Math 221 Worksheet 19
Section 4.3: The Fundamental Theorem of Calculus

1. State the fundamental theorem of calculus.

I. If f is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$ for $x \in [a,b]$, then F is continuous on $[a,b]$ and differentiable on (a,b) with $F'(x) = f(x)$.

II. If f is continuous on $[a,b]$ and if F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

2. Use the fundamental theorem of calculus to evaluate $\displaystyle\int_0^3 x^2 dx$. Compare this to Problem 1 from Worksheet 18.

The function $\frac{1}{3}x^3$ is an antiderivative of x^2, so by FTC part II, we have $\displaystyle\int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}\cdot 3^3 - \frac{1}{3}\cdot 0^3 = 9.$

Much more efficient that evaluating a limit of Riemann sums. However, we can only do this because x^2 has a "nice" antiderivative.

3. Use the fundamental theorem of calculus to determine the following:

(a) $\displaystyle\frac{d}{dx}\left(\int_0^x \sqrt{1-t^2}dt\right)$

$\sqrt{1-x^2}$ by FTC part I

(b) $\displaystyle\frac{d}{dx}\left(\int_x^{-5} t^3 - 2t^2 + 1\;dt\right)$

$$= \frac{d}{dx}\left(-\int_{-5}^x (t^3 - 2t^2 + 1)\,dt\right) = -\frac{d}{dx}\left(\int_{-5}^x (t^3 - 2t^2 + 1)\,dt\right)$$
$$= -(x^3 - 2x^2 + 1)$$

(c) $\displaystyle\frac{d}{dx}\left(\int_2^{7x+3} t^2 dt\right)$

If $F(x) = \int_2^x t^2\,dt$, then we want to find $\frac{d}{dx}[F(7x+3)]$.

Using FTC part I and the chain rule, this is $F'(7x+3)\cdot 7 = (7x+3)^2\cdot 7$.

(d) $\displaystyle\frac{d}{dx}\left(\int_2^{1/x} \arctan t\;dt\right)$

Like in part (c), use FTC part I and the chain rule:

$$\frac{d}{dx}\left(\int_2^{\frac{1}{x}} \arctan(t)\,dt\right) = \arctan\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \arctan\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right).$$

4. Let $F(x) = \int_2^x \frac{1}{1+t+t^2} dt$. Determine the region on which F is concave up.

$F'(x) = \frac{1}{1+x+x^2}$ (by FTC), so $F''(x) = -\frac{1+2x}{(1+x+x^2)^2}$.

So $F''(x) > 0$ when $x < -\frac{1}{2}$.

5. Use the fundamental theorem of calculus to evaluate the following:

(a) $\int_1^4 (2x^4 - 3x^2)dx$

$= \left(\frac{2}{5}x^5 - x^3\right)\Big|_1^4 = \frac{2}{5}4^5 - 4^3 - \left(\frac{2}{5}\cdot 1^5 - 1^3\right)$

(b) $\int_0^4 x\sqrt{x^3}dx$

$= \int_0^4 x^{5/2} dx = \frac{2}{7}x^{7/2}\Big|_0^4 = \frac{2}{7}\cdot 4^{7/2}$

(c) $\int_0^{\frac{\pi}{4}} \sin(x)dx$

$= -\cos(x)\Big|_0^{\pi/4} = -\frac{1}{\sqrt{2}} + 1$

(d) $\int_0^1 (x^3 - 1)^2 dx$

$= \int_0^1 (x^6 - 2x^3 + 1)\,dx = \left(\frac{1}{7}x^7 - \frac{1}{2}x^4 + x\right)\Big|_0^1$

$= \frac{1}{7} - \frac{1}{2} + 1$

6. Compute $\int_{-1}^1 (x + x^3)dx$. Given that you integrated an *odd* function, is there a geometric explanation for your answer?

$\int_{-1}^1 (x+x^3)\,dx = \left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_{-1}^1 = \frac{1}{2} + \frac{1}{4} - \left(\frac{1}{2} + \frac{1}{4}\right) = 0.$

Since $x+x^3$ is odd, the "positive area" under its graph on [0,1] is exactly canceled by "negative area" on [−1,1]. (Sketch the graph!)

7. Let f be a continuous function satisfying $\int_1^5 f(t)dt = 8$.

(a) Let $F(x) = \int_0^x f(t)dt$. Show that $\frac{F(5)-F(1)}{5-1} = 2$.

$$\frac{F(5)-F(1)}{5-1} = \frac{\int_0^5 f(t)\,dt - \int_0^1 f(t)\,dt}{4} = \frac{\int_1^5 f(t)dt}{4} = \frac{8}{4} = 2.$$

(b) Prove that there exists $x \in (1,5)$ such that $f(x) = 2$.

By FTC part I, F is differentiable on $(1,5)$ with $F'(x) = f(x)$.

By the mean value theorem,

$$F'(x) = \frac{F(5)-F(1)}{5-1} = 2$$

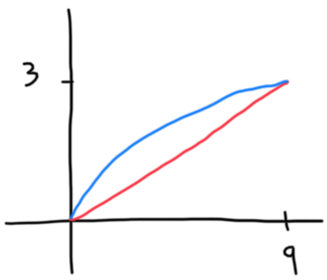
for some $x \in (1,5)$.

8. Let $f(x) = \frac{1}{3}x$ and $g(x) = \sqrt{x}$.

(a) Find all points at which the graphs of f and g intersect.

Solving $\frac{1}{3}x = \sqrt{x}$, we find that the graphs intersect at $(0,0)$ and $(9,3)$.

(b) Find the area of the bounded region enclosed by the graphs of f and g.



$$\text{area} = \int_0^9 \sqrt{x}\,dx - \int_0^9 \frac{1}{3}x\,dx$$
$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{6}x^2\right)\Big|_0^9$$
$$= \frac{2}{3}\cdot 9^{3/2} - \frac{1}{6}\cdot 9^2.$$

9. (**Fun/optional**) Let f be a continuous function and let c be a real number. Prove that

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{c-r}^{c+r} f(x)dx = f(c).$$

Let $F(x) = \int_0^x f(t)\,dt$. Then

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{c-r}^{c+r} f(x)dx = \lim_{r\to 0^+} \frac{F(c+r)-F(c-r)}{2r}$$
$$= \lim_{r\to 0^+}\left(\frac{F(c+r)-F(c)}{2r} - \frac{F(c-r)-F(c)}{2r}\right)$$
$$= \lim_{r\to 0^+}\frac{F(c+r)-F(c)}{2r} + \lim_{s\to 0^-}\frac{F(r+s)-F(c)}{2s}$$
$$= \frac{1}{2}F'(c) + \frac{1}{2}F'(c) = F'(c) \underset{(FTC)}{=} f(c).$$