The Dot Product

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What is a Dot Product?

Definition Type	Mathematical Expression
Geometric Definition	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$
Algebraic Definition (Cartesian Coordinates)	$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$
Matrix Representation	$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$
Integral Definition (Function Inner Product)	$\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

Properties of the Dot Product

Property	Mathematical Expression
Commutativity	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
Distributivity (over vector addition)	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
Associativity	$(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$
Zero Vector Property	$\mathbf{a} \cdot \mathbf{0} = 0$
Orthogonality	If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are orthogonal
Positivity	$\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$
Squared Length Property	$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

Inequalities

Inequality	Mathematical Statement
Cauchy–Schwarz Inequality	$\lvert\mathbf{a}\cdot\mathbf{b}\rvert \leq \lVert\mathbf{a}\rVert\lVert\mathbf{b}\rVert \iff (\mathbf{a}\cdot\mathbf{b})^2 \leq \lVert\mathbf{a}\rVert^2\lVert\mathbf{b}\rVert^2$
Triangle Inequality	$\lVert\mathbf{a}+\mathbf{b}\rVert \leq \lVert\mathbf{a}\rVert + \lVert\mathbf{b}\rVert$
Reverse Triangle Inequality	$\lVert\mathbf{a}-\mathbf{b}\rVert \geq \lvert\lVert\mathbf{a}\rVert - \lVert\mathbf{b}\rVert\rvert$
Minkowski Inequality	$\lVert\mathbf{a}+\mathbf{b}\rVert_p \leq \lVert\mathbf{a}\rVert_p + \lVert\mathbf{b}\rVert_p$ for $1 \leq p \leq \infty$
Holder's Inequality	$\sum_{i=1}^n \lvert x_i y_i\rvert \leq \left(\sum_{i=1}^n \lvert x_i\rvert^p\right)^{1/p}\left(\sum_{i=1}^n \lvert y_i\rvert^q\right)^{1/q}$, where $\frac{1}{p}+\frac{1}{q}=1$

Unit Vector and Normalization

Vector Normalization
Given a nonzero vector, normalization scales it to have a magnitude of 1.

Formula for Normalization
$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\lVert\mathbf{a}\rVert} = \frac{\mathbf{a}}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}$

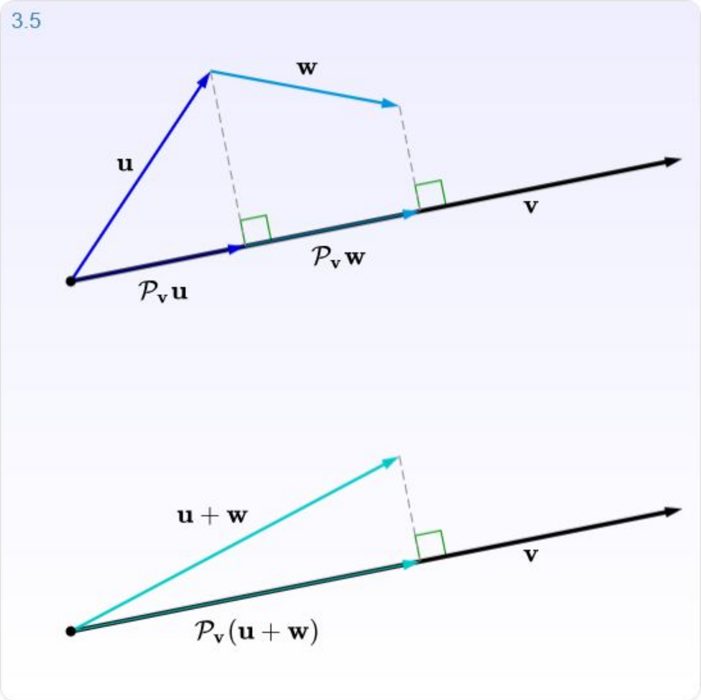
Unit Vector Definition
A unit vector is any vector with a magnitude of 1.

Property of a Unit Vector
$\lVert\hat{\mathbf{a}}\rVert = 1$

Example in 2D
If $\mathbf{a} = (3, 4)$, then $\lVert\mathbf{a}\rVert = \sqrt{3^2 + 4^2} = 5$, so $\hat{\mathbf{a}} = \left(\frac{3}{5}, \frac{4}{5}\right)$.

Example in 3D
If $\mathbf{b} = (1, 2, 2)$, then $\lVert\mathbf{b}\rVert = \sqrt{1^2 + 2^2 + 2^2} = 3$, so $\hat{\mathbf{b}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

Orthogonal Projection



http://immersivemath.com/ila/ch03_dotproduct/ch03.html

For another Interpretation: [link]

Orthogonal Projection
The orthogonal projection of a vector $\mathbf{a}$ onto another vector $\mathbf{b}$ gives the component of $\mathbf{a}$ that lies along $\mathbf{b}$.
Formula for Projection
$\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$
Geometric Interpretation
The projection is the shadow of $\mathbf{a}$ onto $\mathbf{b}$ when a perpendicular is dropped from $\mathbf{a}$ to the line containing $\mathbf{b}$.
Orthogonal Decomposition
Any vector $\mathbf{a}$ can be decomposed as: $\mathbf{a} = \text{proj}_{\mathbf{b}}\mathbf{a} + \mathbf{a}_{\perp}$ where $\mathbf{a}_{\perp}$ is the perpendicular component of $\mathbf{a}$ relative to $\mathbf{b}$.
Perpendicular Component Formula
$\mathbf{a}_{\perp} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}$
Example in 2D
If $\mathbf{a} = (3,4)$ and $\mathbf{b} = (1,2)$, then: $\text{proj}_{\mathbf{b}}\mathbf{a} = \frac{(3,4)\cdot(1,2)}{(1,2)\cdot(1,2)}(1,2)$ $= \frac{3(1)+4(2)}{1^2+2^2}(1,2) = \frac{3+8}{1+4}(1,2) = \frac{11}{5}(1,2)$ $= \left(\frac{11}{5}, \frac{22}{5}\right).$

Orthonormal Basis

Orthonormal Basis
An orthonormal basis for a vector space is a set of mutually orthogonal unit vectors that span the space.
Conditions for an Orthonormal Basis
A set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis if: 1. Orthogonality: $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ 2. Normalization: $\|\mathbf{e}_i\| = 1$ for all i
Example of an Orthonormal Basis in $\mathbb{R}^2$
The standard basis $\{\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1)\}$ is an orthonormal basis of $\mathbb{R}^2$.
Example of an Orthonormal Basis in $\mathbb{R}^3$
The standard basis $\{\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)\}$ forms an orthonormal basis of $\mathbb{R}^3$.
Gram-Schmidt Process
The Gram-Schmidt process converts a set of linearly independent vectors into an orthonormal basis by iteratively orthogonalizing and normalizing them.

A parameterized Line

Parameterized Line
A straight line can be described by a starting point, $\mathbf{p}$, and a direction vector, $\mathbf{v}$.
The parametric equation of the line is:
$\mathbf{r}(t) = \mathbf{p} + t\mathbf{v}$ where the conditions are: $\begin{cases} t \text{ is a scalar parameter,} \\ \mathbf{p} \text{ is a point on the line,} \\ \mathbf{v} \text{ is the direction vector.} \end{cases}$

Explicit Form
A line in explicit form means that all points on the line can be generated directly from the parametric equations by substituting different values of t.
This form helps in calculating or plotting any point on the line using the starting point $\mathbf{p}$ and the direction vector $\mathbf{v}$.

A parameterized Line Examples

Parametric Form in 2D

In $\mathbb{R}^2$, a line passing through (x_1, y_1) with direction vector (v_x, v_y) is described by:

$$x(t) = x_1 + tv_x \text{ and } y(t) = y_1 + tv_y,$$

where t is a scalar parameter.

Parametric Form in 3D

In $\mathbb{R}^3$, a line passing through (x_1, y_1, z_1) with direction vector (v_x, v_y, v_z) is given by:

$$x(t) = x_1 + tv_x,\ y(t) = y_1 + tv_y, \text{ and } z(t) = z_1 + tv_z.$$

Example in 2D

For the line passing through $(2, 3)$ with direction vector $(1, 4)$, the parametric equations are:

$$x(t) = 2 + t \text{ and } y(t) = 3 + 4t.$$

Example in 3D

For the line passing through $(1, 2, 3)$ with direction vector $(4, -1, 2)$, the parametric equations are:

$$x(t) = 1 + 4t,\ y(t) = 2 - t, \text{ and } z(t) = 3 + 2t.$$

Let’s do some practice, once more!

https://images.pexels.com/photos/796603/pexels-photo-796603.jpeg?cs=srgb&dl=hand-desk-notebook-796603.jpg&fm=jpg

Questions?

Next Week -

The Vector Product

Reinforcement Learning

`Goal: Use normalization to have an object move consistently on screen along diagonals`

`Goal: Use OpenMP to parallelize a Dot Product`