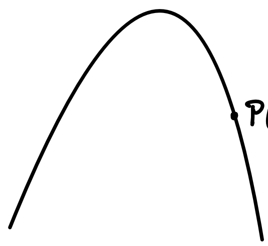
§2.2 空间曲线与曲面

曲线与曲面的描述方式（参数方程，一般方程）

- 点在空间中运动轨迹



$P(t) = (\underbrace{x(t), y(t), z(t)}_{\text{连续}})$.

推广 $\Rightarrow$ 1) 若 $x(t), y(t), z(t)$ 为连续函数，

$$P(t) = (x(t),\ y(t),\ z(t)) \leftarrow \text{参数方程}$$

则 $\{P(t) \mid t \in I\}$ 为一条空间曲线

2) 若 $x(s,t)$, $y(s,t)$, $z(s,t)$ 连续，

$$P(s,t) = (x(s,t),\ y(s,t),\ z(s,t))$$

则 $\{P(s,t) \mid s \in J, t \in I\}$ 为一个空间曲面

- $f(x,y,z)$ 连续函数 （例：$z = x^2 + y^2$）

$$\{(x,y,z) \mid \underbrace{f(x,y,z) = 0}_{\text{该曲面的一般方程}}\} \text{ 为一个曲面}$$

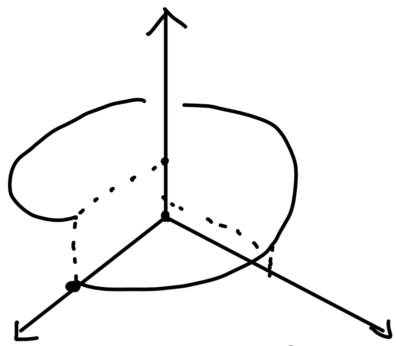
- $f(x,y,z)$, $g(x,y,z)$

相交曲线 $\left\{(x,y,z) \,\middle|\, \underbrace{\begin{matrix} f(x,y,z)=0 \\ g(x,y,z)=0 \end{matrix}}_{\text{该曲线的一般方程}}\right\}$.

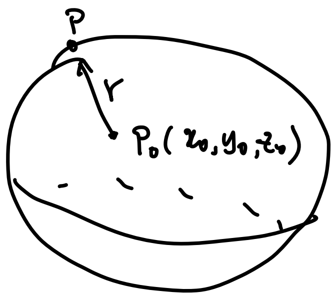
例（螺旋线）

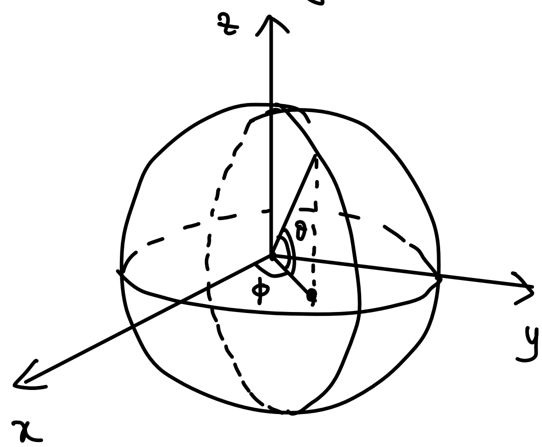
$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \qquad 0 \leq t \leq 2\pi$$

$\Rightarrow$



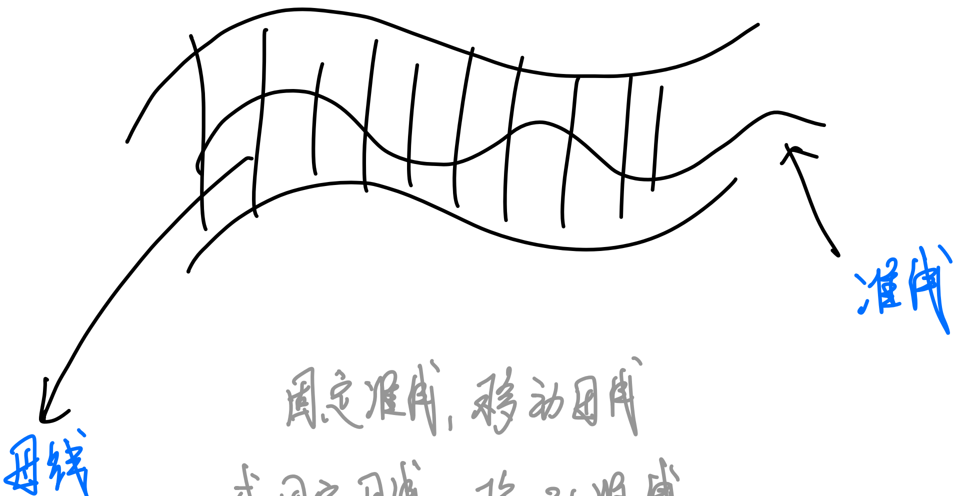
例：（球面）





- $|\overrightarrow{P_0P}| = r$
- $(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$
- $$\begin{cases} x = x_0 + r\cos\theta\cos\phi \\ y = y_0 + r\cos\theta\sin\phi \\ z = r\sin\theta \end{cases} \qquad 0 \leq \phi < 2\pi,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

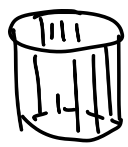
§2.2.2. 柱面 = 一族平行直线形成的曲面



固定准线，移动母线

或固定母线，移动准线

例：圆柱面

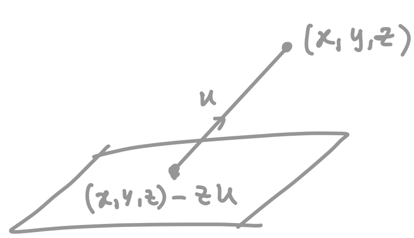


特殊柱面一般方程

- $f(x,y)=0 \rightsquigarrow$ 母线平行于 z 轴
- 准线：$\left\{(x,y,z) \;\middle|\; \begin{array}{l} f(x,y)=0 \\ z=0 \end{array}\right\}$

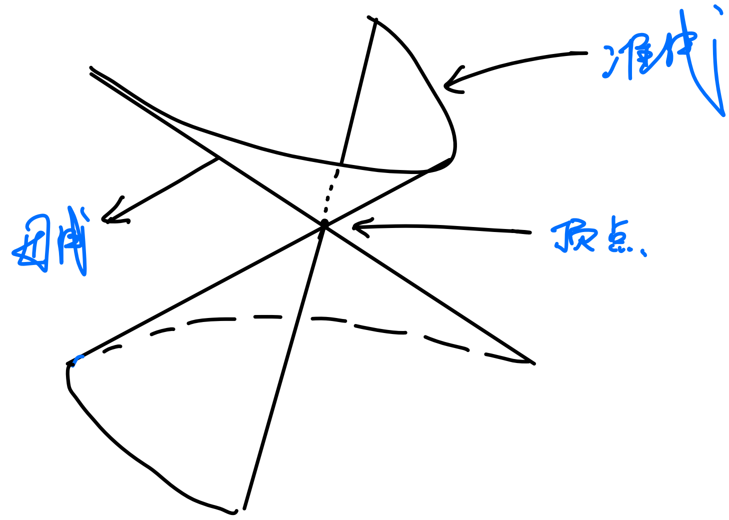
  母线方向 $u=(u_1,u_2,1)$

$\Rightarrow f(x-zu_1,\ y-zu_2)=0$



§2.2.3 锥面

锥面 = 一族经过定点的直线形成的曲面.



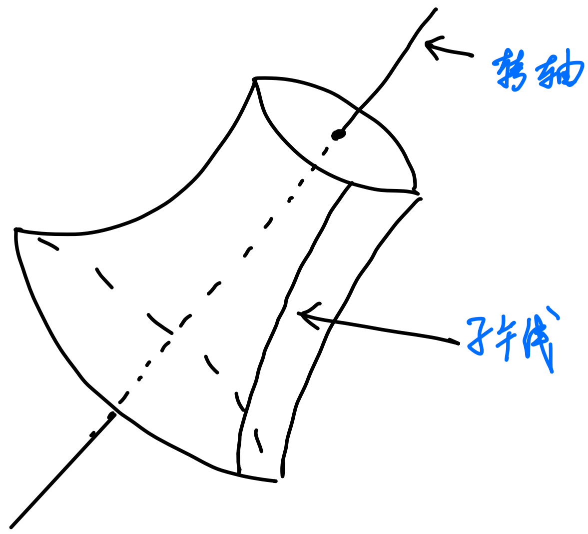
准线参数方程: $Q(t) = (f_1(t), f_2(t), f_3(t))$

$$\Downarrow$$

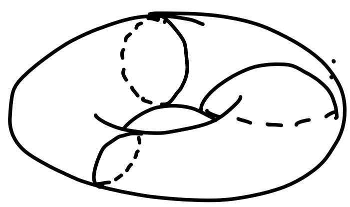
锥面参数方程: $P(s,t) = (1-s)A + sQ(t)$

例: 圆锥面.

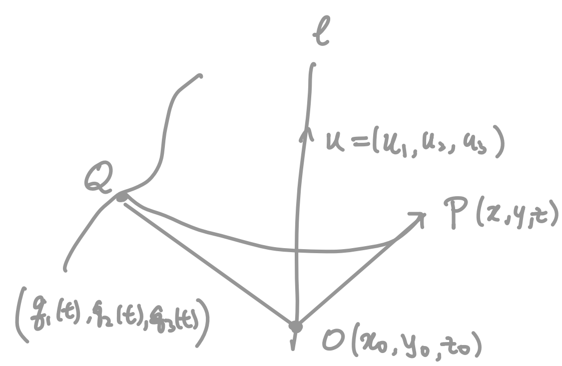
旋转面.



例:



例:



$$\Rightarrow \begin{cases}|OQ|=|OP| \\ PQ \perp l_2\end{cases} \Rightarrow \begin{cases}(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=(g_1(t)-x_0)^2+(g_2(t)-y_0)^2+(g_3(t)-z_0)^2 \\ u_1(x-g_1(t))+u_2(y-g_2(t))+u_3(z-g_3(t))=0\end{cases}$$

$\xRightarrow{\text{消去}t}$ 一般方程.

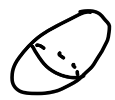
§二次曲面简介.

一般方程:

$$a_1x^2+a_2xy+a_3xz+a_4y^2+a_5yz+a_6z^2+a_7x+a_8y+a_9z+a_{10}=0$$

常见九种 $a,b,c>0$:

1. 椭球面: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



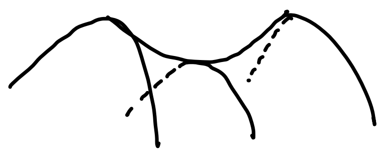
2. 单叶双曲面: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

3. 双叶双曲面: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$

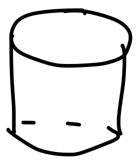
4. 二次锥面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ 2&3的渐近锥面

5. 椭圆抛物面 $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

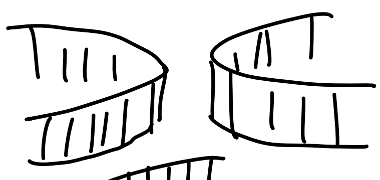
6. 双曲抛物面(马鞍面) $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$



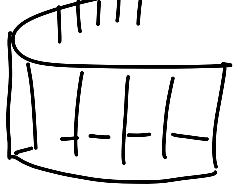
7. 椭圆柱面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



8. 双曲柱面 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$



9. 抛物柱面 $y^2=2px$



- 包含直线 4, 7, 8, 9 ?

- 旋转面 $a=b \Rightarrow 1, 2, 3, 4, 5, 7,$

- 单叶双曲面，双曲抛物面

$\Downarrow$

$$\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{y}{b}-\frac{z}{c}\right)=\left(1+\frac{x}{a}\right)\left(1-\frac{x}{a}\right) \qquad \left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=z$$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$$\begin{cases} \frac{y}{b}+\frac{z}{c}=\lambda\left(1+\frac{x}{a}\right) \\ \frac{y}{b}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{x}{a}\right) \end{cases} \qquad \begin{cases} \frac{x}{a}+\frac{y}{b}=\lambda \\ \frac{x}{a}-\frac{y}{b}=\frac{z}{\lambda} \end{cases}$$

直线族

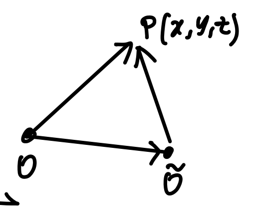
§2.3、坐标变换

二次曲面 $\xRightarrow{\text{坐标变换}}$ 简化形式

§2.3.1 坐标系的平移

$[O; e_1, e_2, e_3] \longrightarrow [\tilde{O}, e_1, e_2, e_3]$

$(x, y, z) \longmapsto (\tilde{x}, \tilde{y}, \tilde{z})$



$\overrightarrow{O\tilde{O}} = (x_0, y_0, z_0)$

$\overrightarrow{OP} = (x, y, z) \qquad \overrightarrow{\tilde{O}P} = (\tilde{x}, \tilde{y}, \tilde{z})$

$\overrightarrow{OP} = \overrightarrow{O\tilde{O}} + \overrightarrow{\tilde{O}P}$

$\Rightarrow$ 变换式 $\begin{cases} x = \tilde{x} + x_0 \\ y = \tilde{y} + y_0 \\ z = \tilde{z} + z_0 \end{cases}$ or $\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0 \\ \tilde{z} = z - z_0 \end{cases}$

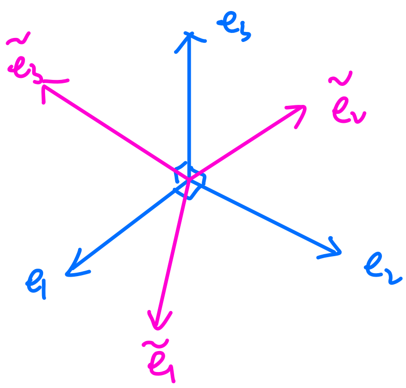
例：$x^2 - y^2 - z^2 + 2x + 2y - 2z = 0 \Rightarrow \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 = -1$

$\begin{cases} \tilde{x} = x+1 \\ \tilde{y} = y-1 \\ \tilde{z} = z+1 \end{cases}$

旋转单叶双曲面

§2.3.2. 坐标系的旋转

不妨设为 直角坐标系



$$[O; e_1, e_2, e_3] \longrightarrow [O; \tilde e_1, \tilde e_2, \tilde e_3]$$

$$\begin{cases} \tilde e_1 = a_1 e_1 + a_2 e_2 + a_3 e_3 \\ \tilde e_2 = b_1 e_1 + b_2 e_2 + b_3 e_3 \\ \tilde e_3 = c_1 e_1 + c_2 e_2 + c_3 e_3 \end{cases}$$

$$x e_1 + y e_2 + z e_3 = \overrightarrow{OP} = \tilde x \tilde e_1 + \tilde y \tilde e_2 + \tilde z \tilde e_3$$

$$= \tilde x (a_1 e_1 + a_2 e_2 + a_3 e_3) + \tilde y (b_1 e_1 + b_2 e_2 + b_3 e_3) + \tilde z (c_1 e_1 + c_2 e_2 + c_3 e_3)$$

$\Rightarrow$ 变换公式 $\begin{cases} x = a_1 \tilde x + b_1 \tilde y + c_1 \tilde z \\ y = a_2 \tilde x + b_2 \tilde y + c_2 \tilde z \\ z = a_3 \tilde x + b_3 \tilde y + c_3 \tilde z \end{cases}$

$|\tilde e_1| = 1$, $\tilde e_1 \perp \tilde e_2$
$|\tilde e_2|^2 = 1$, $\tilde e_2 \perp \tilde e_3$
$|\tilde e_3|^2 = 1$, $\tilde e_3 \perp \tilde e_1$

$\Downarrow$

$a_1^2 + a_2^2 + a_3^2 = 1$ $\quad a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$
$b_1^2 + b_2^2 + b_3^2 = 1$ $\quad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0$
$c_1^2 + c_2^2 + c_3^2 = 1$ $\quad c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

$$\begin{cases} \tilde x = a_1 x + a_2 y + a_3 z \\ \tilde y = b_1 x + b_2 y + b_3 z \\ \tilde z = c_1 x + c_2 y + c_3 z \end{cases}$$