STRICT ARAKELOV INEQUALITY FOR A FAMILY OF VARIETIES OF GENERAL TYPE

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Abstract. Let $f : X \to Y$ be a semistable non-isotrivial family of n-folds over a smooth projective curve with discriminant locus $S \subseteq Y$ and with general fibre F of general type. We show the strict Arakelov inequality

$$\frac{\deg f_*\omega_{X/Y}^{\nu}}{\operatorname{rank} f_*\omega_{X/Y}^{\nu}} < \frac{n\nu}{2} \cdot \deg \Omega_Y^1(\log S),$$

for all $\nu \in \mathbb{N}$ such that the ν-th pluricanonical linear system $|\omega_F^{\nu}|$ is birational. This answers a question asked by Möller, Viehweg and the third named author [MVZ06].

1. Introduction

We always work over the complex number field $\mathbb{C}$. Let Y be a non-singular projective curve, X a projective manifold, and let $f : X \to Y$ be a proper surjective morphism with connected general fibre F. Denote by $S \subseteq Y$ the discriminant divisor of f, i.e., the restricted map

$$f : X \setminus f^{-1}(S) \longrightarrow Y \setminus S$$

is smooth. Recall that f is birationally isotrivial, if $X \times_Y \operatorname{Spec} \overline{\mathbb{C}(Y)}$ is birational to $F \times \operatorname{Spec} \overline{\mathbb{C}(Y)}$. Putting together results due to Parshin-Arakelov, Migliorini, Zhang, Kovacs, Bedulev-Viehweg, Oguiso-Viehweg, Viehweg-Zuo, etc. (see [VZ02] and the references given there), one has

Theorem 1.1. *Let $f : X \to Y$ be a non-isotrivial family of n-folds, with general fibre F. Assume either*

- $\kappa(F) = \dim(F)$, *or*
- *F has minimal model F' with $\omega_{F'}$ semi-ample.*

Then (Y, S) is logarithmic hyperbolic, i.e., $\deg \Omega_Y^1(\log S) > 0$.

Let M_h denote the coarse moduli space of polarized manifolds with semi-ample canonical line bundle and with fixed Hilbert polynomial h. Theorem 1.1 is equivalent to saying that the moduli stack of M_h is algebraic hyperbolic. As a complex analytic version, Viehweg-Zuo have constructed a complex Finsler metric h_f with strictly negative curvature and consequently, the various complex hyperbolicities; for example, the Brody [VZ03b], Kobayashi [TY15] and big Picard hyperbolicities [DLSZ19] have been proven for the moduli stack.

Having such negatively curved Finsler metric on the moduli stack the general principle Yau's form of Schwarz Lemma suggests that there must be an inequality between the logarithmic hyperbolic metric on (Y, S) and some singular Kähler metric on $\overline{M}_h$. In his work on moduli space of polarized manifolds, Viehweg has constructed a series of classes of ample invertible

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sheaves λ_ν on M_h, which are natural in the sense that if $\phi : U \to M_h$ is induced by a family $f : V \to U$ then $\phi^*\lambda_\nu = \det f_*\omega^\nu_{V/U}$. Furthermore, for the cases where ω^ν_F is trivial for some $\nu > 0$ or ω_F is ample Viehweg has constructed a "good" projective compactification $\overline{M}_h \supset M_h$ such that λ_ν extends to numerically effective invertible sheaves $\bar{\lambda}_\nu$ on $\overline{M}_h$ which is again natural in the sense that if ϕ is induced by a semistable family $f : X \to Y$ then for the extension $\bar{\phi} : Y \to \overline{M}_h$ of ϕ one has $\bar{\phi}^*\bar{\lambda}_\nu = \det f_*\omega^\nu_{X/Y}$.

We denote the slope of a vector bundle E over a smooth projective curve Y as

$$\mu(E) = \frac{\deg E}{\operatorname{rank} E}.$$

In [MVZ06, VZ06] one finds the Arakelov inequality for degree of the direct image of pluri relative dualizing sheaf.

Theorem 1.2. *Let $f : X \to Y$ be a non-isotrivial family of n-folds in Theorem 1.1, and assume it is semistable, then*

$$\mu(f_*\omega^\nu_{X/Y}) \leq \frac{n\nu}{2} \cdot \deg \Omega^1_Y(\log S), \quad \forall\, \nu \in \mathbb{N},\ \textit{with}\ f_*\omega^\nu_{X/Y} \neq 0.$$

In general

$$\mu(A) \leq \frac{n\nu}{2} \cdot \deg \Omega^1_Y(\log S), \quad \forall\, \textit{non-zero subbundle}\ A \subseteq f_*\omega^\nu_{X/Y}.$$

If the equality holds, then A is semistable.

The Arakelov inequality in Theorem 1.2 can be interpreted as the a global form of Yau's Schwarz inequality.

$$\deg \bar{\phi}^*\bar{\lambda}_\nu \leq \frac{n\nu}{2} \cdot \operatorname{rank} f_*\omega^\nu_{X/Y} \cdot \deg \Omega^1_Y(\log S),$$

where the constant $\frac{n\nu}{2} \cdot \operatorname{rank} f_*\omega^\nu_{X/Y}$ depends only on the geometry on the generic fibre on the moduli space.

The Arakelov inequality for $\nu = 1$ is well-known since long time. Faltings and Deligne have proven 1-th Arakelov inequality for families of abelian varieties. For a semistable family $f : X \to Y$ of abelian g-folds Faltings [Fal83] and with an improvement by Deligne has shown that

$$\deg f_*\omega_{X/Y} \leq \frac{g}{2} \deg \Omega^1_Y(\log S).$$

As $f_*\omega^\nu_{X/Y} = (f_*\omega_{X/Y})^\nu$ for a semistable family of abelian varieties one obtains immediately the ν-th Arakelov inequality by taking ν-th power of 1-th Arakelov inequality

$$\deg f_*\omega^\nu_{X/Y} \leq \frac{\nu g}{2} \deg \Omega^1_Y(\log S).$$

We shall remark that ν-th Arakelov inequality is sharp and can be an equality for some special families of abelian varieties. In [VZ04] it is shown that the ν-th Arakelov equality holds for a semistable family of abelian varieties if and only if the family is a universal family of Shimura curves of Mumford-Tate type. In general,

Conjecture 1.3. *Let $f : X \to Y$ be as above. Assume that the Arakelov qualities hold, i.e.,*

$$\mu(f_*\omega^\nu_{X/Y}) = \frac{n\nu}{2} \cdot \deg \Omega^1_Y(\log S), \quad \forall\, \nu \in \mathbb{N},\ \textit{with}\ f_*\omega^\nu_{X/Y} \neq 0.$$

Then the general fiber is of Kodaira dimension zero and the family is Shimura.

In contrast, motivated by the Colemen-Oort conjecture we expect the Arakelov inequality holds strictly for families of n-folds of higher Kodaira dimension. Tan [Tan95] and Liu [Liu96] have shown that

$$\deg f_*\omega_{X/Y} < \frac{g}{2} \deg \Omega^1_Y(\log S),$$

for a semistable family $f : X \to Y$ of curves of genus $g \geq 2$. The proof relies on logarithmic Miyaoka-Yau inequality on fibred algebraic surfaces [Miy84] and Xiao's slope inequality [Xia87]. The ν-th Arakelov inequality for families of curves can be deduced from 1-th Arakelov inequality together with the canonical class inequality.

For families of higher dimensional fibres over curves, Viehweg-Zuo [VZ06] have generalized the above strict Arakelov inequality in the following form

Theorem 1.4. *Let $f : X \to Y$ be a semistable non-isotrivial family of n-folds, and let $A \subseteq f_*\omega_{X/Y}$ be a subbundle. Assume that either*

- **a.** *$f^*A \to \omega_{X/Y}$ defines a birational Y-morphism $\eta : X \to \mathbb{P}_Y(A)$,*
- **b.** *or $n = 1$ and rank $A \geq 2$.*

Then

$$\mu(A) < \frac{n}{2} \deg \Omega^1_Y(\log S).$$

Note that the assumption for the statement **a.** seems too strong. For a n-fold of general type the canonical linear system $|\omega|$ could be very small and does not define birational maps, only by taking a sufficiently large power $|\omega^\nu|$ defines a birational map. In this note we show the strict Arakelov inequality for a semistable family of n-folds of general type in the following form, which answers a question asked by Möller, Viehweg and the third named author; see the discussion after [MVZ06, Theorem 0.3].

Theorem 1.5. *Let $f : X \to Y$ be a non-isotrivial semistable family of n-folds, and let $A \subseteq f_*\omega^\nu_{X/Y}$ be a subbundle. Assume that $f^*A \to \omega^\nu_{X/Y}$ defines a birational Y-morphism $\eta : X \to \mathbb{P}_Y(A)$, then*

$$\mu(A) < \frac{n\nu}{2} \cdot \deg \Omega^1_Y(\log S).$$

The above theorem leads some direct consequences for families of manifolds of general type of small dimensions.

1. For a semistable family $f : X \to Y$ of surfaces of general type, then it is known that the ν-th pluricanonical linear system on a general fiber defines a birational map for $\nu \geq 5$ (cf. [Rei88]). Hence we obtain the strictly Arakelov inequality

$$\frac{\deg f_*\omega^\nu_{X/Y}}{\operatorname{rank} f_*\omega^\nu_{X/Y}} < \nu \cdot \deg \Omega^1_Y(\log S), \qquad \forall\ \nu \geq 5.$$

2. M. Chen and Jk. Chen [CC07] have shown for any smooth three-fold F there exists a number $m(3)$ between 27 and 57 and depending on certain classification classes on F such that ω^ν_F defines a birational map for any $v \geq m(3)$. So we obtain the strictly Arakelov inequality of a semistable family of three-folds of general type

$$\frac{\deg f_*\omega^\nu_{X/Y}}{\operatorname{rank} f_*\omega^\nu_{X/Y}} < \frac{3\nu}{2} \cdot \deg \Omega^1_Y(\log S), \qquad \forall\ \nu \geq 57.$$

We would also like to point out that the Arakelov bound on the slope of subbundles in $f_*\omega_{X/Y}^\nu$ is asymptotically optimal in the following sense: there exist semi-stable families of n-folds and subbundles $A_\nu \subseteq f_*\omega_{X/Y}^\nu$ such that $f^*A_\nu \to \omega_{X/Y}^\nu$ defines a birational Y-morphism $\eta : X \to \mathbb{P}_Y(A_\nu)$, and that

$$\lim_{\nu\to\infty} \frac{\mu(A_\nu)}{\frac{n\nu}{2}\cdot \deg \Omega_Y^1(\log S)} = 1. \tag{1-1}$$

Indeed, let $f : X \to Y$ be the universe family over a Teichmüller curve [Möl06]. Then there is a line bundle $L \subseteq f_*\omega_{X/Y}$ such that

$$\mu(L) = \deg(L) = \frac{1}{2}\deg \Omega_Y^1(\log S).$$

Clearly,

$$A_\nu := L^{\nu-1} \otimes f_*\omega_{X/Y} \subseteq f_*\omega_{X/Y}^\nu, \qquad \forall\, \nu \geq 2.$$

Note that $f^*A_\nu \to \omega_{X/Y}^\nu$ defines a birational Y-morphism $\eta : X \to \mathbb{P}_Y(A_\nu)$ if f is non-hyperelliptic, because the morphism defined by $f^*A_\nu \to \omega_{X/Y}^\nu$ is the same as the one defined $f^*f_*\omega_{X/Y} \to \omega_{X/Y}$, which is birational when f is non-hyperelliptic. Moreover, by direct computation, one has $\mu(A_\nu) = (\nu-1)\mu(L)+\mu(f_*\omega_{X/Y})$. Thus (1-1) holds. taking the self-fiber-product, one can get semi-stable families of n-folds with subbundles $A_\nu \subseteq f_*\omega_{X/Y}^\nu$ satisfying (1-1). Nevertheless, after a discussion with C. Simpson on Theorem 1.5, we raise the following question on a sharp form of the Arakelov inequality for families of n-folds of general type:

Problem 1.6 (Simpson)**.** *Let $f : X \to Y$ be a family and $A \subset f_*\omega_{X/Y}^\nu$ a subbundle such that $f^*A \to \omega_{X/Y}^\nu$ defines a birational Y-morphism $\eta : X \to \mathbb{P}_Y(A)$, as in Theorem 1.5. Does there exist a positive number ϵ_n depending only on the fibre dimension such that*

$$\mu(A) \leq \frac{n\nu}{2}\cdot \deg \Omega_Y^1(\log S) - \epsilon_n?$$

The statement **b.** in Theorem 1.4 is a strong result. The proof relies on the theory on Teichmüller theory due to Möller. At the moment we do not know how to prove this type strict inequality for semistable families of higher dimensional varieties. We leave this as a conjecture

Conjecture 1.7. *Let $f : X \to Y$ be a semistable family of n-folds of general type, then for any subbundle $A \subseteq f_*\omega_{X/Y}^\nu$ of $\operatorname{rank} A \geq 2$ it holds*

$$\mu(A) < \frac{n\nu}{2}\cdot \deg \Omega_Y^1(\log S).$$

The structure of the note is organized as follows. In Section 2 we explain the basic properties of polarized variation of Hodge structure: Deligne's quasi canonical extension of a filtered de Rham bundle arising from geoemtry origin, Griffiths' curvature formula and Simpson's result on the semistability of the graded logarithmic Higgs bundle of a canonical extension of a filtered de Rham bundle.

In Section 3 we sketch a proof for Theorem 1.2 for readers' convenience. We recall the notion of the logarithmic deformation Higgs bundle attached to a semistable family. Given a invertible sub sheaf $L \subseteq f_*\omega_{X/Y}^\nu$ and after a base change of Y we may raise the ν-power of L and take the ν-th roots out of the section $s : \mathcal{O}_X \to \omega_{X/Y}^\nu \otimes L^{-\nu}$. Via the new family induced by the ν-cyclic cover $h : W \xrightarrow{\tau} X \xrightarrow{f} Y$, we construct a comparison map between the deformation Higgs bundle twisted with L and the logarithmic graded Higgs bundle of the variation of Hodge structure of the middle cohomology of the new family h. Applying Simpson's semistability to the comparison map for a Higgs subbundle $A \subseteq f_*\omega^{\nu}_{X/Y}$ we finish the proof of Theorem 1.2.

We finish the proof of Theorem 1.5 in Section 4. Simpson's theorem [Sim92] on the formality of the category of semistable vector bundles of degree zero plays a crucial roll in the proof. Given a subbundle $A \subseteq f_*\omega^{\nu}_{X/Y}$ and assuming that the linear subsystem of $f^*A \to \omega^{\nu}_{X/Y}$ defines a birational map

$$\eta : X \to \mathbb{P}_Y(A),$$

we consider the d-multiplication map

$$S^dA \subseteq S^d f_*\omega^{\nu}_{X/Y} \xrightarrow{m_d} f_*\omega^{\nu d}_{X/Y}, \qquad \text{for } d \in \mathbb{N}.$$

Consider the kernel $K_d \subseteq S^d(A)$ of the multiplication map m_d restricted to $S^d(A)$, then sections of $\mathrm{Ker}(m_d)$ over an analytic open set $U \subseteq Y$ are just homogeneous polynomials of degree d in the homogeneous idea defining the fibres of the image family $f' : \eta(X)_U \to U$. If $\mu(A)$ achieves the maximal value

$$\mu(A) = \frac{n\nu}{2} \cdot \deg \Omega^1_Y(\log S),$$

then A is semistable, and hence all $S^d(A)$ is semistable. A simple stability argument on the multiplication maps shows that all kernels $K_d \subseteq S^d(A)$ remain semistable and have the same slope as $S^d(A)$'s. After a base change and twisting A with a line bundle. We may assume A is semistable of degree 0. By solving the Yang-Mills equation, A carries integrable connections. Integrable connections on A are not unique if A is not stable. Thanks to Simpson's theorem we do find a natural integrable connection (A, ∇) such that all $K_d \subseteq S^d(A)$ are preserved by the induced connection $(S^d(A), S^d(\nabla))$. This means that we may take an analytic open set $U \subseteq Y \setminus S$ and find a base $\mathbb{A}$ of $(A, \nabla)(U)$ consisting of flat sections, such that $\mathbb{K}_d(U)$ is spanned by of some flat sections $\mathbb{K}_d \subseteq S^d(\mathbb{A})$ for all d. This implies that the family $f' : \eta(X)_U \to U$ is constant, which is a contradiction since η is birational and our original family is non-isotrivial.

2. Graded Higgs Bundle arising from Geometry

2.1. **System of Hodge Bundles from Hodge Theory.** Throughout this section, we will assume that U is a quasi-projective manifold and compactified by a projective manifold $\overline{Y}$ with $\overline{S} = \overline{Y} \setminus U$ is a normal crossing divisor, and that there is smooth family $f : V \to U$ of n-folds. Leaving out a codimension two subset of $\overline{Y}$ we find a good partial compactification $f : X \to Y$ in the following sense

- X and Y are quasi-projective manifolds, f is flat, $U \subseteq Y$ and $\mathrm{codim}(\overline{Y} \setminus Y) \geq 2$.
- $S = Y \setminus U$ is smooth and $\Delta = f^*S$ is a relative normal crossing divisor over S (i.e. whose components, and all their intersections are smooth over components of S).

Following Griffiths and Simpson, one constructs the most natural Higgs bundle relating to the geometry and topology on the family. The tautological sequence

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0 \tag{2-1}$$

induces the short exact sequence of logarithmic forms of higher degrees

$$0 \to f^*(\Omega^1_Y) \otimes \Omega^{p-1}(\log S) \to \mathrm{gr}\,\Omega^p_X(\log S) \to \Omega^p_{X/Y}(\log S) \to 0, \tag{2-2}$$

where

$$\mathrm{gr}\,\Omega^p_X(\log \Delta) = \Omega^p_X(\log \Delta)/f^*\Omega^2_Y(\log S) \otimes \Omega^{p-2}_X(\log \Delta).$$

The direct sum of the direct image sheaves

$$E^{p,q} = R^q f_* \Omega^p_{X/Y}(\log \Delta), \quad p+q=k$$

endowed with the connecting maps in (2-2)

$$\theta^{p,q} : R^q f_* \Omega^p_{X/Y}(\log \Delta) \xrightarrow{\partial} \Omega^1_Y(\log S) \otimes R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta)$$

forms a so-called system of Hodge bundles of weight-k by Simpson.

$$(E,\theta) = (\bigoplus_{p+q=k} E^{p,q}, \bigoplus_{p+q=k} \theta^{p,q}).$$

Take the dual of(2-1), one has an exact sequence

$$0 \to T_{X/Y}(-\Delta) \to T_X(-\Delta) \to f^* T_Y(-\log S) \to 0.$$

The connecting map of the direct image defines the logarithmic Kodaira-Spencer map

$$\tau : T_Y(-\log S) \to R^1 f_* T_{X/Y}(-\log \Delta).$$

The Higgs field $\theta^{p,q}$ can be also defined as the cup-product with τ

$$\theta^{p,q} : T_Y(-\log S) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) \xrightarrow{\tau \otimes id}$$

$$R^1 f_* T_{X/Y}(-\log \Delta) \otimes R^q f_* \Omega^p_{X/Y}(\log \Delta) \xrightarrow{\cup} R^{q+1} f_* \Omega^{p-1}_{X/Y}(\log \Delta).$$

Proposition 2.1. *The Higgs bundle (E,θ) is the grading of Deligne's quasi-canonical extension of the variation of the polarized Hodge structure on k-th Betti cohomology $R^k_B f_* \mathbb{Z}_V$ of the smooth family $f : V \to U$.*

This result is well-known and due to Griffiths using harmonic forms. We present an algebraic approach due to Katz-Oda. It works also over any characteristic satisfying E_1-degeneration of Hodge to de Rham spectral sequence.

The short exact sequence (2-2) induces a short exact sequence of complexes of log differential forms

$$0 \to f^*(\Omega^1_Y(\log S) \otimes \Omega^\bullet_{X/Y}(\log S)[-1] \to K^0/K^2(\Omega^\bullet_{X/Y}(\log \Delta) \to \Omega^\bullet_{X/Y}(\log S) \to 0. \tag{2-3}$$

The k-th direct image sheaf $V := R^k_{\mathrm{dR}} f_* \Omega^\bullet_{X/Y}(\log S)$ of the relative de Rham complex carries the so-called logarithmic Gauss-Manin connection

$$V \xrightarrow{\nabla} V \otimes \Omega^1_Y(\log S),$$

defined by the connecting map in the long exact sequence

$$\begin{aligned} \nabla : R^k_{\mathrm{dR}} f_* \Omega^\bullet_{X/Y}(\log S) \quad &\to \quad R^{k+1}_{\mathrm{dR}} f_* f^*(\Omega^1_Y(\log S) \otimes \Omega^\bullet_{X/Y}(\log S)[-1] \\ &\simeq \quad \Omega^1_Y(\log S) \otimes R^k_{\mathrm{dR}} f_* \Omega^\bullet_{X/Y}(\log S). \end{aligned}$$

Introduce a decreasing filtration of the truncated sub-complexes

$$\{F^p_{\mathrm{tru}} \Omega^\bullet_{X/Y}(\log \Delta)\}_{0 \le p \le n} \subseteq \Omega^\bullet_{X/Y}(\log \Delta)$$

by

$$0 \to \cdots \to 0 \to \Omega^p_{X/Y}(\log \Delta) \xrightarrow{d} \Omega^{p+1}_{X/Y}(\log \Delta) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X/Y}(\log \Delta).$$

We define the Hodge filtration of V by

$$\{F^p\} = \{\mathrm{Im}(R^k_{\mathrm{dR}} f_* F^p_{\mathrm{tru}} \Omega^\bullet_{X/Y}(\log \Delta) \to R^k_{\mathrm{dR}} f_* \Omega^\bullet_{X/Y}(\log S))\} \subseteq V.$$

The connecting maps on $R^i_{\mathrm{dR}}f_*$ of

(2-4)
$$0 \to f^*\Omega^1_Y(\log S) \otimes F^{p-1}_{\mathrm{tru}}\Omega^\bullet_{X/Y}(\log \Delta)[-1] \to F^p_{\mathrm{tru}}K^0/K^2\Omega^\bullet_X(\log \Delta) \to F^p_{\mathrm{tru}}\Omega^\bullet_{X/Y}(\log \Delta) \to 0$$

is compatible with the Gauss-Manin connection under the natural map induced by the inclusion of short exact sequences (2-4) $\subseteq$ (2-3), and we find the **Griffiths transversality**:

$$\nabla F^p \subset \Omega^1_Y(\log S) \otimes F^{p-1},$$

which gives rise to a filtered logarithmic de-Rham bundle

$$(V, \nabla, F^\bullet),$$

and with the associated graded Higgs bundle

$$\mathrm{Gr}_{F^\bullet}(V,\nabla) = \left(\bigoplus_{p=0}^{k} \frac{F^p}{F^{p+1}}, \quad \bigoplus_{p=0}^{k} \frac{F^p}{F^{p+1}} \xrightarrow{\bar{\nabla}} \Omega^1_Y(\log S) \otimes \frac{F^{p-1}}{F^p}\right).$$

As over the complex number field, the E_1-degeneration of Hodge to de-Rham holds true. It implies that the map induced by the inclusion of the complexes (2-4) $\subseteq$ (2-3) is an isomorphism

$$R^k_{\mathrm{dR}}f_*F^p_{\mathrm{tru}}\Omega^\bullet_{X/Y}(\log \Delta) \simeq F^p$$

and consequently it induces an isomorphism

$$\left(\bigoplus_{p+q=k} E^{p,q}, \bigoplus_{p+q=k} \theta^{p,q}\right) \simeq \mathrm{Gr}_{F^\bullet}(V,\nabla).$$

Finally, by Deligne the filtered logarithmic de Rham bundle $(V, \nabla, F^\bullet)$ is the quasi-canonical extension of the variation of polarized variation of Hodge structure of k-th Betti cohomology of the smooth family $f : V \to U$.

2.2. **Graded Higgs Bundle From Deformation Theory.**

Given a log smooth family

$$f : (X, \Delta) \to (Y, S),$$

what will happen if all Torelli mappings associated to f become trivial? In the joint work of Viehweg and the third named author, we start with the classical Kodaira-Spencer map

$$T_Y(-\log S) \xrightarrow{\tau^{n,0}} R^1f_*T_{X/Y}(-\log \Delta),$$

and define then the extended Kodaira-Spencer map

$$\begin{tikzcd} T_Y(-\log S) \otimes R^qf_*T^q_{X/Y}(-\log \Delta) \arrow[r, "\tau^{p,q}"] \arrow[d, "\tau^{n,0}\otimes \mathrm{Id}"'] & R^{q+1}f_*T^{q+1}_{X/Y}(-\log \Delta) \\ R^1f_*T_{X/Y}(-\log \Delta) \otimes R^qf_*T^q_{X/Y}(-\log \Delta). \arrow[ur, "\cup"] & \end{tikzcd}$$

Putting all individual sheaves

$$F^{p,q} = R^qf_*T^q_{X/Y}(-\log \Delta)/\mathrm{torsion}, \quad p+q=n,$$

$$\left(= R^qf_*(\Omega^p_{X/Y}(\log \Delta) \otimes (\Omega^n_{X/Y}(\log \Delta))^{-1})/\mathrm{torsion}\right),$$

and together with the maps $\tau^{p,q}$ we obtain the so-called *Deformation Higgs bundle (sheaf)* attached to $f : X \to Y$:

$$(F, \tau) := \Big(\bigoplus_{p+q=n} F^{p,q}, \bigoplus_{p+q=n} \tau^{p,q} \Big).$$

One checks that $\tau = \oplus \tau^{p,q}$ satisfies the integrability condition $\tau \wedge \tau = 0$ using the associativity and anti-commutivity of the cup product on Dolbeault cohomology of the tangent sheaves along fibres.

The systems of Hodge bundles and the deformation Higgs bundle will play the central role in our investigation of global properties on moduli spaces of manifolds. For a semi-stable family $f : X \to Y$ of abelian, or Calabi-Yau n-folds with the maximal $\mathrm{Var}(f)$, we find the relation between the deformation Higgs bundle and the Higgs bundle arsing from VHS on the middle cohomology of fibres

$$(F, \tau) = (E, \theta) \otimes \big(E^{n,0}\big)^{-1},$$

where the Hodge (automorphic) line bundle $E^{n,0}$ is ample on Y. In general, for a family $f : X \to Y$ of n-folds with non-negative Kodaira-dimension, Kawamata and Viehweg's positivity theorem on the direct image of the relative pluri canonical sheaf $f_*\omega^{\nu}_{X/Y}$ allows us to construct a non-trivial Higgs map between (F, τ) and a system of Hodge bundles (E, θ) arising from an intermediate geometric situation

$$\rho : (F, \tau) \to (E, \theta) \otimes A^{-1},$$

where A is an ample line bundle on Y.

3. Comparison Between Deformation Higgs Bundle and Systme of Hodge Bundle

3.1. **Positivity of Direct Image Sheaves.** Let $f : X \to Y$ be a family of n-folds with semistable singular fibres Δ over S and with the smooth part of the family

$$f : V = X \setminus \Delta \to Y \setminus S =: U.$$

The following positivity of $f_*\omega^{\nu}_{X/Y}$ plays the fundamental role in study of the Iitaka conjecture:

Theorem 3.1 (Kawamata and Viehweg, cf. [Kaw85, Vie83])**.** *Assume that $f : X \to Y$ has the maximal variation, and $\omega_{V/U}$ is semi-ample. Then $f_*\omega^{\nu}_{X/Y}$ is ample w.r.t. open subsets for all $\nu > 1$ with $f_*\omega^{\nu}_{X/Y} \neq 0$.*

For constructing the comparison map we need actually a slightly stronger form of the positivity for the relative log differential form of the top degree

$$\mathcal{L} := \Omega^n_{X/Y}(\log \Delta) \subset \omega_{X/Y}.$$

Proposition 3.2 (Viehweg-Zuo, cf. [VZ02])**.** *Let $f : X \to Y$ be as in the above theorem. Then*

1. *$f_*(\mathcal{L}^{\nu})$ is is ample on an open subset $U' \subseteq U$ for $\nu \gg 0$.*
2. *Fix an ample line bundle A on Y, by replacing the original family $f : X \to Y$ (a birational modification) by a self fibre product of a higher power $f^{(r)} : X^{(r)} \to Y$, or by a Kawamata base change $\phi : Y' \to Y$ we may assume $\mathcal{L}^{\nu} \otimes f^*(A^{-\nu})$ is globally generated over $f^{-1}(U')$.*

3.2. **The Comparison Map.** One of the motivation of constructing cyclic covers is trying to give a more Hodge theoretical proof for Kodaira-Akizuki-Nakano vanishing theorem, cf. [EV92]. Let X be a projective manifold, $\mathcal{L}$ a line bundle on X and $s \in H^0(X, \mathcal{L}^\nu)$ with the zero divisor $D \subseteq X$. One takes the ν-th cyclic cover

$$\gamma : Z = X(\sqrt[\nu]{s}) \to X$$

with

$$\gamma_* \Omega^p_Z(\log \gamma^* D) = \bigoplus_{i=0}^{\nu-1} \Omega^p_X(\log D) \otimes \mathcal{L}^{-i}.$$

Deligne has shown

$$H^k(Z \setminus \gamma^* D, \mathbb{C}) = \bigoplus_{p+q=k} H^q(Z, \Omega^p_Z(\log \gamma^* D)).$$

Assume D is ample, then $X \setminus D$ is affine, the same holds true for $Z \setminus \gamma^* D$ and hence

$$H^k(Z \setminus \gamma^* D, \mathbb{C}) = 0, \quad \forall\, k > \dim X = n.$$

By the Hodge decomposition

$$0 = H^q(Z, \Omega^p_Z(\log \gamma^* D) = \bigoplus_{i=0}^{\nu-1} H^q(X, \Omega^p_X(\log D) \otimes \mathcal{L}^{-i}).$$

for any $p + q > n$. In particular,

$$H^q(X, \Omega^p_X(\log D) \otimes \mathcal{L}^{-1}) = 0, \forall\, p + q > n.$$

Using the residue map and induction on $\dim X$ we show Kodaira-Akizuki-Nakano vanishing theorem

$$\begin{aligned} &a). \qquad H^q(X, \Omega^p_X \otimes \mathcal{L}) = 0, \quad \forall\, p + q > n \\ &b). \qquad H^q(X, \Omega^p_X \otimes \mathcal{L}^{-1}) = 0, \quad \forall\, p + q < n. \end{aligned}$$

In contrast, the non-vanishing of the middle dimensional cohomology

$$H^q(X, \Omega^p_X \otimes \mathcal{L}^{-1}), \quad p + q = n$$

plays the crucial role in the construction of the comparison map.

We are going to construct a comparison map. Consider a log smooth family $f : X \to Y$, and denote $\mathcal{L} = \Omega^n_{X/Y}(\log \Delta)$. Let A be a line bundle on Y, and assume there is a non-zero section s of $\mathcal{L}^\nu \otimes f^* A^{-\nu}$. Indeed it is always the case if $f : X \to Y$ is a family of n-folds with semi-able canonical sheaf and with maximal $\mathrm{Var}(f)$ and A is a given line bundle. By 2. in Proposition 3.2, after taking a higher power of the self-fibre of product $f^{(r)} : X^{(r)} \to Y$ or by Kawamata base we find a section of $\mathcal{L}^\nu \otimes f^* A^{-\nu}$.

For a family $f : X \to Y$ of n-folds either of good minimal model or of general type. Then Kawamata (for good minimal model) and Kollar (for general type) showed that $f_* \omega^\nu_{X/Y}$ is big for $\nu >> 0$. The main difference between the case of good minimal model and the case of semi-ample is that the linear system of $\omega^\nu_{X/Y}$ in the first case could be not globally generated over $f^{-1}(U_0)$ for any open subset of U, while it is globally generated over $f^{-1}(U_0)$ for some open subset of U in the later case. Popa-Schnell applied the theory of Hodge module to get a comparison similar to what Viehweg-Zuo have done. It has the advantage that one does not care too much about the complication of the singularity appearing in the construction. Below, we propose an approach along the original construction by Viehweg-Zuo which works for all above cases.

Proposition 3.3 (Viehweg-Zuo). *The ν-th cyclic cover defined by a non-zero section s of $\mathcal{L}^\nu \otimes f^*A^{-\nu}$ induces a family*

$$g : Z \to X \xrightarrow{f} Y$$

*with the singular fibres Π over $S+T$, where T is the degeneration locus of the*new*singular fibers arising from the cyclic cover. By blowing up and leaving out a codimension-two sub scheme of Y we may assume*

$$g : (Z, \Pi) \to (Y, S+T)$$

is log smooth.

Taking (E, θ) to be the graded Higgs bundle of Deligne's quasi-canonical extension of VHS on the middle cohomology $R^n g_* \mathbb{Z}_{Z \setminus \Pi}$ on Y, then there exists a Higgs map

$$\rho : (F, \tau) \to (E, \theta) \otimes A^{-1};$$

that is

$$\begin{tikzcd} F^{p,q} \arrow[r, "\tau^{p,q}"] \arrow[d, "\rho^{p,q}"] & F^{p-1,q+1} \otimes \Omega^1_Y(\log S) \arrow[d, "\rho^{p-1,q+1} \otimes \iota"] \\ A^{-1} \otimes E^{p,q} \arrow[r, "\mathrm{id} \otimes \theta^{p,q}"] & A^{-1} \otimes E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T)). \end{tikzcd}$$

where $\iota : \Omega^1_Y(\log S) \hookrightarrow \Omega^1_Y(\log(S+T))$ is the natural inclusion. Note that the comparison map $\rho^{p,q}$ is defined only on Y *a priori*; that is, a morphism between $F^{p,q} = \bar{F}^{p,q}|_Y$ and $E^{p,q} = \bar{E}^{p,q}|_Y$. Since $\bar{F}^{p,q}$ is reflexive and $\mathrm{codim}(\overline{Y} \setminus Y) \geq 2$, the comparison map $\rho^{p,q}$ extends to $\overline{Y}$.

We would like to emphasize the crucial point in the comparison map: although the Higgs field θ on E has singularity along $S+T$, its restriction to $\rho(F)$ has only singularity on the original degeneration locus S.

Sketch the proof of Proposition 3.3. Let D denote the zero divisor of s. Note that the intersection of D with the generic fibres could always be singular.

• Replacing the family $f : X \to Y$ by a blowing up

$$\hat{f} : \hat{X} \xrightarrow{\sigma} X \to Y,$$

one may assume that $\sigma^*(D)$ is a normal crossing divisor.

• Let $T \subseteq Y$ denote the closure of the discriminant of the map

$$\hat{f} : \sigma^* D \cap \sigma^{-1}(V) \to U;$$

that is, the locus where the normal crossing divisor $\sigma^* D$ meets $\hat{f}^{-1}(y)$, $y \in U$ not transversally.

Let $\Sigma = \hat{f}^{-1}(T)$, and we take a further blowing up

$$\delta : X' \xrightarrow{\beta} \hat{X} \xrightarrow{\sigma} X$$

and put $D' + \Delta' + \Sigma' = \delta^*(D + \Delta) + \beta^* \Sigma$. Then the family

$$f' : X' \xrightarrow{\delta} X \xrightarrow{f} Y$$

with leaving out some codim-2 sub scheme in Y is log smooth as a morphism between the log pairs

$$f' : \big(X', (D' + \Delta' + \Sigma')\big) \to (Y, (S+T)).$$

Step 1. Cyclic Cover Defined by s. We write $\mathcal{M} := \delta^*(\mathcal{L} \otimes f^*A^{-1})$ and $D' := \delta^*(D)$, then $\mathcal{M}^\nu = O_{X'}(D')$. One takes the ν-th cyclic cover for the section $\delta^* s \in H^0(X', \mathcal{M}^\nu)$

$$\gamma' : Z' \xrightarrow{\text{normalization}} X'(\sqrt[\nu]{\delta^* s}) \xrightarrow{\gamma} X'.$$

Z' could be singular. By taking a resolution of singularity of Z', and a blowing up at the centers in the fibres over Y we obtain a non-singular variety Z and a generically finite map $\eta : Z \to Z'$. Leaving out a codimension two sub scheme in Y we may assume the induced map

$$g : Z \xrightarrow{\eta} Z' \xrightarrow{\gamma'} X' \xrightarrow{f'} Y$$

is log smooth for the pairs

$$g : (Z, g^{-1}(S+T)) \to (Y, (S+T)).$$

Step 2. Differential Forms On the Cyclic Cover. Recall that the local system $\mathbb{V} = R^n g_* \mathbb{Z}_0$ gives rise to the filtered logarithmic de Rham bundle

$$\nabla : V \to V \otimes \Omega^1_Y(\log(S+T)).$$

as the quasi canonical extension of $\mathbb{V} \otimes \mathcal{O}_{Y\setminus(S+T)}$. Let (E, θ) denote the induced system of Hodge bundles

$$\mathrm{Gr}_F(V, \nabla) = (E, \theta) = \Big(\bigoplus_{p+q=n} E^{p,q}, \bigoplus_{p+q=n} \theta^{p,q} \Big)$$

with

$$E^{p,q} = R^q g_* \Omega^p_{Z/Y}(\log \Pi).$$

The Higgs map

$$\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T))$$

is the edge map of $R^\bullet g_*$ of the exact sequence

$$0 \to g^* \Omega^1_Y(\log(S+T)) \otimes \Omega^{p-1}_{Z/Y}(\log \Pi) \to \mathrm{gr}\, \Omega^p_Z(\log \Pi) \to \Omega^p_{Z/Y}(\log \Pi) \to 0. \tag{3-1}$$

We also consider the pulled back of the deformation Higgs bundle (F, τ) on Y via the blowing up $\delta : X' \to X$

$$\delta^*(F, \tau) = (\bigoplus \delta^* F^{p,q}, \bigoplus \delta^* \tau^{p,q}) = (\bigoplus F'^{p,q}, \bigoplus \tau'^{p,q}),$$

with

$$F'^{p,q} = R^q f'_* (\delta^* \Omega^p_{X/Y}(\log \Delta) \otimes \delta^* \mathcal{L}^{-1}) / \text{torsion}.$$

Note that the Kodaira-Spencer map

$$\tau'^{p,q} : F'^{p,q} \to F'^{p-1,q+1} \otimes \Omega^1_Y(\log S)$$

is the edge map of $R^\bullet f'_*$ of the exact sequence

(3-2)

$$0 \to f'^* \Omega^1_Y(\log S) \otimes \delta^* \Omega^{p-1}_{X/Y}(\log \Delta) \otimes \mathcal{L}'^{-1} \to \delta^* \mathrm{gr}\, \Omega^p_X(\log \Delta) \otimes \mathcal{L}'^{-1} \to \delta^* \Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{L}'^{-1} \to 0.$$

Step 3 Comparison Between two Higgs Bundles. Let $\bullet$ stand either for $\mathrm{Spec}(\mathbb{C})$ or for Y. Then the Galois group $\mathbb{Z}/\nu\mathbb{Z}$ of $\psi : Z \to X'$ acts on $\psi_* \Omega^p_{Z/\bullet}(\log \Pi)$ with the eigenspace decomposition

$$\psi_* \Omega^p_{Z/\bullet}(\log \Pi) = \Omega^p_{X'/\bullet}(\log(\Delta' + \Sigma')) \oplus \bigoplus_{i=1}^{\nu-1} \Omega^p_{X'/\bullet}(\log(\Delta' + \Sigma' + D') \otimes \mathcal{L}'^{-1} \otimes f'^* A,$$

which indues a natural inclusion

$$\begin{tikzcd}
\delta^*\Omega^p_{X/\bullet}(\log\Delta)\otimes\mathcal{L}'^{-1} \arrow[r, hook] \arrow[d, hook] & \psi_*\Omega^p_{Z/\bullet}(\log\Pi)\otimes f'^*(A^{-1}) \\
\delta^*\Omega^p_{X/\bullet}(\log\Delta+\Sigma)\otimes\mathcal{L}'^{-1} \arrow[r, hook] & \Omega^p_{X'/\bullet}(\log\Delta'+\Sigma'+D')\otimes\mathcal{L}'^{-1}\otimes f'^*(A)\otimes f'^*(A^{-1}) \arrow[u, hook]
\end{tikzcd}$$

Via $\psi : Z \to X'$ the above inclusion together with the inclusion $\Omega^1_Y(\log S) \hookrightarrow \Omega^1_Y(\log(S+T))$ induces an inclusion of the exact sequences

$$\psi^*(3\text{-}2) \subseteq (3\text{-}1) \otimes g^*A^{-1}.$$

Finally taking the direct image

$$g_*\left(\psi^*(3\text{-}2) \subseteq (3\text{-}1) \otimes g^*A^{-1}\right),$$

it yields a map

$$\rho^{p,q} : F^{p,q} \to E^{p,q} \otimes A^{-1},$$

commuting with the Higgs fields as the edge maps.

4. Arakelov Inequality

4.1. **Viehweg Line Bundle and Arakelov Inequality.** Let $M = M_h$ denote the coarse moduli space of polarized manifolds of semi-ample canonical line bundle and with a fixed Hilbert polynomial h. We take a good projective compactification and consider a log map $\phi : (Y, S) \to (\overline{M}, S_M)$ from a log curve. Having such a negatively curved Finsler metric on M, the general principle Yau's form of Ahlfors-Schwarz Lemma suggests that there must be an inequality between the logarithmic hyperbolic metric on (Y, S) and some singular Kähler metric on M.

Theorem 4.1 (Arakelov inequality for relative pluri canonical linear system)**.** *Let $f : X \to Y$ be a non-isotrivial semi-stable family of n-folds over a smooth projective curve Y and with the bad reduction over S. Then*

$$\mu(f_*\omega^\nu_{X/Y}) \leq \frac{n\nu}{2} \cdot \deg\Omega^1_Y(\log S), \quad \forall \nu \in \mathbb{N}, \textit{ with } f_*\omega^\nu_{X/Y} \neq 0.$$

In general,

$$\mu(A) \leq \frac{n\nu}{2} \cdot \deg\Omega^1_Y(\log S), \quad \forall \textit{ non-zero subbundle } A \subseteq f_*\omega^\nu_{X/Y}.$$

Before sketching the proof of the theorem we recall some history of the Arakelov inequality:

- Arakelov-Parshin, Faltings and Deligne (sharp form): Arakelov inequality for families of abelian varieties.
- Viehweg-Zuo [VZ04]: Arakelov equality holds for some special families of abelian varieties. They are precisely the families of abelian varieties over Shimura curves with Mumford-Tate group of Mumford-type.
- Tan and Liu [Tan95, Liu96]: Arakelov inequality holds strictly for families of curves and $\nu = 1$.
- Green-Griffiths-Kerr [GGK09], Jost-Zuo [JZ02], Peters [Pet00], Viehweg-Zuo [VZ03a] and a very recent work by Biquard-Collier-Garcia-Prada-Toledo [BCGPT21] have studied Arakelov inequality for systems of Hodge bundles over curves.

• Möller-Viehweg-Zuo [MVZ12], and Lu-Tan-Zuo [LTZ17]: Some results over higher dimensional base.

For readers convenience we sketch the proof, and the details can be found in [MVZ06].

Sketched Proof of Theorem 4.1. The proof contains three main steps.

• Step I. Reduce the proof to the case where A is a line subbundle.

Indeed, given any non-zero subbundle $A \subseteq f_*\omega^{\nu}_{X/Y}$ of rank $A = r$, by taking the determinant and the r-power of self-fibre product of f, we have

$$\det A \subseteq (f_*\omega^{\nu}_{X/Y})^{\otimes r} \cong \tilde{f}_*\omega^{\nu}_{\widetilde{X}/Y},$$

where $\widetilde{X}$ is the desingularization of the r-power of self-fibre $X \times_Y \cdots \times_Y X$ and $\tilde{f} : \widetilde{X} \to Y$ is the induced fibration. Hence we may assume A is a line subbundle.

• Step II. Use the cyclic cover to construct a new Higgs bundle and a comparison between two Higgs bundles.

First, Replacing the original family by suitable unramified base change, we may assume that A is ν-divisible; that is, there exists an invertible sheaf A' on Y such that $A = A'^{\nu}$. In other words, we get an injection $A'^{\nu} \hookrightarrow f_*\omega^{\nu}_{X/Y}$ and hence a non-zero map $f^*A'^{\nu} \hookrightarrow \omega^{\nu}_{X/Y}$. This is equivalent to a non-zero section s of $\omega^{\nu}_{X/Y} \otimes f^*A'^{\nu}$. Thus by Proposition 3.3, we get a new fibration

$$g : (Z, \Pi) \to (Y, S + T),$$

which is log smooth. Moreover, the graded Higgs bundle (E, θ) of Deligne's quasi-canonical extension of VHS on the middle cohomology $R^n g_*\mathbb{Z}_{Z\setminus\Pi}$ admits a comparison with the original deformation Higgs bundle (F, τ) attached to $f : X \to Y$; that is, there exists a Higgs map

$$\rho : (F, \tau) \to (E, \theta) \otimes A'^{-1}.$$

It gives the following commutative diagram:

$$\begin{tikzcd}
F^{p,q} \arrow[r, "\tau^{p,q}"] \arrow[d, "\rho^{p,q}"] & F^{p-1,q+1} \otimes \Omega^1_Y(\log S) \arrow[d, "\rho^{p-1,q+1}\otimes\iota"] \\
A'^{-1} \otimes E^{p,q} \arrow[r, "\mathrm{id}\otimes\theta^{p,q}"] & A'^{-1} \otimes E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T)).
\end{tikzcd}$$

where $\iota : \Omega^1_Y(\log S) \hookrightarrow \Omega^1_Y(\log(S+T))$ is the natural inclusion.

• Step III. The sheaf A' generates a Higgs subbundle

$$\Big(H = \bigoplus_{q=0}^{n} H^{n-q,q}, \theta|_H\Big) \subseteq (E, \theta),$$

where $H^{n,0} = A'$, and

$$H^{n-q-1,q+1} = \mathrm{Im}\big(\theta|_{H^{n-q,q}} : H^{n-q,q} \to E^{n-q-1,q+1} \otimes \Omega^1_Y(\log(S+T))\big) \otimes \Omega^1_Y(\log S)^{-1}.$$

Let $q_0 \leq n$ be the largest number such that $H^{n-q,q} \neq 0$. Then

$$\deg H = \sum_{q=0}^{q_0} \deg H^{n-q,q} = \sum_{q=0}^{q_0} \big(\deg A' - q \deg \Omega^1_Y(\log S)\big) = (q_0+1)\big(\deg A' - \frac{q_0}{2} \deg \Omega^1_Y(\log S)\big).$$

By Simpson's polystability of the Higgs bundle (E, θ), one has $\deg H \leq 0$. Hence

$$\deg A = \nu \cdot \deg A' \leq \nu \cdot \frac{q_0}{2} \deg \Omega^1_Y(\log S) \leq \frac{n\nu}{2} \deg \Omega^1_Y(\log S).$$

This completes the proof of Theorem 3.1.

4.2. **Strict Arakelov Inequality of Families of Varieties of General Type.** In contrast, for families of varieties of general type we show the Arakelov inequality always holds strictly for large ν.

Proof of Theorem 1.5. For simplicity, we prove this for the total direct image sheaf $f_*\omega_{X/Y}^{\nu}$; the proof is similar for subbundles $A \subseteq f_*\omega_{X/Y}^{\nu}$ which defines a birational map as in the theorem. Assume on the contrary that there exists such an $\nu \in \mathbb{N}$ with the Arakelov equality

$$\mu_0 := \mu(f_*\omega_{X/Y}^{\nu}) = \frac{n\nu}{2} \cdot \deg \Omega_Y^1(\log S).$$

As for any subbundle $A \subseteq f_*\omega_{X/Y}^{\nu}$ we have

$$\mu A \leq \frac{n\nu}{2} \cdot \deg \Omega_Y^1(\log S) = \mu_0$$

Hence $f_*\omega_{X/Y}^{\nu}$ is a semi-stable vector bundle over Y.

Consider the d-th multiplication map

$$K_{m_d} \subseteq S^d(f_*\omega_{X/Y}^{\nu}) \xrightarrow{m_d} I_{m_d} \subseteq f_*\omega_{X/Y}^{d\nu},$$

where K_{m_d} is the kernel of the d-th multiplication map. Note that the restriction of K_{m_d} on a general fiber is the subspace of all homogeneous polynomials of degree d in the ideal defining the birational embedding of that fiber.

Applying Arakelov inequality for the image of the d-multiplication $I_{m_d} \subseteq f_*\omega_{X/Y}^{d\nu}$ we show $\mu(I_{m_d}) \leq d \cdot \mu_0$. On the other hand, $S^d(f_*\omega_{X/Y}^{\nu})$ is semi-stable of slope $d \cdot \mu_0$ and I_{m_d} is a quotient bundle we obtain

$$\mu I_{m_d} = d \cdot \mu_0 = \mu S^d(f_*\omega_{X/Y}^{\nu}) = \mu K_{m_d},$$

and hence K_{m_d} is a semi-stable subbundle of $S^d(f_*\omega_{X/Y}^{\nu})$ of the same slope.

After a base change of Y and twisting with a line bundle we may assume $f_*\omega_{X/Y}^{\nu}$ is semi-stable of degree zero and $S^d(f_*\omega_{X/Y}^{\nu})$ contains K_{m_d} as a semi-stable subbundle of degree zero.

Theorem 4.2 (Simpson, cf. [Sim92])**.** *Let $\mathcal{C}_{dR}$ be the category of vector bundles over Y with integrable connections and $\mathcal{C}_{Dol}$ be the category of semi-stable Higgs bundle of degree 0. Then there exists an equivalent functor*

$$\mathcal{F} : \mathcal{C}_{Dol} \to \mathcal{C}_{dR}.$$

We just recall some properties about this functor. Let $(E, 0)$ be a semistable Higgs bundle of degree 0 with the trivial Higgs field. Let $(E', 0)$ be a Higgs subbundle of $(E, 0)$ of degree 0.

(1). The functor $\mathcal{F}$ preserves the tensor products. In particular it also preserves symmetric powers.

(2). The underlying bundle of the bundle $\mathcal{F}((E, 0))$ with the integrable connection is isomorphic to E. We call the connection to be canonical and denote it by $\nabla_{can}(E)$.

(3). The connection $\nabla_{can}(E)$ preserves E' and $\nabla_{can}(E) \mid_{E'} = \nabla_{can}(E')$.

(4) For a semi-stable vector bundle V of degree 0. Then there exists an integrable connection ∇ on V such that for any $d \geq 1$ and any subbundle $K \subseteq S^d(V)$ of degree 0, the connection $S^d(\nabla)$ on $S^d(V)$ preserves K.

Applying (4) in our situation for $f_*\omega^\nu_{X/Y}$ (after twisting with a suitable line bundle), we find an integrable connection $(f_*\omega^\nu_{X/Y}, \nabla)$ such that $S^d(\nabla)$ preserves $K_{m_d} \subseteq S^d(f_*\omega^\nu_{X/Y})$ for any $d \in \mathbb{N}$, i.e. for each point $p \in U$ we find an analytic open neighborhood $U_p \subseteq U$ and a flat base $\mathbb{V}$ for the solutions of $(f_*\omega^\nu_{X/Y}, \nabla)_{U_p}$ and such that $K_{m_d} \subseteq S^d(f_*\omega^\nu_{X/Y})$ is spanned by a flat sub space $\mathbb{K}_{m_d} \subseteq S^d(\mathbb{V})$. This means that we find a base of $f_*\omega^\nu_{X/Y}$ over U_p such that the coefficients of all polynomials in the ideals defining the smooth fibres of $f : X \to Y$ under the birational embedding $X \to \mathbb{P}_Y(f_*\omega^\nu_{X/Y})$ are locally constant. This gives a contradiction since f is non-isortrivial. $\square$

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