PARABOLIC FONTAINE-FALTINGS MODULES AND PARABOLIC HIGGS-DE RHAM FLOWS

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Abstract. In this note, we aim to extend the concepts of the Fontaine-Faltings module and Higgs-de Rham flow to parabolic versions. The crucial aspect of our generalization lies in the construction of parabolic inverse Cartier functors. The twisted versions discussed in Sun-Yang-Zuo's work can be viewed as a special case, wherein the parabolic weights are equal at every infinity point. We note that a modulo p version of parabolic Higgs-de Rham flow was previously established in [KS20].

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1. Parabolic structures

In this section, we undertake a study of de Rham bundles and Higgs bundles equipped with parabolic structures. We adopt the terminology and notation for parabolic structures established in [IS07], see also in [KS20].

1.1. **Introduction to Parabolic vector bundles.** Our intention is not to introduce parabolic objects on arbitrary spaces. Although the methods and results remain applicable to a broader range of scenarios, for the sake of brevity, we restrict our focus to the following specific spaces $(Y, D_Y)/S$.

Let p be an odd prime number and let S be our base space. We assume it is one of the following spaces.

(i) $S = \mathrm{Spec}(K)$, where K is a field of characteristic 0;
(ii) $S = \mathrm{Spec}(W_n(k))$, where $W_n(k)$ is a ring of truncated Witt vectors with coefficients in a finite field k;

(iii) $S = \mathrm{Spec}(\mathcal{O}_K)$, where K is an unramified p-adic number field.
(iv) $S = \mathrm{Spf}(\mathcal{O}_K)$, where K is an unramified p-adic number field.

For a smooth curve Y over S (or a smooth formal curve over S if S is a formal scheme), we define the reduced divisor D_Y by n S-sections $x_i \colon S \to Y$, $i = 1, \cdots, n$, that do not intersect with each other. We denote by $U_Y := Y - D_Y$ and by j_Y the open immersion $j_Y \colon U_Y \to Y$. The irreducible components of D_Y are denoted by $D_{Y,i}$, $i = 1, 2, \cdots, n$, and we have $D_Y = \bigcup_{i=1}^n D_{Y,i}$.

We set $\Omega^1_{Y/S}$ to be the sheaf of relative 1-forms and $\Omega^1_{Y/S}(\log D_Y)$ to be the sheaf of relative 1-forms with logarithmic poles along D_Y. By the smoothness of Y over S, both of these sheaves are line bundles over Y.

Parabolic vector bundles. In this section, we introduce the concept of parabolic vector bundles, which is based on [IS07].

Definition 1.1. A *parabolic sheaf* on $(Y, D_Y)/S$ is a collection of torsion-free coherent sheaves $V = V_\alpha$ on Y, which are flat over S, indexed by multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^n$. The sheaves $\{V_\alpha\}$ are subject to the following conditions:

- (inclusion) $V_\alpha \hookrightarrow V_\beta$ with cokernel flat over S, where $\alpha \leq \beta$ (i.e. where $\alpha_i \leq \beta_i$ for all i).
- (normalization) $V_{\alpha+\delta^i} = V_\alpha(D_{Y,i})$ where $\delta^i = (0, \cdots, 1, \cdots, 0) \in \mathbb{Q}^n$.
- (semicontinuity) for any given α there exists a constant $c = c(\alpha) > 0$ such that $V_{\alpha+\epsilon} = V_\alpha$ for $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in \mathbb{Q}^n$ with $0 \leq \epsilon_i \leq c$.

A *morphism between two parabolic sheaves* from F to F' is a collection of compatible morphisms of sheaves $f_\alpha \colon F_\alpha \to F'_\alpha$.

Remark 1.2. 1). The second condition implies that the quotient sheaves F_α/F_β for $\beta \leq \alpha$ are supported at D_Y.
2). The third condition means that the structure is determined by the sheaves F_α for a finite collections of indices α with $0 \leq \alpha_i < 1$.
3). The extension $j_{Y*}(j_Y^* F_\beta)$ does not depend on the choice of $\beta \in \mathbb{Q}^n$, denote by F_∞. Note that the F_α may all be consider as subsheaves of F_∞.

Definition 1.3. Let F and F' be two parabolic sheaves. Denote

$$(F \otimes F')_\alpha := \sum_{\beta+\gamma=\alpha} F_\beta \otimes F_\gamma \subset F_\infty \otimes F'_\infty.$$

Then $F \otimes F' := \{(F \otimes F')_\alpha\}\}$ forms a parabolic sheaf, which satisfies the universal property of *tensor product* of F and F' in the category of parabolic sheaves. Similarly, one can define the *wedge product*, *symmetric product*, and *determinant* for parabolic vector bundles as usual.

For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^n$, denote

$$\alpha D_Y = \alpha_1 D_{Y,1} + \cdots + \alpha_n D_{Y,n}$$

which is a rational divisor supported on D_Y. Of course, all rational divisor supported on D_Y are of this form. Denote

$$\lfloor \alpha \rfloor := (\lfloor \alpha_1 \rfloor, \lfloor \alpha_2 \rfloor, \cdots, \lfloor \alpha_n \rfloor)$$

where $\lfloor \alpha_i \rfloor$ is the maximal integer smaller than or equal to α_i. In particular $\lfloor \alpha \rfloor D_Y$ is an integral divisor supported on D_Y.

Example 1.4 (trivial parabolic structure)**.** For any torsion-free sheaf E on Y, it may be considered as a parabolic sheaf (we say "with trivial parabolic structure") by setting

$$E_\alpha = E(\lfloor \alpha \rfloor D).$$

Example 1.5. Let L be a line bundle and let $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{Q}^n$ be a rational multi-indices. Then there is a parabolic sheaf denoted

$$\mathcal{L} = L(\gamma D_Y)$$

by setting

$$\mathcal{L}_\alpha := L(\lfloor \alpha + \gamma \rfloor D_Y).$$

Clearly, for any two line bundles L, L' and two multi-indices $\gamma, \gamma' \in \mathbb{Q}^n$, one has

$$L(\gamma D) \otimes L'(\gamma' D') = (L \otimes L')\Big((\gamma + \gamma')D\Big).$$

Then the set of all isomorphic classes of parabolic line bundles forms an abelian group under the tensor product, which contains the Picard group of Y as a subgroup in the natural way.

The fractional part of the real number γ_i is just the parabolic weight of F along $D_{Y,i}$. We denote

$$\deg(\mathcal{L}) = \deg(L) + \sum_{i=1}^n \gamma_i.$$

Definition 1.6. 1). The parabolic sheaves appeared in Example 1.5 are called *parabolic line bundles.*

2). A *parabolic vector bundle* is a parabolic sheaf which is locally isomorphic to a direct sum of parabolic line bundles. (Simpson called this a *locally abelian parabolic vector bundle.*) A *morphism between two parabolic vector bundles* is a morphism between their underlying parabolic sheaves.

Example 1.7. Let V be a vector bundle and let $\gamma \in \mathbb{Q}^n$. Then

$$V(\gamma D) := V \otimes \mathcal{O}(\gamma D)$$

is a parabolic vector bundle.

Quasi-parabolic structures, parabolic weights. Historically, the parabolic vector bundles was defined by using some filtrations. In this subsection, we show the equivalency of the new and old definitions for parabolic vector bundles.

Definition 1.8. Let V be a vector bundle over (Y, D_Y). An *quasi-parabolic structure on V* is a decreasing filtration of direct summands of $V\mid_{D_{Y,i}}$.

$$V\mid_{D_{Y,i}} = \mathrm{QP}^1(V\mid_{D_{Y,i}}) \supsetneqq \mathrm{QP}^2(V\mid_{D_{Y,i}}) \supsetneqq \cdots \supsetneqq \mathrm{QP}^{n_i}(V\mid_{D_{Y,i}}) \supsetneqq 0.$$

A set of *parabolic weights* attached to the quasi-parabolic structure is a set of rational numbers $\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^{n_i}$ satisfying

$$0 \leq \alpha_i^1 < \alpha_i^2 < \cdots < \alpha_i^{n_i} < 1.$$

A *previous parabolic vector bundle* is triple (V, QP, α), which consists a vector bundle, a quasi-parabolic structure, and a set of parabolic weights. A *morphism between two previous parabolic vector bundles* from (V, QP, α) to (W, QP, β), is a morphism of vector bundle $f\colon V \to W$ such that

$$f\mid_{D_{Y,i}} (\mathrm{QP}^j(V\mid_{D_{Y,i}})) \subseteq \mathrm{QP}^k(W\mid_{D_{Y,i}})$$

for any triple of indices (i,j,k) satisfying $\alpha_i^j \geq \beta_i^k$.

Remark 1.9. Since our base space S is connected, for any direct summands W of $V\mid_{D_{Y,i}}$, it is actually a sub vector bundle of $V\mid_{D_{Y,i}}$ over $D_{Y,i}$. Moreover, if one consider the pullback of W along the surjective morphism $V \to V\mid_{D_{Y,i}}$, one gets a subsheaf F_W of V, which is also a vector bundle over Y. In fact, the subsheaf F_W fits in the following morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & V(-D_{Y,i}) \arrow[r] \arrow[d, equal] & F_W \arrow[r] \arrow[d, hook] & W \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & V(-D_{Y,i}) \arrow[r] & V \arrow[r] & V\mid_{D_{Y,i}} \arrow[r] & 0.
\end{tikzcd}$$

Since both W and $V(-D_{Y,i})$ are flat over S, the quasi-coherent sheaf F_W is also flat over S. According [Sta17, Tag 080Q], the local freeness of F_W can be checked fiberwisely.

Example 1.10. Let $F = \{F_\alpha\}$ be a parabolic vector bundle over $(Y, D_Y)/S$. There is a natural previous parabolic vector bundle with underlying vector bundle $V \coloneqq F_0$.

For all $\epsilon \in [0,1)$, denote

$$\mathrm{P}^\epsilon(F_0\mid_{D_{Y,i}}) \coloneqq F_{-\epsilon\delta^i}/F_{-\delta^i} \subseteq F_0/F_{-\delta^i} = F_0\mid_{D_{Y,i}}$$

which form a left continue descending filtration of direct summands index by rational number in $[0,1)$. To show that $\mathrm{P}^\epsilon(F_0\mid_{D_{Y,i}})$ is a direct summand, we may reduce to the case F being a parabolic vector bundle and check that directly. Denote by $\{\alpha_i^j\}_{j=1}^{n_i} \subset [0,1)$ the finite set of all jumping locus of the filtration $\mathrm{P}^\epsilon(F_0\mid_{D_{Y,i}})$. By reordering the index, we may assume $0 \leq \alpha_i^1 < \alpha_i^2 < \cdots < \alpha_i^{n_i} < 1$. Then the filtration is uniquely determined by the following sub filtration

$$F_0\mid_{D_{Y,i}} = \mathrm{P}^{\alpha_i^1}(F_0\mid_{D_{Y,i}}) \supsetneqq \mathrm{P}^{\alpha_i^2}(F_0\mid_{D_{Y,i}}) \supsetneqq \cdots \supsetneqq \mathrm{P}^{\alpha_i^{n_i}}(F_0\mid_{D_{Y,i}}) \supsetneqq 0. \tag{1.1}$$

Clearly, this forms a quasi-parabolic structure of F_0 along $D_{Y,i}$, which we call *the quasi-parabolic structure of F along $D_{Y,i}$*. And the numbers in the set $\{\alpha_i^j\}_{j=1}^{n_i}$ forms a set of parabolic weights, which we call *the parabolic weights of F along $D_{Y,i}$*, or *the parabolic weight of the quasi-parabolic structure of F along $D_{Y,i}$*.

Remark 1.11. By the definition of our parabolic vector bundle, the parabolic weights are all rational.

Lemma 1.12. *The construction in Example 1.10 induces an equivalent functor from the category of all parabolic vector bundles to the category of all previous parabolic vector bundles.*

Proof. Let V be a vector bundle over (Y, D_Y) with a quasi-parabolic structures

$$V\mid_{D_{Y,i}} = \mathrm{QP}^1(V\mid_{D_{Y,i}}) \supsetneqq \mathrm{QP}^2(V\mid_{D_{Y,i}}) \supsetneqq \cdots \supsetneqq \mathrm{QP}^{n_i}(V\mid_{D_{Y,i}}) \supsetneqq 0$$

and parabolic weights

$$0 \leq \alpha_i^1 < \alpha_i^2 < \cdots < \alpha_i^{n_i} < 1$$

for $i = 1, \cdots, n$. We need to show that it associates a unique parabolic vector bundle $F = \{F_\alpha\}$ in the sense under the construction in Example 1.10.

Existence: For any $\epsilon \in [0,1) \cap \mathbb{Q}$, denote

$$\mathrm{P}^{\epsilon}(V\mid_{D_{Y,i}}) \coloneqq \begin{cases} \mathrm{QP}^1(V\mid_{D_{Y,i}}) & \text{if} \epsilon \leq \alpha_i^1 \\ \mathrm{QP}^j(V\mid_{D_{Y,i}}) & \text{if} \alpha_i^{j-1} < \epsilon \leq \alpha_i^j \text{for some} j = 2, \cdots, n_i \\ 0 & \text{if} \alpha_i^{n_i} < \epsilon \end{cases}$$

Denote by $F_{-\epsilon\delta^i}$ the pull back of $\mathrm{P}^{\epsilon}(V\mid_{D_{Y,i}}) \subset V\mid_{D_{Y,i}}$ under the surjective morphisms $V \to V\mid_{D_{Y,i}}$. By Remark 1.9, the $F_{-\epsilon\delta^i}$ is also a vector bundle. For all $\alpha = (\alpha_1, \cdots, \alpha_n) \in \Big((-1,0] \cap \mathbb{Q}\Big)^n$, denote

$$F_\alpha = \bigcap_{i=1}^n F_{\alpha_i\delta^i}.$$

By normalization in the definition of the parabolic vector bundles, we defines F_α for any $\alpha \in \mathbb{Q}^n$. By direct computation, one checks that $\{F_\alpha\}$ forms a parabolic vector bundle associated to the given previous parabolic vector bundle.

Uniqueness: Let $\{F_\alpha\}$ and $\{F'_\alpha\}$ be two parabolic vector bundles associated to the given previous parabolic vector bundle. Then the isomorphism $F_0 \cong V \cong F'_0$ can be restricted to isomorphisms $F_{-\epsilon\delta^i} \cong F'_{-\epsilon\delta^i}$, because of $P^{\epsilon}(F_0\mid_{D_{Y,i}}) \cong P^{\epsilon}(F'_0\mid_{D_{Y,i}})$ for all $\epsilon \in [0,1)$. By normalization, one gets an natural isomorphism between $\{F_\alpha\}$ and $\{F'_\alpha\}$. $\square$

Degrees and semistability. In order to define the degree of a parabolic vector bundle and the semistability of a parabolic vector bundle, we assume Y is projective over S in this subsection.

Let F be a coherent sheaf on Y which is flat over S. By the locally constancy of the Chern classes, the first Chern class of the coherent sheaf F_s over the curve Y_s for any points $s \in S$ does not depend on the choice of the point s. It is well-defined to set

$$\deg(F) \coloneqq c_1(F_s).$$

We define the degree of a parabolic vector bundle as follows.

Definition 1.13. Let F be a parabolic sheaf over $(Y, D_Y)/S$.

- The *degree of* F is defined as
$$\deg(F) \coloneqq \deg(F_0) + \sum_{D_{Y,i}\in D_Y} \sum_{j=1}^{n_i} \alpha_i^j \cdot \mathrm{rank}(\mathrm{P}^{\alpha_i^j}(F_0\mid_{D_{Y,i}})/\mathrm{P}^{\alpha_i^{j+1}}(F_0\mid_{D_{Y,i}}))$$
- The parabolic vector bundle F is called *semistable* (resp. *stable*) if for any proper sub parabolic sheaves $F' \subsetneq F$, one has
$$\frac{\deg(F')}{\mathrm{rank}(F')} \leq \frac{\deg(F)}{\mathrm{rank}(F)} \qquad \Big(\text{resp.} \frac{\deg(F')}{\mathrm{rank}(F')} < \frac{\deg(F)}{\mathrm{rank}(F)}\Big).$$

The pullback of parabolic vector bundles. Let $f\colon (Y', D_{Y'}) \to (Y, D)$ be a morphism of smooth curves with relative normal crossings divisors over S such that $f^{-1}(D) \subset D'$. We recall a definition proposed by Simpson in [IS07, Section 2.2] for the pullback f^*F of a parabolic vector bundle F.

Definition 1.14. (1) If F is a parabolic line bundle, then there exists a line bundle L and a rational divisor B which is supported on D such that $F = L(B)$. We define
$$f^*F \coloneqq (f^*L)(f^*B).$$

(2) In general case, by localization, we reduce to the case of parabolic line bundles.

In the following, we will give an easy-to-use format description of pullback and then extend the definition for parabolic de Rham bundles.

Let $V = \{V_\alpha\}$ be a parabolic vector bundle over $(Y, D_Y)/S$. For any $\gamma \in \mathbb{Q}^n$, we identify the two parabolic vector bundles

$$V = V(-\gamma D) \otimes \mathcal{O}_Y(\gamma D).$$

Clearly, once we identify the usual vector bundle with its associated parabolic vector bundle, then V_0 can be view as a parabolic subsheaf of V

$$V_0 \subseteq V.$$

In particular, for the parabolic bundle $V(-\gamma D)$, we have a parabolic subsheaf

$$V(-\gamma D)_0 \hookrightarrow V(-\gamma D).$$

By pullback along f and tensoring $f^*\Big(\mathcal{O}_Y(\gamma D)\Big)$, we gets a parabolic subsheaf

$$f^*_\gamma(V) \coloneqq f^*\Big(V(-\gamma D)_0\Big) \otimes f^*\Big(\mathcal{O}_Y(\gamma D)\Big) \subseteq f^*\Big(V(-\gamma D)\Big) \otimes f^*\Big(\mathcal{O}_Y(\gamma D)\Big) = f^*(V).$$

Proposition 1.15. $f^*V = \sum\limits_{\gamma \in \mathbb{Q}^n} f^*_\gamma(V)$.

Proof. By localization, we reduce it to parabolic line bundle case, which follows by taking $\mathcal{L}$ to be the parabolic line bundle of its self. $\square$

Remark 1.16. By replace the parabolic line bundle $\mathcal{O}_Y(\gamma D)$ with a general parabolic line bundle $\mathcal{L}$, one can define a parabolic subsheaf

$$f^*_{\mathcal{L}}(V) := f^*\Big((V \otimes \mathcal{L}^{-1})_0\Big) \otimes f^*\mathcal{L} \subset f^*V.$$

1.2. **Parabolic de Rham bundles.**

logarithmic de Rham bundle. A *(logarithmic) connection* on a sheaf V of $\mathcal{O}_Y$-modules over $(Y, D_Y)/S$ is an $\mathcal{O}_S$-linear map $\nabla\colon V \to V \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}(\log D_Y)$ satisfying the Leibniz rule $\nabla(rv) = v \otimes \mathrm{d}\, r + r\nabla(v)$ for any local section $r \in \mathcal{O}_Y$ and $v \in V$. Give a (logarithmic) connection, there are canonical maps

$$\nabla\colon V \otimes \Omega^i_{Y/S}(\log D_Y) \to V \otimes \Omega^{i+1}_{Y/S}(\log D_Y)$$

given by $s \otimes \omega \mapsto \nabla(s) \wedge \omega + s \otimes \mathrm{d}\,\omega$. The curvature $\nabla \circ \nabla$, the composition of the first two canonical maps, is $\mathcal{O}_Y$-linear and contained in $\mathcal{E}\mathrm{nd}(V) \otimes_{\mathcal{O}_Y} \Omega^2_{Y/S}(\log D_Y)$. The connection is called *integrable* and (V, ∇) is called a *(logarithmic) de Rham sheaf* if the curvature vanishes. For a de Rham sheaf, one has a natural *de Rham complex*:

$$\mathrm{DR}(V, \nabla): \qquad 0 \to V \xrightarrow{\nabla} V \otimes \Omega^1_{Y/S}(\log D_Y) \xrightarrow{\nabla} V \otimes \Omega^2_{Y/S}(\log D_Y) \xrightarrow{\nabla} V \otimes \Omega^3_{Y/S}(\log D_Y) \to \cdots$$

A logarithmic de Rham sheaf is called a *logarithmic de Rham bundle*, if the underlying sheaf is a vector bundle over Y. Denote by $\mathrm{MIC}(Y, D_Y)$ the category of all logarithmic de Rham bundle over (Y, D_Y).

In the same way, we define a *(logarithmic) p-connection along D_Y* by replacing the Leibniz rule in the definition of connection with

$$\nabla(rv) = pv \otimes \mathrm{d}\, r + r\nabla(v).$$

Denote by $\widetilde{\mathrm{MIC}}(Y, D_Y)$ the category of vector bundles over Y with integrable logarithmic p-connections along D_Y.

residue of a logarithmic de Rham bundle. Let (V, ∇) be a logarithmic de Rham bundle. On any given sufficient small open subset U, we choose a coordinate functor f_i for each irreducible component $D_{Y,i}$. Let $e = (e_1, \cdots, e_r)$ be the local frame of V, and ω the relative connection matrix of D with respect to e for V on U. Then for each i, the matrix ω can be written as

$$\omega = \sum_{i=1}^{r} R_i \cdot \frac{\mathrm{d}\, f_i}{f_i} + S$$

where R_i is an $r \times r$ matrix with entries in $\mathcal{O}_Y(U)$ and S is a $r \times r$ matrix with entries in $(\Omega^1_{Y/S})(U)$. Denote

$$\mathrm{res}_{Y/S}(\omega, D_{Y,i}) := R_i \mid_{U \cap D_{Y,i}}$$

which is $r \times r$ matrix whose entries are contained in $\mathcal{O}_{D_{Y,i}}(U \cap D_{Y,i})$, and it is independent of choice of f_i. When U run through a fine enough covering of Y, these matrices are glued into a global section

$$\mathrm{res}_{Y/S}(\nabla, D_{Y,i}) \in \mathrm{H}^0(D_{Y,i}, \mathcal{E}\mathrm{nd}_{\mathcal{O}_Y}(V) \mid_{D_{Y,i}}),$$

which is called the *residue of the connection* ∇ *along* $D_{Y,i}$. The residue map of the connection along $D_{Y,i}$ is also represented as an $\mathcal{O}_{D_{Y,i}}$-endomorphism of $V \mid_{D_{Y,i}}$

$$\mathrm{res}_{D_{Y,i}}(\nabla)\colon V \mid_{D_{Y,i}} \longrightarrow V \mid_{D_{Y,i}} .$$

We assume $\mathrm{res}_{D_{Y,i}}(\nabla)$ is quasi-nilpotent and with rational eigenvalues in $[0, 1)$.

Hodge filtration and filtered logarithmic de Rham bundle. In this paper, a filtration $\mathrm{Fil}^\bullet$ on a logarithmic de Rham bundle (V, ∇) over $(Y, D_Y)/S$ will be called a *Hodge filtration of level in* $[a, b]$ if the following conditions hold:

- $\mathrm{Fil}^i V$'s are subbundles of V, with

$$V = \mathrm{Fil}^a V \supset \mathrm{Fil}^{a+1} V \supset \cdots \supset \mathrm{Fil}^b V \supset \mathrm{Fil}^{b+1} V = 0.$$

- Fil satisfies Griffiths transversality with respect to the logarithmic connection ∇.

In this case, the triple $(V, \nabla, \mathrm{Fil})$ is called a *filtered logarithmic de Rham bundle*. We denote by $\mathrm{MCF}(Y, D_Y)$ the category of filtered logarithmic de Rham bundles over (Y, D_Y) and by $\mathrm{MCF}_a(Y, D_Y)$ those with level in $[0, a]$.

The parabolic vector bundles associated to logarithmic de Rham bundles (over $\mathbb{C}$). We first recall the following well know result.

Lemma 1.17. *Let (V, ∇) be a logarithmic de Rham bundle over $(Y, D_Y)/S$. Denote by i the open immersion of $U = Y \setminus D_Y$ into Y. Then*

(1) the connection ∇ on V can be uniquely extended onto $V_\infty := j_{Y*}j_Y^*(V)$ under the natural injection $V \hookrightarrow V_\infty$, and

(2) the extension connection can be restricted onto $V(D')$ for any integral divisor D' supported on D. In particular, if D' is also positive, then the logarithmic de Rham bundle (V, ∇) extends to another one $(V(D'), \nabla)$ with natural injection

$$(V, \nabla) \hookrightarrow (V(D'), \nabla).$$

(3) Let V' be a vector bundle over Y contained in V_∞ as a subsheaf. Then there is at most one connection ∇' onto V' such that $j_Y^*\nabla = j_Y^*\nabla'$.

Denote by $\mathbb{Q}^S$ the maximal subring of $\mathbb{Q}$ such that the natural ring homomorphism $\mathbb{Z} \to \mathcal{O}_S(S)$ can be extended to $\mathbb{Q}^S$

$$\iota\colon \mathbb{Q}^S \to \mathcal{O}_S(S)$$

Then

$$\mathbb{Q}^S = \begin{cases} \mathbb{Q}, & S\text{is in case (i)}; \\ \mathbb{Q}_{(p)} & \text{otherwise.} \end{cases}$$

We say that a logarithmic de Rham bundle (V,∇) over $(Y,D_Y)/S$ *has rational eigenvalues*, if the eigenvalues of the residues are all contained in the image of ι.

In the rest of this subsection, we set $S = \mathrm{Spec}(\mathbb{C})$.

Proposition 1.18 (Iyer-Simpson 2006)**.** *Let (V,∇) be a logarithmic de Rham bundle over (Y,D_Y). Assume the residues have rational eigenvalues. Then*

1). *there exists a unique (locally abelian) parabolic vector bundle F associated to (V,∇). I.e. a parabolic vector bundle $F = \{F_\alpha\}$ together with isomorphisms*

$$F_\alpha \mid_{Y^0} \cong V \mid_{Y^0}$$

such that for each α, the ∇ extends to a logarithmic connection ∇_α on F_α with the residue on the piece $\mathrm{Gr}_\alpha(F) := F_\alpha/F_{\alpha-\varepsilon\delta^i}$ being an operator with eigenvalue $-\alpha_i$.

2). *The eigenvalues of the residue of ∇_α along $D_{Y,i}$ are contained in the interval $[-\alpha_i, 1-\alpha_i)$.*

3). *The construction preserves short exact sequences and restrictions.*

Corollary 1.19. *If the eigenvalues $\{\eta_i^j\}$ of the residues are all located in $[0,1)$, then under suitable reordering of these eigenvalues, for each indices pair (i,j), one has*

$$\eta_i^j = \begin{cases} 0, & \mathit{if} \alpha_i^j = 0; \\ 1-\alpha_i^j, & \mathit{if} \alpha_i^j \neq 0 \end{cases}$$

Remark 1.20. Let (V,∇) be a logarithmic de Rham bundle over (X,D_Y).

- From the construction of the associated parabolic vector bundle,
  - if the eigenvalues of the residues of ∇ are located in $[0,1)$, then

$$(V,\nabla) = (F_0, \nabla_0)$$

  - if the eigenvalues of the residues of ∇ are located in $[-1,0)$, then

$$(V,\nabla) = (F_{(1,\cdots,1)}, \nabla_{(1,\cdots,1)})$$

  - if the eigenvalues of the residues of ∇ are located in $(-1,0]$, then

$$(V,\nabla) = \bigcup_{\varepsilon<1} (F_{(\varepsilon,\cdots,\varepsilon)}, \nabla_{(\varepsilon,\cdots,\varepsilon)}).$$

- F is also the parabolic vector bundle associated to logarithmic de Rham bundle $(F_\alpha, \nabla_\alpha)$.

Parabolic de Rham bundles. Inspired by Proposition 1.18, one can give a definition for parabolic de Rham bundles and parabolic Higgs bundles.

Definition 1.21. A *parabolic de Rham bundle* $(V,\nabla)=\{(V_\alpha,\nabla_\alpha)\}$ over $(Y,D_Y)/S$ is parabolic vector bundle $V=\{V_\alpha\}$ together with integrable connections ∇_α having logarithmic pole along D_Y such that the inclusions $V_\alpha\hookrightarrow V_\beta$ preserves the connections. We call $\nabla:=\{\nabla_\alpha\}$ a *parabolic connection* on the parabolic vector bundle V.

A parabolic de Rham bundle $(V,\nabla)=\{(V_\alpha,\nabla_\alpha)\}$ is called *standard* if all parabolic weights are contained in $\mathbb{Q}^S$, and for any $\alpha\in\left(\mathbb{Q}^S\right)^n$, the residue on the piece $V_\alpha/V_{\alpha-\varepsilon\delta^i}$ of the logarithmic connection ∇_α is an operator with eigenvalue $\iota(-\alpha_i)$.

Remark 1.22. (1). The (V_α,∇_α) may be considered as de Rham subsheaves of

$$(V_\infty,\nabla_\infty):=\varinjlim_{\beta}(V_\beta,\nabla_\beta)=j_{Y*}j_Y^*(V_\alpha,\nabla_\alpha).$$

(2). Tensor product of two parabolic de Rham bundles can be naturally defined: for any two parabolic de Rham bundle (V,∇) and (V',∇'), the underlying parabolic bundle of they tensor product is just $V\otimes V'$ and the parabolic connection is defined as the restrictions of the connection $\left(\nabla_\infty\otimes\mathrm{id}+\mathrm{id}\otimes\nabla'_\infty\right)$ on $V_\infty\otimes V'_\infty$.

Example 1.23. Let $\gamma\in\mathbb{Q}^n$ be a multiple indices and let (V,∇) be a logarithmic de Rham bundle over $(Y,D_Y)/S$. There is a natural parabolic connection, denote by $\nabla(\gamma D)$, on the parabolic vector bundle $V(\gamma D)$ given by

$$\nabla(\gamma D)_\alpha:=\nabla_\infty\mid_{V(\gamma D)_\alpha}.$$

The logarithmic de Rham bundle (V,∇) may be considered as a parabolic de Rham bundle (we say "with trivial parabolic structure") by identifying it with $(V(0D),\nabla(0D))$. Assume $(V,\nabla)=(\mathcal{O}_Y,\mathrm{d})$. We call parabolic de Rham line bundle of form $(\mathcal{O}_Y(\gamma D),\mathrm{d}(\gamma D))$ *shifting parabolic de Rham line bundles.* A shifting parabolic de Rham line bundle $(\mathcal{O}_Y(\gamma D),\mathrm{d}(\gamma D))$ is of standard if and only if $\gamma_i\in\ker\iota$ for all $i=1,\cdots,n$.

Proposition 1.24. *If S has characteristic zero, then any standard parabolic de Rham bundle over (X,D_Y) is semistable and of degree zero.*

Recall that a logarithmic p-connection on a vector bundle V over $(Y,D_Y)/S$ is an $\mathcal{O}_S$-linear mapping

$$\nabla\colon V\to V\otimes\Omega_{Y/S}(\log D_Y)$$

satisfying, for any local section $s\in\mathcal{O}_Y$ and any local section $v\in V$

$$\nabla(sv)=pv\otimes\mathrm{d}\,s+s\nabla(v)$$

We note that the multiplication of a connection with p is always a p-connection and if p is invert in $\mathcal{O}_S(S)$, then all p-connections are coming from this way. Similarly, one can defines parabolic p-connections on parabolic vector bundles, and their tensor products. We left the routine definitions to the readers.

Parabolic de Rham line bundles. By tensor some shifting parabolic de Rham line bundle, a parabolic de Rham bundle can be modified to be with trivial parabolic structure. Thus we have following result.

Lemma 1.25. *Let $(\mathcal{L},\nabla)$ be a parabolic de Rham line bundle over $(Y,D_Y)/S$. Denote by $w_i \in [0,1)\cap\mathbb{Q}$ the parabolic weight of $\mathcal{L}$ along $D_{Y,i}$ and denote $w=(w_1,\cdots,w_n)$. Then*

(1) *we have a decomposition of $(\mathcal{L},\nabla)$ into a tensor product of usual logarithmic de Rham bundle and a shifting parabolic de Rham line bundle*
$$(\mathcal{L},\nabla)=(\mathcal{L}_0,\nabla_0)\otimes(\mathcal{O}_Y(wD),\mathrm{d}(wD)).$$
(2) *Suppose we have another such decomposition*
$$(\mathcal{L},\nabla)=(L,\nabla)\otimes(\mathcal{O}_Y(\gamma D),\mathrm{d}(\gamma D)),$$
then $\omega-\gamma\in\mathbb{Z}^n$ and
$$(L,\nabla)=(\mathcal{L}_0,\nabla_0)\otimes(\mathcal{O}_Y((w-\gamma)D),\mathrm{d}((w-\gamma)D)).$$

Proof. Consider $(\Delta V,\Delta\nabla):=(\mathcal{L},\nabla)\otimes(\mathcal{L}_0,\nabla_0)^{-1}\otimes(\mathcal{O}_Y(wD),\mathrm{d}(wD))^{-1}$. From the definition of tensor product, one checks directly that $\Delta V=\mathcal{O}_Y$, and $\Delta\nabla\mid_{Y\setminus D_Y}=\mathrm{d}\mid_{Y\setminus D_Y}$. Thus $(\Delta V,\Delta\nabla)=(\mathcal{O}_Y,\mathrm{d})$ and (1) holds true. (2) follows (1) directly. □

Remark 1.26. By the same method, one has similar decompositions for parabolic vector bundle with parabolic p-connection.

The pullback of a parabolic de Rham vector bundles. Now, we consider the pullback for parabolic de Rham bundle case. Let (V,∇) be a parabolic de Rham bundle over $(Y,D_Y)/S$. For any $\gamma\in\mathbb{Q}^n$, then $f^*(\gamma D)$ is also a rational divisor over Y' supported on D'. We simply set
$$f^*\Big(\mathcal{O}_Y(\gamma D),\mathrm{d}_Y(\gamma D)\Big):=\Big(\mathcal{O}_{Y'}\big(f^*(\gamma D)\big),\mathrm{d}_{Y'}\big(f^*(\gamma D)\big)\Big).$$
Similarly as the parabolic vector bundle case, for each $\gamma\in\mathbb{Q}^n$, we set
$$f_\gamma^*(V,\nabla):=f^*\Big(\big(V(-\gamma D),\nabla(-\gamma D)\big)_0\Big)\otimes f^*\Big(\mathcal{O}_Y(\gamma D),\mathrm{d}_Y(\gamma D)\Big).$$
Denote $U_Y=Y-D_Y$ and $U_{Y'}=Y'-D_{Y'}$. If we restrict $f_\gamma^*(V,\nabla)$ onto the open subset $U_{Y'}$, then by the construction, one gets
$$\Big(f_\gamma^*(V,\nabla)\Big)\mid_{U_{Y'}}=f^*\Big((V,\nabla)\mid_{U_Y}\Big).$$
By Lemma 1.17, the connections on $f_\gamma^*(V,\nabla)$ for difference choice of γ are coincide with each others over the maximal common subsheaf.

Definition 1.27. Let (V,∇) be a parabolic de Rham bundle over $(Y,D_Y)/S$. We define the pullback of (V,∇) along f as
$$f^*(V,\nabla):=\bigcup_{\gamma\in\mathbb{Q}^n}f_\gamma^*(V,\nabla).$$

1.3. **Parabolic Higgs bundles.**

Graded vector bundles. Let $\{\mathrm{Gr}^\ell E\}_{\ell\in\mathbb{Z}}$ be subbundles of V. The pair (E,Gr) is called *graded vector bundle* over X if the natural map $\oplus_{\ell\in\mathbb{Z}}\mathrm{Gr}^\ell E\cong E$ is an isomorphism.

logarithmic Higgs bundles. Let E be vector bundle over X. Let $\theta\colon E \to E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log D)$ be an $\mathcal{O}_X$-linear morphism. The pair (E, θ) is called *(logarithmic) Higgs bundle* over $(X, D)/S$ if θ is integrable. i.e. $\theta \wedge \theta = 0$. For a Higgs bundle, one has a natural *Higgs complex*

$$\mathrm{DR}(E,\theta): \qquad 0 \to E \xrightarrow{\theta} E \otimes \Omega^1_{X/S}(\log D) \xrightarrow{\theta} E \otimes \Omega^2_{X/S}(\log D) \xrightarrow{\theta} E \otimes \Omega^3_{X/S}(\log D) \to \cdots$$

graded logarithmic Higgs bundles. A *graded (logarithmic) Higgs bundle* over $(X, D)/S$ is a Higgs bundle (E, θ) together with a grading structure Gr on E satisfying

$$\theta(\mathrm{Gr}^\ell E) \subset \mathrm{Gr}^{\ell-1} E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log D).$$

Thus we have subcomplexes of $\mathrm{DR}(E, \theta)$

$$\mathrm{Gr}^\ell \mathrm{DR}(E,\theta): \qquad 0 \to \mathrm{Gr}^\ell E \xrightarrow{\theta} \mathrm{Gr}^{\ell-1} E \otimes \Omega^1_{X/S}(\log D) \xrightarrow{\theta} \mathrm{Gr}^{\ell-2} E \otimes \Omega^2_{X/S}(\log D) \xrightarrow{\theta} \cdots.$$

The following is the main example we will be concerned with.

Example 1.28. Let $(V, \nabla, \mathrm{Fil})$ be a filtered de Rham bundle over V. Denote $E = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{Fil}^\ell V/\mathrm{Fil}^{\ell+1} V$ and $\mathrm{Gr}^\ell E = \mathrm{Fil}^\ell V/\mathrm{Fil}^{\ell+1} V$. By Griffith's transversality, the connection induces an $\mathcal{O}_X$-linear map $\theta\colon \mathrm{Gr}^\ell E \to \mathrm{Gr}^{\ell-1} E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log D)$ for each $\ell \in \mathbb{Z}$. Then (E, θ, Gr) is a graded Higgs bundle. Moreover we have

$$\mathrm{Gr}^\ell \mathrm{DR}(E,\theta) = \mathrm{Fil}^\ell \mathrm{DR}(V,\nabla)/\mathrm{Fil}^{\ell+1} \mathrm{DR}(V,\nabla).$$

Parabolic Higgs bundles.

Definition 1.29. A *parabolic Higgs bundle* $(E, \theta) = \{(E_\alpha, \theta_\alpha)\}$ over (Y, D_Y) is

- a parabolic vector bundle $E = \{E_\alpha\}$, together with
- integrable Higgs fields θ_α having logarithmic pole along D_Y

such that the inclusions $E_\alpha \hookrightarrow E_\beta$ preserves the Higgs fields.

A parabolic Higgs bundle (E, θ) is called *graded*, if there is a grading structure Gr on E satisfying decomposition of the underlying parabolic vector bundle E

$$\theta(\mathrm{Gr}^\ell E) \subset \mathrm{Gr}^{\ell-1} E \otimes_{\mathcal{O}_X} \Omega^1_{X/S}(\log D).$$

Remark 1.30. (1). The $(E_\alpha, \theta_\alpha)$ may be considered as Higgs subsheaves of

$$(E_\infty, \theta_\infty) := \varinjlim_\beta (E_\beta, \theta_\beta) = j_{Y*} j_Y^* (E_\alpha, \theta_\alpha).$$

(2). Tensor product of two parabolic Higgs bundle bundles can be naturally defined: for any two parabolic Higgs bundle (E, θ) and (E', θ'), the underlying parabolic bundle of they tensor product is just $E \otimes E'$ and the parabolic connection is defined as the restrictions of the connection $\left(\theta_\infty \otimes \mathrm{id} + \mathrm{id} \otimes \theta'_\infty\right)$ on $E_\infty \otimes E'_\infty$.

1.4. **Parabolic de Rham bundles and parabolic Higgs bundles of lower rank over projective lines.** In this section, we take $X = \mathbb{P}^1_S$ as the projective line over S and take $D = D_S \subset \mathbb{P}^1_S$ as the divisor given by 4 S-points $\{0, 1, \infty, \lambda\}$. Denote by D_i the reduce and irreducible divisor given by the point x for any $x \in \{0, 1, \infty, \lambda\}$.

Classification of parabolic de Rham line bundles. We first classifies all logarithmic de Rham line bundle over $(\mathbb{P}^1_S, D_S)/S$.

Lemma 1.31. (1) *Let* $e = (e_0, e_1, e_\lambda, e_\infty) \in \Big(\mathcal{O}_S(S)\Big)^4$ *and* $d \in \mathbb{Z}$ *satisfy*

$$e_0 + e_1 + e_\lambda + e_\infty + d = 0 \in \mathcal{O}_S(S).$$

Then up to an isomorphism there is a unique logarithmic de Rham line bundle $(L^{(e,d)}, \nabla^{(e,d)})$ *over* $(\mathbb{P}^1_S, D_S)/S$ *such that*

- *the degree of* $L^{(e,d)}$ *is of degree* d*, and*
- e_x *is the eigenvalue of the residue of* $\nabla^{(e,d)}$ *along* D_x *for all* $x = \{0, 1, \lambda, \infty\}$.

(2) *Any logarithmic de Rham line bundle is of this form.*

Proof. (1). **Existence:** We first take $L^{(e,d)} = \mathcal{O}_{\mathbb{P}^1_S}(-d(\infty))$ and set

$$\nabla^{(e,d)}(1) = 1 \otimes \Big(e_0 \,\mathrm{d}\log t + e_1 \,\mathrm{d}\log(t-1) + e_\lambda \log(t-\lambda)\Big)$$

where t is the parameter of the projective line. clearly, the residues of $\nabla^{(e,d)}$ at $0, 1, \lambda$ are e_0, e_1, e_λ respectively. We only need to show $\nabla^{(e,d)}$ can extends over ∞ and the residue is equal to e_∞. This follows the following explicit computation:

$$\begin{aligned}\nabla^{(e)}\left(t^d\right) &= 1 \otimes \mathrm{d}\left(t^d\right) + t^d \cdot \nabla^{(e)}(1)\\ &= t^d \otimes \Big(-d \cdot \mathrm{d}\log\frac{1}{t} + e_0 \,\mathrm{d}\log t + e_1 \,\mathrm{d}\log(t-1) + e_\lambda \log(t-\lambda)\Big)\\ &= t^d \otimes \Big(-d - e_0 - e_1 \cdot \frac{t}{t-1} - e_\lambda \cdot \frac{t}{t-\lambda}\Big) \cdot \mathrm{d}\log\frac{1}{t}\end{aligned}$$

Uniqueness: Suppose there are two logarithm de Rham line bundles (L, ∇) and (L', ∇') satisfy the conditions. Since our base space is projective line, L and L' has the some degree, they must isomorphic to each other. So we may identify them $L = L'$. Now, on this line bundle there are two logarithmic connections with the same residues. They must coincide with each other, since

$$\nabla - \nabla' \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1_S}}(L, L \otimes \Omega^1_{\mathbb{P}^1_S}) \cong H^0(\mathbb{P}^1_S, \Omega^1_{\mathbb{P}^1_S}) = 0.$$

(2). Suppose (L, ∇) be a logarithmic de Rham line bundle with rational eigenvalues $(e_0, e_1, e_\infty, e_\lambda)$ and degree d. Denote $e'_\infty := -(e_0 + e_1 + e_\lambda + \iota(d))$ and $e' = (e_0, e_1, e_\lambda, e'_\infty)$. By (1), one has logarithmic de Rham bundle $(L^{(e',d)}, \nabla^{(e',d)})$. By similar proof as in (1), we may identify $L^{(e',d)}$ with L, and gets

$$\nabla - \nabla^{(e',d)} \in \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1_S}}(L, L \otimes \Omega^1_{\mathbb{P}^1_S}(\log D_\infty)) \cong H^0(\mathbb{P}^1_S, \Omega^1_{\mathbb{P}^1_S}(\log D_\infty)) = 0.$$

Thus $e'_\infty = e_\infty$ and $(L, \nabla) \cong (L^{(e',d)}, \nabla^{(e',d)})$. □

Lemma 1.32. *Let* $e = (e_0, e_1, e_\lambda, e_\infty) \in \Big(\mathcal{O}_S(S)\Big)^4$ *and* $d \in \mathbb{Z}$ *satisfy*

$$e_0 + e_1 + e_\lambda + e_\infty + d = 0 \in \mathcal{O}_S(S).$$

For any $m = (m_0, m_1, n_\lambda, m_\infty) \in \mathbb{N}^4$*, denote*

$$e' = (e_0 - m_0, e_1 - m_1, e_\lambda - m_\lambda, e_\infty - m_\infty) \quad \text{and} \quad d' = d + m_0 + m_1 + m_\lambda + m_\infty.$$

Then the natural extension $(L^{(e,d)}(D'), \nabla^{(e,d)})$ *(Lemma 1.17) with respect to* $D' = m_0D_0 + m_1D_1 + m_\lambda D_\lambda + m_\infty D_\infty$ *is isomorphic to* $(L^{(e',d')}, \nabla^{(e',d')})$.

Proof. By the uniqueness in Lemma 1.31, we only show that e' is coincide with the residues of $\nabla^{(e,d)}$ on $L^{(e,d)}(D')$. Let f be a local generator of $L^{(e,d)}$ around $t = 0$ and suppose $\nabla^{(e,d)}(f) = f \otimes \omega$. Then $x^{-m_0}f$ is the local generator of $L^{(e,d)}(D')$ and

$$\nabla^{(e,d)}(x^{-m_0}f) = x^{-m_0}f \otimes (-m_0 + \omega).$$

Thus $e_0 - m_0$ is the residue of $\nabla^{(e,d)}$ on $L^{(e,d)}(D')$ along D_0. Similarly, one checks that around the other three points. □

We classifies all parabolic de Rham line bundle over $(\mathbb{P}^1_S, D_S)/S$.

Lemma 1.33. *(1) Let* $w = (w_0, w_1, w_\lambda, w_\infty) \in \Big([0,1) \cap \mathbb{Q}\Big)^4$, $e = (e_0, e_1, e_\lambda, e_\infty) \in \Big(\mathcal{O}_S(S)\Big)^4$*, and* $d \in \mathbb{Z}$ *satisfy*

$$e_0 + e_1 + e_\lambda + e_\infty + d = 0 \in \mathcal{O}_S(S).$$

Then there exists a unique parabolic de Rham line bundle $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}_\alpha$ *such that*

- $d = \deg(L_0^{(e,d,w)})$*;*
- e_x *is the eigenvalue of residue of* $\nabla_0^{(e,d,w)}$ *along* D_x *for* $x \in \{0, 1, \lambda, \infty\}$*, and*
- w_x *is the parabolic weight of the parabolic line bundle* $\{L_\alpha^{(w,d,e)}\}_\alpha$ *along* D_x *for* $x \in \{0, 1, \lambda, \infty\}$*;*

(2) The underlying parabolic line bundle of $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}_\alpha$ *is isomorphic to*

$$\mathcal{O}_{\mathbb{P}^1_S}(d)(w_0D_0 + w_1D_1 + w_\lambda D_\lambda + w_\infty D_\infty).$$

The line bundle $L_\alpha^{(e,d,w)}$ *is of degree*

$$d' := d + [\alpha_0 + w_0] + [\alpha_1 + w_1] + [\alpha_\lambda + w_\lambda] + [\alpha_\infty + w_\infty]$$

and the eigenvalue of the residue of $\nabla_\alpha^{(e,d,w)}$ *along* D_x *is* $e'_x := e_x - [\alpha_x + w_x]$*. In other words,*

$$(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)}) \cong (L^{(e',d')}, \nabla^{(e',d')}).$$

(3) Any parabolic de Rham line bundle over $(\mathbb{P}^1_S, D_S)/S$ *is of the form given in (1).*

Proof. (1) **Existence:** For any $\alpha = (\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) \in \mathbb{Q}^4$, denote $D_\alpha = \alpha_0D_0 + \alpha_1D_1 + \alpha_\lambda D_\lambda + \alpha_\infty D_\infty$. We set

$$(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)}) := (L^{(e,d)}(D_{[\alpha+w]}), \nabla^{(e,d)})$$

where $(L^{(e,d)}, \nabla^{(e,d)})$ is the logarithmic de Rham line bundle given in Lemma 1.31 and the extension $(L^{(e,d)}(D_{[\alpha+w]}), \nabla^{(e,d)})$ is defined as in Lemma 1.17. The natural injections introduced in Lemma 1.17 makes the collection $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}$ forming a parabolic de Rham bundle, which satisfies our requirement clearly.

Uniqueness: Let $(L_\alpha, \nabla_\alpha)$ and $(L'_\alpha, \nabla'_\alpha)$ be two parabolic de Rham line bundles satisfy our requirement. Then we may identify (L_0, ∇_0) and (L'_0, ∇'_0) according Lemma 1.31. Thus

$$(L_\alpha, \nabla_\alpha) = (L_0(D_{[\alpha+w]}), \nabla_0) = (L'_0(D_{[\alpha+w]}), \nabla'_0) = (L'_\alpha, \nabla'_\alpha),$$

where the first and third equalities follow from the normalization and the definition of the weights of parabolic de Rham bundles.

(2) This follows the construction in (1).

(3) For any parabolic de Rham line bundle $\{(L_\alpha, \nabla_\alpha)\}_\alpha$, we have associated datum (w, e, d) consisting of weights, eigenvalues and degree. By the uniqueness in (1), the only thing we need to check is
$$e_0 + e_1 + e_\lambda + e_\infty + d = 0 \in \mathcal{O}_S(S).$$
And this follows Lemma 1.31. $\square$

We classifies all standard parabolic de Rham line bundle over $(\mathbb{P}^1_S, D_S)/S$.

Lemma 1.34. *Let $w = (w_0, w_1, w_\infty, w_\lambda)$ be an element in $\Big([0,1) \cap \mathbb{Q}^S\Big)^4$. Denote*
$$e_{w,x} \coloneqq \iota(w_x) \quad \text{and} \quad e_w \coloneqq (e_{w,0}, e_{w,1}, e_{w,\lambda}, e_{w,\infty}) \in \Big(O_S(S)\Big)^4.$$
For any integer d with
$$e_{w,0} + e_{w,1} + e_{w,\lambda} + e_{w,\infty} + d = 0 \in \mathcal{O}_S(S)$$
the parabolic de Rham line bundle $\{(L_\alpha^{(e_w,d,w)}, \nabla_\alpha^{(e_w,d,w)})\}_\alpha$ is standard. And any standard parabolic de Rham line bundle is of this form.

Proof. Fix a parabolic de Rham bundle $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}_\alpha$. Recall that it is standard if and only if the eigenvalue of the residue of the connection $\nabla_\alpha^{(e,d,w)}$ on the grading piece $V_\alpha/V_{\alpha-\varepsilon\delta_x}$ is $\iota(-\alpha_x)$ for any α. By the definition of the parabolic weights, for any α, the grading piece does not vanish if and only if $\alpha_x + w_x$ is an integer. On the other hand, according (2) in Lemma 1.33, $(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})$ along D_x is $e_x - [\alpha_x + w_x]$. Thus $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}_\alpha$ is standard if and only if for any α with $\alpha_x + w_x$ being an integer one has $e_x - [\alpha_x + w_x] = \iota(-\alpha_x) \in \mathcal{O}_S(S)$.

If $\{(L_\alpha^{(e,d,w)}, \nabla_\alpha^{(e,d,w)})\}_\alpha$ is standard, then taking α such that $\alpha_x = -w_x$ one gets
$$e_x = \iota(w_x) = e_{w,x} \in \mathcal{O}_S(S).$$
In other words, it is of the form given in the Lemma.

Since $e_{w,x} = \iota(w_x)$, for any α with $\alpha_x + w_x$, one has $e_{w,x} - [\alpha_x + w_x] = e_{w,x} - \iota(\alpha_x) - \iota(w_x) = \iota(-\alpha_x)\mathcal{O}_S(S)$. Thus $\{(L_\alpha^{(e_w,d,w)}, \nabla_\alpha^{(e_w,d,w)})\}_\alpha$ is indeed standard. $\square$

Corollary 1.35. (1) *A standard parabolic de Rham line bundle is uniquely determined by its weights and degree.*

(2) *Suppose* $\mathrm{char}(\mathcal{O}_S(S)) = 0$. *Then a standard parabolic de Rham line bundle is uniquely determined by its weights. In this case the degree of the underlying parabolic line bundle is zero.*

Some parabolic de Rham bundles of rank 2. Denote by $M_{\mathrm{dR}\,\lambda}^{\frac{1}{2}}(S)$ the set of all isomorphic classes of rank-2 stable standard parabolic de Rham bundles (V, ∇) of degree zero on $(\mathbb{P}^1_S, D_S)/S$ with all parabolic weights being zero at $\{0, 1, \lambda\}$ and with all parabolic weights being $1/2$ at ∞.

Proposition 1.36. *Let (V, ∇) be a parabolic de Rham bundle in $M_{\mathrm{dR}\,\lambda}^{\frac{1}{2}}(S)$. Then*

(1) *the parabolic de Rham bundle* (V,∇) *has the form*

$$(\mathcal{L}\oplus\mathcal{L}^{-1},\nabla).$$

where $\mathcal{L}=\mathcal{O}(\frac{1}{2}(\infty))$.

(2) *if we take the* parabolic Hodge line bundle *as* $\mathcal{L}$, *then the associated graded parabolic Higgs field is nonzero and is of form*

$$\theta\colon\mathcal{L}\to\mathcal{L}^{-1}\otimes\Omega^1_{X/S}(\log D).$$

In particular, the graded parabolic Higgs bundle $(\mathcal{L}\oplus\mathcal{L}^{-1},\theta)$ *is stable and is of degree zero.*

Proof. By tensoring with $\mathcal{O}_{\mathbb{P}^1_S}(-\frac{1}{2}(\infty))$, one gets a parabolic vector bundle $V(-\frac{1}{2}(\infty))$ of degree -1 and with trivial parabolic weights. Thus it is usual vector bundle of degree -1. We decompose $V(-\frac{1}{2}(\infty))$ into direct sum of $\mathcal{O}_{\mathbb{P}^1_S}(d)$ and $\mathcal{O}_{\mathbb{P}^1_S}(-d-1)$ for some $d\geq 0$. Denote $\mathcal{L}=O(\frac{2d+1}{2}(\infty))$. Then one has

$$V=\mathcal{L}\oplus\mathcal{L}^{-1}.$$

In the following, we use the stability to show $d=0$. Since (V,∇) is stable, one has $\nabla(\mathcal{L})\not\subset\mathcal{L}\otimes\Omega^1_{\mathbb{P}^1_S}(\log D)$. In other words, the graded parabolic Higgs field, which is defined as the composition of following maps

$$\mathcal{L}\xrightarrow{\nabla}V\otimes\Omega^1_{\mathbb{P}^1_S}(\log D)\twoheadrightarrow\mathcal{L}^{-1}\otimes\Omega^1_{\mathbb{P}^1_S}(\log D)$$

is nonzero. Compute the degree on both sides of the nonzero map, one gets

$$\frac{2d+1}{2}\leq-\frac{2d+1}{2}+2.$$

Thus $d\leq 0$. The lemma follows. □

Some graded parabolic Higgs bundles of rank 2. Denote by $\mathrm{HIG}^{\mathrm{gr}\,\frac{1}{2}}_{\lambda}(S)$ the set of all isomorphic classes of rank-2 stable graded parabolic Higgs bundles (E,θ) of degree zero on $(\mathbb{P}^1_S,D_S)/S$ with all parabolic weights being zero at $\{0,1,\lambda\}$ and with all parabolic weights being $1/2$ at ∞.

Proposition 1.37. *Let* (E,θ) *be a graded parabolic Higgs bundle in* $\mathrm{HIG}^{\mathrm{gr}\,\frac{1}{2}}_{\lambda}(S)$. *Then*

$$V=\mathcal{L}\oplus\mathcal{L}^{-1},$$

where $\mathcal{L}=\mathcal{O}(\frac{1}{2}(\infty))$ *and the parabolic Higgs field is nonzero and is of form*

$$\theta\colon\mathcal{L}\to\mathcal{L}^{-1}\otimes\Omega^1_{X/S}(\log D).$$

Proof. By tensoring with $\mathcal{O}_{\mathbb{P}^1_S}(-\frac{1}{2}(\infty))$, one gets a parabolic vector bundle $V(-\frac{1}{2}(\infty))$ of degree -1 and with trivial parabolic weights. Thus it is usual vector bundle of degree -1. We decompose $V(-\frac{1}{2}(\infty))$ into direct sum of $\mathcal{O}_{\mathbb{P}^1_S}(d)$ and $\mathcal{O}_{\mathbb{P}^1_S}(-d-1)$ for some $d\geq 0$. Denote $\mathcal{L}=O(\frac{2d+1}{2}(\infty))$. Then one has

$$V=\mathcal{L}\oplus\mathcal{L}^{-1}.$$

By the stability, $\theta(\mathcal{L})\not\subset\mathcal{L}\otimes\Omega^1_{\mathbb{P}^1_S}(\log D)$. Thus graded parabolic Higgs field nonzero and is of form

$$\theta\colon\mathcal{L}\to\mathcal{L}^{-1}\otimes\Omega^1_{\mathbb{P}^1_S}(\log D).$$

□

Corollary 1.38. *Any parabolic Higgs bundle* $(E,\theta)\in \mathrm{HIG}^{\mathrm{gr}\,\frac{1}{2}}_{\lambda}(S)$ *is uniquely determined by* $(\theta)_0\in\mathbb{P}^1_S(S)$*, the zero of the Higgs field* θ*. One has a natural bijection induced by taking zeros*

$$\mathrm{HIG}^{\mathrm{gr}\,\frac{1}{2}}_{\lambda}(S)\xrightarrow[(E,\theta)\mapsto(\theta)_0]{1:1}\mathbb{P}^1_S(S).$$

2. Parabolic Fontaine-Faltings Modules and parabolic Higgs-de Rham Flows

In order to facilitate use, we introduce the definition of Parabolic Fontaine-Faltings Modules and parabolic Higgs-de Rham Flows. One can also see [KS20], for original definitions and more detailed studies.

2.1. Introduction to Fontaine-Faltings modules.

In this subsection, we set $S=\mathrm{Spf}(W)$. Then $\mathcal{Y}$ is a proper smooth formal scheme over S and $D_{\mathcal{Y}}$ be a reduced relative simple normal crossing S-divisor in $\mathcal{Y}$.

Faltings' tilde functor $\widetilde{(\cdot)}$. We recall a functor

$$\widetilde{(\cdot)}\colon \mathrm{MCF}(\mathcal{Y},D_{\mathcal{Y}})\to\widetilde{\mathrm{MIC}}(\mathcal{Y},D_{\mathcal{Y}})$$

which was introduced by Faltings in [Fal89]. We call it *Faltings' tilde functor* and denote it by $\widetilde{(\cdot)}$. For an object (V,∇,Fil) in $\mathrm{MCF}_a(\mathcal{Y},D_{\mathcal{Y}})$, denote by $\widetilde{V}$ the quotient $\bigoplus_{i=0}^{a}\mathrm{Fil}^i V/\sim$ with $x\sim py$ for any local section $x\in\mathrm{Fil}^i V$ and y the image of x under the natural inclusion $\mathrm{Fil}^i V\hookrightarrow\mathrm{Fil}^{i-1}V$. Then connection ∇ naturally induces a p-connection on $\widetilde{\nabla}$ on $\widetilde{V}$. We use $\widetilde{(V,\nabla,\mathrm{Fil})}$ stands for the pair $(\widetilde{V},\widetilde{\nabla})$. The morphisms under the functor are defined in the obvious way.

If V is p-torsion free, then the tilde functor can be easily described as follows:

$$(\widetilde{V},\widetilde{\nabla}):=\left(\sum_{i\in\mathbb{Z}}\frac{1}{p^\ell}\mathrm{Fil}^i V,p\nabla\right)\subset(V,p\nabla)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

The Frobenius pullback functor $\mathcal{F}$. Faltings also constructed a functor [Fal89]

$$\mathcal{F}\colon\widetilde{\mathrm{MIC}}_a(\mathcal{Y},D_{\mathcal{Y}})\to\mathrm{MIC}(\mathcal{Y},D_{\mathcal{Y}})$$

where $\widetilde{\mathrm{MIC}}_a(\mathcal{Y},D_{\mathcal{Y}})$ is the full subcategory of $\widetilde{\mathrm{MIC}}(\mathcal{Y},D_{\mathcal{Y}})$ consisting of the essential image of $\mathrm{MCF}_a(\mathcal{Y},D_{\mathcal{Y}})$ under Faltings' tilde functor for $a\leq p-2$.[1] We recall the definition as follows.

For small affine open subset $\mathcal{U}$ of $\mathcal{Y}$, there exists endomorphism $F_{\mathcal{U}}$ on $\mathcal{U}$ which lifts the absolute Frobenius on $\mathcal{U}_k$ and is compatible with the Frobenius map F_S on $S=\mathrm{Spec}(W(k))$. Let $(\widetilde{V},\widetilde{\nabla})$ be an object in $\widetilde{MIC}_a(\mathcal{Y},D_{\mathcal{Y}})$. Locally on $\mathcal{U}$, applying the functor $F^*_{\mathcal{U}}$, we get a de Rham bundle over $\mathcal{U}$

$$F^*_{\mathcal{U}}(\widetilde{V}\mid_{\mathcal{U}},\widetilde{\nabla}\mid_{\mathcal{U}})$$

where the underlying bundle is just pullback of $\widetilde{V}\mid_{\mathcal{U}}$ along $F^*_{\mathcal{U}}$ and the connection is the pullback of $\widetilde{\nabla}\mid_{\mathcal{U}}$ along $F^*_{\mathcal{U}}$ dividing p. By Taylor formula, up to a canonical isomorphism, it does not depend on the choice of $F_{\mathcal{U}}$ in case $a\leq p-2$. In particular, on the overlap of two

[1]The condition here is essential, which ensure the functor is globally well-defined.

small affine open subsets, there is an canonical isomorphism of two logarithmic de Rham bundles. By gluing via those isomorphisms, one gets a logarithmic de Rham bundle over $(\mathcal{Y}, D_{\mathcal{Y}})$, we denote it by

$$\mathcal{F}(\widetilde{V}, \widetilde{\nabla}).$$

The morphisms under the functor are defined in the obvious way.

A logarithmic Fontaine-Faltings module. A logarithmic Fontaine-Faltings module[2] over $(\mathcal{Y}, D_{\mathcal{Y}})$ is a quadruple $(V, \nabla, \mathrm{Fil}, \Phi)$ where $(V, \nabla, \mathrm{Fil})$ is a logarithmic de Rham bundle over $(\mathcal{Y}, D_{\mathcal{Y}})$ and

$$\Phi \colon \mathcal{F}(\widetilde{V, \nabla, \mathrm{Fil}}) \cong (V, \nabla)$$

is an isomorphism of logarithmic de Rham bundles over $(\mathcal{Y}, D_{\mathcal{Y}})$. We call Φ the *Frobenius structure* in the Fontaine-Faltings module.

Local logarithmic Fontaine-Faltings modules. Let $Y = \mathrm{Spec}(R)$ be an affine W-scheme with an étale map

$$W[T_1, T_2, \cdots, T_d] \to R,$$

over W,let D be the divisor in Y defined by $T_1 \cdots T_d = 0$,and let U be the complement of D in Y. Therefore, U is a small affine scheme. In this context, we say that (Y, D) is *log small.* Denote by $\mathcal{Y}$ the p-adic formal completion of Y along the special fiber Y_1 and by $\mathcal{Y}_K$ the rigid-analytic space associated to $\mathcal{Y}$,which is an open subset of Y_K^{an}. We construct spaces D_K,$\mathcal{D}$,$\mathcal{D}_K$,U_K, $\mathcal{U}$ and $\mathcal{U}_K$ exactly analogously to those for Y. Denote $\mathcal{Y}_K^\circ := \mathcal{Y}_K - \mathcal{D}_K$. Denote by $\widehat{R}$ the p-adic completion of R, so $\mathcal{Y} = \mathrm{Spf}(\widehat{R})$.

Choose $\Phi : \widehat{R} \to \widehat{R}$ a lifting of the absolute Frobenius on R/pR such that $\Phi(T_i) = unit \cdot T_i^p$. Recall a *logarithmic Fontaine-Faltings module* over the p-adic formal completion $(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}})$ of (Y, D) with Hodge-Tate weights in $[a, b]$ is a quadruple $(V, \nabla, \mathrm{Fil}, \varphi)$, where

- (V, ∇) is a finitely generated locally free[3] de Rham $\widehat{R}$-module with logarithmic poles along $T_1 \ldots T_d = 0$;
- Fil is a Hodge filtration on (V, ∇) of level in $[a, b]$;
- $\widetilde{V}$ is the quotient $\bigoplus_{i=a}^{b} \mathrm{Fil}^i / \sim$ with $x \sim py$ for $x \in \mathrm{Fil}^i V$ with y being the image of x under the natural inclusion $\mathrm{Fil}^i V \hookrightarrow \mathrm{Fil}^{i-1} V$;
- φ is an $\widehat{R}$-linear isomorphism

$$\varphi : \widetilde{V} \otimes_{\Phi} \widehat{R} \longrightarrow V,$$

- The relative Frobenius φ is horizontal with respect to the connections.

In particular, a logarithmic Fontaine-Faltings module may be considered as a filtered logarithmic F-crystal in finite, locally free modules. Denote by $\mathcal{MF}_{[a,b]}^{\nabla,\Phi}((\mathcal{Y}, \mathcal{D}_{\mathcal{Y}})/W)$ the category of logarithmic Fontaine-Faltings modules over $(\mathcal{Y}, \mathcal{D}_{\mathcal{Y}})$ with Hodge-Tate weights in $[a, b]$. For the rest of what follows, we assume that $b - a \leq p - 2$. (It will follow that the resulting category is independent of the choice of Φ, exactly as in the non-logarithmic case.)

[2]We note that the definition is slightly different with the original [Fal89] in textual representation, but they are essentially the same, this can be get from [SYZ22, Section 2.3].

[3]For our application, we only consider the locally free case at here. We note that the general definition does not require such kind condition.

Decomposition of the ring $\mathbb{Z}_{p^f}\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^f}$. Take an generator $\zeta\in\mathbb{Z}_{p^f}$ over $\mathbb{Z}_p$ (in other words, $\mathbb{Z}_{p^f}=\mathbb{Z}_p[\zeta]$) and denote

$$e_i := \frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{f-1}\Big(1\otimes\zeta-\zeta^{\sigma^j}\otimes 1\Big)}{\prod\limits_{\substack{j=0\\ j\neq i}}^{f-1}\Big(\zeta^{\sigma^i}\otimes 1-\zeta^{\sigma^j}\otimes 1\Big)}\in\mathbb{Z}_{p^f}\otimes_{\mathbb{Z}_p}\mathbb{Z}_{p^f}. \tag{2.1}$$

Lemma 2.1. (1) *the decomposition $1=e_0+e_1+\cdots+e_{f-1}$ is an idempotent one,*

(2) $(\sigma\otimes\mathrm{id})(e_i)=e_{i+1}$,

(3) $(1\otimes\zeta)\cdot e_i=(\zeta^{\sigma^i}\otimes 1)\cdot e_i$.

Endomorphism structure on logarithmic Fontaine-Faltings module. Let $(V,\nabla,\mathrm{Fil},\varphi)$ be a logarithmic Fontaine-Faltings module. Recall that a $\mathbb{Z}_{p^f}$*-endomorphism structure* is a ring homomorphism

$$\tau\colon\mathbb{Z}_{p^f}\to\mathrm{End}\big((V,\nabla,\mathrm{Fil},\varphi)\big).$$

In this case, V are endow with both $\widehat{R}$-module structure and $\mathbb{Z}_{p^f}$-module structure. Suppose $\mathbb{F}_{p^f}\subseteq k$. Then one has canonical embedding $\mathbb{Z}_{p^f}\subset W(k)\subset\widehat{R}$. Thus one gets two $\mathbb{Z}_{p^f}$-module structures on V. It is natural to consider the sub-$\widehat{R}$-modules, for all $i\geq 0$

$$V_i := V^{\tau=\sigma^i} := \{v\in V\mid \tau(a)(v)=\sigma^i(a)v,\text{for all} a\in\mathbb{Z}_{p^f}\},$$

where $\sigma\colon\mathbb{Z}_{p^f}\to\mathbb{Z}_{p^f}$ is the lifting of the absolute Frobenius map on $\mathbb{F}_{p^f}$. Clearly $V_i=V_{i+f}$ for any $i\geq 0$ because of $\sigma^f=\mathrm{id}$.

Lemma 2.2. *Let $(V,\nabla,\mathrm{Fil},\varphi)$ be a logarithmic Fontaine-Faltings module over $(\mathcal{Y},\mathcal{Z})$ endowed with a $\mathbb{Z}_{p^f}$-endomorphism structure*

$$\tau\colon\mathbb{Z}_{p^f}\to\mathrm{End}\big((V,\nabla,\mathrm{Fil},\varphi)\big)$$

Assume $\mathbb{F}_{p^f}\subseteq k$. Then

(1) *the connection and filtration can be restricted on V_i for all $i\geq 0$, and there is a decomposition of filtered logarithmic de Rham module*

$$(V,\nabla,\mathrm{Fil})=(V_0,\nabla_0,\mathrm{Fil}_0)\oplus(V_1,\nabla_1,\mathrm{Fil}_1)\oplus\cdots\oplus(V_{f-1},\nabla_{f-1},\mathrm{Fil}_{f-1}).$$

(2) *The restriction φ_i of φ on $\widehat{R}\otimes_\Phi\widetilde{V}_i$ gives an isomorphism of de Rham modules*

$$\varphi_i\colon\widehat{R}\otimes_\Phi\widetilde{V}_i\xrightarrow{\simeq}V_{i+1}.$$

Proof. (1) Since ∇ is $\mathbb{Z}_{p^f}$-linear and Fil consists of sub-$\mathbb{Z}_{p^f}$-modules, they both can be restricted on V_i. We only need to show

$$V=V_0\oplus V_1\oplus\cdots\oplus V_{f-1}.$$

Take an generator $\zeta\in\mathbb{Z}_{p^f}$ over $\mathbb{Z}_p$ (in other words, $\mathbb{Z}_{p^f}=\mathbb{Z}_p[\zeta]$) and denote

$$e_i := \frac{\prod\limits_{\substack{j=0\\ j\neq i}}^{f-1}\Big(\tau(\zeta)-\zeta^{\sigma^j}\Big)}{\prod\limits_{\substack{j=0\\ j\neq i}}^{f-1}\Big(\zeta^{\sigma^i}-\zeta^{\sigma^j}\Big)}\in\mathrm{End}_{\widehat{R}}(V).$$

It is easy to check that $\mathrm{id} = e_0 + e_1 + \cdots + e_{f-1}$ is an idempotent decomposition, and that

$$\tau(a)e_i = a^{\sigma^i} e_i, \qquad \text{for any} a \in \mathbb{Z}_{p^f}.$$

In particular, it induces a decomposition

$$V = e_0 V \oplus e_1 V \oplus \cdots \oplus e_{f-1} V.$$

and $V_i = e_i V$ for all $i = 0, \cdots, f-1$.

(2). Since the endomorphism structure preserves the Frobenius structure φ, we have $\tau(a) \circ \varphi = \varphi \circ (\mathrm{id} \otimes \tau(a))$ for any $a \in \mathbb{Z}_{p^f}$. Thus, for any $v_i \in \mathrm{Fil}^\ell V_i$ and any $a \in \mathbb{Z}_{p^f}$, we have

$$\tau(a)\big(\varphi(1 \otimes_\Phi [v_i])\big) = \varphi \circ (\mathrm{id} \otimes_\Phi \tau(a))(1 \otimes [v_i]) = \varphi(1 \otimes_\Phi a^{\sigma^i}[v_i]) = a^{\sigma^{i+1}} \cdot \varphi(1 \otimes [v_i]).$$

where $[v_i]$ is the image of v_i under the natural morphism $\mathrm{Fil}^\ell V_i \to \widetilde{V}_i$. In other words, $\varphi(1 \otimes_\Phi [v_i]) \in V_{i+1}$. Then (2) follows. $\square$

Constant Fontaine-Faltings modules.

Definition 2.3. an *admissible filtered φ-module of rank r over W* is a triple $(V, \mathrm{Fil}, \varphi)$, where

- V is a finite generated free W-module of rank r,
- Fil is a filtration of direct summands of V of form (for some $a, b \in \mathbb{Z}$)

$$V = \mathrm{Fil}^a V \supseteq \mathrm{Fil}^{a+1} V \supseteq \cdots \supseteq \mathrm{Fil}^b V \supseteq \mathrm{Fil}^{b+1} V = 0$$

- $\varphi\colon \widetilde{V} \to V$ is a σ-semilinear isomorphism (sometimes, we also use φ to stand for the induced W-linear isomorphism $W \otimes_\sigma \widetilde{V} \to V$) where

$$\widetilde{V} = \sum_{\ell=a}^{b} \frac{1}{p^\ell} \mathrm{Fil}^\ell V \subset V_K := V \otimes_W K.$$

The jumping indices of the filtration are called the *levels* of $(V, \mathrm{Fil}, \varphi)$. Denote by $\mathcal{MF}^\varphi(W)$ the category of all admissible filtered φ-module over W. Denote by $\mathcal{MF}^\varphi_{[a,b]}(W)$ the full subcategory of all admissible filtered φ-module over W with levels contained in $[a, b]$. An object in $\mathcal{MF}^\varphi_{[0,p-2]}(W)$ is called a *constant Fontaine-Faltings module*.

Remark 2.4.

1. The third term is equivalent to give a σ-semilinear isomorphism $\varphi_K\colon V_K \to V_K$ such that it is admissible (or, compatible with the filtration) in the sense that its restriction induces isomorphism from $\widetilde{V}$ to V. The admissible condition here is also called strong p-divisibility condition in some other literature.
2. The pair (V_K, φ_K) forms an F-isocrystal over k. One has natural functor

$$\mathcal{MF}^\varphi(W) \to \text{F-Isoc}(k).$$

3. Let Fil_K be the induced filtration on V_K from Fil. Then the triple $(V_K, \mathrm{Fil}_K, \varphi_K)$ is an admissible filtered φ-module over K in the sense [FO22].
4. If the index a appeared in the filtration is zero. Then V is contained in $\widetilde{V}$. The restriction of φ gives a σ-semilinear injection $\varphi_V\colon V \to V$. In other words, the pair (V, φ_V) forms an F-crystal over k.

Example 2.5. For each $w \in K^\times$, it can be write uniquely as $w = p^r u$, for some $r \in \mathbb{Z}$ and $u \in W^\times$. We construct an admissible filtered φ-module $M_w = (V, \mathrm{Fil}, \varphi)$ of rank 1 over W as following: Let V be a free W-module of rank 1 with a generator v. The filtration Fil is given by

$$V = \mathrm{Fil}^r V \supset \mathrm{Fil}^{r+1} V = 0,$$

and the Frobenius structure is given by $\varphi(1 \otimes \frac{v}{p^r}) = uv$. Actually, all admissible filtered φ-modules of rank 1 over W are of this form. For any two $w, w' \in K^\times$, $M_w \simeq M_{w'}$ if and only if there exists some $\xi \in W^\times$ such that $w/w' = \sigma(\xi)/\xi$. In particular, the p-adic values of w and w' are of the same.

Definition 2.6. An *admissible filtered φ-module with $\mathbb{Z}_{p^f}$-endomorphism structure of rank r over* W is a tuple $(V, \mathrm{Fil}, \varphi, \tau)$, where $(V, \mathrm{Fil}, \varphi) \in \mathcal{MF}^\varphi(W)$ is of rank rf and τ is a ring homomorphism (which is called a $\mathbb{Z}_{p^f}$-endomorphism structure on $(V, \mathrm{Fil}, \varphi)$)

$$\tau \colon \mathbb{Z}_{p^f} \to \mathrm{End}((V, \mathrm{Fil}, \varphi)).$$

Denote by $\mathcal{MF}^\varphi(W)_{\mathbb{Z}_{p^f}}$ the category of all admissible filtered φ-modules with $\mathbb{Z}_{p^f}$-endomorphism structures over W. Similarly we define the full subcategory $\mathcal{MF}^\varphi_{[a,b]}(W)_{\mathbb{Z}_{p^f}}$. The objects in $\mathcal{MF}^\varphi_{[0,p-2]}(W)_{\mathbb{Z}_{p^f}}$ are called *constant Fontaine-Faltings modules with $\mathbb{Z}_{p^f}$-endomorphism structures.*

Definition 2.7. The triple (V_K, φ_K, τ) forms an F-isocrystal over k with coefficient $\mathbb{Q}_{p^f}$. Denote

$$P((V, \mathrm{Fil}, \varphi, \tau), t) := P((V_K, \varphi_K, \tau), t), \quad \textit{and} \quad \mathrm{tr}(V, \mathrm{Fil}, \varphi, \tau) := \mathrm{tr}(V_K, \varphi_K, \tau).$$

Call them the *characteristic polynomial and trace of* $(V, \mathrm{Fil}, \varphi, \tau)$.

Remark 2.8. Let $(V, \mathrm{Fil}, \varphi, \tau) \in \mathcal{MF}^\varphi(W)_{\mathbb{Z}_{p^f}}$ be of rank r. Then on V there are two $\mathbb{Z}_{p^f}$-moudle structures, the first is the nature one and the second is given by τ. So we may view V as a $W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f}$-module. By the existence of the Frobenius structure, one actually shows that V is locally free over $W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f}$ of rank r. Since the filtration Fil is direct summands as W-modules and is preserved by the action of $\mathbb{Z}_{p^f}$ via τ, it consists of direct summands as $W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f}$-modules.

Example 2.9. Suppose $\mathbb{F}_{p^f} \subset k$. For any $w \in K \otimes \mathbb{Q}_{p^f}$ it can be uniquely written as

$$w = \sum_{i=0}^{f-1} (p^{r_i} u_i \otimes 1) \cdot e_i,$$

where $r_i \in \mathbb{Z}$ and $u_i \in W^\times$ for all $i = 0, 1, \cdots, f-1$. We construct a rank 1 object $M_w = (V, \mathrm{Fil}, \varphi, \tau) \in \mathcal{MF}^\varphi(W)_{\mathbb{Z}_{p^f}}$ as follows: The V is free of rank 1 over $W \otimes \mathbb{Z}_{p^f}$ with a basis v, denote $v_i = e_i v$ and $V_i = W v_i$. The action τ of $\mathbb{Z}_{p^f}$ given by the second factor. In other words, $\tau(a)(v_i) = (1 \otimes a) \cdot v$. On V, we set the Filtration Fil via

$$(V, \mathrm{Fil}) = (V_0, \mathrm{Fil}_0) \oplus (V_1, \mathrm{Fil}_1) \oplus \cdots \oplus (V_{f-1}, \mathrm{Fil}_{f-1}),$$

where the filtration Fil_i is defined by $(r_f := r_0)$

$$V_i = \mathrm{Fil}_i^{r_{i+1}} V_i \supset \mathrm{Fil}_i^{r_{i+1}+1} V_i = 0 \quad \text{for all} i = 0, 1, \cdots, f-1.$$

The Frobenius structure is given by ($u_f := u_0$ and $v_f := v_0$)

$$\varphi(\frac{v_i}{p^{r_{i+1}}}) = u_{i+1}v_{i+1} \quad \text{for all} i = 0, 1, \cdots, f-1.$$

In this case, the associated F-isocrystal over k with coefficient in $\mathbb{Q}_{p^f}$ can be simply represented as $(K \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^f} \cdot v, F)$ with Frobenius given by

$$F(v) = w \cdot v.$$

Definition 2.10. The constant Fontaine-Faltings module corresponding to $w = p$ is called *cyclotomic Fontaine-Faltings module*.

By direct computation, one easily checks the following result.

Lemma 2.11. *Suppose $\mathbb{F}_{p^f} \subset k$. Then all objects of rank 1 in $\mathcal{MF}^{\varphi}(W)_{\mathbb{Z}_{p^f}}$ are of the form M_w for some $w \in (K \otimes \mathbb{Q}_{p^f})^\times$. And one has*

$$M_w \otimes M_{w'} \cong M_{w \cdot w'}.$$

Corollary 2.12. *Let k be a finite field and denote by k' the quadratic field extension of k. Let $\mathcal{E} \in \mathcal{MF}^{\varphi}_{[0,0]}(W(k))_{\mathbb{Z}_{p^f}}$ be of rank 1. Then there exists a rank 1 object in $\mathcal{E}' = \mathcal{MF}_{[0,0]}(W(k'))_{\mathbb{Z}_{p^f}}$ such that*

$$\mathcal{E} = \mathcal{E}' \otimes \mathcal{E}' \in \mathcal{MF}_{[0,0]}(W(k'))_{\mathbb{Z}_{p^f}}.$$

Proof. Let $\mathcal{E}$ corresponds to $w = \sum_{i=0}^{f-1}(u_i \otimes 1) \cdot e_i$, where $u_i \in W(k)^\times$. Since k' is the quadratic extension of k and u_i is a unit in W, one has $\sqrt{u_i} \in W(k')$. Denote by $\mathcal{E}'$ the object corresponding to $w' := \sum_{i=0}^{f-1}(\sqrt{u_i} \otimes 1) \cdot e_i$. Then the corollary follows Lemma 2.11. $\square$

Definition 2.13. The M_w is called of finite order, if there exists some d such that

$$\underbrace{M_w \otimes M_w \otimes \cdots \otimes M_w}_{d} \cong M_1.$$

The smallest integer d satisfying this condition is called the order of M_w.

Corollary 2.14. *Let $\mathcal{E} \in \mathcal{MF}^{\varphi}_{[0,0]}(W(k))_{\mathbb{Z}_{p^f}}$ be of rank 1. Then there exists a rank 1 object in $\mathcal{E}' = \mathcal{MF}_{[0,0]}(W(k))_{\mathbb{Z}_{p^f}}$ such that*

$$\mathcal{E} \otimes \mathcal{E}' \otimes \mathcal{E}'$$

is of finite order. The order divides $\#k - 1$.

Constant Fontaine-Faltings modules over a base. Let Y be a smooth scheme over W with geometrically connected generic fiber. The pullback operator along the structure morphism $\mathcal{Y} \to \mathrm{Spf}(W)$ induces a natural functor

$$\mathcal{MF}^{\varphi}_{[0,p-2]}(W) \to \mathcal{MF}_{[0,p-2]}(\mathcal{Y}/W).$$

We also call the essential image of this pullback functor *constant Fontaine-Faltings modules*. Similarly for those with endomorphism structures.

Lemma 2.15. *If Y is projective over W, then a Fontaine-Faltings module is constant if and only if its underlying de Rham bundle is direct sum of copies of the trivial de Rham line bundle $(\mathcal{O}_{\mathcal{Y}}, \mathrm{d})$.*

Restriction of Fontaine-Faltings modules on points. Let Y be a smooth scheme over W with geometrically connected generic fiber. Let x be a closed point in Y_k. Denote by k_x the residue field at x, which is a finite extension of k. By the smoothness of Y, there exists a $W(k_x)$-point $\widehat{x}$ on Y, which lifts x. By restriction on $\widehat{x}$, one gets a functor

$$\mathcal{MF}_{[0,p-2]}(\mathcal{Y}/W) \to \mathcal{MF}^{\varphi}_{[0,p-2]}(W(k_x)).$$

In the following, we describe this functor clearly and show that it does not depend on the choice of the lifting $\widehat{x}$.

Let $(V, \nabla, \mathrm{Fil}, \varphi) \in \mathcal{MF}_{[0,p-2]}(\mathcal{Y}/W)$. By shrinking Y, we may assume $Y = \mathrm{Spec}(R)$ is small affine and contains $\widehat{x}$ such that there exists a Frobenius lifting $\Phi\colon \widehat{R} \to \widehat{R}$ preserves the W-point $\widehat{x}$. In other words, we have commutative diagram

$$\begin{tikzcd} \widehat{R} \arrow[r, "\Phi"] \arrow[d, two heads, "\widehat{x}"'] & \widehat{R} \arrow[d, two heads, "\widehat{x}"] \\ W \arrow[r, "\sigma"] & W \end{tikzcd}$$

In this case, V is a finite generated free R-module and Fil is a filtration of direct summands of V. Denote

$$V_{\widehat{x}} := V \otimes_{\widehat{x}} W \quad \text{and} \quad \mathrm{Fil}^{\ell}_{\widehat{x}} V_{\widehat{x}} := \mathrm{Fil}^{\ell} V \otimes_{\widehat{x}} W, \text{for all} \ell.$$

Clearly, one has $\widetilde{V_{\widehat{x}}} = \widetilde{V} \otimes_{\widehat{x}} W$. Taking $\otimes_{\widehat{x}} W$ on the Frobenius structure $\varphi\colon \widehat{R} \otimes_{\Phi} \widetilde{V} \to \widetilde{V}$, one gets isomorphism

$$\varphi_{\widehat{x}}\colon W \otimes_{\sigma} \widetilde{V_{\widehat{x}}} \to V_{\widehat{x}}.$$

Thus one gets a constant Fontaine-Faltings module over $W(k_x)$

$$\left(V_{\widehat{x}}, \mathrm{Fil}_{\widehat{x}}, \varphi_{\widehat{x}}\right) \in \mathcal{MF}^{\varphi}_{[0,p-2]}(W(k_x)).$$

Remark 2.16. Up to a canonical isomorphism, this constant Fontaine-Faltings module does not depend on the choice of Φ. Because up to a canonical equivalent the category of Fontaine-Faltings module do not. The deep-seated reason is that a Fontaine-Faltings module corresponds to an F-crystal after forgetting the Hodge filtration.

Lemma 2.17. *Let $\widehat{x}$ and $\widehat{x}'$ be two lifting of x. Then there exists a canonical isomorphism*

$$\left(V_{\widehat{x}}, \mathrm{Fil}_{\widehat{x}}, \varphi_{\widehat{x}}\right) \simeq \left(V_{\widehat{x}'}, \mathrm{Fil}_{\widehat{x}'}, \varphi_{\widehat{x}'}\right).$$

In other words, the isomorphic class of $\left(V_{\widehat{x}}, \mathrm{Fil}_{\widehat{x}}, \varphi_{\widehat{x}}\right)$ does not depend on the choice of the lifting $\widehat{x}$.

Proof. By the smoothness of Y, after shrinking Y, we may find an automorphism of $\mathcal{Y}$

$$\widehat{\mathrm{id}}\colon \mathcal{Y} \to \mathcal{Y}$$

which lifts the identity map on Y_k such that it sends $\widehat{x}$ to $\widehat{x}'$. By using Taylor formula, one gets an canonical equivalent functor

$$\widehat{\mathrm{id}}^*\colon \mathcal{MF}_{[0,p-2]}(\mathcal{Y}/W) \to \mathcal{MF}_{[0,p-2]}(\mathcal{Y}/W).$$

Choose a Frobenius lifting Φ that preserves $\widehat{x}$. Denote $\Phi' := \widehat{\mathrm{id}}^{-1} \circ \Phi \circ \widehat{\mathrm{id}}$, which is a Frobenius lifting that preserves $\widehat{x}'$. Now one has the following commutative diagram of functors

$$\begin{tikzcd}
\mathcal{MF}^{\Phi}_{[0,p-2]}(\mathcal{Y}/W) \arrow[r, "\widehat{\mathrm{id}}^*"] \arrow[d] & \mathcal{MF}^{\Phi'}_{[0,p-2]}(\mathcal{Y}/W) \arrow[d] \\
\mathcal{MF}^{\varphi}_{[0,p-2]}(W(k_x)) \arrow[r, equal] & \mathcal{MF}^{\varphi}_{[0,p-2]}(W(k_x))
\end{tikzcd}$$

□

2.2. **Introduction to Higgs-de Rham flows.** Lan-Sheng-Zuo have introduced the notion of (periodic) Higgs-de Rham flow over a smooth log scheme (X, D) over $W(k)$ and have shown that the category of f-periodic Higgs bundles is equivalent to the category of log Fontaine-Faltings modules over (X, D) endowed with an $\mathbb{Z}_{p^f}$-endomorphism structure.

Lately, Sun-Yang-Zuo introduced projective Higgs-de Rham flow, which generalized the notion of Higgs-de Rham flow. Actually, it is just the parabolic Higgs-de Rham flow with only one parabolic weight along each component of the boundary divisor.

Functor $\overline{\mathrm{Gr}}$. Denote by $\mathcal{H}_a(Y, D_Y)$ the category of tuples $(E, \theta, V, \nabla, \mathrm{Fil}, \psi)$, where

- (E, θ) is a graded logarithmic Higgs bundle [4] over Y;
- $(V, \nabla, \mathrm{Fil})_a \in \mathrm{MCF}(Y, D_Y)$;
- and $\psi : \mathrm{Gr}(V, \nabla, \mathrm{Fil}) \cong (E, \theta)$ is an isomorphism of logarithmic Higgs bundles over (Y, D_Y).

There is a natural functor from $\mathrm{MCF}_a(Y, D_Y)$ to $\mathcal{H}_a(Y, D_Y)$ given as follows for any object $(V, \nabla, \mathrm{Fil}) \in \mathrm{MCF}(Y, D_Y)$

$$\overline{\mathrm{Gr}}(V, \nabla, \mathrm{Fil}) := (E, \theta, V, \nabla, \mathrm{Fil}, \psi),$$

where $(E, \theta) := \mathrm{Gr}(V, \nabla, \mathrm{Fil})$ is the graded bundle with the graded Higgs field induced by the connection, and ψ is the identifying map $\mathrm{Gr}(V, \nabla, \mathrm{Fil}) \cong (E, \theta)$.

Remark 2.18. There are also truncated version $\overline{\mathrm{Gr}}_n$ of this functor defined in [LSZ19, SYZ22]. For our application, we only use the version defined here, which is the limit functor of $\overline{\mathrm{Gr}}_n$. We note that the functor $\overline{\mathrm{Gr}}$ is actual an equivalence, but the $\overline{\mathrm{Gr}}_n$ is not.

The functor $\mathcal{T}$ and inverse Cartier functor $\mathcal{C}^{-1}$. One defines a natural functor from the category $\mathcal{H}(Y, D_Y)$ to the category $\widetilde{\mathrm{MIC}}(Y, D_Y)$, for any object $(E, \theta, V, \nabla, \mathrm{Fil}, \psi) \in \mathcal{H}(Y, D_Y)$

$$\mathcal{T}(E, \theta, V, \nabla, \mathrm{Fil}, \psi) := (\widetilde{V, \nabla, \mathrm{Fil}}).$$

The inverse Cartier functor is defined to be the composition

$$\mathcal{C}^{-1} \colon \mathcal{H}_a(Y, D_Y) \xrightarrow{\mathcal{T}} \widetilde{\mathrm{MIC}}_a(Y, D_Y) \xrightarrow{\mathcal{F}} \mathrm{MIC}(Y, D_Y)$$

for $a \leq p-2$.

Remark 2.19. (1) Similar there is also truncated version $\mathcal{T}_n$ in [LSZ19, SYZ22], whose definition is much more complicated and its limit is just $\mathcal{T}$.

[4]A Higgs bundle (E, θ) is called graded if E can be written as direct sum of subbundles E^i with $\theta(E^i) \subset E^{i-1} \otimes \Omega^1$. Obviously, a graded Higgs bundle is also nilpotent.

(2) Faltings tilde functor $\widetilde{(\cdot)}$ is just the composition of $\mathcal{T}$ and $\overline{\mathrm{Gr}}$. For $0 \leq a \leq p-2$, one has the following commutative diagram

$$
\begin{tikzcd}
 & \mathcal{H}_a(Y, D_Y) \arrow[d, "\text{forget}"] \arrow[dr, "\mathcal{C}^{-1}"] \arrow[dd, bend left, "\mathcal{T}"] & \\
\mathrm{MCF}_a(Y, D_Y) \arrow[ur, "\overline{\mathrm{Gr}}"] \arrow[r, "\text{grading}"] \arrow[dr, "\widetilde{(\cdot)}"'] & \mathrm{HIG}^{\mathrm{gr}}(Y, D_Y) & \mathrm{MIC}(Y, D_Y) \\
 & \widetilde{\mathrm{MIC}}_a(Y, D_Y) \arrow[u, "\text{grading}"] \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}
\tag{2.2}
$$

where $\mathrm{HIG}^{\mathrm{gr}}(Y, D_Y)$ is the category of graded logarithmic Higgs bundle over (Y, D_Y), and the "forget" and "grading" stand for the nature functors.

Higgs-de Rham flow. Recall [LSZ19] that a *Higgs-de Rham flow* over (Y, D_Y) is a sequence consisting of infinitely many alternating terms of filtered logarithmic de Rham bundles and logarithmic Higgs bundles

$$\left\{(V, \nabla, \mathrm{Fil})_{-1}, (E, \theta)_0, (V, \nabla, \mathrm{Fil})_0, (E, \theta)_1, (V, \nabla, \mathrm{Fil})_1, \cdots\right\},$$

which are related to each other by the following diagram inductively

$$
\begin{tikzcd}[column sep=small]
(V, \nabla, \mathrm{Fil})_{-1} \arrow[dr, "\mathrm{Gr}"] & & (V, \nabla, \mathrm{Fil})_0 \arrow[dr, "\mathrm{Gr}"] & & (V, \nabla, \mathrm{Fil})_1 \arrow[dr, "\mathrm{Gr}"] & \\
 & (E, \theta)_0 \arrow[ur, "\mathcal{C}^{-1}"] & & (E, \theta)_1 \arrow[ur, "\mathcal{C}^{-1}"] & & \cdots
\end{tikzcd}
\tag{2.3}
$$

where

- $(V, \nabla, \mathrm{Fil})_{-1}$ is a filtered de Rham bundle over (Y, D_Y) of level in $[0, p-2]$;
- Inductively, for $i \geq 1$, $(E, \theta)_i$ is the graded Higgs bundle $\mathrm{Gr}\left((V, \nabla, \mathrm{Fil})_{i-1}\right)$,

$$(V, \nabla)_i := \mathcal{C}^{-1}((E, \theta)_i, (V, \nabla, \mathrm{Fil})_{i-1}, \mathrm{id})$$

and Fil_i is a Hodge filtration on $(V, \nabla)_i$ of level in $[0, p-2]$.

Remark 2.20. (1) In [LSZ19], Higgs-de Rham flow is defined over any truncated level, whose definition is much more complicated.

(2) The essential data given in a Higgs-de Rham flow are just

$$V_{-1}, \nabla_{-1}, \mathrm{Fil}_{-1}, \mathrm{Fil}_0, \mathrm{Fil}_1, \mathrm{Fil}_2, \cdots$$

since the other terms can be constructed from these, e.g. E_0, θ_0, V_1, ∇_1, $\cdots$.

The Higgs-de Rham flow is called f-periodic if there exists an isomorphism

$$\Phi\colon (E_f, \theta_f, V_{f-1}, \nabla_{f-1}, \mathrm{Fil}_{f-1}, \mathrm{id}) \cong (E_0, \theta_0, V_{-1}, \nabla_{-1}, \mathrm{Fil}_{-1}, \mathrm{id})$$

such that its induces isomorphisms, for all $i \geq 0$,

$$(E, \theta)_{f+i} \cong (E, \theta)_i \quad \text{and} \quad (V, \nabla, \mathrm{Fil})_{f+i} \cong (V, \nabla, \mathrm{Fil})_i.$$

We simply represent this periodic Higgs-de Rham flow with the following diagram

$$
\begin{tikzcd}[column sep=small]
& (V,\nabla,\mathrm{Fil})_0 \arrow[dr, "\mathrm{Gr}"] & & (V,\nabla,\mathrm{Fil})_1 \arrow[dr, "\mathrm{Gr}"] & & (V,\nabla,\mathrm{Fil})_{f-1} \arrow[dr, "\mathrm{Gr}"] & \\
(E,\theta)_0 \arrow[ur, "\mathcal{C}^{-1}"] & & (E,\theta)_1 \arrow[ur, "\mathcal{C}^{-1}"] & & \cdots \arrow[ur, "\mathcal{C}^{-1}"] & & (E,\theta)_f \arrow[llllll, bend left, "\Phi"]
\end{tikzcd}
\tag{2.4}
$$

Theorem 2.21 (Lan-Shen-Zuo[LSZ19])**.** *There exists an equivalence between the category of logarithmic Fontaine-Faltings modules over (Y, D_Y) and the category of 1-periodic logarithmic Higgs-de Rham flows over (Y, D_Y).*

2.3. **Parabolic Fontaine-Faltings modules and Parabolic Higgs-de Rham flows.**

Parabolic versions of the some categories and functors. In order to define parabolic version Higgs-de Rham flow, we need the parabolic versions of the categories and functors appeared in a diagram from [SYZ22, Section 1.2.1]. The most crucial part is to define the inverse Cartier functor (or, equivalently, the Frobenius pullback functor).

$$
\begin{tikzcd}
& \mathcal{H}(X_n) \arrow[dr, "C_n^{-,1}"] \arrow[dd, "\mathcal{T}_n"] & \\
\mathrm{MCF}_{p-2}(X_n) \arrow[ur, "\overline{\mathrm{Gr}}"] \arrow[dr, "\widetilde{(\cdot)}"'] & & \mathrm{MIC}(X_n) \\
& \widetilde{\mathrm{MIC}}(X_n) \arrow[ur, "F_n=\{F_{\mathcal{U}}^*\}_{\mathcal{U}}"'] &
\end{tikzcd}
\tag{2.5}
$$

In this section, we set $S = \mathrm{Spec}(W(k))$ and $S_n = \mathrm{Spec}(W_n(k))$. Denote by

$$(Y_n, D_{Y_n}) \coloneqq (Y, D_Y) \times_S S_n.$$

Definition 2.22. Denote by $\mathrm{MIC}((Y_n, D_{Y_n})/S_n)$ the category of all parabolic de Rham bundles over $(Y_n, D_{Y_n})/S_n$.

Definition 2.23. A *filtered parabolic de Rham bundle over* $(Y_n, D_{Y_n})/S_n$ is a triple $(V, \nabla, \mathrm{Fil})$, which consists of a parabolic de Rham bundle (V, ∇) over $(Y_n, D_{Y_n})/S_n$ and a filtration Fil of parabolic sub bundles such that $(V_\alpha, \nabla_\alpha, \mathrm{Fil}_\alpha)$ forms a usual filtered de Rham bundle over $(Y_n, D_{Y_n})/S_n$ for each $\alpha \in \mathbb{Q}^n$. Denote by $\mathrm{MCF}_{p-2}((Y_n, D_{Y_n})/S_n)$ the category of all filtered parabolic de Rham bundles over $(Y_n, D_{Y_n})/S_n$ with the levels are contained in $[0, a]$.

Definition 2.24. Denote by $\mathcal{H}((Y_n, D_{Y_n})/S_n)$ the category of tuples $(E, \theta, \overline{V}, \overline{\nabla}, \overline{\mathrm{Fil}}, \overline{\psi})$, consisting of

- a graded parabolic Higgs bundle (E, θ) over $(Y_n, D_{Y_n})/S_n$ with exponent $\leq p - 2$,
- a filtered parabolic de Rham bundle over $(Y_n, D_{Y_{n-1}})/S_{n-1}$, and
- an isomorphism of graded parabolic Higgs bundles over $(Y_n, D_{Y_{n-1}})/S_{n-1}$

$$\overline{\psi} \colon\ \mathrm{Gr}(\overline{V}, \overline{\nabla}, \overline{\mathrm{Fil}}) \cong (E, \theta) \otimes_{W_n(k)} W_{n-1}(k).$$

Definition 2.25. An *parabolic integrable p-connection* $\nabla = \{\nabla_\alpha\}$ on a parabolic vector bundle V over $(Y_n, D_{Y_n})/S_n$ is a collection of compatible integrable p-connections ∇_α on the V_α. I.e. for any $\alpha \leq \beta$, the following diagram commutes

$$\begin{tikzcd} V_\alpha \arrow[r, "\nabla_\alpha"] \arrow[d] & V_\alpha \otimes \Omega_{Y_n/S_n}(\log D_{Y_n}) \arrow[d] \\ V_\beta \arrow[r, "\nabla_\beta"] & V_\beta \otimes \Omega_{Y_n/S_n}(\log D_{Y_n}) \end{tikzcd}$$

Let $\widetilde{\mathrm{MIC}}((Y_n, D_{Y_n})/S_n)$ denote the category of pairs (V, ∇), consisting of a parabolic vector bundle V and a parabolic integrable p-connection ∇ on V.

Functor $\overline{\mathrm{Gr}}$. For an object $(V, \nabla, \mathrm{Fil})$ in $\mathrm{MCF}_{p-2}((Y_n, D_{Y_n})/S_n)$, the functor $\overline{\mathrm{Gr}}$ is given by

$$\overline{\mathrm{Gr}}(V, \nabla, \mathrm{Fil}) = (E, \theta, \overline{V}, \overline{\nabla}, \overline{\mathrm{Fil}}, \overline{\psi}),$$

where $(E, \theta) = \mathrm{Gr}(V, \nabla, \mathrm{Fil})$ is the graded parabolic Higgs bundle, $(\overline{V}, \overline{\nabla}, \overline{\mathrm{Fil}})$ is the modulo p^{n-1}-reduction of $(V, \nabla, \mathrm{Fil})$ and $\overline{\psi}$ is the identifying map

$$\mathrm{id}\colon \mathrm{Gr}(\overline{V}, \overline{\nabla}, \overline{\mathrm{Fil}}) = (E, \theta) \otimes_{W_n(k)} W_{n-1}(k).$$

Faltings tilde functor $\widetilde{(\cdot)}$. For an object $(V, \nabla, \mathrm{Fil})$ in $\mathrm{MCF}_{p-2}(X_n)$, the $\widetilde{(V, \nabla, \mathrm{Fil})}$ will be denoted as the quotient $\bigoplus_{i=0}^{p-2} \mathrm{Fil}^i / \sim$ with $x \sim py$ for any $x \in \mathrm{Fil}^i V$ and y the image of x under the natural inclusion $\mathrm{Fil}^i V \hookrightarrow \mathrm{Fil}^{i-1} V$.

The construction of functor $\mathcal{T}_n$. Let $(E, \theta, \bar{V}, \bar{\nabla}, \overline{\mathrm{Fil}}, \psi)$ be an object in $\mathcal{H}((Y_n, D_{Y_n})/S_n)$. For any $\alpha \in \mathbb{Q}^n$, denote

$$(\widetilde{V}_\alpha, \widetilde{\nabla}_\alpha) \coloneqq \mathcal{T}_n(E_\alpha, \theta_\alpha, \bar{V}_\alpha, \bar{\nabla}_\alpha, \overline{\mathrm{Fil}}_\alpha, \psi_\alpha),$$

where the functor on the right hand side, still denote by $\mathcal{T}_n$, is the usual functor defined as in [SYZ22]. Then the collection $(\{\widetilde{V}_\alpha\}, \{\widetilde{\nabla}_\alpha\})\}$ forms a pair of parabolic vector bundle and a parabolic p-connections. We denote it by

$$\mathcal{T}_n(E, \theta, \bar{V}, \bar{\nabla}, \overline{\mathrm{Fil}}, \psi) \coloneqq \{((\widetilde{V}_\alpha, \widetilde{\nabla}_\alpha))\}$$

The Frobenius pullback functor Φ_n^*. Base on the usual Frobenius pullback functor Φ_n^* for usual bundles with p-connections, we define the Frobenius pullback for parabolic vector bundles with parabolic p-connections, which will be still denoted Φ_n^* by abusing notions.

For any $\gamma \in \mathbb{Q}^n$, we has a canonical parabolic p-connection $p \cdot \mathrm{d}(\gamma D)$ on the parabolic vector bundle $\mathcal{O}_Y(\gamma D)$. We simply set

$$\Phi_n^*\Big(\mathcal{O}_Y(\gamma D), p \cdot \mathrm{d}(\gamma D)\Big) \coloneqq \Big(\mathcal{O}_Y(p\gamma D), \mathrm{d}(p\gamma D)\Big).$$

Similarly as the pullback of parabolic de Rham bundle, we can define the Frobenius pullback functor as follows.

Definition 2.26. For any (V, ∇) in $\widetilde{\mathrm{MIC}}((Y_n, D_{Y_n})/S_n)$, we first denote

$$\Phi_{n,\gamma}^*(V, \nabla) \coloneqq \Phi_n^*\Big(\big((V, \nabla) \otimes (\mathcal{O}_Y(\gamma D), p \cdot \mathrm{d}(\gamma D))^{-1}\big)_0\Big) \otimes \Phi_n^*\big(\mathcal{O}_Y(\gamma D), p \cdot \mathrm{d}(\gamma D)\big).$$

And then set

$$\Phi_n^*(V,\nabla) := \bigcup_{\gamma\in\mathbb{Q}^n} \Phi_{n,\gamma}^*(V,\nabla).$$

Denote by $C_n^{-1} := \Phi_n^* \circ \mathcal{T}_n$ the composition functor, call it *the inverse Cartier functor*.

Parabolic Fontaine-Faltings modules with endomorphism structure. Recall [SYZ22, Lemma 1.1] or [LSZ19, Lemma 5.6]], we extend the definition to parabolic version.

Definition 2.27. A *parabolic Fontaine-Faltings module over* $(Y_n, D_{Y_n})/S_n$ is a tuple $(V,\nabla,\mathrm{Fil},\varphi)$, where

- $(V,\nabla,\mathrm{Fil}) \in \mathrm{MCF}_{p-2}((Y_n, D_{Y_n})/S_n)$ is a filtered parabolic de Rham bundle over $(Y_n, D_{Y_n})/S_n$ of level in $[0,p-2]$;
- $\varphi\colon C_n^{-1}\circ \overline{\mathrm{Gr}}(V,\nabla,\mathrm{Fil}) \to (V,\nabla)$ is an isomorphism of parabolic de Rham bundles.

We denote by $\mathcal{MF}^{\nabla}_{[0,p-2]}((Y_n, D_{Y_n})/S_n)$ the category of all parabolic Fontaine-Faltings modules over $(Y_n, D_{Y_n})/S_n$.

Definition 2.28. Let $M \in \mathcal{MF}^{\nabla}_{[0,p-2]}((Y_n, D_{Y_n})/S_n)$. A ring homomorphism

$$\iota\colon \mathbb{Z}_{p^f} \to \mathrm{End}(M)$$

is called a $\mathbb{Z}_p$*-endomorphism structure*. Denote by $\mathcal{MF}^{\nabla}_{[0,p-2]}((Y_n, D_{Y_n})/S_n)_{\mathbb{Z}_{p^f}}$ the category of all parabolic Fontaine-Faltings modules with $\mathbb{Z}_p$-endomorphism structures over $(Y_n, D_{Y_n})/S_n$.

Let $(V,\nabla,\mathrm{Fil},\varphi)$ be a parabolic Fontaine-Faltings module with level in $[0,a]$ and $(V,\nabla,\mathrm{Fil},\varphi)'$ be a parabolic Fontaine-Faltings module with level in $[0,b]$. Then on the tensor parabolic de Rham bundle

$$(V,\nabla)\otimes(V,\nabla)'$$

Set

$$\mathrm{Fil}^n_{tot}(V\otimes V') := \sum_{i+j=n} \mathrm{Fil}^i V\otimes \mathrm{Fil}^j V \quad \text{for all } n\in\mathbb{Z},$$

it forms a Hodge filtration on the parabolic de Rham $(V,\nabla)\otimes(V,\nabla)'$. Denote

$$(V,\nabla,\mathrm{Fil})\otimes(V,\nabla,\mathrm{Fil})' := (V\otimes V', \nabla\otimes\mathrm{id}+\mathrm{id}\otimes\nabla', \mathrm{Fil}_{tot}).$$

By the definition of parabolic version of inverse Cartier, it naturally preserves tensor products, so we can set a Frobenius structure on $(V,\nabla,\mathrm{Fil})\otimes(V,\nabla,\mathrm{Fil})'$ given by $\varphi\otimes\varphi'$, and then get the tensor product of

$$(V,\nabla,\mathrm{Fil},\varphi)\otimes(V,\nabla,\mathrm{Fil},\varphi)'.$$

For the tensor product of Fontaine-Faltings modules with endomorphism structure, we set the underlying parabolic Fontaine-Faltings module of

$$(V,\nabla,\mathrm{Fil},\varphi,\iota)\otimes(V,\nabla,\mathrm{Fil},\varphi,\iota)'$$

to be

$$\Big((V,\nabla,\mathrm{Fil},\varphi)\otimes(V,\nabla,\mathrm{Fil},\varphi)'\Big)^{\iota=\iota'}.$$

on which the action of ι and ι' are coincide with each other, and we using this common action to define the $\mathbb{Z}_{p^f}$-endomorphism structure.

We also define the symmetric product, wedge product and determinant for parabolic Fontaine-Faltings modules as usual way.

Parabolic Higgs-de Rham flows.

Definition 2.29. a parabolic *Higgs-de Rham flow* over $(Y_n, D_{Y_n}) \subset (Y_{n+1}, D_{Y_{n+1}})$ is a sequence consisting of infinitely many alternating terms of filtered parabolic de Rham bundles and graded parabolic Higgs bundles

$$\left\{(V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}, (E,\theta)_0^{(n)}, (V,\nabla,\mathrm{Fil})_0^{(n)}, (E,\theta)_1^{(n)}, (V,\nabla,\mathrm{Fil})_1^{(n)}, \cdots\right\},$$

which are related to each other by the following diagram inductively

$$\begin{tikzcd}
 & & (V,\nabla,\mathrm{Fil})_0^{(n)} \arrow[dr,"\mathrm{Gr}"] & & (V,\nabla,\mathrm{Fil})_1^{(n)} \arrow[dr,"\mathrm{Gr}"] & & (V,\nabla,\mathrm{Fil})_2^{(n)} \arrow[dr,"\mathrm{Gr}"] & \\
 & (E,\theta)_0^{(n)} \arrow[ur,"\mathcal{C}_n^{-1}"] \arrow[dd, dotted] & & (E,\theta)_1^{(n)} \arrow[ur,"\mathcal{C}_n^{-1}"] & & (E,\theta)_2^{(n)} \arrow[ur,"\mathcal{C}_n^{-1}"] & & \cdots \\
 (V,\nabla,\mathrm{Fil})_{-1}^{(n-1)} \arrow[dr,"\mathrm{Gr}"] & & & & & & & \\
 & \mathrm{Gr}\left((V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}\right) & & & & & &
\end{tikzcd}$$

where

- $(V,\nabla,\mathrm{Fil})_{-1}^{(n-1)} \in \mathrm{MCF}_{p-2}((Y_n, D_{Y_n})/S_n)$;
- $(E,\theta)_0^{(n)}$ is a lifting of the graded parabolic Higgs bundle $\mathrm{Gr}\left((V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}\right)$ over $(Y_n, D_{Y_n})/S_n$, $(V,\nabla)_0^{(n)} := C_n^{-1}((E,\theta)_0^{(n)}, (V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}, \psi)$ and $\mathrm{Fil}_0^{(n)}$ is a parabolic Hodge filtration on $(V,\nabla)_0^{(n)}$ of level in $[0, p-2]$;
- Inductively, for $m \geq 1$, $(E,\theta)_m^{(n)} := \mathrm{Gr}\left((V,\nabla,\mathrm{Fil})_{m-1}^{(n)}\right)$ and
$$(V,\nabla)_m^{(n)} := C_n^{-1}\left((E,\theta)_m^{(n)}, (V,\nabla,\mathrm{Fil})_{m-1}^{(n-1)}, \mathrm{id}\right).$$
Here $(V,\nabla,\mathrm{Fil})_{m-1}^{(n-1)}$ is the reduction of $(V,\nabla,\mathrm{Fil})_{m-1}^{(n)}$ on X_{n-1}. And $\mathrm{Fil}_m^{(n)}$ is a Hodge filtration on $(V,\nabla)_m^{(n)}$.

Definition 2.30. Let

$$\mathrm{Flow} = \left\{(V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}, (E,\theta)_0^{(n)}, (V,\nabla,\mathrm{Fil})_0^{(n)}, (E,\theta)_1^{(n)}, (V,\nabla,\mathrm{Fil})_1^{(n)}, \cdots\right\},$$

and

$$\mathrm{Flow}' = \left\{(V',\nabla',\mathrm{Fil}')_{-1}^{(n-1)}, (E',\theta')_0^{(n)}, (V',\nabla',\mathrm{Fil}')_0^{(n)}, (E',\theta')_1^{(n)}, (V',\nabla',\mathrm{Fil}')_1^{(n)}, \cdots\right\},$$

be two Higgs-de Rham flows over $(Y_n, D_{Y_n}) \subset (Y_{n+1}, D_{Y_{n+1}})$. A morphism from Flow to Flow$'$ is a compatible system of morphisms

$$\{\varphi_{-1}^{(n-1)}, \psi_0^{(n)}, \varphi_0^{(n)}, \psi_1^{(n)}, \varphi_1^{(n)}, \cdots\}$$

between the terms respectively in the following sense that

- $\mathrm{Gr}(\varphi_{-1}^{(n-1)}) = \psi_0^{(n)} \pmod{p^{n-1}}$
- $C_n^{-1}(\psi_m^{(n)}, \varphi_{m-1}^{(n-1)}) = \varphi_m^{(n)}$, (if $m \geq 1$, then here $\varphi_m^{(n-1)} := \varphi_m^{(n)} \pmod{p^{n-1}}$), and

- $\mathrm{Gr}(\varphi_m^{(n)}) = \varphi_{m+1}^{(n)}$.

Remark 2.31. The morphism is uniquely determined by the first two terms $\varphi_{-1}^{(n-1)}, \psi_0^{(n)}$. If $n=1$, then the first morphism is vacuous.

Definition 2.32. Let

$$\mathrm{Flow} = \left\{(V,\nabla,\mathrm{Fil})_{-1}^{(n-1)}, (E,\theta)_0^{(n)}, (V,\nabla,\mathrm{Fil})_0^{(n)}, (E,\theta)_1^{(n)}, (V,\nabla,\mathrm{Fil})_1^{(n)}, \cdots\right\},$$

be a flow, we call the flow

$$\mathrm{Flow}[f] = \left\{(V,\nabla,\mathrm{Fil})_{f-1}^{(n-1)}, (E,\theta)_f^{(n)}, (V,\nabla,\mathrm{Fil})_f^{(n)}, (E,\theta)_{f+1}^{(n)}, (V,\nabla,\mathrm{Fil})_{f+1}^{(n)}, \cdots\right\}$$

the *f-th shifting of* Flow, where $(V,\nabla,\mathrm{Fil})_{f-1}^{(n-1)} := (V,\nabla,\mathrm{Fil})_{f-1}^{(n)} \pmod{p^{n-1}}$.

Definition 2.33. If there is an isomorphism

$$\psi := \{\varphi_{-1}^{(n-1)}, \psi_0^{(n)}, \varphi_0^{(n)}, \psi_1^{(n)}, \varphi_1^{(n)}, \cdots\}$$

from $\mathrm{Flow}[f]$ to Flow, then we call the pair (Flow,ψ) is a *periodic Higgs-de Rham flow*. Note that the ψ is part of data in the periodic Higgs-de Rham flow, which we will call a periodic mapping of Flow. Since ψ is uniquely determined by $\varphi_{-1}^{(n-1)}, \psi_0^{(n)}$, we sometime use $(\mathrm{Flow}, (\varphi_{-1}^{(n-1)}, \psi_0^{(n)}))$ to represent (Flow,ψ) and call . In case $n=1$, the first term in the flow is vacuous, we also use $(\mathrm{Flow}, \psi_0^{(1)})$ to represents (Flow,ψ).

With the same method in [LSZ19], one shows that:

Theorem 2.34. *Suppose $\mathbb{F}_{p^f}$ is contained in k. Then there is an equivalent functor from the category of f-periodic parabolic Higgs-de Rham flows to the category of parabolic Fontaine-Faltings modules with $\mathbb{Z}_{p^f}$-endomorphism structure.*

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