Stationary INARMA(p, q) RJMCMC

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An INARMA(p, q) model of

$$X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \sum_{j=1}^{q} \beta_j \circ Z_{t-j} + Z_t \tag{1}$$

where the operator $a \circ b$ denotes some discretising function of (a, b) and Z_t is a discrete r.v. We take $a \circ b$

$$a \circ b = \begin{cases} Bin(a, b) & b > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

and $Z_t \sim Poisson(\lambda)$ for $\lambda > 0$. As a linear combination of an INAR(p) and an INMA(q) model, the following set of conditions must hold:

1. All counting series $a_i \circ X_{t-i}$ and $\beta_j \circ Z_{t-j}$ for $i = 1, \ldots, p, j = 1, \ldots, q$ are mutually independent and independent of Z_t .
2. $0 \leq \alpha_i < 1$, $i = 1, \ldots, p-1$, and $0 < \alpha_p < 1$.
3. $0 \leq \beta_i < 1$, $i = 1, \ldots, q-1$, and $0 < \beta_q < 1$.
4. For stationarity to hold, $\sum_{i=1}^{p} \alpha_i < 1$ and $\sum_{j=1}^{q} \beta_j < 1$.

The RJMCMC works by first proposing changes in p and q and then updating the parameters $(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, \lambda)$. **We use uninformative conjugate priors for both ARMA parameters and order.** We begin by describing the parameter update scheme:

1 Parameter updates

Rewrite 1 as

$$X_t = \sum_{i=1}^{p} \alpha_i \circ X_{t-i} + \sum_{j=1}^{q} \beta_j \circ Z_{t-j} + Z_t = \sum_{i=1}^{p} Y_{t,i} + \sum_{j=1}^{q} V_{t,i} + Z_t,$$

where $\mathbf{V}$ and $\mathbf{Y}$ are treated as augmented data. With uninformative priors, the conditional posteriors of the parameters are

$$\begin{aligned}
\alpha_i|\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{z} &\sim Beta\left(\sum y_{t,i}+1, \sum(x_{t-i}-y_{t,i})+1\right) \text{ for } 1\le i\le p\ ,\\
\beta_j|\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{z} &\sim Beta\left(\sum v_{t,j}+1, \sum(z_{t-j}-v_{t,j})+1\right) \text{ for } 1\le j\le q\ ,\\
\lambda|\mathbf{x},\mathbf{y},\mathbf{v},\mathbf{z} &\sim Gamma\left(\sum z_t+1, n+1\right) .
\end{aligned} \tag{3}$$

Gibbs sample within Metropolis-Hastings algorithm:

1. Initialise values for $(\boldsymbol{\alpha}^{(0)}, \boldsymbol{\beta}^{(0)}, \lambda^{(0)})$ and for augmented dat components $(\mathbf{v},\mathbf{y},\mathbf{z})$. Sensible values for the components are $(v_t = y_t = 0, z_t = x_t)\ \forall t$.

2. Gibbs sampler:

   (a) Update $\boldsymbol{\alpha}$:

      i. Sample α_i from (3) for all $1 \le i \le p$.

      ii. For stationarity, require $\sum \alpha_k < 1$. If this is not the case, repeat step (a) i.

   (b) Update $\boldsymbol{\beta}$:

      i. Sample β_j from (3) for all $1 \le j \le q$.

      ii. For stationarity, require $\sum \beta_l < 1$. If this is not the case, repeat step (b) i.

   (c) Update λ from (3).

3. For each $t \le n$, update the component set (y_t, v_t, z_t) using the Metropolis-Hastings procedure:

   (a) $\forall 1 \le i \le p$, draw $y'_{t,i}$ from $Bin(x_{t-i}, \alpha_i)$.

   (b) $\forall 1 \le j \le q$, draw $v'_{t,j}$ from $Bin(z_{t-j}, \beta_j)$.

   (c) Set $z'_t = x_t - \sum y'_{t,i} - \sum v'_{t,j}$.
   Note that due to the nature of the algorithm, we must assure that $z_t \ge 0$. If this is not the case, repeat steps a - c.

   (d) Calculate the acceptance probability

   $$A = \min\left\{1, \frac{z_t!}{z'_t!}\lambda^{z'_t - z_t}\frac{z'_t!(z_t - v_{t+1})!}{z_t!(z'_t - v_{t+1})!}\prod_{k=1}^{q\wedge n-t}(1-\beta_k)^{z'_t - z_t}\right\}$$

   for the move. Simulate some $U \sim Uniform(0,1)$. If $U \le A$, accept the move and set $(\mathbf{v_t}, \mathbf{y_t}, \mathbf{z_t}) = (\mathbf{v'_t}, \mathbf{y'_t}, \mathbf{z'_t})$.

2 Order Update

To begin with, we propose to either increase or decrease the AR(p) order by 1, each move being proposed with probability 0.5. The same is then done for the MA (q) order. This is subject to the constraints $0 \leq p \leq p_{max}$ and $0 \leq q \leq q_{max}$.

Before describing the algorithm, note we have

$$\mathbb{E}(X_t) = \frac{1 + \sum \beta_j}{1 - \sum \alpha_i} \lambda \ = m. \tag{4}$$

This algorithm exploits the key feature that m is well estimated from the data with the estimator

$$\hat{m} = \frac{1}{n} \sum_{i=1}^{n} x_i \ .$$

During the order-switching step, we intend on keeping m fixed. This is easily implemented as long as we do not propose to move the AR (MA) order p (q) either to or from $p = 0$ ($q = 0$). i.e. we do not propose a move between different models (INAR(p), INMA(q) or INARMA(p, q)).
We can now begin describing the algorithm for changing the AR order p where neither p nor p' are 0.

2.1 AR order

1. Set $p' = p - 1$.
2. Sample K uniformly at random from $[1, 2, \ldots, p-1]$.
3. For $i \in [1, 2, \ldots, p-1]/K$, set $\alpha'_i = \alpha_i$. Let $\alpha'_K = \alpha_K + \alpha_p$.
4. Update the augmented data terms. Set $y'_{t,i} = y_{t,i}$ for $i \neq K, p$, and let $y'_{t,K} = y_{t,k} + y_{t,p}$.
5. For the reverse movement, set $U = \alpha_K / (\alpha_K + \alpha_p)$.
6. Compute the acceptance probability for the move,

$$A = 1 \wedge \frac{n^{-(p'-p)/2}}{p} \times \prod_{t=1}^{n} \frac{\binom{x_{t-K}}{y'_{t,K}} (\alpha'_K)^{x_{t-K} - y'_{t,K}}}{\binom{x_{t-K}}{y_{t,K}} (\alpha_K)^{y_{t,K}} (1-\alpha_K)^{x_{t-K} - y_{t,K}} \binom{x_{t-p}}{y_{t,p}} (\alpha_p)^{y_{t,p}} (1-\alpha_p)^{x_{t-p} - y_{t,p}}}$$
$$\times \prod_{t=1}^{n} \left(\binom{y'_{t,K}}{y_{t,K}} U^{y_{t,K}} (1-U)^{y'_{t,K} - y_{t,K}} \right).$$

7. Simulate some $a \sim Uniform(0, 1)$. If $a \leq A$, accept the proposal and set $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$ and $\mathbf{y}' = \mathbf{y}$.

2.2 MA order

As above, but with q replacing p and $\boldsymbol{\beta}$ replacing $\boldsymbol{\alpha}$ and $\boldsymbol{v}_t$ replacing $\boldsymbol{y}_t$.

2.3 Moving between INAR(p) and INARMA(p, q)

In order to achieve this, we again exploit the use of m and keep it fixed. It is no longer possible to simply apply a spitting and amalgamation mapping as above, so a different method is proposed. First, consider the move from INAR(p) to INARMA($p, 1$). We need to keep $\boldsymbol{\alpha}$ fixed and vary λ to ensure that m remains constant.

1. Set $q' = 1$.

2. Draw $U \sim Unif(0, 1)$.

3. Set $\beta_1' = U$ and $\lambda' = \frac{\lambda}{1+U}$.

4. Update the augmented data terms sequentially. Draw $v_{t,1}' \sim Bin(z_t \wedge z_{t-1}', U/(1+U))$ and set $z_t' = z_t - v_{t,1}$ for $1 \leq t \leq n$.

5. Compute the acceptance probability for the move,

$$A = 1 \wedge \frac{n^{-(p'-p)/2}}{1+U} \times \prod_{t=1}^{n} \frac{\frac{e^{-\lambda'}}{z_t'!}\lambda'^{z_t'}}{\frac{e^{-\lambda}}{z_t!}\lambda^{z_t}} \binom{z_{t-1}'}{v_{t,1}'} {\beta_1'}^{v_{t,1}'}(1-\beta_1')^{z_{t-1}'-v_{t,1}'} \\ \times \frac{1}{\prod_{t=1}^{n}\left(\binom{z_t \wedge z_{t-1}'}{v_{t,1}'}\left(\frac{U}{1+U}\right)^{v_{t,1}'}\left(\frac{1}{1+U}\right)^{z_t \wedge z_{t-1}' - v_{t,1}'}\right)}$$

6. Simulate some $a \sim Uniform(0, 1)$. If $a \leq A$, accept the proposal and set $\beta_1 = \beta_1'$, $\lambda = \lambda'$ and $\mathbf{v} = \mathbf{v}'$.

Now, consider the reverse movmement from INARMA($p, 1$) to INAR(p).

1. Set $q' = 0$.

2. Set $U = \beta_1$.

3. Set $\lambda' = \lambda(\beta_1 + 1)$.

4. Update the augmented data terms sequentially. Set $z_t' = z_t + v_{t,1}$ for all $0 \leq t \leq n$.

5. Compute the acceptance probability for the move,

$$A = 1 \wedge \frac{n^{-(p'-p)/2}}{1+U} \times \prod_{t=1}^{n} \frac{\frac{e^{-\lambda'}}{z_t'!}\lambda'^{z_t'}}{\frac{e^{-\lambda}}{z_t!}\lambda^{z_t}} \frac{1}{\binom{z_{t-1}}{v_{t,1}}{\beta_1}^{v_{t,1}}(1-\beta_1)^{z_{t-1}-v_{t,1}}} \\ \times \prod_{t=1}^{n}\left(\binom{z_t' \wedge z_{t-1}}{v_{t,1}}\left(\frac{U}{1+U}\right)^{v_{t,1}}\left(\frac{1}{1+U}\right)^{z_t' \wedge z_{t-1} - v_{t,1}}\right)$$

6. Simulate some $a \sim Uniform(0, 1)$. If $a \leq A$, accept the proposal and set $\lambda = \lambda'$ and $\mathbf{z} = \mathbf{z}'$.

2.4 Moving between INMA(q) and INARMA(p, q)

The move between the INMA(q) and INARMA(p, q) models is slightly more difficult. We begin by considering the move betwee INMA(q) and INARMA($1, q$). We require keeping $\boldsymbol{\beta}$ fixed and letting λ and $(\mathbf{v}, \mathbf{z})$ vary.

1. Set $p' = 1$.
2. Draw $U \sim Unif(0,1)$.
3. Set $\alpha_1' = U$ and $\lambda' = \lambda(1-U)$.
4. Update the augmented data terms sequentially. For all $1 \le t \le n$ and $1 \le j \le q$, let $S_{t,j} \sim Bin(v_{t,j}, U)$ and $S_{t,0} \sim Bin(z_t, U)$. Then set

$$y_{t,1}' = \sum_{j=0}^{q} S_{t,j}$$

   and $v_{t,j}' = v_{t,j} - S_{t,j}$ for $1 \le j \le q$ and $z_t' = z_t - S_{t,0}$.
5. For $0 \le K \le q$, set $\gamma_K = \beta_K / (\sum_{j=1}^{q} \beta_j + 1)$.
6. Compute the acceptance probability for the move

$$A((\theta, \mathbf{w}), (\theta', \mathbf{w}')) = 1 \wedge \frac{f(\theta', \mathbf{w}'|\mathbf{x})}{f(\theta, \mathbf{w}|\mathbf{x})} \times \frac{q(\theta', \mathbf{w}' \to \theta, \mathbf{w})}{q(\theta, \mathbf{w} \to \theta', \mathbf{w}')} \times (1-U)$$

   where

$$\frac{f(\theta', \mathbf{w}'|\mathbf{x})}{f(\theta, \mathbf{w}|\mathbf{x})} = n^{-(p'-p)/2} \times \prod_{t=1}^{n} \frac{\frac{e^{-\lambda'}}{z_t'!} \lambda'^{z_t'}}{\frac{e^{-\lambda}}{z_t!} \lambda^{z_t}} \frac{\binom{x_{t-1}}{y_{t,1}'} \alpha_1'^{\,y_{t,1}'} (1-\alpha_1')^{x_{t-1} - y_{t,1}'} \prod_{j=1}^{q} \binom{z_{t-j}'}{v_{t,j}'} \beta_j^{\,v_{t,j}'} (1-\beta_j)^{z_{t-j}' - v_{t,j}'}}{\prod_{j=1}^{q} \binom{z_{t-j}}{v_{t,j}} \beta_j^{\,v_{t,j}} (1-\beta_j)^{z_{t-j} - v_{t,j}}}$$

   and

$$\frac{q(\theta', \mathbf{w}' \to \theta, \mathbf{w})}{q(\theta, \mathbf{w} \to \theta', \mathbf{w}')} = \prod_{t=1}^{n} \frac{y_{t,1}'! \prod_{j=0}^{q} \frac{\gamma_j^{S_{t,j}}}{S_{t,j}!}}{\binom{z_t}{S_{t,0}} U^{S_{t,0}} (1-U)^{z_t - S_{t,0}} \prod_{j=1}^{q} \binom{v_{t,j}}{S_{t,j}} U^{S_{t,j}} (1-U)^{v_{t,j} - S_{t,j}}}.$$

7. Simulate some $a \sim Uniform(0,1)$. If $a \le A$, accept the proposal and set $\alpha_1 = \alpha_1'$, $\lambda = \lambda'$ and $\mathbf{v}, \mathbf{z} = \mathbf{v}', \mathbf{z}'$.

This method takes into account that a decrease in λ will lead to a decrease in $\mathbf{v}$ and $\mathbf{z}$ for fixed $\boldsymbol{\beta}$.
For the reverse move:

1. Set $p' = 0$.
2. Set $U = \alpha_1$ and $\lambda = \frac{\lambda}{(1-U)}$.
3. For $0 \le K \le q$, set $\gamma_K = \beta_K / (\sum_{j=1}^{q} \beta_j + 1)$. Set $\gamma = (\gamma_0, \ldots, \gamma_q)$.

4. Update the augmented data terms sequentially. For all $1 \leq t \leq n$ and $1 \leq j \leq q$, let $S_{t,j} \sim multinomial(y_{t,1}, \gamma)$. Set $v'_{t,j} = v_{t,j} + S_{t,j}$ for $1 \leq j \leq q$ and $z'_t = z_t + S_{t,0}$.

5. Compute the acceptance probability for the move

$$A((\theta, \mathbf{w}), (\theta', \mathbf{w}')) = 1 \wedge \frac{f(\theta', \mathbf{w}'|\mathbf{x})}{f(\theta, \mathbf{w}|\mathbf{x})} \times \frac{q(\theta', \mathbf{w}' \to \theta, \mathbf{w})}{q(\theta, \mathbf{w} \to \theta', \mathbf{w}')} \times (1-U)^{-1}$$

   where

$$\frac{f(\theta', \mathbf{w}'|\mathbf{x})}{f(\theta, \mathbf{w}|\mathbf{x})} = \frac{\pi(p')}{\pi(p)} \times \prod_{t=1}^{n} \frac{\frac{e^{-\lambda'}}{z'_t!} \lambda'^{z'_t}}{\frac{e^{-\lambda}}{z_t!} \lambda^{z_t}} \frac{\prod_{j=1}^{q} \binom{z'_{t-j}}{v'_{t,j}} \beta_j^{v'_{t,j}} (1-\beta_j)^{z'_{t-j} - v'_{t,j}}}{\binom{x_{t-1}}{y_{t,1}} \alpha_1^{y_{t,1}} (1-\alpha_1)^{x_{t-1} - y_{t,1}} \prod_{j=1}^{q} \binom{z_{t-j}}{v_{t,j}} \beta_j^{v_{t,j}} (1-\beta_j)^{z_{t-j} - v_{t,j}}}$$

   and

$$\frac{q(\theta', \mathbf{w}' \to \theta, \mathbf{w})}{q(\theta, \mathbf{w} \to \theta', \mathbf{w}')} = \prod_{t=1}^{n} \frac{\binom{z'_t}{S_{t,0}} U^{S_{t,0}} (1-U)^{z'_t - S_{t,0}} \prod_{j=1}^{q} \binom{v'_{t,j}}{S_{t,j}} U^{S_{t,j}} (1-U)^{v'_{t,j} - S_{t,j}}}{y_{t,1}! \prod_{j=0}^{q} \frac{\gamma_j^{S_{t,j}}}{S_{t,j}!}}.$$

6. Simulate some $a \sim Uniform(0,1)$. If $a \leq A$, accept the proposal and set $\lambda = \lambda'$ and $\mathbf{v}, \mathbf{z} = \mathbf{v}', \mathbf{z}'$.