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Advanced Plasma - Plasma Physics for Fusion: Assignment - James Davies

Question 1

Consider a current, I, being carried by a copper rod in the z-direction. This will produce a magnetic field

$$B = \frac{\mu_0 I}{2\pi R} \underline{e}_\phi$$

in the toroidal direction. This system is shown in figure 1:

Info for figure 1: a.) A current in the $\underline{e}_z$ direction at $R=0$ leads to a toroidal field. b.) Illustration of the toroidal field strength dependence on R

a.) Suppose 1MA of current flows through the central rod, find the magnetic field strength at $R = 1m$ and use this to calculate the larmor radius, r_L, and ~~the~~ ~~gy~~ gyro-frequency, Ω, for an electron with temperature 10keV and an alpha with temperature 3.5MeV. You may assume $V_\perp = V_{th} = \sqrt{2k_B T_s / m}$ and $m_\alpha = 4m_p$, where m_p is the proton mass.

$$B = \frac{1.257\times10^{-6} \times 1\times10^{6}}{2\pi \times 1} = \underline{\underline{0.2\,T}}$$

$$\Omega = \frac{ZeB}{m}$$

For electron: $\Omega = \frac{1.6\times10^{-19} \times 0.2}{9.11\times10^{-31}} = 3.51\times10^{10}\,Hz = \underline{\underline{35.1\,GHz}}$

For α-particle $\Omega = \frac{2\times1.6\times10^{-19} \times 0.2}{4\times1.67\times10^{-27}} = 9.58\times10^{6}\,Hz = \underline{\underline{9.58\,MHz}}$

$V_\perp = V_{th} = \sqrt{2k_B T_k / m}$ $\qquad T(\text{keV}) = 8.62\times10^{-8}\, T(k)$

for electron:

$$V_\perp = \sqrt{2(1.38\times10^{-23})\left(\frac{10}{8.62\times10^{-8}}\right) / 9.11\times10^{-31}} = 59.3\times10^{6}\ ms^{-1}$$

for α-particle $\quad V_\perp = \sqrt{2(1.38\times10^{-23})\left(\frac{3500}{8.62\times10^{-8}}\right) / (4\times1.67\times10^{-27})} = 13.0\times10^{6}\ ms^{-1}$

$$r_L = \frac{V_\perp}{\Omega}$$

for electron:

$$r_L = \frac{59.3\times10^{6}}{3.51\times10^{10}} = \underline{\underline{1.69\times10^{-3}\ m}}$$

for ~~electr~~ α-particle:

$$r_L = \frac{13.0\times10^{6}}{9.58\times10^{6}} = \underline{\underline{1.36\ m}}$$

b.) In this system the ∇-B and curvature drift velocities maybe combined and written as:

$$\underline{V_D} = \frac{m}{RZeB}\left(V_\parallel^2 + \frac{V^2}{2}\right)\underline{e}_z$$

This combined drift, V_D, is often refered to as simply the magnetic drift.

i.) Assuming $v_\parallel^2 + v_\perp^2/2 = v_{th}^2$, calculate this drift velocity for electrons and Deuterium ions ($m_D = 2m_p$) at $R = 1$m with 10keV thermal energy.

$$V_D = \frac{m}{RZeB}\left(v_{th}^2\right)$$

for electrons:

$$V_D = \frac{9.11\times10^{-31}}{1\times1.6\times10^{-19}\times0.2}\left((59.3\times10^{6})^2\right) = \underline{\underline{1.00\times10^{5}\ ms^{-1}}}$$

for ions:

$$V_{th}^2 = \left(2(1.38\times10^{-23})\left(\frac{10}{8.62\times10^{-8}}\right)\right) \Big/ \left(2\times1.67\times10^{-27}\right)$$

$$= 9.59\times10^{11}$$

$$V_D = \frac{2\times1.67\times10^{-27}}{1\times2\times1.6\times10^{-19}\times0.2}\left(9.59\times10^{11}\right) = \underline{\underline{5.00\times10^{4}\ ms^{-1}}}$$

ii.) In a terrestrial system, such as the one under consideration here, there will be a gravitational force, mg, acting on the plasma species in the $-e_z$ direction. This force will lead to a time independent particles drift. Starting from the equation of motion:

$$m\frac{dv}{dt} = -mg\underline{e_z} + Ze(\underline{v}\times\underline{B})$$

derive an expression for the perpendicular velocity of the drift due to the gravitational force, $\underline{V}_{g\perp}$. You may find the following vector identity useful:

$$(\underline{A}\times\underline{B})\times\underline{C} = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{B}\cdot\underline{C})\underline{A}$$

$$m\frac{d\underline{V}}{dt} = -mg\underline{e}_z + Ze(\underline{V}\times\underline{B})$$

Split $\underline{V}$ into $\underline{V}_{gyro} + \underline{V}_g$:

$$\underline{V} = \underline{V}_{gyro} + \underline{V}_g$$

Looking at $\underline{V}_{gyro}$:

$$m\frac{d\underline{V}_{gyro}}{dt} = Ze\,\underline{V}_{gyro}\times\underline{B}$$

Therefore:

$$m\frac{d(\underline{V}_{gyro}+\underline{V}_g)}{dt} = -mg\underline{e}_z + Ze\underline{V}_{gyro}\times\underline{B} + Ze\underline{V}_g\times\underline{B}$$

$$m\frac{d\underline{V}_g}{dt} = -mg\underline{e}_z + Ze\underline{V}_g\times\underline{B}$$

Since acceleration due to gravity is constant: $\frac{d\underline{V}_g}{dt} = 0$

$$\therefore\ mg\underline{e}_z = Ze\underline{V}_g\times\underline{B}$$

Crossing Both Sides with $\underline{B}$

$$mg\underline{e}_z\times\underline{B} = Ze(\underline{V}_g\times\underline{B})\times\underline{B}$$

Using $(\underline{A}\times\underline{B})\times\underline{C} = (\underline{A}\cdot\underline{C})\underline{B} - (\underline{B}\cdot\underline{C})\underline{A}$ on $(\underline{V}_g\times\underline{B})\times\underline{B}$:

$$(\underline{V}_g\times\underline{B})\times\underline{B} = (\underline{V}_g\cdot\underline{B})\underline{B} - (\underline{B}\cdot\underline{B})\underline{V}_g = B^2\left(\frac{(\underline{V}_g\cdot\underline{B})\underline{B}}{B^2} - \underline{V}_g\right)$$

If $\underline{V}_g$ is perpendicular to $\underline{B}$ then $(\underline{V}_g\cdot\underline{B}) = 0$

$$\boxed{\therefore\ \underline{V}_g = \frac{-mg\underline{e}_z\times\underline{B}}{ZeB^2}}$$

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iii.) Calculate the magnitude of the gravitational drift velocity for electrons and deuterium ions on the surface of Earth ($g = 9.81 ms^{-2}$). Compare these values to the magnetic drift velocity in part ii.) and discuss the importance of gravitational drift.

$$V_g = \frac{-mgB}{ZeB^2} \quad \text{in } z \text{ direction}$$

$$= -\frac{9.11\times10^{-31} \times 9.81 \times 0.2}{1.6\times10^{-14}(0.2)^2} = -2.79\times10^{-10} ms^{-1} \quad \text{for electrons}$$

$$= -\frac{2\times1.67\times10^{-27} \times 9.81 \times 0.2}{2\times1.6\times10^{-19}\times(0.2)^2} = -\cancel{2.56}\ 5.12\times10^{-7} ms^{-1} \quad \text{for ions}$$

Question 2

The System in question 1 uses a purely toroidal magnetic field. In this question we will consider introducing a poloidal component of the magnetic field, which can be achieved by driving a toroidal current, I_ϕ, at $R = 1m$. The result of this is that the field lines become helical, orbiting about the location of the current as shown in fig 2.

As particles travel along the now helical field lines, they also move in R and Z. To describe this it is convient for us to introduce the poloidal angle, θ, ~~ast~~ as shown in fig 2. As the particle moves along the field line it will move in both the toroidal and poloidal angles.

a) Describe why the magnetic drift means a purely toroidal field, such as that in question 1., cannot confine a plasma indefinitely

A toroidal magnetic field like the one in question 1 would not ~~Cat~~ confine a plasma due to ∇B drift. As B is proportional to $1/R$, there is a grad B inwards, this generates a drift perpendicular to B. This results in a vertical charge seperation of ions + electrons producing an E field. This then produces an $E \times B$ drift to a large radius, resulting in a loss of confinement!

b) The magnitude of the toroidal component of the magnetic field, B_ϕ, depends on R and as such the total field magnitude, B, will vary along the field line. Let us assume that this variation is described by $B = B_0(1 - a\cos\theta)$, where B_0 and a are some constants and θ is the poloidal angle. Starting from the conservation of energy and ~~magnetic~~ magnetic moment show that a particle at $\theta = 0$ with pitch angle λ (defined through $\sin\lambda = V_{\perp,0}/V_0 \neq 0$) will be reflected at a poloidal angle θ satisfying

$$\cos\theta = \frac{1}{a}\left[1 - \frac{1-a}{\sin^2\lambda}\right]$$

Electrostatic potential ϕ may be neglected!

Looking at the conserved quantities:

$$\mathcal{E} = \frac{1}{2}mv^2 + Ze\phi \quad , \quad \mu = \frac{mV_\perp^2}{2B}$$

Since these are conserved and neglecting ϕ

$$\mu = \frac{mV_{\perp,0}^2}{2B_0} = \frac{mV_\perp^2}{2B}$$

There fore: $V_\perp^2 = BV_{\perp,0}^2 / B_0$ ①

Also:

$$\mathcal{E}/m = V_0^2 = V_{z,0}^2 + V_{\perp,0}^2 = V_z^2 + V_\perp^2$$

$$\Rightarrow V_z^2 = V_0^2 - V_\perp^2$$

Subbing in ①

$$V_z^2 = V_0^2 - BV_{\perp,0}^2 / B_0$$

~~[illegible]~~ Defining pitch angle $\sin\lambda = \frac{V_{\perp,0}}{V_0}$

$$\Rightarrow \boxed{V_z^2 = V_0^2\left[1 - \frac{B}{B_0}\sin^2\lambda\right]}$$

So particle to be reflected with θ: $B = \frac{B_0}{\sin^2\lambda} \Rightarrow \sin\lambda = \sqrt{B_0/B}$

Variation: $B = B_c(1 - a\cos\theta)$; $B_0(\theta=0) = B_c(1-a)$

$$\Rightarrow \sin\lambda = \sqrt{(1-a)/(1-a\cos\theta)}$$

Please turn over

$$\sin^2\lambda = \frac{1-a}{1-a\cos\theta}$$

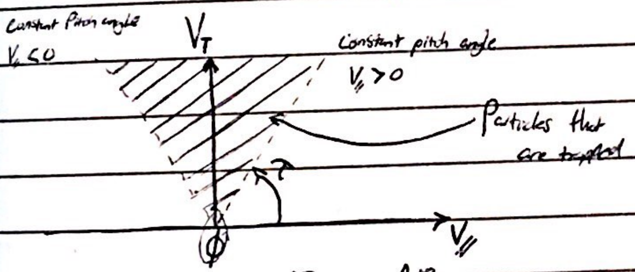
$$1 - a\cos\theta = \frac{1-a}{\sin^2\lambda}$$

$$\Rightarrow \underline{\underline{\cos\theta = \frac{1}{a}\left[1 - \frac{1-a}{\sin^2\lambda}\right]}} \qquad \text{As expected}$$

C.)

A Particle is considered trapped if it is reflected at some point along the field line (i.e. somewhere between $\theta = 0$ and $\theta = \pi$). Assuming a maxwellian velocity distribution, Fn use the result of 2b.) with $a \ll 1$ to show that the ~~fraction~~ fraction of particles which are trapped, f_t is given by:

$$f_t = \sqrt{2a}$$



$$n(v) = A\int_0^{2\pi} d\Phi\, 2\int_\lambda^{\pi/2} \sin\theta\, d\theta \int_0^\infty v^2 \exp(-v^2/v_{th}^2)\, dv$$

$$= 4A\pi\left[-\cos\theta\right]_\lambda^{\frac{\pi}{2}} \underbrace{\int_0^\infty v^2 \exp(-v^2/v_{th}^2)\, dv} \quad \text{use } \int_0^\infty s^2 \exp(-s^2)\, ds$$

$$= +4A\pi\cos\lambda\, V_{th}\left(\frac{\sqrt{\pi}}{4}\right) \quad \text{particles trapped}$$

$$n(v) = A\int_0^{2\pi} d\Phi \int_0^\pi \sin\theta\, d\theta \int_0^\infty v^2 \exp(-v^2/v_{th}^2)\, dv$$

$$= 2\pi A\left[-\cos\theta\right]_0^\pi \int_0^\infty v^2 \exp(-v^2/v_{th}^2)\, dv$$

$$= 4\pi A V_{th}\left(\frac{\sqrt{\pi}}{4}\right) \quad \text{for all particles.} \qquad \Rightarrow \text{fraction} = \cos\lambda$$

Looking at the central angle λ:

$$\sin\lambda = \sqrt{\frac{1-a}{1-a\cos\theta}}$$

At the toroidal limit, poloidal angle $= \pi$

$$\sin\lambda = \sqrt{\frac{1-a}{1+a}} \approx \sqrt{1-2a} \quad \text{for } a \ll 1$$

$$\therefore \lambda = \sin^{-1}\left(\sqrt{1-2a}\right)$$

Subbing into fraction $\Rightarrow$

$$f_t = \cos\left(\sin^{-1}\left(\sqrt{1-2a}\right)\right)$$

Using trig identity: $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$

$$f_t = \sqrt{1-(1-2a)}$$

$$\underline{\underline{f_t = \sqrt{2a}}} \quad \text{as expected!}$$