$$\lim_{N\to\infty} b(i,N,p) = \frac{\lambda^i}{i!}e^{-\lambda}, \qquad \lambda=Np, \qquad i=0,1,2,3,\ldots. \tag{6.81}$$

## 6.4 The Geom/Geom/m/N Queueing System

In chapter 2 a number of state-dependent continuous time queueing systems were presented. In this section a discrete time queueing model whose state transition diagram has the Type A structure of chapter 3 is examined. Thus a simple recursive solution of the equilibrium probabilities is possible. This model is based on the work in [HUAN].

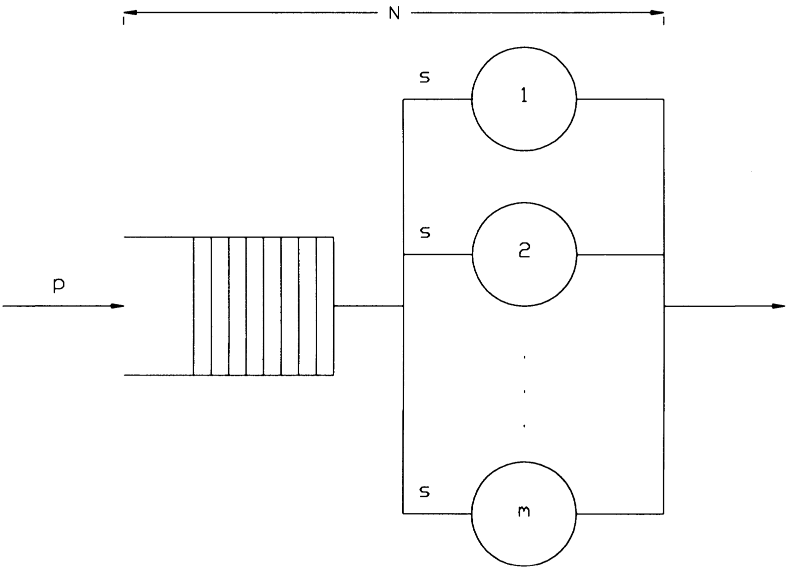


**Fig. 6.4:** Geom/Geom/m/N Queueing System

Consider a discrete time queue as shown in Figure 6.4 where $p$ is the arrival probability of a customer during a time slot and $s$ is the service completion probability of a customer in a server during a slot. Consequently this corresponds to a Bernoulli arrival process and a geometric service time distribution. Also define $N$ as the capacity of the queueing system and $m$ as the number of servers. This is a Geom/Geom/m/N queueing system, which is the discrete time analog of the continuous time M/M/m/N queueing system in discrete time. The system is memoryless so that the state of the queue is defined to be the number of customers in the queue (including the ones in service). We assume that customers can both enter and leave a full queue in the same slot---leading to no net change in state. Figure

6.5 illustrates the state transition diagram of this queueing system. It can be seen to have the Type A nonproduct form structure of section 3.4.

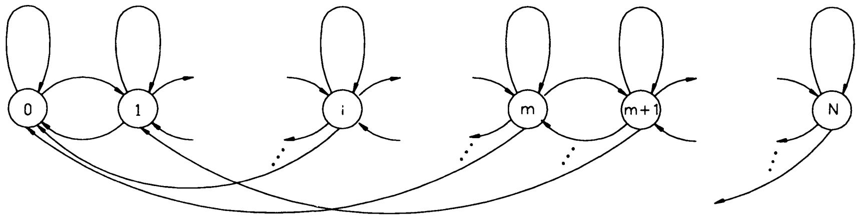


**Fig. 6.5:** Geom/Geom/m/N State Transition Diagram

Our goal is to find the equilibrium state probabilities given $p$,$s$,$m$ and $N$. Since this system's state transition diagram is Type A, a recursive solution for the equilibrium state probabilities is possible by drawing vertical boundaries between adjacent states and equating the flow of probability flux from left to right to that from right to left across the boundaries. Of course, since this is a discrete time Markov chain, there are transition *probabilities*, not continuous time rates, associated with each transition. If $X_k$ is the state at slot $k$ then the transition probability $a_{ij}$ is defined as:

$$a_{ij} = \lim_{k \to \infty} Prob\,(X_{k+1}=j \mid X_k=i), \qquad i,j \; \varepsilon \; [0,N]. \tag{6.82}$$

In one possible case a customer can arrive at the queue with probability $p$, and, after entering a server, it can finish the service in $i$ time slots with probability $s(1-s)^{i-1}$. This means that it is possible for a customer to enter and leave an empty queue in the same slot. To find the transition probabilities, all possible combinations of customers arrivals and departures have to be considered. Two sets of equations can be obtained. First let

$$[n,m]^* = min\,(n,m). \tag{6.83}$$

Also let the combinatorial expressions below equal 0 if the upper term is less than the lower term or if the lower term is less than 0. Then the one-step transition probabilities from state $n$ to $n$-$l$ (except from $N$ to $N$) are

$$a_{n,n-l} = p\begin{bmatrix} [n+1,m]^* \\ l+1 \end{bmatrix} s^{l+1}(1-s)^{[n+1,m]^*-l-1} \tag{6.84}$$

$$+ (1-p)\begin{bmatrix} [n,m]^* \\ l \end{bmatrix} s^{l}(1-s)^{[n,m]^*-l},$$

$$-1 \le l \le m, \qquad n \ne N,$$

and the transition probability from state $N$ to $N$ is:

$$a_{N,N} = p[\binom{m}{1} s(1-s)^{m-1} + (1-s)^m] + (1-p)(1-s)^m. \tag{6.85}$$

Now in a second case if a customer cannot enter and leave an empty queue in the same slot, there will be a different set of state transition probabilities. This delay can be called the synchronization time. One example where this might arise is when time is required to process a header in a packet network node. The one-step transition probabilities from state $n$ to $n$-$l$ (except from $N$ to $N$) in this case are

$$a_{n,n-l} = p\binom{[n,m]^*}{l+1} s^{l+1}(1-s)^{[n,m]^*-l-1} \tag{6.86}$$

$$+ (1-p)\binom{[n,m]^*}{l} s^l(1-s)^{[n,m]^*-l},$$

$$-1 \le l \le m, \qquad n \ne N.$$

The transition probability from state $N$ to $N$ is

$$a_{N,N} = p[\binom{m}{1} s(1-s)^{m-1} + (1-s)^m] + (1-p)(1-s)^m. \tag{6.87}$$

To solve this Markov chain one can make use of the fact that in equilibrium the flow of probability flux across a vertical boundary between state $i$-1 and $i$ balances in both directions. For the state transition diagram of Figure 6.5 it can be expressed as

$$a_{i-1,i}P_{i-1} = \sum_{n=i}^{m+i-1} \sum_{j=n-m}^{i-1} a_{n,j}P_n, \qquad i = N, \ldots 1. \tag{6.88}$$

Here $P_i$ is the equilibrium state probability of there being $i$ customers in the queueing system. Note that in chapter 2 a lower case $p$ was used to describe the same quantity. The (unnormalized) state probabilities can be evaluated starting with state $N$-1 and proceeding toward state 0. One can then choose $P_N$ so that

$$\sum_{n=0}^{N} P_n = 1 \tag{6.89}$$

and then compute the normalized equilibrium state probabilities. The complete procedure is

(1) Let $P_N = 1.0$.

(2) Initialize $a_{i,j}$'s.

(3) $i=N$-1

(4) $P_i = \frac{1}{a_{i,i+1}} \sum_{n=i+1}^{i+m} \sum_{j=n-m}^{i} a_{n,j} P_n$ .

(5) $i=i-1$

(6) Repeat step (4) and (5) until $i < 0$.

(7) Find $\sum P_i$ .

(8) Divide all $P_i$'s acquired in steps (1) and (4) by the sum of step (7). This produces the normalized equilibrium state probabilities.

Overflow problems are possible with the use of these equations for large $N$. However, one can use scaling techniques to scale up or down intermediate results automatically. It should also be noted that a reflected version ($n=N$-$n$) of the state transition diagram of this section can be used to model a discrete time queue with multiple arrivals and single departures [SZYM].

Of course, once the equilibrium state probabilities are calculated various performance measures, which are functions of these state probabilities, can also be calculated:

$$P_b = Prob(Arriving\ Customer\ is\ Blocked), \tag{6.90}$$

$$P_b = Prob(Queue\ Full) \times Prob(No\ Service\ Completion),$$

$$P_b = P_N s_0 \quad (where\ s_0 = (1-s)^m);$$

$$\overline{Y} = Mean\ Throughput, \tag{6.91}$$

$$\overline{Y} = Prob\,(Customer\ Arrival) \times Prob\,(No\ Blocking),$$

$$\overline{Y} = p(1 - P_b);$$

$$\overline{n} = Mean\ Queue\ Length, \tag{6.92}$$

$$\overline{n} = \sum_{n=1}^{N} n P_n;$$

$$\overline{\tau} = Mean\ Delay, \tag{6.93}$$

$$\overline{\tau} = \frac{\overline{n}}{\overline{Y}} \quad (Little's\ Law).$$