Summation and Integrals

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What is an Integral?

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

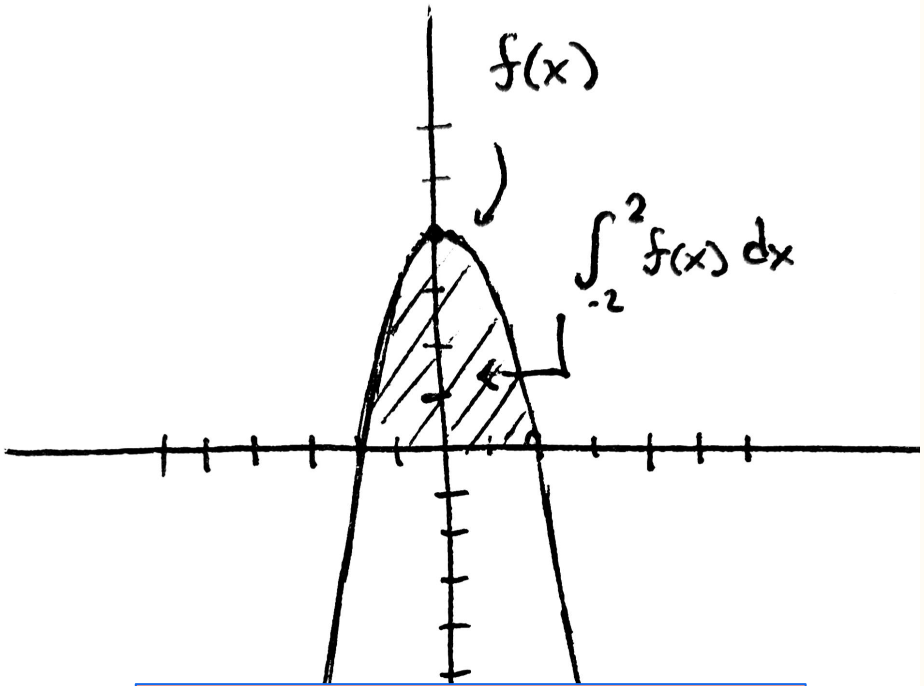
Essentially, an Integral is like taking the inverse of a derivative.

Where derivatives isolate the rate of change of a function at a given point x, integrals allow you to find the area UNDER a function across a given interval

$$f(x) = -x^2 + 4$$

$$\int_{-2}^{2} f(x)\,dx$$

$= 10.6666666667$



To better understand HOW this works, we're going to have to backtrack a little bit...

Summation and Riemann Sums

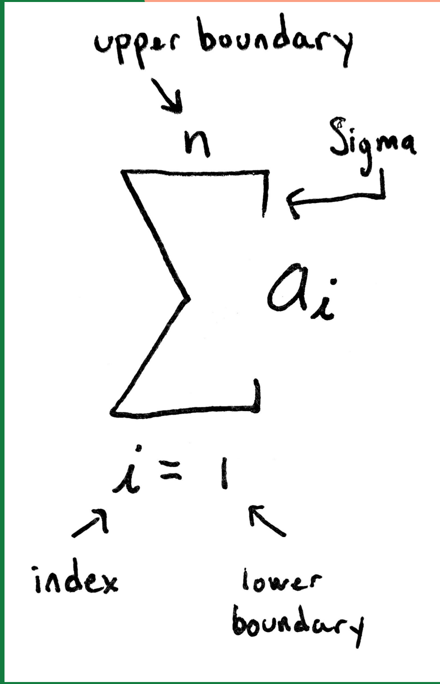
In order to understand how Riemann Sums approximate the area under a curve much like integrals do, we first have to understand *SIGMA NOTATION*, otherwise known as summation.

Theorem 1.7. Summation Properties. *For c constant:*

1. $\sum_{i=1}^{n} c = c \cdot n$
2. $\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$
3. $\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$
4. $\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$

Theorem 1.8. Summation Formulas.

1. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
2. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
3. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$



https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/sec_riemann.html

Telescoping sums

A telescoping sum is a series where all inner terms cancel out when expanded in a summation. For example, consider the sum:

$$S = \sum_{k=1}^{n} (a_k - a_{k+1})$$

When expanded, it becomes:

$$S = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}).$$

After cancellation, only the first and last terms remain:

$$S = a_1 - a_{n+1}.$$

https://en.wikipedia.org/wiki/Telescoping_series

Triangle Inequality

The triangle inequality states that for any real numbers a and b:

$$|a+b| \leq |a| + |b|.$$

In the context of vectors, for vectors $\mathbf{u}$ and $\mathbf{v}$, the inequality is written as:

$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Equality holds if and only if $\mathbf{u}$ and $\mathbf{v}$ are collinear and point in the same or opposite directions.

https://en.wikipedia.org/wiki/Triangle_inequality

Simple Summation example

Using the rules on previous slides, a fairly simple summation looks like this:

$$\sum_{i=1}^{6}(3i-4) = \sum_{i=1}^{6}3i - \sum_{i=1}^{6}4$$

$$= 3\sum_{i=1}^{6}i - 24$$

$$= 3\,\frac{6(6+1)}{2} - 24$$

$$= 63 - 24$$

$$= 39$$

Also, with a power →

$$\sum_{i=1}^{4} -i^2 + 4 = \sum_{i=1}^{4}(-i^2) + \sum_{i=1}^{4}(4)$$

$$= -\sum_{i=1}^{4}(i^2) + 16$$

$$= -\frac{4(4+1)(2(4)+1)}{6} + 16$$

$$= -30 + 16$$

$$= -14$$

Riemann Sums

A little more particular, a little closer to accuracy

We're equating a function to a series of rectangles approximating the area under the curve.

This is getting us closer and closer to a correct answer

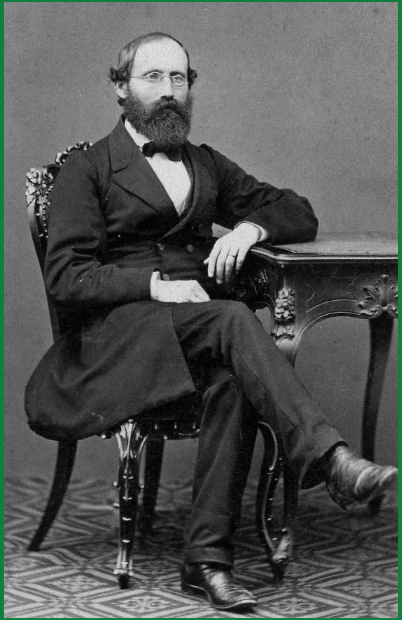
Definition 1.12. Riemann Sum. Let $f(x)$ be defined on the closed interval $[a, b]$ and let $P = \{x_1, x_2, \ldots, x_{n+1}\}$ be a partition of $[a, b]$, with

$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

Let Δx_i denote the length of the i^{th} subinterval $[x_i, x_{i+1}]$ and let c_i denote any value in the i^{th} subinterval. The sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

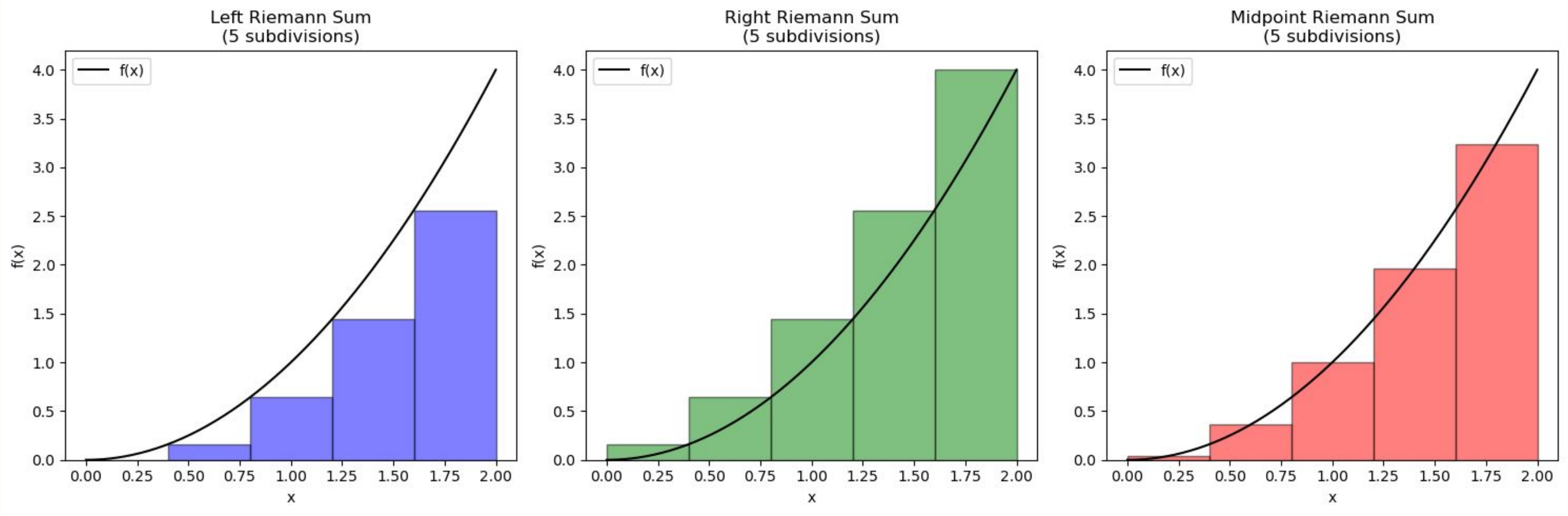
is a ***Riemann sum*** of $f(x)$ on $[a, b]$.



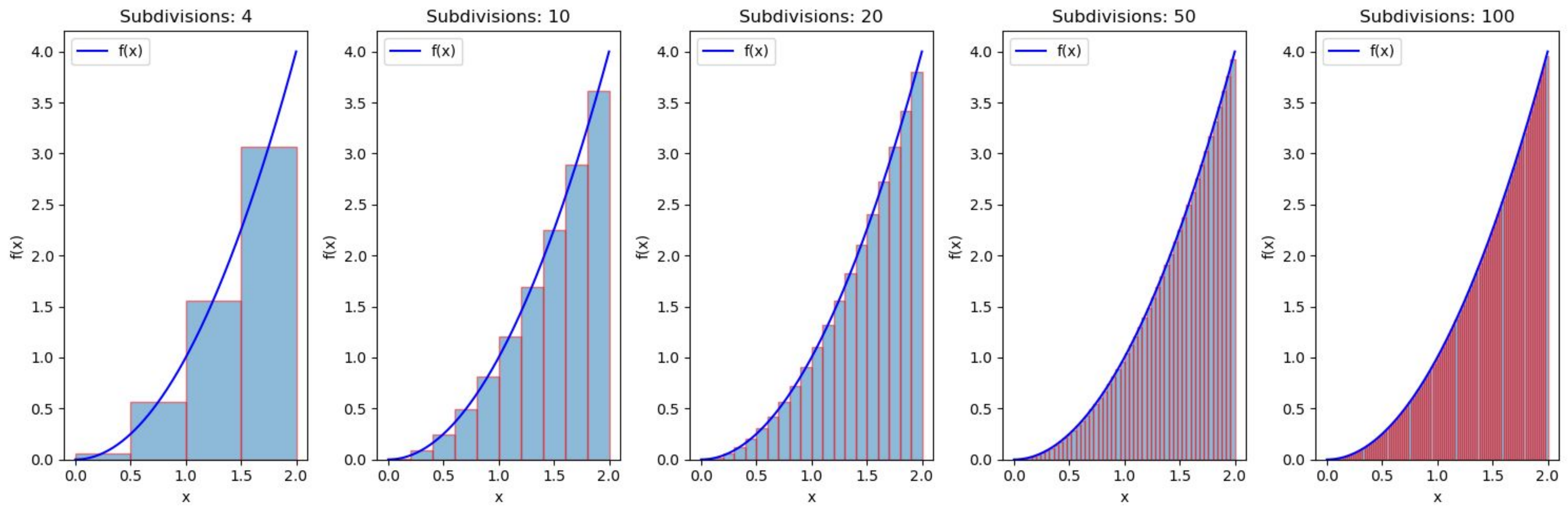
Bernhard Riemann(1826-1866)

https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/sec_riemann.html

Left, Right, and Center

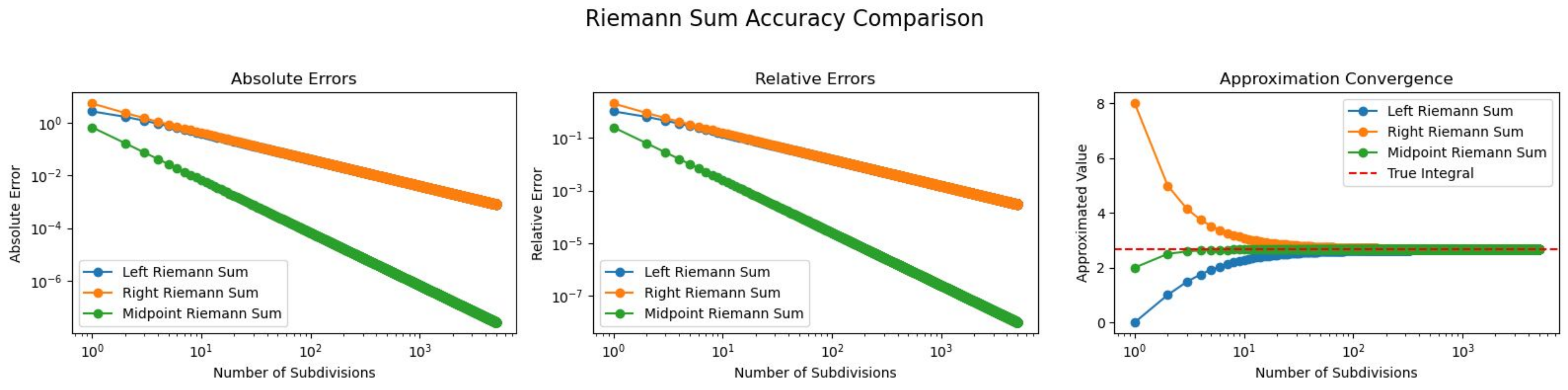


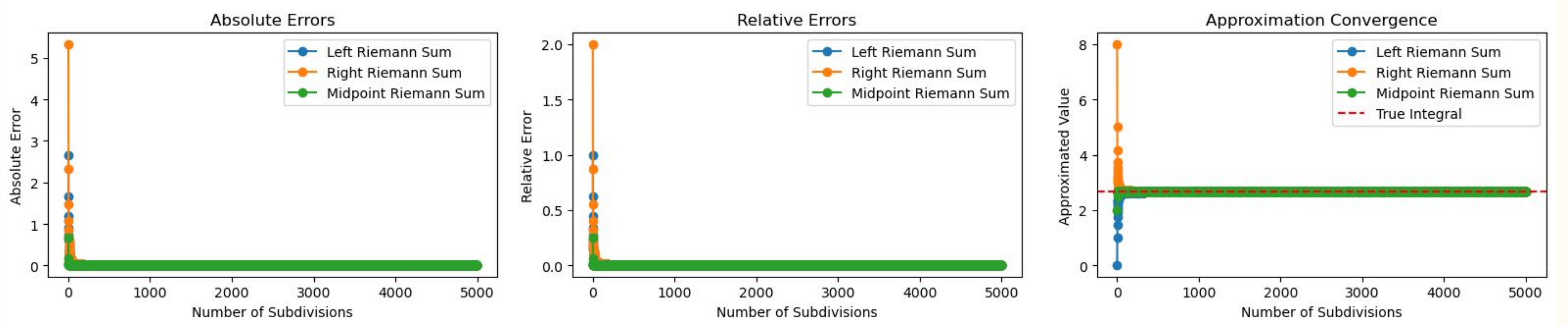
Riemann sums as n increases



Error Analysis

Upper: log space
Lower: Linear space





Other numerical Integration methods

Method	Accuracy	Pros	Cons
Left/Right Riemann Sum	**Local:** $\mathcal{O}(h)$ **Global:** $\mathcal{O}(1)$	Simple to implement.	Poor accuracy, depends on function monotonicity.
Midpoint Rule	**Local:** $\mathcal{O}(h^2)$ **Global:** $\mathcal{O}(h)$	Better accuracy than Left/Right Riemann sums.	Requires calculation of midpoints.
Trapezoidal Rule	**Local:** $\mathcal{O}(h^2)$ **Global:** $\mathcal{O}(h)$	Intuitive geometric interpretation, good accuracy for smooth functions.	Struggles with sharp discontinuities.
Simpson's Rule	**Local:** $\mathcal{O}(h^4)$ **Global:** $\mathcal{O}(h^3)$	High accuracy for polynomials and smooth functions.	Requires even number of subintervals.
Gaussian Quadrature	**Local:** $\mathcal{O}(h^{2k+1})$ **Global:** $\mathcal{O}(h^{2k})$	Extremely efficient for smooth functions, very high accuracy.	Requires pre-computation of weights and nodes.
Monte Carlo Integration	**Accuracy:** $\mathcal{O}(1/\sqrt{n})$ (independent of dimension for smooth functions).	Handles high-dimensional problems well.	Slow convergence, less suitable for low-dimensional integrals.

Cool Reading:
https://math.stackexchange.com/questions/603830/why-does-trapezoidal-rule-have-potential-error-greater-than-midpoint

Riemann Sum Example

Following these steps, we can evaluate the following function

Riemann Sums Using Rules (Left - Right - Midpoint).

Consider a function $f(x)$ defined on an interval $[a, b]$. The area under this curve is approximated by

$$\sum_{i=1}^{n} f(c_i)\Delta x_i.$$

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \frac{b-a}{n}$.
2. The i^{th} term of the partition is $x_i = a + (i-1)\Delta x$. (This makes $x_{n+1} = b$.)
3. The Left Hand Rule summation is: $\sum_{i=1}^{n} f(x_i)\Delta x$.
4. The Right Hand Rule summation is: $\sum_{i=1}^{n} f(x_{i+1})\Delta x$.
5. The Midpoint Rule summation is: $\sum_{i=1}^{n} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x$.

$f(x) = (3x - 4)$ Midpoint Rule

on the interval $[-2, 2]$ 10 spaces

$a = -2 \quad b = 2 \quad n = 10$

$$\Delta x = \frac{2-(-2)}{10} = \frac{2}{5}$$

$$x_i = -2 + (i-1)(2/5) = \frac{2i}{5} - \frac{12}{5}$$

$$x_{i+1} = \frac{2i+2}{5} - \frac{12}{5}$$

$$\frac{x_i + x_{i+1}}{2} = \frac{\left(\frac{2i}{5} - \frac{12}{5}\right) + \left(\frac{2i+2}{5} - \frac{12}{5}\right)}{2} = \frac{2i-11}{5}$$

$$\sum_{i=1}^{10} f\left(\frac{x_i + x_{i+1}}{2}\right)\Delta x = \sum_{i=1}^{10} f\left(\frac{2i-11}{5}\right)\Delta x$$

$f(x) = 3x - 4$

$$= \sum_{i=1}^{10}\left(3\left(\frac{2i-11}{5}\right) - 4\right)\Delta x$$

$$= \Delta x\left(\sum_{i=1}^{10}\frac{6i-33}{5} - \sum_{i=1}^{10} 4\right)$$

$$\frac{2}{5}(0 - 40) = \frac{-80}{5} = -16$$

What is an Integral? (Part Deux)

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of** $f(x)$ **from** a **to** b is

Coming back to this, we can now much more easily see how taking the limit of a Riemann sum eliminates the variance in the rectangular areas, and allows us to find a much more precise area under that function.

Where a and b come from make more sense now too?

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Proof Time!

FUNDAMENTAL THEOREM OF CALCULUS PART ONE AND TWO

$$I.\ \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

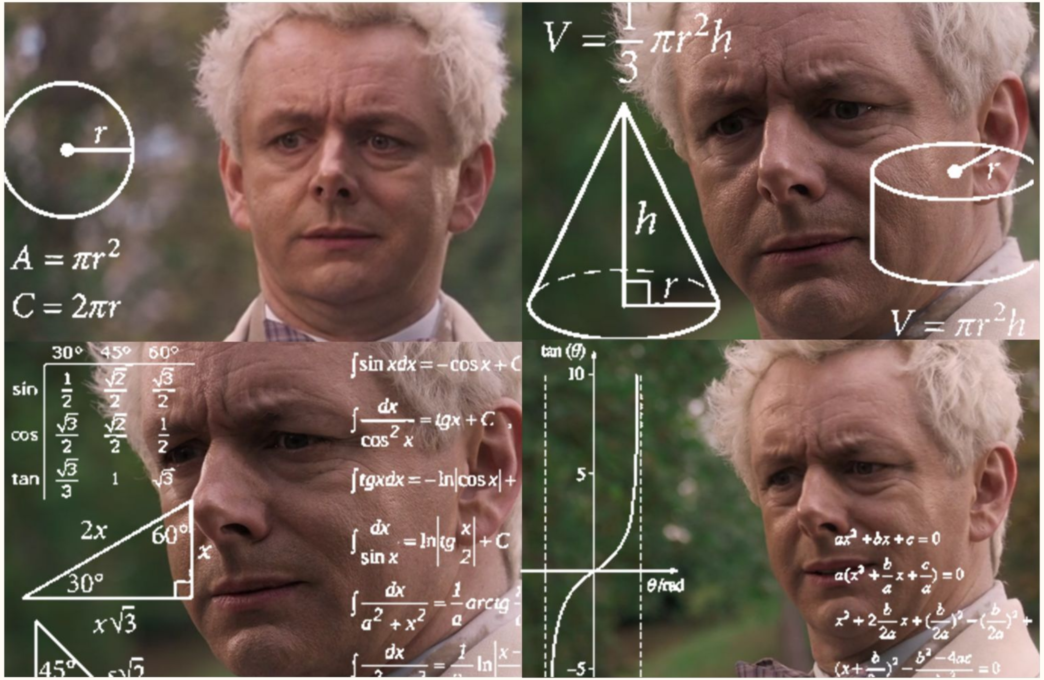
$$II.\ \int_a^b f(x)dx = F(b) - F(a)$$

May Newton have mercy upon your souls

Questions?

Next Week -

Prove Integrals. That's it. It's hard.



Code Project -

```
Goal: Implement one of the numerical integration methods on slide 13
Push your code as [Your_name]_project_3.[file_extention]
```