FINAL EXAM

CALCULUS III

Due Date: 07/30/23

MATH 209
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You may use a TI-83 or TI-84 calculator and the trigonometry formula sheet (last page).
Write in the space provided. Show all your work and explain where you used your calculator to get full credit.
The exam is printed on both sides of the paper. The text has 10 problems on 13 pages for a total of 100 points.

Problem 1 (10 points) Vectors.
Consider the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. Find the following:

i) Find the sum **u** + **v** and the difference **u** - **v**

$u+v = \langle 2+5, 3+(-4), -1+2 \rangle$

$= \boxed{7i - j + k}$

$u - v = u + (-v)$

$-v = \langle -5, 4, -2 \rangle$

$u + (-v) = \langle 2+(-5), 3+4, -1+(-2) \rangle$

$= \boxed{-3i + 7j - 3k}$

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ii) Find the dot product $\mathbf{u} \cdot \mathbf{v}$.

$u \cdot v = \langle 2\cdot 5, 3\cdot(-4), (-1)\cdot(2) \rangle$

$= 10 + (-12) + (-2)$

$= \boxed{-4}$

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iii) Find $|\mathbf{u}|$ and $|\mathbf{v}|$ and the angle between $\mathbf{u}$ and $\mathbf{v}$.

$u = \langle 2, 3, -1 \rangle$

$|u| = \sqrt{(2)^2+(3)^2+(-1)^2}$

$= \sqrt{4+9+1}$

$= \boxed{\sqrt{14}}$

$v = \langle 5, -4, 2 \rangle$

$|v| = \sqrt{(5)^2+(-4)^2+(2)^2}$

$= \sqrt{25+16+4} = \boxed{\sqrt{45}}$

iv) Find the cross product $\mathbf{u} \times \mathbf{v}$. What is the area of the parallelogram that has these two vectors as sides? Explain.

$u = 2i+3j-K \quad , \quad v = 5i-4j+2K$

$$u \times v = \begin{vmatrix} i & j & K \\ 2 & 3 & -1 \\ 5 & -4 & 2 \end{vmatrix} = (6-4)i + (4-(-5))j + (-8-15)K$$

$= \boxed{2i - 9j - 23K}$

v) (EXTRACREDIT: 5 points) Find the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$. What is your conclusion? Explain.

$u \times v = 2i - 9j - 23K$

$= \langle 2, -9, -23 \rangle$

$v = \langle 5, -4, 2 \rangle$

$(u \times v) \cdot v = \langle 2, -9, -3 \rangle \cdot \langle 5, -4, 2 \rangle$

$= \langle 5 \cdot 2, (-9) \cdot (-4), (-3) \cdot 2 \rangle$

$= 10 + 36 - 6$

$= \boxed{40}$

Problem 2. (10 points) Lines and planes in the space.

i) Find the parametric equation of the line passing through the points $P = (\overset{x_0}{1}, \overset{y_0}{2}, \overset{z_0}{-1})$ and $Q = (1, -1, -2)$.

$V = \vec{PQ} = \langle 1-1, -1-2, -2-(-1) \rangle = \langle \underset{a}{0}, \underset{b}{-3}, \underset{c}{-1} \rangle$

$$\boxed{\begin{aligned} x &= x_0 + at = 1 \\ y &= y_0 + bt = 2-3t \\ z &= z_0 + ct = -1-t \end{aligned}}$$

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ii) Find the component equation for the plane that passes through the points $P = (\overset{x_0}{1}, \overset{y_0}{2}, \overset{z_0}{-1})$, $Q = (1, -1, -2)$, and $R = (3 - 2, 5)$.

$\vec{PQ} = \langle 0, -3, -1 \rangle$ $\quad a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\vec{PR} = \langle 3-1, -2-2, 5-(-1) \rangle = \langle 2, 0, 6 \rangle$

$\vec{PQ} \times \vec{PR} = N \begin{vmatrix} i & j & k \\ 0 & -3 & -1 \\ 2 & 0 & 6 \end{vmatrix} = (-18-0)i - (0-(-2))j + (0-(-6))k$

$= -18i - 2j + 6k = \langle \underset{a}{-18}, \underset{b}{-2}, \underset{c}{6} \rangle$

$-18(x-1) - 2(y-2) + 6(z+1) = 0$

$-18x + 18 - 2y + 4 + 6z + 6 = 0$

$= \boxed{-18x - 2y + 6z = -28}$

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iii) Find the distance from the point $(1, 1, 1)$ to the plane found in ii).

$D = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$ $\qquad \underset{a}{-18}x \underset{b}{-2}y + \underset{c}{6}z = \underset{-d}{-28}$

$D = \dfrac{|-18(1) - 2(1) + 6(1) + 28|}{\sqrt{18^2 + 2^2 + 6^2}} = \dfrac{-18-2+6+28}{\sqrt{324+4+36}} = \boxed{\dfrac{14}{\sqrt{364}}}$

Problem 3. (10 points) Curve in the space.

Consider the curve $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j} + 2t\mathbf{k}$.

i) Find the velocity **v** and the acceleration **a** of the curve **r**.

$v = r'(t)$

$\boxed{v = -5\sin(t)i + 5\cos(t)j + 2k}$

$a = v' = r''(t)$

$\boxed{a = -5\cos(t)i - 5\sin(t)j}$

ii) Find the unit tangent vector to the curve **T**.

$T = \frac{r'(t)}{|r'(t)|} = \frac{v}{|v|}$ $\qquad r'(t) = v$ $\qquad |r'(t)| = \sqrt{-5\sin^2(t) + 5\cos^2(t) + 2^2} = \sqrt{29}$

$= \frac{\langle -5\sin(t), 5\cos(t), 2\rangle}{\sqrt{-5\sin^2(t) + 5\cos^2(t) + 2^2}} = \frac{\langle -5\sin(t), 5\cos(t), 2\rangle}{\sqrt{29}}$

$T = \boxed{\frac{1}{\sqrt{29}}\langle -5\sin(t), 5\cos(t), 2\rangle}$

iii) Find the curvature $\kappa(t)$ and the radius de curvature ρ. $\qquad \rho = \frac{1}{K}$

$K = \frac{1}{|v|}\cdot\frac{dT}{dt} = \frac{|T'(t)|}{|r'(t)|}$

$|T'(t)| = \frac{1}{\sqrt{29}}\sqrt{25\sin^2(t) + 25\cos^2(t) + 4^2} = \frac{5}{\sqrt{29}}$

$K = \frac{\frac{5}{\sqrt{29}}}{\sqrt{29}} = \boxed{\frac{5}{29}}$ $\qquad \rho = \frac{1}{\frac{5}{29}} = \boxed{\frac{29}{5}}$

iv) Find the unit normal vector to the curve **N**.

$N = \frac{T'(t)}{|T'(t)|}$

$T'(t) = \frac{1}{\sqrt{29}}\langle -5\cos(t), -5\sin(t), 0\rangle$

$N = \frac{\sqrt{29}}{5}\cdot\frac{1}{\sqrt{29}}\langle -5\cos(t), -5\sin(t), 0\rangle = \boxed{\langle -\cos, -\sin, 0\rangle}$

v) Write the acceleration in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

$$a_T = \frac{d|v|}{dt} = \frac{r'(t)\cdot r''(t)}{|r'(t)|} = \frac{(-5\sin(t))(-5\cos(t)) + (5\cos(t))(-5\sin(t))}{\sqrt{29}}$$

$$= \frac{(25\sin(t)\cos(t) - (25\sin(t)\cos(t))}{} = 0$$

$$a_N = K|v|^2 = \frac{|r'(t)\times r''(t)|}{|r'(t)|} = \frac{5\sqrt{29}}{\sqrt{29}} = 5 \quad \sqrt{29}$$

$$a = 0\cdot T + 5\cdot N = \boxed{5\langle -\cos(t), -\sin(t), 0\rangle = 5N = a \\ = \langle -5\cos(t), -5\sin(t), 0\rangle}$$

vi) Find the unit binormal vector to the curve $\mathbf{B}$.

$$B = T \times N = \frac{r'(t)\times r''(t)}{|r'(t)\times r''(t)|}$$

$$T = \frac{1}{\sqrt{29}}\langle -5\sin(t), 5\cos(t), 2\rangle$$

$$N = \langle -\cos, -\sin, 0\rangle$$

$$T\times N = \frac{1}{\sqrt{29}}\langle -5\sin(t), 5\cos(t), 2\rangle \times \langle -\cos, -\sin, 0\rangle$$

$$= \boxed{\frac{1}{\sqrt{29}}\langle 2\sin(t), -2\cos(t), 5\rangle}$$

vii) Find the arc length of this curve over the interval $0 \le t \le \pi$.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^\pi \sqrt{(-5\sin(t))^2 + (5\cos(t))^2 + (2)^2}\, dt$$

$$= \boxed{\sqrt{29}\,\pi}$$

Problem 4. (10 points) Surfaces in the space
Consider the surface $z = f(x, y) = \sqrt{25 - x^2 - y^2}$.

i) What is the domain of the function?

$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 25\}$

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ii) What is the range of the function?

$\{f \in \mathbb{R} : 0 \leq f \leq 5\}$ or

$[0, 5]$

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iii) Sketch the level curves of the surface for $z =$ -1, 0, 0.5, 1, 2 and clearly label which curve corresponds to which z value (or indicate that the curve does not exist).

Problem 5. (10 points) Limits and continuity
Find the following limits

i) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x^2+y^2+1}$

plug in

$$\lim_{(x,y)\to(0,0)} \frac{0^2+y^3}{0^2+0^2+1} = \frac{0}{1} = \boxed{0}$$

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ii) $\lim_{(x,y)\to(-1,1)} \sqrt{3x^2+2y^2+2}$

$$\lim_{(x,y)\to(-1,1)} \sqrt{3(-1)^2+2(1)^2+2}$$

$$= \lim_{(x,y)\to(-1,1)} \sqrt{3+2+2} = \boxed{\sqrt{7}}$$

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iii) $\lim_{(x,y)\to(2,2),x+y\neq 4} \frac{x+y-4}{\sqrt{x+y}-2}$

$\boxed{\text{DNE}}$

multiply by the conjugate

simplify

$$\lim_{(x,y)\to(2,2)} \left(\frac{x+y-4}{\frac{x+y-4}{\sqrt{x+y}+2}} \right) = \lim_{(x,y)\to(2,2)} (\sqrt{x+y}+2)$$

plug in

$$\lim_{(x,y)\to(2,2)} \sqrt{2+2}+2 = \lim_{(x,y)\to(2,2)} 2+2 = 4$$

$= 4$

but $x+y\neq 4$

iv) At what points (x, y) in the plane is the function $f(x, y) = \cos \frac{1}{x^2+y^2}$ is continuous?

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v) (EXTRA-CREDIT: 5 points) By considering different path of approach, show that the function

$$f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$$

is not continuous at $(0, 0)$.

The function is not continuous at (0,0) because it is not defined here.

Since f is a rational function, it is continuous on its domain, which is the set $D = \{(x,y) \mid (x,y) \neq (0,0)\}$

Problem 6. Partial derivatives and chain rule

Find all the second order partial derivatives of the following functions

i) $f(x, y) = 2x^2y^2 + 3e^x \sin 2y$

$f_x(x,y) = 4xy^2 + 3e^x \sin(2y)$

$f_{xx}(x,y) = 4y^2 + 3e^x \sin(2y)$

$f_{xy}(x,y) = 8xy + 6e^x \cos(2y)$

$f_y(x,y) = 4x^2y + 6e^x \cos(2y)$

$f_{yy}(x,y) = 4x^2 - 12e^x \sin(2y$

$f_{yx}(x,y) = 8xy + 6e^x \cos(2y)$

ii) $f(x, y) = \ln(x^2 + y^2)$

$f_x(x,y) = \dfrac{2x}{x^2+y^2}$

$f_{xx}(x,y) = \dfrac{-2x^2+2y^2}{(x^2+y^2)^2}$

$f_{xy}(x,y) = \dfrac{-4xy}{(x^2+y^2)^2}$

$f_y(x,y) = \dfrac{2y}{x^2+y^2}$

$f_{yy}(x,y) = \dfrac{2x^2-2y^2}{(x^2+y^2)^2}$

$f_{yx}(x,y) = \dfrac{-4xy}{(x^2+y^2)^2}$

iii) FInd by using the chain rule $\frac{\partial z}{\partial u}$ if $z = f(x, y) = 3e^{-2x} \ln y^2$ and $x = \ln(u \cos v)$ and $v = u \sin v$.

Problem 7. (10 points) Tangent planes directional derivatives and gradients.

i) Find the gradient vector and the tangent plane to the surface $x^2+2xy-y^2+z^2=7$ at the point $(1,-1,3)$

$x_0 \quad y_0 \quad z_0$

$f(x,y,z) = x^2+2xy-y^2+z^2-7$

$\nabla f(x,y,z) = \langle 2x, 2x-2y, 2z \rangle$

$\boxed{\nabla f(1,-1,3) = \langle 2, 4, 6 \rangle}$

$a \quad b \quad c$

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$2(x-1)+4(y+1)+6(z-3)=0$

$2x-2+4y+4+6z-18=0$

$2x+4y+6z=2-4+18$

$\boxed{2x+4y+6z=16}$

iii) Find the derivative of the function $f(x,y,z)=2x^3+z\sin y$ at the point $(-1,1,1)$ in the direction of $2i-j+2k$.

Problem 8. (10 points) Maximun and minimum

i) Find the local maxima, local minima and saddle points of the following function

$$f(x, y) = x^3 - y^3 - 2xy + 6.$$

$f_x = 3x^2 - 2y$

$f_y = -3y^2 - 2x$

$f_{xx} = 6x$

$f_{yy} = -6y$

$f_{yx} = -2$

$\boxed{\begin{array}{l} \text{Max}: (x,y) = \left(-\frac{2}{3}, \frac{2}{3}\right) = \frac{170}{27} \\ \text{Min}: \text{DNE} \\ \text{saddle point}: (x,y) = (0,0) \end{array}}$

$D\left(-\frac{2}{3}, \frac{2}{3}\right) = 12 > 0$

$f_{xx}\left(-\frac{2}{3}, \frac{2}{3}\right) = -4 < 0$

$D(0,0) = -4 < 0$

ii) Find the maximum and the minimum value of the function

$$f(x, y) = 2x^2 + 6y^2$$

subject to the constraint $x^4 + 3y^4 = 1$.

$g(x,y) = x^4 + 3y^4 - 1$

$\boxed{\begin{array}{l} \text{Max} = 4 \\ \text{Min} = 2 \end{array}}$

$f_x = 4x \quad , \quad f_x = \lambda g_x \quad , \quad = 4x\lambda$

$f_y = 12y \quad , \quad f_y = \lambda g_y \quad , \quad = 12y\lambda$

$\nabla f = \lambda \nabla g \quad , \quad g = 1$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 4$

$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4$

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 4$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4$

$f(-1,0) = 2$

$f(1,0) = 2$

Problem 9. (10 points) Double and Triple integrals

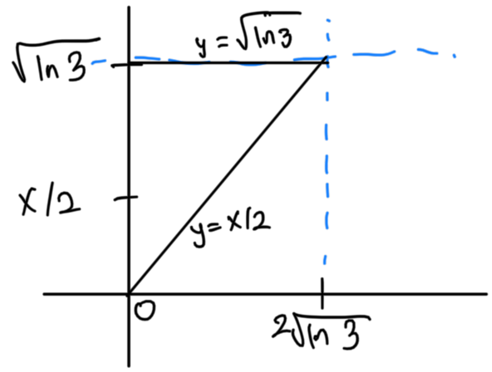
i) Given the double integral

$$\int_0^{2\sqrt{\ln 3}} \int_{x/2}^{\sqrt{\ln 3}} e^{y^2}\, dy\, dx.$$

Sketch the region of integration

$$\int_{x=0}^{x=2\sqrt{\ln 3}} \int_{y=x/2}^{y=\sqrt{\ln 3}} e^{y^2}\, dy\, dx$$

$\downarrow$



$$\int_{y=0}^{y=\sqrt{\ln 3}} \int_{x=0}^{x=2y} e^{y^2}\, dx\, dy$$

ii) Write an equivalent double integral with the order of integration reversed and evaluate the double integral.

$$\int_0^{\sqrt{\ln 3}} \int_0^{2y} e^{y^2}\, dx\, dy = \int_0^{2y} e^{y^2}\, dx = 2ye^{y^2}$$

$$= \int_0^{\sqrt{\ln 3}} 2ye^{y^2}\, dy = \boxed{2}$$

iii) Evaluate the triple integral

$$\int_0^1 \int_0^{2-x} \int_0^{2-x-y} x + y + z\, dz\, dy\, dx$$

$$\int_0^{2-x-y} x+y+z\, dz = -x^2 - 2xy + 2x + \frac{(-x-y+2)^2}{2} - y^2 + 2y$$

$$= \int_0^{2-x} \left(-x^2 - 2xy + 2x + \frac{(-x-y+2)^2}{2} - y^2 + 2y\right) dy$$

$$= x^2 - 4x + \frac{(-x+2)^3}{6} - \frac{(-x+2)^3}{3} + 4$$

$$= \int_0^1 \left(x^2 - 4x + \frac{(-x+2)^3}{6} - \frac{(-x+2)^3}{3} + 4\right) dx = \boxed{\frac{41}{24}}$$

Not enough space to write everything down

Problem 10. (10 points) Line integrals and Green's theorem

i) Evaluate the line integral

$$\int_C (x^2 + y^2 + z^2)ds$$

where C is a curve parametrized as $x = t,\ y = \cos 2t,\ z = 2\sin 2t,\ 0 \le t \le 2\pi.$

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ii) Given the vector field $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = t^3\mathbf{i} + -t^2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$. Also find curl $\mathbf{F}$ and div$\mathbf{F}$. Is $\mathbf{F}$ conservative? Explain.

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iii) Evaluate the line integral by two methods: directly and using Green's theorem.

$$\oint_C xydx + xx^2dy$$

and C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

Trigonometry — Formula Sheet

Trigonometric Identities: The following are true for all x and y, provided that both sides are defined at the chosen x and y.

Sum Formulas	Difference Formulas
$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y)$	$\sin(x-y)=\sin(x)\cos(y)-\cos(x)\sin(y)$
$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$	$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$
$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
Double Angle Formulas	**Half Angle Formulas**
$\sin(2x)=2\sin(x)\cos(x)$	$\sin(\frac{x}{2})=\pm\sqrt{\frac{1-\cos(x)}{2}}$
$\cos(2x)=\cos^2(x)-\sin^2(x)$	$\cos(\frac{x}{2})=\pm\sqrt{\frac{1+\cos(x)}{2}}$
$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$	$\tan(\frac{x}{2})=\pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$
Power-Reducing Formulas	
$\sin^2(x)=\frac{1-\cos(2x)}{2}$	$\cos^2(x)=\frac{1+\cos(2x)}{2}$
$\tan^2(x)=\frac{1-\cos(2x)}{1+\cos(2x)}$	
Sum-to-Product Formulas	
$\sin(x)+\sin(y)=2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$	$\cos(x)+\cos(y)=2\cos(\frac{x+y}{2})\cos(\frac{x-y}{2})$
$\sin(x)-\sin(y)=2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$	$\cos(x)-\cos(y)=-2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$
Product-to-Sum Formulas	
$\sin(x)\sin(y)=\frac{1}{2}[\cos(x-y)-\cos(x+y)]$	$\cos(x)\cos(y)=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$
$\sin(x)\cos(y)=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$	$\cos(x)\sin(y)=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$

Calculus **Formula Sheet**

Substitution in Definite Integrals

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Integration by Parts

$$\int_a^b f(x)\, g'(x)\, dx = f(x)g(x)]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

Parametric Equations

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Area enclosed

$$\int y dx$$

Arc Length

$$L = \int_a^b \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\, dt$$

Area of Surface Revolution

$$S = \int_a^b 2\pi y \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\, dt \text{ (around } x\text{-axis } y \geq 0)$$

$$S = \int_a^b 2\pi x \sqrt{(\tfrac{dx}{dt})^2 + (\tfrac{dy}{dt})^2}\, dt \text{ (around } y\text{-axis } x \geq 0)$$

Polar coordinates

$$x = r\cos\theta \; y = r\sin\theta \qquad r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

Slope for the curve $r = f(\theta)$,

$$\frac{dy}{dx}\Big|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Area of a region $0 \leq r_1(\theta) \leq r \leq r_2(\theta),\ \alpha \leq \theta \leq \beta$

$$A = \int_\alpha^\beta \frac{1}{2}(r_2^2 - r_1^2) d\theta.$$

Arc Length in Polar coordinates

$$L = \int_a^b \sqrt{r^2 + (\tfrac{dr}{d\theta})^2}\, d\theta$$

Calculus	**Formula Sheet**

Velocity, Speed and aceleration

$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$, Speed $=|\mathbf{v}|$, $\quad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$

Unit tangent vector and **Principal unit normal vector**

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

Curvature, radius of curvature and**Arc Length**

$$\kappa = \frac{1}{|\mathbf{v}|}\frac{d\mathbf{T}}{dt}| \; \rho = \frac{1}{\kappa} \qquad L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$$

Binormal vector

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

Tangential and normal scalar components of acceleration

$$a_T = \frac{d|\mathbf{v}|}{dt}, \; a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}.$$

Chain rule

If $w = f(x, y)$, $x = x(t)$, $y = y(t)$ then for the composite $w = f(x(t), y(t))$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

If $w = f(x, y, z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$ then for the composite $w = f(x(t), y(t), z(t))$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

If $w = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$, then for the composite

$$w = f(x(u, v), y(tu, v)),$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

If $w = f(x, y, z)$, $x = x(u, v)$, $y = y(tu, v)$, $z = z(u, v)$ then for the composite

$$w = f(x(u, v), y(u, v), z(u, v)),$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

Directional Derivatives and Gradient Vectors

The gradient vector (gradient) of $f(x, y)$ at (x_0, y_0) is defined as $\nabla f(x_0, y_0) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.
The directional derivative of $f(x, y)$ in the direction $\mathbf{u}$ at (x_0, y_0) can be computed as

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

Calculus **Formula Sheet**

Sumary of Min-Max Test

The extreme values of $f(x,y)$ can occur only at **boundary points** of the domain of f or at **critical points** (interior points where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ or points where they fail to exists)

Second Derivative Test

At a critical point (a,b) of f such that $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$

i) If $\frac{\partial^2 f}{\partial x^2} < 0$ and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial x}^2 > 0$ at (a,b) then (a,b) local maximun

ii) If $\frac{\partial^2 f}{\partial x^2} < 0$ and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial x}^2 < 0$ at (a,b) then (a,b) local minimumn

iii)$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial x}^2 < 0$ at (a,b) then (a,b) saddle point

iv)$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial x}^2 = 0$ at (a,b) the test is inconclusive.

Double integrals

$$\int\int_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy,$$

if $R = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

$$\int\int_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx,$$

if $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

$$\int\int_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy,$$

if $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

Triple integrals

$$\int\int\int_R f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^g f(x,y,z)dz\,dy\,dx = \int_c^d \int_a^b \int_e^g f(x,y,z)dz\,dx\,dy,$$

if $R = \{(x,y,z) : a \le x \le b,\ c \le y \le d, e \le z \le g\}$

$$\int\int_R f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dz\,dy\,dx,$$

if $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x), h_1(x,y) \le z \le h_2(x,y)\}$