Math 221 Worksheet 17
October 29, 2020
Section 4.1: Area and Riemann Sums; Section 3.9: Antiderivatives

1. Evaluate the following sums:

(a) $\sum_{i=0}^{3} 2^i$

$$2^0 + 2^1 + 2^2 + 2^3 = 15$$

(b) $\sum_{n=1}^{4} n^2$

$$1^2 + 2^2 + 3^2 + 4^2 = 30$$

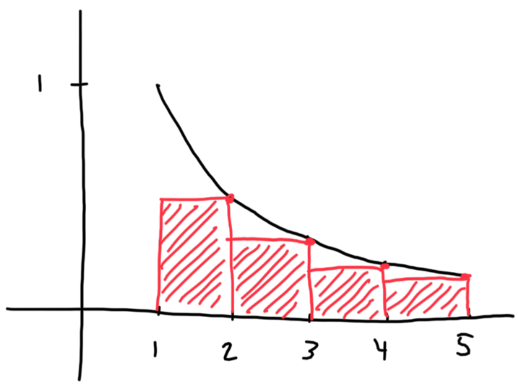
(c) $\sum_{j=10}^{100} (-1)^j$

$$(-1)^{10} + (-1)^{11} + (-1)^{12} + (-1)^{13} + \cdots + (-1)^{99} + (-1)^{100}$$

$$= 1 + (-1) + 1 + (-1) + \cdots + (-1) + 1$$

$= 1$ (because the number of 1's is one more than the number of -1's)

2. (a) Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$ by forming a Riemann sum of four rectangles using the right endpoints. Is your estimate too high or too low?



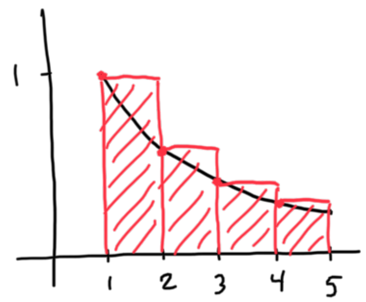
$$\sum_{i=1}^{4} \frac{1}{1+i\,\frac{5-1}{4}} \cdot \frac{5-1}{4} = \sum_{i=1}^{4} \frac{1}{i+1}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$= \frac{77}{60}.$$

This is an underestimate.

(b) Repeat part (a) using the left endpoints.



$$\sum_{i=1}^{4} \frac{1}{1+(i-1)\frac{5-1}{4}} \cdot \frac{5-1}{4} = \sum_{i=1}^{4} \frac{1}{i}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$= \frac{25}{12}.$$

This is an overestimate.

3. (a) Write the area under the graph of $f(x) = x^3$ from $x = 0$ to $x = 3$ as a limit of Riemann sums. (You do not need to evaluate the limit.)

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \left(0 + i \cdot \frac{3-0}{n}\right)^3 \cdot \frac{3-0}{n}$$

(These Riemann sums used right endpoints. You could also use left endpoints, etc.)

(b) Write the area under the graph of $f(x) = \frac{2x}{x^2+1}$ from $x = 1$ to $x = 3$ as a limit of Riemann sums. (You do not need to evaluate the limit.)

$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2\left(1 + i \cdot \frac{3-1}{n}\right)}{\left(1 + i \cdot \frac{3-1}{n}\right)^2 + 1} \cdot \frac{3-1}{n}$$

(Same comment as above)

4. (a) Using the fact that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(2n+1)(n+1)n}{6}$, evaluate $\lim_{n \to \infty} \sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n}$.

$$\sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n} = \sum_{j=1}^{n} j^2 \cdot \frac{8}{n^3}$$

$$= \frac{8}{n^3} \sum_{j=1}^{n} j^2 = \frac{8}{n^3} \cdot \frac{(2n+1)(n+1)n}{6}.$$

So $$\lim_{n \to \infty} \sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n} = \lim_{n \to \infty} \frac{8(2n+1)(n+1)n}{6n^3} = \frac{16}{6} = \frac{8}{3}.$$

(b) Explain why the limit from part (a) is equal to the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 2$.

$\sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n} = \sum_{j=1}^{n} \left(0 + j\frac{2-0}{n}\right)^2 \frac{2-0}{n}$ is a Riemann sum for $f(x) = x^2$ over $[0,2]$ using right endpoints. So the limit as $n \to \infty$ gives the area.

5. Use the idea of area to evaluate $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \sqrt{1 - \left(\frac{j}{n}\right)^2}$. (Hint: Think about the function $f(x) = \sqrt{1-x^2}$.)

$\frac{1}{n} \sum_{j=1}^{n} \sqrt{1-\left(\frac{j}{n}\right)^2} = \sum_{j=1}^{n} \sqrt{1-\left(0+j\frac{1-0}{n}\right)^2}\, \frac{1-0}{n}$ is a Riemann sum for $f(x) = \sqrt{1-x^2}$ over $[0,1]$. So the limit is the area under the graph of $\sqrt{1-x^2}$ on $[0,1]$, namely $\frac{\pi}{4}$ since the graph is a quarter unit circle.

6. A stone is dropped off a cliff and hits the ground at a speed of 120 feet per second. Assuming the acceleration due to gravity is 32 feet per second, what is the height of the cliff?

Let $s(t)$ be the height of the stone above the ground at time t. We want to find $s(0)$, the height of the cliff. We know that acceleration is given by

$$s''(t) = -32.$$

So

$$s(t) = -16t^2 + Ct + C'$$

for some constants C, C'. Since the stone has initial velocity zero, we know that $0 = s'(0) = C$. We also know that $s'(t) = -120$ when $s(t) = 0$. This occurs when $t = \frac{\sqrt{C'}}{4}$. (Note that $C' \geq 0$ because $C' = s(0)$, the height of the cliff.) So we solve

$$-120 = s'\left(\frac{\sqrt{C'}}{4}\right) = -32 \cdot \frac{\sqrt{C'}}{4}$$

and find that $C' = 225$. Thus, $s(0) = 225$ feet.