MSBD 5004, Lecture Notes

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1 Fourier Transforms

A function is called period if there exits a positive real number T such that

$$f(x) = f(x + T)$$

for all x in the domain of f. The smallest T satisfying the condition is called the period of f. It is useful the normalize the scale the function input by T so that

$$g(x) = f(T \times x)$$

has a period of 1.

1.1 Sinusoidals

The sinusoid function has the form:

$$\begin{aligned} S(t) &= A\sin(2\pi\nu t + \phi) \\ &= A\sin(2\pi\nu t)\cos\phi + A\cos(2\pi\nu t)\sin\phi \end{aligned}$$

where the period is $\frac{1}{\nu}$

1.2 Fundamental Theorem of Fourier Analysis

Let $f(t) = f(t + 1)$ and $\int_0^1 |f(t)|^2 dt < \infty$ We define that integral in terms of "energy".

Then f can be written in:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nt + b_n \sin 2\pi nt)$$

Using euler exponentials, we can get the notations:

$$\begin{aligned} a_0 &= 2\int_0^1 f(t)dt \\ a_n &= 2\int_0^1 f(t)\cos 2\pi ntdt \\ b_n &= 2\int_0^1 f(t)\sin 2\pi ntdt \end{aligned}$$

Other intuition of those derivation of notation can be achieved from the integrals.

$$\begin{aligned}\int_0^1 f(t) &= \int_0^1 \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos 2\pi nt + b_n \sin 2\pi nt)dt \\ &= \frac{a_0}{2}\end{aligned}$$

This theorem basically rewrites stuff in sinusoidal form.

1.3 Euler exponentials

$$\begin{aligned}e^{2\pi\iota t} &= \cos(2\pi t) + \iota \sin(2\pi t) \\ e^{-2\pi\iota t} &= \cos(-2\pi t) + \iota \sin(-2\pi t) \\ &= \cos(2\pi t) - \iota \sin(2\pi t)\end{aligned}$$

Doing manipulations we get:

$$\begin{aligned}\cos(2\pi t) &= \frac{1}{2}e^{2\pi\iota t} + \frac{1}{2}e^{-2\pi\iota t} \\ \sin(2\pi t) &= \frac{1}{2\iota}e^{2\pi\iota t} + \frac{1}{2\iota}e^{-2\pi\iota t}\end{aligned}$$

Euler exponentials are derived from Taylor series.
Now, we can rewrite $f(t)$ in terms of these complex numbers, which looks like:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi\iota nt}$$

c_n are complex.
It is worth noting that,

$$\int_0^1 e^{2\pi\iota mt}dt = 0$$

when $n \neq 0$.
Also, we have a fact that,

$$c_n = \int_0^1 f(t)e^{-2\pi\iota mt}dt$$

Proof of that fact is,

$$
\begin{aligned}
f(t)e^{-2\pi\iota t} &= e^{-2\pi\iota t}\sum_{n=-\infty}^{\infty} c_n e^{2\pi\iota nt} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{2\pi\iota(n-m)t}
\end{aligned}
$$

If you integrate it both sides, you will get:

$$c_m = \int_0^1 f(t)e^{-2\pi\iota mt}dt$$

1.4 Step Function

The step function is given by,

$$f(t) = \begin{cases} 1 & 0 \le t \le \frac{1}{2} \\ -1 & \frac{1}{2} \le t \le 1 \end{cases}$$

We know these two formulae,

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi\iota nt} \\
c_m &= \int_0^1 f(t)e^{-2\pi\iota mt}dt
\end{aligned}
$$

So, we will plug the values here,

$$
\begin{aligned}
c_m &= \int_0^1 f(t)e^{-2\pi\iota mt}dt \\
&= \int_0^{\frac{1}{2}} f(t)e^{-2\pi\iota mt}dt + \int_{\frac{1}{2}}^1 f(t)e^{-2\pi\iota mt}dt \\
&= \int_0^{\frac{1}{2}} e^{-2\pi\iota mt}dt - \int_{\frac{1}{2}}^1 e^{-2\pi\iota mt}dt \\
&= \frac{1}{2\iota\pi m}[e^{-2\pi\iota m\frac{1}{2}} - e^0] - \frac{1}{2\iota\pi m}[e^{-2\pi\iota m} - e^{-2\pi\iota m\frac{1}{2}}]
\end{aligned}
$$

After solving it completely, we will get,

$$
\begin{aligned}
c_m &= \frac{1}{\iota\pi m}\begin{cases} 2 & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even} \end{cases} \\
c_0 &= 0
\end{aligned}
$$

which can be written as,

$$f(t) = \sum_{n \text{ is odd, } n=-\infty}^{\infty} \frac{2}{\iota \pi m} e^{2\pi \iota n t}$$

Some key points here:

1. Lower frequencies have higher energy
2. Higher frequencies have lower energy
3. Higher frequencies will decay slowly, in the exponential form

If we plot this function, by taking a range of frequencies, we can reconstruct the step function in the form of sinusoidals. However, there will always be a sharp perturbation at the corners. Hence, in practice, we always apply some kind of smoothing function to avoid those perturbations.

If we plug, $t = \frac{1}{4}$ in $f(t)$,

$$\begin{aligned} 1 &= \sum_{n \text{ is odd, } n=-\infty}^{\infty} \frac{2}{\iota \pi m} e^{2\pi \iota n \frac{1}{4}} \\ &= \frac{4}{\pi}(1 - 1/3 + 1/5 \ldots) \end{aligned}$$

Thus, the expansion of $\frac{\pi}{4}$ could be derived from that.

1.5 Converse Theorem

These transform functions have one-to-one correspondence. Here, we will try to do the converse, converting fourier series into a function.

The theorem states that if the function is finite, the coefficients of fourier will be finite means, if $\int |f(t)|^2 dt < \infty$ implies $\sum |c_n|^2 < \infty$

In other words, we have a fact that energy of discrete coefficients would be equal to the energy of the continuous function.

$$\int_0^1 |f|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof of that is,

$$\begin{aligned}
|f(t)|^2 &= f(t)\overline{f(t)} \\
&= \left(\sum_n c_n e^{2\pi \iota n t}\right)\left(\sum_m \overline{c_m} e^{-2\pi \iota m t}\right) \\
&= \sum_n \sum_m c_n \overline{c_m} e^{2\pi \iota n t} e^{-2\pi \iota m t} \\
&= \sum_n \sum_m c_n \overline{c_m} e^{2\pi \iota (n-m) t} \\
\int_0^1 |f(t)|^2 &= \sum_n \sum_m \int_0^1 c_n \overline{c_m} e^{2\pi \iota (n-m) t} \\
&= \sum_n c_n \overline{c_m} \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

1.6 Fourier Transform

The idea of fourier transform starts from here. We don't care about rest of interval because the function is periodic.

$$c_n = \int_0^1 f(t) e^{-2\pi \iota m t} dt$$

Writing in other way, we get,

$$\widehat{f(\xi)} = \int_{-\infty}^{\infty} f(t) e^{-2\pi \iota \xi t} dt$$

Conversely,

$$f(t) = \sum_{n=-\infty}^{\infty} \widehat{f(t)} e^{2\pi \iota n t}$$

where $supp f \subseteq [0,1]$

Notes

1. These notes are only meant for MSBD 5004 class.
2. For any typos, kindly message Pranav on WeChat
3. If you find these notes useful, message Pranav a "thank you" on WeChat as well!