Math 221 Worksheet 3

September 10, 2020

Topics: Section 1.6 - Calculating Limits Using the Limit Laws

1. Given that $\lim_{x\to -4} f(x) = 6$, $\lim_{x\to -4} g(x) = 0$, and $\lim_{x\to -4} h(x) = 1$, find the limits below. If the limit does not exist, explain why.

   (a) $\lim_{x\to -4} [f(x) - 4h(x)]$

   $$= \lim_{x\to -4} f(x) - 4 \lim_{x\to -4} h(x) = 6 - 4\cdot 1 = 2$$

   (b) $\lim_{x\to -4} \frac{g(x)}{3h(x)}$.

   $$= \frac{\lim_{x\to -4} g(x)}{3 \lim_{x\to -4} h(x)} = \frac{0}{3\cdot 1} = 0$$

   (c) $\lim_{x\to -4} \frac{h(x)}{2g(x)}$.

   Does not exist.

   numerator → 1, while denominator → 0.

2. Evaluate each limit and justify each step by indicating the appropriate Limit Laws.

   (a) $\lim_{a\to 2} \frac{a^4 - 8a + 4}{3a^2 + 16}$

   $$\lim_{a\to 2} (3a^2+16) = 3\left(\lim_{a\to 2} a\right)^2 + 16 = 3\cdot 2^2 + 16 = 28 \ (\neq 0)$$

   $$\text{So } \lim_{a\to 2} \frac{a^4 - 8a + 4}{3a^2+16} = \frac{\lim_{a\to 2}(a^4 - 8a + 4)}{\lim_{a\to 2}(3a^2+16)} = \frac{\left(\lim_{a\to 2} a\right)^4 - 8 \lim_{a\to 2} a + 4}{28}$$

   $$= \frac{2^4 - 8\cdot 2 + 4}{28} = \frac{1}{7}$$

   (b) $\lim_{u\to -1} \sqrt{\frac{2u+5}{3u+11}}$.

   $$\lim_{u\to -1} (3u+11) = 3(-1)+11 = 8 \ (\neq 0),$$

   $$\text{so } \lim_{u\to -1} \frac{2u+5}{3u+11} = \frac{\lim_{u\to -1}(2u+5)}{\lim_{u\to -1}(3u+11)} = \frac{2(-1)+5}{3(-1)+11} = \frac{3}{8}$$

   $$\text{So } \lim_{u\to -1} \sqrt{\frac{2u+5}{3u+11}} = \sqrt{\lim_{u\to -1} \frac{2u+5}{3u+11}} = \sqrt{\frac{3}{8}}$$

3. Evaluate the following limit, if it exists. If the limit does not exist, explain why. If you use a theorem, clearly state which theorem you are using.

   (a) $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

   $$= \lim_{x\to 1} \frac{^{\star}\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \lim_{x\to 1} (x^2+x+1) = 3$$

   Synthetic division

   $$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x-1 & x^3 - 1 \\ & -x^3 - x^2 \\ \hline & x^2 - 1 \\ & -x^2 - x \\ \hline & x - 1 \\ & -x - 1 \\ \hline & 0 \end{array} \longleftrightarrow \begin{array}{r|rrrr} 1 & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

   ✩ how to find this?
   - difference of cubes (ew)
   - "factor th^m" says $f(a)=0 \iff f(x)=(x-a)g(x)$

   ↑ find this by long division

(b) $\lim_{v\to\frac{1}{2}} \frac{|2v-1|}{2v-1}$.

$$\lim_{v\to\frac{1}{2}^+} \frac{|2v-1|}{2v-1} = \lim_{v\to\frac{1}{2}^+} \frac{2v-1}{2v-1} = \lim_{v\to\frac{1}{2}^+} 1 = 1$$

$$\lim_{v\to\frac{1}{2}^-} \frac{|2v-1|}{2v-1} = \lim_{v\to\frac{1}{2}^-} \frac{-(2v-1)}{2v-1} = \lim_{v\to\frac{1}{2}^-} -1 = -1$$

Right- and left-hand limits do not agree, so the limit does not exist.

(c) $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right)$

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$ for every $x \neq 0$.

So $-x^4 \le x^4 \cos\left(\frac{1}{x}\right) \le x^4$ since $x^4 \ge 0$.

Since $\lim_{x\to 0} -x^4 = 0 = \lim_{x\to 0} x^4$, the squeeze theorem implies that $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right) = 0$.

(d) $\lim_{u\to -3} \frac{2-\sqrt{u^2-5}}{u+3}$ (Hint: multiply by the conjugate).

$$= \lim_{u\to -3} \frac{2-\sqrt{u^2-5}}{u+3} \cdot \frac{2+\sqrt{u^2-5}}{2+\sqrt{u^2-5}}$$

$$= \lim_{u\to -3} \frac{9-u^2}{(u+3)(2+\sqrt{u^2-5})} = \lim_{u\to -3} \frac{3-u}{(2+\sqrt{u^2-5})}$$

$$= \frac{6}{2+2} = \frac{6}{4} = \frac{3}{2}$$

4. Is there a number a such that $\lim_{x\to -2} \frac{3x^2+ax+a+3}{x^2+x-2}$ exists? If so, find the value of a and the value of the limit.

$x^2+x-2 = (x+2)(x-1)$. For the limit to exist, we need $(x-2)$ to be a factor of the numerator (why?) This happens if + only if -2 is a root of $3x^2+ax+a+3$.

We solve:
$$0 = \lim_{x\to -2} (3x^2+ax+a+3)$$
$$= 3(-2)^2 + a(-2) + a + 3$$
$$= 15 - a$$
$$\Rightarrow a = 15.$$

$$\lim_{x\to -2} \frac{3x^2+15x+15+3}{x^2+x-2} = \lim_{x\to -2} \frac{(x+2)(3x+9)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{3x+9}{x-1} = \frac{3}{-3} = -1$$

5. True or False.

(a) If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist. If the answer is false, give a counterexample (that is, an example that satisfies the hypothesis but not the conclusion).

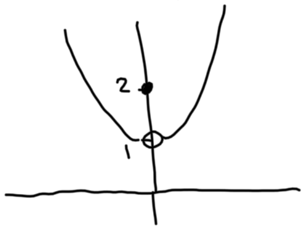
False.

Counterexample: $f(x) = g(x) = x - 5$

(b) If $f(x) > 1$ for all x and if $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$. If the answer is false, give a counterexample.

False

Counterexample: $f(x) = \begin{cases} x^2+1, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$



(c) If $\lim_{x\to 6} f(x)g(x)$ exists, then the limit must be $f(6)g(6)$. If the answer is false, give a counterexample.

False

Counterexample: $f(x) = g(x) = \begin{cases} 0, & x \neq 6 \\ 1, & x = 6 \end{cases}$

$\lim_{x\to 6} f(x) = \lim_{x\to 6} g(x) = 0$

(d) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0}[f(x) - g(x)] = 0$. If the answer is false, give a counterexample.

False.

Counterexample: $f(x) = \frac{1}{x^2} + 1$, $g(x) = \frac{1}{x^2}$