Mathematical model of multicopters

This document aims to establish the mathematical model of multicopters which involves the multicopter control model and multicopter aerodynamic drag model. Note: You can obtain the necessary parameters through our website www.flyeval.com, and be sure to contact our lab (http://rfly.buaa.edu.cn/) if you have any suggestions or opinions.

I. Input variables

The mathematical model is established based on some input variables.

- m : multicopter mass.
- g : acceleration of gravity.
- J_{xx}, J_{yy}, J_{zz} : central principal moments of inertia.
- d : distance between the body center and the rotor.
- C_{T} : lumped parameter thrust coefficient. It can be easily determined from static thrust tests.
- C_{M} : lumped parameter torque coefficient. It can be easily determined from static thrust tests.
- C_{R} : constant parameter. It can represent the slope from the throttle to the motor speed.
- ϖ_{b} : constant parameter. It is the constant term from the throttle to the motor speed.
- J_m : total moments of inertia of the entire rotor and the propeller about the axis of rotation.
- T_{m} : time constant of motor&propeller. It can determine the dynamic response of motors.
- C_{d} : drag coefficient. It can be used to determine the drags applied to the rotor blades. The definition of C_{d} is the drags divided by the square of the multicopter velocities.

II. Coordinate Frame

Before jumping into the mathematical model, appropriate coordinate frames need to be established. The multicopter is regarded as a rigid body whose attitude in space mainly describes the rotation between the Aircraft-Body Coordinate Frame (ABCF) and the Inertial Reference Frame (IRF).

The IRF $o_{\mathrm{e}}x_{\mathrm{e}}y_{\mathrm{e}}z_{\mathrm{e}}$ is used to study a multicopter's motion state relative to the Earth's surface and to determine its three-dimensional (3D) position. The earth curvature is ignored, namely the Earth's surface is flat. The initial position of the multicopter or the center of the Earth is often set as the coordinate origin o_{e}. The $o_{\mathrm{e}}x_{\mathrm{e}}$ axis points to a certain direction in the horizontal plane and the $o_{\mathrm{e}}z_{\mathrm{e}}$ axis points perpendicularly to the ground. Then, the $o_{\mathrm{e}}y_{\mathrm{e}}$ axis is determined according to the right-hand rule.

The ABCF $o_{\mathrm{b}}x_{\mathrm{b}}y_{\mathrm{b}}z_{\mathrm{b}}$ is fixed to the multicopter. The Center of Gravity (CoG) of the multicopter is chosen as the origin o_{b} of $o_{\mathrm{b}}x_{\mathrm{b}}y_{\mathrm{b}}z_{\mathrm{b}}$. The $o_{\mathrm{b}}x_{\mathrm{b}}$ axis points to the nose direction in the symmetric plane of the multicopter (nose direction is related to the plus-configuration multicopter or the X-configuration multicopter). The $o_{\mathrm{b}}z_{\mathrm{b}}$ axis is in the symmetric plane of the multicopter, pointing downwards, perpendicular to the $o_{\mathrm{b}}x_{\mathrm{b}}$ axis. The $o_{\mathrm{b}}y_{\mathrm{b}}$ axis is determined according to the right-hand rule.

Define the following unit vectors

$$\mathbf{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\mathbf{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\mathbf{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}. \tag{1.1}$$

In the IRF, the unit vectors along the $o_{\mathrm{e}}x_{\mathrm{e}}$ axis, $o_{\mathrm{e}}y_{\mathrm{e}}$ axis and $o_{\mathrm{e}}z_{\mathrm{e}}$ axis are expressed as $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$, respectively. In the ABCF, the unit vectors along the $o_{\mathrm{b}}x_{\mathrm{b}}$ axis, $o_{\mathrm{b}}y_{\mathrm{b}}$ axis and $o_{\mathrm{b}}z_{\mathrm{b}}$ axis satisfy the following relationship

$$^{\mathrm{b}}\mathbf{b}_1=\mathbf{e}_1,\ ^{\mathrm{b}}\mathbf{b}_2=\mathbf{e}_2,\ ^{\mathrm{b}}\mathbf{b}_3=\mathbf{e}_3.$$

In the IRF, the unit vectors along the $o_b x_b$ axis, $o_b y_b$ axis and $o_b z_b$ axis are expressed as ${}^{e}\mathbf{b}_1$, ${}^{e}\mathbf{b}_2$, ${}^{e}\mathbf{b}_3$, respectively.

III. Notation

Some discussion of notation is needed. Due to the complexity of a system with 6 degrees of freedom, various methods of notation have been developed and are required in order to sufficiently describe the critical variables. Shown below is an example of the notation we have chosen:

$$ {}^{b}\boldsymbol{\omega} $$

Here, the base variable is angular velocity, or $\boldsymbol{\omega}$. The top left superscript, b, tells us that the velocity is given in terms of ABCF vector components. These components are denoted as $\omega_{x_b}, \omega_{y_b}, \omega_{z_b}$. The subscripts x_b, y_b, z_b represent the ABCF axes. Here is another example:

$$ {}^{e}\mathbf{v} $$

Here, the base variable is linear velocity, or $\mathbf{v}$. The top left superscript, e, tells us that the velocity is given in terms of IRF vector components. These components are denoted as $v_{x_e}, v_{y_e}, v_{z_e}$. The subscripts x_e, y_e, z_e represent the IRF axes.

IV. Basic Concepts

A. Euler angles

The Euler angles are an intuitive way to represent the attitude. Based on Euler's theorem, the rotation of a rigid body around one fixed point can be regarded as the composition of several finite rotations around the fixed point. The ABCF can be achieved by three elemental rotations of the IRF around one fixed point. During these elemental rotations, each rotation axis is one of the coordinate axes of the rotating coordinate frame and each rotation angle is one of the Euler angles. Thus, the attitude matrix is closely related to the sequence of three elemental rotations and it can be represented by a product of three elemental rotation matrices. Intuitively, let the IRF align with the ABCF. Then the yaw angle ψ, the pitch angle θ and the roll angle ϕ are shown in Figure 1 with their directions determined by the right-hand rule.

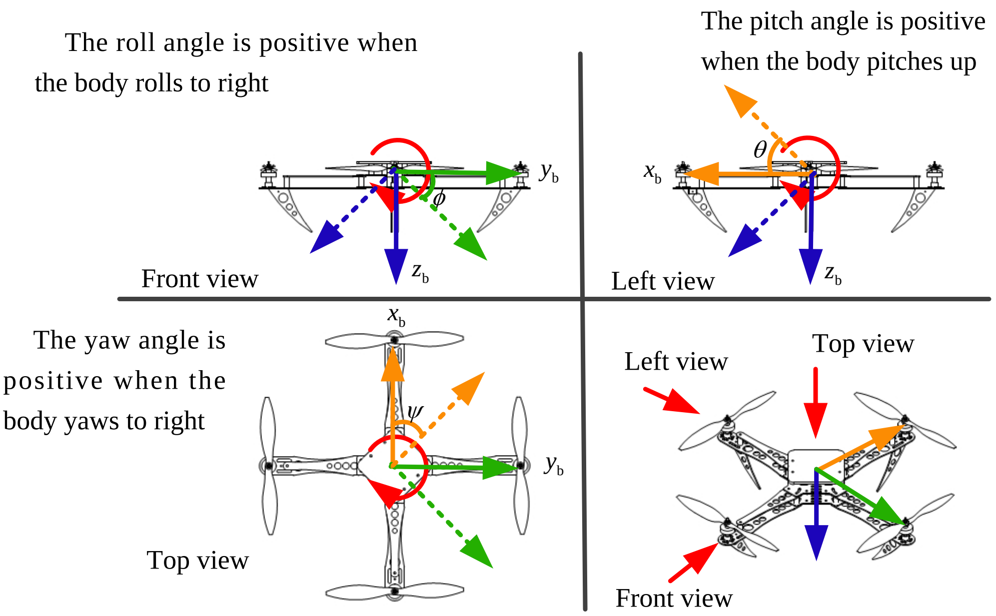


Figure 1. Intuitive representation of the Euler angles

B. Rotation matrix

Rotation matrix is another attitude representation method which can be used to represent the rotation between the ABCF and the IRF. The rotation matrix $\mathbf{R}_b^e$, which represents the rotation from the ABCF to the IRF, is expressed as

$$\mathbf{R}_b^e = \begin{bmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}. \quad (1.2)$$

C. Moments of Inertia

For objects with geometric symmetry like multicopters, moments of inertia are expressed as

$$\mathbf{J} = \begin{bmatrix} J_{xx} & 0 & 0 \\ 0 & J_{yy} & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \tag{1.3}$$

where J_{xx}, J_{yy}, J_{zz} are central principal moments of inertia.

D. Thrust Coefficient

The motors' thrust is the driving force behind all multicopter maneuvers and thus is integral to control design and simulation. The thrust, T, provided by a single propeller can be calculated as follows

$$T = C_{\mathrm{T}} \varpi^2 \tag{1.4}$$

where $C_{\mathrm{T}} \in \mathbb{R}_+$ is the lumped parameter thrust coefficient. It should be noted that the thrust provided by propellers points along the negative direction of the $o_{\mathrm{b}} z_{\mathrm{b}}$ axis.

E. Torque Coefficient

In order to understand motor effect on yaw, the reaction torque created by propellers must also be determined, and can be done in a similar fashion to that of the thrust tests. The related lumped parameter equation is shown below

$$M = C_{\mathrm{M}} \varpi^2. \tag{1.5}$$

In this case, M is the torque created by the motor and $C_{\mathrm{M}} \in \mathbb{R}_+$ is the lumped parameter torque coefficient. This torque provides a force that acts to yaw the system about the $o_{\mathrm{b}} z_{\mathrm{b}}$ axis.

F. Gyroscopic Torques

Gyroscopic precession is a phenomenon that occurs when the axis of rotation of a rotating body is changed, and the results are typically non-intuitive to those unfamiliar with its effects. The gyroscopic forces resulting on the body are governed by the total moments of inertia of the entire rotor and the propeller about the axis of rotation ($J_m \in \mathbb{R}_+$), the rolling and pitching rates (ω_{x_b} and ω_{y_b}), as well as the speed of each motor ($\varpi_i \in \mathbb{R}_+, i=1,\cdots,n_\mathrm{r}$, n_r is the number of propellers). The gyroscopic torques created by the motors for pitch and roll actions are shown below

$$\begin{aligned}
G_{\mathrm{a},\phi} &= \sum_{i=1}^{n_\mathrm{r}} J_m \omega_{y_\mathrm{b}} (-1)^{i+1} \varpi_i \\
G_{\mathrm{a},\theta} &= \sum_{i=1}^{n_\mathrm{r}} J_m \omega_{x_\mathrm{b}} (-1)^{i} \varpi_i \\
G_{\mathrm{a},\psi} &= 0.
\end{aligned} \tag{1.6}$$

As shown above, the gyroscopic torque does not exist in the yaw channel.

V. Multicopter Control Model

The multicopter modeling mainly includes four parts.

A. Rigid Body Kinematic Model

Let the vector of the multicopter barycenter be ${}^\mathrm{e}\mathbf{p} = [p_{x_\mathrm{e}}, p_{y_\mathrm{e}}, p_{z_\mathrm{e}}] \in \mathbb{R}^3$, then

$${}^\mathrm{e}\dot{\mathbf{p}} = {}^\mathrm{e}\mathbf{v} \tag{1.7}$$

where ${}^\mathrm{e}\mathbf{v} = [v_{x_\mathrm{e}}, v_{y_\mathrm{e}}, v_{z_\mathrm{e}}] \in \mathbb{R}^3$ represents the velocity of the multicopter.

Let the angular velocity of the aircraft body's rotation be ${}^\mathrm{b}\boldsymbol{\omega} = \begin{bmatrix} \omega_{x_\mathrm{b}} & \omega_{y_\mathrm{b}} & \omega_{z_\mathrm{b}} \end{bmatrix}^\mathrm{T} \in \mathbb{R}^3$, then the relationship between the attitude rates and the angular velocity of the aircraft body's rotation is expressed as

$$\dot{\boldsymbol{\Theta}} = \mathbf{W} \cdot {}^\mathrm{b}\boldsymbol{\omega} \tag{1.8}$$

where $\boldsymbol{\Theta} = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^\mathrm{T}$ and $\mathbf{W} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$.

B. Position Dynamic Model

The multicopter is only under gravity and propeller thrust. Therefore, one has

$$^{\mathrm{e}}\dot{\mathbf{v}} = g\mathbf{e}_3 - \frac{f}{m}\mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot \mathbf{e}_3 \tag{1.9}$$

where $f \in \mathbb{R}_+$ represents the total propeller thrust, $m \in \mathbb{R}_+$ represents the multicopter mass, $g \in \mathbb{R}_+$ represents the acceleration of gravity and $\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}$ represents the rotation matrix. After some deductions, equation (1.9) can be further expressed as

$$^{\mathrm{b}}\dot{\mathbf{v}} = -\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times} {}^{\mathrm{b}}\mathbf{v} + g\mathbf{R}^{\mathrm{T}}\mathbf{e}_3 - \frac{f}{m}\mathbf{e}_3 \tag{1.10}$$

where

$$\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times} = \begin{bmatrix} 0 & -\omega_{z_{\mathrm{b}}} & \omega_{y_{\mathrm{b}}} \\ \omega_{z_{\mathrm{b}}} & 0 & -\omega_{x_{\mathrm{b}}} \\ -\omega_{y_{\mathrm{b}}} & \omega_{x_{\mathrm{b}}} & 0 \end{bmatrix}.$$

C. Attitude Dynamic Model

The attitude dynamic equation in the ABCF is established as

$$\mathbf{J}\,{}^{\mathrm{b}}\dot{\boldsymbol{\omega}} = -{}^{\mathrm{b}}\boldsymbol{\omega} \times \left(\mathbf{J}\,{}^{\mathrm{b}}\boldsymbol{\omega}\right) + \mathbf{G}_{\mathrm{a}} + \boldsymbol{\tau} \tag{1.11}$$

where $\boldsymbol{\tau} = \begin{bmatrix}\tau_x & \tau_y & \tau_z\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^3$ represents the moments generated by the propellers in the body axes, $\mathbf{J} \in \mathbb{R}^{3\times 3}$ represents the multicopter moments of inertia and $\mathbf{G}_{\mathrm{a}} = \begin{bmatrix}G_{\mathrm{a},\phi} & G_{\mathrm{a},\theta} & G_{\mathrm{a},\psi}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^3$ represents the gyroscopic torques.

D. Multicopter Flight Control Rigid Model

By combining the rigid body kinematic model, the position dynamic model and the attitude dynamic model, the multicopter flight control rigid model is expressed as

$$\begin{cases} {}^{\mathrm{e}}\dot{\mathbf{p}} = {}^{\mathrm{e}}\mathbf{v} \\ {}^{\mathrm{e}}\dot{\mathbf{v}} = g\mathbf{e}_3 - \dfrac{f}{m}\mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot \mathbf{e}_3 \\ \dot{\mathbf{\Theta}} = \mathbf{W} \cdot {}^{\mathrm{b}}\boldsymbol{\omega} \\ \mathbf{J}\,{}^{\mathrm{b}}\dot{\boldsymbol{\omega}} = -{}^{\mathrm{b}}\boldsymbol{\omega} \times \left(\mathbf{J}\,{}^{\mathrm{b}}\boldsymbol{\omega}\right) + \mathbf{G}_{\mathrm{a}} + \boldsymbol{\tau} \end{cases} \tag{1.12}$$

E. Control Allocation Model

The flight of the multicopter is driven by multi-propellers. The propellers' angular velocity $\varpi_i, i = 1, 2, \cdots n_{\mathrm{r}}$ will determine the total thrust f and moments $\boldsymbol{\tau}$. The desired thrust f_{d} and desired moments $\boldsymbol{\tau}_{\mathrm{d}}$ are firstly determined according to position and attitude controller. Then the desired speed $\varpi_{i\mathrm{d}}$ for each propeller is determined by the control allocation model.

For multicopters, in order to implement the control allocation, positions of all motors in the ABCF need to be established. For a multicopter, the propellers are marked in clockwise fashion from $i = 1$ to $i = n_{\mathrm{r}}$, as shown in Figure 2. The angle between the $o_{\mathrm{b}}x_{\mathrm{b}}$ axis and the supported arm of each rotor is denoted by $\varphi_i \in \mathbb{R}_+$. The distance between the body center and the ith rotor is denoted by $d_i \in \mathbb{R}_+, i = 1, 2, \cdots, n_r$. Then the thrust and moments created by propellers are expressed as

$$\begin{bmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \underbrace{\begin{bmatrix} C_{\mathrm{T}} & C_{\mathrm{T}} & \cdots & C_{\mathrm{T}} \\ -d_1 C_{\mathrm{T}} \sin\varphi_1 & -d_2 C_{\mathrm{T}} \sin\varphi_2 & \cdots & -d_{n_r} C_{\mathrm{T}} \sin\varphi_{n_{\mathrm{r}}} \\ d_1 C_{\mathrm{T}} \cos\varphi_1 & d_2 C_{\mathrm{T}} \cos\varphi_2 & \cdots & d_{n_r} C_{\mathrm{T}} \cos\varphi_{n_{\mathrm{r}}} \\ C_{\mathrm{M}}\delta_1 & C_{\mathrm{M}}\delta_2 & \cdots & C_{\mathrm{M}}\delta_{n_{\mathrm{r}}} \end{bmatrix}}_{\mathbf{M}_{n_{\mathrm{r}}}} \begin{bmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \vdots \\ \varpi_{n_{\mathrm{r}}}^2 \end{bmatrix} \tag{1.13}$$

where $\mathbf{M}_{n_{\mathrm{r}}} \in \mathbb{R}^{4 \times n_r}, \delta_i = (-1)^{i+1}, i = 1, \cdots, n_{\mathrm{r}}$.

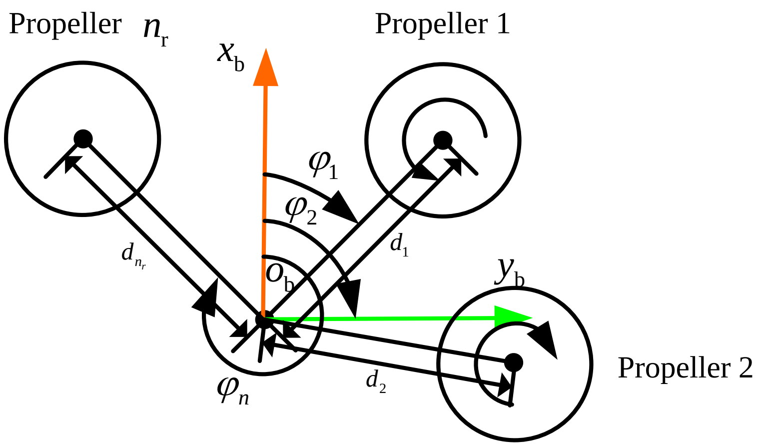


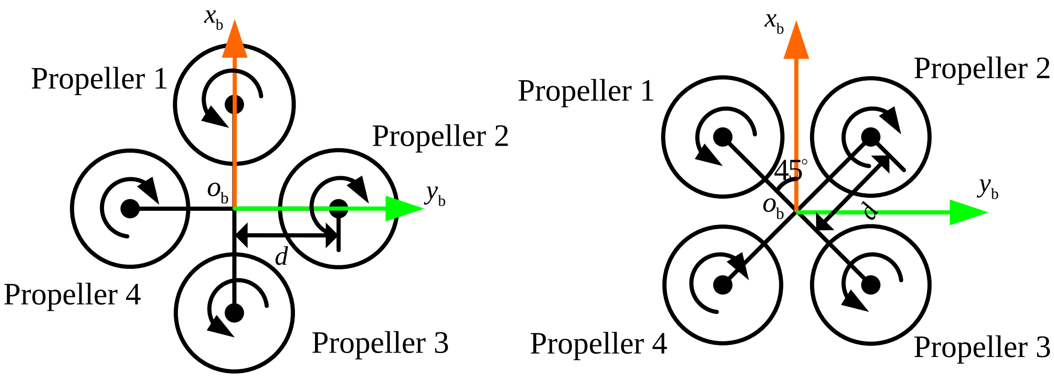
Figure 2. Airframe configuration parameters of a multicopter

As shown in Figure 3 and Figure 4, the common quadcopters (plus-configuration and X-configuration) and hexacopters are presented. For a plus-configuration quadcopter, the thrust and moments are expressed as

$$\begin{bmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} C_\mathrm{T} & C_\mathrm{T} & C_\mathrm{T} & C_\mathrm{T} \\ 0 & -dC_\mathrm{T} & 0 & dC_\mathrm{T} \\ dC_\mathrm{T} & 0 & -dC_\mathrm{T} & 0 \\ C_\mathrm{M} & -cC_\mathrm{M} & C_\mathrm{M} & -C_\mathrm{M} \end{bmatrix} \begin{bmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \end{bmatrix} \tag{1.14}$$

where $d \in \mathbb{R}_+$ represents the distance between the body center and the rotor. For an X-configuration quadcopter, the thrust and moments are expressed as

$$\begin{bmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} C_\mathrm{T} & C_\mathrm{T} & C_\mathrm{T} & C_\mathrm{T} \\ \frac{\sqrt{2}}{2}dC_\mathrm{T} & -\frac{\sqrt{2}}{2}dC_\mathrm{T} & -\frac{\sqrt{2}}{2}dC_\mathrm{T} & \frac{\sqrt{2}}{2}dC_\mathrm{T} \\ \frac{\sqrt{2}}{2}dC_\mathrm{T} & \frac{\sqrt{2}}{2}dC_\mathrm{T} & -\frac{\sqrt{2}}{2}dC_\mathrm{T} & -\frac{\sqrt{2}}{2}dC_\mathrm{T} \\ C_\mathrm{M} & -C_\mathrm{M} & C_\mathrm{M} & -C_\mathrm{M} \end{bmatrix} \begin{bmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \end{bmatrix} \tag{1.15}$$



(a) Plus-configuration (b) X-configuration

Figure 3. Quadcopter propellers' rotation direction

For a hexacopter, the thrust and moments are expressed as

$$\begin{bmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} = \begin{bmatrix} C_T & C_T & C_T & C_T & C_T & C_T \\ 0 & -\frac{\sqrt{3}}{2}dC_T & -\frac{\sqrt{3}}{2}dC_T & 0 & \frac{\sqrt{3}}{2}dC_T & \frac{\sqrt{3}}{2}dC_T \\ dC_T & \frac{1}{2}dC_T & -\frac{1}{2}dC_T & -dC_T & -\frac{1}{2}dC_T & \frac{1}{2}dC_T \\ -C_M & C_M & -C_M & C_M & -C_M & C_M \end{bmatrix} \begin{bmatrix} \varpi_1^2 \\ \varpi_2^2 \\ \varpi_3^2 \\ \varpi_4^2 \\ \varpi_5^2 \\ \varpi_6^2 \end{bmatrix} \quad (1.16)$$

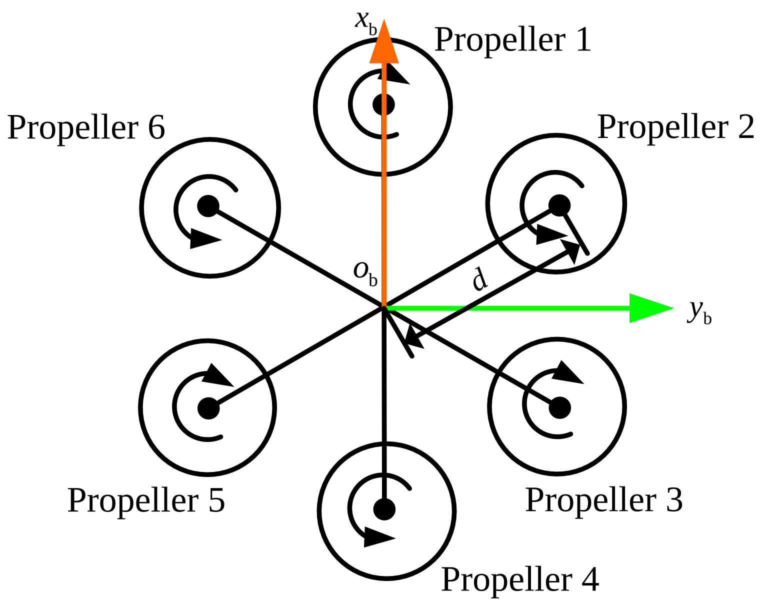


Figure 4. Hexacopter propellers' rotation direction

F. Propulsor Model

As shown in Figure 5, the propulsor model is a whole power mechanism that includes not only the brushless Direct Current (DC) motor but also the Electronic Speed Controller (ESC) and propellers. Throttle command σ is an input signal between 0 and 1, while the battery output voltage U_{b} cannot be controlled.

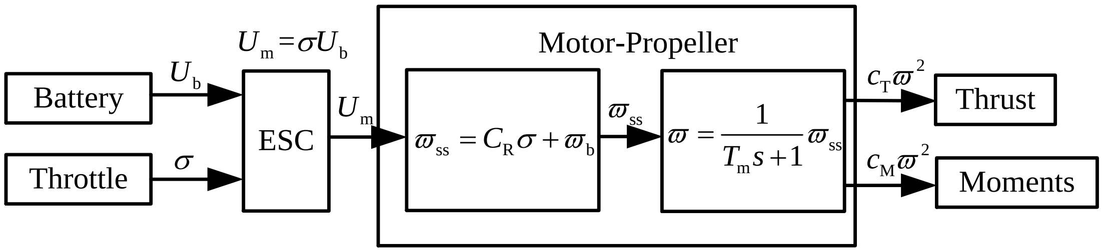


Figure 5. Signal transmission of the motor-propeller model

The complete propulsor model is expressed as

$$\varpi=\frac{1}{T_{\mathrm{m}}s+1}(C_{\mathrm{R}}\sigma+\varpi_{\mathrm{b}}) \tag{1.17}$$

where T_{m} represents the time constant, s is the Laplace operator and $C_{\mathrm{R}},\varpi_{\mathrm{b}}$ are constant parameters. The input of this model is the throttle σ and the output is motor speed ϖ .

VI. Multicopter Aerodynamic Drag Model

According to equation (1.10), the position dynamic model is expressed as

$$\begin{aligned}\dot{v}_{x_{\mathrm{b}}}&=v_{y_{\mathrm{b}}}\omega_{z_{\mathrm{b}}}-v_{z_{\mathrm{b}}}\omega_{y_{\mathrm{b}}}-g\sin\theta\\ \dot{v}_{y_{\mathrm{b}}}&=v_{z_{\mathrm{b}}}\omega_{x_{\mathrm{b}}}-v_{x_{\mathrm{b}}}\omega_{z_{\mathrm{b}}}+g\cos\theta\sin\phi\end{aligned} \tag{1.18}$$

where $v_{x_{\mathrm{b}}},v_{y_{\mathrm{b}}},v_{z_{\mathrm{b}}}\in\mathbb{R}$ are the multicopter velocities along the body axes $o_{\mathrm{b}}x_{\mathrm{b}},o_{\mathrm{b}}y_{\mathrm{b}},o_{\mathrm{b}}z_{\mathrm{b}}$, respectively. For the multicopter, drags applied to rotor blades are in the direction of the body axes. Due to the symmetry of the multicopter, the drags are simply expressed as

$$\begin{aligned}f_x&=-C_{\mathrm{d}}v_{x_{\mathrm{b}}}^2\\ f_y&=-C_{\mathrm{d}}v_{y_{\mathrm{b}}}^2\end{aligned} \tag{1.19}$$

where $f_x,f_y\in\mathbb{R}$ are respectively drags along the body axes $o_{\mathrm{b}}x_{\mathrm{b}},o_{\mathrm{b}}y_{\mathrm{b}}$ and $C_{\mathrm{d}}\in\mathbb{R}_+$ is the drag coefficient. Consequently, the multicopter aerodynamic drag model (1.18) becomes

$$\begin{aligned}\dot{v}_{x_{\mathrm{b}}} &= v_{y_{\mathrm{b}}}\omega_{z_{\mathrm{b}}} - v_{z_{\mathrm{b}}}\omega_{y_{\mathrm{b}}} - g\sin\theta - \frac{C_{\mathrm{d}}}{m}v_{x_{\mathrm{b}}}^2 \\ \dot{v}_{y_{\mathrm{b}}} &= v_{z_{\mathrm{b}}}\omega_{x_{\mathrm{b}}} - v_{x_{\mathrm{b}}}\omega_{z_{\mathrm{b}}} + g\cos\theta\sin\phi - \frac{C_{\mathrm{d}}}{m}v_{y_{\mathrm{b}}}^2\end{aligned} \tag{1.20}$$