Section 5

We will cover the fluid treatment of plasmas:

- Moments of the kinetic equations to derive equations for:
  - density in terms of the flow
  - flow in terms of the pressure

- “Closure” – additional physics input to provide an equation for pressure in terms of density

- Combined with Maxwell’s equations for the electromagnetic fields

The difficulty

Earlier, we considered how individual charged particles move in $\boldsymbol{E}$ and $\boldsymbol{B}$ fields

But if charged particles move relative to each other:

1. There is a change in charge density that can modify the electric field being applied

2. The moving charges correspond to currents that can modify the magnetic field being applied

In a plasma, the particle motions and fields are ***not*** independent

The self-consistent problem:

Coupling particle motion and fields

To understand the interaction of many charged particles with electromagnetic fields requires the solution of many coupled equations:

- Equation of motion for each particle:

$$m\frac{d\underline{v}}{dt} = Ze\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

- ***E*** is then given by:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad \rho = \left(Zn_i - n_e\right) e$$

- and ***B*** is given by

$$\underline{\nabla} \times \underline{B} = \mu_0 \left[\underline{J} + \epsilon_0 \frac{\partial \underline{E}}{\partial t}\right]$$

In general, ***E*** and ***B*** calculated from (2) and (3) will not equal those used to calculate particle positions

The fluid approximation

A plasma might typically have ~10^{13} particles in a cubic centimetre (many more for inertial fusion)

$\Rightarrow$ solving for each individual particle's trajectory is clearly a major challenge (impossible in a fusion device like JET)

A great simplification is provided by considering the plasma to be a fluid:

$\Rightarrow$ average over the velocity distribution to produce physical quantities associated with fluids (the "moments")

$f\left(\underline{r},\underline{v}\right)$

- Zeroth moment gives density:

$$n\left(\underline{r}\right) = \oint f\left(\underline{r},\underline{v}\right) d\underline{v}$$

- First moment gives flow:

$$n\left(\underline{r}\right)\underline{u}\left(\underline{r}\right) = \oint \underline{v} f\left(\underline{r},\underline{v}\right) d\underline{v}$$

$$\Longrightarrow n\left(\underline{r}\right) u_i\left(\underline{r}\right) = \oint v_i f\left(\underline{r},\underline{v}\right) d\underline{v}$$

- Second moment is related to pressure: a tensor:

$$\underline{\underline{P}} = m \oint f\left(\underline{r},\underline{v}\right)\left(\underline{v}\underline{v} - \underline{u}\underline{u}\right) d\underline{v}$$

$$v_1 = v_x \quad v_2 = v_y \quad v_3 = v_z$$

- etc

$$P_{ij} = m \oint f\left(\underline{r},\underline{v}\right)\left(v_i v_j - u_i u_j\right) d\underline{v}$$

$$d\underline{v} = v_1 v_2 v_3$$

Note the distinction between the fluid flow, $\boldsymbol{u}$ and the particle velocity $\boldsymbol{v}$. Compare with water in a river:

$\boldsymbol{u}$ is equivalent to the flow of the river

$\boldsymbol{v}$ is equivalent to the velocity of individual water moelcules

The fluid equations describe how these quantities evolve in time

It is an approximation to the real situation: the approximation comes in restricting the number of fluid quantities we consider by a so-called "closure relation" (see later)

Starting point: kinetic theory

We define the number of particles at a given position ***r***, with velocity ***v*** in terms of a distribution function:

$$dn = f\left(\underline{r}, \underline{v}\right) d\underline{v}$$

Two things can modify a distribution function:

In the absence of a source:

$$\frac{df}{dt} = C\left(f\right)$$

Let us neglect collisions

$$\frac{df}{dt} = 0$$

$$f = f\left(\underline{r}, \underline{v}; t\right) = f\left(x, y, z, v_x, v_y, v_z, t\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial f}{\partial v_x}\frac{\partial v_x}{\partial t} + \frac{\partial f}{\partial v_y}\frac{\partial v_y}{\partial t} + \frac{\partial f}{\partial v_z}\frac{\partial v_z}{\partial t}$$

$$\underline{v} = \left(v_x, v_y, v_z\right)$$

$$\underline{a} = \left(a_x, a_y, a_z\right)$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla_v f = \left(\frac{\partial f}{\partial v_x}, \frac{\partial f}{\partial v_y}, \frac{\partial f}{\partial v_z}\right)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f + \frac{1}{m}\underline{F} \cdot \underline{\nabla}_v f = 0$$

$$\underline{F} = m\underline{a}$$

$$\underline{F} = Ze\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

Continuity Equation: example of a fluid equation

$$\frac{\partial f}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) f + \frac{1}{m} (\boldsymbol{F} \cdot \boldsymbol{\nabla}_v) f = 0$$

The density is defined as:

$$n(\underline{r}) = \oint f(\underline{r}, \underline{v})\, d\underline{v}$$

The flow velocity of a fluid element is defined as:

$$n\underline{u} = \oint \underline{v} f(\underline{r}, \underline{v}) \qquad \underline{v} \cdot \underline{\nabla} f = \underline{\nabla} \cdot (f\underline{v})$$

$$\oint \frac{\partial f}{\partial t} d\underline{v} = \frac{\partial}{\partial t} \oint f d\underline{v} = \frac{\partial n}{\partial t}$$

$$\oint \underline{v} \cdot \underline{\nabla} f d\underline{v} = \oint \underline{\nabla} \cdot (f\underline{v})\, d\underline{v} = \underline{\nabla} \cdot \oint (f\underline{v})\, d\underline{v} = \underline{\nabla} \cdot (n\underline{u})$$

$$\underline{\nabla}_v \cdot \underline{F} = Ze\underline{\nabla}_v \cdot (\underline{E} + \underline{v} \times \underline{B}) = Ze\left[\frac{\partial}{\partial v_x}(v_y B_z - v_z B_y) + \cdots\right]$$

$$\oint \boldsymbol{F} \cdot \boldsymbol{\nabla}_v f \; \boldsymbol{dv} = \oint [\boldsymbol{\nabla}_v \cdot (f\boldsymbol{F}) \quad - \quad f\boldsymbol{\nabla}_v \cdot \boldsymbol{F}]\, \boldsymbol{dv}$$

$$\oint \underline{\nabla}_v \cdot (f\underline{F})\, d\underline{v} = \int f\underline{F} \cdot d\underline{S}$$

$\Rightarrow$ continuity equation: $$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n\underline{u}) = 0$$

This represents particle conservation:
Consider a volume V

$$\frac{\partial N}{\partial t} + \oint \underline{\nabla} \cdot (n\underline{u})\, dV = 0$$

$$\frac{\partial N}{\partial t} + \oint_S (n\underline{u}) \cdot d\underline{S} = 0$$

Force balance

$$\frac{\partial f}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla}) f + \frac{1}{m}(\boldsymbol{F}\cdot\boldsymbol{\nabla}_v) f = 0$$

$$\underline{vF}\cdot\underline{\nabla}_v f = \underline{\nabla}_v\cdot(f\underline{vF}) - f\underline{\nabla}_v\cdot(\underline{Fv})$$

$$m\frac{\partial}{\partial t}(n\boldsymbol{u}) + m\boldsymbol{\nabla}\cdot\oint \boldsymbol{vv}f\,d\boldsymbol{v} - \oint f\boldsymbol{\nabla}_v\cdot(\boldsymbol{Fv})\,d\boldsymbol{v} = 0$$

$$[\boldsymbol{\nabla}_v\cdot(\boldsymbol{Fv})]_i = \sum\frac{\partial}{\partial v_i}(v_iF_j) = \sum\delta_{ij}F_j + v_i\boldsymbol{\nabla}_v\cdot\boldsymbol{F} = F_i$$

$$[\underline{\nabla}_v\cdot(\underline{Fv})]_x = \frac{\partial}{\partial v_x}(F_xv_x) + \frac{\partial}{\partial v_y}(F_yv_x) + \frac{\partial}{\partial v_z}(F_zv_x) = F_x + v_x\underline{\nabla}_v\cdot\underline{F}$$

$$\Rightarrow \boldsymbol{\nabla}_v\cdot(\boldsymbol{vF}) = \boldsymbol{F}$$

$$\oint f\boldsymbol{\nabla}_v\cdot(\boldsymbol{Fv}) = \oint f\boldsymbol{F}d\boldsymbol{v} = Ze\oint f(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$$

$$= Ze[n\boldsymbol{E} + n\boldsymbol{u}\times\boldsymbol{B}]$$

$$\Rightarrow m\boldsymbol{u}\frac{\partial n}{\partial t} + mn\frac{\partial\boldsymbol{u}}{\partial t} + m\boldsymbol{\nabla}\cdot\oint\boldsymbol{vv}f\,d\boldsymbol{v} - nZe(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}) = 0$$

Subtract $\boldsymbol{u}\times$ continuity equation:

$$m\boldsymbol{u}\frac{\partial n}{\partial t} + m\boldsymbol{u}\boldsymbol{\nabla}\cdot(n\boldsymbol{u}) = 0$$

$$\Rightarrow m\boldsymbol{u}\frac{\partial n}{\partial t} + m\{n\boldsymbol{\nabla}\cdot(\boldsymbol{uu}) - n(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + [(\boldsymbol{u}\cdot\boldsymbol{\nabla})n]\boldsymbol{u}\} = 0$$

$$mn\left[\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] + \boldsymbol{\nabla}\cdot\underline{\underline{P}} - nZe(\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}) = 0$$

$$\underline{\underline{P}} = m\oint f(\underline{r},\underline{v})[\underline{vv} - \underline{uu}]\,d\underline{v}$$

$$P_{ij} = m\oint f(\underline{r},\underline{v})[v_iv_j - u_iu_j]\,d\underline{v}$$

Force balance (cont.)

For a Maxwellian: $f\left(\underline{r},\underline{v}\right) = \frac{n}{\pi^{3/2}v_{th}^3}\exp\left(-\frac{v^2}{v_{th}^2}\right)$

Pressure tensor:

$$P_{ij} = m\int_{-\infty}^{\infty}dv_1\int_{-\infty}^{\infty}dv_2\int_{-\infty}^{\infty}dv_3\frac{n}{\pi^{3/2}v_{th}^3}\exp\left(-\frac{v_1^2+v_2^2+v_3^2}{v_{th}^2}\right)v_iv_j$$

If $i \neq j$ (e.g. $i=x$, $j=y$) two of the integrals are odd in $\boldsymbol{v} \Rightarrow P_{ij}=0$.
Therefore, we only need to consider cases with $i=j$.
Suppose $i=j=x$ and let $s = v_1/v_{th} \quad t = v_2/v_{th} \quad u = v_3/v_{th}$

$$P_{11} = \frac{mnv_{th}^2}{\pi^{3/2}}\int_{-\infty}^{\infty}ds\,s^2\exp\left(-s^2\right)\int_{-\infty}^{\infty}dt\exp\left(-t^2\right)\int_{-\infty}^{\infty}du\exp\left(-u^2\right)$$

$$P_{11} = \frac{mn}{2}\frac{2k_BT}{m} = nk_BT = p$$

$$P_{ij} = p\delta_{ij}$$

$$\left[\underline{\nabla}\cdot\underline{\underline{P}}\right]_i = \sum_j\frac{\partial}{\partial x_j}P_{ji} = \sum_j\frac{\partial p}{\partial x_j}\delta_{ij} = \frac{\partial p}{\partial x_i} = \left[\underline{\nabla}p\right]_i$$

$$\underline{\nabla}\cdot\underline{\underline{P}} = \underline{\nabla}p$$

$$mn\left[\frac{\partial\underline{u}}{\partial t} + \left(\underline{u}\cdot\underline{\nabla}\right)\underline{u}\right] = -\underline{\nabla}p + nZe\left(\underline{E}+\underline{u}\times\underline{B}\right)$$

Fluid "moments" (no collisions)

We started with the kinetic equation:

$$\frac{\partial f}{\partial t} + (\boldsymbol{v} \cdot \nabla) f + \frac{1}{m} (\boldsymbol{F} \cdot \nabla_v) f = 0$$

$$n = \oint f \, \boldsymbol{d}^3 v \qquad\qquad n\boldsymbol{u} = \oint \boldsymbol{v} f \, \boldsymbol{d}^3 v$$

Zeroth moment [$\oint \cdots \boldsymbol{d}^3 v$] $\Rightarrow$ continuity

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n\underline{u}) = 0$$

First moment [$\oint \boldsymbol{v} \cdots \boldsymbol{d}^3 v$] $\Rightarrow$ force balance

$$mn \left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} \right] = -\underline{\nabla} p + nZe \left(\underline{E} + \underline{u} \times \underline{B} \right)$$

Next moment generates equation for p, but involves third moment, etc.

We need a "closure" rule to close the system (ie an equation for p without introducing a new variable). This is where the approximation comes in,
One approximation, often used, is to assume adiabatic behaviour (cf ideal gas pV^γ=constant; $\gamma = C_p / C_v$=ratio of specific heats)

$$\frac{d}{dt} \left(p n^{-\gamma} \right) = 0 \qquad\qquad \left[\frac{\partial}{\partial t} + \underline{u} \cdot \nabla \right] \left(p n^{-\gamma} \right) = 0$$

$$p = C n^\gamma$$

Full set of 2-fluid equations (ions and e)

Continuity:

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \left(n_j \boldsymbol{u}_j \right) = 0$$

Force balance:

$$m_j n_j \left[\frac{\partial \boldsymbol{u}_j}{\partial t} + \left(\boldsymbol{u}_j \cdot \nabla \right) \boldsymbol{u}_j \right] = -\nabla p_j + n_j q_j \left(\boldsymbol{E} + \boldsymbol{u}_j \times \boldsymbol{B} \right)$$

Adiabatic pressure law (other approximations are used):

$$\left[\frac{\partial}{\partial t} + \left(\boldsymbol{u}_j \cdot \nabla \right) \right] \left(p_j n_j^{-\gamma} \right) = 0$$

Combined with Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0} \left(n_i q_i + n_e q_e \right)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left[n_i q_i \boldsymbol{u}_i + n_e q_e \boldsymbol{u}_e + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right]$$

$\Rightarrow$ Closed set of equations for n_j, $\boldsymbol{u}_j$, p_j, $\boldsymbol{B}$, $\boldsymbol{E}$

What you need to know:

You need to understand that the fundamental plasma physics problem involves solving the Lorentz force balance equation for each particle, together with Mazwell's equations for the E and B fields, which themselves depend on the particle positions (charge density) and motion (currents). You need to appreciate that this full problem is generally not solvable

You must have an understanding of the plasma distribution function, and how to calculate the moments such as density, flow and the pressure tensor. You must be able to distinguish between the fluid flow and the motion of individual particles. You need to learn the definitions of density, flow and pressure in terms of the distribution function

You must learn the Vlasov equation, and be able to describe what it represents

You should be able to derive the continuity equation describing the density in terms of the fluid flow (from the Vlasov equation). You need to learn the continuity equation, and be able to demonstrate that it represents conservation of number of particles

You are not expected to be able to derive the force balance equation, but you should be able to quote it. You should be able to describe the physics of each of the terms.

You should understand the concept of a closure relation, to close the system of fluid equations, and that this is where the approximation comes in. You should learn the adiabatic pressure law as a definite example of a closure.

You should be able to write down Maxwell's equations for the electromagnetic fields in terms of the particle densities and flows.