (2)$$

$$(F_T)_z = -T h_w R d\theta \frac{\partial^2 R}{\partial z^2} \frac{\partial R}{\partial z} dz. \quad (3)$$

Surfaces III and IV have areas  $A_{III} = A_{IV} = h_w dz$ . For a vessel with circumferential extending strain  $e_{\theta\theta} = \Delta R/R$ , the circumferential extending stress is  $\sigma_{\theta\theta} = E_{\theta\theta} \Delta R/R$ . The net circumferential extending force  $F_C$  acting on the surfaces III and IV of the vessel

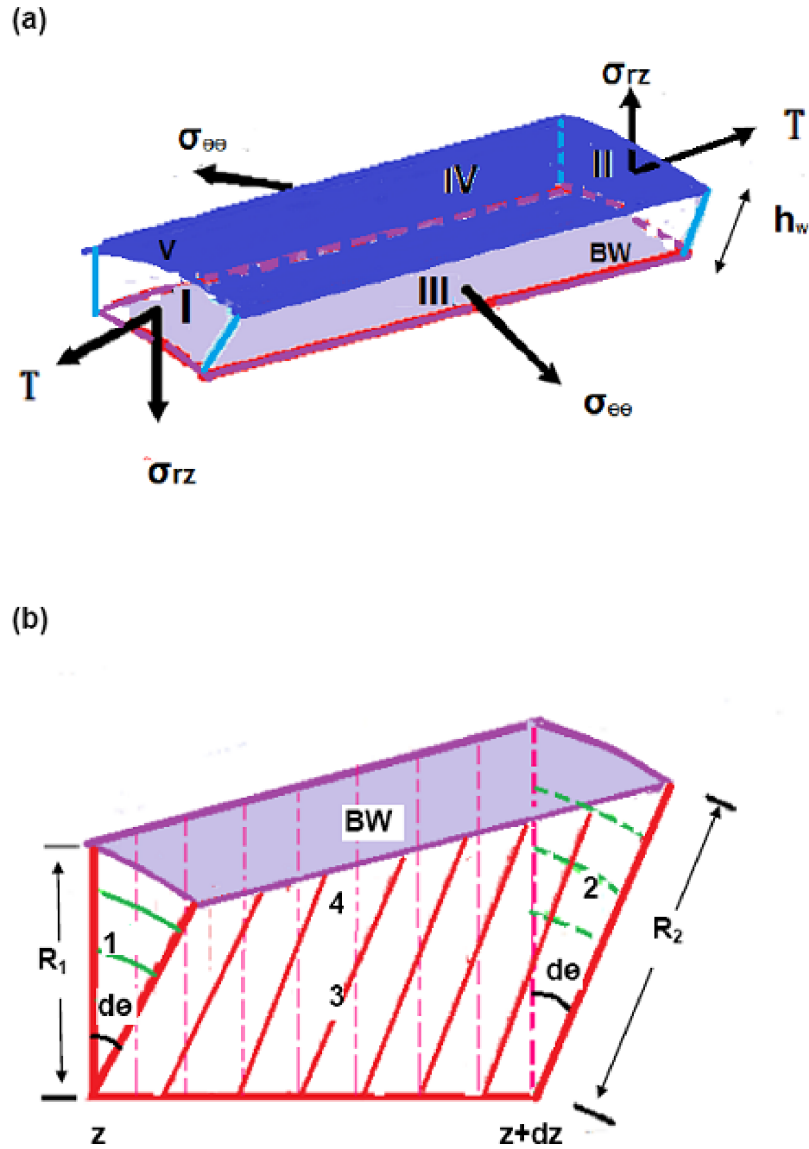


FIG. 2: The vessel element (a) and the wedge-shaped fluid element (b) of axial length  $dz$  and azimuth angle  $d\theta$ , taken from the arterial segment shown in Figure 1. V is the upper surface. BW is the contact surface of the blood fluid and the vessel.

element by the neighboring adjacent vessels is in the radial direction and can be expressed as

$$F_C = -2\sigma_{\theta\theta}h_w dz \sin(d\theta/2) \cong -h_w\sigma_{\theta\theta}dzd\theta. \quad (4)$$

On the surface V with area  $A_V = Rd\theta dz$  (see Figure 2a), the external pressure acts as a pressure force  $F_O = P_e A_V$  normal to the surface, and its radial component  $(F_O)_r$  and axial

component  $(F_O)_z$  are

$$(F_O)_r = F_O \left[ 1 - \left( \frac{\partial R}{\partial z} \right)^2 \right]^{1/2} \cong -P_e R d\theta dz, \quad (5)$$

$$(F_O)_z = P_e \frac{\partial R}{\partial z} R dz d\theta, \quad (6)$$

respectively.

On surface BW, the contact force  $F_{BW}$  acting on the vessel element by the adjacent fluid can be decomposed into  $(F_{BW})_r$  in the radial direction and  $(F_{BW})_z$  in the axial direction.

For the vessel element of radial velocity  $u_{wr} = \frac{\partial R}{\partial t}$ , the radial damping force  $(F_d)_r$  can be expressed in terms of the damping constant of the wall  $\beta_w$  as

$$(F_d)_r = -\beta_w \frac{\partial R}{\partial t} R d\theta dz, \quad (7)$$

and the momentum of the vessel element in the radial direction  $(p_w)_r$  is given as

$$(p_w)_r = \frac{\partial R}{\partial t} \rho_w h_w R d\theta dz. \quad (8)$$

Taken all the forces into account, we obtain the momentum equation for the vessel element in the radial direction as

$$\frac{d(p_w)_r}{dt} = (F_T)_r + F_S + F_C + (F_{BW})_r + (F_O)_r + (F_d)_r, \quad (9)$$

and the axial momentum equation for the vessel element with axial momentum  $(p_w)_z$  as

$$\frac{d(p_w)_z}{dt} = (F_T)_z + (F_{BW})_z + (F_O)_z. \quad (10)$$

## II-2. Momentum equations for the wedge-shaped fluid element

The fluid element (see Figure 2b) is in contact with the neighboring fluid (via surfaces 1, 2, 3, and 4) and the layer vessel (via surface BW). The Newton momentum equations of this fluid element are also determined by the forces acting on these five surfaces and the viscous force associated with its movement.

On surface 1 (at  $z$ ) and surface 2 (at  $z + dz$ ), the neighboring adjacent blood will exert normal pressure forces  $F_{b1} = P_{i1} R_1^2 d\theta/2$  and  $F_{b2} = P_{i2} R_2^2 d\theta/2$  on the element. The sum of these two forces is a net force  $(F_P)_z$  in the axial direction:

$$(F_P)_z = -\frac{1}{2} \left( R^2 \frac{\partial P_i}{\partial z} + P_i \frac{\partial R^2}{\partial z} \right) d\theta dz. \quad (11)$$

The area of surfaces 3 and 4 can be expressed as  $A_3 = A_4 = R dz$ . Pressure forces  $F_{b3} = A_3 P_i$  and  $F_{b4} = A_4 P_i$  acting normally on surface 3 and 4, respectively, by the adjacent fluid will contribute a net force  $(F_P)_r$  in the radial direction:

$$(F_P)_r = (A_3 P_i + A_4 P_i) \sin(d\theta/2) = P_i R dz d\theta. \quad (12)$$

On surface BW, the contact forces acting on the fluid system by the neighboring vessel element in the radial and the axial direction are  $(F_{WB})_r$  and  $(F_{WB})_z$ , respectively.

Thus the momentum equations of the fluid element with radial momentum  $(p_b)_r$  and axial momentum  $(p_b)_z$  can be written as

$$\frac{d(p_b)_r}{dt} = (F_P)_r + (F_{WB})_r + (F_v)_r, \quad (13)$$

$$\frac{d(p_b)_z}{dt} = (F_P)_z + (F_{WB})_z + (F_v)_z. \quad (14)$$

Here,  $(F_v)_r$  and  $(F_v)_z$  are the viscous force in the radial and the axial directions, respectively, and

$$(F_v)_r = -\beta_b \frac{\partial R}{\partial t} R d\theta dz, \quad (15)$$

with  $\beta_b$  as the viscous coefficient of the blood in the radial direction.

The momentum of the fluid element in the axial direction can be written as

$$(p_b)_z = \int_{r=0}^{r=R} \rho_b u_{bz} r d\theta dr dz, \quad (16)$$

and the momentum of the fluid element in the radial direction can be written as

$$(p_b)_r = \int_{r=0}^{r=R} \rho_b u_{br} r d\theta dr dz. \quad (17)$$

Here  $\rho_b$  is the fluid density;  $u_{bz}$  and  $u_{br}$  are the fluid velocity in the axial and the radial directions, respectively. Near the contact surface, the radial velocity of the fluid and the wall are continuous [15]; we may define  $h_b$  as the equivalent thickness of the blood layer that moves together with the wall with the wall velocity  $u_{wr}$ . Hence, from Equation (17), we have the momentum of the fluid element in the radial direction  $(p_b)_r$  as

$$(p_b)_r = \rho_b h_b u_{wr} R d\theta dz = \rho_b h_b \frac{\partial R}{\partial t} R d\theta dz. \quad (18)$$

The value of  $h_b$  depends on the radial velocity profile of the blood and is of the same order as the Stokes layer [16].

### III. RESULT

Equations (9) and (10) are the general momentum equations for the vessel element, while Equations (13) and (14) are those for the fluid elements. We may utilize these four equations to derive the momentum equations in large arteries.

#### III-1. The generalized axial momentum equations for large arteries in vivo

In vivo, the arterial vessel is strongly constrained in the axial direction and the axial motion of the wall is negligibly small [17]; which implies that  $d(p_w)_z/dt$  is zero. Hence, the axial momentum equation for the vessel element (10) becomes as

$$d(p_w)_z/dt = (F_T)_z + (F_{BW})_z + (F_O)_z = 0. \quad (19)$$

Thus, by Newton's third law, the axial component of the contact force acting on the blood by the vessel  $(F_{WB})_z$  can be expressed explicitly as

$$(F_{WB})_z = -(F_{BW})_z = (F_T)_z + (F_O)_z. \quad (20)$$

Due to the axial symmetry, we may integrate Equation (14) over the angle  $\theta$ , and by Equation (20), a generalized momentum equation associated with the axial motion of the blood is then given by

$$\rho \frac{dQ}{dt} = -\pi R^2 \frac{\partial P_i}{\partial z} - P_i \frac{\partial(\pi R^2)}{\partial z} + P_e \frac{\partial(\pi R^2)}{\partial z} - 2\pi R h_w T \frac{\partial^2 R}{\partial z^2} \frac{\partial R}{\partial z} + f_{vz} \quad (21)$$

Here the total fluid flux  $Q(z, t)$  at axial position  $z$ , or the volume rate of flow, is the integration of the fluid axial velocity  $u_{bz}$  across the lumen of the tube [2]. On the left hand side, the spatial convective term is also included.

#### III-2. The radial momentum equations for large arteries in vivo

From the radial momentum equation of the fluid element (Equation (13)), the radial component of the contact force acting on the blood by the arterial wall is

$$(F_{WB})_r = \frac{d(p_b)_r}{dt} - (F_P)_r - (F_v)_r.$$

And by Newton's third law again, the radial component of the contact force  $(F_{BW})_r$  acting on the wall by the blood becomes

$$(F_{BW})_r = -(F_{WB})_r = -\frac{d(p_b)_r}{dt} + (F_P)_r + (F_v)_r. \quad (22)$$

Substituting Equation (22) into Equation (9), the momentum equation of the arterial wall in the radial direction becomes

$$\frac{d(p_w)_r}{dt} = -\frac{d(p_b)_r}{dt} + (F_P)_r + (F_O)_r + F_C + (F_T)_r + F_S + (F_d)_r + (F_v)_r. \quad (23)$$

By inserting Equations (18), (12), (5), (4), (2), (1), (7), and (15) into Equation (23), then integrating over the angle  $\theta$  we obtain the radial momentum equation of the arterial wall as

$$\rho_w h_w \frac{\partial^2 R}{\partial t^2} + \rho_b h_b \frac{\partial^2 R}{\partial t^2} = (P_i - P_e) - \frac{h_w}{R} \sigma_{\theta\theta} - (\beta_w + \beta_b) \frac{\partial R}{\partial t} + h_w (T_w + E_{rz}) \frac{\partial^2 R}{\partial z^2}. \quad (24)$$

Without coupling with the axial momentum equation or the continuity equation of the blood, a pressure wave equation follows directly from Equation (24) by using the chain rule and the definition of Peterson's elastic modulus, [18]  $E_P = R(\partial P / \partial R)$ ,

$$M_L \frac{\partial^2 P}{\partial t^2} + \beta_L \frac{\partial P}{\partial t} - \frac{E_P}{R_0} \kappa_L = \tau_L \frac{\partial^2 P}{\partial z^2} + \frac{E_P}{R_0} F_{ext}(z, t), \quad (25)$$

with

$$M_L = 2\pi R(\rho_w h_w + \rho_b h_b), \quad (26)$$

$$\kappa_L = 2\pi R \left[ (P_i - P_e) - \frac{h_w}{R} \sigma_{\theta\theta} \right], \quad (27)$$

$$\tau_L = 2\pi R h_w (T + E_{rz}), \quad (28)$$

$$\beta_L = 2\pi R(\beta_w + \beta_b). \quad (29)$$

Here  $F_{ext}(z, t)$  is any additional external force, such as the input from the heart. Equation (25) may be called the pressure-radius (PR) wave equation.

## IV. DISCUSSION

Equations (21) and (24) are the two major momentum equations for large arteries derived directly from Newton's Law. We may compare these equations with other equations in the literature which have been widely utilized to describe the arterial system.

### IV-1. The deficiency of utilizing axial NS equation in large arteries

For most of the PQ wave models which take the NS equation of motion relevant to the longitudinal blood flow as the primary momentum equation, the pressure gradient force  $f_{PG} = -\pi R^2 \frac{\partial P_i}{\partial z}$  on the right hand side of Equation (21) is the only lowest order term to be considered, while the second term  $f_{iAG} = -P_i \frac{\partial(\pi R^2)}{\partial z}$  has been neglected.

Equation (20) shows that, via the contacting surface BW, the surrounding external pressure force  $f_{OZ} = -P_e \frac{\partial(\pi R^2)}{\partial z}$  and the force associated with the longitudinal tension  $T$  of the wall  $f_{TZ} = -2\pi R h_w T \frac{\partial^2 R}{\partial z^2} \frac{\partial R}{\partial z}$  are acting on the blood. Since for all the PQ wave



models, the arterial wall was taken only as the boundary of the blood system, these two forces cannot be considered at the first step.

By combining the two forces associated with the area gradient, we may define the area gradient force [19] as

$$f_{AG} = f_{iAG} + f_{OZ} = -P \frac{\partial(\pi R^2)}{\partial z} = -2\pi R P \frac{\partial R}{\partial z}, \quad (30)$$

with  $P(z, t) = P_i(z, t) - P_e(z)$ .

By the Peterson elastic modulus [18]  $E_P$ , the ratio of the area gradient force to the pressure gradient force can be evaluated by

$$\frac{f_{AG}}{f_{PG}} = \frac{2P}{E_P}. \quad (31)$$

For the large arteries in vivo [2, 3], the ratio of the area gradient force to the pressure gradient force is more than 50%.

In a preliminary study [20], we have shown that for the  $n$ th harmonic mode of the pressure wave the ratio of  $f_{TZ}$  to  $f_{PG}$ , or  $(f_{TZ}/f_{PG})_n \approx -2n^2\%$  for  $n = 1, 2, 3, \dots$

Both  $f_{AG}$  and  $f_{TZ}$  are unable to be taken into account if one starts from the NS equation, and both of them are of comparable order with the major considered force term  $f_{PG}$ ; this implies that the axial NS equation cannot describe realistically the blood motion for large arteries in vivo.

#### IV-2. The inertial and the longitudinal extending stress terms shall be retained simultaneously in the radial equation of motion for the vessel system

To derive solitary waves in large blood vessels, Yomosa [11] has coupled the NS equation of motion relevant to the longitudinal flow with a radial equation of motion for the arterial wall as follows:

$$\rho_w h_w \frac{\partial^2 R}{\partial t^2} = (P_i - P_e) - \frac{h_w}{R} \sigma_{\theta\theta}.$$

Defining  $\sigma_{zz}$  as the Cauchy stress tensor in the axial direction and  $u$  as the radial displacement of the wall, Demiray [21] used  $\sigma_{zz} \frac{\partial^2 u}{\partial z^2}$  to consider the force associated with the longitudinal tension. Yet, to derive a solitary wave in a pre-stressed elastic tube, he discarded the inertia term  $\rho_w h_w \frac{\partial^2 R}{\partial t^2}$  in the first order expansion solution.

According to Laplace's law [22],  $(P_i - P_e) - \frac{h_w}{R} \sigma_{\theta\theta} \approx 0$ , hence the inertial and the longitudinal extending stress terms become the two most important non-zero terms, and both of them should be retained in the radial equation of motion for the lowest order. As a matter of fact, by retaining them simultaneously, we have obtained directly an ordinary pressure wave equation (25).

#### IV-3. The PR wave equation is an improvement of the PS wave equation

In a previous study [6], we have derived a similar pressure-area (PS) wave equation by taking the arterial wall and the blood together as one system, and an approximation

that the thickness of the wall  $h_w$  is constant was made. The present PR wave equation (25) is derived from the radial momentum equations by taking the arterial wall and the blood as two separate but mutual interacting systems, and, more accurately,  $Rh_w$  is taken as constant to follow the conservation law of the wall mass.

#### **IV-4. The PR wave equation is the primary pressure wave equation in large arteries**

It was reported that axial fluid kinetic power takes only 2 to 7% of the total ventricular output [23]. Thus, in terms of the relevance to the power, the induced axial blood flow wave  $Q$  is only a minor parameter as compared with the induced pressure wave  $P$  or the associated radial wave  $R$  in large arteries. Furthermore, since most of the side branch arteries are connected perpendicularly to the axial direction of the main arteries [24], the axial flow wave in large arteries is a passive effect, and better be reduced as regard to the transportation of blood to the connected side branches [25]. Hence, the PR wave equation (25) is the primary pressure wave equation in large arteries.

#### **IV-5. The PR wave equation makes the study of blood-vessel interaction simpler**

The PR wave equation has been derived with only the assumption that the artery is of rotational symmetry; it is applicable to arteries which are curved, bifurcate, and have nonlinear elastic properties. The method for solving the PR wave equation is similar to that of solving the transverse string wave or the EM wave; only the sites that have discontinuity in wave velocity shall be counted as connecting sites. The boundary conditions at the connecting sites refer to the pressure only and not to the flow, while the axial flow  $Q$  can be obtained afterwards by substituting the pressure wave  $P$  into the generalized momentum equation (21) for the axial motion of the blood. This approach makes the study of the arterial fluid-structure interaction (FSI) simpler, since we need not solve the complicated boundary value problems associated with both the pressure and the flow along the whole arterial tube.

#### **IV-6. The PR wave equation can be used as a basis for developing quantitative methods to study the arterial system in a collective manner**

It was found that almost all of the work done in distending the arteries is returned later in each cycle of the heart beat, due to the relatively small viscosity of the vascular wall [2, 3, 26]. Hence we may deduce that the magnitude of  $\beta_L$  in Equation (25) for large arteries is small enough to cause only a light damping for the radial oscillatory motion. The low dissipation character of the PR wave equation not only lowers the loading of the heart, it also makes the resonance behavior [27, 28] of the main arterial system feasible, so that the ventricular-arterial system can perform collectively.

We thus conclude that, due to the strong interaction between the arterial wall and the enclosed blood, starting directly from Newton's law and not from the NS equations is a must method for the hemodynamic studies. The resultant realistic PR wave equation (25) can be

used as a basis for developing quantitative methods [7–10] to cure cardiovascular diseases in a collective manner, such as the diagnose and treatment in the traditional Chinese medicine.

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