§6.2. 线性变换的矩阵

$V=n$ 维 F-线性空间

$\mathscr{A}: V\to V$ 线性变换

取定 V 的一组基：$\alpha_1,\alpha_2,\cdots,\alpha_n$.

$\forall i, \mathscr{A}(\alpha_i)\in V \Rightarrow \forall i\ \exists a_{1i},a_{2i},\cdots,a_{ni}\in F$ s.t.

$$\mathscr{A}(\alpha_i)=a_{1i}\alpha_1+a_{2i}\alpha_2+\cdots+a_{ni}\alpha_n.$$

改写

$$\Rightarrow \big(\mathscr{A}(\alpha_1),\mathscr{A}(\alpha_2),\cdots,\mathscr{A}(\alpha_n)\big)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

并不是数量矩阵，但乘法有意义.

记 $\mathscr{A}(\alpha_1,\alpha_2,\cdots,\alpha_n):=\big(\mathscr{A}(\alpha_1),\mathscr{A}(\alpha_2),\cdots,\mathscr{A}(\alpha_n)\big)$, $A=(a_{ij})_{n\times n}$, 则

$$\mathscr{A}(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)A$$

线性变换 $\mathscr{A}$ 在基 $\alpha_1,\cdots,\alpha_n$ 下的矩阵

注：① A 由 $\mathscr{A}$ 及基 $\alpha_1,\cdots,\alpha_n$ 唯一确定.

② A 的第 j 列为 $\mathscr{A}(\alpha_j)$ 在 $\alpha_1,\cdots,\alpha_n$ 下的坐标.

例：任给 $A\in F^{n\times n}$ 定义 F^n 上线性变换 $\mathscr{A}: F^n\to F^n$ $x\mapsto Ax$.

则 $\mathscr{A}$ 在自然基下的矩阵为 A.

证：$\mathscr{A}(e_1,\cdots,e_n):=(\mathscr{A}e_1,\cdots,\mathscr{A}e_n):=(Ae_1,\cdots,Ae_n)=A(e_1,\cdots,e_n)$

$=A\cdot I_n=A=I_n\cdot A=(e_1,e_2,\cdots,e_n)\cdot A$ □

例: $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}\in V=F^{2\times 2}$ $\forall M\in V$ $\mathcal{A}M:=AM$.

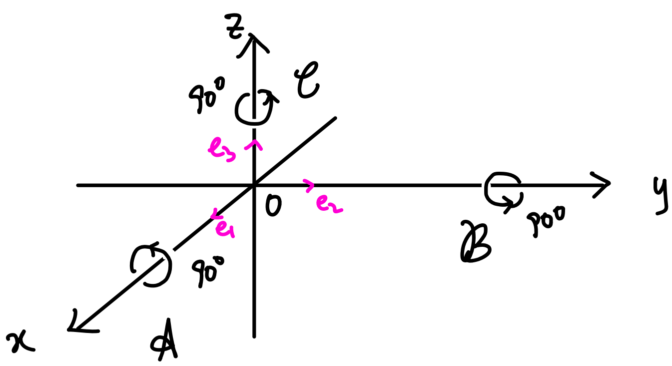
求 $\mathcal{A}$ 在 $e_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $e_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ 下的矩阵

解: $A\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}1&0\\3&0\end{pmatrix}=e_{11}+3e_{21}$ $\quad A\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&1\\0&3\end{pmatrix}=e_{12}+3e_{22}$

$A\begin{pmatrix}0&0\\1&0\end{pmatrix}=\begin{pmatrix}2&0\\4&0\end{pmatrix}=2e_{11}+4e_{21}$ $\quad A\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\0&4\end{pmatrix}=2e_{12}+4e_{22}$

$$\Rightarrow \mathcal{A}(e_{11},e_{12},e_{21},e_{22})=(e_{11},e_{12},e_{21},e_{22})\begin{pmatrix}1&0&2&0\\0&1&0&2\\1&0&4&0\\0&3&0&4\end{pmatrix}$$

例: OXYZ $\mathbb{R}^3$ 中直角坐标系.



0, 1 0 → 0 0 1

解: $\mathcal{A}(x,y,z)=(x,-z,y)$ $\quad \mathcal{B}(x,y,z)=(z,y,-x)$

$\mathcal{C}(x,y,z)=(-y,x,z)$

$\mathcal{A}(e_1,e_2,e_3)=(e_1,e_3,-e_2)$

$\mathcal{B}(e_1,e_2,e_3)=(-e_3,e_2,e_1)$

$\mathcal{C}(e_1,e_2,e_3)=(e_2,-e_1,e_3)$ $\qquad \mathcal{A}^2\mathcal{B}^2=\mathcal{B}^2\mathcal{A}^2$

问若旋转 θ 角度呢?.

x 与 $\mathscr{A}x$ 在同一基下的坐标之间的关系.

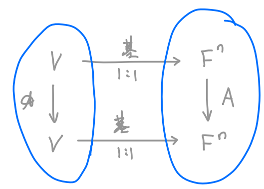
定理：设 $\mathscr{A}: V \to V$ 在基 $\alpha_1, \cdots, \alpha_n$ 下的矩阵为 A.

$x, y \in V$ 在 $\alpha_1, \cdots, \alpha_n$ 下的坐标为 $X, Y \in F^n$，则

$$y = \mathscr{A}(x) \quad \Rightarrow \quad Y = AX.$$

证明：

$$\begin{cases} \mathscr{A}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) A \\ x = (\alpha_1, \cdots, \alpha_n) X \qquad\qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ y = (\alpha_1, \cdots, \alpha_n) Y \end{cases}$$



$$\Rightarrow \quad y = \mathscr{A}(x) = \mathscr{A}\big((\alpha_1 \cdots \alpha_n) X\big) = \big(\mathscr{A}(\alpha_1, \cdots, \alpha_n)\big) X$$

$$= \big((\alpha_1, \cdots, \alpha_n) A\big) X = (\alpha_1, \cdots, \alpha_n)(AX)$$

坐标的唯一性 $\Rightarrow$ $Y = AX$.

例：$\mathscr{A}(\overset{\alpha_1}{(2,3,5)^T}) = \overset{\beta_1}{(1,2,0)^T}$ $\quad \mathscr{A}(\overset{\alpha_2}{(0,1,2)^T}) = \overset{\beta_2}{(2,4,-1)^T}$

$\mathscr{A}(\overset{\alpha_3}{(1,0,0)^T}) = \overset{\beta_3}{(3,0,5)^T}$

(1) $\mathscr{A}$ 在 $\alpha_1, \alpha_2, \alpha_3$ 下的矩阵 (2). $\mathscr{A}$ 在自然基下的矩阵.

解：(1) 设 $\mathscr{A}$ 在 $\alpha_1, \alpha_2, \alpha_3$ 下的矩阵为 A. 则

$$(\beta_1, \beta_2, \beta_3) = (\alpha_1, \alpha_2, \alpha_3) A$$

$$\Rightarrow \quad A = (\alpha_1, \alpha_2, \alpha_3)^{-1}(\beta_1, \beta_2, \beta_3) = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 9 & -5 \\ -10 & -23 & 15 \\ -7 & -16 & 13 \end{pmatrix}$$

(2). 设 $\mathscr{A}$ 在自然基下的矩阵为 B，则

定理 $\Rightarrow$ $\beta_i = B\alpha_i \qquad i = 1, 2, 3$

$$\Rightarrow (\beta_1, \beta_2, \beta_3) = B(\alpha_1, \alpha_2, \alpha_3)$$

$$\Rightarrow B = (\beta_1\ \beta_2\ \beta_3)(\alpha_1, \alpha_2, \alpha_3)^{-1} = \begin{pmatrix} 3 & -20 & 11 \\ 0 & -16 & 10 \\ 5 & -15 & 7 \end{pmatrix}$$

两种坐标变换公式

线性变换的矩阵

$$\begin{tikzcd}
V \arrow[r, "(\alpha_1\cdots\alpha_n)", "1:1"'] \arrow[d, "\nu \mapsto \mathscr{A}(\nu)"'] \arrow[dr, phantom, "\mathscr{A}(\alpha_1\cdots\alpha_n)=(\alpha_1\cdots\alpha_n)A"] & F^n \arrow[d, "X \mapsto AX"] \\
V \arrow[r, "(\alpha_1,\cdots,\alpha_n)", "1:1"'] & F^n
\end{tikzcd}$$

一个向量与它在$\mathscr{A}$下的像在同一基下的坐标之间关系

基变换，坐标变换公式

$$\begin{tikzcd}
V \arrow[r, "(\alpha_1\cdots\alpha_n)", "1:1"'] \arrow[d, "\mathrm{id}\ \ \nu \mapsto \nu"'] \arrow[dr, phantom, "(\beta_1,\cdots,\beta_n)=(\alpha_1\cdots\alpha_n)A"] & F^n \arrow[d, "X \mapsto AX"] \\
V \arrow[r, "(\beta_1,\cdots,\beta_n)", "1:1"'] & F^n
\end{tikzcd}$$

同一个向量在不同基下坐标之间的关系

设 $\alpha_1,\cdots,\alpha_n$ 为 n 维 F-线性空间 V 的一组基.

定理：$\mathscr{A} \mapsto A$.

反之，给定 n 阶矩阵 $B=(b_{ij})$，我们可如下定义线性变换、

$\forall\ \alpha = x_1\alpha_1+\cdots+x_n\alpha_n \in V$,

$$\mathscr{B}(\alpha) := \sum_{i=1}^n\Big(\sum_{j=1}^n b_{ij}x_j\Big)\alpha_i = (\alpha_1\cdots\alpha_n)B\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}$$

首先：$\mathscr{B}$ 为 V 上的线性变换.

- $\mathscr{B}(\alpha+\beta)=\mathscr{B}(\alpha)+\mathscr{B}(\beta)$
- $\mathscr{B}(\lambda\alpha)=\lambda\mathscr{B}(\alpha)$.

其次：$\mathscr{B}(\alpha_1,\cdots,\alpha_n)=(\alpha_1\cdots\alpha_n)B$.

总结：$\{V上的线性变换\} \xrightarrow[1:1]{\alpha_1\cdots\alpha_n} F^{n\times n}$

$\mathscr{A} \longmapsto \mathscr{A}$在$\alpha_1\cdots\alpha_n$下的矩阵.

§ 线性变换在不同基下的矩阵.

定理: 设 V 上的线性变换 $\mathscr{A}: V \to V$ 在两组基 $\alpha_1, \cdots, \alpha_n$ 和 $\beta_1 \cdots \beta_n$ 下的矩阵为 A 和 B. 设 $\alpha_1, \cdots \alpha_n$ 到 $\beta_1 \cdots \beta_n$ 的过渡矩阵为 T. 则

$$B = T^{-1}AT$$

证:
$$\left.\begin{array}{l} \mathscr{A}(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n)A \\ \mathscr{A}(\beta_1 \cdots \beta_n) = (\beta_1 \cdots \beta_n)B \\ (\beta_1 \cdots \beta_n) = (\alpha_1 \cdots \alpha_n)T \end{array}\right\} \Rightarrow$$

$$\mathscr{A}(\beta_1 \cdots \beta_n) = \mathscr{A}\big((\alpha_1 \cdots \alpha_n)T\big) = \big(\mathscr{A}(\alpha_1 \cdots \alpha_n)\big)\cdot T = (\alpha_1 \cdots \alpha_n)AT$$

$$= (\beta_1 \cdots \beta_n)T^{-1}AT$$

$$\Rightarrow B = T^{-1}AT. \qquad \square$$

例. $\mathscr{A}: F^3 \to F^3$ 满足

$$\mathscr{A}\left(\begin{pmatrix}2\\3\\5\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right) = \left(\begin{pmatrix}2\\3\\5\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right)\underbrace{\begin{pmatrix}4 & 9 & -5\\-10 & -23 & 15\\-7 & -16 & 13\end{pmatrix}}_{=A}$$

$\mathscr{A}$ 在 $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\2\end{pmatrix}, \begin{pmatrix}-2\\0\\-1\end{pmatrix}$ 下的矩阵 $C = ?$

解: $$\begin{pmatrix}1 & 2 & -2\\0 & 1 & 0\\1 & 2 & -1\end{pmatrix} = \begin{pmatrix}2 & 0 & 1\\3 & 1 & 0\\5 & 2 & 0\end{pmatrix}T \Rightarrow T = \begin{pmatrix}-1 & 0 & 1\\3 & 1 & -3\\3 & 2 & -4\end{pmatrix}$$

$$\Rightarrow C = T^{-1}AT = \begin{pmatrix}-10 & 2 & 3\\10 & 4 & -10\\-2 & 1 & 0\end{pmatrix}$$

易得：$\mathcal{A}(e_1,e_2,e_3)=(e_1,e_2,e_3)\begin{pmatrix}3&-20&11\\0&-16&10\\5&-15&7\end{pmatrix}=B$

$$\begin{pmatrix}1&2&-2\\0&1&0\\1&2&-1\end{pmatrix}=(e_1,e_2,e_3)\begin{pmatrix}1&2&-2\\0&1&0\\1&2&-1\end{pmatrix}=:S$$

$$\Rightarrow C=S^{-1}BS=\begin{pmatrix}-10&2&3\\10&4&-10\\-2&1&0\end{pmatrix}$$

§ 矩阵的相似

定义：设 $A,B\in F^{n\times n}$，若 ∃ 可逆阵 $T\in F^{n\times n}$ 使得 $B=T^{-1}AT$，则称 A 与 B 相似，记为 $A\sim B$.

性质：相似为等价关系，即

(1) 反身性：

(2) 对称性：

(3) 传递性：

证：...

例：同余为等价关系 $m\equiv n \bmod 3$

将 $\mathbb{Z}$ 分成 3 个等价类

$\bar{0}=\{\cdots,-6,-3,0,3,6,\cdots\}$

$\bar{1}=\{\cdots-5,-2,1,4,7,\cdots\}$

$\bar{2}=\{\cdots-4,-1,2,5,8,\cdots\}$

$\Rightarrow \mathbb{Z}=\bar{0}\cup\bar{1}\cup\bar{2}$.

根据相似关系将 $F^{n\times n}$ 分为若干类.

相似类，代表元

定理 ⇒ 不同基下矩阵相似. 反之，属于该相似类的矩阵，均为该线性变换在不同基下对应的矩阵.

$\mathscr{A}(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n) A$

B 与 A 相似 $\Rightarrow$ $B = T^{-1}AT$ （其中 T 可逆）

$(\beta_1 \cdots \beta_n) := (\alpha_1 \cdots \alpha_n) T$ 则 $\beta_1 \cdots \beta_n$ 为基.

$\Rightarrow \mathscr{A}(\beta_1 \cdots \beta_n) = (\beta_1 \cdots \beta_n) B$.

相似不变量. 例：行列式，秩

问题：1) 两矩阵相似的条件？ 相抵关系：1) rank

2) 最简代表元？ 2) 相抵标准形