Math 221 Worksheet 8
September 29, 2020
Sections 2.4 and 2.5 - Trigonometric Functions and Chain Rule

1. Compute the derivative of each of the following functions:

   (a) $f(x) = x\sin(x)$

   $f'(x) = 1 \cdot \sin(x) + x \cdot \cos(x)$

   (b) $g(t) = \dfrac{4t^2}{\cos(t)}$

   $$g'(t) = \frac{\cos(t) \cdot 8t - 4t^2(-\sin(t))}{\cos^2(t)}$$

   (c) $f(x) = \tan(x)$

   $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, so

   $$f'(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \sec^2(x)$$

   (d) $g(v) = v^3\sec(v)$

   $v^3\sec(v) = \dfrac{v^3}{\cos(v)}$, so

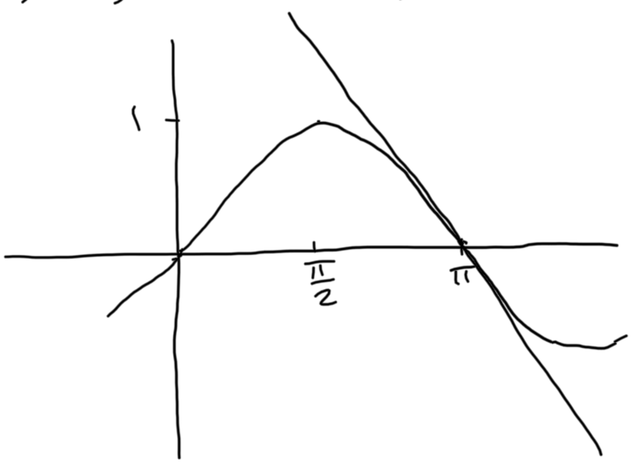
   $$g'(v) = \frac{\cos(v) \cdot 3v^2 - v^3(-\sin(v))}{\cos^2(v)}$$

2. Let $f(x) = \sin(x)$. Find the equation for the line tangent to the graph of f at the point $(\pi, f(\pi))$. Sketch the graph and tangent line.

   Slope is $f'(\pi) = \cos(\pi) = -1$,

   passes through $(\pi, 0)$, so the equation is

   $y = -(x - \pi)$



3. Evaluate the following limits:

(a) $\lim_{x\to 0} \frac{\sin x}{x2^x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{2^x}$ (both limits exist)

$= 1 \cdot 1 = \boxed{1}$

(b) $\lim_{x\to 0} \frac{\tan x}{x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = \boxed{1}$

(c) $\lim_{\theta\to 0} \frac{\sin(6\theta)}{3\theta}$

$= 2 \lim_{\theta\to 0} \frac{\sin(6\theta)}{6\theta}$

$= 2 \cdot 1 = \boxed{2}$

(d) $\lim_{x\to 0} \frac{1-\cos(x)}{4x\sin(x)}$

$= \lim_{x\to 0} \frac{1-\cos(x)}{4x\sin(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)}$

$= \lim_{x\to 0} \frac{1-\cos^2(x)}{4x\sin(x)} \cdot \frac{1}{1+\cos(x)}$

$= \lim_{x\to 0} \frac{1}{1+\cos(x)} \lim_{x\to 0} \frac{1-\cos^2(x)}{4x\sin(x)}$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin^2(x)}{4x\sin(x)}$

$= \frac{1}{8} \lim_{x\to 0} \frac{\sin(x)}{x}$

$= \boxed{\frac{1}{8}}$

4. Recall that a function f is *even* if $f(-x) = f(x)$ for all x and *odd* if $f(-x) = -f(x)$ for all x. Show that if f is even, then f' is odd.

Differentiate both sides of the equation $f(-x) = f(x)$:

$\underbrace{f'(-x)(-1)}_{\text{from chain rule}} = f'(x).$

5. Use the chain rule to find the derivative of each of the following functions:

(a) $f(x) = (2x+1)^2$

$f'(x) = 2(2x+1)\cdot 2$

(b) $f(x) = \sin(4x)$

$f'(x) = \cos(4x)\cdot 4$

(c) $f(x) = \sqrt{2+x^2} + (2+x^2)^3$

$$f'(x) = \frac{1}{2}(2+x^2)^{-\frac{1}{2}} \cdot 2x + 3(2+x^2)^2 \cdot 2x$$

(d) $f(x) = \sqrt{\dfrac{x-1}{x+1}}$

$$f'(x) = \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}} \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}$$

6. Let $g(x) = f(\frac{1}{x^2})$, where f is a differentiable function satisfying $f(3) = 5$, $f(\frac{1}{9}) = 7$, $f'(3) = 11$, and $f'(\frac{1}{9}) = 13$. Find the equation for the line tangent to the graph of g at the point $(3, g(3))$.

slope is $g'(3) = f'\left(\frac{1}{3^2}\right) \cdot \frac{-2}{3^3} = 13 \cdot \frac{-2}{27} = -\frac{26}{27}$, passes through $(3, g(3)) = (3, 7)$. So the equation is

$$y - 7 = -\frac{26}{27}(x-3).$$

7. Find the 100th derivative of the function $f(x) = \cos(2x+1)$.

By the chain rule, $f'(x) = -\sin(2x+1)\cdot 2$,

$f''(x) = -\cos(2x+1)\cdot 2^2$

$f'''(x) = \sin(2x+1)\cdot 2^3$

$\vdots$

So $f^{(100)}(x) = \cos^{(100)}(2x+1)\cdot 2^{100} = \boxed{\cos(2x+1)\cdot 2^{100}}$

since 100 is a multiple of 4.

8. Suppose that f is a twice-differentiable function satisfying $f(x^2) = f(x) + x^2$. What are $f'(1)$ and $f''(1)$?

Differentiate: $f'(x^2)\cdot 2x = f'(x) + 2x$

So $f'(1)\cdot 2 = f'(1) + 2$ which gives $\boxed{f'(1) = 2.}$

Differentiate again: $f''(x^2)\cdot 2x\cdot 2x + f'(x^2)\cdot 2 = f''(x) + 2$

So $f''(1)\cdot 4 + f'(1)\cdot 2 = f''(1) + 2$, which gives $\boxed{f''(1) = -\frac{2}{3}.}$

9. Suppose that f is a differentiable function satisfying $f(x)^3 = x - 1 - f(x^2)$. What is $f'(1)$?

Differentiate: $3f(x)^2 f'(x) = 1 - f'(x^2)\cdot 2x$

So $3f(1)^2 f'(1) = 1 - f'(1)\cdot 2$. We also know that $f(1)^3 = -f(1)$, so $f(1) = 0$. Thus $\boxed{f'(1) = \frac{1}{2}.}$