MSBD 5007 Homework 1

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Problem 1.

The data given in the problem for solving $A\mathbf{x} = b$ is:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{x^{(0)}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the matrix A is not diagonally dominant, the rows of A and b has to be swapped, in order to converge.

Thus we get,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \mathbf{x^{(0)}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x^{(true)}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

a. Jacobi solution:

Given the initial guess as $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\mathbf{k = 0:} \quad x_1 = 0, \quad x_2 = 0, \quad ||x^{(0)} - x^{(true)}||_\infty = \left|\left| \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right|\right|_\infty = 2$$

$$\mathbf{k = 1:} \quad x_1 = \frac{3 - x_2}{2} = \frac{3}{2}, \quad x_2 = \frac{0 - x_1}{2} = 0 \quad ||x^{(1)} - x^{(true)}||_\infty = \left|\left| \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right|\right|_\infty = 1$$

$$\mathbf{k = 2:} \quad x_1 = \frac{3 - x_2}{2} = \frac{3}{2}, \quad x_2 = \frac{0 - x_1}{2} = \frac{-3}{4} \quad ||x^{(2)} - x^{(true)}||_\infty = \left|\left| \begin{bmatrix} \frac{3}{2} \\ \frac{-3}{4} \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right|\right|_\infty = \frac{1}{2}$$

It is observed that the absolute error decreases with proceeding iterations. Thus, the solution would converge with further iterations.

b. Gauss Seidel solution:

Given the initial guess is $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$\textbf{k = 0:} \quad x_1 = 0, \quad x_2 = 0, \quad ||x^{(0)} - x^{(true)}||_\infty = \left|\left|\begin{bmatrix}0\\0\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix}\right|\right|_\infty = 2$$

$$\textbf{k = 1:} \quad x_1 = \frac{3 - x_2}{2} = \frac{3}{2}, \quad x_2 = \frac{0 - x_1}{2} = \frac{-3}{4} \quad ||x^{(0)} - x^{(true)}||_\infty = \left|\left|\begin{bmatrix}\frac{3}{2}\\\frac{-3}{4}\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix}\right|\right|_\infty = \frac{1}{2}$$

$$\textbf{k = 2:} \quad x_1 = \frac{3 - x_2}{2} = \frac{15}{8}, \quad x_2 = \frac{0 - x_1}{2} = \frac{-15}{16} \quad ||x^{(0)} - x^{(true)}||_\infty = \left|\left|\begin{bmatrix}\frac{15}{8}\\\frac{-15}{16}\end{bmatrix} - \begin{bmatrix}2\\-1\end{bmatrix}\right|\right|_\infty = \frac{1}{8}$$

It is observed that the absolute error decreases with proceeding iterations. Thus, the solution would converge with further iterations.

Problem 2.1.

At the lower bound of k to satisfy the problem, we have, the relative error ϵ is given by,

$$\begin{aligned}
\frac{||x^{(k)} - x^{(true)}||}{||x^{(true)}||} &\leq \epsilon && \textit{(k is the lower bound to satisfy)} \\
\frac{\rho||x^{(k-1)} - x^{(true)}||}{||x^{(true)}||} &\leq \epsilon && \textit{(Ratio of consecutive iterative errors is given by } \rho\textit{)} \\
\frac{\rho^k||x^{(k-k)} - x^{(true)}||}{||x^{(true)}||} &\leq \epsilon && \textit{(Moving down to k steps)} \\
\frac{\rho^k||x^{(0)} - x^{(true)}||}{||x^{(true)}||} &\leq \epsilon && (x^{(0)} = 0) \\
\rho^k &\leq \epsilon && \textit{(Simplifying)} \\
k \log \rho &\leq \log \epsilon && \textit{(Taking log on both sides)} \\
k &\geq \log_\rho \epsilon && (\log \rho < 0)
\end{aligned}$$

Thus k should be made at least as large as $\log_\rho \epsilon$

Problem 2.2.

From the 2.1 problem, we had $k \geq \frac{\log \epsilon}{\log \rho}$. Substituting the value of ρ as $1 - O(\frac{1}{n})$ and making use of Taylor series, we get:

$$
\begin{aligned}
k &\geq \frac{\log \epsilon}{\log \rho} && \textit{(From the 2.1 problem)} \\
k &\geq \frac{\log \epsilon}{\log(1 - O(\frac{1}{n}))} && \textit{(Substituiting the value of } \rho\textit{)} \\
k &\geq \frac{\log \epsilon}{O(\frac{1}{n})} && \textit{(From Taylor's series, } O(1/n) = log(1 - O(1/n))\textit{)} \\
O(k) &= O\left(\frac{\log \epsilon}{O(\frac{1}{n})}\right) && \textit{(Taking upper bound on both sides as Big O notation)} \\
O(k) &= O\left(n\right) && \textit{(Largest polynomial order would be in order of } n\textit{)}
\end{aligned}
$$

Thus the order of k is same as order of n as both grow at the same rate. This is trivially true because $\lim_{k,n\to\infty} \frac{k}{n} = c$, where c is the constant.

Problem 3.

Instead of projecting a vector, we would project a matrix (solution plane) this time. Let $A = [a_{i_1}, a_{i_2}]^T$ and $B = [b_{i_1}, b_{i_2}]^T$ Here we are given a $x^{(k)}$, we need to project on hyperplane defined by $A^T x = B$.

Assuming that projection is Y:

1. $x^{(k)} - Y$ is parallel to the normal vector of the hyperplane of A. Hence, $x^{(k)} - Y = \alpha A$
2. y is the solution to the hyperplane of the current iterative guess. Thus, $A^T Y = B$

The objective here is to find an update equation as Y. The process is shown as follows:

$$
\begin{aligned}
Y &= x^{(k)} - \alpha A && \textit{(From the first point)} && (1) \\
A^T Y &= B && \textit{(From the second point)} && (2) \\
A^T(x^{(k)} - \alpha A) &= B && \textit{(Substituting Y from (1))} && (3) \\
A^T x^{(k)} - \alpha A^T A &= B && \textit{(Opening and expansion)} && (4) \\
\alpha A^T A &= \alpha A^T x^{(k)} - B && \textit{(Term rearrangement)} && (5) \\
\alpha &= (A^T A)^{-1}(A^T x^{(k)} - B) && (A^T A \textit{ is invertible)} && (6) \\
Y &= x^{(k)} - (A^T A)^{-1} A(A^T x^{(k)} - B) && \textit{(Substituting in (1))} && (7)
\end{aligned}
$$

Thus the update equation is $x^{(k+1)} = x^{(k)} - (A^T A)^{-1} A(A^T x^{(k)} - B)$ or $x^{(k+1)} = x^{(k)} - A^*(A^T x^{(k)} - B)$, where A^* is a pseudoinverse.

Notes

1. This given solution of the assignment follows the HKUST honour code. Although assignment has been discussed with other peers, the solutions are my own.
2. Kindly give feedback on this assignment on how to write up the solutions more elegantly.