Problems and Solutions, Sterrenstelsels, 2015

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Contents

1 Werkcollege, Sterrenstelses, Week 1

This is the first set of assignments for the course *Sterrenstelsels*.
Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 1.3** below.

1.1 S&G Probl. 2.2

Show that an error or uncertainty of 0.1 magnitudes in the distance modulus is roughly equivalent to a 5% error in the distance, d

1.2 Rotation of "Spiral Nebulae"

In 1914, V. M. Slipher deduced from spectroscopic observations of the Sombrero galaxy (NGC 4594) a rotational velocity of about 100 km/s (at $20''$ from the nucleus). Slipher had also measured positive radial velocities for many spiral "nebulae", often several hundred km/s.

Around the same time, Adriaan van Maanen compared several images of M101 taken over a period of about 15 years and measured an annual rotation of $0.022''$ at a distance of $5'$ from the centre (meaning that, according to van Maanen's measurement, a point located $5'$ from the centre would move $0.022''$ in a year). Van Maanen's measurement was used by Harlow Shapley in the "great debate" as one argument against the idea that spiral nebulae are external galaxies similar to the Milky Way.

Let us now explore some of the implications of these measurements:

1. Based on van Maanen's measurement, what is the rotation period of M101 (in years)?

2. Shapley had estimated that the Sun is located about 15 kpc from the centre of the Milky Way. If the Sun is orbiting around the centre of the Milky Way with the same period as van Maanen's measurement implied for M101, what would be the speed of the Sun? In km/s? In units of c, the speed of light? Would you agree with Shapley that this is unreasonable?

3. If, on the other hand, M101 rotates as fast as NGC 4594 (100 km/s), what would be the distance of M101? Does this seem more reasonable? Why / why not?

Both Slipher's and van Maanen's observations were extremely challenging at the time. An angle of $0.022''$ is tiny. G. W. Ritchie had already measured two of van Maanen's plates before and found no rotation. The spectroscopic measurements were based on exposures that had to extend over many hours, and not everybody believed Slipher's radial velcities, either.

4. The "plate scale" on the photographs used by van Maanen was about $30''$ mm^{-1}. For two observations made 15 years apart, what is the shift measured by van Maanen in mm?

5. If you had been attending the debate and knew what was known then, what would you have concluded about the galactic or extragalactic nature of spiral nebulae?

1.3 Radiation Pressure and Radial Velocities

In the "great debate", neither Shapley nor Curtis had a good explanation for the positive radial velocities of the nebulae. Today we know that this is due to the expansion of the Universe itself, but cosmology was still in its infancy in the 1920s and most people believed in a static Universe. Shapley suggested, somewhat hand-wavingly, that the nebulae might be accelerated by radiation pressure from the Milky Way. However, Henry Norris Russell was quick to demonstrate this cannot plausibly work. In this assignment we examine some of Russell's arguments.

Russell made a few simple assumptions:

1. Masses of the nebulae can be estimated from their rotation, assuming the standard Newtonian formula for circular rotation (but note that, strictly speaking, this assumes a spherically symmetric mass distribution). In 1921, such measurements were available for two nebulae: M31 and NGC 4594.

2. The plane of a nebula is perpendicular to the line-of-sight towards the Milky Way.

3. A nebula absorbs all the radiation from the Milky Way that falls upon it.

4. As seen from a nebula, the Milky Way occupies half the sky.

5. Seen from a nebula, the intensity of the light from the Milky Way is similar to that seen from Earth.

6. The intensity of the Milky Way corresponds to 3.5% of the flux from a 1st magnitude star per square degree (this number came from measurements by the Dutch astronomer Pieter van Rhijn, a student of Kapteyn). Such a star is a factor of $10^{0.4\times(1+26.7)} = 1.2\times10^{11}$ times fainter than the Sun.

7. Two measures of the "radius" of a nebula were considered: 1) an "inner" radius r, containing the majority of the mass, and 2) an "outer" radius R that represents the maximum area on which the radiation pressure would act.

The momentum of a photon (or a collection of photons) with energy E is $p = E/c$. Also, recall that pressure is force per area.

- Start by calculating the radiation pressure from a square degree of the Milky Way, seen from a nebula. Show that this pressure is

$$\mathcal{P} = 2.3 \times 10^{-14} \frac{L_\odot}{c(1\mathrm{AU})^2}$$

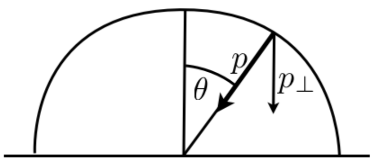
where 1 AU = 1 astronomical unit = the distance from the Sun to the Earth, $L_\odot$ is the luminosity of the Sun, and c is the speed of light.

- Next, show that the force on the nebula due to radiation pressure from by a whole hemisphere is

$$\mathcal{F} = 7.5 \times 10^{-10} \frac{D^2 R^2 L_\odot}{c(1\mathrm{AU})^2}$$

for distance D.

Hint: For radiation originating somewhere on the hemisphere, only the component of the momentum vector perpendicular to the surface of the nebula ($p_\perp$ in the figure below) contributes to the acceleration. The integral $\int_0^{\pi/2} \sin\theta \cos\theta \, \mathrm{d}\theta = \frac{1}{2}$.



- Finally, show that the acceleration produced by radiation pressure is then

$$A = 7.5 \times 10^{-10} \frac{L_\odot G}{c(1\mathrm{AU})^2} \frac{DR^2}{rv^2}$$

for "inner" radius r, "outer" radius R, circular velocity v at r. G is the gravitational constant.

Some of the assumptions made here (e.g. #4) may seem very unrealistic today, but it is important to keep the context of this calculation in mind. Russell's aim was to examine whether radiation pressure could significantly affect the kinematics of nebulae, given Shapley's view that the Milky Way was very large, and the nebulae all essentially part of the Milky Way.

One of the few nebulae for which the necessary observations were available in 1921 was the "Sombrero galaxy", NGC 4594. NGC 4594 has a radial velocity of +1000 km/s. For r and R, values of $r = 150''$ and $R = 210''$ may be assumed, as well as a rotational velocity of $v = 415$ km/s. The distance was very uncertain, but Russell assumed a distance of 1.43 Mpc or 4.4×10^{22} m.

- Under the above assumptions, calculate the current acceleration of the Sombrero galaxy due to radiation pressure
- If the acceleration had remained constant, and the Sombrero were initially at rest, how long would it have taken to accelerate to the current radial velocity?
- How far would the Sombrero have moved in this time?

Of course, the calculation above is extremely simplified. Which effects have been ignored? How would the calculation change (qualitatively) if these were included?

2 Werkcollege, Sterrenstelsels, Week 2

These are the assignments for the second week of the course *Sterrenstelsels*. Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 2.12** from Sparke & Gallagher.

2.1 Thin and thick disc

According to data from the Sloan Digital Sky Survey, the thin disc has an exponential scale height of 270 pc, the thick disc has a scale height of 1200 pc, and about 4% of the stars near the Sun belong to the thick disc.

Assuming that both the thin and thick disc are symmetric around the Sun in the vertical direction, what fraction of the total surface density of disc stars is due to the thick disc?

2.2 Extinction in magnitudes

Show that, if τ is the optical depth at wavelength λ, then the extinction in magnitudes is

$$A_\lambda = 1.09\,\tau_\lambda$$

2.3 Interstellar absorption

One of the main difficulties which plagued early attempts to determine the dimensions of the Milky Way was *interstellar absorption*. In 1930, Robert Trumpler (at Lick Observatory) studied the relation between apparent diameters and distances of open star clusters. Trumpler determined *photometric distances* from the apparent brightness of stars of known luminosity in the clusters. He could then use the distances to estimate the linear diameters of the clusters. Assuming that all clusters have (on average) the same linear dimensions, the apparent diameter should be inversely proportional to the distance. However, Trumpler instead found a non-linear relation, and correctly concluded that the discrepancy was due to systematic dimming of the light from the more distant clusters, causing the photometric distances to be too large. He was then able to give a quantitative estimate of the average amount of interstellar absorption for the first time. In this assignment we will re-examine some aspects of Trumpler's analysis.

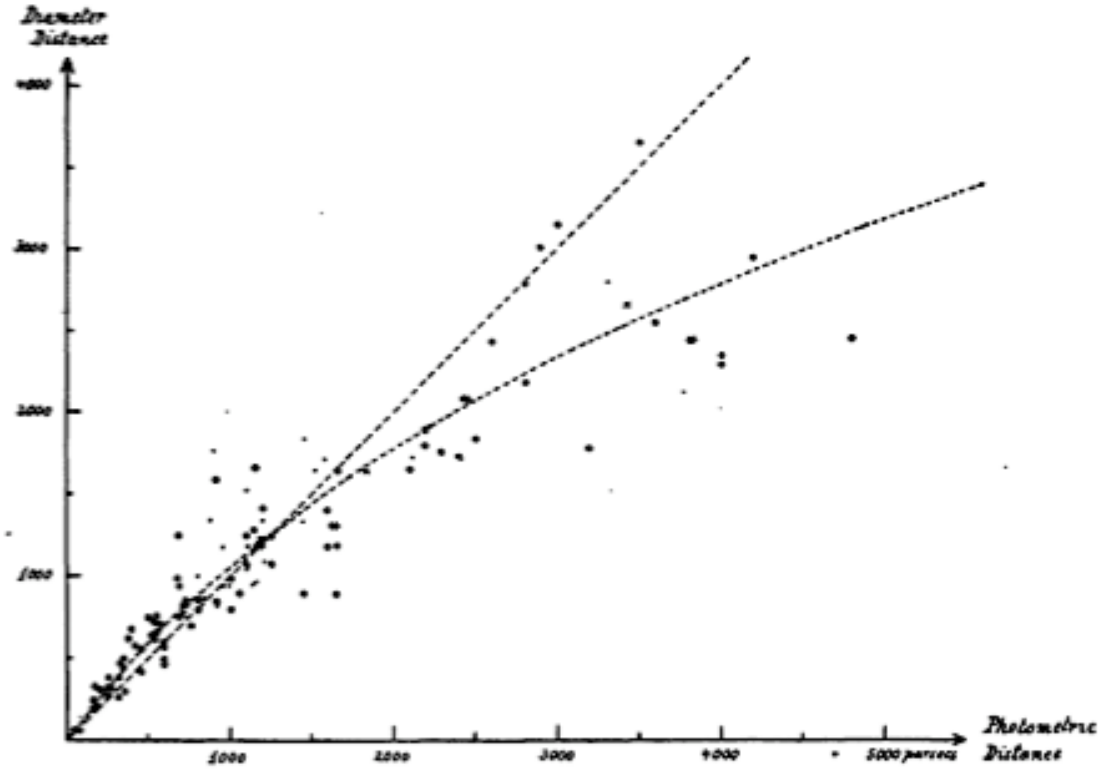


Fig 1: Trumpler's plot of Diameter distance versus Photometric distance (Trumpler 1930, PASP 42, 214). The curved line is Trumpler's estimated relation between the two distance estimates.

1. Show that, in a uniformly distributed absorbing medium, the absorption A, measured in magnitudes, is directly proportional to the distance d: $A = k\,d$.

2. Let us define the photometric distance $d_{\rm phot}$ as the distance one would infer from observations of the apparent brightness of sources with known luminosity, *neglecting extinction*. Further, define $d_{\rm true}$ as the actual distance. Then show that $d_{\rm phot}$ and $d_{\rm true}$ are related as

$$d_{\rm phot} = d_{\rm true} 10^{0.2k\,d_{\rm true}}$$

Fig. 1 shows Trumpler's plot of "Diameter distance" versus "Photometric distance". Note that the two curves *intersect* at $\approx$ 1300 pc. This is because the true linear diameters of the clusters are unknown *a priori* and must be estimated, in the first approximation, by use of the photometric distances. At some mean distance, the photometric and diameter distances will be equal (both greater than the true distance), at even greater distances the photometric distances are systematically too large, and at smaller distances the photometric distances approach the true distances. Thus, the Diameter distance d_{diam} is *proportional*, but not identical, to the true distance: $d_{\mathrm{diam}} = s\, d_{\mathrm{true}}$.

3. From Fig. 1, we estimate $d_{\mathrm{diam}} = d_{\mathrm{phot}}$ at 1300 pc, and $d_{\mathrm{diam}} = 2800$ pc at $d_{\mathrm{phot}} = 4000$ pc. Using these estimates, what is the average absorption in magnitudes per kpc?

More modern studies suggest that the absorption estimated by Trumpler is too low. For example, Binney & Merrifield give an absorption of $A_V = 1.9$ mag kpc^{-1} for a typical line of sight in the Galactic disk.

4. The brightest individual stars reach absolute magnitudes of $M_V \approx -9$. What would be the apparent magnitude of such a star, located in the Galactic plane, at a distance of 8 kpc?
5. Assuming a distance of 760 kpc and a *total* absorption of $A_V = 0.2$ towards the Andromeda galaxy (M31), what would be the apparent magnitude of a similar star there?
6. Comment on the relative ease / difficulty of studying stars, open star clusters, and globular clusters in our own Galaxy and in M31.

2.4 Sparke & Gallagher, Problem 2.12

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4 Werkcollege, Sterrenstelsels, Week 4

These are the assignments for the fourth week of the course *Sterrenstelsels*. Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 4.2** below.

4.1 Oort constants

The two Oort constants are defined as

$$A = -\frac{1}{2}\left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] \tag{4.1.1}$$

$$B = -\frac{1}{2}\left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} + \frac{v_0}{R_0}\right] \tag{4.1.2}$$

for circular velocity $v(R)$. The subscript 0 denotes the corresponding quantities for a circular orbit with the same radius as the Sun's current distance from the Galactic centre, i.e. the "Local System of Rest" (LSR).

In the lecture we saw that the radial velocity of a star on a circular orbit with radius R, relative to the LSR, is given by

$$v_r = [\Omega(R) - \Omega_0]\, R_0 \sin l \tag{4.1.3}$$

where Ω is the angular frequency, $\Omega = v/R$ and l is the Galactic longitude. Near the Sun (i.e., for distance $d \ll R_0$) this can be approximated as

$$v_r \approx -\frac{1}{2}\left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] d\, \sin 2l \tag{4.1.4}$$

and hence

$$v_r \approx A\, d\, \sin 2l \tag{4.1.5}$$

- Show that the equivalent expression for the *tangential* velocity (i.e. the velocity in the plane of the sky) is

$$v_t = [\Omega(R) - \Omega_0]\, R_0 \cos l - \Omega(R) d \tag{4.1.6}$$

- Show that the corresponding approximate expression for small d is

$$\begin{aligned} v_t &\approx -\left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] d\, \cos^2 l - \frac{v}{R} d \\ &\approx -\left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] d\, \cos^2 l - \frac{v_0}{R_0} d \end{aligned}$$

- Then show that, in terms of the Oort constants, this can be written as

$$v_t \approx Ad \cos 2l + Bd \tag{4.1.7}$$

  Hint: you may find the following trigonometric identities useful:
  $\cos^2 l + \sin^2 l = 1$
  $\cos^2 l - \sin^2 l = \cos 2l$.

4.2 Kinematic distances

An observer is located in the disc of a spiral galaxy, 10 kpc from the centre of the galaxy. The galaxy has a flat rotation curve with $v_{\rm circ} = 200$ km s^{-1}. At a galactic longitude of $l = 150°$, a star cluster with a radial velocity of -20 km s^{-1} is observed.

- Assuming that both the cluster and the observer are on circular orbits, calculate the distance from the centre of the galaxy to the cluster.

- Calculate the distance from the observer to the cluster

5 Werkcollege, Sterrenstelsels, Week 5

These are the assignments for the fifth week of the course *Sterrenstelsels*.
Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 5.3** below.

5.1 Motions of stars near the Sun

A star is currently observed at distance R_0 from the Galactic centre. Let $R_0 + \rho$ be the radial distance at which the centre of the star's epicycle is orbiting, where $\rho \ll R_0$.

- Show that the difference between the velocity of the star in the azimuthal direction and that of a star orbiting on a circular orbit at R_0, is

$$v_\chi(\rho) \approx -2B\rho$$

where B is the second Oort constant,

$$B = -\frac{1}{2}\left(\frac{\partial v_c}{\partial R} + \frac{v_c}{R}\right)$$

Hint: Use the conservation of angular momentum to calculate the azimuthal velocity of the star at R_0. Then use a first-order Taylor expansion of the rotation curve to express the circular velocity at $R_0 + \rho$ in terms of that at R_0.

5.2 Epicycle motion of the Sun

By analysing the motion of the Sun with respect to other stars, one finds that the Sun currently has an *inwards* velocity of 9 km/s towards the Galactic centre (i.e., $v_\rho = -9$ km/s) and a *forwards* velocity of 12 km/s with respect to the Local System of Rest ($v_\chi = +12$ km/s). The Sun completes one epicyclic oscillation in 170×10^6 years (this follows from the Oort constants: the epicyclic frequency is $\kappa^2 = -4B(A - B)$).

- Assume that the Sun is currently at the midpoint of its epicyclic oscillation. Estimate the amplitude of the Sun's excursion in the radial direction relative to a circular orbit. Does your result represent a minimum or maximum estimate of the actual deviation?

5.3 Chemical evolution

Recall that, in the closed-box model for chemical evolution, the change in metallicity δZ during a small time step is

$$\delta Z = -p\delta M_g / M_g \tag{5.3.1}$$

for yield p and gas mass M_g. If the yield is assumed to be constant and the initial metallicity of the system (when all the mass is in the form of gas) is $Z(t = 0) = 0$, then Eq. (5.3.1) implies

$$Z = -p \ln f_{\rm gas} \tag{5.3.2}$$

for gas fraction $f_{\rm gas}$.
In the more general case, we may define the *effective yield* as:

$$p_{\rm eff} \hat{=} - Z/ \ln f_{\rm gas} \tag{5.3.3}$$

Suppose the yield depends on metallicity in the following way:

$$p(Z) = 0.002 + 0.6\, Z \tag{5.3.4}$$

- Show that for $p(Z)$ given by Eq. (5.3.4), the effective yield is

  $$p_{\rm eff} = \frac{0.6Z}{\ln(1 + 300Z)} \tag{5.3.5}$$

  (assuming, as before, that $Z = 0$ initially)

7 Werkcollege, Sterrenstelsels, Week 7

These are the assignments for the seventh week of the course *Sterrenstelsels*. Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 7.1** below.

7.1 Chemical evolution with inflow

Adapted from Sparke & Gallagher, problem 4.10

Suppose that the inflow of metal-poor gas is proportional to the rate at which new stars form, so that

$$\delta M_s + \delta M_g = \nu \delta M_s \tag{7.1.1}$$

for some constant $\nu > 0$.

- *Verify Eq. (7.1.1)*
- *Show that S&G Equation 4.14*,

  $$\delta Z = \delta\left(\frac{M_h}{M_g}\right) = \frac{p\delta M_s - Z\left[\delta M_s + \delta M_g\right]}{M_g}$$

  becomes

  $$\delta Z = \frac{p - \nu Z}{\nu - 1}\frac{\delta M_g}{M_g} \tag{7.1.2}$$

  so that the metallicity of the gas is

  $$Z = \frac{p}{\nu}\left[1 - \left(\frac{M_g(t)}{M_g(0)}\right)^{\nu/(1-\nu)}\right] \tag{7.1.3}$$

7.2 Mass of the Local Group

In the lecture we discussed the Kahn-Woltjer estimate of the mass of the Local Group. This method takes advantage of the specific dynamical situation in the Local Group, which is dominated by two galaxies (the Milky Way and M31).

A more general method is to rely on the *virial theorem*, which is valid for any self-gravitating system in equilibrium. The virial theorem states that

$$T = -\frac{1}{2}U \tag{7.2.1}$$

for kinetic energy T and (gravitational) potential energy U.

- Show that the virial theorem also applies to the "one-body problem" where a planet of mass m orbits around a star of mass M in a circular orbit, for $m \ll M$.

In the more general case of a system consisting of a large number of particles, we can still write the kinetic energy as

$$T = \frac{1}{2} M \langle v^2 \rangle \tag{7.2.2}$$

where $\langle v^2 \rangle$ is the mean squared difference of the velocities with respect to the centre of gravity and M is the total mass.

- Show that the potential energy for a uniform sphere of radius R and mass M is given by
$$U = -\frac{3}{5} \frac{GM^2}{R} \tag{7.2.3}$$
(hint: start by calculating the potential energy of a thin shell at radius r for a sphere of density ρ. Then integrate over all such shells and eliminate the density.)

- Hence show that
$$M = \frac{5}{3} \frac{\langle v^2 \rangle R}{G} \tag{7.2.4}$$

For the Local Group, the velocity dispersion and radius may be estimated as $\langle v^2 \rangle^{1/2} \approx 106$ km/s and $R = 1.2 \times 10^6$ pc, respectively (van den Bergh 1999).

- Using these values, estimate the mass of the Local Group. Compare with the estimate based on the relative motion of the Milky Way and M31 alone ($M \sim 4 \times 10^{12}\, M_\odot$). You will need the following constants:

  1 pc = 3.08×10^{16} m
  G = 6.673×10^{-11} m^3 kg^{-1} s^{-2}
  1 $M_\odot$ = 1.989×10^{30} kg

8 Werkcollege, Sterrenstelsels, Week 8

These are the assignments for week 8 of the course *Sterrenstelsels*.
Every week, one of the problems provides credit towards the final exam. If at least **five** of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 8.1** below.

8.1 Two-body relaxation

- Chapter 3.2.2: Fill in the missing steps leading up to (3.50)

We saw that the two-body relaxation time scale can be written as

$$t_{\rm rel} = \frac{V^3}{8\pi G^2 m^2 n \ln\Lambda} \tag{8.1.1}$$

for relative velocities V, particle mass m and particle density n. Being slightly more precise about the meaning of V, we rewrite this as

$$t_{\rm rel} = \frac{\langle V^2\rangle^{3/2}}{8\pi G^2 m^2 n \ln\Lambda} \tag{8.1.2}$$

For a system in *virial equilibrium*, the velocity dispersion $\langle V^2\rangle$, mass M and radius R of the system are related as

$$M = \eta\frac{\langle V^2\rangle R}{G} \tag{8.1.3}$$

The constant η depends on the radial structure of the system and how exactly the radius R is defined (note the similarity of this relation to the corresponding relation for a circular orbit).

- For a system of uniform density, total mass M and radius R, show that the two-body relaxation time scale for a virialized system can be expressed as

$$t_{\rm rel} = \frac{M^{1/2}R^{3/2}}{6G^{1/2}\eta^{3/2} m \ln\Lambda} \tag{8.1.4}$$

- With M and R defined as above, the constant $\eta = 5/3$. Now calculate $t_{\rm rel}$ for the following systems (you can assume a mean stellar mass of $m = 1M_\odot$ and $\ln\Lambda = 10$ in all cases):
  - An open cluster of stars ($M = 10^3\, M_\odot$, $R = 3$ pc)
  - A globular cluster ($M = 10^6\, M_\odot$, $R = 10$ pc)
  - A giant elliptical galaxy ($M = 10^{12}\, M_\odot$, $R = 10$ kpc)

8.2 Spiral structure: Lindblad resonances

In the theory of spiral structure, the *global pattern speed* Ω_{GP} is the overall pattern speed of the spiral density wave. In general, this may differ from the *local pattern speed* for a two-armed spiral pattern, Ω_{LP}, that follows from the condition $\Omega_{\mathrm{LP}} = \Omega - \frac{1}{2}\kappa$. However, where the two are equal, we talk about an *Inner Lindblad Resonance*.
A spiral galaxy is observed to have a flat rotation curve with circular speed $v_c = 250\ \mathrm{km\ s^{-1}}$ from near the centre of the galaxy to very large radii. The co-rotation radius of the spiral pattern is at $R = 5$ kpc.

- Calculate the location of the Inner ($m = 2$) Lindblad Resonance. To do this, you will need to know how to compute the epicyclic frequency from the rotation curve. This would be a topic for a more advanced course, so here we simply quote the result:

$$\kappa^2 = -4B(A - B) \tag{8.2.1}$$

  for Oort constants A and B (see chapter 3.3 in S&G).

- Assuming that the mass distribution is dominated by a spherical dark matter halo, calculate the density profile $\rho(R)$ of the halo.

9 Werkcollege, Sterrenstelsels, Week 9

These are the assignments for the ninth (and last) week of the course *Sterrenstelsels*.
Every week, one of the problems provides credit towards the final exam. If at least 5 of these problems are handed in and approved, one question on the final exam may be skipped. The hand-in assignment for this week is **Problem 9.2** below.

9.1 Tully-Fisher / Faber-Jackson relations (from 2013 Exam)

The *Faber-Jackson* relation for elliptical galaxies is the equivalent of the Tully-Fisher relation for spiral galaxies. According to the Faber-Jackson relation, the luminosity L and the velocity dispersion σ of an elliptical galaxies are related according to the following expression:

$$L \propto \sigma^4. \tag{9.1.1}$$

The following relation applies to a gravitationally bound system in virial equilibrium:

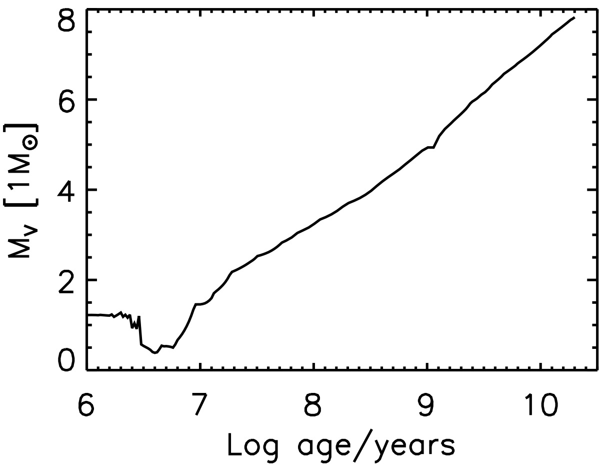
$$M = \eta \frac{\sigma^2 R}{G} \tag{9.1.2}$$

where M is the mass, R is the radius and σ is the velocity dispersion. η is a constant that depends on the geometry of the system.

1. Show that the Faber-Jackson relation follows from Eq. (9.1.2) under similar assumptions about the mass-to-light ratio and intensity as those that lead to the Tully-Fisher relation for disc galaxies.

2. Explain how the Faber-Jackson relation can be used for distance determination. What must be measured? How does the distance follow from the measurements?

9.2 Elliptical galaxies, mergers and globular clusters



Model calculation for the M_V magnitude per solar mass as a function of time after a burst of star formation.

Many elliptical galaxies are surrounded by large numbers of globular clusters (GCs). A convenient way to quantify the richness of the globular cluster population around a galaxy is the GC *specific frequency*, S_N, which is defined as

$$S_N = N_{\rm GC} \times 10^{0.4\times(M_V+15)} \tag{9.2.1}$$

where $N_{\rm GC}$ is the number of globular clusters and M_V the absolute visual magnitude of the galaxy. An early argument that was put forward against the idea that elliptical galaxies are the result of spiral-spiral mergers was the fact that elliptical galaxies can have much higher GC specific frequencies than spiral galaxies. This is not easily explained if the GC population of the merging galaxy is simply the combination of the GC populations in the two individual galaxies. The counterargument is that new GCs might be formed during the merger.

In this exercise we look at these arguments.

Suppose that two Milky Way-like spiral galaxies are just about to merge. Prior to the merger, each galaxy has an absolute visual magnitude $M_V = -21.0$. Each galaxy contains 5×10^9 $M_\odot$ of gas, which is instantaneously turned into stars during the merger. Each galaxy also hosts 200 old globular clusters.

The figure above shows model calculations for the absolute M_V magnitude, normalised to a stellar mass of 1 $M_\odot$ for a single burst of star formation, as a function of time after the burst. At 10^8 years, the burst has a magnitude of M_V = 3.3 mag per Solar mass.

a. Calculate the total M_V magnitude of the merged galaxies at 10^8 years after the merging, and give the specific frequency if no globular clusters are formed (or destroyed) in the merger. Assume that the magnitude of the pre-existing stellar populations in the original galaxies do not change. Also, ignore extinction.

A typical specific frequency for an elliptical galaxy is $S_N = 3$ and the average mass of a globular cluster can be taken to be $10^5 M_\odot$.

b. What fraction of the gas would have be turned into new globular clusters in order to get $S_N = 3$? If you did not find an answer in **(a)**, you can assume $M_V = -22$ but note that this is *not* the exact answer.

9.3 Projected axis ratio for prolate spheroid

We have seen that the distribution of projected minor/major axis ratios q for *oblate* spheroids is given by

$$\frac{\mathrm{d}N}{\mathrm{d}q} = \frac{q}{\sqrt{(1 - B^2/A^2)(q^2 - B^2/A^2)}} \tag{9.3.1}$$

where A and B are the three-dimensional major and minor axes of the oblate spheroid.

Show that the corresponding formula for a *prolate* spheroid can be written as

$$\frac{\mathrm{d}N}{\mathrm{d}q} = \frac{1}{q^3 \sqrt{(1 - A^2/B^2)(q^{-2} - A^2/B^2)}} \tag{9.3.2}$$

1 Werkcollege, Sterrenstelsels, Week 1 - Hints and Solutions

1.1 S&G Probl. 2.2

Can be shown in a couple of ways.
1) The distance modulus is

$$m - M = 5 \log D/10\text{pc}$$

.

So for distances D_1 and D_2, the distance moduli are

$$(m - M)_i = 5 \log D_i/10\text{pc}$$

and the difference is

$$(m - M)_1 - (m - M)_2 = 5 \log D_1/D_2$$

so if $(m - M)_1 - (m - M)_2 = 0.1$, then the ratio $D_1/D_2 \approx 1.05$, so a 5% error on the distance.
2) We can also use standard error propagation:

$$\begin{aligned} \delta(m - M) &= \frac{\partial(5 \log D/10\text{pc})}{\partial D} \delta D \\ &= \frac{5}{\ln 10} \frac{\delta D}{D} \end{aligned}$$

Hence, $\delta D/D = \delta(m - M)\frac{1}{5} \ln 10 \approx 0.5\delta(m - M)$

1.2 Rotation of "Spiral Nebulae"

1. Rotation period = $2\pi\ 5 \times 60''/(0.022''\text{yr}^{-1}) \approx 86000$ yr
2. Speed = $2\pi\times$ 15 kpc / 86000 yr $\approx$ 1.1 pc/yr = 1.07×10^9 m/s = $3.6\,c$!
3. Proper motion on the sky = $0.022''$ yr^{-1}. Absolute velocity = 100 km/s = 1.0×10^{-4} pc yr^{-1}.
   Distance where $0.022''$ corresponds to 10^{-4} pc: 10^{-4} pc / D = $\tan 0.022''$ i.e. $D \approx 960$ pc. Well within Shapley's Galaxy.
4. Plate scale = $30''$ mm^{-1}. Shift = $15 \times 0.022'' = 0.33'' \approx 0.01$ mm

1.3 Radial velocities and radiation pressure

As per assumption (1), masses are

$$M = \frac{r_p v^2}{G} = \frac{r_a D v^2}{G} \tag{1.3.1}$$

where r_p is the physical radius, r_a is the angular radius, D the distance and v the rotational velocity.

According to assumption (4), the radiation pressure is distance-independent. The momentum of a photon is

$$p = E/c \tag{1.3.2}$$

for photon energy E. Hence, the radiation pressure from a star of magnitude 1 is

$$\mathcal{P} = \frac{L_\odot}{4\pi(1\mathrm{AU})^2}\frac{1}{1.2\times 10^{11}c} \tag{1.3.3}$$

and the radiation pressure from one square degree

$$\begin{aligned}\mathcal{P} &= 0.035\times\frac{L_\odot}{4\pi(1\mathrm{AU})^2}\frac{1}{1.2\times 10^{11}c} && (1.3.4)\\ &= 2.92\times 10^{-13}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} && (1.3.5)\end{aligned}$$

There are $2\pi(180/\pi)^2 = 20626$ square degrees in a hemisphere. If all the light came from one point, along a line-of-sight perpendicular to the plane of a nebula, the pressure would then be

$$\mathcal{P} = 6.0\times 10^{-9}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} \tag{1.3.6}$$

Since it is distributed over a hemisphere, the actual pressure is half that,

$$\mathcal{P} = 3.0\times 10^{-9}\frac{L_\odot}{4\pi c(1\mathrm{AU})^2} \tag{1.3.7}$$

Hence, the force on the nebula is

$$\begin{aligned}F &= \mathcal{P}\times\pi R_p^2 && (1.3.8)\\ &= 3.0\times 10^{-9}\frac{D^2R_a^2L_\odot}{4c(1\mathrm{AU})^2} && (1.3.9)\end{aligned}$$

where R_p and R_a are the "outer" physical and angular radii. Hence, the acceleration is

$$\begin{aligned}A = F/M &= 3.0\times 10^{-9}\frac{D^2R_a^2L_\odot}{4c(1\mathrm{AU})^2}\frac{G}{r_aDv^2} && (1.3.10)\\ &= 7.5\times 10^{-10}\frac{L_\odot G}{c(1\mathrm{AU})^2}\frac{DR_a^2}{r_av^2} && (1.3.11)\end{aligned}$$

- For $D = 4.4 \times 10^{22}$ m, $r = 150'' = 7.27 \times 10^{-4}$ rad, $R = 210'' = 1.02 \times 10^{-3}$ rad, we get $A = 1.04 \times 10^{-15}$ m/s^2.
- Time to accelerate to 1000 km/s = 3×10^{13} years.
- Distance travelled:

  $$D = \int_0^t v(\tau)\mathrm{d}\tau = \int_0^t A\,\tau \mathrm{d}\tau = \frac{1}{2} A\, t^2$$

  Inserting $t = 9.62 \times 10^{20}$ s and $A = 1.04 \times 10^{-15}$ m/s^2, we find $D = 4.8 \times 10^{26}$ m = 1.6×10^{10} pc. Leads to many evident inconsistencies.

The most obvious effect that has been ignored is, of course, gravity from the Milky Way. This would counteract the acceleration from radiation pressure, although the exact gravitation of the Milky Way was difficult to estimate in 1921 as the mass of the Milky Way was very poorly constrained.

Also, it is clearly not realistic to assume that the Milky Way occupies half the sky, as seen from a distant galaxy. This would further reduce the radiation pressure.

2 Werkcollege, Sterrenstelsels Week 2 - Hints and Solutions

2.1 Thin and thick disc

First note that

$$\rho_{\rm tot,0} = \rho_{\rm thin,0} + \rho_{\rm thick,0}$$

where $\rho_{\rm thick,0} = 0.04\rho_{\rm tot,0}$ and $\rho_{\rm thin,0} = 0.96\rho_{\rm tot,0}$.
We just have to integrate over all Z:

$$\Sigma_{\rm thin} = 2 \times 0.96 \times \rho_{\rm tot,0} \int_0^\infty e^{-Z/h_{\rm thin}} \, {\rm d}Z = 1.92\rho_{\rm tot,0} h_{\rm thin}$$

$$\Sigma_{\rm thick} = 2 \times 0.04 \times \rho_{\rm tot,0} \int_0^\infty e^{-Z/h_{\rm thick}} \, {\rm d}Z = 0.08\rho_{\rm tot,0} h_{\rm thick}$$

where $\rho_{\rm thin,0}$ is the local volume density of thin disc stars. The fraction of the surface density that is due to thick disc stars is then

$$\begin{aligned} f_{\rm thick} &= \frac{\Sigma_{\rm thick}}{\Sigma_{\rm thin} + \Sigma_{\rm thick}} \\ &= \frac{0.08 h_{\rm thick}}{1.92 h_{\rm thin} + 0.08 h_{\rm thick}} \\ &\approx 16\% \end{aligned}$$

2.2 Extinction in magnitudes

In flux units:

$$F = F_0 \exp(-\tau_\lambda)$$

In magnitudes:

$$\begin{aligned} m &= -2.5 \log_{10} F \; + \; {\rm const} \\ &= -2.5 \log_{10} [F_0 \exp(-\tau_\lambda)] \; + \; {\rm const} \\ &= -2.5 [\log_{10} F_0 + \log_{10} \exp(-\tau_\lambda)] \; + \; {\rm const} \\ &= m_0 + 2.5\tau_\lambda \log_{10} e \\ &= m_0 + 1.09\tau_\lambda \end{aligned}$$

2.3 Interstellar absorption

1. This is easily seen from 2.2.

   Alternatively, unreddened flux = F_0, reddened flux = F, distance = d. If light is dimmed by factor f per unit distance, then total dimming is f^d so $F = F_0 f^{-d}$. Per definition, $m = -2.5 \log F$ (leaving out the zero-point), so $m = -2.5 \log(F_0 f^{-d}) = m_0 + d\, 2.5 \log f \equiv m_0 + k\, d$. That is, $A = k\, d$.

2. In the absence of reddening, $F_0 = \frac{L}{4\pi d^2}$ or $d = \sqrt{\frac{L}{4\pi F_0}}$ where L is the luminosity of the source. Define $d_{\rm phot} \equiv \sqrt{\frac{L}{4\pi F}}$. From above, $F = F_0\, 10^{-0.4k\, d_{\rm true}}$ so $d_{\rm phot} = \sqrt{\frac{L}{4\pi F_0} 10^{0.4k\, d_{\rm true}}}$. Given that $d_{\rm true} = \sqrt{\frac{L}{4\pi F_0}}$, we now have $d_{\rm phot} = d_{\rm true} 10^{0.2k\, d_{\rm true}}$.

3. Insert $d_{\rm true} = d_{\rm diam}/s$ above. We get $4\,{\rm kpc} = 2.8\,{\rm kpc}/s \times 10^{0.2\,(2.8\,{\rm kpc})\,k/s}$ and $1.3\,{\rm kpc} = 1.3\,{\rm kpc}/s \times 10^{0.2\,(1.3\,{\rm kpc})\,k/s}$. Divide the two expressions by each other: $1.43 = 10^{0.30\,k/s}$ so $k/s \approx 0.52$. Inserting this above, $s \approx 1.363$ and $k \approx 0.7$ mag kpc^{-1}.

4. Dimming due to extinction = 8 kpc $\times$ 1.9 mag kpc^{-1} = 15.2 mag. Distance modulus = $m - M = 5 \log d/10\,{\rm pc}$ = 14.5. $m_V = 20.7$.

5. Distance modulus = 24.4 mag, $m_V = 15.6$.

6. Galaxy: reddening a problem, especially in the disk. M31: Greater distance, can see the whole disk, but less spatial resolution.

2.4 Sparke & Gallagher, Problem 2.12

The range in apparent magnitude for Figure 2.16 was chosen to separate stars of the thin disk cleanly from those in the halo. To see why this works, use Figure 2.2 to represent the stars of the local disk, and assume that the color-magnitude diagram for halo stars is similar to that of the metal-poor globular cluster M92, in Figure 2.14

(a) What is the absolute magnitude M_V of a disk star at $B - V = 0.4$? How far away must it be to have $m_V = 20$? In M92, the bluest stars still on the main sequence have $B - V \approx 0.4$. Show that, if such a star has apparent magnitude $m_V = 20$, it must be at $d \approx 20$ kpc.

(b) What absolute magnitudes M_V could a disk star have, if it has $B-V = 1.5$? How far away would that star be at $m_V = 20$? In M92, what is M_V for the reddest stars, with $B - V \approx 1.2$? How distant must these stars be if $m_V = 20$?

(c) Explain why the reddest stars in Figure 2.16 are likely to belong to the disk, while the bluest stars belong to the halo.

Answers (absolute magnitudes, distances are approximate):

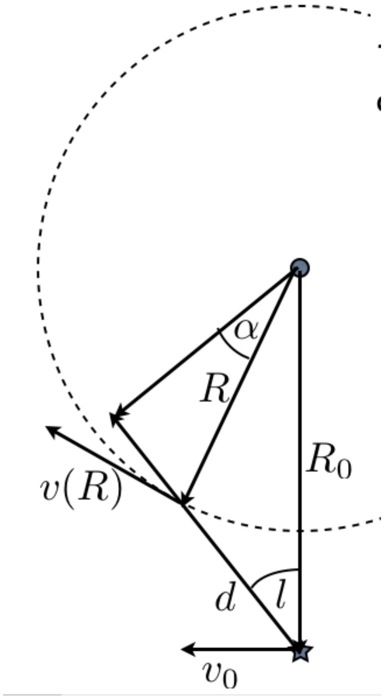
a. Disk star: $M_V \approx +3$ at $B - V \approx 0.4$. For $m_V = 20$, distance modulus $m - M \approx 17$, so $d \approx 25$ kpc.
M92: For $B - V = 0.4$, $M_V \approx +4$. For $m_V = 2-$, we have $m - M \approx 16$ so $d \approx 16$ kpc.

b. For $B - V = 1.5$, a disk star could have $M_V \approx +10$ (if on the main sequence) or $M_V \approx -1$ (if it is a giant). For $m_V = 20$, the distance moduli would be $m - M \approx 10$ (MS) or $m - M \approx 21$ (RG). Distances are thus ≈ 1 kpc or ≈ 160 kpc.
In M92, the reddest stars have $M_V \approx -2$, so for $m_V = 20$ we get $m - M \approx 22$ and $d \approx 250$ kpc.

c. If the reddest stars were halo stars, they would have to be either dwarfs (main sequence stars) or giants. In the former case, the distances would be ≈ 1 kpc, but at these small distances disk stars dominate. In the latter case, the distances would have to be ≈ 250 kpc. At such distances (≈ 5 times the distance to the Large Magellanic Cloud!) the density of halo stars will be extremely low. It is therefore much more likely that these stars are dwarf disk stars with distances of ≈ 1 kpc.
From (a), the bluest stars would have distances that are, in any case, too large for disk membership. However, their apparent magnitudes and colours are quite consistent with them being halo stars near the turn-off at $d \approx 16$ kpc.

4 Werkcollege, Sterrenstelsels, Week 4 - Hints and Solutions

4.1 Oort constants



- The derivation is largely equivalent to that for v_r given in the lecture slides. From the figure above we have

  $$v_t = v(R)\sin\alpha - v_0\cos l$$

  For $\Omega = v/R$ we then have

  $$v_t = \Omega(R)R\sin\alpha - \Omega_0 R_0\cos l$$

  Noting that

  $$R\sin\alpha = R_0\cos l - d$$

  we have

  $$\begin{aligned} v_t &= \Omega(R)(R_0\cos l - d) - \Omega_0 R_0\cos l \\ &= [\Omega(R) - \Omega_0]\,R_0\cos l - \Omega(R)d \end{aligned}$$

- As in the expression for v_t, we first make a first-order Taylor expansion of Ω:

$$\Omega(R) - \Omega_0 \simeq \left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)_{R_0} (R - R_0)$$

  where

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) = \left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)\frac{1}{R} - \frac{v}{R^2}$$

  Then

$$v_t \simeq \left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0}\frac{1}{R_0} - \frac{v_0}{R_0^2}\right](R - R_0)R_0 \cos l - \Omega d$$

  Using

$$R - R_0 \approx -d\cos l$$

  (the negative sign because $l = 0$ in the direction of the Galactic centre) this becomes

$$v_t \simeq \left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] d\cos^2 l - \Omega d$$

  To first order, we can approximate $\Omega \approx \Omega_0$ in the last term, i.e.

$$v_t \simeq \left[\left(\frac{\mathrm{d}v}{\mathrm{d}R}\right)_{R_0} - \frac{v_0}{R_0}\right] d\cos^2 l - \Omega_0 d$$

- From the definition of the Oort constants and the above,

$$v_t \simeq 2Ad\cos^2 l - (A - B)d$$

  We now use

$$\cos^2 l + \sin^2 l = 1$$

  and find

$$\begin{aligned} v_t &\simeq 2Ad\cos^2 l - Ad(\cos^2 l + \sin^2 l) + Bd \\ &\simeq Ad\cos^2 l - Ad\sin^2 l + Bd \end{aligned}$$

  Then, using

$$\cos^2 l - \sin^2 l = \cos 2l$$

  we can write

$$v_t \simeq Ad\cos 2l + Bd$$

4.2 Kinematic distance

- We have:

$$v_r = [\Omega(R) - \Omega(R_0)]\, R_0 \sin l$$

In this case

$$\sin l = \sin 150^\circ = 0.5$$

$$\Omega(R_0) = 200\,\mathrm{km\,s^{-1}}/10\,\mathrm{kpc} = 20\ \mathrm{km\ s^{-1}\ kpc^{-1}}$$

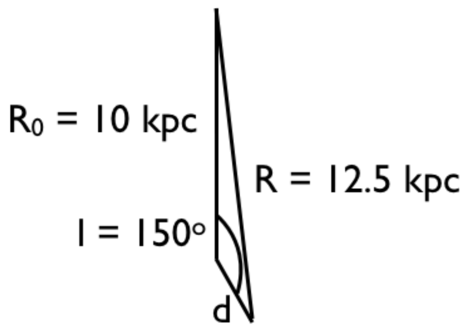
$$v_r = -20\,\mathrm{km\ s^{-1}}$$

We can the solve for

$$\Omega(R) = v_{\mathrm{circ}}/R = v_r/(R_0 \sin l) + \Omega_0 = 16\,\mathrm{km\ s^{-1}\ kpc^{-1}}$$

so

$$\boxed{\mathrm{R} = 12.5\ \ \mathrm{kpc}}$$



- This is now pure geometry. We have a triangle with one known angle and two known sides. Call these $C = 150^\circ$ and $a = 10$, $c = 12.5$. (note: the angle l is shown with the wrong sign in the figure). We then have

$$c^2 = a^2 + b^2 - 2ab\cos C$$

which we need to solve for b

$$12.5^2 = 10^2 + b^2 - 2 \times 10 \times \cos 150^\circ \times b$$

$$b^2 + 17.32b - 56.25 = 0.$$

i.e.

$$b = \frac{-17.32 \pm \sqrt{17.32^2 + 4 \times 56.25}}{2}$$

so distance $\boxed{d = 2.8\ \mathrm{kpc}}$

5 Werkcollege, Sterrenstelsels, Week 5 - Hints and Solutions

5.1 Motions of stars near the Sun

Assume that a star's orbit is described by an epicycle whose centre follows a circular orbit with radius $R_0 + \rho$. At this radius, the circular velocity is v_c, and the specific angular momentum thus

$$J = v_c \times (R_0 + \rho)$$

Currently, the star is observed at radius R_0. From conservation of angular momentum, its velocity in the tangential direction is then

$$v_\theta = v_c \times (R_0 + \rho)/R_0$$

The star's velocity in the tangential direction, relative to a circular orbit with radius R_0, is thus

$$\begin{aligned} v_\chi &= v_\theta - v_c(R_0) \\ &= v_c(R_0 + \rho) \times (R_0 + \rho)/R_0 - v_c(R_0) \end{aligned}$$

Now, to first order

$$v_c(R_0 + \rho) \approx v_c(R_0) + \frac{\partial v_c}{\partial \rho}\rho$$

We insert this above:

$$\begin{aligned} v_\chi &\approx \left(v_c(R_0) + \frac{\partial v_c}{\partial \rho}\rho\right)\frac{R_0 + \rho}{R_0} - v_c(R_0) \\ &= v_c(R_0)\left(\frac{R_0 + \rho}{R_0}\right) + \frac{\partial v_c}{\partial \rho}\rho\left(\frac{R_0 + \rho}{R_0}\right) - v_c(R_0) \\ &= \rho\frac{v_c(R_0)}{R_0} + \rho\frac{\partial v_c}{\partial \rho} + \rho\frac{\rho}{R_0}\frac{\partial v_c}{\partial \rho} \end{aligned}$$

For $\rho \ll R_0$, we have

$$v_\chi \approx \rho\left(\frac{\partial v_c}{\partial \rho} + \frac{v_c}{R}\right) = -2B\rho$$

5.2 Epicycle motion of the Sun

- *Derivation of oscillation amplitude assuming the Sun is at the midpoint:* Oscillation around midpoint in the radial direction: $\rho(t) = A_R \sin \kappa t$ where

A_R is the amplitude of the oscillation, which we want to determine and κ is the epicyclic frequency (which follows from the Oort constants). Derivative $\dot{\rho}(t) = A_R \kappa \cos \kappa t$. Define $t = 0$ at midpoint so that $\rho(0) = 0$. Then $\dot{\rho}(0) = -9$ km/s for the Sun and we can solve for $A_R = \dot{\rho}(0)/\kappa$. For epicyclic frequency $\kappa = 2\pi(170 \times 10^6 \text{ years})^{-1}$ we get $A_R = 250$ pc.

5.3 Chemical evolution

Start with

$$\delta Z = -p\delta M_g / M_g \tag{5.3.1}$$

which is valid also for non-constant yield. For $p(Z) = 0.002 + 0.6\,Z$, we can write

$$(0.002 + 0.6\,Z)^{-1} \delta Z = -M_g^{-1} \delta M_g \tag{5.3.2}$$

Now integrate

$$\int_0^Z (0.002 + 0.6\,Z')^{-1}\, \mathrm{d}Z' = -\int_{M_{g,0}}^{M_{g,t}} M_g^{-1}\, \mathrm{d}M_g \tag{5.3.3}$$

The right-hand side, as before, is

$$-\int_{M_{g,0}}^{M_{g,t}} M_g^{-1}\, \mathrm{d}M_g = -\ln M_{g,t}/M_{g,0} = -\ln f_{\rm gas} \tag{5.3.4}$$

For the left-hand side, substitute

$$u = 0.002 + 0.6Z' \tag{5.3.5}$$

$$\mathrm{d}u = 0.6\,\mathrm{d}Z' \tag{5.3.6}$$

so that

$$\int_0^Z (0.002 + 0.6\,Z')^{-1}\, \mathrm{d}Z' = \frac{1}{0.6} \int_{u(0)}^{u(Z)} u^{-1}\, \mathrm{d}u \tag{5.3.7}$$

$$= \frac{1}{0.6} \ln \frac{0.002 + 0.6Z}{0.002} \tag{5.3.8}$$

Then,

$$-\ln f_{\rm gas} = \frac{1}{0.6} \ln \frac{0.002 + 0.6Z}{0.002} \tag{5.3.9}$$

Now use the definition of the effective yield:

$$p_{\rm eff} = -Z/\ln f_{\rm gas} \tag{5.3.10}$$

$$= 0.6 \frac{Z}{\ln \frac{0.002+0.6Z}{0.002}} \tag{5.3.11}$$

$$= \frac{0.6Z}{\ln(1 + 300\,Z)} \tag{5.3.12}$$

7 Werkcollege, Sterrenstelsels, Week 7 - Hints and Solutions

7.1 Sparke & Gallagher, problem 4.10

Suppose that the inflow of gas (δM_a) is proportional to the rate of star formation (δM_s), i.e., $\delta M_a = \nu \delta M_s$ for $\nu > 0$. Then we have

$$\delta M_g = \delta M_a - \delta M_s = \nu \delta M_s - \delta M_s \tag{7.1.1}$$

for the change in the amount of gas or

$$\delta M_s + \delta M_g = \nu \delta M_s \tag{7.1.2}$$

Eq. 4.14 in S&G is the same relation derived in the viewgraphs,

$$\delta Z = \frac{1}{M_g}\left(p\delta M_s - Z\delta M_s - Z\delta M_g\right) \tag{7.1.3}$$

which is generally valid. From (7.1.2) we have

$$\delta M_s = \delta M_g/(\nu - 1) \tag{7.1.4}$$

$$\begin{aligned}
\delta Z &= \frac{1}{M_g}\left(p\delta M_g/(\nu-1) - Z\delta M_g/(\nu-1) - Z\delta M_g\right) && (7.1.5)\\
&= \frac{1}{M_g}\left(\frac{p-Z}{\nu-1}\delta M_g - \frac{\nu-1}{\nu-1}Z\delta M_g\right) && (7.1.6)\\
&= \frac{1}{M_g}\left(\frac{p-Z-Z(\nu-1)}{\nu-1}\delta M_g\right) && (7.1.7)\\
&= \frac{1}{M_g}\left(\frac{p-Z-Z(\nu-1)}{\nu-1}\delta M_g\right) && (7.1.8)\\
\delta Z &= \frac{p-Z\nu}{\nu-1}\frac{\delta M_g}{M_g} && (7.1.9)
\end{aligned}$$

which is the desired relation. The metallicity of the gas can now again be found by integration:

$$\frac{\nu-1}{p-\nu Z}\delta Z = \frac{\delta M_g}{M_g} \tag{7.1.10}$$

so

$$\int_0^Z \frac{\nu-1}{p-\nu Z'}\mathrm{d}Z' = \int_{M_g(0)}^{M_g(t)} (M_g')^{-1}\mathrm{d}M_g' \tag{7.1.11}$$

The right-hand side is straight forward,

$$\int_{M_g(0)}^{M_g(t)} (M_g')^{-1} \mathrm{d}M_g' = \ln\left(\frac{M_g(t)}{M_g(0)}\right) \tag{7.1.12}$$

For the left-hand side, substitute

$$u = p - \nu Z' \tag{7.1.13}$$

so

$$\mathrm{d}Z' = -\mathrm{d}u/\nu \tag{7.1.14}$$

then

$$\int_0^Z \frac{\nu - 1}{p - \nu Z'} \mathrm{d}Z' = -\frac{\nu - 1}{\nu} \int_p^{p-\nu Z} u^{-1} \mathrm{d}u \tag{7.1.15}$$

$$= \frac{1-\nu}{\nu} \ln\left(\frac{p - \nu Z}{p}\right) \tag{7.1.16}$$

Combining the left-hand and right-hand sides, we have

$$\ln\left(\frac{M_g(t)}{M_g(0)}\right) = \frac{1-\nu}{\nu} \ln\left(\frac{p - \nu Z}{p}\right) \tag{7.1.17}$$

i.e.

$$\left(\frac{M_g(t)}{M_g(0)}\right)^{\nu/(1-\nu)} = \frac{p - \nu Z}{p} = 1 - \frac{\nu Z}{p} \tag{7.1.18}$$

so that

$$Z = \frac{p}{\nu}\left[1 - \left(\frac{M_g(t)}{M_g(0)}\right)^{\nu/(1-\nu)}\right] \tag{7.1.19}$$

7.2 Mass of the Local Group

- Since $m \ll M$, we treat the star as stationary. The kinetic energy is then

  $$T = \frac{1}{2} m v^2 \tag{7.2.1}$$

  where v is the orbital velocity of the planet, which is given by

  $$v^2 = \frac{GM}{r} \tag{7.2.2}$$

  for a circular orbit with radius r. Combining, we find

  $$T = \frac{1}{2}\frac{GMm}{r} \tag{7.2.3}$$

The potential energy is

$$U = -\frac{GMm}{r} \tag{7.2.4}$$

and we thus see that

$$T = -\frac{1}{2}U \tag{7.2.5}$$

- The mass of a shell of thickness dr and radius r is

$$\mathrm{d}M = 4\pi r^2 \rho \mathrm{d}r \tag{7.2.6}$$

were ρ is the density. To calculate the potential energy of the sphere, we "remove" layers starting from the outside. Hence, the potential energy of a shell is

$$\mathrm{d}U(r) = -G\frac{M(r)\mathrm{d}M}{r} \tag{7.2.7}$$

For a uniform sphere,

$$M(r) = \frac{4}{3}\pi r^3 \rho \tag{7.2.8}$$

so

$$\mathrm{d}U(r) = -G\frac{\left(\frac{4}{3}\pi r^3 \rho\right)\left(4\pi r^2 \rho \mathrm{d}r\right)}{r} = -\frac{1}{3}G(4\pi)^2\rho^2 r^4 \mathrm{d}r \tag{7.2.9}$$

Integrating over all r, we find

$$U(R) = -\frac{1}{3}G(4\pi)^2\rho^2 \int_0^R r^4 \mathrm{d}r = -\frac{(4\pi)^2 G\rho^2}{15}R^5 \tag{7.2.10}$$

Now eliminate the density:

$$\rho = \frac{3M}{4\pi R^3} \tag{7.2.11}$$

to find

$$U = -\frac{3}{5}\frac{GM^2}{R} \tag{7.2.12}$$

- Using the virial theorem, we have

$$M\langle v^2\rangle = \frac{3}{5}\frac{GM^2}{R} \tag{7.2.13}$$

or

$$M = \frac{5}{3}\frac{\langle v^2\rangle R}{G} \tag{7.2.14}$$

- Inserting the numbers, one finds $M = 5.1 \times 10^{12} M_\odot$. This agrees quite well with the estimate from the relative motion of the Milky Way and M31, considering the crudeness of the approximations.

8 Werkcollege, Sterrenstelsels, Week 8 - Hints and Solutions

8.1 Two-body relaxation

- The acceleration depends on where exactly the particle is with respect to the deflecting mass. If $l = Vt$ is the distance from the closest encounter, then the total acceleration of the particle is

$$F = \frac{GmM}{r^2} = \frac{GmM}{b^2 + l^2} \tag{8.1.1}$$

The component of this perpendicular to V is

$$F_\perp = F\frac{b}{r} = \frac{GmM}{b^2 + V^2t^2}\frac{b}{\sqrt{b^2 + V^2t^2}} \tag{8.1.2}$$

$$= \frac{GmMb}{(b^2 + V^2t^2)^{3/2}} \tag{8.1.3}$$

The total velocity change is found by integrating over all l, i.e.

$$v_\perp = \int_{-\infty}^{\infty} a_\perp \mathrm{d}t = \int_{-\infty}^{\infty} \frac{Gmb}{(b^2 + V^2t^2)^{3/2}}\mathrm{d}t \tag{8.1.4}$$

$$= 2\frac{Gm}{Vb} \tag{8.1.5}$$

- We use

$$t_{\mathrm{rel}} = \frac{\langle V^2\rangle^{3/2}}{8\pi G^2 m^2 n \ln\Lambda} \tag{8.1.6}$$

and

$$M = \eta\frac{\langle V^2\rangle R}{G} \tag{8.1.7}$$

$$\langle V^2\rangle = \frac{GM}{\eta R} \tag{8.1.8}$$

Eliminating $\langle V^2\rangle$, we find

$$t_{\mathrm{rel}} = \frac{\left(\frac{GM}{\eta R}\right)^{3/2}}{8\pi G^2 m^2 n \ln\Lambda} \tag{8.1.9}$$

The mean density is

$$\rho = nm = \frac{M}{\frac{4}{3}\pi R^3} \tag{8.1.10}$$

so that we can also eliminate nm:

$$t_{\rm rel} = \frac{\left(\frac{GM}{\eta R}\right)^{3/2}}{8\pi G^2 m \ln\Lambda}\frac{\frac{4}{3}\pi R^3}{M} \tag{8.1.11}$$

$$= \frac{\left(\frac{GM}{\eta R}\right)^{3/2}}{6G^2 m \ln\Lambda}\frac{R^3}{M} \tag{8.1.12}$$

$$= \frac{M^{1/2}\eta^{-3/2}R^{3/2}}{6G^{1/2}m\ln\Lambda} \tag{8.1.13}$$

- - Open cluster: $t_{\rm rel} \approx 19$ Myr
  - Globular cluster: $t_{\rm rel} \approx 3.6$ Gyr
  - Elliptical galaxy: $t_{\rm rel} \approx 10^{17}$ yr

8.2 Spiral structure and Lindblad resonances

- ILR: where $\Omega_{LP} = \Omega_{GP}$, i.e. $\Omega - \frac{1}{2}\kappa = \Omega_{GP}$. We infer Ω_{GP} from the corotation radius (5 kpc), i.e. $\Omega_{GP} = 250$ km s^{-1} / (5 kpc) = 50 km s^{-1} kpc^{-1}.

  We have $\Omega = v/R$ and $\kappa^2 = -4B(A-B)$. $A = -\frac{1}{2}\left(\frac{dv_c}{dR} - \frac{v_c}{R}\right)$ and $B = -\frac{1}{2}\left(\frac{dv_c}{dR} + \frac{v_c}{R}\right)$.

  Flat part of rotation curve: $dv_c/dR = 0$ so $B = -\frac{1}{2}\frac{v_c}{R}$ and $A = \frac{1}{2}\frac{v_c}{R}$, i.e. $\kappa^2 = 2\frac{v_c}{R}\frac{v_c}{R}$. Hence, for ILR we require $\Omega_{GP} = \frac{v_c}{R} - \frac{1}{2}\sqrt{2}\frac{v_c}{R}$ i.e. $\Omega_{\rm GP} = \frac{v_c}{R}(1-\sqrt{1/2})$. For $\Omega_{GP} = 50$ km s^{-1} kpc^{-1} and $v_c = 250$ km s^{-1} we find $\boxed{R_{\rm ILR} = 1.5 \text{ kpc}}$.

- For circular motion, $M = \frac{Rv^2}{G}$, so for constant $v = v_c$ we have $\frac{dM}{dR} = \frac{v_c^2}{G}$. We also have $dM = 4\pi R^2\rho\, dR$. Then $\frac{v_c^2}{G} = 4\pi R^2 \rho$ so $\boxed{\rho(R) = \frac{v_c^2}{4\pi G R^2}}$

9 Werkcollege, Sterrenstelsels, Week 9 - Hints and Solutions

9.1 Tully-Fisher / Faber-Jackson relations

1. The calculation goes exactly as for the T-F relation, except that V is replaced with σ: From
$$M \propto \frac{\sigma^2 R}{G} \tag{9.1.1}$$
and defining the mass-to-light ratio, $\Upsilon = M/L$, we get
$$L \propto \frac{\sigma^2 R}{\Upsilon G} \tag{9.1.2}$$
For intensity
$$I \propto L/R^2 \tag{9.1.3}$$
we get
$$L \propto \frac{\sigma^4}{G\Upsilon I} \tag{9.1.4}$$
i.e., if Υ and I are constant we get the desired relation. These are the same assumptions that lead to the T-F relation for spiral galaxies.

2. The F-J relation contains relates the two quantities σ and L to each other. σ can be measured without knowledge of the distance to a galaxy, and this gives L. If we then measure the flux F received from the galaxy, we can derive the distance via the inverse square law, $F = L/(4\pi D^2)$ for distance D.

9.2 Elliptical galaxies, mergers and globular clusters

9.3 Projected axis ratio for prolate spheroid

If we adopt the x-axis as the major axis, then the surface of a prolate spheroid is given by
$$m^2 = \frac{x^2}{A^2} + \frac{y^2 + z^2}{B^2} \tag{9.3.1}$$
where $A > B$. For an observer located in the $x - z$ plane, the projected semi-minor axis b is then
$$m^2 = \frac{b^2}{B^2} \Rightarrow b = mB \tag{9.3.2}$$

Going through the same calculation that gave us the projected semi-minor axis for the oblate spheroid, we now get the projected semi-*major* axis for the prolate spheroid (see slides and S&G Fig. 6.8):

$$a = OQ \sin i = \frac{1}{z} B^2 m^2 \sin i \tag{9.3.3}$$

so that

$$q = \frac{b}{a} = \frac{zmB}{B^2 m^2 \sin i} = \frac{z}{Bm \sin i} \tag{9.3.4}$$

In order to eliminate m and z, we use

$$m^2 = \frac{x^2}{A^2} + \frac{z^2}{B^2} \Rightarrow \tag{9.3.5}$$

$$\frac{m^2 B^2}{z^2} = \frac{B^2}{A^2}\frac{x^2}{z^2} + 1 \Rightarrow \tag{9.3.6}$$

$$\frac{Bm}{z} = \left(\frac{B^2}{A^2}\frac{x^2}{z^2} + 1\right)^{1/2} \tag{9.3.7}$$

and

$$\tan i = -\frac{A^2}{B^2}\frac{z}{x} \Rightarrow \frac{x}{z} = -\frac{A^2}{B^2} \cot i \tag{9.3.8}$$

Inserting this in (9.3.4), we find

$$q = \left(\frac{B^2}{A^2}\frac{x^2}{z^2} + 1\right)^{-1/2} \frac{1}{\sin i} \tag{9.3.9}$$

$$= \left(\frac{B^2}{A^2}\frac{A^4}{B^4} \cot^2 i + 1\right)^{-1/2} \frac{1}{\sin i} \tag{9.3.10}$$

$$= \left(\frac{A^2}{B^2}\frac{\cos^2 i}{\sin^2 i} + 1\right)^{-1/2} \frac{1}{\sin i} \tag{9.3.11}$$

or

$$q^2 = \left(\frac{A^2}{B^2}\frac{\cos^2 i}{\sin^2 i} + 1\right)^{-1} \frac{1}{\sin^2 i} \tag{9.3.12}$$

$$= \left[\left(\frac{A^2}{B^2}\frac{\cos^2 i}{\sin^2 i} + 1\right) \sin^2 i\right]^{-1} \tag{9.3.13}$$

$$= \left(\frac{A^2}{B^2} \cos^2 i + \sin^2 i\right)^{-1} \tag{9.3.14}$$

(Note: this is slightly different than Eq. (6.10) in S&G. Here we have kept the same meaning of i as in Fig. 6.8 and kept A and B as the major and minor axes.).

We can now proceed to find the distribution of observed q values:

$$\frac{\mathrm{d}N}{\mathrm{d}q} = \frac{\mathrm{d}N}{\mathrm{d}i}\frac{\mathrm{d}i}{\mathrm{d}q} \tag{9.3.15}$$

Now, since there is only one way to view a prolate spheroid along the major axis, but 2π ways to view it along the minor axis, we have

$$\mathrm{d}N \propto \cos i \, \mathrm{d}i \tag{9.3.16}$$

and

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}i} &= \frac{\mathrm{d}}{\mathrm{d}i}\left(\frac{A^2}{B^2}\cos^2 i + \sin^2 i\right)^{-1/2} && (9.3.17)\\
&= -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}i}\frac{A^2}{B^2}\cos^2 i + \sin^2 i}{\left(\frac{A^2}{B^2}\cos^2 i + \sin^2 i\right)^{-3/2}} && (9.3.18)\\
&= -\frac{q^3}{2}\frac{\mathrm{d}}{\mathrm{d}i}\left(\frac{A^2}{B^2}\cos^2 i + \sin^2 i\right) && (9.3.19)\\
&= -\frac{q^3}{2}\left(-2\frac{A^2}{B^2}\cos i \sin i + 2\sin i \cos i\right) && (9.3.20)\\
&= q^3 \cos i \sin i \left(\frac{A^2}{B^2} - 1\right) && (9.3.21)
\end{aligned}$$

so that

$$\frac{\mathrm{d}N}{\mathrm{d}q} = \left(q^3(A^2/B^2 - 1)\sin i\right)^{-1} \tag{9.3.22}$$

From

$$q^{-2} = \frac{A^2}{B^2}\cos^2 i + \sin^2 i \tag{9.3.23}$$

we find

$$\begin{aligned}
q^{-2} &= \frac{A^2}{B^2}(1 - \sin^2 i) + \sin^2 i && (9.3.24)\\
&= A^2/B^2 + (1 - A^2/B^2)\sin^2 i && (9.3.25)\\
\sin^2 i &= (q^{-2} - A^2/B^2)(1 - A^2/B^2)^{-1} && (9.3.26)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}q} &= \left(q^3(A^2/B^2 - 1)(q^{-2} - A^2/B^2)^{1/2}(1 - A^2/B^2)^{-1/2}\right)^{-1} && (9.3.27)\\
&= -\left(q^3(1 - A^2/B^2)^{1/2}(q^{-2} - A^2/B^2)^{1/2}\right)^{-1} && (9.3.28)\\
&= -\frac{1}{q^3\sqrt{(1 - A^2/B^2)(q^{-2} - A^2/B^2)}} && (9.3.29)
\end{aligned}$$