Math 221 Worksheet 9
October 1, 2020
Section 2.6: Implicit Differentiation

1. For each of the following equations find $\frac{dy}{dx}$:

   (a) $x^2 + xy = y^2$

$$2x + 1\cdot y + xy' = 2yy'$$

$$2x + y = y'(2y - x) \rightsquigarrow \boxed{y' = \frac{2x+y}{2y-x}}$$

   (b) $\sqrt{xy} = \cos(x+y)$

$$\frac{1}{2\sqrt{xy}}(1\cdot y + xy') = -\sin(x+y)(1+y')$$

$$\frac{y}{2\sqrt{xy}} + \sin(x+y) = y'\left(-\sin(x+y) - \frac{x}{2\sqrt{xy}}\right)$$

$$\rightsquigarrow \boxed{y' = \frac{\frac{y}{2\sqrt{xy}} + \sin(x+y)}{-\sin(x+y) - \frac{x}{2\sqrt{xy}}}}$$

   (c) $\sin(x)\sin(y) = xy^2$

$$\cos(x)\sin(y) + \sin(x)\cos(y)\,y' = y^2 + x\cdot 2yy'$$

$$y'(\sin(x)\cos(y) - 2xy) = y^2 - \cos(x)\sin(y)$$

$$\rightsquigarrow \boxed{y' = \frac{y^2 - \cos(x)\sin(y)}{\sin(x)\cos(y) - 2xy}}$$

   (d) $\tan(xy^2) = x$

$$\sec^2(xy^2)(y^2 + x\cdot 2yy') = 1$$

$$\rightsquigarrow \boxed{y' = \frac{1 - \sec^2(xy^2)\,y^2}{\sec^2(xy^2)\cdot 2xy}}$$

2. The equation $\cos(x^2y) = 3xy^2 + y$ defines a curve. Find the line tangent to it at the point $(0, 1)$.

Its slope is $\left.\frac{dy}{dx}\right|_{(0,1)}$.

Implicitly differentiate:

$$-\sin(x^2y)(2xy + x^2y') = 3(y^2 + x\cdot 2yy') + y'$$

Evaluate at $(x,y) = (0,1)$:

$$0 = 3 + y'|_{(0,1)} \rightsquigarrow y'|_{(0,1)} = -3$$

Equation: $\boxed{y - 1 = -3x}$

3. Suppose that f is an invertible function, and let g be its inverse. Suppose additionally that f and g are differentiable, and let $y = f(x)$. What is $g'(y)$?

We have $g(y) = x$ since g is the inverse of f.

So $g'(y)y' = 1$, giving $\boxed{g'(y) = \frac{1}{y'} = \frac{1}{f'(x)}}$.

4. For each of the following equations find $\frac{d^2y}{dx^2}$:

(a) $xy = x^2 + 1$

$$y + xy' = 2x \rightsquigarrow y' = \frac{2x - y}{x}$$

$$y' + y' + xy'' = 2$$

$$\text{So } y'' = \frac{2 - 2y'}{x} = \boxed{\frac{2 - 2\cdot\frac{2x-y}{x}}{x}}$$

(b) $\sin(y) = xy$

$$\cos(y)y' = y + xy', \text{ so } y' = \frac{y}{\cos(y) - x}$$

$$-\sin(y)y'y' + \cos(y)y'' = y' + y' + xy'',$$

$$\text{So } y'' = \frac{2y' + \sin(y)(y')^2}{\cos(y) - x} = \boxed{\frac{2\cdot\frac{y}{\cos(y)-x} + \sin(y)\left(\frac{y}{\cos(y)-x}\right)^2}{\cos(y) - x}}$$

5. The equation $x^2 + y^2 + xy = 1$ defines an ellipse. Among all points (x, y) on this ellipse, which one has the largest y-value and which one has the smallest?

These are the points with horizontal tangent lines, i.e. where $\frac{dy}{dx} = 0$.

Implicitly differentiate:

$$2x + 2yy' + y + xy' = 0$$

$$y' = -\frac{y + 2x}{2y + x}$$

This is 0 when $y = -2x$. Plug this into the equation for the ellipse:

$$\underbrace{x^2 + (-2x)^2 + x(-2x)}_{3x^2} = 1 \rightsquigarrow x = \pm\frac{1}{\sqrt{3}}$$

$\boxed{\text{Highest } y\text{-value: } \left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \quad \text{Lowest: } \left(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)}$

6. The equation $y^2 = x^3 + x + 2$ defines a curve. At which point(s) does it have a vertical tangent line?

These are the points where $\frac{dy}{dx}$ does not exist.

$$2yy' = 3x^2 + 1 \rightsquigarrow y' = \frac{3x^2+1}{2y}$$

y' does not exist if and only if $y = 0$.

Plug in $y = 0$ to the equation for the curve: $0 = x^3 + x + 2$

The only (real number) solution is $x = -1$.

So the only point with a vertical tangent is $\boxed{(-1, 0)}$

7. Let L be the line defined by $4y - 3x = 1$. Find a circle of unit radius that contains the point $(1, 1)$ and whose tangent line at $(1, 1)$ is L.

Let (a, b) be the center of the circle; we need to find it.

The equation for the circle is $(x-a)^2 + (y-b)^2 = 1$.

It contains $(1,1)$; thus

$$(1-a)^2 + (1-b)^2 = 1. \qquad (*)$$

Also, by implicit differentiation,

$$y' = -\frac{x-a}{y-b}.$$

Since L is tangent to the circle at $(1,1)$, we have

$$-\frac{1-a}{1-b} = y'\Big|_{(1,1)} = \frac{3}{4} \qquad (**)$$

slope of L

Equations $(*)$ and $(**)$ together imply that

$$\left(-\tfrac{3}{4}(1-b)\right)^2 + (1-b)^2 = 1,$$

or $(1-b)^2 = \frac{16}{25}$. Two solutions — let's choose $b = \frac{1}{5}$.

Then $(**)$ implies that $a = \frac{8}{5}$. Our circle is

given by $\boxed{\left(x - \frac{8}{5}\right)^2 + \left(y - \frac{1}{5}\right)^2 = 1}$

(Had we chosen the other solution for b, we would get a different circle. It makes sense that there are two.)