Equivalence Relations

R is **reflexive** if for every x, xRx
R is **symmetric** if for every x and y, xRy implies yRx
R is **transitive** if for every x, y, z, $xRy \wedge yRz \rightarrow xRz$

Boolean Logic

$\neg$ - Not
$\wedge$ - And
$\vee$ - Or
$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$
$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$
For any two sets A and B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Finite Automata

DFA

Each state must have exactly one transition arrow for every item in the alphabet, and it may only occupy a single state at a time. $(Qx\Sigma \rightarrow Q)$, q_0 is the start state $\in Q$, and F is the set of accept states, which may may be the empty set, $\subset Q$.

Formally described by $(Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of states (as $\{q_0, q_1, q_2\}$), Σ is the alphabet, δ is the transition function with domain $Qx\Sigma$ and range Q

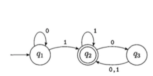


FIGURE 1.6
The finite automaton M_1

We can describe M_1 formally by writing $M_1 = (Q, \Sigma, \delta, q_1, F)$, where

1. $Q = \{q_1, q_2, q_3\}$,
2. $\Sigma = \{0,1\}$,
3. δ is described as

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

4. q_1 is the start state, and
5. $F = \{q_2\}$.

Complementation

Flip acceptance states to get the complement.

Combining

Let's say there are two DFAs, $A = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $B = (Q_2, \Sigma, \delta_2, q_2, F_2)$. Let the intersection of the two, $C = (Q_1 \times Q_2, \Sigma, \delta_3, (q_1, q_2), F_1 \times F_2)$, where $\delta_3((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))$. The union of the two is given by closure properties: $A \cup B = \overline{\bar{A} \cap \bar{B}}.F_3 = \overline{\bar{F_1} \times \bar{F_2}}$

Regular Operations

Union ($\cup$): Returns a set containing all elements that appear in either set.
Concatenation (.): Returns a set containing all combinations of an element from set A and an element from set B
Star ($*$): Returns a set containing all permutations of each element in a given language. This returns an infinite set. Also all words over the alphabet.

EXAMPLE 1.24

Let the alphabet Σ be the standard 26 letters $\{a, b, \ldots, z\}$. If $A = \{good, bad\}$ and $B = \{boy, girl\}$, then

$A \cup B = \{good, bad, boy, girl\}$,

$A \circ B = \{goodboy, goodgirl, badboy, badgirl\}$, and

$A^* = \{\varepsilon, good, bad, goodgood, goodbad, badgood, badbad, goodgoodgood, goodgoodbad, goodbadgood, goodbadbad, \ldots\}$.

Non-Deterministic

A more generalized form of a DFA. Each state does not need a transition arrow for each element in the alphabet. May have more than one active state, may also have more than one transition arrow for a given element in the alphabet. Have a special symbol ϵ which is followed when present as a transition and does not "eat" a character from the string.
Try all legal transitions in parallel. On choice, pick/guess best transition towards acceptance. Accept if there is some path from start to accept.

Let A be the language consisting of all strings over $\{0,1\}$ containing a 1 in the third position from the end (e.g., 000100 is in A but 0011 is not). The following four-state NFA N_2 recognizes A.

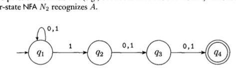


FIGURE 1.31
The NFA N_2 recognizing A

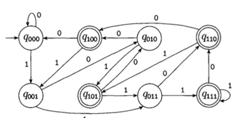


FIGURE 1.32
A DFA recognizing A

Closure and Projection

Closure: The idea that any of the regular operations performed on two regular languages will result in another regular language.
Projection: Think of projection like a shadow, so $\{AxBxC\}$ projected with $\{AxC\}$ yields $\{AxC\}$. It can also be considered the intersection where all elements in both sets will be present in the new set.

Regular Expressions

Formal definition (R_n are regexps):

- α for some a in the alphabet Σ
- ϵ
- $\emptyset$
- $(R_1 \cup R_2)$
- $(R_1.R_2)$
- (R_1^*)

α and ϵ represent the languages $\{\alpha\}, \{\epsilon\}$ respectively, and $\emptyset$ is the empty language. The remaining show what happens when a closure property is applied on the base languages.
Symbols:

- Σ: Any symbol in the alphabet
- $*$: Repeat the previous character 0 or more times
- $+$: Repeat the previous character 1 or more times.

Definitions:

- $R \cup \emptyset = R$ - Adding the empty language to any other language will not change it
- $R.\epsilon = R$ - Joining the empty string to any other string is the same string
- $A^* = \{\epsilon\} \cup A \cup AA \cup AAA \cup \ldots$
- $|A \cup B| <= |A| + |B|$
- $|AB| <= |A|.|A|$
- $\emptyset^* = \{\epsilon\}$. If $A \neq \emptyset$, then A^* is infinite
- $L(\emptyset) = \emptyset$
- $L(\epsilon) = \{\epsilon\}$
- $L((a \cup b)\epsilon) = \{a, b\}$
- $L((a \cup b)\emptyset) = \emptyset$

NFA $\rightarrow$ DFA

Given a set $R \subseteq Q$ of states in A_1, let $E(R)$ be the states reachable from R by following 0 or more ϵ-transitions. Given NFA $A_1 = (Q, \Sigma, \delta, q_0, F)$, construct DFA $A_2 = (Q', \Sigma', \delta', q_0', F')$. $Q' = P(Q)$, $q_0' = E(\{q_0\})$ (states reachable from q_0 by ϵ-archs. For $R \in Q'$ and $a \in \Sigma$, let $\delta'(R, a) = \{q \in Q | \exists r \in Rq \in E(\delta(r, a))\}$ (states reachable from some $r \in R$ by an a arc followed by any number of ϵ arcs.
$F' = \{R \in Q' | R$ contains an accept state of $A_1\}$

Regex $\rightarrow$ NFA

EXAMPLE 1.58

In Figure 1.59, we convert the regular expression $(a \cup b)^*aba$ to an NFA. A few of the minor steps are not shown.

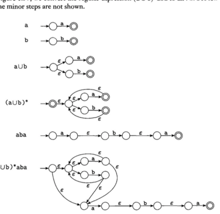


FIGURE 1.59
Building an NFA from the regular expression $(a \cup b)^*aba$

NFA $\rightarrow$ Regex

Remove states one at a time, maintain equivalence, until we get two states with one regex transition.

- Add new start state with an ϵ arc to old start state
- Add new single ccpet state, connected with ϵ arcs from old accept states
- No arcs into start state or out of accept state
- All other arcs present (with $\emptyset$ label if necessary)

DFA $\rightarrow$ Regex

This is aided by a **generalized nondeterministic finite automaton** defined as the 5-tuple $(Q, \Sigma, \delta, q_{start}, q_{accept})$. Q is the finite set of states, Σ is the input alphabet, $delta : (Q - \{q_{accept}\})x(Q - \{q_{start}\}) \rightarrow R$ is the transition function.

CONVERT(G):

1. Let k be the number of states of G.
2. If $k = 2$, then G must consist of a start state, an accept state, and a single arrow connecting them and labeled with a regular expression R. Return the expression R.
3. If $k > 2$, we select any state $q_{rip} \in Q$ different from q_{start} and q_{accept} and let G' be the GNFA $(Q', \Sigma, \delta', q_{start}, q_{accept})$, where

$$Q' = Q - \{q_{rip}\},$$

and for any $q_i \in Q' - \{q_{accept}\}$ and any $q_j \in Q' - \{q_{start}\}$ let

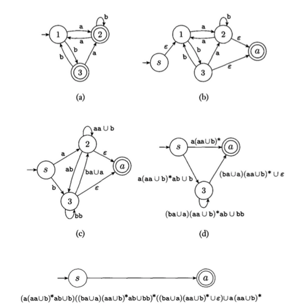
$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$

for $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$, and $R_4 = \delta(q_i, q_j)$.
4. Compute CONVERT(G') and return this value.

And of course the obligatory example problem:

EXAMPLE 1.68

In this example we begin with a three-state DFA. The steps in the conversion are shown in the following figure.

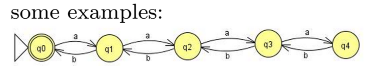


Nonregular Languages

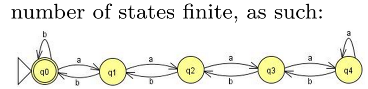
Any language that can not be determined in a finite amount of states on a DFA or NFA is nonregular. Good examples are languages that require the knowledge of a previous count to prove them, such as a language that has some number of 0 followed by the same number of 1s.

Pumping Lemma

The pumping lemma is a simple proof to show that a language is not regular, meaning a FSM can not be built for it. The canonical example is the language $(a^n)(b^n)$. It is trivial to build a FSM for some examples:



Such an FSM will work all the way up to $n = 4$, but the language does not constrain n. No matter how many states are added to the FSM, there will be an input where n is the number of states plus 1, and the machine will fail. Therefore, we must add a loop in the FSM to keep the number of states finite, as such:



This will work. However, after the first 4 as, the machine loses count of how many have bene input because it stays in the same state. Therefore, this erroneously accepts the string $aaaa(a^*)bbbb$. We can say that (a^*) can be pumped. The fact that the FSM is finite and n is not bounded guarantees that any machine which accepts all strings in the language must have this property – the machine must loop at some point, and when it loops, the language can be pumped. Therefore, no FSM can be built for this language, thus it is not regular.
Mathematically, let L be a regular language. Then there exists an integer $p >= 1$ depending only on L such that every string w in L of length $>= p$ can be written as $w = wyx$, i.e. w can be divided into 3 substrings, such that:

- $|y| >= 1$
- $xy <= p$
- $\forall i >= 0, xy^iz \in L$

y is the substring that can be **pumped** (removed or repeated any number of times, and the resulting string is always in L). (1) means the loop y to be pumped must be of length at least 1, (2) means the loop must occur within the first p characters. $|x|$ must be smaller than p by (1) and (2); apart from that there is no restriction on x and z.
In other words, for any regular language L, any sufficiently long word $w \in L$ can be split into 3 parts, $w = xyz$, such that all the strings xy^kz for $k >= 0$ are also in L.

Game

Consider $ADD = \{``a + b = c"\}$, e.g. $2 + 3 = 5 \in ADD$, but $2 + 3 = 6 \notin ADD$.
Can a 17-state DFA M recognize this language? **No**. Consider $1, 2, 3, ..., 18$ as the value for a. Two different numbers, e.g. 2 and 5, get M to the same state. Then $2 + 1 = 3, 5 + 1 = 3$ take M to the same state. **CONTRADICTION**.
Paula wants to prove that a language is regular; skeptical Victor wants to verify.
Paula: "This 17-state M recognizes ADD
Victor: "Oh, really? What if we start with $1, 2, ..., 18$?
Victor wins; ADD is not regular
Show $A = \{ww : w \in \Sigma^*\}$ is not regular.
Paula: p
Victor: $w = 10^{p+1}$
Paula: $ww = xyz$ with $|y| > 0, |xy| <= p$
Victor: $i = 0$
$y \in 0^+$ has exactly one 1. $x =$"1", $y =$ "00", $z =$ "0010000" $\rightarrow w =$ "1000010000". Alternatively, $w = 1000|010|000$. So xz has unequal 0 fields or an odd number of 1s. $xz \notin A \rightarrow$ contradiction.

General for Regular Languages

If a language L is regular, then there exists a number $p >= 1$ such that every string $uwv \in L$ with $|w| >= p$ can be written in the form $uwv = wxyzv$ with strings x, y, z such that $|xy| <= p, |y| >= 1$, and $uxy^izv \in L \forall i >= 0$.

Converse

The converse of the pumping lemma is not true; a language that satisfies the conditions may still be non-regular. Both the original and general versions of the lemma necessary **but not sufficient condition** for the language to be regular.

Turing Machines

Hypothetical devices that manipulate symbols on a strop of tape according to some table of rules. Consists of:

- A **tape** divided into cells, one next to the other. Each cell contains a symbol from some finite alphabet. The alphabet contains some special blank symbol and one or more other symbols. The tape is assumed to be arbitrarily extendable to the left and right. Cells that have no content are assumed to be filled with the blank symbol.
- A **head** that can read and write symbols on the tape and move the tape left and right one cell at a time.
- An **action table or transition function** of instructions that, given the state the machine is currently in and the current symbol being read from the tape, tells the machine to do the folliwing in sequence:
  - Either erase or write a symbol, and then
  - Move the head, and then
  - Assume the same or a new state as prescribed

Note that every part of the machine and its actions is finite, discrete and distinguishable.

Formal Definition

7-tuple $M = (Q, \Gamma, \beta, \Sigma, \delta, q_0, F)$, where:

- Q is a finite, non-empty set of states
- Γ is a finite, non-empty set of the tape alphabet/symbols
- $\beta \in \Gamma$ is the blank symbol (the only symbol allowed to occur on the table indefinitely often at any step during the computation)
- $\Sigma \subseteq (\Gamma - \{\beta\})$ is the set of input symbols
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of final/accepting states
- $\delta : (Q - \{Fx\Gamma\}) \rightarrow Qx\Gamma x\{L, R\}$ is a partial function called the

transition function, where L is left shift, R is right shift, N is no shift.

3-State Busy Beaver

A turing machine that attains the maximum number of steps performed or number of nonblank symbols finally on the tape among all Turing machines of a certain class.

- $Q = \{A, B, C, HALT\}$
- $\Gamma = \{0, 1\}$
- $\beta = 0$ ("blank")
- $\Sigma = \{1\}$
- $q_0 = A$ (initial state)
- $F = \{HALT\}$
- δ is given on attached page

Decidable Languages

A decidable language is:

- A recursive subset in the set of all possible words over the alphabet of the language
- A formal language for which there exists a Turing machine which will, when presented with any finite input string, halt and accept if the string is in the language, and halt and reject otherwise. The Turing machine always halts; it is known as a decider and is said to decide the language.

Decider

A decider is a machine that halts for every input. Also known as a total turing machine. $L(M)$ is decidable.

Closure Properties

Decidable languages are closed under the following operations (if L and P are two decidable languages, then the following are also decidable):

- Kleene star L^*
- Concatenation $L.P$
- Union $L \cup P$
- Intersection $L \cap P$
- Complement $\bar{L}$
- Set difference $L - P$ (can be expressed in terms of intersection and complement)

Decidability

Let $A_{DFA} = \{< B, w > | B$ is a DFA that accepts input string $w\}$

- Need to present a Turing machine M that decides A_{DFA}
- $M =$"On input $< B, w >$, where B is a DFA and w is a string:
  - Simulate B on input w
  - If the simulation ends in an accept state, accept. Otherwise, reject
- When M receives its input, it checks whether the input is actually a DFA and string. If not, reject.
- M then carries out the simulation, keeping track of B's state and position on its tape
- When M finishes processing the last symbol of w, it accepts if B is in an accepting state; rejects otherwise.

Undecidable Languages

Assume that $A_{TM} = \{< M, w >: M(w)$ enters "accept"$\}$ is undecidable. The following are undecidable:

- $\{< M, w >: M(w)$ enters state $q_5\}$
- $\{< M, w >: M(w)$ prints $5\}$
- $\{< M, w >: M(w)$ enumerates $0^n 1^n\}$
- $\{< M >: M()$ enumerates $0^n 1^n\}$
- $\{< M >: M()$ enumerates a regular language$\}$
- $\{< M >:$ For M as an enumerator, $L(M)$ is regular$\}$
- $\{< M >:$ For M as a recognizer, $L(M)$ is regular$\}$
- $\{< M >:$ For M as a recognizer, $L(M)$ is ...$\}$

Halting Problem

$A_{TM} = \{< M, w > | M$ is a TM and M accepts $w\}$
Theorem: A_{TM} is undecidable.
Proof:

- Assume A_{TM} is decidable. Suppose H is a decider for A_{TM}: $H(< M, w >) = \{$accept if M accepts w; reject otherwise$\}$
- Construct a new TM D that calls H to determine what M does when M is its own input. $D =$"On input $< M >$:
  - Run H on input $< M, < M >>$
  - Output the opposite of what H outputs
- Example of D: $D(< M >) = \{$ accept if M does not accept $< M >$, reject if M accepts $< M >\}$
- Run D with itself as input: $D(< D >) = \{$accept if D does not accept $< D >$; reject if D accepts $< D >\}$
- No matter what D does, it is forced to do the opposite $\rightarrow$ contradiction. Thus, neither D nor H can exist. Therefore, A_{TM} is undecidable.

Relationships

- If L is decidable, then L is recognizabl. **Given decider D, let R simulate D. If D is about to enter REJECT, then R instead loops**
- The decidable languages are closed under complement. **Exchange REJECT and ACCEPT states.**
- The recognizable languages are closed under projection. If $\{(x, y) : P(x, y)\}$ is some decidable language of pairs, then $\{x : \exists y P(x, y)\}$ is recognizable. **Given R for the pairs language, build R' to try all y, breadth-first, and simulate** $R(x, y)$
- The recognizable languages are not closed under complement, so some recognizable language is not decidable
- L has a recognizer iff L has an enumerator
- L has a decider iff L has an enumerator that outputs **in lexicographic order**

Equivalencies

- If L has a recognizer R, then a simulating enumerator E tries all strings $w = \epsilon, 0, 1, 00, ...$ breadth-first and prints w if $R(w)$ accepts. Then $L(R) \subseteq L(E)$, $L(E) \subseteq L(R)$
- If L has an enumerator E, then a simulating recognizer R on input w runs E. R accepts if R sees that E prints w
- If L has a decider D, then a simulating emulator E tries all strings $w = \epsilon, 0, 1, 00, ...$ depth-first
- If L is finite, L has a decider. Otherwise, if L has a sorted-order enumerator E, then a simulating decider D on winput w runs E. D accepts w if D sees that E pritns w, and D rejects w if D sees that E prints anything after w
- If L is recognizable and $\bar{L}$ is recognizable, then L is decidable
- Some language is neither recognizable nor co-recognizable
- If A_{TM} were co-recognizable, A_{TM} would be decidable. On input $< M, w >$, interleave recognizers for A_{TM} and $\overline{A_{TM}}$. When one of the recognizers halts, decide $< M, w > \in A_{TM}$. So $\overline{A_{TM}}$ is not recognizable.
- Similarly, if L is recognizable and co-recognizable, then L is decidable.

Recognizers

Recognizers either reach **accept** or loop forever. $L(M) = \{w : M(w)$ reaches ACCEPT$\}$ is recognizable.

Enumerators

An enumerator is a variant of a TM with an attached printer; used as an output device to print strings. Every time the TM wants to add a string to the list of recognized strings it sends it to the printer.
An enumerator starts with a blank input tape. If it does not halt, it may print an infinite list of strings.
The language recognized by the enumerator is the collection of strings that it eventually prints out. These strings may be generated in any order, possibly with repetitions.
Theorem: A language A is Turing-recognizable iff some enumerator enumerates it.
if: If A is recognizable it means that there is some TM M that recognizes A. Then we can construct an enumerator E for A. For this, consider $s_1, s_2, ...$, the list of all possible strings in Σ^*, where Σ is the alphabet of M.
$E =$ "Ignore the input.

- Repeat for $i = 1, 2, 3, ...$
- Run M for i steps on each input $s_1, s_2, ..., s_i$
- If any computation accepts, print the corresponding s_j

Note that if M accepts s, eventually it will appear on the list generated by E. It will appear infinitely many times because M runs from the beginning on each string for each repetition of step 1 – it appears that it runs in parallel on all possible input strings.
only if: If we have an enumerator E that enumerates a language A then a TM M recognizes A. M operates on input w:

- Run E on w. Every time E outputs a string, compare it with w
- If W ever appears in the output of E, accept.

Clearly, M accepts those strings that appear on E's list.

Double Indefinite Tape

A Turing machine with double indefinite tape has indefinite tape to the left and right. Assume that the tape is initially filled with blanks except for the portio that contains the input. Show that a TM D with double infinite tape can simulate an ordinary TM M and M can simulate a TM D with double infinite tape.

Simulating M by D

A TM D with double infinite tape can simulate M by marking the left-hand side of the input to detect and prevent the head from moving off that end.

Simulating D by M

First, we show how to simulate D with a 2-tape TM M

- The first tape of M is written with the input string; second tape is blank
- Cut the tape pf the double infinite tape TM into two parts at the starting cell of the input string
- The portion with the input string and all blank spaces to its right appears on the first tape of the 2-tape TM. The other portion appears on the second tape in reverse order.

Multi-Tape Turing Machines

Theorem: Every multitape Turing machine has an equivalent single tape Turing machine.
Proof: We show how to convert a multitape TM M into a single tape TM S. Assume that M has k tapes. S will simulate the effect of k tapes by storing their information on its single tape. S uses a symbol "#" as a delimeter to separate the contents of the tapes, and keeps track of the locations of the heads by marking the symbols where the heads should be with a dot.
$S =$"On input $w = w_1 w_2 ... w_n$

- Put $S(tape)$ in the format that represents $M(tapes)$, described above
- Scan the tape from the first "#" (representing the left-hand end) to the $(k+1)$st "#" (right-hand end) to determine the symbols under the virtual heads. Then S makes a second pass to update it according to M's δ function
- If S moves one of the virtual heads to the right of a "#", M has moved on the corresponding tape to unread blank contents. So S shifts the tape contents from this cell until the rightmost "#" and writes a blank value on the freed tape cell. It continues the simualtion.

Rice's Theory

A property of recognizable languages is itself a recognizable language. Because of this, "regularity" is a property. If language L has a nontrivial property P, such that $\{< M >: L(M) \in P\}$, L is not decidable. This indicates that **if a language has some nontrivial property, it is not decidable**.
Note that the theorem does not stipulate whether a language is **recognizable**

Non-Trivial Property

A **property** is a set of recognizable languages, e.g. regularity, "two as", etc.
A **non-trivial property** P is a language which contains and avoids at least one recognizable language. Simply, P if any property which requires more computing capability than can effectively be used, thus can not be decided.

Countability

A set is considered countable if it either has a finite number of elements or it has the same size as the set of natural numbers ($\{0, 1, 2, 3, ...\}$). **The real numbers are uncountable**

Misc Theorems

- The class of regular languages is closed under the union operation
- The class of regular languages is closed under the concatenation operation
- The class of regular languages is closed under the star operation
- Every nondeterministic finite automaton has an equivalent deterministic finite automaton
- A language is regular iff some nondeterministic finite automaton recognizes it
- A language is regular iff some regular expression describes it
- If a language is described by a regular expression, then it is regular
- Every multitape Turing machine has an equivalent single tape turing machine
- A language is Turing-recognizable iff some multitape Turing machine recognizes it
- A language is Turing-recognizable iff some enumerator enumerates it
- Regular expressions, NFAs, and DFAs are decidable
- Every context-free language is decidable
- The set of real numbers is uncountable
- Some languages are not Turing-recognizable
- A language is decidable iff it is Turing-recognizable and co-Turing recognizable
- Regular languages are undecidable
- If M is a linear bounded automaton (Turing machine where tape head states inside of the input) where $L(M) = \emptyset$, then the machine M is undecidable
- If G is a context-free grammar and $L(G) = \Sigma^*$, then the machine G is undecidable

3-State Busy Beaver State Table

Tape Symbol	Current State A			Current State B			Current State C		
	Write Symbol	Move Tape	Next State	Write Symbol	Move Tape	Next State	Write Symbol	Move Tape	Next State
0	1	R	B	1	L	A	1	L	B
1	1	L	C	1	R	B	1	R	**HALT**