Proving set properties & Boolean algebra

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3.3 and 3.4

These Chapters:

1) Element-wise Proofs to show that one set is a subset of another

2) Converting Set and Logic notation into Boolean algebra

Chapter 3.3
Proving Set Properties

Element-wise Proofs

When we're interested in proving that one set is a subset of another set, we use an element-wise proof. For discrete sets, this is easy:

For sets B = { 2, 4, 6, 8, 10 } and C = { 2, 4 }, we can see that C is a subset of B by checking every element of C: (1) $2 \in B$**, and (2)** $4 \in B$ **.**

Element-wise Proofs

But for some sets, we cannot possibly check every element because it has infinite elements, such as…

$$B = \{k \in \mathbb{Z} : k \, is \, even\}$$

We would have to take a slightly different (but still familiar) approach to an element-wise proof.

Element-wise Proofs

Example: **Let A be the set** $A = \{10k : k \in \mathbb{N}\}$

And B be the set $B = \{k \in \mathbb{Z} : k \text{ is even}\}$

Prove that $A \subseteq B$

Element-wise Proofs

<u>Example:</u> Let A be the set $A=\{10k : k\in\mathbb{N}\}$

And B be the set $B=\{k\in\mathbb{Z} : k\, is\, even\}$

Prove that $A\subseteq B$

1. First, we must build an implication (if-then). What does it mean for A to be a subset of B?

It means all of A’s elements are also in B…

Element-wise Proofs

Example: Let A be the set $A = \{10k : k \in \mathbb{N}\}$
And B be the set $B = \{k \in \mathbb{Z} : k \text{ is even}\}$
Prove that $A \subseteq B$

1. If *x* is an element of A, $(x \in A)$
   Then *x* must also be an element of B. $(x \in B)$

Hypothesis: $(x \in A)$

Conclusion: $(x \in B)$

Element-wise Proofs

<u>Example:</u> Let A be the set $A = \{10k : k \in \mathbb{N}\}$

And B be the set $B = \{k \in \mathbb{Z} : k\, is\, even\}$

Prove that $A \subseteq B$

1. $x \in A \rightarrow x \in B$

2. We know that our <u>hypothesis</u> must be true for our proof, so we can say that $x = 10k$

Element-wise Proofs

Example: **Let A be the set** $A = \{10k : k \in \mathbb{N}\}$

And B be the set $B = \{k \in \mathbb{Z} : k \, is \, even\}$

Prove that $A \subseteq B$

1. $x \in A \rightarrow x \in B$

2. $x = 10k$

3. We want to show that the conclusion is also true, so we can rewrite our alias for x:

$$x = 10k = 2(5k)$$

By the definition of an even integer

Element-wise Proofs

Example: **Let A be the set** $A=\{10k : k\in\mathbb{N}\}$

And B be the set $B=\{k\in\mathbb{Z} : k \text{ is even}\}$

Prove that $A\subseteq B$

1. $x\in A \rightarrow x\in B$
2. $x=10k$
3. $x=10k=2(5k)$

Element-wise Proof

<u>Practice:</u> For the set A = { a, s, d, f }
And the set of the English alphabet
Z = { a, b, c, …, x, y, z }
Prove that A is a subset of Z, by showing that each element of A is also in Z.

Element-wise Proof

<u>Practice:</u> For the set A = { a, s, d, f }
And the set of the English alphabet
Z = { a, b, c, …, x, y, z }
Prove that A is a subset of Z, by showing that each element of A is also in Z.

$a \in Z \qquad s \in Z \qquad d \in Z \qquad f \in Z$

Element-wise Proof

<u>Practice:</u> Let there be the sets

$$A = \{4n + 1 : n \in \mathbb{Z}\} \qquad B = \{2m + 1 : m \in \mathbb{Z}\}$$

Prove that $A \subseteq B$

1. Come up with an implication regarding element *x*

2. The hypothesis is true, so rewrite so that the conclusion is proven.

Element-wise Proof

Practice: **Let there be the sets**

$$A = \{4n+1 : n \in \mathbb{Z}\} \qquad B = \{2m+1 : m \in \mathbb{Z}\}$$

Prove that $A \subseteq B$

$$x \in A \rightarrow x \in B$$

$$x = 4n+1$$

$$\boxed{x = 4n+1 = 2(2n)+1}$$

Chapter 3.4
Boolean Algebra

Switching between Logic and Sets

The operations that we do with Logic and with Sets are actually very similar...

a	Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
b	Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
c	Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
d	Identity	$p \wedge t \equiv p$	$p \vee c \equiv p$
e	Negation	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
f	Double negative	$\neg(\neg p) \equiv p$	
g	Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
h	DeMorgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
i	Universal bound	$p \vee t \equiv t$	$p \wedge c \equiv c$
j	Absorption	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
k	Negations of t and c	$\neg t \equiv c$	$\neg c \equiv t$

Switching between Logic and Sets

The operations that we do with Logic and with Sets are actually very similar...

a	Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
b	Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
c	Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
d	Identity	$p \wedge t \equiv p$	
e	Negation	$p \vee \neg p \equiv t$	
f	Double negative	$\neg(\neg p) \equiv p$	
g	Idempotent	$p \wedge p \equiv p$	
h	DeMorgan's laws	$\neg(p \wedge q) \equiv$	
i	Universal bound	$p \vee t \equiv t$	
j	Absorption	$p \wedge (p \vee q)$	
k	Negations of t and c	$\neg t \equiv c$	

a	Commutative	$A \cap B = B \cap A$	$A \cup B = B \cup A$
b	Associative	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
c	Distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
d	Identity	$A \cap U = A$	$A \cup \emptyset = A$
e	Negation	$A \cup A' = U$	$A \cap A' = \emptyset$
f	Double negative	$(A')' = A$	
g	Idempotent	$A \cap A = A$	$A \cup A = A$
h	DeMorgan's laws	$(A \cap B)' = A' \cup B'$	$(A \cup B)' = A' \cap B'$
i	Universal bound	$A \cup U = U$	$A \cap \emptyset = \emptyset$
j	Absorption	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
k	Complements of U and $\emptyset$	$U' = \emptyset$	$\emptyset' = U$

Switching between Logic and Sets

We can actually convert a Logic expression to a Set expression (and vice versa) by swapping specific parts:

A, B, C, $\leftrightarrow$ p, q, r	$=$ $\leftrightarrow$ $\equiv$	
$\cap$ $\leftrightarrow$ $\wedge$	$\cup$ $\leftrightarrow$ $\vee$	$'$ $\leftrightarrow$ $\neg$
U $\leftrightarrow$ t	$\emptyset$ $\leftrightarrow$ c	

$$A \cap B \rightarrow p \wedge q$$

$$p \vee q \rightarrow A \cup B$$

Switching between Logic and Sets

The same can be done to convert Logic and Sets to Boolean algebra:

	Logical	Sets	Boolean Algebra
Variables	p, q, r	A, B, C	a, b, c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c, t	$\emptyset, U$	$0, 1$

$$(p \wedge q) \rightarrow a \cdot b$$

$$(A \cup B) \rightarrow a + b$$

Switching between Logic and Sets

We use Boolean algebra to abstract problems, which makes it easier to discover and understand properties of more concrete systems.

(At least, that's according to the book)

Switching between Logic and Sets

For this chapter, we're mostly just concerned with converting between Logic or Set expressions to Boolean algebra

	Logical	Sets	Boolean Algebra
Variables	p, q, r	A, B, C	a, b, c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c, t	$\emptyset, U$	$0, 1$

Switching between Logic and Sets

<u>Practice:</u> Using the table,

	Logical	Sets	Boolean Algebra
Variables	p, q, r	A, B, C	a, b, c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c, t	$\varnothing, U$	$0, 1$

Convert $(p \wedge \neg q) \vee p \equiv p$

To Boolean Algebra

Switching between Logic and Sets

<u>Practice:</u> Using the table,

	Logical	Sets	Boolean Algebra
Variables	p, q, r	A, B, C	a, b, c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c, t	$\emptyset, U$	0, 1

Convert $(p \wedge \neg q) \vee p \equiv p$

To Boolean Algebra

$$(a \cdot b') + a = a$$

Switching between Logic and Sets

<u>Practice:</u> Using the table,

	Logical	Sets	Boolean Algebra
Variables	p,q,r	A,B,C	a,b,c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c,t	$\emptyset,U$	0, 1

Convert $(A-B)'=A'\cup(A\cap B)$

To Boolean Algebra

Switching between Logic and Sets

Practice: Using the table,

	Logical	Sets	Boolean Algebra
Variables	p, q, r	A, B, C	a, b, c
Operations	$\wedge$, $\vee$, $\neg$	$\cap$, $\cup$, $'$	$\cdot$, $+$, $'$
Special elements	c, t	$\emptyset, U$	$0, 1$

Convert $(A-B)'=A'\cup(A\cap B)$
To Boolean Algebra

$$(a\cdot b')'=a'+(a\cdot b)$$

That’s basically it…