Proof by Contradiction

Written by Rachel J. Morris, last updated 2016-10-02

Proof by Contradiction

This Chapter:

1) Proof by contradiction

2) Existence/non-existence proofs

Proof by Contradiction

From the book:

“One way to think about direct proof and proof by contrapositive is that they both demonstrate, in different ways, that **there cannot possibly be a counterexample to the theorem.** Thus, we know the theorem must be true for all elements of the domain.

Proof by Contradiction

These proof techniques are based on the two properties that a potential counterexample to a given implication must possess:

1) It must make the hypothesis of the implication true.
2) It must make the conclusion of the implication false.

Proof by Contradiction

In a **direct proof**, we choose an element satisfying property 1, and that element cannot satisfy property 2, so we cannot find a counterexample.

When we show the **contrapositive**, if we choose an element satisfying property 2, then the same element cannot satisfy property 1.

In **proof by contradiction**, we assume a counterexample is found, and we show that it cannot possibly be true because it would lead to a contradiction – a statement that we know to be false. Thus we are showing that properties 1 and 2 are logically incompatible.

Example

"Prove by contradiction: If n^2 is even, then n is even."

1) Assume we've found a case that results in the proposition being ***not true.***

We have found some n^2 that is even, for n being odd.

2) Figure out how to represent the statement mathematically:

$n^2 = 2K$ $\quad$ $n = 2L + 1$

So we represent the proposition with:

$$(2L+1)^2 = 2K$$

3) Simplify:

$$4L^2 + 4L + 1 = 2K$$

$$1 = 2K - 4L^2 - 4L$$

$$\frac{1}{2} = K - 2L^2 - 2L$$

4) Result:

The result is a fraction, not an integer, therefore no counterexample exists.

Proof by Contradiction

Let’s work out some more examples…

Existence Proofs

An **existence proof** is a proof of a theorem characterized as "There exists *x* in some domain D such that P(x)". Likewise, the proof of a theorem characterized as "There does not exist an x in some domain D such that P(x)" is called a **nonexistence proof**.

Example

"There exists a positive rational number *r* such that $6r^2+11r=35$"

With this one, we can solve by getting the roots of the polynomial.

1)Rewrite: $6r^2+11r-35=0$

2)Factor: $(3r-5)(2r+7)=0$

3)Find roots: $r=-\left(\frac{7}{2}\right)$ $r=\frac{5}{3}$

4)Result: *We have found a positive rational root, so the proposition is true.*

In other cases, this type of proof only requires that we find one value that makes the existence (there exists at least one…) true, and don't necessarily require much computation.