Quantum Mechanics and Path Integrals Selected Book Solutions (R.P. Feynman and A.R. Hibbs)

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1 Introduction

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

1.1 Probability in Quantum Mechanics

2 The Quantum Mechanical Law of Motion

2.1 The Classical Action

Problem 2-1 For a free particle $L = (m/2)\dot{x}^2$. Show that the action S_{cl} corresponding to the classical motion of a free particle is

$S_{cl} = \frac{m}{2}\frac{(x_b-x_a)^2}{(t_b-t_a)}$

Solution:

Since L is given by

$$L = \frac{m}{2}\dot{x}^2$$

we use the Euler-Lagrange equation,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) = 0$$

we first calculate using L the partial derivative $\frac{\partial L}{\partial x}$,

$$\frac{\partial L}{\partial x} = 0$$

because L does not depend on x explicitly. Next we find $\frac{\partial L}{\partial \dot{x}}$:

$$\frac{\partial L}{\partial \dot{x}} = \frac{2}{2} m\dot{x} = m\dot{x}$$

next we take the derivative with respect to t which gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\frac{d}{dt}(\dot{x}) = m\frac{d^2x}{dt^2} = m\ddot{x}$$

Plugging both equations above into out Euler Lagrange equation simply gives

$$0 - m\frac{d^2x}{dt^2} = 0$$

so that after dividing by $-m$ we are left with the simple second-order ordinary differential equation

$$\frac{d^2x}{dt^2} = 0$$

In physical terms, all this equation is saying is that the acceleration is 0. Therefore, if there is no acceleration, the particle is moving with a constant velocity, in other words, $\dot{x} = const$, in which case we can just set $\dot{x}$ to be the average velocity,

$$\dot{x} = \frac{x(t_b) - x(t_a)}{t_b - t_a}$$

Therefore, to find S_{cl}, we must use

$$S_{cl} = \int_{t_a}^{t_b} L(x, \dot{x}, t)dt$$

and so

$$S_{cl} = \int_{t_a}^{t_b} L(x, \dot{x}, t)dt = \int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2 dt = \frac{m}{2}\dot{x}^2 \int_{t_a}^{t_b} dt$$

where the inside term containing $\dot{x}$ and all constants were brought outside because as just discussed, $\dot{x}$ is itself a constant (constant velocity). Plugging in our previous equation obtained and doing the integral just gives

$$S_{cl} = \frac{m}{2}\dot{x}^2 \int_{t_a}^{t_b} dt = \frac{m}{2}\Big(\frac{x(t_b) - x(t_a)}{t_b - t_a}\Big)^2 (t_b - t_a) = \frac{m}{2}\frac{(x(t_b) - x(t_a))^2}{t_b - t_a}$$

where the square term at the bottom was reduced by one power because of the time difference at the top that resulted from the integral.

Alternatively integration by parts can be used in the second last step but we will not demonstrate it here. The reader is encouraged to try it for themselves if they feel so inclined. However, it does not appear to provide much new insight, if any, that the author is aware of currently.

2.2 The Quantum-Mechanical Amplitude

2.3 The Classical Limit

2.4 The sum over paths

Question 2-6 Refer to the solutions given by images at the end of this document

3 Developing the Concepts with Special Examples

3.1 This is a

4 The Schrodinger Description of Quantum Mechanics

4.1 The Schrodinger Equation

Question 4-3 Show that the complex conjugate function ψ^* (all i's changed to $-i$'s) satisfies

$$\frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

Solution: Using (4.14),

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} H\psi$$

and taking the complex conjugate of both sides yields

$$\left(\frac{\partial \psi}{\partial t}\right)^* = \left(-\frac{i}{\hbar} H\psi\right)^* \Leftrightarrow \frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H\psi)^*$$

Question 4-4 Show

$$\frac{\partial^2}{\partial x^2} x = x\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}$$

Solution:

Hint: (4.21) (below) is very helpful.

$$\frac{\partial}{\partial x} x = x\frac{\partial}{\partial x} + 1$$

Full solution:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} x = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} x\right) = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} + 1\right) = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial x} \\ = x\frac{\partial^2}{\partial x^2} + (1)\frac{\partial}{\partial x} + \frac{\partial}{\partial x} = x\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}\end{aligned}$$

where the distributive property was used in the third step and the product rule in the fourth.

To answer the second part, just write out the full form of H using (4.15) and insert that into the left side of (4.23) then use the newly proven (4.22) to simplify the equation to the right hand side of (4.23)

Note here that $H = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$. Thus,

$$\begin{aligned}Hx - xH &= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} x + V(x,t)x - x\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - xV(x,t) \\ &= \frac{-\hbar^2}{2m}\left(x\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}\right) + V(x,t)x - x\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - xV(x,t) \\ &= \frac{-\hbar^2}{2m} 2\frac{\partial}{\partial x} \\ &= \frac{-\hbar^2}{m}\frac{\partial}{\partial x}\end{aligned}$$

where we have assumed that $[V(x,t), x] = 0$

Bonus: What about $\frac{\partial^3}{\partial x^3}x$? We use the result above obtained for the case $\frac{\partial^2}{\partial x^2}x$.

$$\begin{aligned}
\frac{\partial^3}{\partial x^3}x &= \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}x\right) \\
&= \frac{\partial}{\partial x}\left(x\frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}\right) \\
&= \frac{\partial}{\partial x}(x\frac{\partial^2}{\partial x^2}) + 2\frac{\partial}{\partial x}(\frac{\partial}{\partial x}) \\
&= x\frac{\partial^3}{\partial x^3} + (1)\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x^2} \\
&= x\frac{\partial^3}{\partial x^3} + 3\frac{\partial^2}{\partial x^2}
\end{aligned}$$

where the product rule was used to obtain the fourth line. One can conjecture and probably prove the operator formula

$$(\frac{\partial^n}{\partial x^n})x = x\frac{\partial^n}{\partial x^n} + n\frac{\partial^{n-1}}{\partial x^{n-1}}$$

where n refers to the n-th derivative wrt x, or in proper operator notation using D_x to stand for the differentiation operator with respect to x,

$$D_x^n x = xD_x^n + nD_x^{n-1}$$

where again we emphasize that on the left hand side, since these are operators, we have the operator that applies the n-th derivative *multiplying* the operator that takes some state/function/etc ψ and multiplies it by x. The operator D_x is *not* acting on x. If ψ is, for example, some function of x and we call the operator M_x the operator that multiplies something by x (for example, $M_x\psi(x) = x\psi(x)$), then we get using the same notation as before for the operator equation on ψ as

$$\big(D_x^n M_x\big)\psi(x) = \big(M_x D_x^n + nD_x^{n-1}\big)\psi(x)$$

The relation can be simply be summarized as

$$[D_x^n, M_x] = nD_x^{n-1}$$

where again the square brackets refer to the commutator, ie $[x, y] = xy - yx$.

Although this is one case, let's take a look at what happens if we switch the roles and instead of modifying the derivative operator, we change the multiplication operator to be "multiply by x^2" instead of x.

As Feynman originally presented on page 80, let's modify his approach and let A be the operator that takes the first derivative, ie $A = D_x = \frac{\partial}{\partial x}$ and B is the opertor that multiplies by x^2.

We first need to modify Feynman's result, ie that $\frac{\partial}{\partial x}(x\psi) = x\frac{\partial \psi}{\partial x} + \psi$. Since we now instead multiply by x^2, the result becomes:

$$\frac{\partial}{\partial x}(x^2\psi) = x^2\frac{\partial \psi}{\partial x} + 2x\psi$$

Using this, similar to Problem 4-4, we ask the following:

Problem: Bonus $A^2B = \frac{\partial^2}{\partial x^2}x^2 = ?$

Solution:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}x^2 &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}x^2\right) \\
&= \frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x} + 2x\right) \\
&= \frac{\partial}{\partial x}(x^2\frac{\partial}{\partial x}) + 2\frac{\partial}{\partial x}x \\
&= x^2\frac{\partial^2}{\partial x^2} + 2x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial x}x
\end{aligned}$$

while it may not look obvious at first, re-writing this in the differential operator notation proper gives the relation:

$$\left(D_x^2 M_{x^2}\right) = \left(M_{x^2}\left(D_x^2\right) + 2M_x\left(D_x\right) + 2\left(D_x\right)M_x\right)$$

(where M_{x^2} refers to the operator that multiplies by x^2) but after factoring out the 2 and moving the first term on the right hand side to the left, we obtain

$$D_x^2 M_{x^2} - M_{x^2}\left(D_x^2\right) = 2\left(M_x\left(D_x\right) + \left(D_x\right)M_x\right)$$

but looking closely, on the left we just have the commutator and on the right we have the well-known *poisson brackets* or *anticommutator* ie $\{a, b\} = ab + ba$. Thus we may re-write the operator equation as

$$[D_x^2, M_{x^2}] = 2\{M_x, D_x\}$$

or

$$[D_x^2, M_{x^2}] = 2\{D_x, M_x\}$$

since we have assumed commutativity in the addition of the operators, ie that $AB + BA = BA + AB$.

Further interesting properties may be derived from this point, either by looking at an operator that multiplies by the n-th power of the variable or mixing the number of derivatives and the powers. Actually, this relation speaks to a deeper aspect of both the differentiation operator and the multiplication operator. A good starting guide can be found in section 10.7 of Kreyszig's "Introductory Functional Analysis with Applications" and Helmberg's "Introduction to Spectral Theory in Hilbert Space" Chapter IV, Section 18.[2] Due to the author's laziness, he leaves it to the reader to try it for themselves using this question as a starting point.

Question 4-5 Using the relation

$$K(b, a) = \int_{-\infty}^{\infty} K(b, c)K(c, a)dx_c$$

with $t_c - t_a = \epsilon$, an infinitesimal, show that if $t_b > t_a$ the kernel K satisfies

$$\frac{\partial}{\partial t_a} = K(b, a) = +\frac{i}{\hbar}H_a^* K(b, a)$$

Solution:

Hints:

1. $K(c, a) = K^*(a, c)$ (where K^* is the complex conjugate)
2. $[\frac{\partial}{\partial t_a}, K(b, c)] = 0$ (i.e. they commute since it's independent of a and [] is the commutator, $[x, y] = xy - yx$)
3. Following (4.21),
   if $\frac{\partial \psi^*}{\partial t} = +\frac{i}{\hbar}(H_a\psi)^*$ then $\frac{\partial}{\partial t_a}K^*(a, c) = +\frac{i}{\hbar}\Big(H_a K(a, c)\Big)^*$

Full Solution:

$$K(b,a) = \int_{-\infty}^{\infty} K(b,c)K(c,a)dx_c \Longleftrightarrow$$
$$\frac{\partial}{\partial t_a}K(b,a) = \frac{\partial}{\partial t_a}\int_{-\infty}^{\infty} K(b,c)K(c,a)dx_c$$

bringing the partial derivative inside and using hints 1 and 2 we have

$$\frac{\partial}{\partial t_a}K(b,a) = \int_{-\infty}^{\infty} K(b,c)\frac{\partial}{\partial t_a}K^*(a,c)dx_c$$

so that using hint 3 yields

$$\frac{\partial}{\partial t_a}K(b,a) = \int_{-\infty}^{\infty} K(b,c)\Big(\frac{i}{\hbar}\Big(H_a K(a,c)\Big)^*\Big)dx_c$$

and applying the complex conjugate operator gives

$$\frac{\partial}{\partial t_a}K(b,a) = \int_{-\infty}^{\infty} K(b,c)\Big(\frac{i}{\hbar}K^*(a,c)H_a^*\Big)dx_c$$
$$= \frac{i}{\hbar}H_a^*\int_{-\infty}^{\infty} K(b,c)K(c,a)dx_c = \frac{i}{\hbar}H_a^*K(b,a)$$

where constants have been taken out of the integral (i.e. $i, \hbar$), H_a^* has been taken out since it is independent of the integration variable (recall (4.24)) and hint 1 has been used but to go back to the original kernel K.

Question 4-7 Show that $\int K^*(b,a)K(b,c)dx_b = K^*(c,a)$ with the usual $t_b > t_c > t_a$ convention holding.

Solution:

Using (4.38),

$$\int_{-\infty}^{\infty} K^*(b,c)K(b,a)dx_b = K(c,a)$$

we take the complex conjugate of both sides such that

$$\Big(\int_{-\infty}^{\infty} K^*(b,c)K(b,a)dx_b\Big)^* = \Big(K(c,a)\Big)^*$$
$$\Longleftrightarrow \int_{-\infty}^{\infty} ((K(b,a))^*(K^*(b,c))^*dx_b = K^*(c,a)$$
$$\Longleftrightarrow \int_{-\infty}^{\infty} K^*(b,a)K(b,c)dx_b = K^*(c,a)$$

where the complex conjugate of the complex conjugate is the kernel itself.

5 Measurements and Operators

5.1 Measurement of quantum mechanical variables

Problem 5-3 Assume $\int_{-\infty}^{\infty} f^*(x)f(x)dx$, which is the probability that a particle of wave function $f(x)$ is somewhere, has been normalized to the value 1. Under this constraint, show that the state $f(x)$ which has the highest probability of having the property G is $f(x) = g(x)$

Solution: Note that if $f(x) = g(x)$, then $f^*(x) = g^*(x)$, where $*$ denotes the complex conjugate. Then the probability of having some property ξ is

$$P(\xi) = \left| \int_{-\infty}^{\infty} f^*(x)f(x)dx \right|^2$$

using (5.32),

$$P(G) = \left| \int_{-\infty}^{\infty} g^*(x)f(x)dx \right|^2$$

where $g^*(x)$ is probability of having property G but since $f^*(x) = g^*(x)$

$$P(G) = \left| \int_{-\infty}^{\infty} f^*(x)f(x)dx \right|^2$$

which occurs when $f(x) = g(x)$.

Problem 5-4 Suppose the wave function for a system is $\psi(x)$ at time t_a. Suppose further that the behaviour of the system id described by the kernel $K(x_b, t_b; x_a, t_a)$ for motions in the interval $t_b \geq t \geq t_a$. Show that the probability that the system is found to be in the state $\chi(x)$ at time t_b is given by the square of the integral

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi^*(x_b)K(x_b, t_b; x_a, t_a)\psi(x_a)dx_a dx_b$

We call this integral the *transition amplitude* to go from state $\psi(x)$ to state $\chi(x)$

Solution: We use (5.32), ie

$$P(G) = \left| \int_{-\infty}^{\infty} g^*(x)f(x)dx \right|^2$$

and identify that $g^*(x)$ is $\int_{-\infty}^{\infty} \chi^*(x_b)dx_b$ and $f(x)$ is $\int_{-\infty}^{\infty} K(x_b,t_b;x_a,t_a)\psi(x_a)dx_a$. In other words,

$$P(\chi^*(x_b)) = \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi^*(x_b)K(x_b,t_b;x_a,t_a)\psi(x_a)dx_a dx_b \right|^2$$

5.2

5.3 Operators

Problem 5-8 Note that equation (5.44) implies $G_A^*(x,x') = G_A(x',x)$. With this in mind show that for any two wave functions $g(x)$ and $f(x)$, both of which approach 0 as x goes to $\pm\infty$,

$$\int_{-\infty}^{\infty} g^*(x)\mathcal{A}f(x)dx = \int_{-\infty}^{\infty} [\mathcal{A}g(x)]^* f(x)dx$$

Any operator, such as $\mathcal{A}$, for which Eq. (5.47) holds is called *hermitian* (see Eq. 4.30)

Solution:

Note: use (5.45) where $R = \mathcal{A}f$, where using (5.43), $R(x) = \int_{-\infty}^{\infty} G_A(x,x')f(x')dx'$

Also note that using $G_A^*(x,x') = G_A(x',x) \Leftrightarrow G_A^*(x',x) = G_A(x,x')$

Hints:

1. Use (5.45) where $R = \mathcal{A}f$, where using (5.43), $R(x) = \int_{-\infty}^{\infty} G_A(x,x')f(x')dx'$
2. $G_A^*(x,x') = G_A(x',x) \Leftrightarrow G_A^*(x',x) = G_A(x,x')$
3. Use for operators A and B, $(AB)^* = B^*A^*$

$$\begin{aligned}\int_{-\infty}^{\infty} g^*(x)\mathcal{A}f(x)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)G_A(x,x')f(x')dx'dx \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)G_A^*(x',x)f(x')dx'dx\end{aligned}$$

where in the last step we used hint 2, and then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^*(x)G_A^*(x',x)f(x')dx'dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G_A(x,x')g(x'))^* f(x)dx'dx$$

using hint 3 and we now notice that $(G_A(x,x')g(x'))^*$ has the form $R^* = [\mathcal{A}g]^*$ so that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(G_A(x,x')g(x'))^* f(x)dx'dx = \int_{-\infty}^{\infty}[\mathcal{A}g(x)]^* f(x)dx$$

6

7 Transition Elements

Problem 7-1 If $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$, show that, for any s inside the range t_a to t_b,

$\frac{\delta S}{\delta x(s)} = -\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x}$

where the partial derivatives are evaluated at $t = s$

Solution:

In this case we use the first variation of a functional given by (7.24)

$$\delta F = \int \frac{\delta F}{\delta x(s)}\delta x(s)ds$$

or re-written in terms of the action S,

$$\delta S = \int \frac{\delta S}{\delta x(s)}\delta x(s)ds$$

and assuming that since $S[x(t)] = \int_{t_a}^{t_b} L(\dot{x}, x, t)dt$, we are dealing with the same ideas of the Lagrangian as introduced in section 2-1, ie things like $\delta x(t)$ vanishing at the end points such that $\delta x(t_a) = \delta x(t_b) = 0$.

We use the arguments of chapter 2-1, and start from (2.6) by recognizing that $\delta F = \delta S$ in that case/equation and so

$$\delta S = \delta x(t)\frac{\partial L}{\partial x(t)}\Big|_{t_a}^{t_b} - \int_{t_a}^{t_b}\delta x(t)\Big[\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x}\Big]dt$$

the first term on the right is 0 simply by virtue of the condition that $\delta x(t_a) = \delta x(t_b) = 0$ and distributing the negative sign inside the second term gives

$$\delta S = \int_{t_a}^{t_b}\Big[-\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{x}}\Big) + \frac{\partial L}{\partial x}\Big]\delta x(t)dt$$

but for $s \in [t_a, t_b]$ with $t = s \to dt = ds$, $\delta x(t) \to \delta x(s)$ and $\frac{d}{dt} \to \frac{d}{ds}$

$$\delta S = \int_{t_a}^{t_b} \Big[-\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}}\Big) + \frac{\partial L}{\partial x}\Big] \delta x(s) ds$$

but this is the same form as (7.24) and we immediately see from the term in the brackets then that

$$\frac{\delta S}{\delta x(s)} = -\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot{x}}\Big)\Big|_{t=s} + \frac{\partial L}{\partial x}\Big|_{t=s}$$

Problem 7-2

8 Harmonic Oscillator

9 Quantum Electrodynamics

10 Statistical Mechanics

11 The Variational Method

12 Other Problems in Probability

12.1 Part 1

12.2 Part 2

12.3 Part 3

12.4 Part 4

12.5 Part 5

12.6 Brownian Motion

Problem 12-2 Show that the constant required to normalize the probability function $P(D, \theta) dD d\theta$ is

$const = \sqrt{\frac{6}{\pi R T^3}} \sqrt{\frac{1}{2\pi R T}}$

Solution: It is helpful to consult the table of integrals given in the Appendix, especially (A.1)

We will make many uses of the following Gaussian integral:

$$\int_{-\infty}^{\infty} \exp\bigl(-a(x+b)^2\bigr)dx = \sqrt{\frac{\pi}{a}}$$

where a is a non-zero real number. This result is derived and/or proven in a myriad of different physics and mathematics textbooks, so we won't go into a detailed explanation of this integral here.

The normalized distribution for P should satisfy

$$\frac{1}{const}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(D,\theta)dDd\theta = 1$$

Therefore, solving the integral and inverting will give the required constant.

Using P as given in (12.76),

$$P(D,\theta) = \exp\Bigl\{-\frac{6}{RT^3}(D-\frac{\theta T}{2})^2 - \frac{\theta^2}{2RT}\Bigr\}$$

we perform the integration

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(D,\theta)dDd\theta &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\Bigl\{-\frac{6}{RT^3}(D-\frac{\theta T}{2})^2 - \frac{\theta^2}{2RT}\Bigr\}dDd\theta \\ &= \int_{-\infty}^{\infty}\Bigl(\int_{-\infty}^{\infty} \exp\Bigl\{-\frac{6}{RT^3}(D-\frac{\theta T}{2})^2\Bigr\}dD\Bigr)\exp\Bigl\{-\frac{\theta^2}{2RT}\Bigr\}d\theta\end{aligned}$$

where we have interchanged the integrals by using Fubini's theorem to interchange the order of integration.

First, we evaluate the inner integral in brackets, noting that we will use the "Gaussian Integral" mentioned above by setting $a = \frac{6}{RT^3}$ and $b = -\frac{\theta T}{2}$ giving

$$\begin{aligned}\int_{-\infty}^{\infty} \exp\Bigl\{-\frac{6}{RT^3}(D-\frac{\theta T}{2})^2\Bigr\}dD &= \sqrt{\frac{\pi}{(6/RT^3)}} \\ &= \sqrt{\frac{\pi RT^3}{6}}\end{aligned}$$

Our integral is now

$$\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\exp\Big\{-\frac{6}{RT^3}(D-\frac{\theta T}{2})^2\Big\}dD\Big)\exp\Big\{-\frac{\theta^2}{2RT}\Big\}d\theta=\sqrt{\frac{\pi RT^3}{6}}\int_{-\infty}^{\infty}\exp\Big\{-\frac{\theta^2}{2RT}\Big\}d\theta$$

Next, we evaluate the second integral. This integral is also Gaussian and we again make use of the Guassian integral mentioned above, this time by setting $a=\frac{1}{2RT}$ and $b=0$ to get

$$\begin{aligned}\int_{-\infty}^{\infty}\exp\Big\{-\frac{\theta^2}{2RT}\Big\}d\theta &= \sqrt{\frac{\pi}{(1/2RT)}}\\ &= \sqrt{2\pi RT}\end{aligned}$$

Thus our integral ultimately evaluates to

$$\frac{1}{const}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P(D,\theta)dDd\theta=\sqrt{\frac{\pi RT^3}{6}}\sqrt{2\pi RT}$$

so that $\frac{1}{const}$ is given by

$$\frac{1}{const}=\sqrt{\frac{\pi RT^3}{6}}\sqrt{2\pi RT}$$

and upon inverting gives

$$const=\sqrt{\frac{6}{\pi RT^3}}\sqrt{\frac{1}{2\pi RT}}$$

which is the required constant for normalization.

13 Conclusion

"I always thought something was fundamentally wrong with the universe" [1]

14 Ya

References

[1] D. Adams. *The Hitchhiker's Guide to the Galaxy.* San Val, 1995.

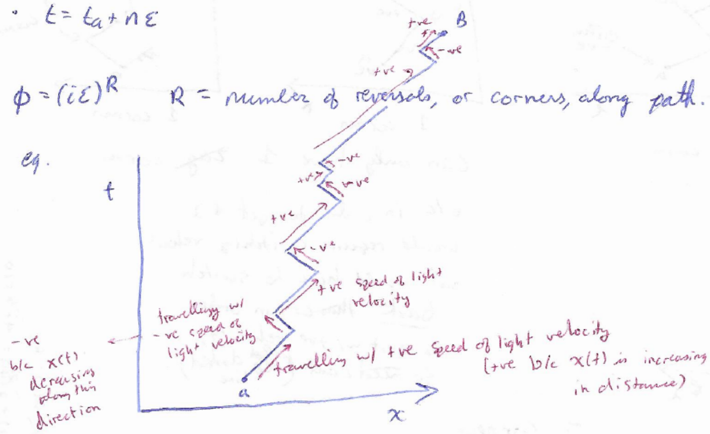
[2] E. Kreyszig. *Introductory Functional Analysis with Applications.* San Val, 1986.

2-6 (2.15) $\rightarrow \phi[x(t)] = \text{const.}\, e^{(i/\hbar)S[x(t)]}$

- Can only go back & forth @ speed of light
- $t = t_a + n\varepsilon$

$\phi = (i\varepsilon)^R$ R = number of reversals, or corners, along path.

eg.



(2.27) $K(b,a) = \sum_R N(R)(i\varepsilon)^R$ add together contributions to paths for 1 corner, 2 corners, etc.

$K_{++}(b,a)$ → Starting @ 'a' w/ a positive velocity & coming into 'b' w/ a positive velocity

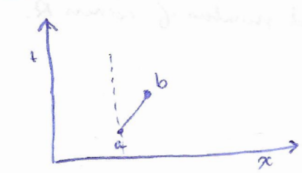
$K_{+-}(b,a)$ → Start @ 'a' w/ a -ve velocity, arrive @ 'b' w/ a ~~pos~~ positive velocity (+ve)

$K_{-+}(b,a)$ → Start @ 'a' w/ a +ve velocity, arrive @ 'b' w/ a -ve velocity

$K_{--}(b,a)$ → start & end w/ negative velocity.

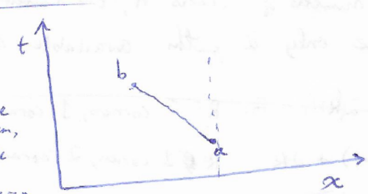
1 ~~to~~ corner/zag:

$K_{++}(b,a)$ → Start @ w/ +ve, end w/ +ve



there cannot be any corners no matter ~~b/c then~~ how close b is b/c that would require changing from +ve to -ve velocity;

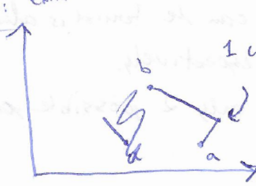
$K_{--}(b,a)$: -ve to -ve



Same problem, ~~Can~~ b/c cannot change sign, so no corner

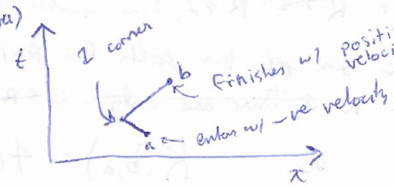
K_{-+}
~~K_{+-}~~ (b,a): enters +ve exits -ve

1 corner

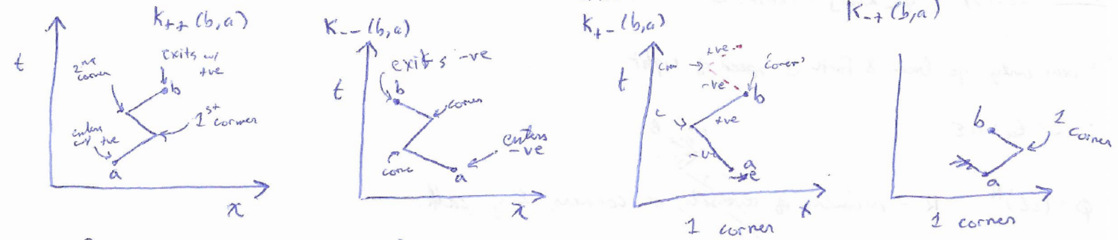


~~$K_{-+}(b,a)$~~
$K_{+-}(b,a)$
enters -ve exit +ve

1 corner



2 corners / zig-zags



2 corners

2 corners

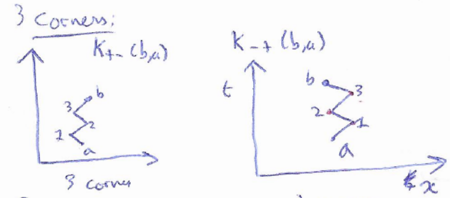
1 corner

1 corner

Can only have 1 ~~zag~~ corner

b/c in order to get to 2 would require switching velocities but would have to switch back ~~then~~ in order to exit w/ +ve velocity, so ~~need~~ can't (red dashed line)

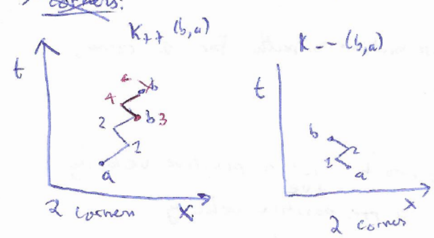
3 Corners:



3 corners

3 corners

~~3 corners~~:

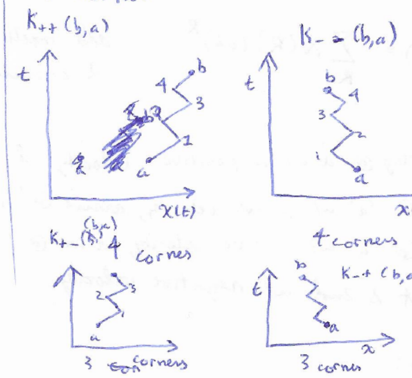


2 corners

2 corners

2 corners permitted, again, to have 3 corners requires entering b w/ a -ve velocity, which can't do

4 corners



~~So for the numbers of corners n, the number of paths available is~~

There are only 2 paths available each time for the specified numbers of corners R.

~~$2R + 2(R+1)$~~ for $R = 0$ corners, 1 corners, 2 corners, etc.

~~$2(R-1) + 2R$~~ $R = 1$ corner, 2 corner, 3 corners, ...

~~$2n - 1$~~ ~~$2n - 2$~~

If ~~$R > 1$~~ $R \geq 2$, the number of possible paths that can be found is always ~~4~~ 2.

B/c you get two paths from $R+1$ & 2 path from R respectively.

For ~~$R = 1$, there are only~~ ~~$0 \leq R$~~ $R = 0$ or 1, there are only 2 possible paths w/ that many corners

So $K(b,a) = 4(i\epsilon)^R$

The number of possible paths for a specified R is 2. Notice in going from R to $R+1$, either kernel $K_{+-}(b,a)$ & $K_{-+}(b,a)$ stay same b/c can only have that many corners as previous or else can gain another corner only from going from R to $R+2$.

Same applies for $K_{++}(b,a)$ & $K_{--}(b,a)$

Thus $K(b,a) = \sum_{R} N(R)(i\varepsilon)^R = 2(i\varepsilon)^R$ where R is the number of ~~path~~ corners such a path must have.