Syzygies, Castelnuovo-Mumford Regularity, and Koszul Cohomology

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Abstract.

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1 Introduction

1.1 Personal Aside

I wanted to write about koszul cohomology and Mark Green's framework, and as I see it there are two main ways that I can motivate it:

i **(Conceptually)** It imports one of the most central commutative algebra tools (syzygies) to algebraic geometry. This way is super clean conceptually and allows me to get into the details of koszul cohomology faster. I believe this is closer to the reason that Green might give you for defining it.

ii **(Historically)** It generalizes coarse information you get from castelnuovo-mumford regularity, and simplifies a bunch of techniques that were already happening at the time. This way is not very clean, but I think it places it more accurately in history.

Naturally, I've chosen the worse messier second approach to go down because I want to do a more general survey of the landscape.

1.2 Topic Structure

This needs to be rewritten at the end, so that it reflects what I actually ended up writing about. Currently, it seems like I need to introduce two questions that lead into section 2 and 3, and then conclude with the paper information.

These notes are framed around motivating and explaining the results of the famous[1] paper ”On a theorem of Castelnuovo, and the equations defining space curves" by Gruson, Lazarsfeld, and Peskine, whose main result is the following

Theorem 1.1 (Gruson-Lazarsfeld-Peskine [GLP83], 1983). *If $X \subset \mathbb{P}^r$ is a reduced and irreducible curve of degree d, not contained in a hyperplane, then X is $(d+2-r)$-regular.*

In particular, this theorem implies that that the degrees of the polynomials needed to generate the ideal corresponding to X are bounded by $d+2-r$. The theorem is important, but perhaps even more important are the techniques used in the proof. The result is accomplished using *very few ideas* – all techniques are very general homological methods which lead to various generalizations. Alongside Green's syzygy framework, the ideas in this paper seemingly set off a revolution in how we organize information about curves and their moduli.

Helpful Resources

Much of the discussion here is mostly a patchwork of different sources and should not be considered original. First, in addition to having a really nice book [AN10], Marian Aprodu gave a nice talk on Koszul Cohomology thats on Youtube. Robert Lazarsfeld also has a really nice book [Laz17] and has a whole series of talks called the "The equations defining Projective Varieties" on Youtube.

[1] There is an entire chapter in “Geometry of Syzygies” devoted to this.

2 Castelnuovo-Mumford Regularity

2.1 Motivation

A lot of the following setup can be relaxed or relativized, but I will try to be as concrete as possible for a while.Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P} = \mathbb{P}^r_{\mathbb{C}}$.[2] I won't really care about the dimension for a while, so I will omit it. Recall Serre's Vanishing Theorem, which tells us that any cohomological peculiarities associated to $\mathscr{F}$ disappears after twisting enough.

Theorem 2.1 (Serre's Vanishing over $\mathbb{P}$). *There exists some $n_0 = n_0(\mathscr{F})$ depending on $\mathscr{F}$ such that for all $n \geq n_0$ then*

i *$\mathscr{F}(n)$ is globally generated, i.e. the evaluation morphism $H^0(\mathscr{F}(n)) \otimes \mathcal{O}_{\mathbb{P}} \twoheadrightarrow \mathscr{F}(n)$ is surjective*

ii $H^i(\mathbb{P}, \mathscr{F}(n)) = 0$ *for all* $i > 0$.

Lets first talk concretely about why we care about (i) and (ii).

Example 2.2. Consider $X \subseteq \mathbb{P}$ a variety defined by $\mathscr{I}_X \subseteq \mathcal{O}_{\mathbb{P}}$ the ideal sheaf. Then Serre's vanishing 2.1 tells us that there is a n_0 depending on X such that for all $n \geq n_0$:

(i) X is cut out (as a scheme) by hypersurfaces of degree $\leq n_0$:

$$\begin{aligned} \mathscr{I}_X(n_0) \text{ is g.g.} &\iff \text{we have a surjection } \mathcal{O}_{\mathbb{P}}^{\oplus k} \twoheadrightarrow \mathscr{I}_X(n_0) \text{ where } r = H^0(X, \mathscr{I}_X(n_0)) \\ &\iff \text{we have a surjection } \mathcal{O}_{\mathbb{P}}^{\oplus k}(-n_0) \twoheadrightarrow \mathscr{I}_X \text{ where } r = H^0(X, \mathscr{I}_X(n_0)) \end{aligned}$$

The point here is that $\mathcal{O}_{\mathbb{P}}(-n_0)$ is the ideal cutting out hypersurfaces of degree n_0. The structure sheaf is $\mathcal{O}_{\mathbb{P}}(n_0)$ (these correspond to degree n_0 equations in the homogeneous coordinate ring S), and the ideal cutting this out is the dual $\mathcal{O}_{\mathbb{P}}(-n_0)$.

(ii) Hypersurfaces of degree $\geq n_0$ trace out a complete linear series on X.

We have the following exact sequence:

$$0 \longrightarrow \mathscr{I}_X(n) \longrightarrow \mathcal{O}_P(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow 0$$

Running the long exact sequence yields

$$0 \longrightarrow H^0(\mathscr{I}_X(n)) \longrightarrow H^0(\mathcal{O}_P(n)) \longrightarrow H^0(\mathcal{O}_X(n)) \longrightarrow H^1(\mathscr{I}_X(n)) \longrightarrow \cdots$$

Condition (ii) is equivalent to saying that $H^1(\mathbb{P}, \mathscr{I}_X(n)) = 0$, or that the restriction maps $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \twoheadrightarrow H^0(X, \mathcal{O}_X(n)))$ are surjective for $n \geq n_0$.[3] Sections in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ correspond to equations vanishing on hypersurfaces of degree

[2] Some results here can be extended, e.g. to characteristic p.

[3] Note that this is the same as $\mathrm{Sym}^m H^0(\mathscr{L}) \to H^0(\mathscr{L}^{\otimes m})$, where $\mathscr{L}$ gives the embedding of $X \hookrightarrow P(H^0(\mathscr{L}))$.

n, and this surjection is telling me that when restricted to X, they generate $\mathcal{O}_X$ in degree n.[4]

Definition 2.3. X is said to be n-normal if the restriction maps $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) \twoheadrightarrow H^0(X, \mathcal{O}_X(n)))$ is surjective. If X is n-normal for all $n > 0$, then X is *projectively normal*. This is equivalent to saying that the affine cone over C is a normal variety.

There is an explanation that this corresponds to checking the degree n part of the condition for normality of S_X.

Remark 2.4. The importance of the projectively condition is that it reduces the problem of computing the number of hypersurfaces of degree m passing through X ($= \dim(\ker \rho_n)$) to Riemann-Roch.

Explain

Remark 2.5 (Globally generated $\implies$ Line Bundles Resolutions). Note that $(*)$ is what allows us to resolve coherent sheaves by direct sums of line bundles. This helps us link cohomological questions about coherent sheaves to the case of line bundles.[5]

Remark 2.6. In the previous example, we are comparing how an older classical language compares to a more modern language – instead of asking about hypersurfaces cutting out something, we talk about vanishing of some H^1.

Okay, so hopefully you are convinced that knowing this is useful. A natural first question is when does this happen – can we bound n?

Question 2.7. *How do I make Serre's Vanishing effective? How do I find n_0?*

Our answer is given by Castelnuovo-mumford regularity.

2.1.1 Some questions about curves ★

This is a discussion that is pursued in the paper, but may have been covered implicitly

To understand the curve X better, there are a few things we can ask:

(A_n) The line bundle $\mathcal{O}_X(n)$ is non-special, i.e. $h^1(X, \mathcal{O}_X(n)) = 0$
In this case, by Riemann-Roch and the fact that $h^1(X, \mathcal{O}_X(n)) = h^0(\omega_X \otimes \mathcal{O}_X(-n))$ we know $h^0(X, \mathcal{O}_X(n)) = nd - g + 1$:

$$\begin{aligned} h^0(X, \mathcal{O}_X(n)) &= \deg \mathcal{O}_X(n) - g + 1 \\ &= n \deg \mathcal{O}_X(1) - g + 1 \\ &= nd - g + 1. \end{aligned}$$

[4] Why the complete linear series language? The hypersurfaces of degree $\geq n_0$ make up a linear subspace V of $\mathbb{P}$, namely $H^0(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(n))$. These are a complete linear series if $|V| = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(n)))$ is the same as the space $\mathbb{P}$ we embedded X into. I think our surjection checks this as long as $\mathbb{P}(H^0(X, \mathcal{O}_X(n))) = \mathbb{P}$.

[5] Example 1.2.22 in [Laz17] to see how Beilinson's resolution of the diagonal is used to give a theorem about multiplying Cartier Divisors.

Why is this important?

(B_n) (X **is n-normal**) The homomorphism $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^n}(n)) \to H^0(X, \mathcal{O}_X(n))$ is surjective.

Lemma 2.8. *A_{n-2} and B_{n-1} imply that X is cut out by hypersurfaces of degree n and the homogeneous ideal of X is generated in degrees $\geq n$ by its component of degree n.*

Proof. ■

Lemma 2.9. *X is n-regular if and only if X satisfies A_{n-2} and B_{n-1}*

Proof. ■

Corollary 2.10. *If X is n-regular, then X is cut out by hypersurfaces of degree n and the homogeneous ideal of X is generated in degrees $\geq n$ by its component of degree n.*

2.2 Definitions

Definition 2.11 (Castelnuovo-Mumford Regularity, [MB66] Lecture 14)**.** A coherent sheaf $\mathscr{F}$ on a projective space $\mathbb{P}^r$ is said to be *n-regular* if $H^i(\mathbb{P}^r, \mathscr{F}(n-i)) = 0$ for $i > 0$.

The following lemma is due to Castelnuovo (according to Mumford); in reference to the setup in section , it tells us that we can take $n_0 = \operatorname{reg} \mathscr{F}$.

fill this in

Theorem 2.12. *Let $\mathscr{F}$ be an n-regular sheaf on $\mathbb{P}^r$. Then for every $k \geq 0$,*

i *$\mathscr{F}(n)$ is generated by its global sections.*

ii *The maps*
$$H^0(\mathscr{F}(n)) \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(l)) \to H^0(\mathscr{F}(n+l))$$
are surjective for $l \geq 0$.

iii *$\mathscr{F}$ is $(n+k)$-regular.*

Should I include a proof here?

I want to link this to being n-normal.

So as soon as $\mathscr{F}$ is n-regular, it is N-regular for all $N > n$. The least n for which $\mathscr{F}$ is n-regular is its regularity $\operatorname{reg} \mathscr{F}$.

Remark 2.13. Something in Daniel's research I'm somewhat thinking about is how we generalize these notions to spaces more general than projective spaces – perhaps to toric varieties. For instance, the projective space $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_d}$ has a coordinate ring that is *multigraded* by $\mathbb{Z}^d$. Here, the regularity generalizes to a region and we can ask all these question about choosing line bundles that correspond to points inside and outside the region.

Relation to Cohen-Macaulay: Pasting this here to absorb later: Exercise 18.16 in Eisenbud's Commutative Algebra book asks to show the following two things:
1) The homogeneous coordinate ring of a curve with ideal sheaf I is Cohen-Macaulay iff $H^1(\mathbb{P}^r, I(n)) = 0$ for all n iff X is linearly normal and $\bigoplus_m H^0(\mathbb{P}^r, \mathcal{O}_X(n))$ is generated as an algebra by $H^0(\mathbb{P}^r, \mathcal{O}_X(1))$.
2) The homogeneous coordinate ring of a projective variety of pure dimension d with ideal sheaf I is Cohen-Macaulay iff $H^1(\mathbb{P}^r, I(n)) = 0$ for all n and $H^i(\mathbb{P}^r, \mathcal{O}_X(n)) = 0$ for $0 < i < d$ and all n.

The Castelnuovo-Mumford regularity of a coherent sheaf on a projective space is an important measure of it's *algebraic complexity*.

Example 2.14. Suppose $X \subset \mathbb{P}^r$ is a projective scheme corresponding to the coherent ideal sheaf $\mathscr{I}_X$. We say that $X \subseteq \mathbb{P}^r$ is n-regular if its ideal sheaf $\mathscr{I}_X$ is. The direct implication of Theorem 2.12 is that $\mathscr{I}_X$ is generated in degrees $\leq \deg X$ and that hypersurfaces of degree $\geq \deg X - 1$ traces out a complete linear series on X. However, it turns out that knowing the regularity of X is the same thing as having bounds on the degrees of generators of all the modules of syzygies of I_X – we get data about the minimal free resolution.

Its weird that I do this before formally defining Syzygies.

To be more specific, given a coherent sheaf $\mathscr{F}$ on projective space $\mathbb{P}^r$, let $F = \oplus_k H^0(\mathbb{P}^r, \mathscr{F}(k))$ be the corresponding graded S-module. For simplicity, assume that F is finitely generated[6] – then it has some minimal free graded resolution $F_\bullet$:

$$0 \longrightarrow F_{r+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0$$

where $F_i = \oplus_j S(-a_{i,j})^{\beta_{i,j}}$. These are the syzygy modules of F – the twists $a_{i,j}$ keep track of the degrees of the generators of these modules. The main fact here is that $\mathscr{F}$ is n-regular if and only if $a_{i,j} \leq i + m$ for all (i,j). In light of this fact, the supremum of the numbers $a_{i,j} - i$ is called the regularity of F.

Could put in a fact about how important betti numbers are – they determine even MORE than the hilbert function! You can straight up recover the hilbert function.

Basepoint-free pencil trick: It might be cute to insert the original idea of Castelnuovo that led Mumford to define what we call Castelnuovo-Mumford regularity. This was called the basepoint-free pencil trick. This is Exercise 4E.13 in [Eis05].

Should I talk about how to do arguments using hyperplane sections? This notion is well-adapted to running inductive arguments by taking hyperplane sections.

Lazarsfeld notes (drawing from "What can be computed in AG") there for nice X (e.g. smooth), the regularity is bounded linearly in terms of geometric input data. But for arbitrary schemes, the regularity can grow very fast. He says that people don't really understand this – last week, were you saying that the dividing line between nice varieties and arbitrary things has to do with some sort of secant variety thing?

2.3 Regularity as a Coarse Invariant

Let X be a projective scheme of dimension n and $\mathscr{L} = \mathscr{L} \cong \mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r}(1)$ a very ample line bundle on X giving us an embedding $X \hookrightarrow \mathbb{P}(H^0(\mathscr{L}))$. The point of the following theorem is this: As long as we choose $\mathscr{L}$ to be sufficiently positive (e.g. $\mathscr{L}^{\otimes d}, d >> 0$) (and thus getting different embeddings $X \hookrightarrow \mathbb{P}$), you are not going to get anything interesting by looking at the regularity.

Lemma 2.15. *If $\mathscr{L} >> 0$ then there exists some n so that $n + 1 \leq \operatorname{reg}(X) \leq n + 2$.*

Proof. Assume $\mathscr{L} >> 0$. Its a classical theorem that $H^1(\mathscr{I}_X(m)) = 0$ for all m (that is, X is projectively normal). If $i \geq 2$, then by running the long exact sequence on

How do we know this?

$$0 \to \mathscr{I}_X(m-i) \to \mathcal{O}_{\mathbb{P}^r}(m-i) \to \mathcal{O}_X(m-i) \to 0$$

[6]The problem comes about if and only if $\mathscr{F}$ has 0-dimensional associated points – I find the reason for this hard to describe briefly, but it is related to local cohomology [Eis05]. We could equivalently assume that $H^0(\mathbb{P}^r, \mathscr{F}(k)) = 0$ for $k << 0$, or just suck it up and work with any truncation $\oplus_{k \geq k_0} H^0(\mathbb{P}^r, \mathscr{F}(k))$.

yields

$$\begin{aligned} H^i(\mathbb{P}^r, \mathscr{I}_X(m-i)) &= H^{i-1}(X, \mathcal{O}_X(m-i)) \\ &= H^{i-1}(X, \mathscr{L}^{\otimes m-i}) \\ &= H^{i-1}(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-i)) \end{aligned}$$

Take $i = n+1$, the last group becomes $H^n(X, \mathscr{L}^{\otimes m-n-1})$. When does this vanish? Certainly, we want the exponent be nonnegative thus $n+1 < m$, then $H^n(X, \mathscr{L}) = 0$ ∎

I dont understand this

Moral: If you want to fix a variety and look at very positive embeddings defined by a complete linear series, then you are not going to get anything interesting from regularity.

3 Syzygies

So now we have an effective bound on which coherent sheaves have all of these nice properties. Consider any line bundle $\mathscr{L}$ on a projective scheme X – the regularity tells us when $\mathscr{L}(n)$ is globally generated, or in other words, when $\mathscr{L}$ gives a map from X into projective space. Serre's vanishing tells us when we have a map into projective space.

3.1 Introduction

Theorem 3.1 (Castelnuovo [Cas93])**.** *Let C be a smooth projective curve of genus g, and $\mathscr{L}$ a line bundle on C of some degree d.*

i If $d \geq 2g+1$ we already know that $\mathscr{L}$ is very ample, but furthermore it gives an embedding of $C \hookrightarrow \mathbb{P}^r$ [7] where C is projectively normal.

Explain this.

ii If $d \geq 2g+2$ then $\mathscr{I}_C$ is generated by quadrics.[8]

Apparently, Mumford Kempf looked at analogous questions for abelian varieties. Mumford wanted to construct an algebraic theory of theta functions to construct moduli (why would this help?) so he has a series of papers on equations defining abelian varieties – the first volume of invencionnes.

Although the following question was seemingly not investigated classically, it brings us to the questions that led Mark Green to syzygies.

Question 3.2. *What happens when $d \geq 2g+3$? How about higher?*

3.2 Set up

Let $S = \operatorname{Sym} H^0(\mathscr{L})$ be the homogeneous coordinate ring of $\mathbb{P}(H^0(\mathscr{L})) = \mathbb{P}^r$, the space we are embedding X in. Likewise, look at the graded ring associated to $\mathscr{L}$: $R = R(L) = \bigoplus_{m \geq 0} H^0(mL)$ – this is like the homogeneous coordinate ring of

[7] $\mathscr{L}$ is nonspecial thus $h^1(\mathscr{L}) = 0$ and by Riemann-Roch we have $h^0(\mathscr{L}) = d+1-g \implies \dim|\mathscr{L}| = d-g$ after modding out by k^*.

[8] This is the best possible case. If it were generated by linears, then it would be contained in a hyperplane and thus is degenerate.

$\mathcal{O}_X$. This is a finitely generated graded S-module, and like any f.g. module over a polynomial ring, it has a minimal graded free resolution $F_\bullet$.

We want to start by picking generators of R to build F^1. In degree zero $R_0 = H^0(\mathcal{O}_X) = k$, we have a canonical generator since there we have the constant functions.

$$\begin{array}{ccccccccccccc} 0 & \longleftarrow & R & \longleftarrow & \bigoplus S(-a_{0,j}) & \longleftarrow & \bigoplus S(-a_{1,j}) & \longleftarrow & \cdots & \longleftarrow & \bigoplus S(-a_{r,j}) & \longleftarrow & 0 \\ & & & & F_0 & & F_1 & & \cdots & & F_r & & \end{array}$$

Minimality here means that the image of the map is in the maximal ideal of the next term. More concretely, it means that the matrices of homogeneous polynomials defining the maps don't have any nonzero constants inside of them. If we have a nonzero constant in our matrix at some step, this would be mean that the target has a generator that we've immediately set equal to zero in the next step. This is the sense in which minimality detects irrelevant information.

Fact 3.3. *This minimal free resolution $F_\bullet$ is both unique and guaranteed to end by r steps. Furthermore, all other free resolutions of M are quasi-isomorphic to $F_\bullet$ – any other free resolution of $F_\bullet$ is isomorphic to a direct sum of $F_\bullet$ and some trivial complex.*

Of course, it might end sooner. In general the length is between r and $r - n$. If its Cohen-Macaulay, then its as small as possible – $r - n$.

Remark (Moral). Since the minimal free resolution is unique, we can make many invariants out of a free resolution of a module.

Lemma 3.4. $E_0 = S \iff X \subset \mathbb{P}^r$ *is projectively normal.*

Finish this

Proof. If E_0 is S, this means that all the sections of the higher powers of $\mathscr{L}$ come from taking homogeneous polynomials coming from degree 1 generators. ∎

In the case when X is projectively normal, the kernel of the first map is just the ideal of X, so $F_\bullet$ becomes a resolution of the ideal defining X:

$$0 \longleftarrow I_X \longleftarrow F_1 \longleftarrow \cdots$$

Remark 3.5. All $a_{0,j} \geq 1$. More generally, for all $p \geq 1$, all $a_{p,j} \leq p + 1$. All I am saying here is that at each step in the resolution, our shifts are going up by at least 1. If it weren't, we wouldn't have a minimal free resolution. For example, why are the generators of F_1 all above degree 2? If F_1 had a generator of degree 1, this would be like an injection of $V(I_X) = X$ into a linear space $V(L_1)$.

3.2.1 Two examples

Put in twisted cubic and elliptic curve.

3.3 N_p Conditions

The next natural question is to ask (Like Green did): When are the first k terms in $F_\bullet$ generated in the lowest possible degree?

Definition 3.6 (N_p conditions). $\mathscr{L}$ satisfies (N_p) if $E_0 = S$ and if for all $1 \le p \le k$ $E_p = \oplus S(-p-1)$

$$0 \longleftarrow R \longleftarrow S \longleftarrow \bigoplus S(-2) \longleftarrow \cdots \longleftarrow \bigoplus S(-r) \longleftarrow 0$$

So the conditions (N_k) organizes a bunch of classical properties for us:

(N_0) $\iff$ $\mathscr{L}$ normally generated

(N_1) $\iff$ (N_0) and I_X is generated by quadrics

(N_2) $\iff$ (N_1) and the relations between those quadric generators have degree 1 coefficients

Returning to our two example:

Fill this in

Theorem 3.7 (Green). *Consider a curve C of genus g, and $\mathscr{L}$ a line bundle of degree d. If $d \ge 2g+1+k$ then $\mathscr{L}$ satisfies (N_k).*

Discuss how this theorem predicts what happens in the examples, explore what it says for $k = 1, \dots, n$.

The key idea of the proof is to interpret the syzygies as koszul cohomology groups.

4 Koszul Cohomology

4.1 Algebraic Introduction

Greens 1984 paper did not say too much that was new – what was important was the definition! The indexing here gets a bit confusing, don't know of a way to fix that.

Definition 4.1. Let $K_{p,q}(X, \mathscr{L})$ be the degree $(p+q)$ minimal generators of p^{th} module of syzygies of $R(\mathscr{L})$.

The index p is telling us is telling us the *weight*, or how far back we are in the resolution. Then the index q is telling us something about the degree of the generators there. So in terms of our setup, we have

$$F_p = \oplus_q K_{p,q} \otimes S(-p-q)$$

Refer back to twisted cubic example. Now is the time to introduce betti tables

Example 4.2. We can now summarize the (N_k) for $k \geq 1$ conditions easier: An ample line bundle $\mathscr{L}$ on a projective variety is said to satisfy the property (N_k) if $K_{p,q}(X, L) = 0$ for all $q \neq 1$ and $1 \leq p \leq k$.

To reiterate: for $k \geq 1$, (N_k) means that $\mathscr{L}$ is normally generated, the ideal of X in the embedded is generated by quadratics, and all the syzyges up to order k are linear. In terms of betti tables, (N_k) means that after the degree zero generator, everything is on the first row for k steps.

But how do we actually get compute these things? The next theorem says that they are the cohomology groups of some koszul-like complex.

Theorem 4.3. $K_{p,q}(X, \mathscr{L})$ *is the middle cohomology of*

$$\bigwedge^{p+1} H^0(\mathscr{L}) \otimes H^0(\mathscr{L}^{\otimes q-1}) \longrightarrow \bigwedge^{p} H^0(\mathscr{L}) \otimes H^0(\mathscr{L}^{\otimes q}) \longrightarrow \bigwedge^{p-1} H^0(\mathscr{L}) \otimes H^0(\mathscr{L}^{\otimes q}$$

where

$$\partial(v_1 \wedge \cdots \wedge v_p \otimes s) = \sum_i (-1)^i v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \otimes v_i \cdot s$$

Proof. Let $k = S/\mathfrak{m}$ be the residue field of S, where $\mathfrak{m}$ is the irrelevant ideal – all positively graded pieces of S. Since $E_\bullet$ is a minimal resolution,

$$K_{p,q} = (E_p)_q = \mathrm{Tor}_p^S(R, k)_{p+q}.$$

This is by the definition of Tor – $E_\bullet$ is a minimal free resolution of R and then when we tensor by k, our maps in this resolution are zero.This is because $E_p \otimes k = S^n \otimes S/\mathfrak{m} = k^n$ and maps between fields are either injective or zero.[9]

So we computed Tor by resolving k and tensoring by R. On the other hand, we can compute these Tor by resolving $\mathbb{C}$ and tensoring by R. But we can resolve $\mathbb{C}$ by the Koszul complex:

$$0 \longleftarrow kS \longleftarrow H^0(\mathscr{L}) \otimes S(-1) \longleftarrow \wedge^2 H^0(\mathscr{L}) \otimes S(-2) \longleftarrow \cdots$$

Tensoring by R gives you the complex in the theorem. ■

So far everything here has been some sort of algebraic object – although built by line bundles and whatnot, our objects are algebraic.

4.2 Geometric Introduction

The point of this chapter is that we can reinterpret this Koszul complex in terms of a vector bundle that lives on any variety. Given X and $\mathscr{L}$ (still very ample, for later) we have an associated evaluation map $ev : H^0(\mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L}$ that for any open U sends

[9] In fact, $E_\bullet$ is minimal if and only if tensoring by k gives zero maps.

$$\begin{aligned} ev(U) : H^0(\mathscr{L}) \otimes \mathcal{O}_X(U) &\longrightarrow \mathscr{L}(U) \\ s \otimes f &\longmapsto f \cdot s|_U \end{aligned}$$

As long as $\mathscr{L}$ is globally generated (which it is, since it is very ample), this is a surjection.

Definition 4.4 (Kernel bundle)**.** Suppose that $\mathscr{L}$ is globally generated. The kernel $M_{\mathscr{L}}$ of the surjective map

$$ev : H^0(\mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L}$$

is called the *kernel bundle* of $\mathscr{L}$. It is a vector bundle of rank $h^0(X, \mathscr{L}) - 1$.

Remark 4.5. The short exact sequence

$$0 \to M_{\mathscr{L}} \to H^0(\mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L} \to 0$$

is the pull back via the morphism defined by $\mathscr{L}$ of the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}}(1) \to H^0(\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_P(1) \to 0$$

on the projective space $X \hookrightarrow \mathbb{P}(H^0(\mathscr{L}))$.

Proposition 4.6. *Assume that* $H^i(\mathscr{L}^{\otimes k}) = 0$ *when* $i > 0$ *and* $k \geq 1$. *Then for* $q > 2$,

highlight when this happens

$$K_{p,q}(X, \mathscr{L}) = H^1(\bigwedge^{p+1} M_{\mathscr{L}} \otimes \mathscr{L}^{\otimes q-1})$$

In particular, $(N_k) \iff H^1(\bigwedge^{p+1} M_{\mathscr{L}} \otimes \mathscr{L}^{\otimes q-1}) = 0$ for $q \geq 2$.

So checking (N_k) has been reduced to checking the vanishing of some vector bundles.

We can take exterior powers of

$$0 \to M_{\mathscr{L}} \to H^0(\mathscr{L}) \otimes \mathcal{O}_X \to \mathscr{L} \to 0$$

sequence to obtain the exact sequence

$$0 \to \bigwedge^p M_{\mathscr{L}} \to \bigwedge^p H^0(\mathscr{L}) \otimes \mathcal{O}_X \to \bigwedge^{p-1} M_{\mathscr{L}} \otimes \mathscr{L} \to 0$$

Then these can be threaded together:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \bigwedge^p M_{\mathscr{L}} \arrow[r] & \bigwedge^p H^0(\mathscr{L}) \otimes \mathcal{O}_X \arrow[r] \arrow[dr, dashed] & \bigwedge^{p-1} M_{\mathscr{L}} \otimes \mathscr{L} \arrow[r] \arrow[d] & 0 \\
 & & & \bigwedge^{p-1} H^0(\mathscr{L}) \otimes \mathscr{L} \arrow[d] & \\
 & & & \bigwedge^{p-2} M_{\mathscr{L}} \otimes \mathscr{L} \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

We can keep doing this to build a long exact sequence

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 \\
 & & & \bigwedge^{p-1} M_{\mathscr{L}} \otimes L^{\otimes q+1} \arrow[ur] \arrow[dr] & \\
\bigwedge^{p+1} H^0(\mathscr{L}) \otimes \mathscr{L}^{\otimes q-1} \arrow[rr] \arrow[dr] & & \bigwedge^{p} H^0(\mathscr{L}) \otimes \mathscr{L}^{\otimes q} \arrow[rr] \arrow[ur] & & \bigwedge^{p-1} H^0(\mathscr{L}) \otimes \mathscr{L}^{\otimes q+1} \\
 & \bigwedge^p M_{\mathscr{L}} \otimes L^{\otimes q} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

So $K_{p,q}(X, L)$ is the cohomology of H^0 of the middle term.

4.2.1 Cech cohomology is a type of Koszul Cohomology ★

On a reasonable scheme (quasi-compact and quasi-separated, or Noetherian), Cech cohomology of a quasi-coherent sheaf is really a type of Koszul cohomology. Of course, this is where Cech cohomology is usually defined anyways – (quasi)-separated is needed so that intersections of affine opens are (finite unions of) affine opens, and quasi-compact is needed to get a finite cover.[10]

More accurately, Cech cohomology is obtained as a direct limit of Koszul cohomologies. This allows us to compute things in algebraic geometry using properties of the Koszul complex. For instance, it lets us compute the cohomology of an affine space, and then go on to compute projective space– this is how Grothendieck does is in EGA III, and it has the advantage (unlike the argument in Hartshorne) of applying to non-Noetherian rings.

Can finish Koszul-Cech connection.

[10]Reference: Akhil Mathew's blog

FURTHER READING

For now, I will be collecting related references that I find along the way here.

- Lazarsfeld, "on the projective normality of syzygies of algebraic curves"
- Mumford "What can be computed in Algebraic geometry?"

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NOTES