MSBD 5007 Homework 5

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December 5, 2017

Problem 1.

For the given problem, I will proceed through GramSchmidt process for the QR decomposition.

$$A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & \frac{-1}{2} \\ 0 & \sqrt{2} \end{bmatrix},$$

Step 1: Operations on first vertical vector and normalization.

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$r_{11} = \sqrt{1+1}$$

,

$$q_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

Step 2: Operations of orthonormalization to get second vector.

$$r_{12} = q_1^T a_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}^T \begin{bmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ \sqrt{2} \end{bmatrix} = \frac{\sqrt{2}}{2}$$

$$u_2 = a_2 - r_{12} q_1 = \begin{bmatrix} \frac{3}{2} \\ \frac{-1}{2} \\ \sqrt{2} \end{bmatrix} - \frac{\sqrt{2}}{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

$$r_{22} = \sqrt{1+1+2} = 2$$

$$q_2 = \frac{u_2}{r_{22}} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Therefore, aggregating the results together:

$$A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & \frac{-1}{2} \\ 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{-1}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & 2 \end{bmatrix} = QR$$

Problem 2.

I will approach this problem by using the fact that singular values are square roots of eigenvalues of AA^T and A^TA

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$AA^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

Getting the characteristic equation for this matrix and solving for it, we get:

$$(1-\lambda)^2 - 1 = 0$$
$$\lambda_1 = 2, \lambda_2 = 0$$
$$\sigma_1 = \sqrt{2}, \sigma_2 = 0$$

Using these eigenvalues, the eigenvectors obtained are:

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

After normalizing we get,

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Now we will proceed the same for the $A^T A$:

$$A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(2-\lambda)^2 = 0$$
$$\lambda_1 = \lambda_2 = 2$$

Eigenvectors are:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

After normalizing we get,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence the SVD is after aggregating and solving,

$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Problem 3.

For this problem, I firstly I would SVD of the given matrix. Since eigenvector and power methods were getting too complex to compute, I unfortunately used Wolfram Alpha to compute the decomposition.

First singular vector is then given by,

$$\sigma^{(1)} u^{(1)} v^{(1)T} = \sqrt{1+M^2} \begin{bmatrix} \frac{1+M^2}{\sqrt{1+M^4+3M^2}} \\ \frac{M}{\sqrt{1+M^4+3M^2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1+M^2}} & \frac{M}{\sqrt{1+M^2}} \end{bmatrix}$$

Then the best rank-1 approximation is given by

$$\frac{1}{\sqrt{1+M^4+3M^2}} \begin{bmatrix} 1+M^2 & (1+M^2)M \\ M & M^2 \end{bmatrix}$$

Problem 4.

For this firstly I will prove that $||(P_S x + y) - x||_2^2 \geq ||P_S x - x||_2^2$ for any $y \in S$.

Expanding $||(P_S x + y) - x||_2^2$ I get,

$$\begin{aligned} ||(P_S x + y) - x||_2^2 &= ||P_S x + y||_2^2 + ||x||_2^2 - 2\langle (P_S x + y), x\rangle \\ &= ||P_S x||_2^2 + ||y||_2^2 + 2\langle P_S x, y\rangle + ||x||_2^2 - 2\langle (P_S x + y), x\rangle \end{aligned}$$

Expanding $||P_S x - x||_2^2$ I get,

$$||P_S x - x||_2^2 = ||P_S x||_2^2 + ||x||_2^2 - 2\langle P_S x, x\rangle$$

Subtracting the above two expansions we get that,

$$\begin{aligned} ||(P_S x + y) - x||_2^2 - ||P_S x - x||_2^2 &= ||y||_2^2 + 2\langle P_S x, y\rangle - 2\langle P_S x + y, x\rangle + 2\langle P_S x, x\rangle \\ &= ||y||_2^2 + 2\langle P_S x, y\rangle - 2\langle P_S x, x\rangle - 2\langle y, x\rangle + 2\langle P_S x, x\rangle \\ &= ||y||_2^2 + 2\langle P_S x - x, y\rangle \end{aligned}$$

But we know that,

$$\begin{aligned} ||y||_2^2 &\geq 0 \\ 2\langle P_S x - x, y\rangle &= 0 \end{aligned}$$

Hence,

$$||(P_S x + y) - x||_2^2 \geq ||P_S x - x||_2^2$$

for any $y \in S$

Thus we conclude that minimum distance y between orthogonal projection and x is $P_S x$.

Problem 5.

I will split the A into symmetric and skew-symmetric matrix.

Let $H = \frac{A+A^T}{2}$ and $S = \frac{A-A^T}{2}$.

It can be proved that

$$||A - B||_F^2 = ||H - B||_F^2 + ||S||_F^2$$

Thus the answer would be doing the SVD of H, then best symmetric rank-1 approximation would be

$$H_1 = \sigma_1 u_1 v_1^T$$

Notes

1. This given solution of the assignment follows the HKUST honour code. Although assignment has been discussed with other peers, the solutions are my own.
2. Kindly give feedback on this assignment on how to write up the solutions more elegantly.