Geometry of Schemes: Week 5

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Problem 0.1. (II-22) To find the length of the primary component of (xy, y^2) at the origin, we need to find the largest ideal of *finite* length in the ring $(K[x,y]/(xy,y^2))_{(x,y)}$ and find its length.

Note that the ideal $\overline{(x,y)}$ in the ring is *not* of finite length since we can find a chain $\overline{(x,y)} \supset \overline{(x^2,y)} \supset \overline{(x^3,y)} \supset \dots$ in $(K[x,y]/(xy,y^2))_{(x,y)}$.

Also, keep in mind that the ideal we are looking at pulls back to an ideal in $K[x,y]$, say J, such that $(x^2, xy) \subset J \subset (x,y)$.

We claim that the largest ideal of finite length is $\overline{(y)}$. Suppose $\bar{\mathfrak{a}}$ is an ideal of finite length and suppose $\bar{\mathfrak{a}} \not\subseteq \overline{(y)}$. Then its pullback $\mathfrak{a} \in K[x,y]$ is such that $\mathfrak{a} \subset (x,y)$ and must contain a polynomial $f(x,y)$ which consists of monomials of pure powers of x. Otherwise $\mathfrak{a} \subset (y)$. So $f(x,y) \in \mathfrak{a}$ "looks like" $x^n + x^{n-1} + \dots x + yg(x,y)$. Therefore, we can find a chain of arbitrary length: $\bar{\mathfrak{a}} \supset \overline{(f(x,y))} \supset \overline{(f(x,y)^2)} \supset \dots$ (the pure powers of x don't die in the quotient ring). So $\mathfrak{a}$ does not have finite length, a contradiction.