A FINITENESS THEOREM FOR GEOMETRICALLY ABSOLUTELY IRREDUCIBLE LOGARITHMIC CRYSTALLINE REPRESENTATIONS

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Abstract. Let K be an unramified p-adic local field and let W be the ring of integers of K. Let $(X,S)/W$ be a smooth projective scheme together with a relative simple normal crossings divisor and fix positive integers r and f. We show that the set of absolutely irreducible representations $\pi_1(X_{\bar{K}}) \to \mathrm{GL}_r(\mathbb{Z}_{p^f})$ that come from log crystalline $\mathbb{Z}_{p^f}$-local systems over (X_K, S_K) of rank r is finite. The proof uses p-adic nonabelian Hodge theory and a finiteness result due Abe/Lafforgue.

1. Introduction

To state our main theorem, the following setup will be convenient.

Setup 1.1. Let r and f be positive integers. Let p be an odd prime and let k be a finite field containing $\mathbb{F}_{p^f}$. Set $W := W(k)$ to be the ring of Witt vectors of k and $K := \mathrm{Frac}(W)$. Let $(X,S)/W$ be a smooth projective scheme together with a relative simple normal crossings divisor over W. Set $U := X \backslash S$. Let x be a $\bar{K}$ point of U. For a positive integer $n \geq 1$ and a scheme T over W, the notation T_n refers to the reduction of T modulo p^n.

The following is our main result, in which the base point is suppressed.

Theorem 1.2. *Notation as in Setup 1.1. Then the following set*

$$\left\{ \rho\colon \pi_1^{et}(U_K) \to \mathrm{GL}_r(\mathbb{Z}_{p^f}) \;\middle|\; \begin{array}{l} \rho \text{ is log crystalline} \\ \text{with HT weights in } [a, a+p-1] \text{ for some } a \in \mathbb{Z} \\ \text{and } \rho^{geo}\colon \pi_1^{\acute{e}t}(U_{\bar{K}}) \to \mathrm{GL}_r(\mathbb{Q}_{p^f}) \text{ absolutely irreducible.} \end{array} \right\} \Big/ \begin{array}{l} \rho_1 \sim \rho_2 \text{ if} \\ \rho_1|_{\pi_1^{\acute{e}t}(U_{\bar{K}})} \cong \rho_2|_{\pi_1^{\acute{e}t}(U_{\bar{K}})} \end{array}$$

is finite.

Note that $\rho_1 \sim \rho_2$ in Theorem 1.2 if and only if there exists a character $\chi\colon \mathrm{Gal}(\bar{K}/K) \to \mathbb{Z}_{p^f}^{\times}$ such that $\rho_1 \cong \rho_2 \otimes \chi$ because ρ_1 and ρ_2 are assumed to be geometrically absolutely irreducible.

Crystalline representations are a p-adic analog of polarized variations of Hodge structures. Therefore Theorem 1.2 is an arithmetic analog of a theorem of Deligne [Del87]. See also the very recent work of Litt for a finiteness result in a different spirit [Lit18].

2. Preliminaries

Our main technique is p-adic nonabelian Hodge theory, which rests on the fundamental work of Ogus-Vologodsky [OV07]. First of all, we reduce Theorem 1.2 to the case of curves.

Lemma 2.1. *Notation as in Setup 1.1. Then there exists a smooth projective relative curve $C \subset X$ over W that intersects S transversely, with the property that $\pi_1(C_K \cap U_K) \to \pi_1(U_K)$ is surjective.*

Therefore, to prove Theorem 1.2, it suffices to consider the case when X/W has relative dimension 1.

Proof. We claim that there exists a smooth ample relative divisor $D \subset X$ over W that intersects S transversely. Indeed, pick some ample line bundle L on X; then for all $m \gg 0$, the map $H^0(X, L^m) \to H^0(X_1, L_1^m)$ is surjective. On the other hand, for $m \gg 0$, the vector space $H^0(X_1, L_1^m)$ has a section s_1 whose zero locus $V(s_1)$ is smooth and intersects S_1 transversely by Poonen's Bertini theorem [Poo04, Theorem 1.3]. Take any lift $s \in H^0(X, L^m)$ of s_1; then the zero locus $V(s)$ is smooth over W and intersects S transversely. Finally, it is well known that the map on fundamental groups $\pi_1(D_K \cap U_K) \to \pi_1(U_K)$ is surjective because $D_K \subset X_K$ is ample and D_K intersects S_K transversely. Proceed by induction.

Now, as $\pi_1(C_K \cap U_K) \to \pi_1(U_K)$ is surjective, it follows that to prove Theorem 1.2, it suffices to prove it for the pair $(C, S \cap C)$, i.e., we may reduce to the case of curves. □

In this article, a *de Rham bundle* is a vector bundle together with an integrable connection. We now recall the basic definitions we need for this article, including those of convergent F-isocrystals, overconvergent F-isocrystals and convergent log-F-isocrystals from [Ked18b, Definition 2.1, Definition 2.4 and Definition 7.1] in the context of maintaining Setup 1.1.

- Let K be a field of characteristic 0 and let $\mathcal{C}$ be a K-linear abelian category. Let L/K be a finite extension. We denote by $\mathcal{C}_L$, the extension of scalars, i.e., the category of pairs (X, ι) where $X \in \mathrm{Ob}(\mathcal{C})$ and $\iota \colon L \to \mathrm{End}_{\mathcal{C}}(X)$ is a K-algebra homomorphism. Morphisms are defined in the natural way. If $\mathcal{C}$ is a Tannakian category, so is $\mathcal{C}_L$. For any algebraic extension M/K, we set $\mathcal{C}_M$ to be the 2-colimit of the categories $\mathcal{C}_L$, indexed over finite subextensions $K \subset L \subset M$.
- Suppose there exists a lifting $\sigma \colon \mathcal{U} \to \mathcal{U}$ of the absolute Frobenius on U_1. A convergent F-isocrystal over U_1 is a de Rham bundle $\mathcal{E}$ over the Raynaud generic fiber $\mathcal{U}_K$ of the formal completion $\mathcal{U}$ of U along the special fiber U_1 together with an isomorphism $F \colon \sigma^*\mathcal{E} \to \mathcal{E}$ of de Rham bundles. Denote by $\mathbf{F}\text{-}\mathbf{Isoc}(U_1)$ the category of all convergent F-isocrystals over U_1. Up to canonical equivalence, this category does not depend on the choice of the lifting σ. In general, there may not exist a global lifting of the absolute Frobenius on U_1, but one can still define the category $\mathbf{F}\text{-}\mathbf{Isoc}(U_1)$ (see [Ked18b, definition 2.1]). One way to do this is as follows: we can find local liftings of absolute Frobenius on U_1, define local categories by using these local liftings as above, and use the canonical equivalences between local categories to glue them into a global one.
- A convergent F-isocrystal is called overconvergent if it can be extended to a strict neighborhood of $\mathcal{U}_K$ in X_K^{an}. Denote by $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(U_1)$ the category of all overconvergent F-isocrystal over U_1.
- A convergent log-F-isocrystal is a logarithmic de Rham bundle over X_K^{an} together with an isomorphism F of logarithmic de Rham bundles similar as that in the definition of convergent F-isocrystal (see e.g. [Ked18b, Definition 7.1]). For such objects, the residues of the underlying logarithmic isocrystal are automatically nilpotent. We denote the category of such objects by $\mathbf{F}\text{-}\mathbf{Isoc}_{\log}^{\mathrm{nilp}}(X_1, S_1)$.

We note that a convergent log-F-isocrystals can be algebraicalized to a vector bundle over X_K together with an integral logarithmic connection and a parallel semilinear action, because in our case X_K is proper. To a logarithmic crystalline representation, we may attach an overconvergent F-isocrystal. For a logarithmic crystalline representation $\rho\colon \pi_1(U_K) \to \mathrm{GL}_r(\mathbb{Z}_{p^f})$, according Faltings' definition of crystalline representation [Fal89], there exists an attached Fontaine-Faltings module $(V, \nabla, \mathrm{Fil}, \varphi, \iota)$[1] Forgetting the filtration and tensoring $\mathbb{Q}_p$, one gets the attached overconvergent F-isocrystal $(V, \nabla, \varphi, \iota)_{\mathbb{Q}_p}$.

We now show that a logarithmic crystalline representation being irreducible implies that the attached overconvergent F-isocrystal is also irreducible. While this is not strictly useful for the rest of the article, it seems to be of independent interest and is new to the best of our knowledge.

Lemma 2.2. *Notation as in Setup 1.1. Let $\rho\colon \pi_1(U_K) \to \mathrm{GL}_r(\mathbb{Z}_{p^f})$ be a logarithmic crystalline representation with associated logarithmic Fontaine-Faltings module $(V, \nabla, \mathrm{Fil}, \varphi, \iota)$ with endomorphism structure ι. If $\rho_{\mathbb{Q}}$ is irreducible then the overconvergent F-isocrystal $(V, \nabla, \varphi, \iota)_{\mathbb{Q}}$ in $\textbf{\textit{F-Isoc}}^\dagger(U_1)_{\mathbb{Q}_{p^f}}$ is irreducible.*

Proof. First of all, it follows from [Ked07, Theorem 6.4.5] that it suffices to check that $(V, \nabla, \varphi, \iota)_{\mathbb{Q}}$ is irreducible in $\mathbf{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X_1, S_1)_{\mathbb{Q}_{p^f}}$, the category of convergent logarithmic F-isocrystals together with nilpotent residues and multiplication by $\mathbb{Q}_{p^f}$. Our proof will proceed by contradiction.

Let Φ be a local lifting of the absolute Frobenius on (X_1, S_1). For the logarithmic Fontaine-Faltings module, locally the φ-structure can be represented as an isomorphism

$$\varphi\colon \Phi^*(\widetilde{V, \nabla, \mathrm{Fil}}) \xrightarrow{\simeq} (V, \nabla),$$

where $\widetilde{(\cdot)}$ is Faltings' tilde functor. In the case when V is p-torsion free, one may describe this as follows:

$$(\widetilde{V, \nabla, \mathrm{Fil}}) = \sum_i \frac{\mathrm{Fil}^i(V, \nabla)}{p^i} \subset (V, \nabla, \mathrm{Fil})_{\mathbb{Q}}.$$

That φ is an isomorphism encodes the strong divisibility in the definition of a Fontaine-Faltings module. After shifting the filtration, we assume $\mathrm{Fil}^0 V = V$. In this case, $(V, \nabla) \subset (\widetilde{V, \nabla, \mathrm{Fil}})$. and φ can be restricted on $\Phi^*(V, \nabla)$,

$$\varphi\colon \Phi^*(V, \nabla) \to (V, \nabla).$$

This yields the underlying logarithmic F-crystal.

Suppose the F-isocrystal $(V, \nabla, \varphi, \iota)_{\mathbb{Q}}$ is not irreducible in $\mathbf{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X, S)_{\mathbb{Q}_{p^f}}$. Let $(\mathcal{V}', \nabla', \varphi', \iota')$ be a proper sub F-isocrystal of $(V, \nabla, \varphi, \iota)_{\mathbb{Q}}$ in $\mathbf{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X, S)_{\mathbb{Q}_{p^f}}$. In particular, $\mathcal{V}'$ is stable under the $\mathbb{Q}_{p^f}$-action ι, the connection ∇, and φ. There is a natural choice of lattice:

$$V' = \mathcal{V}' \cap V$$

Then the restriction of φ induces a map

$$\varphi'\colon \Phi^*(V', \nabla') \to (V', \nabla'), \tag{2.2.1}$$

since $\varphi(\Phi^*((V', \nabla'))) \subset \varphi(\Phi^*((\mathcal{V}', \nabla')) \cap \varphi(\Phi^*((V, \nabla))) \subset (\mathcal{V}', \nabla') \cap (V, \nabla) = (V', \nabla')$. Denote Fil' the restriction of Fil on V'. The endomorphism structure clearly restricts on the quadruple $(V', \nabla', \mathrm{Fil}', \varphi')$. In the following, we show that $(V', \nabla', \mathrm{Fil}', \varphi')$ forms a sub-Fontaine-Faltings

[1] Faltings' original definition is for $\mathbb{Z}_p$-representations. It can be easily extended to $\mathbb{Z}_{p^f}$-representations by adding an endomorphism structures ι on the side of Fontaine-Faltings modules. More precisely, see [LSZ19].

module of $(V,\nabla,\mathrm{Fil},\varphi)$. As the triple (V',∇',φ') is a logarithmic F-crystal in finite, locally free modules, we must check that the pair (Fil',φ') is strongly divisible, i.e., that the isogeny $\varphi\colon \Phi^*(V',\nabla')\to(V',\nabla')$ extends to an isomorphism

$$\Phi^*\widetilde{(V',\nabla',\mathrm{Fil}')}\to(V',\nabla')$$

Denote

$$(V'',\nabla'',\mathrm{Fil}''):=(V,\nabla,\mathrm{Fil})/(V',\nabla',\mathrm{Fil}').$$

We constructed the embedding $(V',\nabla',\mathrm{Fil}')\hookrightarrow(V,\nabla,\mathrm{Fil})$ to be saturated and strict with respect to the filtrations. Therefore the triple $(V'',\nabla'',\mathrm{Fil}'')$ is a filtered logarithmic de Rham bundle.

Applying Faltings' tilde functor, one has short exact sequence

$$0\to\widetilde{V'}\longrightarrow\widetilde{V}\longrightarrow\widetilde{V''}\to 0.$$

Locally, one has the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Phi^*\widetilde{V'} \arrow[r] \arrow[d, "\varphi'"] & \Phi^*\widetilde{V} \arrow[r] \arrow[d, "\varphi", "\cong"'] & \Phi^*\widetilde{V''} \arrow[r] \arrow[d, dotted, "\exists\varphi''"] & 0 \\
0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0
\end{tikzcd}$$

where φ' (by abusing notation as in (2.2.1)) is the restriction of φ on $\Phi^*\widetilde{V'}$, which extends the φ' in (2.2.1). The image $\varphi'(\Phi^*\widetilde{V'})$ is contained in V', because

$$\varphi(\Phi^*(\widetilde{V'}))\subset\varphi(\Phi^*(\mathcal{V}')\cap\varphi(\Phi^*(\widetilde{V}))\subset\mathcal{V}'\cap V=V'.$$

Since φ is surjective, φ'' is also surjective. On the other hand $\Phi^*\widetilde{V''}$ and V'' are bundles with the same rank, so φ'' is actually an isomorphism. By the snake lemma φ' is also an isomorphism. This proves the strong divisibility, as desired.

By the Fontaine-Lafaille-Faltings correspondence, to $(V',\nabla',\mathrm{Fil}',\varphi',\iota')$ one may attach a (log crystalline) subrepresentation ρ' of ρ of strictly smaller rank [Fal89, Theorem 2.6* (i)]. It follows that $\rho_{\mathbb{Q}}$ is not irreducible, contradicting our hypothesis. □

The following is a version of Lemma 2.2 with the stronger hypothesis that $\rho_{\mathbb{Q}}$ is *geometrically absolutely irreducible.* With these assumptions, we show something much stronger than the conclusion of Lemma 2.2. This will be essential in the proof of Theorem 1.2.

Lemma 2.3. *Notation as in Setup 1.1 and suppose X/W is a curve. Let $\rho\colon\pi_1(U_K)\to GL_r(\mathbb{Z}_{p^f})$ be a logarithmic crystalline representation such that $\rho_{\mathbb{Q}}$ is geometrically absolutely irreducible. Let $(V,\nabla,\mathrm{Fil},\Phi,\iota)$ be the associated logarithmic Fontaine-Faltings module with endomorphism structure ι. Let $(V,\nabla)^{(0)}$ be the identity eigenspace of the ι-action. Then the following holds.*

1. *The logarithmic de Rham bundle $(V,\nabla)^{(0)}_{\mathbb{Q}_p^{\mathrm{unr}}}$ on $(X,S)_{\mathbb{Q}_p^{\mathrm{unr}}}$ admits no proper de Rham subbundle.*
2. *the object $(V,\nabla,\Phi,\iota)_{\mathbb{Q}}\in$ **F-Isoc**$^\dagger(U_k)_{\mathbb{Q}_{p^f}}$ is absolutely irreducible. That is, after extending the coefficient field from $\mathbb{Q}_{p^f}$ to $\bar{\mathbb{Q}}_p$, the object is still irreducible.*

Proof. We first prove the first statement. Fist of all, $(V,\nabla)^{(0)}_K$ is a semistable de Rham bundle of degree 0. Suppose for contradiction that there is a logarithmic de Rham subbundle $(V',\nabla')_{K'}$ of $(V,\nabla)_{K'}$ for some finite unramified extension K'/K. Set $\mathcal{O}'$ to be the ring of integers and k' to be the residue field. Then $(V',\nabla')_{K'}$ automatically has nilpotent residues and hence also has degree 0 and is therefore semistable. We claim that we may find a logarithmic de Rham subbundle (W,∇) of $(V,\nabla)^{(0)}_{\mathcal{O}'}$ over $(X,S)_{\mathcal{O}'}$ such that $(W,\nabla)_{K'}\cong(V',\nabla')_{K'}$ and $(W,\nabla)_{k'}$ is semistable of degree 0. First of all, there is clearly an extension to a torsion-free logarithmic de Rham subsheaf (W,∇). By [KYZ20c, Section 6], the degree of $W_{k'}$ is 0; therefore $(W,\nabla)_{k'}$ is a degree 0 logarithmic de Rham subsheaf of $(V,\nabla)_{k'}$. Note that $(W,\nabla)_{k'}\subset(V,\nabla)^{(0)}_{k'}$; as the latter is semistable of degree 0, it follows that the inclusion $(W,\nabla)_{k'}\subset(V,\nabla)^{(0)}_{k'}$ is saturated and hence is a de Rham subbundle.[2] As both de Rham bundles have degree 0 and $(V,\nabla)_{k'}$ is semistable, it follows that $(W,\nabla)_{k'}$ is semistable.

Let $\mathcal{HDF}$ be the (logarithmic) Higgs-de Rham flow attached to ρ. Run the Higgs-de Rham flow over X with initial term (W,∇), where the Hodge filtrations are chosen to be the restrictions of Fil on V. One obtains a sub Higgs-de Rham flow $\mathcal{HDF}'$ of $\mathcal{HDF}$. In general, this sub Higgs-de Rham is *not preperiodic*. Nonetheless, we claim that $\mathcal{HDF}'$ is preperiodic over each truncated level $W_m(k')$. This holds because $(V,\nabla)_{W_m(k')}$ has only finitely many subbundles of degree 0 for each m.

Note that because there are only finitely many (isomorphism classes of) Higgs terms in $\mathcal{HDF}$ by the periodicity. Therefore we may inductively shift the index of $\mathcal{HDF}'$, to find a sequence of sub Higgs-de Rham flows $\mathcal{HDF}'_{W_m(k')}\subset\mathcal{HDF}_{W_m(k')}$ which are periodic with periodicity f_m, and satisfying $\mathcal{HDF}'_{W_{m+1}(k')}\equiv\mathcal{HDF}'_{W_m(k')} \pmod{p^m}$ and $f_m\mid f_{m+1}$.

To each of these truncated periodic Higgs-de Rham flows, there is an associated torsion logarithmic crystalline representation

$$\rho'_m\colon \pi_1(U_{\mathbb{Q}_p^{\mathrm{unr}}})\to GL_s(W_m(\overline{k})).$$

Recall that $\widehat{\mathcal{O}_{\mathbb{Q}_p^{\mathrm{unr}}}}$, the p-adic completion of $\mathcal{O}_{\mathbb{Q}_p^{\mathrm{unr}}}$, is equal to $W(\overline{k})$. Taking the inverse limit over m, one obtains a sub-representation

$$\rho'\colon \pi_1(U_{\mathbb{Q}_p^{\mathrm{unr}}})\to GL_s(\widehat{\mathcal{O}_{\mathbb{Q}_p^{\mathrm{unr}}}})$$

of $\rho\colon \pi_1(U_{\mathbb{Q}_p^{\mathrm{unr}}})\to GL_r(\widehat{\mathcal{O}_{\mathbb{Q}_p^{\mathrm{unr}}}})$. We claim this is in contradiction with the fact that $\rho\otimes\mathbb{Q}_p$ is geometrically absolutely irreducible. Indeed,$\rho'\mid_{\pi_1(U_{\bar{K}})}\otimes\mathbb{C}_p$ is a non-trivial sub-representation of $\rho\mid_{\pi_1(U_{\bar{K}})}\otimes\mathbb{C}_p$; on the other hand, the fact that $\rho\mid_{\pi_1(U_{\bar{K}})}\otimes\overline{\mathbb{Q}}_p$ is irreducible implies that $\rho\mid_{\pi_1(U_{\bar{K}})}\otimes\mathbb{C}_p$ is also irreducible.

We now prove the second part. First of all, the functor $\mathbf{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X_k,S_k)\to\mathbf{F\text{-}Isoc}^{\dagger}(U_k)$ is fully faithful and reflects subobjects. Therefore the same is true for the base change to any algebraic field extension $L/\mathbb{Q}_p$.

Now, note that an object $\mathcal{E}$ in $\mathbf{F\text{-}Isoc}^{\dagger}(U_k)_{\mathbb{Q}_{p^f}}$ is absolutely irreducible if and only if $\mathcal{E}_{\mathbb{Q}_p^{\mathrm{unr}}}\in\mathbf{F\text{-}Isoc}^{\dagger}(U_k)_{\mathbb{Q}_p^{\mathrm{unr}}}$ is irreducible. This follows from local class field theory.[3]

[2] Note that because X_k is a smooth curve, any torsion-free sheaf is automatically a vector bundle.

[3] Indeed, the object $\mathcal{E}$ is absolutely irreducible if and only if the natural map $\mathbb{Q}_{p^f}\hookrightarrow\mathrm{End}(\mathcal{E})$ is an isomorphism. Suppose that $\mathcal{E}$ has rank r and assume without loss of generality that $\mathcal{E}$ is irreducible in $\mathbf{F\text{-}Isoc}^{\dagger}(U_1)$ (otherwise there

Our goal is to prove that $\mathcal{E} := (V, \nabla, \Phi, \iota)_{\mathbb{Q}} \in \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(U_k)_{\mathbb{Q}_{p^f}}$ is absolutely irreducible. Replacing f be some multiple, it suffices to prove that $\mathcal{E} := (V, \nabla, \Phi, \iota)_{\mathbb{Q}} \in \mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(U_k)_{\mathbb{Q}_{p^f}}$ is irreducible. Suppose there exists an overconvergent sub-F-isocrystal $\mathcal{E}'$ of $\mathcal{E}$ in $\mathbf{F}\text{-}\mathbf{Isoc}^{\dagger}(U_k)_{\mathbb{Q}_{p^f}}$. Then $\mathcal{E}'$ may be represented by a quadruple $(V', \nabla, \Phi, \iota)_K$ on $\mathcal{U}_K$. By the above, this is automatically a logarithmic sub F-isocrystal, and hence (V', ∇') has nilpotent residues. Let $(V', \nabla')^{(0)}_K$ denote the identity eigenspace of ι on $(V', \nabla')_K$. Then this is a proper de Rham subbundle of $(V, \nabla)^{(0)}_K$, contradicting the first part of this lemma.

□

We come to the following crucial definition.

Definition 2.4. Notation as in Setup 1.1. Let $(\mathcal{V}, \nabla, \varphi, \iota)$ be an object of $\mathbf{F}\text{-}\mathbf{Isoc}^{\mathrm{nilp}}_{\log}(X_1, S_1)_{\mathbb{Q}_{p^f}}$. An *extension* of $(\mathcal{V}, \nabla, \varphi, \iota)$ is an logarithmic F-crystal in finite, locally free modules $(V, \nabla, \varphi, \iota)$ with $\mathbb{Z}_{p^f}$-structure such that $(V, \nabla, \varphi, \iota)_{\mathbb{Q}_p} \cong (\mathcal{V}, \nabla, \varphi, \iota)$. An extension $(V, \nabla, \varphi, \iota)$ is said to be *semistable* if the logarithmic flat connection $(V, \nabla)_1$ on (X_1, S_1) is semistable.

Recall that a rank-1 F-crystal over k with $\mathbb{Z}_{p^f}$-structure can be identified with a pair (L, φ) where L is a finite free $W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f}$-module and $\varphi \colon L \to L$ is an injective $\sigma \otimes 1$-semilinear map where $\sigma \colon W \to W$ is the canonical lift of Frobenius. For any element $r \in \left(K \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^f}\right)^{\times} \cap (W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f})$, we denote $L_r = W \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^f} \cdot e$ with $\varphi_{L_r}(e) = re$. Conversely, for any rank-1 F-crystal over W with $\mathbb{Z}_{p^f}$-structure is isomorphic to some L_r. By tensoring $\mathbb{Q}_p$, one gets an rank-1 F-isocrystal $\mathcal{L}_r = L_r \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ over k with $\mathbb{Q}_{p^f}$-structure.

Let $(V, \nabla, \varphi, \iota)$ be a logarithmic F-crystal in finite, locally free modules over (X_1, S_1) with $\mathbb{Z}_{p^f}$-structure. Locally one can view φ as a $\Phi \otimes 1$-semilinear map of the $\mathcal{O}_X \otimes \mathbb{Z}_{p^f}$-modules. We define the twist of $(V, \nabla, \varphi, \iota)$ by L_r to be: $(V, \nabla, \varphi, \iota) \otimes L_r = (V, \nabla, r \cdot \varphi, \iota)$.

Remark 2.5. *Twisting by a constant rank 1 object does not change the underlying de Rham bundle.*

Lemma 2.6. *Let $(\mathcal{V}, \nabla, \varphi, \iota)$ be an object of $\boldsymbol{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X_1, S_1)_{\mathbb{Q}_{p^f}}$. Let L be a rank-1 F-crystal over k with $\mathbb{Z}_{p^f}$-structure and let $\mathcal{L} = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Denote $\varphi' = \varphi \otimes \varphi_L$. Then tensoring L induces an injection*

$$\{\textit{extension of } (\mathcal{V}, \nabla, \varphi, \iota)\} \to \{\textit{extension of } (\mathcal{V}, \nabla, \varphi', \iota)\}$$

Proof. Since an extension of an F-isocrystal is uniquely determined by the extension of the underlying de Rham bundle and twisting a constant rank-1 object doesn't change the underlying de Rham bundle, the map is injective.

□

Lemma 2.7. *Notation as in Setup 1.1, and suppose X/W is a curve. Let $(\mathcal{V}, \nabla, \varphi, \iota)$ be an irreducible object of $\boldsymbol{F\text{-}Isoc}^{\mathrm{nilp}}_{\log}(X_1, S_1)_{\mathbb{Q}_{p^f}}$. Then there exists only finitely many isomorphism classes of semistable extensions $(V, \nabla, \varphi, \iota)$ of $(\mathcal{V}, \nabla, \varphi, \iota)$.*

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is nothing to prove). Then $\mathrm{End}(\mathcal{E})$ is a division algebra over $\mathbb{Q}_{p^f}$, represented by some Brauer class in $\frac{1}{r}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$ by local class field theory. Then we claim that that $\mathcal{E}$ is absolutely irreducible if and only if $\mathcal{E}_{\mathbb{Q}_{p^{fr}}}$ is irreducible; this is because for any finite extension L/K of degree r of p-adic local fields, the induced map on Brauer groups $\mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is given by multiplication by r. The result readily follows.

Proof. We note that an extension of $(\mathcal{V},\nabla,\varphi,\iota)$ is an extension (V,∇) of $(\mathcal{V},\nabla)$ which is preserved by the two structures φ and ι. Thus the intersections, sums and projective limits of extensions are also preserved by the two structures. Assume there are infinitely many isomorphism classes of semistable extensions, choose a representative from each isomorphism class, and enumerate them as follows $\{T_i=(V,\nabla,\varphi,\iota)_i\mid i\in I\}$. We will construct an infinite descending chain $T_{i_0}\supset T_{i_1}\supset T_{i_2}\dots$ such that the intersection yields a proper logarithmic sub-F-isocrystal, contradicting out original assumption.

Fix one element $i_0\in I$, and set $I_0=I$ and $J_0=I_0-\{i_0\}$. By assumption, we may embed each (V,∇,φ,ι) as a lattice in $(\mathcal{V},\nabla,\varphi,\iota)$. By multiplying by a suitable power of p on each T_i, for $i\in J_0$, we may assume that

$$T_i\subset T_{i_0}\quad\text{ and }\quad T_i\not\subset pT_{i_0}.$$

This is equivalent to saying that the image of V_i in V_{i_0}/pV_{i_0} is a non-zero proper submodule. In fact, we claim that the image, namely $(V_i+pV_{i_0})/pV_{i_0}$, together with the induced logarithmic flat connection, is a semistable logarithmic de Rham bundle on (X_1,S_1). Indeed, both V_{i_0}/pV_{i_0} and V_i/pV_i admit semistable flat connections of degree zero. Therefore the image $(V_i+pV_{i_0})/pT_{i_0}$ has degree 0 and hence, when equipped with the induced connection, is semistable. Finally, we claim that $(V_i+pV_{i_0})/pV_{i_0}$ is a subbundle of V_{i_0}/pV_{i_0} (as opposed to merely a subsheaf). If not, the saturation would be a subbundle; but any non-trivial saturation increases the degree. As $(V_i+pV_{i_0})/pV_{i_0}$ with the induced flat connection is semi-stable, so is the saturation; this contradicts semistability of V_{i_0}/pV_{i_0}.

Consider the map

$$\begin{aligned} f_0\colon J_0 &\longrightarrow \Sigma_0:=\{\text{ proper sub bundles of } V_{i_0}/p\cdot V_{i_0}\text{ of degree } 0\}\\ i &\longmapsto \Big(V_i+pV_{i_0}\Big)/pV_{i_0}\end{aligned}$$

The initial set is infinite by assumption. On the other hand, the terminal set is finite; indeed, this follows because the set all subbundles with fixed degree of a given bundle forms a bounded family and our base field is finite.

Thus there exists a submodule $\overline{M}_0$ of V_{i_0}/pV_{i_0} such that $I_1:=f_0^{-1}(\overline{M}_0)$ is infinite. For any fixed $i\in I_1$, the submodule $V_i+pV_{i_0}$ is the inverse image of $\overline{M}_0$ under surjective map $V_{i_0}\to V_{i_0}/pV_{i_0}$; hence, the submodule $V_i+pV_{i_0}$ does not depend on the choice of $i\in I_1$. We further claim that for each $i\in I_0$, the module $V_i+pV_{i_0}$ together with the induced logarithmic flat connection, Frobenius structure, and endomorphism structure, yields a semistable extension of $(\mathcal{V},\nabla,\varphi,\iota)$. This follows from the fact that V_i/pV_i and V_0/pV_0, equipped with their flat connections, are semistable de Rham bundles of degree 0. Thus there exists $i_1\in I_1$ such that

$$T_{i_1}=T_i+pT_{i_0}\text{ for all } i\in I_1.$$

Denote $J_1=I_1\setminus\{i_1\}$. Then for all $i\in J_1$ one has

$$T_i\subsetneqq T_{i_1}\subsetneqq T_{i_0}\quad\text{ and }\quad T_i\not\subset pT_{i_1}.$$

Repeating the process, one can find a sequence of extensions

$$\cdots\subsetneqq T_{i_3}\subsetneqq T_{i_2}\subsetneqq T_{i_1}\subsetneqq T_{i_0}$$

satisfying $T_{i_m}\not\subset pT_{i_n}$ for all $m,n\geq 0$.

Denote $T_\infty = \cap_{k=0}^\infty T_{i_k}$. In the following we show that $(T_\infty)_{\mathbb{Q}_p}$ is a proper logarithmic sub F-isocrystal of $(\mathcal{V},\nabla,\varphi,\iota)$. Thus we get a contradiction with the irreducibility of $(\mathcal{V},\nabla,\varphi,\iota) \in \mathbf{F\text{-}Isoc}_{\log}^{\mathrm{nilp}}(X_1,S_1)_{\mathbb{Q}_{p^f}}$.

Locally, we may assume T_{i_k} are free modules of the same rank r over a regular local ring R satisfying

$$\cdots \subsetneq T_{i_3} \subsetneq T_{i_2} \subsetneq T_{i_1} \subsetneq T_{i_0}.$$

Since $T_\infty = \cap_{k=0}^\infty T_{i_k}$ is torsion-free and finitely generated over R, we may choose a free sub R-module of $T'_\infty \subseteq T_\infty$ with maximal rank $r_\infty \le r$. We only need to show

$$T_\infty \neq 0 \text{ and } r_\infty \neq r.$$

We first show that $T_\infty \neq 0$. For a given positive integer n, consider the descending sequence

$$\left(\left(T_{i_k} + p^n T_{i_0}\right)/p^n T_{i_0}\right)_k.$$

We claim the sequence stabilizes for $k \gg 0$. Each term is contained in $T_{i_0}/p^n T_{i_0}$. Let's consider the index between T_{i_0} and $p^n T_{i_0}$, which has only p-primary part and is finite. Thus the increasing sequence of the index $[T_{i_0} : T_{i_k} + p^n T_{i_0}]$ with upper bound $[T_{i_0} : p^n T_{i_0}]$ is stable. This implies $[T_{i_0} : T_{i_k} + p^n T_{i_0}] = [T_{i_0} : T_{i_{k+1}} + p^n T_{i_0}]$ for sufficiently large k, So $T_{i_k} + p^n T_{i_0} = T_{i_{k+1}} + p^n T_{i_0}$ for $k \gg 0$. Denote

$$\overline{T}_0^{(n)} := \bigcap_{k=0}^{\infty} \left(T_{i_k} + p^n T_0\right)/p^n T_0 = \left(T_N + p^n T_0\right)/p^n T_0 \neq 0, \text{ for } N >> 0.$$

Thus one has an surjective inverse system

$$\cdots \twoheadrightarrow \overline{T}_0^{(3)} \twoheadrightarrow \overline{T}_0^{(2)} \twoheadrightarrow \overline{T}_0^{(0)} \twoheadrightarrow \overline{T}_0^{(0)}$$

whose inverse limit $\varprojlim_n \overline{T}_0^{(n)}$ is non-empty. By the left exactness of inverse limits, the inclusion maps

$$\left(\overline{T}_0^{(n)} \subset \left(T_{i_k} + p^n T_0\right)/p^n T_0\right)_n$$

induce an injective map

$$\varprojlim_n \overline{T}_0^{(n)} \hookrightarrow \varprojlim_n \left(T_{i_k} + p^n T_0\right)/p^n T_0 = T_{i_k}.$$

Thus $\varprojlim_n \overline{T}_0^{(n)} \subset T_\infty = \bigcap_k T_{i_k}$. This implies that $T_\infty \neq 0$. The fact that $T_\infty \neq 0$ immediately implies that T_∞ yields a logarithmic F-crystal in finite modules on (X_1,S_1).

We now show that $r_\infty \neq r$. By étale localization, we reduce to the following setup in linear algebra. Let $A = W < x >$ be the p-adic completion of a polynomial ring in a single variable over W, and let $M_0 \supsetneq M_1 \supsetneq \dots$ be an infinite nested collection of finite free modules of fixed rank r, that is strictly decreasing and such that $M_j \not\subset pM_k$ for $j,k \ge 0$. Set $M_\infty := \cap_{j=0}^\infty M_j$. We wish to prove that M_∞ has rank smaller than r; equivalently, that it does not contain a lattice L_∞ in $M_0 \otimes \mathrm{Frac}(A)$. If it did, then M_∞ would have finite, p-primary index in M_0. However, the index of M_j in M_0 gets arbitrarily large; indeed, if $[M_j : M_k] = 1$, then $M_j = M_k$. As index is multiplicative, we obtain a contradiction.

□

Lemma 2.8. *Notation as in Setup 1.1. Let $(\mathcal{V},\nabla,\varphi,\iota)$ be an irreducible object of $\boldsymbol{F}$-$\boldsymbol{Isoc}^{\mathrm{nilp}}_{\log}(X_1,S_1)_{\mathbb{Q}_{p^f}}$. Let (V,∇,φ,ι) be an extension of $(\mathcal{V},\nabla,\varphi,\iota)$ to (X,S). Then there exists only finitely many Hodge filtrations Fil_1 on $(V,\nabla,\iota)_1 := (V,\nabla,\iota) \pmod p$ with $\mathrm{Fil}^0_1V_1 = V_1$ and $\mathrm{Fil}^p_1V_1 = 0$ such that there exists φ_1 rendering the quintuple $(V,\nabla,\mathrm{Fil},\varphi,\iota)_1$ a logarithmic Fontaine-Faltings module with endomorphism structure over (X_1,S_1).*

Proof. By the Fontaine-Lafaille-Faltings correspondence [Fal89, Theorem 2.6*(i)], the category of p-torsion logarithmic Fontaine-Faltings modules (with endomorphism structure) on (X_1,S_1) is equivalent to the category of logarithmic crystalline representations of $\pi_1^{\text{ét}}(U_K)$ with coefficients in $\mathbb{F}_{p^f}$. The étale fundamental group $\pi_1^{\text{ét}}(U_K)$ is topologically finitely generated. Therefore the set of isomorphism classes of $\mathrm{GL}_r(\mathbb{F}_{p^f})$ representations of $\pi_1^{\text{ét}}(U_K)$ is finite. In particular, there are only finitely many isomorphism classes of crystalline $\mathrm{GL}_r(\mathbb{F}_{p^f})$ representations. Forgetting the φ-structure, it follows that the set of isomorphism classes of de Rham bundles (with endomorphism structure) which underlie a Fontaine-Faltings module (with endomorphism structure) over (X_1,S_1) is also finite.

Suppose there are infinitely many distinct Hodge filtrations $\mathrm{Fil}_1^{(i)}$ $(i = 1,2,\cdots)$ on $(V,\nabla,\iota)_1$ such that for each i, there exists $\varphi^{(i)}$ rendering the quintuple $(V,\nabla,\mathrm{Fil},\varphi,\iota)_1$ a Fontaine-Faltings module. By the pigeonhole principle, there are infinitely i such that there exists a log Fontaine-Faltings module $(V,\nabla,\mathrm{Fil}^{(i)},\varphi^{(i)},\iota)_1$ whose isomorphism class is independent of i. In particular, one deduces $\mathrm{Aut}(V_1)$ is a infinite set. But this contradicts the finiteness of $\mathrm{Aut}(M)$ for any vector bundle M over X_1, as our base field is finite. □

Lemma 2.9. *Notation as in 1.1. Let $(\mathcal{V},\nabla,\varphi,\iota)$ be an irreducible object of $\boldsymbol{F}$-$\boldsymbol{Isoc}^{\mathrm{nilp}}_{\log}(X_1,S_1)_{\mathbb{Q}_{p^f}}$. Let (V,∇,φ,ι) be an extension $(\mathcal{V},\nabla,\varphi,\iota)$. Let Fil_1 be a Hodge filtration on $(V,\nabla,\iota)_1 := (V,\nabla,\iota) \pmod p$ with $\mathrm{Fil}^0_1V_1 = V_1$ and $\mathrm{Fil}^p_1V_1 = 0$ such that there exists φ_1 rendering the quintuple $(V,\nabla,\mathrm{Fil},\varphi,\iota)_1$ a logarithmic Fontaine-Faltings module with endomorphism structure over X_1. Assume there exists two liftings Fil and Fil' of the Hodge filtration Fil_1.*

Then there exists an automorphism $f\colon (V,\nabla,\iota)\to(V,\nabla,\iota)$ such that

$$\mathrm{Fil} = f^*(\mathrm{Fil}').$$

In other words, one has an isomorphism $f\colon (V,\nabla,\mathrm{Fil},\iota)\to(V,\nabla,\mathrm{Fil}',\iota)$.

Proof. We first consider the trivial endomorphism structure case. Since the data $(V,\nabla,\mathrm{Fil},\varphi)_1$ forms a Fontaine-Faltings module, the graded Higgs bundle (E_1,θ_1) initiates the attached 1-periodic Higgs de Rham flow. According [KYZ20a, Lemma 6.1], the Hodge-de Rham spectral sequence attached to $(V,\nabla,\mathrm{Fil})_1$ degenerated at E_1. Now the assumption in [KYZ20a, Theorem 1.6(2)] is fulfilled, thus there exists such an isomorphism $f\colon (V,\nabla,\mathrm{Fil})\to(V,\nabla,\mathrm{Fil}')$.

For the general case, we consider the f-periodic Higgs de Rham flow attaching to $(V,\nabla,\mathrm{Fil},\varphi,\iota)_1$. We note that all filtered de Rham bundles appearing in the flow are just those eigenspaces of the action of ι on $(V,\nabla,\mathrm{Fil})_1$. To find an isomorphism $f\colon (V,\nabla,\mathrm{Fil},\iota)\to(V,\nabla,\mathrm{Fil}',\iota)$, we only need to find one between the corresponding eigenspaces. By the periodicity and similar proof as [KYZ20a, Lemma 6.1], the Hodge-de Rham spectral sequence attached to each filtered de Rham bundle appeared in the flow degenerates at E_1. So the result follows from [KYZ20a, Theorem 1.6(2)]. □

3. The Proof

The proof of the main theorem of this article is diagrammatically sketched below; the definition of the various terms will follow. Here is a two sentence summary of the proof. The Langlands correspondence implies that $\mathrm{MF}^{\mathrm{twist}}_{\textbf{F-Isoc}}$ is finite. By following the following diagram from $\mathrm{MF}^{\mathrm{twist}}_{\textbf{F-Isoc}}$ (on the right) to $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar{\Gamma})$ (on the bottom left), it follows that $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar{\Gamma})$ is finite.

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma) \arrow[ddd, "\infty:1"', "\text{restriction}"] & \mathrm{MF}_{\textbf{FF}} \arrow[l, "\mathbb{D}"', "1:1"] \arrow[r, two heads] \arrow[dd, "\infty:1"] \arrow[rrr, bend left, "\text{Lemma 2.2}"] & \mathrm{MF}_{\textbf{F-Cris}} \arrow[r, two heads, "\text{Lemma 2.7}", "n:1"'] \arrow[d, "\infty:1"] & \mathrm{MF}_{\textbf{F-isoc}} \arrow[d, "\infty:1"] \\
 & & \mathrm{MF}^{\mathrm{twist}}_{\textbf{F-Cris}} \arrow[r, two heads, "\text{Lemma 3.1}", "n:1"'] \arrow[d, "\text{Remark 2.5}"] & \mathrm{MF}^{\mathrm{twist}}_{\textbf{F-Isoc}} \\
 & \mathrm{MF}_{\textbf{FdR}} \arrow[r, two heads, "\text{Lemma 2.8,2.9}", "n:1"'] \arrow[d, "\mathrm{Gr}"] & \mathrm{MF}_{\textbf{dR}} & \\
\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar{\Gamma}) & \mathrm{MF}_{\textbf{Higgs}} \arrow[l, two heads, "\text{Faltings' } p\text{-adic}", "\text{Simpson corr.}"'] & &
\end{tikzcd}$$

We explain all of the terms in the above diagram.

- $\Gamma = \pi_1^{\text{ét}}(U_K, x)$ and $\bar{\Gamma} = \pi_1^{\text{ét}}(U_{\bar{K}}, x)$.
- $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma)$ is the set of isomorphism classes of logarithmic crystalline representations $\rho\colon \Gamma \to \mathrm{GL}_r(\mathbb{Z}_{p^f})$ whose Hodge Tate weights are located in $[0, p-1]$ such that $\rho_{\mathbb{Q}}\colon \Gamma \to \mathrm{GL}_r(\mathbb{Q}_{p^f})$ is geometrically absolutely irreducible.
- $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar{\Gamma})$ is the set of isomorphism classes of representations of $\bar{\Gamma}$ that come from $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma)$ under the restriction map induced by the natural embedding map $\bar{\Gamma} \hookrightarrow \Gamma$. Thus one has a surjective map
$$\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma) \twoheadrightarrow \mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar{\Gamma}).$$
- $\mathrm{MF}_{\textbf{FF}}$ is the set of isomorphism classes of logarithmic Fontaine-Faltings module $(V, \nabla, \mathrm{Fil}, \varphi)$ with endomorphism structure $\iota\colon \mathbb{Z}_{p^f} \hookrightarrow \mathrm{End}(V, \nabla, \mathrm{Fil}, \varphi)$ associated to representations in $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma)$ via [Fal89]. Thus one has an bijection
$$\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma) \xrightarrow{1:1} \mathrm{MF}_{\textbf{FF}}.$$
- $\mathrm{MF}_{\textbf{FdR}}$ is the image of $\mathrm{MF}_{\textbf{FF}}$ under the map that sends an isomorphism class of a logarithmic Fontaine-Faltings module $[(V, \nabla, \mathrm{Fil}, \varphi, \iota)]$ to the isomorphism class of the quadruple

$(V, \nabla, \mathrm{Fil}, \iota)$ in the additive category of filtered de Rham bundles equipped with a $\mathbb{Z}_{p^f}$-endomorphism structure. Thus one has a surjective map

$$\mathrm{MF}_{\mathbf{FF}} \twoheadrightarrow \mathrm{MF}_{\mathbf{FdR}}.$$

- $\mathrm{MF}_{\mathbf{F\text{-}Cris}}$ is the image of $\mathrm{MF}_{\mathbf{FF}}$ under the map that sends an isomorphism class of a logarithmic Fontaine-Faltings module to the isomorphism class of the underlying logarithmic F-crystal in locally free modules with endomorphism structure: $[(V, \nabla, \mathrm{Fil}, \varphi, \iota)] \mapsto [(V, \nabla, \varphi, \iota)]$. Thus one has a surjective map
$$\mathrm{MF}_{\mathbf{FF}} \twoheadrightarrow \mathrm{MF}_{\mathbf{F\text{-}Cris}}.$$
- $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Cris}}$ is the set of equivalence classes of the set $\mathrm{MF}_{\mathbf{F\text{-}Cris}}$ modulo the equivalence relations defined by twisting a by constant rank 1 Fontaine-Faltings modules (with endomorphism structure).
- $\mathrm{MF}_{\mathbf{dR}}$ is the image of $\mathrm{MF}_{\mathbf{F\text{-}Cris}}$ under the map that sends an isomorphism class of a logarithmic F-crystal to the isomorphism class of the underlying logarithmic de Rham *bundle* with endomorphism structure: $[(V, \nabla, \varphi, \iota)] \mapsto [(V, \nabla, \iota)]$. Thus one has surjective maps
$$\mathrm{MF}_{\mathbf{FdR}} \twoheadrightarrow \mathrm{MF}_{\mathbf{dR}} \twoheadleftarrow \mathrm{MF}_{\mathbf{F\text{-}Cris}}.$$
By Remark 2.5 the second surjective map factors through $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Cris}}$.
- $\mathrm{MF}_{\mathbf{Higgs}}$ is the image of $\mathrm{MF}_{\mathbf{FdR}}$ under the map that sends an isomorphism class of filtered de Rham bundle with endomorphism structure to the isomorphism class of the associated Higgs bundle with endomorphism structure: $[(V, \nabla, \mathrm{Fil}, \iota)] \mapsto [\mathrm{Gr}(V, \nabla, \mathrm{Fil}, \iota)]$. Thus one has a surjective map
$$\mathrm{MF}_{\mathbf{FdR}} \twoheadrightarrow \mathrm{MF}_{\mathbf{Higgs}}.$$
- $\mathrm{MF}_{\mathbf{F\text{-}isoc}}$ is the image of $\mathrm{MF}_{\mathbf{FF}}$ under the map that sends an isomorphism classs of a logarithmic Fontaine-Faltings module to the isomorphism class of the associated overconvergent F-isocrystal and multiplication by $\mathbb{Q}_{p^f}$, i.e., the isomorphism class of an object of $\mathbf{F\text{-}Isoc}^{\dagger}(U_1)_{\mathbb{Q}_{p^f}}$.
- $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Isoc}}$ is the set of equivalence classes of the set $\mathrm{MF}_{\mathbf{F\text{-}isoc}}$ modulo the equivalence relations defined by twisting a constant rank-1 F-isocrystal.

Every constant rank 1 F-isocrystal comes from a constant rank 1 Fontaine-Faltings module. Therefore the natural map $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Cris}} \to \mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Isoc}}$ is surjective.

If the reader is uncomfortable with carrying around the endomorphism structure ι, we introduce the following notation: if $(V, \nabla, \mathrm{Fil}, \varphi, \iota)$ is a logarithmic Fontaine-Faltings modules, then $(V, \nabla, \mathrm{Fil})^{(0)}$ denotes the identity eigenspace of the action of ι on $(V, \nabla, \mathrm{Fil})$, i.e., for any $v \in V$,

$$v \in V^{(0)} \quad \text{iff} \quad \iota(r)v = rv \text{ for all } r \in \mathbb{Z}_{p^f}.$$

One may replace all objects above with the identity eigenspaces of ι (together with φ^f, if φ shows up). This will yield equivalent categories because $\mathbb{F}_{p^f} \subset k$ and hence $\mathbb{Z}_{p^f} \subset W(k)$.

We have one final preliminary result, using the above notation.

Lemma 3.1. *The map*
$$\mathrm{MF}^{\mathrm{twist}}_{\boldsymbol{F\text{-}Cris}} \twoheadrightarrow \mathrm{MF}^{\mathrm{twist}}_{\boldsymbol{F\text{-}Isoc}}$$
is finite-to-one.

Proof. Fix an object $\mathcal{E} = (\mathcal{V}, \nabla, \varphi, \iota)$ in $\mathrm{MF}_{\mathbf{F\text{-}isoc}}$. Recall that any twisting of $\mathcal{E}$ by a constant rank-1 F-isocrystal is of the form $\mathcal{E}_\lambda = (\mathcal{V}, \nabla, \lambda\varphi, \iota)$ for some $\lambda \in (K \otimes \mathbb{Q}_{p^f})^\times$. Set

$$\mathcal{T}_\lambda := \left\{ T \in \mathrm{MF}_{\mathbf{F\text{-}Cris}} \mid T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathcal{E}_\lambda \right\},$$

i.e., the set of isomorphism classes of F-crystals which underlie Fontaine-Faltings modules and are also extensions of $\mathcal{E}_\lambda$. Note that if $(V, \nabla, \varphi, \iota)$ comes from a Fontaine-Faltings module, then V is automatically semistable.[4] Therefore T_λ is a finite set by Lemma 2.7 Under this notation, The lemma claims that the following set

$$\bigcup_\lambda \mathcal{T}_\lambda \Big/ \sim$$

is finite, where the equivalence relation "$\sim$" is given as follows: $(V, \nabla, \varphi, \iota) \sim (V', \nabla', \varphi', \iota')$ if and only if they are twists by a constant, rank 1 Fontaine-Faltings module (with endomorphism structure).

Let $(\mathcal{V}, \nabla) = \bigoplus_{i=0}^{f-1} (\mathcal{V}_i, \nabla_i)$ be the eigen decomposition of $\iota \colon \mathbb{Q}_{p^f} \to \mathrm{End}(\mathcal{V}, \nabla)$. That is $\mathcal{V}_i$ consists of elements $v \in \mathcal{V}$ satisfying $\iota(a)v = \sigma^i(a)v$ for any $a \in \mathbb{Q}_{p^f}$, where σ is the Frobenius in the Galois group $\mathrm{Gal}(\mathbb{Q}_{\mathrm{p^f}}/\mathbb{Q}_{\mathrm{p}})$. Since it maps one eigenspace to one another, the structure φ can be decomposed as semi-linear maps

$$\varphi_i \colon \mathcal{V}_i \to \mathcal{V}_{i+1} \text{ for } i = 0, \cdots, f-2 \quad \text{and} \quad \varphi_{f-1} \colon \mathcal{V}_{f-1} \to \mathcal{V}_0,$$

More precisely, this is because $\iota(a)\big(\varphi(v_i)\big) = \varphi(\iota(a)(v_i)) = \varphi\big(\sigma^i(a) \cdot v_i\big) = \sigma^{i+1}(a) \cdot \varphi(v_i)$ and the last equality due to the fact that φ is σ-semi-linear. Analogously, for any $\lambda = (\lambda_i) \in (K \otimes \mathbb{Q}_{p^f})^\times = \prod_{i=0}^{f-1} K^\times$, the isogeny $\lambda \cdot \varphi$ decomposes as

$$\lambda_i\varphi_i \colon \mathcal{V}_i \to \mathcal{V}_{i+1} \text{ for } i = 0, \cdots, f-2 \quad \text{and} \quad \lambda_{f-1}\varphi_{f-1} \colon \mathcal{V}_{f-1} \to \mathcal{V}_0.$$

We denote

$$v_p(\lambda) := \frac{1}{f} \sum_{i=0}^{f-1} v_p(\lambda_i) \in \frac{1}{f}\mathbb{Z}.$$

The finiteness will then follow from the following two claims:

Claim 1. $\mathcal{T}_\lambda = \emptyset$, if $v_p(\lambda) > p-1$ or $v_p(\lambda) < 1-p$.

Claim 2. $\mathcal{T}_\lambda = \mathcal{T}_{\lambda'}$ as subsets of $\mathrm{MF}^{twist}_{\mathbf{F\text{-}Cris}}$, if $v_p(\lambda) = v_p(\lambda')$.

Let $(V, \nabla, \mathrm{Fil}, \varphi, \iota) \in \mathrm{MF}_{\mathbf{FF}}$ be an object mapping to $\mathcal{E}$. After twisting by a constant rank 1 Fontains-Faltings modules, we may assume that the filtration satisfies $\mathrm{Fil}^p = 0$. Therefore, one has $p^{p-1}\widetilde{V}_i \subset V_i \subset \widetilde{V}_i$, where $\widetilde{(\cdot)}$ is Faltings' tilde functor. By strong divisibility, $\varphi_i(\widetilde{V}_i)$ generates V_{i+1} (or V_0, if $i = f-1$). Thus

$$p^{p-1}V_0 \subset \langle \varphi_{f-1}(V_{f-1}) \rangle \subset V_0 \quad \text{and} \quad p^{p-1}V_{i+1} \subset \langle \varphi_i(V_i) \rangle \subset V_{i+1} \quad \text{for } i = 0, 1 \cdots, f-2.$$

Considering the composition, one gets

$$p^{(p-1)f}V_0 \subset \langle \varphi_{f-1} \circ \cdots \circ \varphi_0(V_0) \rangle \subset V_0.$$

[4] One way of seeing this is that a strict p-torsion Fontaine-Faltings module corresponds to a periodic Higgs-de Rham flow. The Higgs bundles in the flow are all semistable by [LSZ19, Proposition 6.3]. Because C^{-1} preserves semistability, this implies that (V, ∇) is semistable.

Choose a basis of V_0 and write $\varphi_{f-1}\circ\cdots\circ\varphi_0$ as a matrix in terms of this basis. Then the p-adic valuation of the determinant of this matrix is well-defined; one has

$$0\leq v_p\Big(\det(\varphi_{f-1}\circ\cdots\circ\varphi_0)\Big)\leq (p-1)f\cdot\mathrm{rank}(V_0). \tag{3.1.1}$$

Suppose $\mathcal{T}_\lambda\neq\emptyset$ for some $\lambda=(\lambda_i)\in(K\otimes\mathbb{Q}_{p^f})^\times=\prod_{i=0}^{f-1}K^\times$. Then by precisely analagous reasoning, one has

$$0\leq v_p\Big(\det((\lambda_{f-1}\varphi_{f-1})\circ\cdots\circ(\lambda_0\varphi_0))\Big)\leq (p-1)f\cdot\mathrm{rank}(V_0) \tag{3.1.2}$$

Since $v_p\Big(\det((\lambda_{f-1}\varphi_{f-1})\circ\cdots\circ(\lambda_0\varphi_0))\Big)=v_p(\lambda)\cdot f\cdot\mathrm{rank}(V_0)+v_p\Big(\det(\varphi_{f-1}\circ\cdots\circ\varphi_0)\Big)$, by (3.1.1) and (3.1.2), one has

$$1-p\leq v_p(\lambda)\leq p-1.$$

Thus the Claim 1 follows.

We now show Claim 2. By replacing φ with $\lambda'\varphi$, one may reduce the claim to case $\lambda'=1$; in this setting, as $v_p(\lambda)=v_p(\lambda')$, it follows that $v_p(\lambda)=0$. Write $\lambda\in(K\otimes\mathbb{Q}_{p^f})^\times$ as $\lambda=(\lambda_i)_i$ as above, set $n_i=v_p(\lambda_i)$, and set $m_i=n_1+n_2+\cdots+n_{i-1}$ for all $i=0,1,\cdots,f-1$. Consider the following map

$$F_{1,\lambda}\colon\mathcal{T}_1\to\mathcal{T}_\lambda$$

which maps $(\oplus_iV_i,\oplus_i\nabla_i,\oplus_i\varphi_i,\iota)$ to $(\oplus_ip^{m_i}V_i,\oplus_i\nabla_i,\oplus_i\lambda_i\varphi_i,\iota)$ for any $\lambda\in(K\otimes\mathbb{Q}_{p^f})^\times$ with $v_p(\lambda)=0$. The Claim 2 is reduced to showing that this map

(1) is well-defined;
(2) is bijective and
(3) preserves the twisted classes.

We first show it is well-defined. Suppose (λ_i) are $K^\times$ with $\sum_{i=0}^{f-1}v_p(\lambda_i)=0$. Again, denote by $n_i=v_p(\lambda_i)$ and $m_i=n_1+n_2+\cdots+n_{i-1}$ for all $i=0,1,\cdots,f-1$. Suppose Fil_i is a Hodge filtration on (V_i,∇_i) such that $(\oplus_iV_i,\oplus_i\nabla_i,\oplus_i\mathrm{Fil}_i,\oplus_i\varphi_i,\iota)$ forms an object in $\mathrm{MF}_{\mathbf{FF}}$. Then $(\oplus_ip^{m_i}V_i,\oplus_i\nabla_i,\oplus_i\mathrm{Fil}_i,\oplus_i\lambda_i\varphi_i,\iota)$ is also contained in $\mathrm{MF}_{\mathbf{FF}}$, because the pair $(\oplus_i\mathrm{Fil}_i,\oplus_i\lambda_i\varphi_i)$ also satisfies strong divisibility on $\oplus_ip^{m_i}V_i$:

$$(\lambda_i\varphi_i)(p^{m_i}\widetilde{V}_i)=\lambda_ip^{m_i}\varphi_i(\widetilde{V}_i)=\lambda_ip^{m_i}V_{i+1}=p^{m_{i+1}}V_{i+1}$$

and

$$(\lambda_{f-1}\varphi_{f-1})(p^{m_{f-1}}\widetilde{V_{f-1}})=\lambda_{f-1}p^{m_{f-1}}\varphi_{f-1}(\widetilde{V_{f-1}})=\lambda_{f-1}p^{m_{f-1}}V_0=p^{m_0}V_0,$$

in the last equality we used the fact that $n_{f-1}+m_{f-1}=f\cdot v_p(\lambda)=0=m_0$. Thus $F_{1,\lambda}$ is well-defined. The map $F_{1,\lambda}$ is bijective, because one can define its inverse map $\mathcal{T}_\lambda\to\mathcal{T}_1$ in similar manner by sending $(\oplus_iV_i,\oplus_i\nabla_i,\oplus_i\varphi_i,\iota)$ to $(\oplus_ip^{-m_i}V_i,\oplus_i\nabla_i,\oplus_i\lambda_i^{-1}\varphi_i,\iota)$.

Now we only need to show $(\oplus_iV_i,\oplus_i\nabla_i,\oplus_i\mathrm{Fil}_i,\oplus_i\varphi_i,\iota)$ and $(\oplus_ip^{m_i}V_i,\oplus_i\nabla_i,\oplus_i\mathrm{Fil}_i,\oplus_i\lambda_i\varphi_i,\iota)$ differ by twisting a constant rank 1 Fontaine-Faltings module. We will construct such a constant, rank 1 Fontaine-Fatlings module. Denote $c_i=\lambda_i/p^{n_i}\in W^\times$ for $i=0,1,\cdots,f-1$ and consider the rank-1 Fontaine-Faltings module with endomorphism structure

$$\mathcal{L}=\Big(\bigoplus_{i=0}^{f-1}W\cdot e_i,\mathrm{Fil}_{tri},\phi,\iota\Big)$$

where $\iota(a)\colon \bigoplus_{i=0}^{f-1} W\cdot e_i \to \bigoplus_{i=0}^{f-1} W\cdot e_i$ is the W-linear map sending e_i to $\sigma^i(a)e_i$ for each $i=0,1,\cdots,f-1$ and for any $a\in\mathbb{Z}_{p^f}$ and $\phi\colon \bigoplus_{i=0}^{f-1} W\cdot e_i \to \bigoplus_{i=0}^{f-1} W\cdot e_i$ is the unique σ-semilinear map satisfying $\phi(e_{f-1}) = c_{f-1}e_0$ and $\phi(e_i) = c_ie_{i+1}$ for $i=0,1,\cdots,f-2$. Since $v_p(c_i)=0$ for all $i=0,1,\cdots,f-1$, this is a well-defined constant Fontaine-Faltings module. Twisting the first Fontaine-Faltings module $(\oplus_i V_i, \oplus_i\nabla_i, \oplus_i \mathrm{Fil}_i, \oplus_i\varphi_i, \iota)$ via $\mathcal{L}$, one gets a new module $(\oplus_i V_i, \oplus_i\nabla_i, \oplus_i \mathrm{Fil}_i, \oplus_i c_i\varphi_i, \iota)$. In the following, we only need to check the the isomorphism between de Rham bundles $\oplus_i(p^{m_i}\mathrm{id}_{V_i})\colon (\oplus_i V_i,\oplus_i\nabla_i)\to(\oplus_i p^{m_i}V_i,\oplus_i\nabla_i)$ is actually an isomorphism between Fontaine-Faltings modules

$$(\oplus_i V_i, \oplus_i\nabla_i, \oplus_i \mathrm{Fil}_i, \oplus_i c_i\varphi_i, \iota) \to (\oplus_i p^{m_i}V_i, \oplus_i\nabla_i, \oplus_i \mathrm{Fil}_i, \oplus_i \lambda_i\varphi_i, \iota).$$

This is equivalent to checking that the following diagrams commute

$$\begin{tikzcd}
V_i \arrow[r, hook] \arrow[d, "p^{m_i}\cdot\mathrm{id}"] & \widetilde{V_i} \arrow[r, "\simeq", "c_i\varphi_i"'] \arrow[d, "p^{m_i}\cdot\mathrm{id}"] & V_{i+1} \arrow[d, "p^{m_{i+1}}\cdot\mathrm{id}"] \\
p^{m_i}V_i \arrow[r, hook] & p^{m_i}\widetilde{V_i} \arrow[r, "\simeq", "\lambda_i\varphi_i"'] & p^{m_{i+1}}V_{i+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
V_{f-1} \arrow[r, hook] \arrow[d, "p^{m_{f-1}}\cdot\mathrm{id}"] & \widetilde{V_{f-1}} \arrow[r, "\simeq", "c_{f-1}\varphi_{f-1}"'] \arrow[d, "p^{m_{f-1}}\cdot\mathrm{id}"] & V_0 \arrow[d, "p^{m_0}\cdot\mathrm{id}"] \\
p^{m_{f-1}}V_{f-1} \arrow[r, hook] & p^{m_{f-1}}\widetilde{V_{f-1}} \arrow[r, "\simeq", "\lambda_{f-1}\varphi_{f-1}"'] & p^{m_0}V_0
\end{tikzcd}$$

This follows from the fact that $m_{i+1}=m_i+n_i$ and $m_0=0=m_{f-1}+n_{f-1}$ and $\lambda_i = p^{n_i}c_i$. □

Proof of Theorem 1.2. By Faltings' definition of a crytalline representation, the Hodge-Tate weights are in an interval of length $p-1$. Since Tate twisting of a crystalline representation ρ does not change the isomorphism class of $\rho\mid_{\bar\Gamma}$, it suffices to prove the finiteness of set

$$\left\{\rho\colon \pi_1^{et}(U_K)\to \mathrm{GL}_r(\mathbb{Z}_{p^f}) \;\middle|\; \begin{array}{l}\rho \text{ is log crystalline}\\ \text{with HT weights in } [0,p-1] \text{ and}\\ \rho^{\mathrm{geo}}\colon \pi_1^{\text{ét}}(U_{\bar K})\to \mathrm{GL}_r(\mathbb{Q}_{p^f}) \text{ absolutely irreducible.}\end{array}\right\}\Big/ \rho_1\sim\rho_2 \text{ if } \rho_1|_{\pi_1^{\text{ét}}(U_{\bar K})}\cong\rho_2|_{\pi_1^{\text{ét}}(U_{\bar K})}$$

Equivalently, to prove Theorem 1.2 we will show that $\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar\Gamma)$ is finite.

Since the Faltings p-adic Simpson's correspondence [Fal05] is compatible with his $\mathbb{D}$-functor [Fal89], one has following commutative diagram of surjective maps between sets

$$\begin{tikzcd}
\mathrm{MF}_{\mathbf{FF}} \arrow[r, two heads, "{(V,\nabla,\mathrm{Fil},\varphi,\iota)\mapsto \mathrm{Gr}\left((V,\nabla,\mathrm{Fil},\iota)\right)}"] \arrow[d, "\mathbb{D}", "\simeq"'] & \mathrm{MF}_{\mathbf{Higgs}} \arrow[d, "\text{Faltings' } p\text{-adic Simpson correspondence}"] \\
\mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma) \arrow[r, two heads, "\text{restriction}"] & \mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar\Gamma)
\end{tikzcd}$$

Since the two horizontal arrows and the left vertical arrow are surjective, the right vertical arrow is also surjective. One has a surjective composition

$$\mathrm{MF}_{\mathbf{FdR}} \twoheadrightarrow \mathrm{MF}_{\mathbf{Higgs}} \twoheadrightarrow \mathrm{Rep}^{\mathrm{irr.crys}}_{\mathrm{GL}_r(\mathbb{Z}_{p^f})}(\Gamma,\bar\Gamma).$$

To prove Theorem 1.2, we only need to show the finiteness of $\mathrm{MF}_{\mathbf{FdR}}$.

Firstly, we claim that $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Isoc}}$ is finite. By Lemma 2.3, all elements in $\mathrm{MF}_{\mathbf{F\text{-}isoc}}$ are of absolutely irreducible. Then the set of equivalence classes of absolutely irreducible objects of $\mathbf{F\text{-}Isoc}^\dagger(U_1)_{\mathbb{Q}_{p^f}}$

up to twisting by a constant rank 1 F-isocrystal is finite by [Ked18a, Corollary 2.1.5]. Secondly, we claim that $\mathrm{MF}^{\mathrm{twist}}_{\mathbf{F\text{-}Cris}}$ is finite. This follows from Lemma 3.1. Finally, the map $\mathrm{MF}_{\mathbf{FdR}} \twoheadrightarrow \mathrm{MF}_{\mathbf{dR}}$ is finite-to-one by Lemma 2.8 and Lemma 2.9. Thus $\mathrm{MF}_{\mathbf{FdR}}$ is finite. □

4. Theorem 1.2 is false over $k = \bar{\mathbb{F}}_p$

Let $\lambda \in W$ with $\lambda \not\equiv 0, 1 \pmod p$. Let $(X, S) = (\mathbb{P}^1, \{0, 1, \infty, \lambda\})$. Let $\mathcal{M}_\lambda$ denote the moduli space of semi-stable graded logarithmic Higgs bundles (E, θ) of rank 2 and degree 1 over (X, S). A Higgs bundle in this moduli space may be written as

$$(E, \theta) = (\mathcal{O} \oplus \mathcal{O}(-1),\, \theta\colon \mathcal{O} \xrightarrow{\theta} \mathcal{O}(-1) \otimes \Omega^1_X(\log S)).$$

We attach a parabolic structure at one of the four points of D with parabolic weight $(\frac{1}{2}, \frac{1}{2})$, so that (E, θ) is parabolically stable and of parabolic degree zero. Let $\mathcal{M}^{par}_\lambda$ denote the moduli space of those type logarithmic Higgs bundles with a parabolic structure at one of those four cusps. Forgetting the parabolic structure, one defines an isomorphism

$$\mathcal{M}_\lambda \simeq \mathbb{P}^1_{W(k)}$$

by sending (E, θ) to the zero locus $(\theta)_0 \in \mathbb{P}^1$. Consider the self map $\varphi_{\lambda, \mathbb{F}_p} = \mathrm{Gr} \circ \mathcal{C}^{-1}_{1,2}$ on $\mathcal{M}_\lambda \otimes_W k \simeq \mathbb{P}^1_k$ induced by Higgs-de Rham flow $(\otimes \mathcal{O}_{\mathbb{P}^1}((1-p)/2))$. Since $C^{-1}_{1,2}$ is a factor of the composition, $\varphi_{\lambda, \mathbb{F}_p}$ factors through the Frobenius map, i.e., there exists a rational function $\psi_\lambda \in k(z)$ such that $\varphi_{\lambda, \mathbb{F}_p}(z) = \psi_\lambda(z^p)$. In this note we call ψ_λ the *Verschiebung part* of the self map $\varphi_{\lambda, \mathbb{F}_p}$. The periodic points of the self map $\varphi_{\lambda, \mathbb{F}_p}$ are naturally corresponding to the twisted periodic Higgs-de Rham flows [SYZ17, Section 4] on $(X, S)_k$. Conjecturally this self map is related to the multiplication by p map on the elliptic curve

$$C_\lambda \colon y^2 = x(x-1)(x-\lambda).$$

Conjecture 4.1 (Conjecture 5.8 in [SYZ17])**.** *The following diagram commutes*

(4.1.1)
$$\begin{tikzcd} C_{\lambda k} \arrow[r, "{[p]}"] \arrow[d, "\pi"] & C_{\lambda,k} \arrow[d, "\pi"] \\ \mathbb{P}^1_k \arrow[r, "\varphi_{\lambda,\mathbb{F}_p}"] & \mathbb{P}^1_k \end{tikzcd}$$

Let $(E, \theta)_n \in \mathcal{M}_\lambda(W_n)$ be a periodic Higgs bundle over $(X, S)_n$. Consider the self map $\mathrm{Gr} \circ C^{-1}$ on the deformation space $\mathrm{Def}_{(E,\theta)_n}(W_{n+1})$. Since $\mathrm{Def}_{(E,\theta)_n}(W_{n+1})$ is an $\mathbb{A}^1$-torsor space, we may identify a self map on $\mathrm{Def}_{(E,\theta)_n}(W_{n+1})$ with a self map on $\mathbb{A}^1$. Under this identification, $\mathrm{Gr} \circ C^{-1}$ is just a polynomial.

Lemma 4.2. *Notation as above. Then*

(1) *Let $(E, \theta)_n \in \mathcal{M}_\lambda(W_n(\mathbb{F}_{q^h}))$ be a periodic Higgs bundle over $(X, S)_n$. Then the polynomial associated to the self map* $\mathrm{Gr} \circ C^{-1}$ *on* $\mathrm{Def}_{(E,\theta)_n}(W_{n+1})$ *is*

$$\mathbb{A}^1(\mathbb{F}_q) \to \mathbb{A}^1(\mathbb{F}_q), \quad z \to a \cdot z^p + b,$$

where $a, b \in \mathbb{F}_q$. Consequently if $a \neq 0$, then by solving the Artin-Schreier equation $az^p - z + b = 0$ one obtains p different 1-periodic liftings over $W_{n+1}(\mathbb{F}_{q^{ph}})$. Moreover the constant a

is the derivative of the Verschiebung part of $\varphi_{\lambda,\mathbb{F}_p}$ at the point associated to $(E,\theta)_n \mid_{(X,S)_k}$. In particular the value of a only depends on the reduction modulo p of $(E,\theta)_n$.

(2) *For $p < 50$, if the torsion point associated to $(E,\theta)_n \mid_{(X,S)_k}$ is not of order 2, then the coefficient $a \neq 0$ if and only if the associated elliptic curve $C_{\lambda,k}$ is not supersingular.*

Proof. The fact that the self map $\mathrm{Gr} \circ C^{-1}$ on $\mathrm{Def}_{(E,\theta)_n}(W_{n+1})$ is of form $az^p + b$ follows from [SYZ17, Proposition 8.5, Corrollary 8.7]. By an explicit computation similar as in [SYZ17, Section 7], the coefficient a is just the derivation of the Verschiebung part of the self map $\varphi_{\lambda,\mathbb{F}_p}$ at the point associated to $(E,\theta)_n \mid_{(X,S)_k}$.

Conjecture 4.1 was checked directly by explicit computation for $p < 50$. The Verschiebung part of the multiplication by p map $[p]$ is either a Frobenius map or an etale map of degree p, depending on $C_{\lambda,k}$ being either supersinglar or ordinary. After taking the $2:1$ map $\pi\colon C_{\lambda,k} \to \mathbb{P}^1_k$, one gets the second part.

□

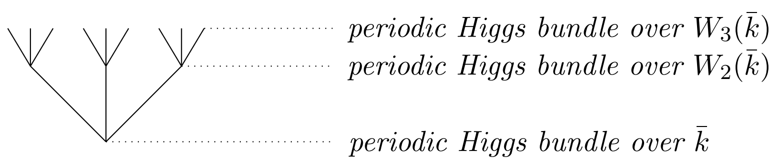
Let $\pi\colon (X', S') \to (X, S)$ be the double cover that is ramified at the parabolic point and one other point. Using [SYZ17, Theorem 4.6] together with Lemma 4.2, one obtains the following corollary.

Corollary 4.3. *Suppose $p < 50$ and the elliptic curve $C_{\lambda,k}$ is not supersingular. Then there exists infinitely many log crystalline $\mathrm{GL}_2(\mathbb{Z}_p)$ local systems on $(X', S')_{\mathbb{Q}_p^{unr}}$ whose residual representation is absolutely geometrically irreducible.*

Remark 4.4. *Let S/k be a smooth variety over a finitely generated field of characteristic 0. Let L be an ℓ-adic local field and $n \geq 1$. In* [Lit18]*, Litt proves that there are only finitely many continuous semisimple arithmetic representations $\pi_1(S_{\bar{k}}) \to GL_n(L)$. However, in* [Lit18, Remark 1.1.7]*, he notes that this is* false *if one replaces L with $\bar{L}$, i.e., there can be infinitely many continuous semisimple arithmetic representations $\pi_1(S_{\bar{k}}) \to GL_n(\bar{L})$. This fact is orthogonal to Corollary 4.3. Indeed, Corollary 4.3 is ensured by extending the ground field and keeping the coefficient ring fixed to be $\mathbb{Z}_p$.*

We end this section with a final remark.

Remark 4.5. *We consider the ordinary case, i.e. $a \neq 0$. Then to lift a periodic Higgs bundle over $(X,S)_n$ to $(X,S)_{n+1}$ is equivalent to find an invariant point of $z \mapsto a \cdot z^p + b$. In particular if we ignore the extension of the base field, there are always p lifts in each lifting step.*



The finiteness in Theorem 1.2 implies that for any finite extension k'/k, almost all liftings don't live in $W(k')$ (but they of course live in $W(\bar{k}) \supset W(k') \supset W(k)$). This means that the solutions of the Artin-Schreier equation in (2) *must lie in the degree p-extension of $\mathbb{F}_q(b)$ at almost all lifting steps. But, it seems difficult to prove that directly, because the constant coefficient b depends on $(E,\theta)_n$ and it varies when we consider different lifting steps!*

5. Some speculations on a uniform upper bound

In Section 4 we made an identification

$$M_\lambda \simeq \mathbb{P}^1$$

by sending (E,θ) to the zero θ_0 of the Higgs field θ and let $\pi\colon C_\lambda \to \mathbb{P}^1$ branched along S.

Conjecture 5.1 (Sun-Yang-Zuo)**.** *A Higgs bundle in $\mathcal{M}_\lambda$ over a finite unramified extension is f-periodic if and only if $\pi^*(\theta_0)$ is a $(p^f \pm 1)$-torsion point in C_λ.*

Conjecture 5.1 is a consequence of Conjecture 4.1. It imples that the number of $\mathrm{PGL}_2(\mathbb{Z}_{p^f})$-crystalline local systems over (X,S) over finite unramified extensions of $\mathbb{Q}_p$ is exactly $p^{2f}+1$. Conjecture 5.1 has been checked for the following cases.

(1) When we only work modulo p and $p \leq 50$.
(2) When $C_{\lambda,k}$ is supersingular and $p \leq 50$.
(3) For all p, when the torsion point has order 1, 2, 3, 4 and 6.

Remark 5.2. *When $C_{\lambda,k}$ is supersingular, any GL_2-crystalline local system corresponding to Higgs bundles in $\mathcal{M}_\lambda(\mathbb{Q}_p^{ur})$ automatically descends to a local system over a* finite, unramified *extensions of $\mathbb{Q}_p$. This contrasts with the situation when $C_{\lambda,k}$ is an ordinary elliptic curve; we expect that most GL_2-crystalline local systems corresponding to Higgs bundles in $\mathcal{M}_\lambda(\mathbb{Q}_p^{ur})$ over $(X',S')/\mathbb{Q}_p^{ur}$ do not descend to finite unramified extension of $\mathbb{Q}_p$.*

We end by posing a conjecture, the first part of which is in the spirit of the Fontaine-Mazur conjecture and the second of which is analogous to a theorem of Litt [Lit18].

Conjecture 5.3. *Let (X,S) be a log pair over $W(\mathbb{F}_q)$ with $p^f \mid q$.*

(1) *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two $\mathbb{Z}_{p^f}$-crystalline local systems over $(X,S)/W(\mathbb{F}_q)$. If $\mathbb{L}_1 \simeq \mathbb{L}_2$ mod p, then $\mathbb{L}_1 \simeq \mathbb{L}_2$.*
(2) *The number of isomorphism classes $\mathrm{GL}_r(\mathbb{Z}_{p^f})$-crystalline local systems over $(X,S)_{W(\mathbb{F}_{q^h})}$, as we let h range through the positive integers, is finite.*

.

The motivation of Conjecture 5.3 is Conjecture 5.1 for the case of rank-2 irreducible logarithmic crystalline local systems on $\mathbb{P}^1, \{0,1,\infty,\lambda\})$ and with parabolic weight $(1/2,1/2)$ at one punctured point. There, we consider the moduli space $\mathcal{M}_\lambda$ of rank-2 stable logarithmic Higgs bundles over $(\mathbb{P}^1,\{0,1,\infty,\lambda\})$ and at one punctured point with parabolic weight $(1/2,1/2)$. By taking the zero locus of the Higgs field we identify $\mathcal{M}_\lambda \simeq \mathbb{P}^1$, and by taking the two double cover

$$\pi : C_\lambda \to \mathbb{P}^1 \simeq \mathcal{M}_\lambda$$

we conjecture that a Higgs bundle $(E,\theta) \in \mathbb{P}^1_k$ corresponding to a crystalline $\mathbb{F}_{p^f}$-local system if and only if the zero of the Higgs field is a torsion point on the associated elliptic curve $\pi : C_\lambda \to \mathbb{P}^1$ over k. Moreover, we conjecture that this local system lifts to $\mathbb{Z}_{p^f}$-crystalline local system if and only the corresponding Higgs bundle under the above the associated map $\pi : C_\lambda \to \mathbb{P}^1 \simeq \mathcal{M}_\lambda$ lifts to a torsion point in C_λ over $W(k)$ uniquely. For the even more extreme case where C_λ is a supersingular elliptic curve and $p < 50$ one checks directly any rank-2 irreducible logarithmic $\mathbb{Z}_{p^f}$-crystalline local systems over $(\mathbb{P}^1,\{0,1,\infty,\lambda\})$ over $\mathbb{Q}_p^{ur}$ is, in fact, defined over some $\mathbb{Q}_{p^q}$ and is uniquely lifted from a $\mathbb{F}_{p^f}$-crystalline local system.

6. A brief discussion on the number of $\mathrm{GL}_2(\mathbb{Z}_p)$-local systems on $\mathbb{P}^1$ minus four points over a finite unramified extension of $\mathbb{Q}_p$ and with eigenvalues -1 around one of the four puncture points.

Maintain notation as in Section 4; in particular, $(X,S) = (\mathbb{P}^1, \{0,1,\infty,\lambda\})$. It seems possible that all of the local systems in Section 4 could come from families of abelian varieties of Hilbert modular type over $(X,S)/W(\mathbb{F}_q)$. We also conjecture that they correspond to $(p\pm 1)$-torsion points on the elliptic curve $C_\lambda/W(\mathbb{F}_q)$.

We have checked this conjecture for torsion-points of orders $1,\ 2,\ 3,\ 4$ and 6, when K is a number field. There exists exactly 26 elliptic curves of $(X,S)/\mathcal{O}_K$ such that the arithmetic local systems attached to those families correspond to arithmetic 1-periodic Higgs bundles from $\mathcal{M}_\lambda/\mathcal{O}_K$ and the zero locus of the Higgs fields are torsion points of order $1,\ 2,\ 3,\ 4$ and 6. in $C_\lambda/\mathcal{O}_K$ under the pull back of π.

Let $(\mathcal{M}_\lambda)^{\mathrm{periodic}}$ consists of those Higgs bundles in $\mathcal{M}_\lambda$ which are periodic (without specifying the periodicity map) and $\mathcal{M}_\lambda^{\ell\text{-adic}}$ is the set of equivalence classses of (geometrically) irreducible systems $\mathrm{GL}_2(\bar{\mathbb{Q}}_\ell)$ -local systems over (X_k, S_k) with prescribed local monodromy on the cusp points, up to twisting by a character on $\mathbb{F}_q$. We will show that there is an inclusion:

$$(\mathcal{M}_\lambda)^{\mathrm{periodic}} \hookrightarrow \mathcal{M}_\lambda^{\ell\text{-adic}}. \tag{6.0.1}$$

Indeed, given a p-adically periodic Higgs bundle in $\mathcal{M}_\lambda$, we may forget the filtration to obtain an overconvergent F-isocrystal $\mathcal{E}$ on U_k; moreover, this gives an injective map from the set of equivalence classes of such Higgs bundles (without specifying the periodicity map) to the set of isomorphism classes of (overconvergent) F-isocrystals up to twisting by a character on $\mathbb{F}_q$ by [KYZ20b, Proposition 3.11]. Picking a field isomorphism $\sigma\colon \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$, we may take the σ-companion of $\mathcal{E}$ to obtain a lisse l-adic sheaf on U_k, whose local monodromy around S_k matches with that of the F-isocrystal.

If we choose any of the four points for the parabolic structure, then we may take the associated elliptic curve $\tau : (C_\lambda, S') \to (X,S)$ over $\mathbb{F}_q$ to kill the -1 eigenvalues in the local monodromy. In this way we obtain p^2+1 $\mathrm{GL}_2(\mathbb{Q}_\ell)$-local systems over $(C_\lambda, S')/\mathbb{F}_q$, which just corresponds to the $(p\pm 1)$-torsion points on $C_\lambda/\mathbb{F}_q$.

Question 6.1. *Setup as above.*

1. *Can we intrinsically characterize the image of Equation 6.0.1?*
2. *Is there numerical evidence for the conjecture using the trace formula? More specifically, one may transform the question of counting $GL_2(\mathbb{Q}_\ell)$ local systems on (X_k, S_k) with perscribed monodromy (in our case, eigenvalues of -1 around the parabolic point, with a non-trivial Jordan block and principal unipotent monodromy at the other punctures) to a question about counting certain types of automorphic forms via the Langlands correspondence. Drinfeld, and then Deligne-Flicker have a method to compute such numbers via the trace formula and have fully worked out this number in the case when the sheaves are supposed to have principal unipotent monodromy around each puncture* [Dri82, DF13, Fli15]. *Can we see the number $p^{2f}+1$ from the trace formula? Can we see the expected group law on the zeroes of the Higgs field from $(\mathcal{M}_\lambda)^{periodic}$ via automorphic forms?*

Acknowledgements. We thank Kiran Kedlaya and Atsushi Shiho several very helpful exchanges. We also thank anonymous referees for very useful suggestions and comments, which helped us clarify some parts of the manuscript. R.K gratefully acknowledges support from NSF Grant No. DMS-1344994 (RTG in Algebra, Algebraic Geometry and Number Theory at the University of Georgia)

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