Math 221 Worksheet 20
November 10, 2020
Section 4.4: Indefinite Integrals and Net Change Theorem; Section 4.5: Substitution Rule

1. Determine the following indefinite integrals:

(a) $\int x^{3/2}\,dx$

$$\boxed{\frac{2}{5}x^{5/2} + C}$$

(b) $\int \cos(x+3)\,dx$

$$\boxed{\sin(x+3) + C}$$

(Could do a substitution with $u = x+3$)

(c) $\int 2x\cos(x^2)\,dx$

Let $u = x^2$. Then $du = 2x\,dx$. So

$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du = \sin(u) + C$$
$$= \boxed{\sin(x^2) + C}$$

(d) $\int 5x\sqrt{x^2+1}\,dx$

Let $u = x^2+1$. Then $du = 2x\,dx$. So

$$\int 5x\sqrt{x^2+1}\,dx = \int \frac{5}{2}\sqrt{u}\,du = \frac{5}{3}u^{3/2} + C$$
$$= \boxed{\frac{5}{3}(x^2+1)^{3/2} + C}$$

2. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $t \in [0, 50]$ is the number of minutes since the water began flowing. Find the amount of water that flows out of the tank during the first ten minutes.

This is given by $\int_0^{10} r(t)\,dt$, which is

$$\int_0^{10}(200-4t)\,dt = \left(200t - 2t^2\right)\Big|_0^{10} = 200\cdot 10 - 2\cdot 10^2$$
$$= \boxed{1800 \text{ liters}}$$

3. A particle is moving with an acceleration of $a(t) = 2t + 5$ meters per second squared at time t. The initial velocity of the particle is $v(0) = 4$. Find the velocity $v(t)$ at time t, as well as the total distance traveled over the first 10 seconds.

Since $v'(t) = a(t) = 2t+5$, we have $v(t) = t^2 + 5t + C$ for some C. Plugging in $t=0$ shows that $C = 4$.

Total distance is $\int_0^{10} v(t)\,dt = \left(\frac{1}{3}t^3 + \frac{5}{2}t^2 + 4t\right)\Big|_0^{10}$

$$= \boxed{\frac{1}{3}\cdot 1000 + \frac{5}{2}\cdot 100 + 4\cdot 10 \text{ m}}$$

4. Evaluate the following definite integrals:

(a) $\int_{-\pi/4}^{0} \sin(2x)\,dx$

Let $u = 2x$. Then $du = 2\,dx$. So

$$\int_{-\frac{\pi}{4}}^{0} \sin(2x)\,dx = \int_{-\frac{\pi}{2}}^{0} \frac{1}{2}\sin(u)\,du = -\frac{1}{2}\cos(u)\Big|_{-\frac{\pi}{2}}^{0}$$

$$= \boxed{-\frac{1}{2}}$$

(b) $\int_{0}^{\sqrt{\pi/2}} 2x\cos(x^2)\,dx$

Let $u = x^2$. Then $du = 2x\,dx$. So

$$\int_{0}^{\sqrt{\pi/2}} 2x\cos(x^2)\,dx = \int_{0}^{\pi/4} \cos(u)\,du = \sin(u)\Big|_{0}^{\pi/4} = \boxed{\frac{1}{\sqrt{2}}}$$

(c) $\int_{-1}^{1} 5x\sqrt{1-x^2}\,dx$

By symmetry: $5x\sqrt{1-x^2}$ is an odd function, and we're integrating over $[-1,1]$, so the integral is $\boxed{0.}$

By substitution: $u = 1-x^2 \rightarrow du = -2x\,dx$

$$\int_{-1}^{1} 5x\sqrt{1-x^2}\,dx = \int_{0}^{0} -\frac{5}{2}\sqrt{u}\,du = \boxed{0}$$

(d) $\int_{0}^{3\pi/4} \sin(x)\cos(x)$

Let $u = \sin(x)$. Then $du = \cos(x)\,dx$. So

$$\int_{0}^{3\pi/4} \sin(x)\cos(x)\,dx = \int_{0}^{\frac{1}{\sqrt{2}}} u\,du = \frac{1}{2}u^2\Big|_{0}^{\frac{1}{\sqrt{2}}} = \boxed{\frac{1}{4}}$$

(e) $\int_{\pi/6}^{\pi/3} \frac{\sec^2(x)}{\sqrt{\tan(x)}}\,dx$

Let $u = \tan(x)$. Then $du = \sec^2(x)\,dx$. So

$$\int_{\pi/6}^{\pi/3} \frac{\sec^2(x)}{\sqrt{\tan(x)}}\,dx = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}\Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \boxed{2\left(3^{\frac{1}{4}} - 3^{-\frac{1}{4}}\right)}$$

5. Evaluate $\int_{-100}^{100} \left[\cos(x)^{101}\sin(x)^{101} + \sqrt[101]{\tan\left(\frac{x}{100}\right)}\right] dx$ and explain your answer. (Hint: use symmetry)

$\sin(x)$ is an odd function and $\cos(x)$ is an even function.

Thus,

$$\cos(-x)^{101}\sin(-x)^{101} = \cos(x)^{101}(-\sin(x))^{101}$$
$$= -\cos(x)^{101}\sin(x)^{101}.$$

Similarly,

$$\sqrt[101]{\tan\left(\frac{-x}{100}\right)} = \sqrt[101]{\frac{\sin\left(-\frac{x}{100}\right)}{\cos\left(-\frac{x}{100}\right)}} = \sqrt[101]{\frac{-\sin\left(\frac{x}{100}\right)}{\cos\left(\frac{x}{100}\right)}} = -\sqrt[101]{\frac{\sin\left(\frac{x}{100}\right)}{\cos\left(\frac{x}{100}\right)}}$$
$$= -\sqrt[101]{\tan\left(\frac{x}{100}\right)}.$$

So we're integrating an odd function over $[-100, 100]$, and thus the answer is $\boxed{0}$

6. Let f be a continuous function satisfying $\int_0^1 f(x)\,dx = 3$. Prove that there exists an $x \in (0,1)$ such that $f(x) = 3$.

There are multiple ways to do this:

① Let m be the minimum value of f on $[0,1]$ and M the maximum value. Then

$$m \le \int_0^1 f(x)\,dx \le M,$$

So $m \le 3 \le M$. Therefore, by the intermediate value theorem, $f(x) = 3$ for some x.

② Let $F(x) = \int_0^x f(t)\,dt$. Then $F(0) = 0$ and $F(1) = 3$, and by the fundamental theorem of calculus, F is differentiable with $F' = f$. So by the mean value theorem,

$$3 = \frac{F(1) - F(0)}{1 - 0} = F'(x) = f(x)$$

for some $x \in (0,1)$.