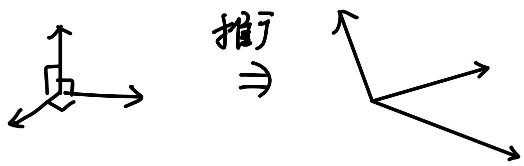
§1.2 坐标系

§1.2.1 仿射坐标系



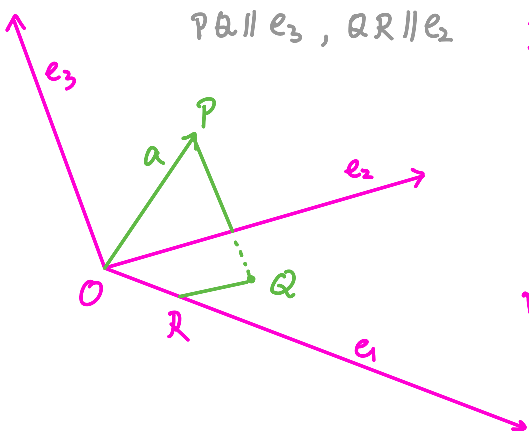
不共面.

定理1.2.1 设 e_1, e_2, e_3 不共面. 则 $\forall$ 向量 a $\exists!$ (x_1, x_2, x_3) s.t.

$$a = x_1 e_1 + x_2 e_2 + x_3 e_3$$

证:

$PQ \parallel e_3$, $QR \parallel e_2$



存在性:

$$a = \overrightarrow{OP} = OR + RQ + QP$$
$$= x_1 e_1 + x_2 e_2 + x_3 e_3$$

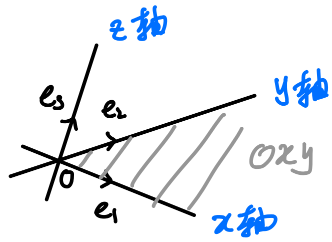
唯一性:

$$x_1 e_1 + x_2 e_2 + x_3 e_3 = a = y_1 e_1 + y_2 e_2 + y_3 e_3$$
$$\Rightarrow (x_1 - y_1) e_1 + (x_2 - y_2) e_2 + (x_3 - y_3) e_3 = 0$$
$$\Rightarrow x_1 - y_1 = x_2 - y_2 = x_3 - y_3 = 0$$
$$\Rightarrow x_1 = y_1 \ \& \ x_2 = y_2 \ \& \ x_3 = y_3 \quad \square$$

定义1.2.1 称三个有序不共面向量 e_1, e_2, e_3 为空间的一组**基**. 称 (x_1, x_2, x_3) 为 $a = x_1 e_1 + x_2 e_2 + x_3 e_3$ 在 e_1, e_2, e_3 下的**仿射坐标**或简称**坐标**

定义 1.2.2 仿射坐标系 $=$ 点 O $+$ 基 e_1, e_2, e_3 记为 $[O; e_1, e_2, e_3]$

$$\underset{\text{坐标原点.}}{\uparrow} \qquad \underset{\text{坐标向量}}{\uparrow}$$



坐标平面: Oxy

Oyz

Ozx

(一一对应) 给定 $[O; e_1, e_2, e_3]$

$$\{P \mid \text{空间中的点}\} \overset{1:1}{\longleftrightarrow} \{\text{向量 } \vec{OP}\} \overset{1:1}{\longleftrightarrow} \{(x_1, x_2, x_3)\}$$

例: 1) e_1, e_2, e_3 为基 $\Rightarrow$ $a = e_1 + e_2 + e_3$, $b = e_1 - e_2 + e_3$, $c = e_1 + e_2 - e_3$ 为基

2) 求 $d = a + 2b - 3c$ 在 e_1, e_2, e_3 下的坐标.

证明 1): $xa + yb + zc = 0 \Rightarrow (x+y+z)e_1 + (x-y+z)e_2 + (x+y-z)e_3 = 0$

$$\Rightarrow \begin{cases} x+y+z=0 \\ x-y+z=0 \\ x+y-z=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow \checkmark$$

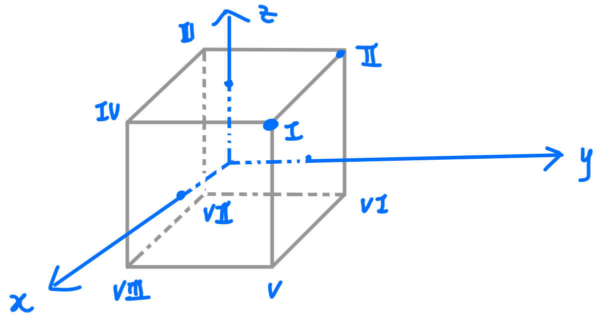
解 2): $d = a + 2b - 3c = (e_1 + e_2 + e_3) + 2(e_1 - e_2 + e_3) - 3(e_1 + e_2 - e_3)$

1-2-2 $= -4e_2 + 6e_3 \Rightarrow$ 坐标为 $(0, -4, 6)$ □

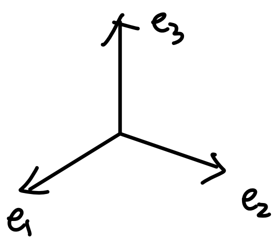
八个卦限：

Ⅰ(+,+,+)，Ⅱ(−,+,+)，Ⅲ(−,−,+)，Ⅳ(+,−,+)

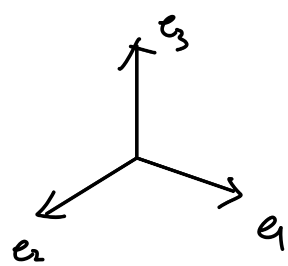
Ⅴ(+,+,−)，Ⅵ(−,+,−)，Ⅶ(−,−,−)，Ⅷ(+,−,−)



左(右)手仿射坐标系：



右



左

§1.2.2 向量的坐标运算

向量的运算 转化为 坐标运算

取定 $[O;e_1,e_2,e_3]$

$$\{\text{向量}\} \xleftrightarrow{1:1} \{\text{坐标}\}$$

$\Rightarrow$ 用坐标 (x_1,x_2,x_3) 表示向量 $x_1e_1+x_2e_2+x_3e_3$

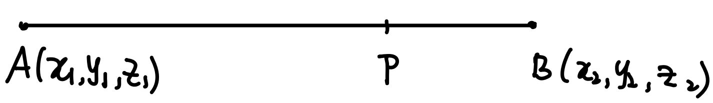
(计算公式)
$$\begin{cases} (x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,\ x_2+y_2,\ x_3+y_3) \\ \lambda(x_1,x_2,x_3) = (\lambda x_1, \lambda x_2, \lambda x_3). \end{cases}$$

Pf: $(x_1,x_2,x_3)+(y_1,y_2,y_3) = (x_1e_1+x_2e_2+x_3e_3)+(y_1e_1+y_2e_2+y_3e_3)$

$= (x_1+y_1)e_1+(x_2+y_2)e_2+(x_3+y_3)e_3 = (x_1+y_1,\ x_2+y_2,\ x_3+y_3)$

similar for λa. □

例 1.2.2



$\overrightarrow{AP} = \lambda\overrightarrow{PB} \Rightarrow$ P坐标为 ?

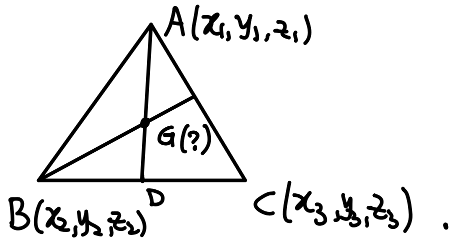
解: $\overrightarrow{AP} = \lambda\overrightarrow{PB} \Rightarrow \overrightarrow{OP}-\overrightarrow{OA} = \lambda(\overrightarrow{OB}-\overrightarrow{OP})$

$\Rightarrow \overrightarrow{OP} = \frac{1}{1+\lambda}\overrightarrow{OA} + \frac{\lambda}{1+\lambda}\overrightarrow{OB}$

1-2-4 $= \left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}, \frac{z_1+\lambda z_2}{1+\lambda}\right)$ □

$\Rightarrow$ 中点 $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

例 1.2.3（重心坐标）



解：$\overrightarrow{OG} \overset{1.1.3}{\underset{1.2.2}{=}} \frac{2}{3}\overrightarrow{OD} + \frac{1}{3}\overrightarrow{OA} = \frac{2}{3}\left(\frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}\right) + \frac{1}{3}\overrightarrow{OA}$

$$= \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

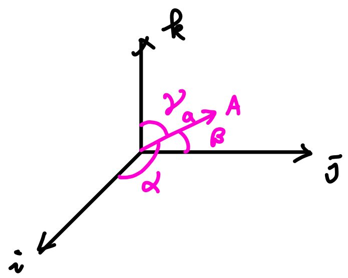
§1.2.3 直角坐标系

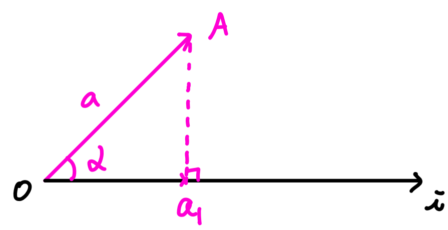
直角坐标系 = 仿射坐标系 + 坐标向量两两垂直 + 坐标向量为单位向量.

表示为: i, j, k.

优点: 容易计算模长与夹角.

(模长, 方向余弦) 设 $[O; i, j, k]$ 为直角坐标系, $a = a_1 i + a_2 j + a_3 k$





则
$$
\begin{cases}
|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} \\
(\cos\alpha, \cos\beta, \cos\gamma) = \left(\dfrac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}, \dfrac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}, \dfrac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}} \right) = \dfrac{a}{|a|}
\end{cases}
$$

方向余弦

例: $P(1,2,3)$, $Q(2,4,-1)$ $\Rightarrow$ $\overrightarrow{PQ}$ 的方向余弦为 $\left(\frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}\right)$.

解: $\overrightarrow{PQ} = (1, 2, -4)$, $|\overrightarrow{PQ}| = \sqrt{21}$. □