MSBD 5007 Homework 4

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Problem 1.

For this given problem,

$$\min_x \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda||\mathbf{z}||_1 \qquad \text{s.t. } \mathbf{Bx} = \mathbf{z}$$

The augmented lagrangian is given by,

$$L_\gamma(x, z, \mu) = \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda||\mathbf{z}||_1 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2$$

In ADMM, we need to solve,

$$\begin{aligned}
\arg\min_x L_\gamma(x, z, \mu) &= \arg\min_x \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda||\mathbf{z}||_1 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2 \\
&= \arg\min_x \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2
\end{aligned}$$

Taking the gradient and setting to zero, we get,

$$\begin{aligned}
x - y + B^T\mu + \gamma B^T(Bx - z) &= 0 \\
x + \gamma B^T Bx &= \gamma B^T z - B^T\mu \\
x &= (I + \gamma B^T Bx)^{-1}(\gamma B^T z - B^T\mu)
\end{aligned}$$

We also need to solve,

$$\begin{aligned}
\arg\min_z L_\gamma(x, z, \mu) &= \arg\min_z \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda||\mathbf{z}||_1 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2 \\
&= \arg\min_z \lambda||\mathbf{z}||_1 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2 \\
&= \arg\min_z \lambda||\mathbf{z}||_1 + \mu^T(Bx - z) + \frac{\gamma}{2}||Bx - z||_2^2 \\
&= \arg\min_z \lambda||\mathbf{z}||_1 + \frac{\gamma}{2}||Bx - z + \frac{\mu}{\gamma}||_2^2 \\
&= \arg\min_z \frac{\lambda}{\gamma}||\mathbf{z}||_1 + \frac{1}{2}\left\|z - \left(Bx + \frac{\mu}{\gamma}\right)^2\right\|_2^2
\end{aligned}$$

So the optimal solution is,

$$z = T_{\frac{\lambda}{\gamma}}\left(Bx + \frac{\mu}{\gamma}\right)$$

where T is the thresholding operator.

Therefore the ADMM steps are:

$$\begin{aligned}
x^{(k+1)} &= (I + \gamma B^T B x^{(k)})^{-1}(\gamma B^T z^{(k)} - B^T \mu) \\
z^{(k+1)} &= T_{\frac{\lambda}{\gamma}}\left(Bx^{(k+1)} + \frac{\mu}{\gamma}\right) \\
\mu^{(k+1)} &= \mu^k + \alpha_k \left(Bx^{(k+1)} - z^{(k+1)}\right)
\end{aligned}$$

Problem 2.1.

Given the equation,

$$\min_{x,u,v} \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda \mathbf{1}^T(\mathbf{u} + \mathbf{v}) \qquad \text{s.t. } \mathbf{Bx} = \mathbf{u} - \mathbf{v}, \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$$

Since, $\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$, I can rewrite it as,

$$\min_{x,u,v} \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{u} + \mathbf{v}||_1 \qquad \text{s.t. } \mathbf{Bx} = \mathbf{u} - \mathbf{v}, \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$$

Since, $\mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$, using the triangular inequality, $||\mathbf{u} - \mathbf{v}||_1 \leq ||\mathbf{u} + \mathbf{v}||_1$, Thus the problem can be rewritten as,

$$\min_{x,u,v} \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{u} - \mathbf{v}||_1 \qquad \text{s.t. } \mathbf{Bx} = \mathbf{u} - \mathbf{v}, \mathbf{u} \geq \mathbf{0}, \mathbf{v} \geq \mathbf{0}$$

But, $\mathbf{Bx} = \mathbf{u} - \mathbf{v}$, hence the problem can be rewritten as,

$$\min_{x} \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{Bx}||_1$$

which is equivalent to original problem.

Problem 2.2.

Given the problem,

$$\min_{x,z} \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{z}||_1 \qquad \text{s.t. } \mathbf{Bx} = \mathbf{z}$$

The Lagrangian becomes,

$$\begin{aligned}
L(\mathbf{x}, \mathbf{z}, \alpha) &= \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{z}||_1 + \alpha^T(\mathbf{Bx} - \mathbf{z}) \\
&= \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \lambda ||\mathbf{z}||_1 + \alpha^{\mathbf{T}}\mathbf{Bx} - \alpha^{\mathbf{T}}\mathbf{z}
\end{aligned}$$

The dual function $g(\alpha)$ is the infimum of L over x and z. I noticed that if $||\alpha||_\infty > \lambda$, then $g(\alpha) = -\infty$. Otherwise, infimum over z of $\lambda||\mathbf{z}||_1 - \alpha^{\mathbf{T}}\mathbf{z}$ is equal to zero.

Let me consider the case where $||\alpha||_\infty \leq \lambda$. Here the value of $g(\alpha)$ would depend only on x infimum, which is given by,

$$\begin{aligned} g(\alpha) &= \frac{1}{2}||\mathbf{x} - \mathbf{y}||_2^2 + \alpha^{\mathbf{T}}\mathbf{B}\mathbf{x} \\ &= -\frac{1}{2}||\mathbf{y} - \mathbf{B}^{\mathbf{T}}\alpha||_2^2 \qquad \text{s.t. } ||\alpha||_\infty \leq \lambda \end{aligned}$$

Finally, the dual problem is to find the maximum of $g(\alpha)$ over all α. I can wrap all of the conditions up nicely by stating the dual problem as:

$$\begin{aligned} \min_{\alpha} &= \frac{1}{2}||\mathbf{y} - \mathbf{B}^{\mathbf{T}}\alpha||_2^2 \qquad \text{s.t. } ||\alpha||_\infty \leq \lambda \\ &= \frac{1}{2}||\mathbf{y} - \mathbf{B}^{\mathbf{T}}\alpha||_2^2 \qquad \text{s.t. } \lambda\mathbf{1} \leq \alpha \leq \lambda\mathbf{1} \end{aligned}$$

Problem 2.3.

It can be shown that the dual is strictly convex if $\text{rank}(B) = m$. Then it can be shown that each coordinate for $i = 1, \ldots, m$ of B is given by

$$\alpha_i \in \begin{cases} -\lambda & \text{if } Bx < 0, \\ +\lambda & \text{if } Bx > 0 \\ [-\lambda, +\lambda] & \text{if } Bx = 0 \end{cases}$$

I will try to solve this using coordinate descent algorithm. For the first coordinate, I can set the $\lambda_0 = \infty$, solve for α_0 using least squares, move on to the next α.

Problem 2.4.

If the strong duality holds, then I would solve for every λ_i for which solution is given by $x^* = y - B^T\alpha^*$

Notes

Problem 3.

If I try to form the lagragian of the function, which is $L(x, \lambda) = \frac{1}{2}\left\|x - y\right\|^2 + \lambda x^T\gamma$, and try to find the dual from here, minimum of the function would be undefined as in the extreme cases. It could be either $\frac{1}{2}\left\|\beta - y\right\|^2 + \lambda x^\gamma$ or $\frac{1}{2}\left\|\alpha - y\right\|^2 + \lambda x^\gamma$ which depends on their signs. Once we know the signs explicity, the strong duality holds. Then the dual function is given by, $\max_{\beta,\lambda}\min_y \frac{1}{2}\left\|\beta - y\right\|^2 + \lambda x^T\gamma$

1. The given solutions of the assignment follows the HKUST honour code. Although assignment has been discussed with other peers, the solutions are my own.

2. Kindly give feedback on this assignment on how to write up the solutions more elegantly. Your feedback is more important to us than grades.