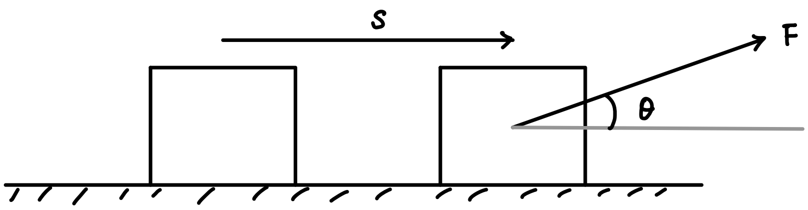
§1.3 向量的数量积

§1.3.1. 数量积的定义与性质



力F所作的功为：　$W = |F|\cdot|s|\cdot\cos\theta$

定义 1.3.1. a与b的**数量积（内积）**

a与b的夹角.

$$a\cdot b := |a|\cdot|b|\cdot\cos\theta .$$

注：$a \perp b \Leftrightarrow a\cdot b = 0$.

命题 1.3.1.

$a\cdot b = b\cdot a$ ✓

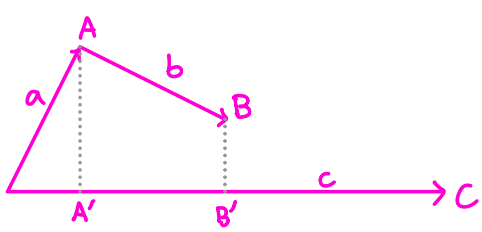
$(a+b)\cdot c = a\cdot c + b\cdot c$?

$(\lambda a)\cdot b = \lambda(a\cdot b) = a\cdot(\lambda b)$ ✓

$a^2 := a\cdot a \geq 0$ ✓

等号成立当且仅当 $a=0$.

Pf:



$\left.\begin{aligned}\overrightarrow{OA'} &= xc\\ \overrightarrow{A'B'} &= yc\end{aligned}\right\} \Rightarrow OB' = (x+y)c$

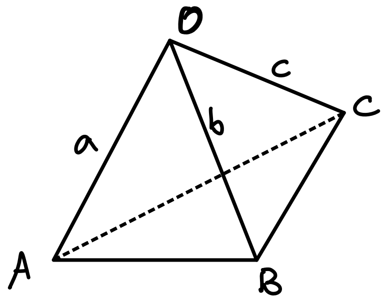
$(a+b)\cdot c := (x+y)|c|^2 = x|c|^2 + y|c|^2 = a\cdot c + b\cdot c$ □

推论： 1) $(a\pm b)^2 = a^2 \pm 2a\cdot b + b^2$ ✓

2) $|a|+|b| \ge |a+b|$

Pf： $|a+b|^2 = (a+b)^2 = a^2+b^2+2|a|\cdot|b|\cdot\cos\overset{\text{夹角}}{\theta} \le a^2+b^2+2|a|\cdot|b| = (|a|+|b|)^2$ □

例



$$\left.\begin{array}{l}\vec{OA}\perp\vec{BC}\\ \vec{OB}\perp\vec{AC}\end{array}\right\} \Rightarrow \vec{OC}\perp\vec{AB}$$

Pf：

$$\left.\begin{array}{l}\vec{OA}\perp\vec{BC} \Rightarrow a\cdot(c-b)=0 \Rightarrow a\cdot c = a\cdot b\\ \vec{OB}\perp\vec{AC} = b\cdot(c-a)=0 \Rightarrow b\cdot c = b\cdot a\end{array}\right\}$$

$$\Rightarrow a\cdot c = b\cdot c \Rightarrow (a-b)\cdot c = 0 \Rightarrow \vec{OC}\perp\vec{AB}$$ □

§1.3.2 直角坐标系下数量积的计算

$[O; i, j, k]$ = 直角坐标系.

$$\Rightarrow \begin{cases} i\cdot i = j\cdot j = k\cdot k = 1 \\ i\cdot j = j\cdot k = k\cdot i = 0 \end{cases}$$

内积夹角公式：$a = a_1 i + a_2 j + a_3 k$, $b = b_1 i + b_2 j + b_3 k$, 则

$$a\cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

$$\cos\theta = \frac{a\cdot b}{|a||b|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$$

例 1.3.2：求 $\vec{x}$ s.t. $|\vec{x}| = 1$, $\vec{x} \perp a = -i + 2j + k$, $\vec{x} \perp b = i + 3k$

解：设 $\vec{x} = x_1e_1 + x_2e_2 + x_3e_3$. 则

$$\left.\begin{array}{l} |\vec{x}| = 1 \Rightarrow x_1^2 + x_2^2 + x_3^2 = 1 \\ \vec{x} \perp \vec{a} \Rightarrow -x_1 + 2x_2 + x_3 = 0 \\ \vec{x} \perp b \Rightarrow x_1 + 3x_3 = 0 \end{array}\right\} \Rightarrow (x_1, x_2, x_3) = \pm\frac{1}{\sqrt{14}}(3i + 2j - k)$$

例 (Cauchy 不等式) $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$

Pf: $LHS = (a\cdot b)^2 = (|a|\cdot|b|\cos\theta)^2 \leq a^2b^2 = RHS$. □