Math 221 Worksheet 13
October 15, 2020
Section 3.4: Limits at Infinity and Horizontal Asymptotes

1. Evaluate the following limits (some may be ∞ or $-\infty$).

   (a) $\lim_{x\to\infty} \frac{2x+1}{3x+4}$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3+\frac{4}{x}} = \frac{2+\lim_{x\to\infty}\frac{1}{x}}{3+\lim_{x\to\infty}\frac{4}{x}} = \boxed{\frac{2}{3}}$$

   (b) $\lim_{x\to\infty} \frac{x+3}{2x^2-10}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{3}{x^2}}{2-\frac{10}{x^2}} = \frac{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{3}{x^2}}{2-\lim_{x\to\infty}\frac{10}{x^2}} = \boxed{0}$$

   (c) $\lim_{x\to-\infty} \frac{x^2+1}{10x^2-x+1}$

$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x^2}}{10-\frac{1}{x}+\frac{1}{x^2}} = \cdots = \boxed{\frac{1}{10}}$$

   (d) $\lim_{x\to-\infty} \frac{3x^2+4}{x-2}$

$$= \lim_{x\to-\infty} \frac{3x+\frac{4}{x}}{1-\frac{2}{x}}$$

So the limit is $\boxed{-\infty}$

numerator $\to -\infty$
denominator $\to 1$.

2. The limit laws we learned also apply to limits at infinity. That being said, what is wrong with the following?

$$1 = \lim_{x\to\infty} 1 = \lim_{x\to\infty} \frac{1}{x}\cdot x = \lim_{x\to\infty} \frac{1}{x}\cdot \lim_{x\to\infty} x = 0 \cdot \lim_{x\to\infty} x = 0$$

Does not exist!
(as a number)

3. Evaluate $\lim_{x\to\infty} \frac{x^2+\cos(x)}{2x^2+4x+1}$.

$$= \lim_{x\to\infty} \frac{1 + \frac{\cos(x)}{x^2}}{2 + \frac{4}{x} + \frac{1}{x^2}}.$$

Since $1 - \frac{1}{x^2} \le 1 + \frac{\cos(x)}{x^2} \le 1 + \frac{1}{x^2}$, the squeeze theorem implies that the numerator approaches 1. The denominator approaches 2, so the limit is $\boxed{\frac{1}{2}}$.

4. Evaluate $\lim_{x\to-\infty} \sqrt{9x^2-x}+3x$.

$$= \lim_{x\to-\infty} \frac{(\sqrt{9x^2-x}+3x)(\sqrt{9x^2-x}-3x)}{\sqrt{9x^2-x}-3x}$$

$$= \lim_{x\to-\infty} \frac{-x}{\sqrt{9x^2-x}-3x}$$

$$= \lim_{x\to-\infty} \frac{1}{\sqrt{9-\frac{1}{x}}+3} = \boxed{\frac{1}{6}}$$

5. Evaluate $\lim_{x\to\infty} \frac{4x+1}{\sqrt{x^2+2}}$.

$$= \lim_{x\to\infty} \frac{4+\frac{1}{x}}{\sqrt{1+\frac{2}{x^2}}}$$

$$= \boxed{4}$$

6. Evaluate $\lim_{x\to-\infty}(\sqrt[3]{x-8}-\sqrt[3]{x})$.

We can make things cancel by creating a difference of cubes:

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

(This is a cubic version of "multiplying by the conjugate".)

$$\lim_{x\to-\infty}\left((x-8)^{\frac{1}{3}}-x^{\frac{1}{3}}\right) = \lim_{x\to-\infty} \frac{\left((x-8)^{\frac{1}{3}}-x^{\frac{1}{3}}\right)\left((x-8)^{\frac{2}{3}}+(x-8)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}{(x-8)^{\frac{2}{3}}+(x-8)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}}$$

Could also do this problem using mean value theorem!

$$= \lim_{x\to-\infty} \frac{-8}{(x-8)^{\frac{2}{3}}+(x-8)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}} = \boxed{0}$$

7. Evaluate $\lim_{x\to-\infty}\cos(\frac{\pi x^2+1}{4x^2-3})$.

$$= \cos\left(\lim_{x\to-\infty}\frac{\pi x^2+1}{4x^2-3}\right) = \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}}$$

because cosine is continuous

8. Find all vertical and horizontal asymptotes of the function $f(x) = \frac{5x^2}{x^2-4}$. Justify your answer.

Vertical

- $x = 2$ because $\lim_{x\to 2^+} f(x) = \infty$
- $x = -2$ because $\lim_{x\to -2^+} f(x) = -\infty$
- No others because f is continuous everywhere except $x = \pm 2$.

Horizontal

- $y = 5$ because $\lim_{x\to\infty} f(x) = 5$
- No others, because $\lim_{x\to-\infty} f(x)$ is also 5.

9. Find all vertical and horizontal asymptotes of the function $f(x) = \frac{x^2+x-2}{x^2-1}$. Justify your answer.

Vertical

- $x = -1$ because $\lim_{x\to -1^+} f(x) = \infty$
- No others because $\lim_{x\to 1} f(x) = \frac{3}{2}$ and f is continuous everywhere except $x = \pm 1$.

Horizontal

- $y = 1$ because $\lim_{x\to\infty} f(x) = 1$.
- No others because $\lim_{x\to-\infty} f(x)$ is also 1.

10. Find all vertical and horizontal asymptotes of the function $f(x) = \frac{x+2}{\sqrt{x^2+1}}$. Justify your answer.

Vertical

- None, because f is continuous.

Horizontal

- $y = 1$ because $\lim_{x\to\infty} f(x) = 1$.
- $y = -1$ because $\lim_{x\to-\infty} f(x) = -1$.

(• No others possible)