Gröbner Complexes and Tropical Bases

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Before dealing with the slightly generalized version of Gröbner bases that Maclagan and Sturmfelds [1] use, I wish to motivate the discussion by recalling some of the results about usual Gröbner bases which I originally learned from Ideals, Varieties, and Algorithms [2]. Broadly speaking, a Gröbner basis helps us solve problems about polynomial ideals in an algorithmic fashion. Some basic questions might be:

i Given $f \in k[x_1, \dots, x_n]$ and an ideal $I = \langle f_1, \dots, f_n \rangle$, is $f \in I$?

ii Find all common solutions in k^n of a system of polynomial equations

$$f_1(x_1, \dots, x_n) = \cdots = f_s(x_1, \dots, x_n) = 0$$

iii Given a $V \subset k^n$ given parametrically, can we find a system of polynomial equations the defines the variety?

All of the problems above are easily dispatched in the single variable case. For instance, if $n = 1$ we can use the division algorithm to see if $f \in I$ since this occurs if and only if the remainder is zero since $k[x]$ is a principal ideal domain. To answer (i) for general n, we might divide f by f_1, then take the remainder r_1 and divide by f_2, and repeat for all f_i in the basis of I. We end up with

$$f = q_1 f_1 + q_2 f_2 + \cdots + q_k f_k + r.$$

If $r = 0$, then we can definitely say that $f \in I$ but if $r \neq 0$ we cannot exclude the possibility that $f \in I$. This is because in general the result depends on the order of the basis. The concept of Gröbner bases "fixes" this issue with multivariable polynomial division:

Definition 1. The subset $G = \{g_1, \dots, g_n\} \subset I$ is said to be a Gröbner basis for I with respect to a monomial order $<$ if $\langle \mathrm{LT}(g_1), \cdots, \mathrm{LT}(g_t) \rangle = \langle \mathrm{LT}(I) \rangle$, where LT stands for the leading term.

As long as you choose a Gröbner basis of I to divide by, then the ordering of the basis does not matter and $f \in I$ if and only if $r = 0$. Further, with a description of how to solve (i), Gröbner bases can be used to prove the Weak Nullstellensatz[a]. If all of your polynomials are linear, finding a Gröbner basis is equivalent to the process of doing Gaussian elimination.

Now, Gröbner bases are not unique for a fixed monomial ordering, but you may make a minimal choice called the *reduced* Gröbner basis. Still, reduced Gröbner bases are not unique among monomial orderings. The *Gröbner fan* of an ideal I is a collection of polyhedral cones in $\mathbb{R}^n$ that are in bijection with the reduced Gröbner bases of I – through study of Gröbner fans one can learn the surprising fact that there are only finitely many reduced Gröbner bases for a given ideal I, even with infinitely many choices of monomial orderings. [3] It also gives a way to produce *universal* Gröbner bases for ideals, which are simultaneously Gröbner bases for all possible monomial orderings. When these concepts have been appropriately generalized to the tropical setting, the Gröbner fan becomes a Gröbner complex and the universal Gröbner basis becomes a tropical basis. Further, a tropical variety can be shown to be a certain subcomplex of a Gröbner complex.

[a] In order to prove that Weak nullstellensatz, we must be able to prove that if $V(I) = \varnothing$, then $1 \in I$.

What sort of generalization am I referring to? For ease of thought, let $k = \overline{k}$ be a field with a nontrivial valuation whose value group Γ_{val} is dense in $\mathbb{R}^n$. If $\text{val}(a) \geq 0$, a is in the valuation ring R_K of K. Let $\overline{a}$ denote the image of a in R_K. Every polynomial $f \in k[x_1, \dots, x_n]$ has a tropicalization $\text{trop}(f)$ where the operations $(+, \times)$ are swapped out for $(\oplus, \odot)$, which are classical minimum and multiplication operation respectively:

$$c_1x^{a_1} + c_2x^{a_2} + \cdots c_sx^{a_n} \longrightarrow \text{val}(c_1) \odot x^{\cdot a_1} \oplus \text{val}(c_2) \odot x^{\odot a_2} \oplus \cdots \oplus \text{val}(c_n) \odot x^{\odot a_n}$$
$$\min(\text{val}(c_i) + x \cdot a_i)$$

Given a weight vector $u \in \Gamma_{val}$, the *initial form* $\text{in}_u(f)$ is the subsum of the terms $\overline{c_ix^{a_i}}$ where $\text{val}(c_i) \odot u^{\cdot a_i}$ is minimal, and $\overline{c_i}$ is the term of lowest order in c_i. Using this, the initial ideal of an ideal $I \subset k[x_1, \dots, x_n]$ is simply the collection of initial forms of all the elements in the ideal. This notation yields a very similar looking definition for Gröbner bases:

Definition 2. The subset $G = \{g_1, \dots, g_n\} \subset I$ is said to be a Gröbner basis for I with respect to a weight vector u if $\langle \text{in}_u(g_1), \cdots, \text{in}_u(g_t)\rangle = \langle \text{in}_u(I)\rangle$.

Note that if our valuation is trivial, then this definition is actually equivalent to definition 1. Indeed, if w is sufficiently generic then $\text{in}_w(f)$ is the leading monomial of f with respect to the term order determined by $-w$.
This has many of the same properties as the original Gröbner bases. In particular, it "fixes" multivariable polynomial division as before (Proposition 2.4.5). Further, we learn that if we wiggle the weight vector in a small neighborhood, we do not change the initial ideal (Lemma 2.4.4). This leads us to the question: what is the structure of the initial ideals as we vary u?

Definition 3. A polyhedral decomposition of $\mathbb{R}^n$ (a polyhedral complex whose support is $\mathbb{R}^n$) is a Gröbner complex for I if $\text{in}_w(I) = \text{in}'_w(I)$ whenever w and w' are in the same relatively open cell.

The surprising fact here is that the structure is a finite polyhedral complex[b]. By picking a representative weight vector w in each cell of this polyhedra complex and computing a Gröbner basis with respect to each of these weight vectors, we can take the union to obtain a universal Gröbner basis.
However, some of our discussion is worthless in the primary case that we care about. In order to discuss tropical varieties and hypersurfaces, we want to expand our consideration to the Laurent polynomials $k[x^{\pm}] = k[x_1^{\pm}, \dots, x_n]$, since elements $f \in k[x^{\pm}]$ define hypersurfaces in the algebraic torus $(k^*)^n$. In this setting, most choices of our weight vector u leads to $\text{in}_u(f)$ being a unit in $k[x^{pm}]$. Consequently the initial ideal $\text{in}_u(I)$ where $f \in I$ is the entirety of $k[x^{\pm}]$. So we are primarily concerned with weight vectors where the initial ideal is actually a proper ideal in $k[x^{\pm}]$! An important fact is that these happen to correspond exactly with the *tropical zeros*[c] of $\text{trop}(f)$:

Lemma. *For $f \in k[x^{\pm}]$ and $u \in \mathbb{R}^n$, $\text{in}_u(f)$ is not a unit in $k[x^{\pm}]$ (i.e. not a monomial) if and only if u is a tropical zero of trop(f).*

By restricting a classical variety $V(I)$ to the roots that are also tropical roots for all $f \in I$ gives us a tropical variety, which the fundamental theorem says is the image of the classical tropical variety under the coordinate-wise valuation. By carefully selecting a universal Gröbner basis $\mathcal{G}$ so that the initial ideal $\text{in}_u(I)$ contains a unit if and only if $\text{in}_u(G)$ contains a unit, we can show that every ideal in $K[x^{\pm}]$ has a tropical basis.

References

[1] Bernd Sturmfels Diane Maclagan, *Introduction to Tropical Geometry*, Vol. 161, American Mathematical Society, 2015.

[2] David A. Cox and John Little and Donal O'Shea, *Ideals, Varieties, and Algorithms*, Springer International Publishing, 2015.

[3] Anders Nedergaard Jensen, *Computing Gröbner Fans and Tropical Varieties in Gfan*, Software for Algebraic Geometry, 2008, pp. 33–46.

[b] A finite collection of polyhedra satisfying two conditions: every face is a polyhedron, and all intersections between polyhedra is a common face

[c] These are values $x = u$ such among $\{c_iu^{a_i}\}$ (the terms of $f(u)$), the minimum is obtained at least twice. This definition makes sense since any classical root x of f, $\text{val}(x)$ is a root of $\text{trop}(f)$.