More About Induction

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This Chapter:

1) Sums as recursive sequences

2) Other uses of induction

3) The Fundamental Theorem of Arithmetic

Mathematical Induction

Theorem 1:
<u>Fundamental Theorem of Arithmetic</u>

Every integer greater than 1 can be expressed as the product of a list of prime numbers.

Mathematical Induction

Theorem 1:

Fundamental Theorem of Arithmetic

Every integer greater than 1 can be expressed as the product of a list of prime numbers.

Example:

$P(4) \rightarrow 2 \times 2$

$P(5) \rightarrow 5$

$P(6) \rightarrow 2 \times 3$

Sums as recursive sequences

In the last chapter, we were proving that the result of a summation at each step was equivalent to some closed formula (e.g., $\sum_{i=1}^{n}(2i-1)=n^2$)

This time, we will be finding a recursive formula (aka “recurrence relation”) for summations.

Sums as recursive sequences

Example 1 from the book:

Consider the sum

$$\sum_{i=1}^{n} (2i-1)$$

Which is the same as
1 + 3 + 5 + … + (2n-1).

Use the notation S_n to denote this sum.
For example, S_5 means 1 + 3 + 5 + 7 + 9.

Find a recursive description of S_n.

Sums as recursive sequences

Example 1 from the book:
Find a recurrence relation for $\sum_{i=1}^{n}(2i-1)$

So remember that we need to get all the values from 1 to n-*1* for the summation. In this case, that would be $s_{(n-1)}$

So we express the summation as the sum of the first *n-1* terms, plus the final term:

$$s_n = \sum_{i=1}^{n}(2i-1)$$

$$= \underbrace{[1+3+5+\ldots+(2n-3)]}_{\text{Sum from 1 to n-1}} + \underbrace{(2n-1)}_{\text{Final term}}$$

$$= \underbrace{s_{(n-1)} + (2n-1)}_{\text{Written in terms of s}}$$

Sums as recursive sequences

Example 1 from the book:
Find a recurrence relation for $\sum_{i=1}^{n}(2i-1)$

From this work:

$$s_n=\sum_{i=1}^{n}(2i-1)$$
$$=[1+3+5+\ldots+(2n-3)]+(2n-1)$$
$$=s_{(n-1)}+(2n-1)$$

The result is that the recurrence relation is

$$s_n=s_{(n-1)}+(2n-1)$$

But remember – we also need a starting value.

Sums as recursive sequences

Example 1 from the book:
Find a recurrence relation for $\sum_{i=1}^{n}(2i-1)$

Recurrence Relation (Recursive Formula):

$$s_n = s_{(n-1)} + (2n-1)$$

If we look at s_1 for the summation,

$$\sum_{i=1}^{1}(2i-1) = 2\cdot1-1 = 1$$

$$= 2\cdot1-1$$

$$= 1$$

So $s_1 = 1$

Sums as recursive sequences

Example 1 from the book:

RESULT!

The Recursive Formula for $\sum_{i=1}^{n}(2i-1)$ is:

$s_n = s_{(n-1)} + (2n-1) \quad s_1 = 1$

Other uses of induction

Example 6 from the book:

Show that n^3+2n is divisible by 3 for all positive integers *n*.

First, let's look at D(1) to see if it is true:

$$D(n)=n^3+2n$$

$$D(1)=1^3+2\cdot 1$$

$$D(1)=1+2=3$$

True!

Other uses of induction

Example 6 from the book:

Show that n^3+2n is divisible by 3 for all positive integers *n*.

Next, let's say that some positive integer *m* is given, such that we've been able to check D(1) through D(m-1) are true.

So for D(m-1):

$$D(n)=n^3+2n$$

$$D(m-1)=(m-1)^3+2(m-1)$$

$$D(m-1)=m^3-3m^2+3m-1+2m-2$$

$$D(m-1)=m^3-3m^2+3m+2m-3$$ Only combining the constants

$$D(m-1)=(-3m^2+3m-3)+m^3+2m$$ Rearranging terms

Other uses of induction

Example 6 from the book:

At this point:

$$D(m-1)=(-3m^2+3m-3)+m^3+2m$$

we've rearranged the terms so that we separate out m^3+2m .

Remember that: $D(n)=n^3+2n$

So now we can say that…

$$D(m-1)=(-3m^2+3m-3)+D(m)$$

Or if we rewrite to isolate D(m)...

$$D(m)=D(m-1)-(-3m^2+3m-3)$$

$$D(m)=D(m-1)+3m^2-3m+3$$

Other uses of induction

Example 6 from the book:

Now we have:

$$D(m)=D(m-1)+3m^2-3m+3$$

We know previously that D(1) through D(m-1) are true for the statement "divisible by 3 for all positive integers", so we will rewrite D(m-1) as 3K and sub it out.

$$D(m)=3K+3m^2-3m+3$$

$$D(m)=3(K+m^2-m+1)$$ Factor out 3

And the result is our proof, that D(m) is divisible by 3.

Other uses of induction

Let’s work some examples from
the textbook problems.

Mathematical Induction

Let’s work some problems