MSBD 5004 Homework 1

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Problem 1.

If f is the periodic function with the period $T = 2L$. Then the fourier series of f is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where,

$$\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx
\end{aligned}$$

The given function $f(t) = t$ is an odd function. Hence, $a_0 = a_n = 0$
Calculating the b_n coefficients, we get:

$$\begin{aligned}
b_n &= \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt \\
&= 2 \int_{\frac{1}{2}}^{\frac{-1}{2}} t \sin(2n\pi t) dt \\
&= 2 \int_{\frac{1}{2}}^{\frac{-1}{2}} t \sin(2n\pi t) dt \\
&= 2 \left(\left[\frac{t}{2n\pi t} \cos(2n\pi t) \right]_{\frac{1}{2}}^{\frac{-1}{2}} - \frac{1}{2n\pi t} \int_{\frac{1}{2}}^{\frac{-1}{2}} \cos(2n\pi t) dt \right) \\
&= \frac{-\pi n \cos(\pi n)}{\pi^2 n^2} \\
&= \frac{-\cos(\pi n)}{\pi n}
\end{aligned}$$

$$= \begin{cases} \frac{-1}{n\pi}, & n = \text{odd} \\ \frac{1}{n\pi} & n = \text{even} \end{cases}$$
$$= \frac{(-1)^{n+1}}{n\pi}$$

Substuiting the b_n values, we get the fourier series,

$$f(t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin 2\pi nt$$

Problem 2.

For this problem, firstly I will compute the fourier series of t^2. Since it is an even function, the b_n terms need not to be computed.

From the definition, a_n terms are given by,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

Integrating from $-\pi$ to π, we get,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\ dt$$

Integrating twice by parts, we get for $n \neq 0$,

$$\begin{aligned} a_n &= \frac{-2}{n}\int_{-\pi}^{\pi} t\sin nt\frac{dx}{\pi} \\ &= -\int_{-\pi}^{\pi}\frac{2\cos nx}{\pi n^2}dx + \left[\frac{2x\cos nx}{\pi n^2}\right]_{-\pi}^{\pi} \\ &= (-1)^n\frac{4}{n^2} \end{aligned}$$

For the $n = 0$ component, we get:

$$\begin{aligned} a_0 &= \int_{-\pi}^{\pi} t^2\frac{dt}{\pi} \\ &= \frac{2\pi^2}{3} \end{aligned}$$

Putting it together, we get the fourier series,

$$t^2 = \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nt}{n^2}$$

According to the Parseval's formula,

$$||f||^2 = \sum_{n=-\infty}^{\infty} |c_k|^2$$

Here,

$$\begin{aligned} ||f||^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 dt \\ &= \frac{\pi^4}{5} \end{aligned}$$

Plugging the values in the formula, we get,

$$\begin{aligned} ||f||^2 &= \sum_{n=-\infty}^{\infty} |c_k|^2 \\ \frac{\pi^4}{5} &= \left(\frac{2\pi^2}{3}\right)^2 + \sum_{k=1}^{\infty}\left((-1)^n \frac{4}{n^2}\right)^2 \\ \frac{\pi^4}{5} - \frac{4\pi^4}{9} &= 16\sum_{k=1}^{\infty}\frac{1}{k^4} \\ \sum_{k=1}^{\infty}\frac{1}{k^4} &= \frac{\pi^4}{90} \\ \sum_{k=2}^{\infty}\frac{1}{k^4} &= \frac{\pi^4}{90} - 1 \end{aligned}$$

Problem 3.

The problem asks to find the function $g(t)$, which is in time domain, but asks us to filter the frequencies. Clearly, inverse fourier transform is the only way.

We know that $f * g = \hat{f}\hat{g}$. Thus we need to answer the inverse fourier transform of low pass filter.

In the frequency domain, the low pass filter is defined by,

$$h(s) = \begin{cases} 1 & -1 < s < 1 \\ 0 & \text{otherwise} \end{cases}$$

Inverse fourier transform of $h(s)$ is the sinc function, which is given by,

$$\mathcal{F}^{-1}[h(s)] = 2sinc(2t) = g(t)$$

Thus, $g(t) = 2sinc(2t)$

Problem 4.

Given the function,

$$f(t) = \begin{cases} 1 - |t|, & -1 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

The fourier transform can be obtained by,

$$X(\omega) = \int_0^\infty f(t)e^{-j\omega t}dt$$

For simplicity, I have assumed $\omega = 2\pi s$, where s is the frequency.
Plugging the values, we get,

$$\begin{aligned}
X(\omega) &= \int_{-1}^{1} (1 - |t|)e^{-j\omega t}dt \\
&= \int_{-1}^{0} (1 + t)e^{-j\omega t}dt + \int_{0}^{1} (1 - t)e^{-j\omega t}dt \\
&= \left[(t+1)\frac{e^{-j\omega t}}{-j\omega} - \int \frac{e^{-j\omega t}}{-j\omega}dt\right]_{-1}^{0} + \left[(-t+1)\frac{e^{-j\omega t}}{-j\omega} - \int \frac{-e^{-j\omega t}}{-j\omega}dt\right]_{0}^{1} \\
&= \left[\frac{-1}{j\omega} + \frac{1}{\omega^2} - \frac{e^{j\omega}}{\omega^2}\right] + \left[\frac{1}{j\omega} + \frac{1}{\omega^2} - \frac{e^{j\omega}}{\omega^2}\right] \\
&= \frac{2}{\omega^2} - \frac{1}{\omega^2}\left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right] \\
&= \frac{2}{\omega^2}\left[1 - \cos\omega\right] \\
&= \frac{2}{\omega^2} 2\sin^2\frac{\omega}{2} \\
&= \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2 \\
&= sinc^2(\pi s)
\end{aligned}$$

Problem 5.

In general, the entries for DFT are computed by,

$$F[n] = \sum_{k=0}^{N-1} f[k]e^{-\iota\frac{2\pi}{N}nk}$$

For the sequence of four, the corresponding transform matrix, A_n will be,

$$A_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\iota & -1 & \iota \\ 1 & -1 & 1 & -1 \\ 1 & \iota & -1 & -\iota \end{bmatrix}$$

The f vector is given by, $f = [1, 1, 2, 2]^T$
The DFT is then

$$A_n f = [6, -1 + \iota, 0, -1 - \iota]^T$$

Problem 6.

To prove, $A_n(f * g) = (A_n f) \cdot (A_n g)$

$$\begin{aligned}
A_n(f * g) &= \frac{1}{N} \sum_{m=0}^{N-1} \left(\sum_{k=0}^{N-1} f[k] g[m-k] \right) e^{\frac{-\iota 2\pi mn}{N}} && \text{(From definition)} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} f[k] \left(\sum_{m=0}^{N-1} g[m-k] e^{\frac{-\iota 2\pi(m-k)n}{N}} e^{\frac{-\iota 2\pi kn}{N}} \right) && \text{(Splitting exponentials)} \\
&= \frac{1}{N} \left(\sum_{k=0}^{N-1} f[k] e^{\frac{-\iota 2\pi kn}{N}} \right) \left(\sum_{k=0}^{N-1} g[m-k] e^{\frac{-\iota 2\pi(m-k)n}{N}} \right) && \text{(Rearrangement)} \\
&= (A_n f) \cdot (A_n g) && \text{(Using Fourier's definition)}
\end{aligned}$$

Problem 7.

The right shift indicates this,

$$\tau(f) = f[n-1]$$

We will take the fourier transform on both sides.

$$\begin{aligned}
\tau(f) &= f[n-1] && \text{(The shift equation)} \\
F(\tau(f)) &= \sum_{n=0}^{N-1} f[n-1] e^{\frac{-\iota 2\pi nk}{N}} && \text{(Taking fourier transform)} \\
&= \sum_{m=-1}^{N-1-1} f[m] e^{\frac{-\iota 2\pi(m+1)k}{N}} && \text{(Change of variables)} \\
&= \sum_{m=0}^{N-1} f[m] e^{\frac{-\iota 2\pi mk}{N}} e^{\frac{-\iota 2\pi k}{N}} && \text{(Splitting the exponential term)} \\
&= e^{\frac{-\iota 2\pi k}{N}} \sum_{m=0}^{N-1} f[m] e^{\frac{-\iota 2\pi mk}{N}} && \text{(Rearrangement)} \\
&= e^{\frac{-\iota 2\pi k}{N}} F(f) && \text{(Using Fourier's definition)}
\end{aligned}$$

Notes

1. This given solution of the assignment follows the HKUST honour code. Although assignment has been discussed with other peers, the solutions are my own.

2. Kindly give feedback on this assignment on how to write up the solutions more elegantly.