Number Puzzles and Sequences

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Topics:

1) Figuring out formulas from sequences of numbers

2) Recursive Formulas

3) Closed Formulas

4) Summations

Sequences of Numbers

Sometimes when we look at a sequence of numbers, we can easily tell what comes next:

1, 3, 5, 7, __

10, 20, 30, 40, __

5, 10, 15, 20, __

Sequences of Numbers

Sometimes the pattern may not be so obvious, and we need to analyze the sequence further in order to figure it out.

1, 2, 6, 24, 120, 720, ___

1, 9, 17, 25, 33, 41, ___

1, 4, 9, 16, 25, 36, ___

Sequences of Numbers

There are two ways we can write this pattern down:

What the value of a number is based on the previous value…

Or the value of a number based on its position in the sequence…

Sequences of Numbers

Definition: Recursive Formula

A recursive formula for a sequence is a formula where each term is described in relation to a previous term (or terms) of the sequence.

Definition: Closed Formula

A closed formula for a sequence is a formula where each term is described only in relation to its position in the list.

Sequences of Numbers

Definition: Recursive Formula

Example: $a_1 = 2$ $\qquad$ $a_n = a_{n-1} + 2$

Result: 2, 4, 6, 8, 10, …

Definition: Closed Formula

Example: $a_n = 2n$

Result: 2, 4, 6, 8, 10, …

Sequences of Numbers

Definition: Recursive Formula

Example: $a_1 = 2$ $\qquad$ $a_n = a_{n-1} + 2$

Result: 2, 4, 6, 8, 10, …

With a recursive sequence, because each term is written based on the *last* term, we must specify the first element of the sequence (The item at position 1.)

Definition: Closed Formu

Example: $a_n = 2n$

Result: 2, 4, 6, 8, 10, …

Sequences of Numbers

Definition: Recursive Formula

Example: $a_1 = 2$ $\qquad$ $a_n = a_{n-1} + 2$

Result: 2, 4, 6, 8, 10, …

Definition: Closed Formula

Example: $a_n = 2n$

With a closed formula, the value is based on whatever the element's *position* is in the list, such as item #1, item #2, item #3, and so on...

Result: 2, 4, 6, 8, 10, …

Sequences of Numbers

Definition: Recursive Formula

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Stepping through…

a_1 Is already given as part of the formula.

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Sequences of Numbers

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$a_3 = a_2 + 2 = 4 + 2 = 6$

Sequences of Numbers

Definition: Recursive Formula

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Stepping through…

a_1 Is already given as part of the formula.

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$a_3 = a_2 + 2 = 4 + 2 = 6$

And so on…

Sequences of Numbers

Definition: Closed Formula

Example: $a_n = 2n$

Stepping through…

$a_1 = 2 \times 1 = 2$

Sequences of Numbers

Definition: Closed Formula

Example: $a_n = 2n$

Stepping through…

$a_1 = 2 \times 1 = 2$

$a_2 = 2 \times 2 = 4$

Sequences of Numbers

Definition: Closed Formula

Example: $a_n = 2n$

Stepping through…

$a_1 = 2 \times 1 = 2$

$a_2 = 2 \times 2 = 4$

$a_3 = 2 \times 3 = 6$

Sequences of Numbers

Definition: Closed Formula

Example: $a_n = 2n$

Stepping through…

a_1 = 2 x 1 = 2

a_2 = 2 x 2 = 4

a_3 = 2 x 3 = 6

And so on…

Figuring out the Formula

It can be a little tricky to figure out the formula based on a sequence of numbers. The main things to keep in mind are…

- **Is addition involved?**
- **Is subtraction involved?**
- **Is multiplication involved?**
- **Are exponents involved?**

The more practice you get with this, the easier it will be.

Figuring out the Formula

Example 1

Find the closed formula and recursive formula for the sequence

1, 3, 5, 7, 9, …

Figuring out the Formula

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Find the closed formula and recursive formula for the sequence

1, 3, 5, 7, 9, …

For the closed formula, it might help to associate each number with its index to try to uncover a pattern that way…

$a_1=1$ $a_2=3$ $a_3=5$ $a_4=7$ $a_5=9$

Figuring out the Formula

Example 1

Find the closed formula and recursive formula for the sequence

1, 3, 5, 7, 9, …

It might also be useful to look at the differences between each element…

$$a_1=1 \qquad a_2=3 \qquad a_3=5 \qquad a_4=7 \qquad a_5=9$$

+2 +2 +2 +2

Figuring out the Formula

Example 1

Find the closed formula and recursive formula for the sequence

1, 3, 5, 7, 9, …

And to think of any patterns that might look similar, such as "2, 4, 6, 8, 10"… What is the offset?

2(1) – 1	*2(2) – 1*	*2(3) – 1*	*2(4) – 1*	*2(5) – 1*
$a_1=1$	$a_2=3$	$a_3=5$	$a_4=7$	$a_5=9$
	+2	*+2*	*+2*	*+2*

Figuring out the Formula

Example 1

Find the closed formula and recursive formula for the sequence

1, 3, 5, 7, 9, …

Closed Formula: $a_n = 2n - 1$

Recursive Formula: $a_1 = 1$

$a_n = a_{n-1} + 2$

2(1) – 1	*2(2) – 1*	*2(3) – 1*	*2(4) – 1*	*2(5) – 1*
$a_1 = 1$	$a_2 = 3$	$a_3 = 5$	$a_4 = 7$	$a_5 = 9$
	+2	*+2*	*+2*	*+2*

Figuring out the Formula

Sometimes you have to try a few different approaches before you find the pattern – that's ok; you don't have to know what it is automatically. That's why we *analyze it*.

Keep in mind that there could be multiplication, addition, subtraction, and/or exponents involved!

Figuring out the Formula

Example 2 – Try to solve it before I do in the video

Sequence: 1, 9, 17, 25, 33, 41, ____

Closed formula? Recursive formula?

Figuring out the Formula

Example 3 – Try to solve it before I do in the video

Sequence: 1, 4, 9, 16, 25, 36, ____

Closed formula? Recursive formula?

Figuring out the Formula

Example 4 – Try to solve it before I do in the video

Sequence: 2, 4, 8, 16, 32, 64, ____

Closed formula? Recursive formula?

Figuring out the Formula

Example 5 – Try to solve it before I do in the video

Sequence: 1, 2, 6, 24, 120, 720, ___

Closed formula? Recursive formula?

Summations

Definition:
For a sequence of numbers a_k **with** $k \geq 1$ **,**
we use the notation

$$\sum_{k=1}^{n} a_k$$

To denote the sum of the first *n* terms of the sequence.
This is called *sigma notation* for the sum.

Summations

Example – Evaluate the following sums:

(a) $\sum_{k=1}^{6} (2k-1)$

(b) $\sum_{k=0}^{4} 3^k$

(c) $\sum_{k=3}^{3} k^2$

(d) $\sum_{k=1}^{5} \frac{1}{k(k+1)}$

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Bonus: If you have any experience with programming, you can write a program to solve summations for you.