§1.6 复数

§1.6.1 复数的四则运算

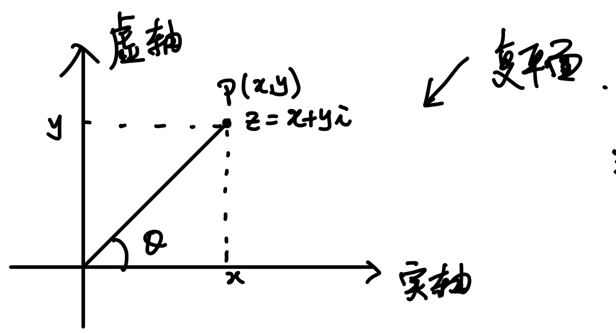
$$z = \underset{\substack{\uparrow \\ \text{实部} \\ \mathrm{Re}\, z}}{x} + \overset{\sqrt{-1}}{i}\,\underset{\substack{\uparrow \\ \text{虚部} \\ \mathrm{Im}\, z}}{y}$$

"+" : $(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$

"·" : $(x_1 + iy_1) \cdot (x_2 + iy_2) \overset{i^2=-1}{=} (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

"÷" : $\dfrac{x_2 + iy_2}{\underset{(0\neq)}{x_1 + iy_1}} = \dfrac{(x_2 + iy_2)(x_1 - iy_1)}{(x_1 + iy_1)(x_1 - iy_1)} = \dfrac{(x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1)}{x_1^2 + y_1^2}$

§1.6.2. 复数的几何表示



z 用 $\vec{OP}$ 表示 ($z=\vec{OP}$)

z的模长： $|z| := |\vec{OP}| = \sqrt{x^2+y^2}$

z的辐角： $\arg z :=$ x轴(逆时针)旋转到 $\vec{OP}$ 的角度 $= \theta+2k\pi$.

(一般规定 $0\le \arg z < 2\pi$ 主值)

性质： · $z_1+z_2 = \vec{OP_1}+\vec{OP_2}$

· $|z_1+z_2| \le |z_1|+|z_2|$

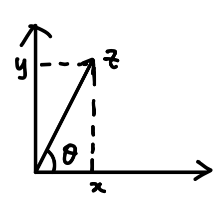
$z=x+iy$ 的共轭复数定义为：

$$\bar{z} = x-iy$$

性质： $|\bar{z}|=|z|$, $\arg\bar{z}+\arg z = 2\pi$

$|z|^2 = z\bar{z}$, $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1\cdot\bar{z}_2$

三角表示： $z = r(\cos\theta + i\sin\theta)$



性质： $z_1 = r_1(\cos\theta_1+i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2+i\sin\theta_2)$ 则

1-6-2 $|z_1z_2| = r_1r_2$ & $\arg(z_1z_2) = \arg z_1 + \arg z_2$

Pf: $\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$ □

复数乘法的几何解释. $w = r(\cos\theta + i\sin\theta)$

$$z \xrightarrow{\text{伸缩}} rz \xrightarrow{\text{旋转}} wz$$

Euler 公式 $e^{i\theta} = \cos\theta + i\sin\theta$.

↑ 暂时看成一个记号

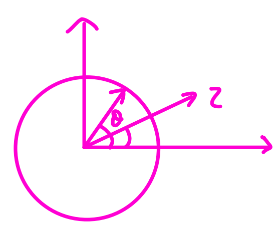
性质: 1) $r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}$

2) (de Moive) $(re^{i\theta})^n = r^n \cdot e^{in\theta}$, $\forall n \in \mathbb{Z}$

例: 求 $z = 1+\cos\theta + i\sin\theta \ (-\pi \le \theta < \pi)$ 的三角形式.

解: $|z| = 2\cos\frac{\theta}{2}$

$\arg z = \arccos \dfrac{1+\cos\theta}{2\cos\frac{\theta}{2}} = \frac{\theta}{2}$



$\Rightarrow \quad z = 2\cos\frac{\theta}{2} \cdot \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$

例: 逆时针旋转 $z = x+iy$ 角度 $\frac{\pi}{2}$.

解: $z e^{\frac{\pi}{2}i} = (x+iy)i = -y + ix$.

例：解方程：$z^n = a$.

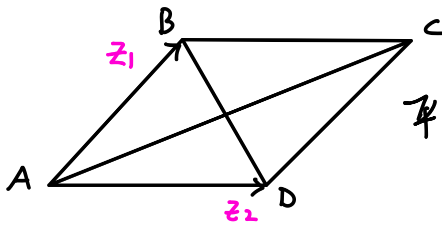
解：设 $a = re^{i\theta}$, $r \ge 0$, $0 \le \theta < 2\pi$.

设 $z = se^{i\phi}$. 则

$$\left.\begin{matrix} s^n = r \\ n\phi = \theta + 2k\pi \end{matrix}\right\} \Rightarrow \begin{cases} s = \sqrt[n]{r} \\ \phi = \frac{\theta + 2k\pi}{n} \end{cases} \Rightarrow z = \sqrt[n]{r}\, e^{i\frac{\theta+2k\pi}{n}}$$

□

例：



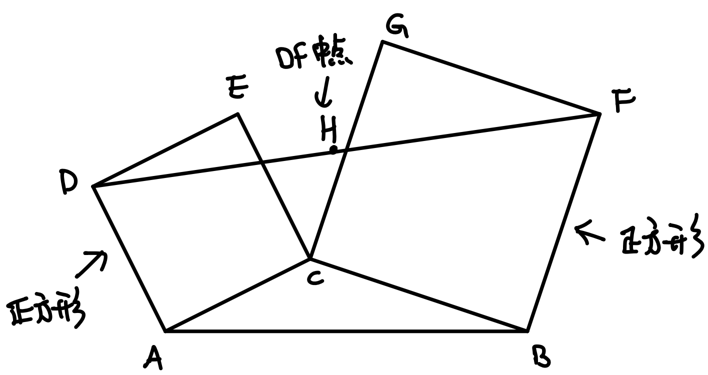
平行四边形. 则

$$|AC|^2 + |BD|^2 = |AB|^2 + |BC|^2 + |CD|^2 + |AD|^2$$

Pf：$LHS = |z_1+z_2|^2 + |z_1-z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2 = 2|z_1|^2 + 2|z_2|^2 = RHS$ □

例：



则 H与C无关.

证：$\overrightarrow{AH} = \frac{1}{2}\overrightarrow{AD} + \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BF})$

$= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}\, e^{\frac{\pi i}{2}} + \frac{1}{2}\overrightarrow{CB}\, e^{\frac{\pi i}{2}} = \frac{1+i}{2} \cdot \overrightarrow{AB}$ □