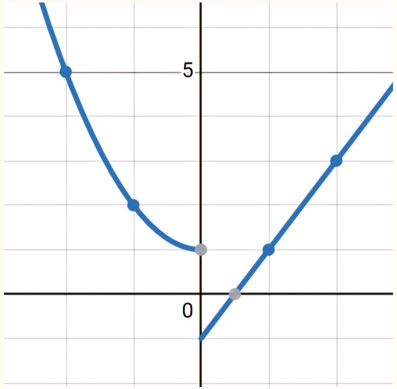
Limits and Continuity

John Hohman and Nathan James

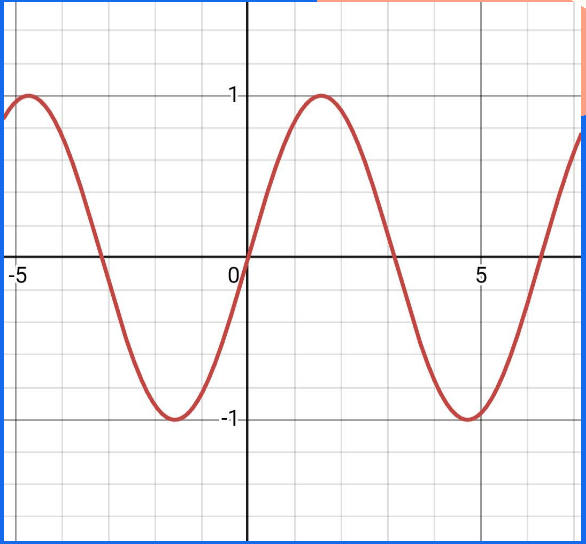
Table of Contents



Continuity and Discontinuity

What *IS* Continuity?

Continuity is described as the quality of a function being "continuous" or unbroken, from one or both directions, over the constraints of the equation.



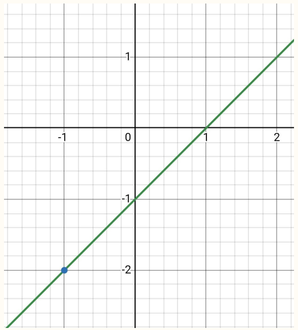
Sets of Continuous Functions

Continuously differentiable $\subset$ Lipschitz continuous $\subset$ α-Hölder continuous $\subset$ uniformly continuous $=$ continuous

Discontinuity - The Three Types

Removeable

Discontinuity where a function has a defined limit, but a single point on the line of the function does not match the limit

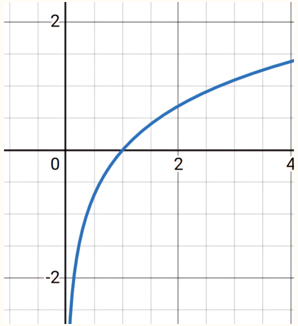


$$f(x) = \frac{x^2 - 1}{x + 1}$$

x	$f(x)$
-2	-3
-1	$undefined$
0	-1
1	0
2	1

Essential/Infinite

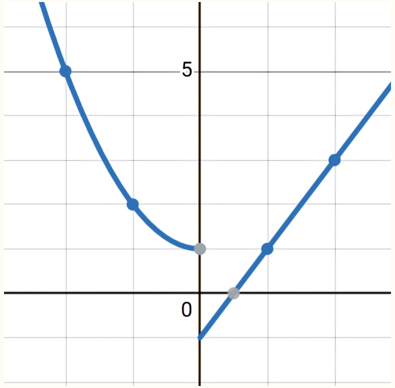
Discontinuity where one or both sides of the function trend toward infinity at a point



$$f(x) = \ln(x)$$

Jump

Discontinuity where the value of a function drastically changes at a given point



$$f(x) = \{x > 0 : 2x - 1, x \le 0 : x^2 + 1$$

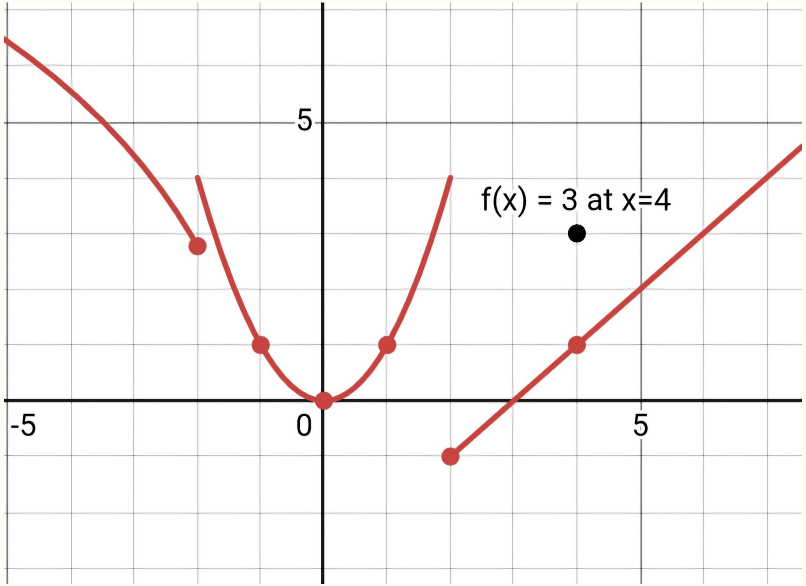
x	$f(x)$
-2	5
-1	2
0	1
1	1
2	3

There is a fourth type of discontinuity called an "Oscillation" - but the circumstances required for those encompass functions we wont be covering for a good while - so for now, act like these are the only 3.

Continuity and Limits

- Continuity and Discontinuity definitions helps define:
  - Where your bounds are to look for limits
  - Which side of the equation to look at to help you narrow down a limit
  - Whether or not there CAN be a limit for the given equation

$$f(x)\begin{cases} x \leq -2, & \ln(x^2) \\ -2 < x < 2, & x^2 \\ x \geq 2, & x-3 \\ x = 4, & 3 \end{cases}$$



Note: [1,5] is Inclusive, (1,5) is Exclusive

What is a Limit?

In Basic Terms:

- A Limit allows us to explore the behavior of a function as it approaches a value

Limits Proof

Taken from Libretexts.org

Definition: The Epsilon-Delta Definition of the Limit

Let $f(x)$ be defined for all $x \neq a$ over an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L \qquad (2.7.1)$$

if, for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

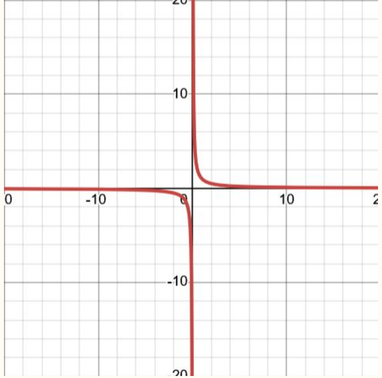
Includes a universal quantifier Epsilon, an existential quantifier Delta, and a set of constraints

- Similar to an asymptote

Directional Limits

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$



Consider the piecewise function:

$$f(x) = \begin{cases} x^2, & x < 1, \\ 3, & x = 1, \\ 2x - 1, & x > 1. \end{cases}$$

Left-Hand Limit:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1$$

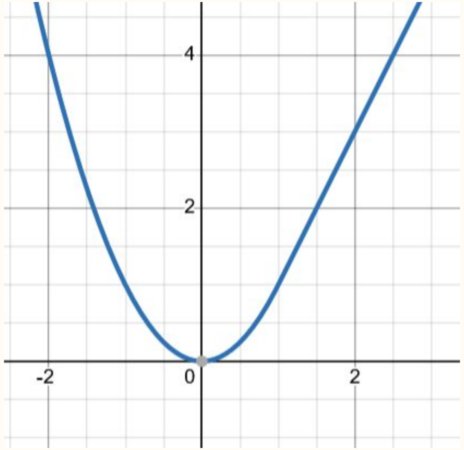
Right-Hand Limit:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 1) = 2(1) - 1 = 1$$

Since the left-hand limit and the right-hand limit are equal:

$$\lim_{x \to 1} f(x) = 1$$

Note that $f(1) = 3$, so the value of the function at $x = 1$ does not affect the limit.



Simple Limits

Most can be solved with algebraic properties like substitution, conjugation, and things like polynomial expansion (factoring)

Let's work through some of these examples quickly

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

$$\lim_{x \to 3} x^2 + 2x - 1$$

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$$

$$\lim_{x \to 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5}$$

$$\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}$$

$$\lim_{x \to 9} \frac{\frac{1}{\sqrt{x}} - \frac{1}{3}}{x - 9}$$

Special Rules and Trig Identities:

If the Limit of f(x) exists and n is an integer

1. $$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, \quad \text{for all real values of } n$$
2. $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$
3. $$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$$
4. $$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$
5. $$\lim_{\theta \to 0} \cos\theta = 1$$
6. $$\lim_{x \to 0} e^x = 1$$
7. $$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$
8. $$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

Tip: PAY ATTENTION TO THESE IDENTITIES
They become increasingly useful when doing some kinds of integration and complex derivatives. Do your best to memorize this table (or at least print it out to reference)

Limits Toward Infinity

When handling limits where the constraint is trending towards infinity, positive or negative, there are four distinct solutions that you can reach (disregarding problems that numerically cancel to a value - in that case when all instances of x cancel out of the solution, the numerical limit is correct):

1. Infinity
2. Negative Infinity
3. Zero
4. DNE

DNE is shorthand for "The Limit Does Not Exist"

Limits are generally considered not to exist if there is a discontinuity at the asymptote of the limit

There are rare exceptions, like certain Jump Discontinuities, where the limit DOES exist, but they are few and far between.

There are two different classifications of infinite limit problems:

1. Bottom Heavy
   a. These problems feature a larger degree/power on the BOTTOM of the fraction in the function, and trend toward zero when an infinite value is substituted in for x
2. Top Heavy
   a. These problems contain a larger degree/power on the TOP of the fraction in the function, and will trend toward infinitiy when an infinite value is substituted in for x

$$\lim_{x\to\infty} \frac{5x^3 - 2x + 4}{-2x^2 - 5} \rightarrow \frac{5x^3}{-2x^2}$$

BIG NUMBER

$$\lim_{x\to\infty} -\frac{5x}{2} = -\infty$$

Limit Properties

If the Limit of f(x) and g(x) exists and n is an integer

These laws allow you to find the limits of multiple co-dependent functions, as long as they share the same constraint:

$$\lim_{x \to 3} [f(x) + g(x)] = \left[\lim_{x \to 3} x^2 + 2\right] + \left[\lim_{x \to 3} x - 4\right] = 10$$

$$f(x) = x^2 + 2$$
$$g(x) = x - 4$$

$$11 + (-1)$$

Law of Addition: $$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Law of Subtraction: $$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Law of Multiplication: $$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Law of Division: $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{where } \lim_{x \to a} g(x) \neq 0$$

Law of Power: $$\lim_{x \to a} c = c$$

Additional Properties/names

1. Limit of a Constant Function

$$\lim_{x \to a} c = c$$

2. Limit of the Identity Function

$$\lim_{x \to a} x = a$$

3. Limit of a Power Function

$$\lim_{x \to a} x^n = a^n$$

4. Limit of a Root Function

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

5. Constant Multiple Rule

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x) = cK$$

6. Addition and Subtraction Rule

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = K \pm L$$

7. Multiplication Rule

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = KL$$

8. Division Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{K}{L}, \quad \text{provided } L = \lim_{x \to a} g(x) \neq 0$$

9. Limit of a Function Raised to a Power

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n = K^n, \quad \text{where } n \text{ is any real number}$$

10. Special Exponential Limit

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

11. Exponential Rule

$$\lim_{x \to 0} e^x = 1$$

12. Logarithmic Limit

$$\lim_{x \to 0^+} \ln(x) = -\infty, \quad \lim_{x \to \infty} \ln(x) = \infty$$

Note that: $\lim_{x \to a} f(x) = K$ $\lim_{x \to a} g(x) = L$

Additional Properties continued

13. Squeeze Theorem If $h(x) \leq f(x) \leq g(x)$ for all x near a and

$$\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = L \to \lim_{x \to a} f(x) = L$$

14. Infinite Limits with Polynomial Functions If $P(x)$ and $Q(x)$ are polynomials,

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \begin{cases} 0, & \text{if degree of } P(x) < \text{degree of } Q(x) \\ \infty, & \text{if degree of } P(x) > \text{degree of } Q(x) \\ \text{leading coefficient ratio}, & \text{if degrees are equal.} \end{cases}$$

Source : https://tutorial.math.lamar.edu/Classes/CalcI/LimitProofs.aspx

Limit Examples

Proof Time!

Let us prove the Limit of a constant function(1)

1. Limit of a Constant Function

$$\lim_{x \to a} c = c$$

Questions?

Next Week -

We're going to dive into the wonderful world of Derivatives, and see how far that takes us!

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$
nth Derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}f(x)$