MSBD 5004 Homework 3

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Problem 1.1.

Let $r^{(0)} = b$
For the first iteration,

$$\begin{aligned} j_1 &= 4 \\ \omega^{(1)} &= \{4\} \\ r^{(1)} &= [0, 0, 1]^T \end{aligned}$$

For the second iteration,

$$\begin{aligned} j_2 &= 3 \\ \omega^{(2)} &= \{3, 4\} \end{aligned}$$

So we find best approximation of b using $[a_3, a_4]$, by solving

$$\min_x ||b - [a_3, a_4]^T [x_3, x_4]||_2^2$$

Thus $[x_3, x_4] = [a_3, a_4]^\dagger b$, where $[a_3, a_4]^\dagger$ is the pseudoinverse, which is $[x_3, x_4] = [1, \sqrt{2}]$

The result is $x = [0, 0, 1, \sqrt{2}]^T$.

Problem 1.2.

Let $x^{(0)} = [0, 0, 0, 0]^T$
For the first iteration,

$$\begin{aligned} \alpha^{(0)} &= 0.5558 \\ x^{(1)} &= [0.556, 0, 0, 0.78567]^T \end{aligned}$$

For the second iteration,

$$\begin{aligned} \alpha^{(1)} &= 0.91928 \\ x^{(2)} &= [0, 0, 0.9192, 1.0023]^T \end{aligned}$$

Problem 2.

(It is assumed that the question is asking us to solve for mutual coherence, not mutual incoherence.)

Given the matrix $\mathbf{A} = [\mathbf{I}\ \mathcal{F}]$, the inner product $\mathbf{A}^T\mathbf{A}$ will be analyzed, which generates the following block matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}^T\mathbf{I} & \mathbf{I}^T\mathcal{F} \\ \mathcal{F}^T\mathbf{I} & \mathcal{F}^T\mathcal{F} \end{bmatrix}$$

Here we are looking for largest off-diagonal entry, because the mutual coherence looks at the argmax of the dot products of two vectors.

Clearly the vector in $\mathcal{F}$ which will be largest would be $\frac{1}{\sqrt{N}}[1, 1, 1, \cdots, 1]$. When this vector is multiplied by any of the $\mathbf{I}$ vector, the result will be $\frac{1}{\sqrt{N}}$

Hence, $\mu(\mathbf{A}) = \frac{1}{\sqrt{N}}$, where N is the dimension of the $\mathcal{F}$ matrix.

Problem 3.

The given objective function is a non-smooth convex function. Hence, the subgradient approach will be taken.

Analyzing subdifferential at $x \neq A^{\dagger}b$ we get ($A^{\dagger}$ stands for pseudoinverse),

$$\partial f(x) = \frac{A^T(Ax - b)}{||Ax - b||_2}$$

Clearly, this subdifferential won't exist at $x \neq A^{\dagger}b$.

However, the supremum of this subdifferential is $||A||$, hence the subgradient at that point will not exceed this supremum.

So, subdifferential at $x = A^{\dagger}b$ is,

$$\partial f(x) = \{g|\ ||g||_2 < ||A||\}$$

Now, projected subgradient approach will be taken.

Algorithm 1 Projected Subgradient Descent

for $k = 1, 2, 3 \cdots$ **do**
  $y^{(k)} \leftarrow x^{(k)} - \alpha_k g_k$
  $x^{(k+1)} \leftarrow \max(0, y^{(k)})$

Here,

1. α_k is the step size.

2. g_k is the subgradient

3. max is the entrywise-max operator (which is the projection on $x \succeq 0$)

Notes

1. This given solution of the assignment follows the HKUST honour code. Although assignment has been discussed with other peers, the solutions are my own.
2. Kindly give feedback on this assignment on how to write up the solutions more elegantly.