Modelling the extremes of spatial aggregates of precipitation using conditional methods

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Motivation

- River flooding caused by high intensity rainfall over spatial catchment area
- For spatial process $\{Y(s) : s \in \mathcal{S}\}$ observed at sampling locations $\mathbf{s} = (s_1, \ldots, s_d)$
- Interested in behaviour of aggregate $R_{\mathcal{A}} = \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} Y(s)\mathrm{d}s$ or $\int_{\mathcal{A}} Y(s)\mathrm{d}s$ for $\mathcal{A} \subseteq \mathcal{S}$ (or equivalent sum)
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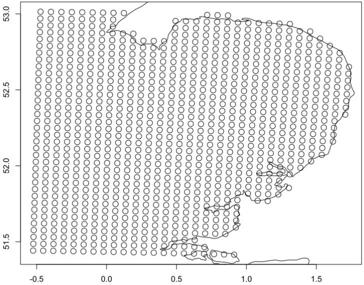

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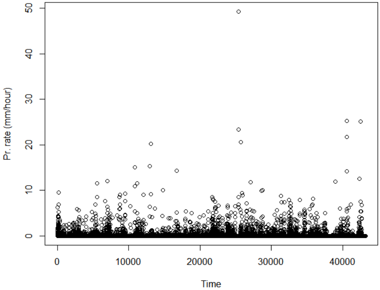

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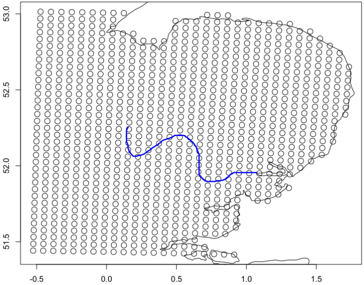
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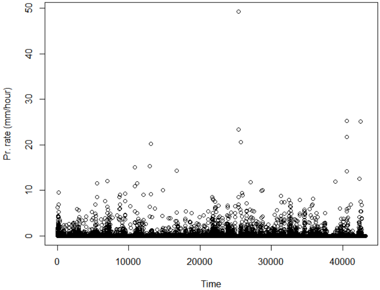
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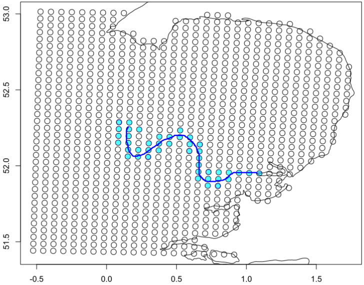
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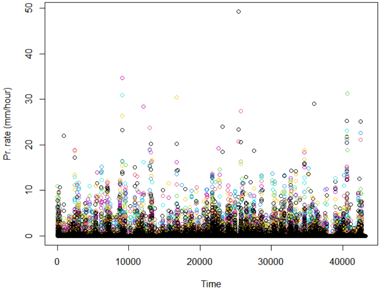
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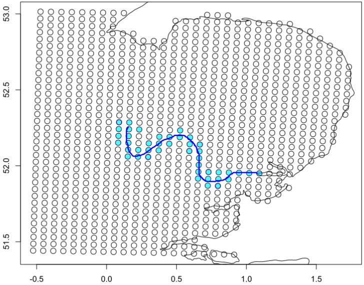

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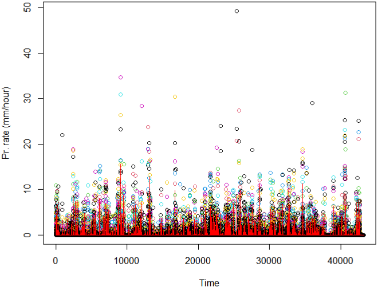

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- Can just take sample aggregate $R_{\mathcal{A}}$ - use univariate methods
- Wasteful - Less data for modelling
- Loses information from the underlying marginal process or dependence structure $\rightarrow$ Could potentially lead to inconsistent inference, i.e., no natural ordering of estimated return levels where physically appropriate
- Idea - model entire process $\{Y(s) : s \in \mathcal{S}\}$, simulate, aggregate, use realisations for inference

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- Model for marginal extremes and extremal dependence
- Focus on modelling extremes of process - largest values of underlying process produce largest values of aggregate
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Dependence model - general outline

General outline for spatial conditional extremes [Wadsworth and Tawn, 2019]

- Spatial extension of [Heffernan and Tawn, 2004]
- Spatial process $\{X(s) : s \in \mathcal{S}\}$ with standard exponential upper-tailed margins (standard Laplace)
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Dependence model

Assume there exists normalising functions $\{a_{s-s_0} : \mathbb{R} \to \mathbb{R}, s \in \mathcal{S}\}$, with $a_0(x) = x$, and $\{b_{s-s_0} : \mathbb{R} \to (0,\infty), s \in \mathcal{S}\}$, such that

$$\left(\left\{ \frac{X(s_i) - a_{s_i - s_0}\{X(s_0)\}}{b_{s_i - s_0}\{X(s_0)\}} \right\}_{i=1,\dots,d}, X(s_0) - u \right) |X(s_0) > u$$

$$\xrightarrow{d} \left(\left\{ Z^0(s_i) \right\}_{i=1,\dots,d}, E \right), \text{ as } u \to \infty,$$

- $E \sim \text{Exp}(1)$
- (Residual) process $\{Z^0(s) : s \in \mathcal{S}\}$ independent of E, satisfies $Z^0(s_0) = 0$ almost surely
- Stationarity assumption; same relationships for any s_0

Modelling choices - Normalising functions

Assume limit holds for $X(s_0) > u$. We have

$$\{X(s) : s \in \mathcal{S}\} = a_{s-s_0}\{X(s_0)\} + b_{s-s_0}\{X(s_0)\}\{Z^0(s) : s \in \mathcal{S}\}.$$

Modelling choices for normalising functions:

- Let $a_{s-s_0}(x) = x\alpha(\|s - s_0\|)$ for $\alpha : \mathbb{R}_+ \to [0, 1]$ and let $b_{s-s_0}(x) = x^{\beta(\|s-s_0\|)}$ for $\beta : \mathbb{R}_+ \to [0, 1]$.
- As $a_0(x) = x$, require $\alpha(0) = 1$.
- Extremal dependence between $X(s)$ and $X(s_0)$ decreases with $\alpha(\|s - s_0\|)$ and $\beta(\|s - s_0\|)$ for $s \in \mathcal{S}$.

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Residual process - Dependence

Reminder that we have

$$\{X(s) : s \in \mathcal{S}\} = \alpha(\|s - s_0\|)X(s_0) + X(s_0)^{\beta(\|s-s_0\|)}\{Z^0(s) : s \in \mathcal{S}\}.$$

Modelling choice for dependence in $\{Z^0(s)\}$:

- Start with a standard GP $\{W(s) : s \in \mathcal{S}\}$ with stationary correlation structure
- Set $\{Z^0(s) : s \in \mathcal{S}\} = \{W(s) | W(s_0) = 0 : s \in \mathcal{S}\}$

Residual process - Margins

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Modelling choice for margins of $\{Z^0(s) : s \in \mathcal{S}\}$:

- Set margins of $\{Z^0(s) : s \in \mathcal{S}\}$ to delta-Laplace

$$f(z) = \frac{\delta}{2k\sigma\Gamma\left(\frac{1}{\delta}\right)} \exp\left\{-\left|\frac{z-\mu}{k\sigma}\right|^{\delta}\right\},$$

  with $\Gamma(\cdot)$ as the standard gamma function and $\mu \in \mathbb{R}, \sigma > 0, \delta > 0$ and $k^2 = \Gamma(1/\delta)\Gamma(3/\delta)$.
- More flexible than Laplace ($\delta = 1$) or Gaussian ($\delta = 2$)

Residual process - Margins cont.

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Modelling choice for margins of $\{Z^0(s) : s \in \mathcal{S}\}$:

- Parametrise μ, σ and δ as functions of $\|s - s_0\|$.
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To achieve independence between $X(s)$ and $X(s_0)$ as $\|s - s_0\| \to \infty$:

- $\alpha(\|s - s_0\|) \to 0$ and $\beta(\|s - s_0\|) \to 0$
- For standard Laplace margins at large enough distances - need $\mu(\|s - s_0\|) \to 0, \sigma(\|s - s_0\|) \to \sqrt{2}$ and $\delta(\|s - s_0\|) \to 1$ as $\|s - s_0\| \to \infty$.

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Inference/Simulation

- Use triple-wise pseudo-likelihood for inference
- Censoring in inference and simulation to account for zeroes (no rain)
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Full details in paper

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Application

- From UKCP18 climate projections - values assigned to 934 spatial grid-boxes rather than point locations - Require sum for $R_{\mathcal{A}}$, rather than integral
- Hourly precipitation rate (mm/hour), Summer (JJA), 1980-2000
- CPM - Spatial resolution $5km \times 5km$ in East-Anglia

Dependence Model - Parametric function choice

To get an idea of the functional forms of the dependence parameters, we fit a simple dependence model

- No dependence in residual process $\{Z^0(s) : s \in \mathcal{S}\}$
- Individual parameter estimates, rather than fitted functions i.e., sequence of α_{s_i} for $i = 1, \ldots, d$, not $\alpha(\|s - s_0\|)$
- Done for several conditioning sites spread out over domain

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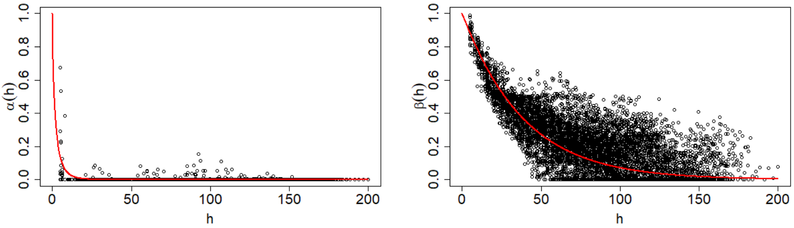
Alpha/Beta



α(h)
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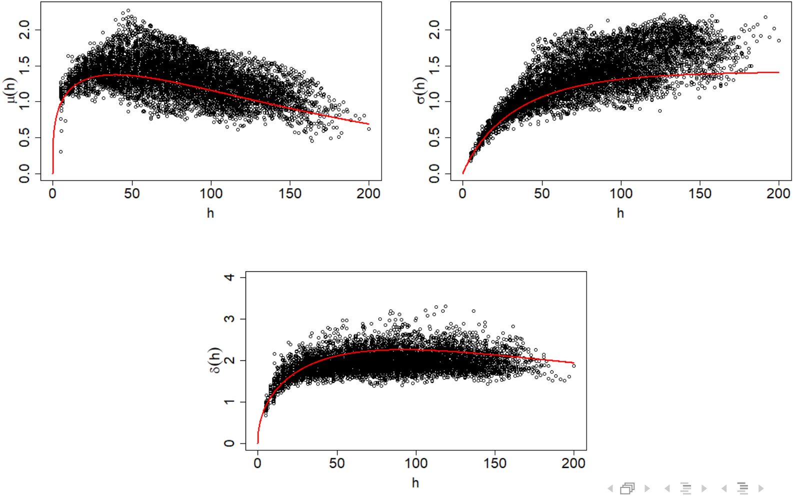
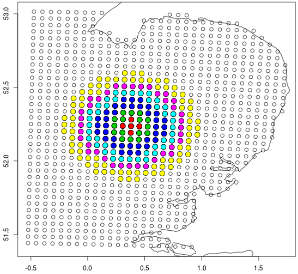
Mu/Sigma/Delta



Aggregate diagnostics

- How well do replications from $\{Y(s) : s \in \mathcal{S}\}$ capture tail behaviour of $R_{\mathcal{A}}$?
- Simulate over entire domain, roughly 200 years of events
- Aggregate over increasingly large regions (coloured points and interior) $(125 - 5425) - km^2$
- Compare model quantiles against data



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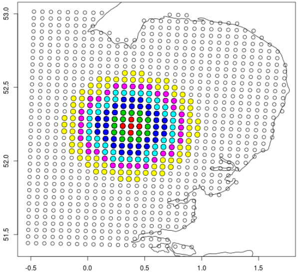

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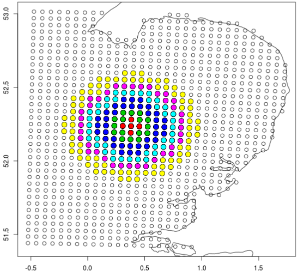

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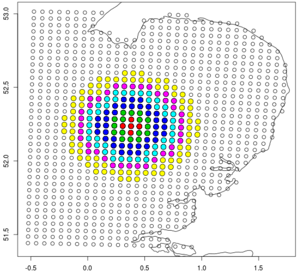
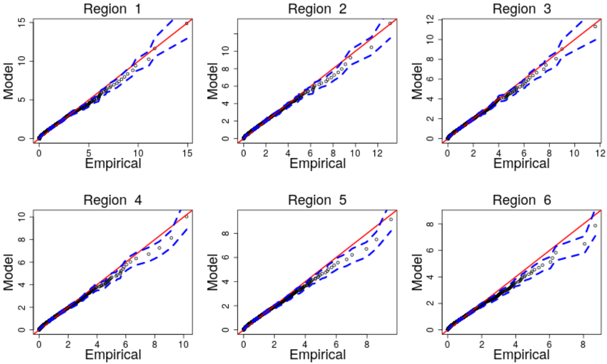


Model diagnostics

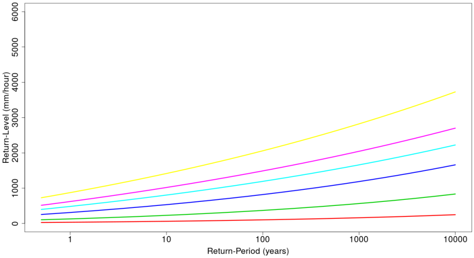
Q-Q plots (blue 95% confidence bands) for high quantiles:

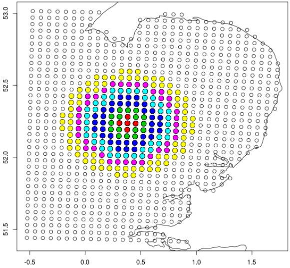
- Regions increase in size with label (i.e., 1 smallest - 6 biggest)
- Largest quantile corresponds to a 20 - year return level



Results

- GPD modelling for return levels of total volume of rainfall
- Guaranteed consistent estimates, i.e., no overlap





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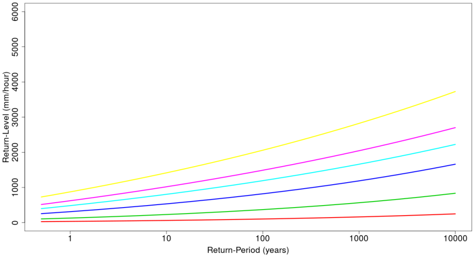

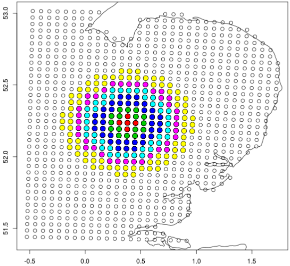

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- Can simulate from model for $\{Y(s) : s \in \mathcal{S}\}$ to produce inference on spatial aggregates $R_{\mathcal{A}}$
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Recap

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Thanks for listening.

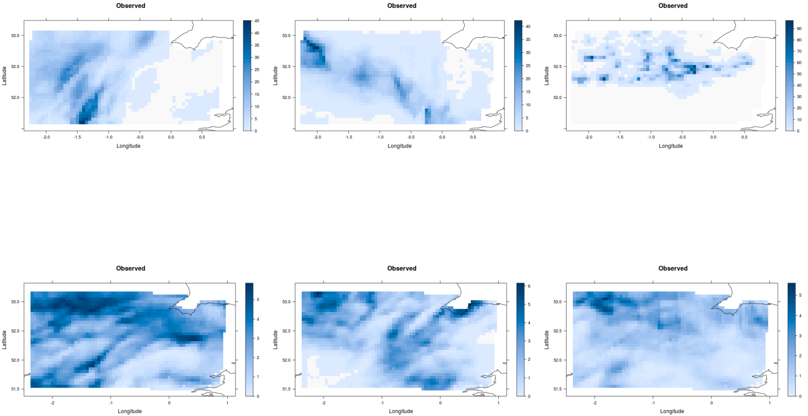
Extensions/Current work

Overview

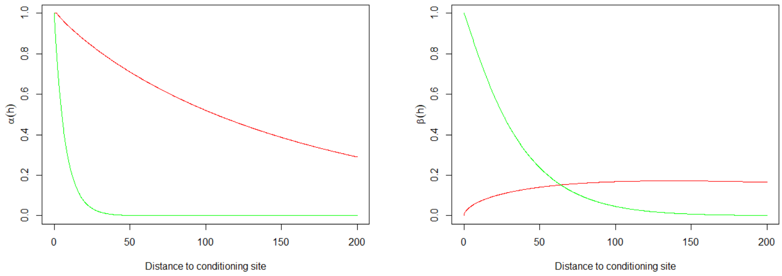
- Mixture modelling - Split data into convective and non-convective events
- Identification algorithm developed at Hadley Centre, UK Met Office - Deterministic, looks at gradients of surface of an observed field
- Fit separate model to each process - Simulate from both models and combine for aggregate
- Also consider higher spatial resolution, $(2.2)km^2$, and much larger spatial domain

Example observations

Top: Convective. Bottom: Non-convective



Comparison of fitted dependence models



- Green: Convective. Red: Non-convective.
- Observe much faster decay of extremal dependence for convective events.

Simulated fields

Top: Convective. Bottom: Non-convective