Local Cohomology

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1 Motivation

As we've seen, many of our theorems in commutative algebra are motivated by geometric understanding via the (anti)equivalence of categories

$$\left\{ \begin{array}{r} \text{graded (integral) rings} \\ (R, R_+ = \oplus_{d \geq 1} R_d) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{r} \text{projective (varieties) schemes} \\ X = \operatorname{Proj} S \end{array} \right\}$$

$$\text{graded } R\text{-modules } M \longmapsto \text{quasi-coherent sheaves } \widetilde{M}$$

$$\text{Koszul Cohomology} \overset{???}{\rightsquigarrow} \text{Sheaf Cohomology}$$

On the left side, we have Koszul cohomology which tells us things about depth, dimension, etc. On the right side, we have sheaf cohomology. It seems obvious that they are related in some way – you can get algebraic data from sheaf cohomology and geometric data from Koszul cohomology.

Despite this, I don't think its obvious how exactly to get from one to the other. My goal is to explain how in a sense, understanding local cohomology fills in this connection – Čech complexes come from limits of Koszul complexes, and this connection allows us to make local-to-global arguments.

Remark 1.1. The Koszul Cohomology theory on graded modules gives a Koszul Cohomology theory on quasi-coherent sheaves. Exploring this relationship is the main topic of the famous Green 1984 paper, and continues to be an area of active research – in particular, in both Daniel's and Michael's research!

2 An algebraic definition

Given any ring R and a choice of elements $\overline{x} = (x_1, \dots, x_n) \in R^n$ we have an associated Koszul complex

$$\begin{array}{lcccccc} K^\bullet(\overline{x}) : 0 \longleftarrow R \longleftarrow & R^n \longleftarrow & \bigwedge^2 R^n \longleftarrow \cdots \longleftarrow & \bigwedge^{n-1} R^n \longleftarrow & \bigwedge^n R^n \longleftarrow 0 \\ \text{index} \qquad\qquad n & n-1 & \cdots & 2 & 1 & 0 \end{array} \qquad (*)$$

with a differential that is the dual of $- \wedge \overline{x}$. The basic fact is that when $\overline{x}$ is a regular sequence, $K(\overline{x})$ is a minimal free resolution of $R/(x_1, \dots, x_n)$. Koszul cohomology is defined by $H^i(K(\overline{x}))$. We can define the Koszul cohomology of an R-module M with respect to $\overline{x}$ by tensoring $(*)$ by M before taking cohomology.

Our Goal: Given a Koszul complex $K(\overline{x})$, we will associate to it a new related complex coming from a limit. The cohomology of this complex defines local cohomology.[1]

Consider the simplest possible Koszul complex – choose a single element x in a ring R. This gives rise to the Koszul complex

$$K^\bullet(x) : 0 \longleftarrow R \xleftarrow{x} R \longleftarrow 0$$

In this setting, x is a regular sequence if and only if x is a non-zero divisor, or if $H^0(K^\bullet(x)) = 0$. We have a natural map from $K^\bullet(x)$ to $K^\bullet(x^2)$ built from multiplying by x as follows:

$$\begin{tikzcd}
K^\bullet(x) : 0 & R \arrow[l] \arrow[d, "x"] & R \arrow[l, "x"'] \arrow[d, equal] & 0 \arrow[l] \\
K^\bullet(x^2) : 0 & R \arrow[l] & R \arrow[l, "x^2"'] & 0 \arrow[l]
\end{tikzcd}$$

Continuing this process gives us a tower of maps

$$\begin{tikzcd}
K^\bullet(x) : 0 & R \arrow[l] \arrow[d, "x"] & R \arrow[l, "x"'] \arrow[d, equal] & 0 \arrow[l] \\
K^\bullet(x^2) : 0 & R \arrow[l] \arrow[d, "x"] & R \arrow[l, "x^2"'] \arrow[d, equal] & 0 \arrow[l] \\
K^\bullet(x^3) : 0 & R \arrow[l] \arrow[d, "x"] & R \arrow[l, "x^3"'] \arrow[d, equal] & 0 \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

Define $K^\bullet(x^\infty)$ by taking the direct limit of the vertical arrows. One can check that

$$\varinjlim \left(R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots \right) = R_x$$
$$\varinjlim \left(R = R = R = \cdots \right) = R$$

and that

$$K^\bullet(x^\infty) : \qquad 0 \longleftarrow R_x \longleftarrow R \longleftarrow 0$$

is the standard localization map – lets call this the local cohomology complex with respect to x.

Definition 2.1. The *local cohomology module* $H^i_x(M)$ of an R-module M with respect to the ideal (x) is the cohomology of $K^\bullet(x^\infty) \otimes M$.

[1] This is a description of local cohomology due to Serre.

This definition can be bootstrapped up to get us the more general definition for any ideal I. Since $\otimes_R$ commutes with direct limits, we can get the local cohomology complex with respect to $\overline{x} = (x_1, x_2, \ldots, x_n) \in R^n$ by taking the tensor complex of all the $K^\bullet(x_i^\infty)$:

$$\begin{aligned}
K^\bullet(\overline{x}^\infty) &= \bigotimes_{i=1}^{n}(0 \leftarrow R_x \leftarrow R \leftarrow 0) \\
&= 0 \leftarrow R_{\overline{x}} \leftarrow \bigoplus_{i=1}^{n} R_{x_1 \ldots \widehat{x_i} \ldots x_n} \leftarrow \cdots \leftarrow R_{x_1} \oplus R_{x_2} \oplus \cdots \oplus R_{x_n} \leftarrow R \leftarrow 0
\end{aligned}$$

Definition 2.2. The *local cohomology module* of M with respect to $I = (x_1, \ldots, x_n)$ is

$$H^i_I(M) = H^i(K^\bullet(\overline{x}^\infty) \otimes M).$$

Example 2.3. Let $R = k[x, y]$, choose the ideal $\mathfrak{m} = (x, y)$ and consider $M = R/(y^2, xy)$. We will compute $H^i_{\mathfrak{m}}(M)$. Following the above definition, we get the complex

$$0 \longleftarrow M_{xy} \xleftarrow{(1,-1)} M_x \oplus M_y \xleftarrow{\begin{pmatrix}1\\1\end{pmatrix}} M \longleftarrow 0$$

Note that both y and xy are nilpotent, thus $M_{xy} = 0$ and $M_y = 0$. So our complex simplifies to the localization map

$$0 \longleftarrow M_x \longleftarrow M \longleftarrow 0$$

So $H^0_{\mathfrak{m}}(M)$ is the set of elements of M annihilated by a power of x – this only consists of y, so

$$H^0_{\mathfrak{m}}(M) = (y)/(y^2, xy)$$

which as a vector space over k is simply $k \cdot y$.

Similarly, for $H^1_{\mathfrak{m}}(M) = M_x/\operatorname{im}(M \to M_x)$, we know that $M_x = (k[x,y]/(y^2, xy))_x = k[x]_x$, and we've just shown that the only thing killed in the map $M \to M_x$ is (y) (leaving (x)), thus

$$H^1_{\mathfrak{m}}(M) = (k[x]_x)/(x) \cong k \cdot x^{-1} \oplus k \cdot x^{-2} \oplus \cdots$$

From this definition, we can already get a bunch of important properties, inherited from properties we've already shown about Koszul complexes.

Fact 2.4. *Given a ring* R *and ideal* I*, the local cohomology modules* $H^i_I(R)$ *satisfy*

(i) *(H^i_I is geometric)* $H^i_I(R) = H^i_{\sqrt{I}}(R)$.

(ii) *(H^i_I detects depth) For any finitely generated* R*-module* M*,*

$$\operatorname{depth}(I, M) = \min\{r \mid H^r_I(M) \neq 0\}$$

(iii) *(H^i_I detects dimension) Let $(R, \mathfrak{m})$ be local. For any finitely generated R-module M,*

$$\dim(M) = \max\{r \mid H^r_{\mathfrak{m}}(M) \neq 0\}$$

(iv) *(H^i_I is supported on I) Every element of $H^i_I(M)$ is killed by a power of I.*

To reiterate, you can imagine that these are properties of the Koszul cohomology that survive this limit process (taking direct limits is exact) and thus become properties of local cohomology.

So what are we getting that is new? Why study local cohomology in addition to Koszul cohomology? We can answer these questions after we see a few more equivalent definitions.

3 A Geometric Definition

This is how Grothendieck originally defined local cohomology. Given a topological space X, we can define a sheaf $\mathscr{F}$ on X (of abelian groups, or whatever) – for the uninitiated, by definition this is something that assigns an algebraic object $\Gamma(U, \mathscr{F})$ to every open in our space X in a consistent way. Abstractly, sheaf cohomology comes from taking the right derived functors of the left exact global section functor $\Gamma(X, -)$.

Now, suppose we had a closed subset $Y \hookrightarrow X$. Since $X \setminus Y$ is in open in X, we have an induced restriction map

$$\Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F}).$$

This map has a kernel, call it $\Gamma_Y(X, \mathscr{F})$, so that we have an exact sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F})$$

Fact 3.1. *$\Gamma_Y(X, \mathscr{F})$ is exactly the global sections of $\mathscr{F}$ on X with support in Y and is left exact.*

Proof. Left exactness is induced via left exactness of Γ and

$$\begin{aligned} H^0_Y(X, \mathscr{F}) &= \ker\left(H^0(X, \mathscr{F}) \to H^0(X \setminus Y, \mathscr{F})\right) \\ &= \{s \in H^0(X, \mathscr{F}) \mid s_x = 0 \text{ for every } x \in X \setminus Y\} \\ &= \{s \in H^0(X, \mathscr{F}) \mid \operatorname{supp}(s) \subseteq Y\}. \end{aligned}$$

■

Remark 3.2. Taking $Y = X$ recovers the usual definition for sheaf cohomology – we put no restriction on where the support needs to be!

So we can get a new cohomology theory from letting $H^i_Y(X, \mathscr{F})$ be the right derived functor of $H^0_Y(X, -)$, and it's local in the sense that it comes from restricting a global cohomology theory.

The idea (which should not be clear at this point) is that this purely geometric thing is computed via the algebraic complexes we've already defined. To be more precise, let

$X = \operatorname{Spec} R$ be an affine scheme and $Y = V(I)$ for some ideal $I \subseteq R$. Since elements of M are global sections of $\tilde{M}$ on $\operatorname{Spec} R$,

$$H^0_I(M) = \Gamma_{V(I)}(X, \widetilde{M})$$

can be thought of as the set of sections with support on $\operatorname{Spec} R/I$. Furthermore, the cohomology of the complex

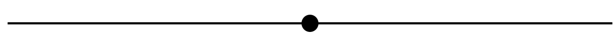
$$K^\bullet(\overline{x}^\infty) \otimes M : 0 \leftarrow M_{\overline{x}} \leftarrow \bigoplus_{i=1}^{n} M_{x_1 \ldots \widehat{x_i} \ldots x_n} \leftarrow \cdots \leftarrow \bigoplus_{i=1}^{n} M_{x_i} \leftarrow M \leftarrow 0$$

computes $H^i_I(M) = H^i_{V(I)}(X, \widetilde{M})$.

Example 3.3. In our previous example we computed

$$H^i_{\mathfrak{m}}(R/(y^2, xy)) = \begin{cases} k \cdot y, & i = 0 \\ k \cdot x^{-1} \oplus k \cdot x^{-2} \oplus \cdots, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

where $R = k[x, y]$ and $\mathfrak{m} = (x, y)$. This is the same as the sheaf cohomology of $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ with sections whose support is in $V(y^2, xy)$, which we can imagine as a line with a "fat" point[2] since $(y^2, xy) = (y) \cap (x, y)^2$



A SEEMING COINCIDENCE

In practice, sheaf cohomology is computed using a Čech complex with respect to a "nice enough" cover.[3] As before, let $X = \operatorname{Spec} R$, $I = (f_1, \ldots, f_n)$ be an ideal of R, and $\widetilde{M}$ be the quasi-coherent sheaf associated to a R-module M. Then $X - V(I)$ is a scheme covered by $X_{f_i} = X - V(f_i)$. By definition, the Čech complex with respect to the covering $\{X_{f_i}\}$ is

$$0 \leftarrow \Gamma(X_{\overline{f}}, \widetilde{M}) \leftarrow \bigoplus \Gamma(X_{f_1 \cdots \widehat{f_i} \cdots f_n}, \widetilde{M}) \leftarrow \cdots \leftarrow \bigoplus \Gamma(X_{f_i}, \widetilde{M}) \leftarrow 0 \qquad (*)$$

Its an extremely general[4] fact that

$$\Gamma(X_f, \mathscr{F}) = \Gamma(X, \mathscr{F})_f.$$

[2]This is *not* Cohen-Macaulay because we have an embedded point.

[3]This is yet another cohomology theory dependent on a choice of covering. If you pick a nice enough cover, we can prove it agrees with sheaf cohomology.

[4]Indeed, all you need is that X is quasi-separated and quasi-compact, just to get unions of finitely many affines and that intersections of affines can be covered again by finitely many affines! The proof is very formal, and just comes from localizing the sequence

$$0 \to \Gamma(X, \mathscr{F}) \to \prod \Gamma(U_i, \mathscr{F}) \to \prod_{i,j} \prod_{k} \Gamma(U_{ijk}, \mathscr{F})$$

Thus our opens are of the form $X_{f_i} = \operatorname{Spec} R_{f_i}$. Thus $(*)$ becomes

$$\mathcal{C}^\bullet(X, \widetilde{M}) : 0 \leftarrow M_{\overline{x}} \leftarrow \bigoplus_{i=1}^{n} M_{x_1 \ldots \widehat{x_i} \ldots x_n} \leftarrow \cdots \leftarrow \bigoplus_{i=1}^{n} M_{x_i} \leftarrow 0$$

But notice! This is just a subcomplex of our local cohomology complex

$$K^\bullet(\overline{x}^\infty) \otimes M : \underbrace{0 \leftarrow M_{\overline{x}} \leftarrow \bigoplus_{i=1}^{n} M_{x_1 \ldots \widehat{x_i} \ldots x_n} \leftarrow \cdots \leftarrow \bigoplus_{i=1}^{n} M_{x_i}}_{\mathcal{C}^\bullet(X, \widetilde{M})} \leftarrow M \leftarrow 0$$

This gives us an exact sequence of complexes

$$0 \to \mathcal{C}^\bullet(X, \widetilde{M}) \to K^\bullet(\overline{x}^\infty \otimes M) \to M \to 0$$

whose long exact sequence gives the following theorem immediately.

Theorem 3.4. *There is an exact sequence of* R*-modules*

$$0 \to H^0_I(M) \to M \to H^0(X - V(I), \widetilde{M}) \to H^1_I(M) \to 0$$

and for every $i \geq 2$

$$H^i_I(M) = H^{i-1}(X - V(I), \widetilde{M})$$

Remark 3.5. Given that we know

$$H^0_I(M) = \ker\left(H^0(X, \widetilde{M}) \to H^0(X - V(I), \widetilde{M})\right)$$

it should be possible to express the local cohomology complex in terms of the Cech complexes on X and $X - V(I)$, and indeed it is!

Going through the Čech complex formalism on the trivial cover $X_0 = \operatorname{Spec} R$, we have the complex

$$0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow M \leftarrow 0.$$

This has a natural map to the complex $\mathcal{C}^\bullet(X, \widetilde{M})$

$$\begin{array}{ccccccccccccc}
M & & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & M & \longleftarrow & 0 \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\mathcal{C}^\bullet(X, \widetilde{M}) & & 0 & \longleftarrow & M_{\overline{x}} & \longleftarrow & \bigoplus_{i=1}^{n} M_{x_1 \ldots \widehat{x_i} \ldots x_n} & \longleftarrow & \cdots & \longleftarrow & \bigoplus_{i,j} M_{x_i x_j} & \longleftarrow & \bigoplus_{i=1}^{n} M_{x_i} & \longleftarrow & 0
\end{array}$$

The cone of this map is the local cohomology complex

$$K^\bullet(\overline{x}^\infty) \otimes M : 0 \leftarrow M_{\overline{x}} \leftarrow \bigoplus_{i=1}^{n} M_{x_1 \ldots \widehat{x_i} \ldots x_n} \leftarrow \cdots \leftarrow \bigoplus_{i=1}^{n} M_{x_i} \leftarrow M \leftarrow 0$$

giving us the exact triangle

$$\begin{tikzcd} \mathcal{C}^\bullet(X, \widetilde{M}) \arrow[rr] & & \mathcal{C}^\bullet(X - V(I), \widetilde{M}) \arrow[dl, "{[1]}"] \\ & K^\bullet(\overline{x}^\infty) \otimes M \arrow[ul] & \end{tikzcd}$$

That is, this corresponds into our sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X \setminus Y, \mathscr{F}) \xrightarrow{[1]}$$

4 Local Cohomology and Global Cohomology

Let S be a graded ring and $X = \operatorname{Proj} S$, with irrelevant ideal $\mathfrak{m}$. Any graded S-module M gives a quasi-coherent sheaf $\widetilde{M}$ on X.

Fact 4.1. [5] *The Čech complex for $\widetilde{M}$ on X is the degree 0 part of the Čech complex for $\operatorname{Spec} S - V(\mathfrak{m})$. In particular*

$$H^i(\operatorname{Proj} S, \widetilde{M}) = \left(H^i(\operatorname{Spec} S - V(\mathfrak{m}), \widetilde{M})\right)_0.$$

Proof. This is essentially because of the fact that $\operatorname{Spec} S$ can be thought of as the affine cone over $\operatorname{Proj} S$ – removing $V(\mathfrak{m})$ removes the cone point, and taking the zeroth degree dehomogenizes. ■

In fact, in general,

$$H^i(\operatorname{Proj} S, \widetilde{M}(d)) = \left(H^i(\operatorname{Spec} S - V(\mathfrak{m}), \widetilde{M})\right)_d.$$

With this in mind, Theorem (3.4) becomes

Theorem 4.2. *There is an exact sequence of S-modules*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{d \geq 0} H^0(\operatorname{Proj} S, \widetilde{M}(d)) \to H^1_{\mathfrak{m}}(M) \to 0$$

and for every $i \geq 2$

$$H^i_{\mathfrak{m}}(M) = \bigoplus_{d \geq 0} H^{i-1}(X - V(\mathfrak{m}), \widetilde{M}(d))$$

[5] I'm lying a bit here, I think although this is certainly true if $S = k[x_0, \dots, x_n]$. Really, I think have

$$H^i(\operatorname{Proj} S, \widetilde{M}) = \left(H^i(\operatorname{Spec} A - V(\mathfrak{m}), \widetilde{M})\right)_0$$

where $A = \bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}$. In general, $H^i(\operatorname{Proj} R, \widetilde{M}) = \left(H^i(\operatorname{Spec} A - V(I), \widetilde{M})\right)_0$ where $A = \bigoplus_k I^k/I^{k+1}$, the Artin-Rees algebra. Intuitively, you can imagine that this is because if you just wrote the right side down in isolation, it may not have a natural grading, but this strange algebra does and it comes from a natural filtration. I'd have to double check this.

This can help us understand when $\Gamma_*(M) = \bigoplus_{d\geq 0} H^0(\widehat{M}(d))$ is finitely generated – it tells us that in some sense, the only obstruction happens at zero-dimensional associated points and is measured by local cohomology.

Example 4.3. Let $\mathcal{O}_{V(x)}$ on $\mathbb{P}^1$, and $S = k[x,y]$. We want to say that $\mathcal{O}_X = \widetilde{S/(x)}$. However, $S/(x) = k[y]$ is not the same as $\bigoplus_{d\geq 0} H^0(\mathcal{O}_X(d))$ since $H^0(\mathcal{O}_X(d)) = H^0(\mathcal{O}_X) = k$.

The following Corollary classifies our problem.

Corollary 4.4. *Let M be a finitely generated graded S-module. The natural map*

$$M \to \bigoplus_{d\geq 0} H^0(X, \widehat{M}(d))$$

is an isomorphism if and only if $\operatorname{depth} M \geq 2$. *That is,* $\operatorname{depth}(\mathfrak{m}, M) \leq 1$ *if and only if Ass*(M) *contain points of* X.

LOCAL DUALITY

One of the most important results about sheaf cohomology is Serre duality, which says that for a sheaf $\mathscr{F}$ on a projective space of dimension d we have

$$H^i(X, \mathscr{F}) \cong \operatorname{Ext}_X^{d-i}(\mathscr{F}, \mathcal{O}_X(-d-1))^\vee.$$

When $\mathscr{F}$ is a line bundle, this becomes the more familiar

$$H^i(X, \mathscr{F}) \cong H^{d-i}(X, \mathscr{F}^{-1} \otimes \omega_X)^\vee.$$

This has a local version (this is not the most general version, but I think what I've written is correct):

Theorem 4.5. *If $(S, \mathfrak{m})$ is a graded ring with irrelevant ideal $\mathfrak{m}$ of dimension d, and M is an S-module then*

$$H_I^i(M) \cong \operatorname{Ext}_S^{d-i}(M, S(-d-1))^\vee.$$

We can use this to calculate $H_I^i(M)$.

Example 4.6. Let $S = k[x,y]$ and return to our previous example where we computed $H^i_{\mathfrak{m}}(S/(y^2, xy))$. Since (y^2, xy) just has two elements, its not hard to just write down a minimal free resolution of $S/(y^2, xy)$

$$0 \longrightarrow S(-3) \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} S^2(-2) \xrightarrow{(x^2,\, xy)} S \longrightarrow S/(y^2, xy) \longrightarrow 0$$

The cohomology of the dual complex (after twisting by $S(-2)$)

$$0 \longleftarrow S(1) \xleftarrow{(y,\, -x)} S^2 \xleftarrow{\begin{pmatrix} x^2 \\ xy \end{pmatrix}} S \longleftarrow S(-2) \longleftarrow 0$$

computes $\mathrm{Ext}^i_S(M, S)$. One can compute that

$$\mathrm{Ext}^i_S(M,S) = \begin{cases} k(-1) \oplus k(-2) \oplus \cdots, & i = 1 \\ k(1), & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

By dualizing and reindexing we see that

$$\mathrm{Ext}^{1-i}_S(M,S)^\vee = \begin{cases} k(-1), & i = 1 \\ k(1) \oplus k(2) \oplus \cdots, & i = 2 \\ 0, & \text{otherwise.} \end{cases}$$

which agrees with $H^i_{\mathfrak{m}}(S/(y^2, xy))$ as desired.

5 Addendum

The derived functor formulation of sheaf cohomology can be used to define local cohomology as well. Given a ring R, an ideal $I = \langle x_1, \dots, x_s \rangle$ of R, and an R-module M, we can define the 0-th local cohomology module of M with support in I to be

$$\begin{aligned} H^0_I(M) &:= \{m \in M \mid mI^n = 0 \text{ for some } n\} \\ &= \bigcup_n (0 :_M I^n) \\ &= \lim_{n\to\infty} \mathrm{Hom}_R(R/I^n, M) \qquad \text{since } (0 :_M I^n) = \mathrm{Hom}_R(R/I^n, M) \end{aligned}$$

Equivalently, this is the saturation of (0) by I. Although it looks pretty different, this agrees with our previous definition.

Lemma 5.1. $H^0_I(M) = H^0(K^\bullet(\overline{x}^\infty) \otimes M)$.

Proof. The right side is obtained by taking the middle cohomology of

$$M_{x_1} \oplus M_{x_2} \oplus \cdots \oplus M_{x_n} \leftarrow M \leftarrow 0,$$

which is just the kernel of the localization map $M \to \bigoplus_{i=1}^n M_{x_i}$. An element r goes to zero under the localization of every x_i if and only if $x_i^n m = 0$ for some n. ■

Note that since Hom is left-exact, H^0_I is also left-exact, so we can take the right-derived functors of it. If $M \to A^\bullet$ is an injective resolution, then we define $H^i_I(M)$ for $i \geq 1$ to be the ith cohomology of the complex $H^0_I(A^\bullet)$. Using an argument about the universality of the derived functor, one can show that these higher local cohomology modules can be defined in terms of the derived functor of Hom, i.e. Ext, or in terms of Koszul cohomologies:

$$\begin{aligned} H^i_I(M) &= \lim_{n\to\infty} \mathrm{Ext}^i_R(R/I^n, M) \\ &= H^i(K^\bullet(\overline{x}^\infty) \otimes M). \end{aligned}$$

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