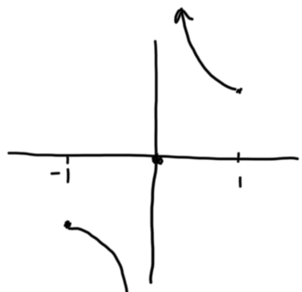
Math 221 Worksheet 11
October 8, 2020
Section 3.1: Maximum and Minimum Values

1. State the extreme value theorem. Find a function f that satisfies the following: (i) the domain of f is $[-1, 1]$, (ii) f is discontinuous at exactly one point in its domain, (iii) f attains neither a maximum nor a minimum value.

EVT: If f is continuous on $[a,b]$, then f attains both a minimum and a maximum on $[a,b]$.

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{x}, & \text{if } x \in [-1,1] \ \& \ x \neq 0. \end{cases}$$



2. Find all critical points of the function $f(x) =$ [redacted]

$x^4 + 2x^2 + 8x$

f' exists everywhere, so the critical points are the values of x such that $f'(x) = 0$.

$$f'(x) = 4x^3 + 4x + 8$$
$$= 4(x+1)(x^2 - x + 2).$$

The only solution to $f'(x) = 0$ is $\boxed{x = -1}$

3. Find all critical points of the function $f(x) = \sin(x)\cos(x)$ in the interval $[0, 2\pi]$.

Again, f' exists everywhere, so we just need to solve $f'(x) = 0$ for $x \in [0, 2\pi]$.

$f'(x) = \cos^2(x) - \sin^2(x)$, so $f'(x) = 0$ when $\cos(x) = \pm\sin(x)$.

In $[0, 2\pi]$, this happens for $\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.}$

4. Find the global minimum and maximum of the function $f(x) = 3x^2 - 12x + 5$ on the interval $[0, 3]$.

$f'(x) = 6x - 12$, so $x = 2$ is the only critical point.

Test at critical points and endpoints: $f(2) = -7$, $f(0) = 5$, $f(3) = -4$.

$\boxed{\begin{array}{l} \text{min. is } -7 \\ \text{max. is } 5 \end{array}}$

5. Find the global minimum and maximum of the function $f(x) = \sin(x) + \cos(x)$ on the interval $[0, \pi]$.

$f'(x) = \cos(x) - \sin(x)$, so the only critical point in $[0, \pi]$ is $x = \frac{\pi}{4}$.

Test at critical points and endpoints: $f(\frac{\pi}{4}) = \sqrt{2}$, $f(0) = 1$, $f(\pi) = -1$.

$\boxed{\begin{array}{l} \text{min. is } -1 \\ \text{max. is } \sqrt{2} \end{array}}$

6. Find the global minimum and maximum of the function $f(x) = x^3 + 5x^2 - 8x + 2$ on the interval $[-1, 2]$.

$f'(x) = 3x^2 + 10x - 8$. This is 0 when $x = \frac{2}{3}$ or $x = -4$, but only $x = \frac{2}{3}$ lies in $[-1, 2]$.

$f(\frac{2}{3}) = -\frac{22}{27}$, $f(-1) = 14$, $f(2) = 14$

$\boxed{\text{min. is } -\frac{22}{27} \quad \text{max. is } 14}$

7. Find the global minimum and maximum of the function $f(x) = \frac{x}{x^2+1}$ on the interval $[-2, 2]$.

$f'(x) = \frac{(x^2+1)\cdot 1 - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$, so $x = 1, -1$ are the critical points.

$f(1) = \frac{1}{2}$, $f(-1) = \frac{-1}{2}$, $f(-2) = \frac{-2}{5}$, $f(2) = \frac{2}{5}$

$\boxed{\text{min. is } -\frac{1}{2} \quad \text{max. is } \frac{1}{2}}$

8. Find all critical points of the function $f(x) = \sin(\cos(x))$. Does f have a global maximum? Why or why not?

$f'(x) = -\cos(\cos(x))\sin(x)$. This is 0 when $x = k\pi$ for some integer k or when $\cos(x) = m\pi + \frac{\pi}{2}$ for some integer m.

The second situation never happens because $-1 \le \cos(x) \le 1$.

So the c.p. are $\boxed{x = k\pi \ (k \text{ integer})}$.

Yes, f has a global max. One reason for this is that f is a 2π-periodic function, so it attains all of its values on $[0, 2\pi]$.

9. Which point on the parabola defined by $y = x^2$ is closest to the point ([redacted] 3, 0)?

The square of the distance between (x, x^2) and $(3, 0)$ is $f(x) = (x-3)^2 + x^4$.

Its graph "opens upward" so it has a global minimum. We want to find the x where this occurs.

$f'(x) = 2(x-3) + 4x^3$
$= 2(x-1)(2x^2+2x+3)$.

The only critical point is $x = 1$. So $\boxed{(1,1) \text{ is the closest point.}}$

10. (Optional) Let P and Q be polynomials of degree 10 such that $P(0) = 0$ and $Q(0) = P'(0) = 1$. Show that the function $\frac{P}{Q}$ has at most 29 critical points.

$\left(\frac{P}{Q}\right)' = \frac{QP' - PQ'}{Q^2}$. This is undefined only when $Q = 0$, which can occur at ≤ 10 points since $\deg Q = 10$.

This is zero only when $QP' - PQ' = 0$. Differentiating lowers the degree of a polynomial by 1, so $\deg(QP' - PQ') = 19$. Plugging in 0, we see that $QP' - PQ'$ is not the zero polynomial, so it has ≤ 19 roots. Altogther, there can be at most $10 + 19 = 29$ critical points.