Integrals

John Hohman and Nathan James

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What is an Integral?

Given a function $f(x)$ that is continuous on the interval $[a, b]$ we divide the interval into n subintervals of equal width, Δx, and from each interval choose a point, x_i^*. Then the **definite integral of $f(x)$ from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Simple Explanations

FUNDAMENTAL THEOREM OF CALCULUS PART ONE AND TWO

$$I.\ \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$II.\ \int_a^b f(x)dx = F(b) - F(a)$$

1. Integrals and derivatives are invariably linked, and are in fact inverses

2. The area under a curve can be approximated between two points using the integral of their derivative function

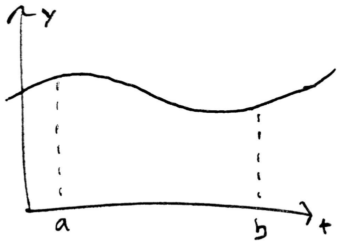
It is important to mention that F(x) is generally notated as the ANTIderivative of function f(x).

Good Reading: https://mathscholar.org/2019/02/simple-proofs-the-fundamental-theorem-of-calculus/

The Fundamental Theorem of Calculus - Part 1.

The first real step is defining our parameters.

f is continuous on $[a,b]$



$$F(x) = \int_a^x f(t)\,dt$$

where $a \leq x \leq b$

Here we're stating that F(x) is the area under the curve between a and x, as long as x is located between a and b where the function is defined as continuous.

We don't need to be specific about our function f(t), because this is true for ANY function as long as it is continuous between a and b

Now, knowing what we do about derivatives, we are able to postulate:

We know that

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

Therefore;

Where h = delta x, since h represents the next incremental step away from x in the derivative definition

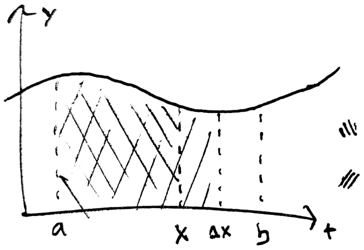
$$F'(x) = \lim_{\Delta x \to 0} \left(\frac{F(x+\Delta x) - F(x)}{\Delta x} \right)$$

or

$$F'(x) = \lim_{\Delta x \to 0} \left(\frac{\int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt}{\Delta x} \right)$$

Substitution with known formulae

f is continuous on $[a,b]$



$\backslash\backslash\backslash = \int_a^x f(t)dt$

$/// = \int_a^{x+\Delta x} f(t)dt$

So, using our understanding of the previous graph, we can show the following:

$$F'(x) = \lim_{\Delta x \to 0} \left(\frac{1}{\Delta x} \int_{x}^{x+\Delta x} f(t)\,dt \right)$$

Which allows us to use the Mean Value Theorem of Definite Integrals (M.V.T.) to SIGNIFICANTLY simplify this whole mess

M.V.T.: (there exists a c where $x \le c \le x+\Delta x$)

so, $f(c)\,\Delta x = \int_{x}^{x+\Delta x} f(t)\,dt$

$$f(c) = \frac{1}{\Delta x} \int_{x}^{x+\Delta x} f(t)\,dt$$

And now, we can express:

there exists a c in $[x, x+\Delta x]$ where

$$F'(x) = \lim_{\Delta x \to 0} f(c)$$

After which, we apply the squeeze theorem to further define f(c)

c as a function of Δx

$(x \le c \le x+\Delta x) = (x \le c(\Delta x) \le x+\Delta x)$

so we take limits: $\lim_{\Delta x \to 0} x = x$, $\lim_{\Delta x \to 0} x+\Delta x = x$, $\lim_{\Delta x \to 0} c(\Delta x) = x$

Allowing us to say:

$c \to x$ as $\Delta x \to 0$,

$f(c) \to f(x)$ as $\Delta x \to 0$,

Which brings us to, finally:

We Bring Back the First Idea

$$I.\ \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Wow!

Defining f(c) as approaching f(x) as delta x approaches 0, allows us to finally express:

$$F'(x) = \lim_{\Delta x \to 0} f(c) = f(x)$$

or

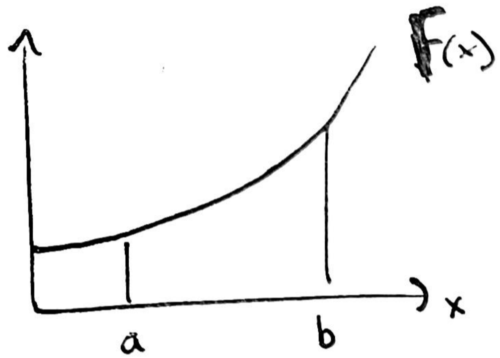
$$F'(x) = f(x) \quad \text{or} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

since $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt$

The Fundamental Theorem of Calculus Part 2.

$$II. \int_a^b f(x)dx = F(b) - F(a)$$

This is the part that links Integrals and the area under a curve USING Riemann sums.



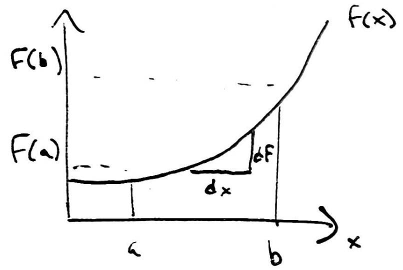
$F(x)$ position as a function of x

$F(b) - F(a) =$ change in position btw a & b

We now have our boundaries with which to compare to the derivative of this function

All we're trying to do is prove that the area under the curve can be expressed using a definite integral

We’re trying to approximate the area using small rectangular areas equal to dF/dx - inside the range [a,b]

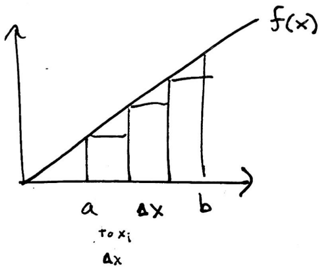


$$\frac{dF}{dx} = F'(x) = f(x)$$

$$II. \int_a^b f(x)dx = F(b) - F(a)$$

Look familiar?!!?!

So, let's graph f(x), assuming F(x) is a parabolic function.



"exact" change in position btw a & b

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i-1}) \Delta x = \int_a^b f(x)\,dx$$

where

$$\Delta x = \frac{b-a}{n}$$

All we’re doing here is trying to find the area under the derivative function f(x) using Riemann sums, and then equating that to a definite Integral using the definition.

One simple substitution later...

And there it is - Part II. MUCH simpler than the first, but they don't work without each other.

$$II. \int_a^b f(x)dx = F(b) - F(a)$$

$$\text{so, } \int_a^b f(x)\,dx = F(b) - F(a)$$

Both parts are defined as the "change in position between a and b" - and so can be determined to be equivalent

Any questions about the proofs?

To recap,

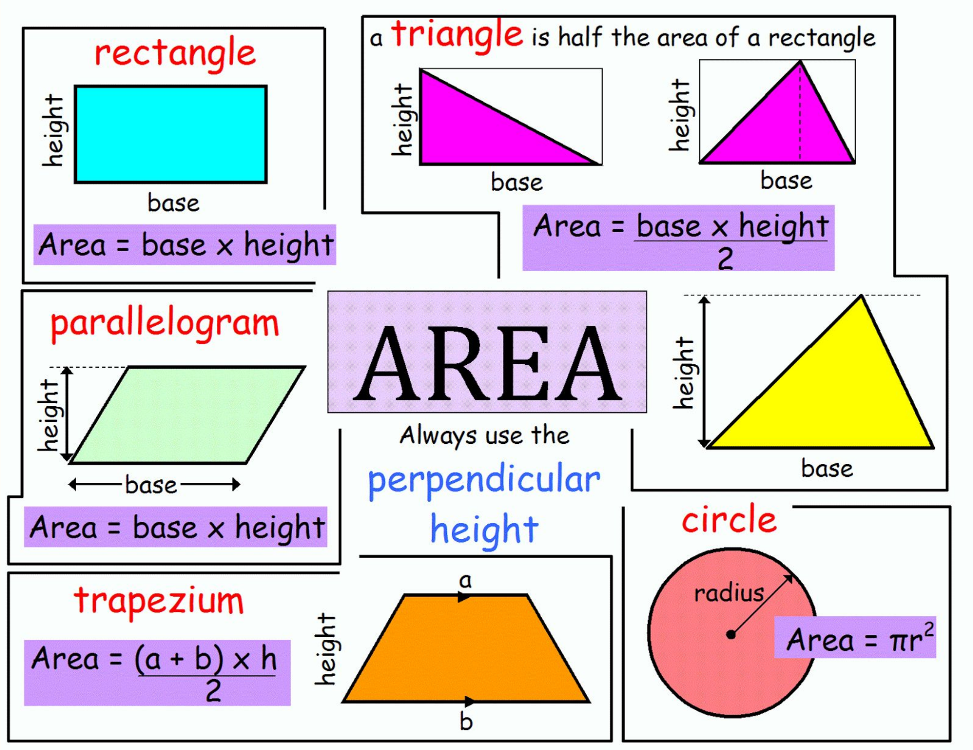
1. Both proofs rely on underlying knowledge of derivative and integral properties, but do not require more than advanced algebra to prove.
2. Both proofs reciprocate, requiring each other to be true in order to function. Antiderivative understanding from Part 1 allows for Part 2; Area approximation from Part 2 allows for Part 1

Notation:

Integral Notation	Description	Use
$\int f(x)\,dx$	Indefinite integral: antiderivative of $f(x)$	Finding general solutions of differential equations
$\int_a^b f(x)\,dx$	Definite integral: area under the curve from $x=a$ to $x=b$	Calculating total distance, probability, or accumulated quantities
$\iint_R f(x,y)\,dx\,dy$	Double integral over a two-dimensional region R	Computing areas, surface mass, or flux in 2D regions
$\iiint_V f(x,y,z)\,dx\,dy\,dz$	Triple integral over a three-dimensional volume V	Finding volumes or mass in 3D spaces
$\int_C f(x,y)\,ds$	Path integral along a curve C	Calculating arc lengths or mass distribution along curves
$\int_C \mathbf{F}\cdot d\mathbf{r}$	Line integral of a vector field along a curve C	Evaluating work done by a force field along a path
$\iint_S f(x,y,z)\,dS$	Surface integral over a surface S	Finding the total mass or flux across a surface
$\iint_S \mathbf{F}\cdot d\mathbf{S}$	Surface integral of a vector field over a surface S	Calculating fluid flux or electromagnetic field effects
$\iiint_V f(x,y,z)\,dV$	Volume integral over a volume V	Determining total mass or charge within a volume
$\int_R f(r,\theta) r\,dr\,d\theta$	Polar coordinates integral	Calculating areas in polar coordinate systems
$\iiint_V f(r,\theta,z) r\,dr\,d\theta\,dz$	Cylindrical coordinates integral	Evaluating volumes of cylindrical objects
$\iiint_V f(\rho,\theta,\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$	Spherical coordinates integral	Computing volumes in spherical regions
$\oint_C f(z)\,dz$	Contour integral in the complex plane	Analyzing complex functions and residue calculations
$\int_X f\,d\mu$	Lebesgue integral with respect to a measure μ	Handling more general forms of integration (probability theory)
$\int_a^b f(x)\,dg(x)$	Stieltjes integral	Weighted integration for applications in probability and finance
$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt$	Fractional integral (Riemann–Liouville)	Fractional calculus in signal processing or dynamic systems

Integration Techniques

1. Indefinite Integration
2. Definite Integration
3. u-substitution
4. Integration by Parts
5. Partial Fraction Decomposition



Source: http://samaminath.blogspot.com/2015/09/area-formula-for-grade-6-7-students.html

Basic Integration

Indefinite Integrals	Definite Integrals
$\int f(x)dx = F(x) + C$	$\int_a^b f(x)dx = F(b) - F(a)$
Example	**Example**
$\int x^2 dx = \frac{1}{3}x^3 + C$	$\int_1^2 x^2 dx = \frac{1}{3}2^3 - \frac{1}{3}1^3 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Advance techniques

Integration by Substitution

Also known as u-substitution

Step	Explanation
Theoretical Framework	The substitution method, also known as u-substitution, is used when the integral contains a composite function. We substitute $u = g(x)$ to transform the integral $\int f(g(x))g'(x)\,dx$ into $\int f(u)du$.
Given Integral	$\int x\cos(x^2)dx$
Step 1: Choose u	Let $u = x^2$ because its derivative, $du/dx = 2x$, appears in the integral.
Step 2: Compute du	Differentiate $u = x^2$: $du = 2xdx$, which implies $xdx = \frac{du}{2}$.
Step 3: Substitute and Simplify	Substitute u and rewrite the integral: $\int x\cos(x^2)dx = \int \cos(u)\frac{du}{2} = \frac{1}{2}\int \cos(u)du$
Step 4: Integrate	$\frac{1}{2}\int \cos(u)du = \frac{1}{2}\sin(u) + C$
Step 5: Substitute u back	Replace u with x^2: $\frac{1}{2}\sin(x^2) + C$
Final Answer	$\int x\cos(x^2)dx = \frac{1}{2}\sin(x^2) + C$

Advance techniques cont.

Integration by parts:

Step	Explanation
Theoretical Framework	The integration by parts formula is derived from the product rule of differentiation: $\int u\,dv = uv - \int v\,du$. The goal is to select u and dv such that the resulting integral is easier to solve.
Given Integral	$\int xe^x\,dx$
Step 1: Choose u and dv	Select $u = x$ and $dv = e^x\,dx$ based on the rule: choose u to be a function that simplifies when differentiated and dv to be easy to integrate.
Step 2: Compute du and v	Differentiate $u = x$: $du = dx$. Integrate $dv = e^x\,dx$: $v = e^x$.
Step 3: Apply the Formula	Using the formula $\int u\,dv = uv - \int v\,du$, substitute the values: $$\int xe^x\,dx = xe^x - \int e^x\,dx.$$
Step 4: Integrate the Remaining Integral	$\int e^x\,dx = e^x$. Thus, we now have: $$xe^x - e^x + C.$$
Final Answer	$\int xe^x\,dx = e^x(x-1) + C$

Advance techniques cont.

Partial Fraction Decomposition:

Step	Explanation
Theoretical Framework	Partial fraction decomposition is used to express a rational function as the sum of simpler fractions. The idea is to break down the fraction $\frac{P(x)}{Q(x)}$ into simpler fractions based on the factorization of $Q(x)$. The integrals of these simpler fractions are easier to handle.
Given Integral	$\int \frac{2x+3}{x^2-x-6}\,dx$
Step 1: Factor the Denominator	Factor the quadratic expression in the denominator: $$x^2 - x - 6 = (x-3)(x+2).$$
Step 2: Set Up the Partial Fraction Decomposition	Express the rational function as a sum of partial fractions: $$\frac{2x+3}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}.$$ Here, A and B are constants to be determined.
Step 3: Multiply by the Denominator	Multiply both sides by the denominator $(x-3)(x+2)$ to eliminate the denominators: $$2x + 3 = A(x+2) + B(x-3).$$
Step 4: Solve for A and B	Expand and collect like terms: $$2x + 3 = A(x+2) + B(x-3) = Ax + 2A + Bx - 3B.$$ Now, group terms in x: $$2x + 3 = (A+B)x + (2A - 3B).$$ Equate the coefficients of like powers of x: - For x: $A + B = 2$, - For the constant: $2A - 3B = 3$. Solve this system of equations to get $A = \frac{15}{5} = 3$ and $B = \frac{-1}{5}$.
Step 5: Rewrite the Integral	Substitute A and B into the partial fractions: $$\frac{2x+3}{(x-3)(x+2)} = \frac{3}{x-3} - \frac{1}{x+2}.$$ Now the integral becomes: $$\int \left(\frac{3}{x-3} - \frac{1}{x+2} \right) dx.$$
Step 6: Integrate	Integrate each term: $$\int \frac{3}{x-3}\,dx = 3\ln\lvert x-3\rvert + C_1, \quad \int \frac{-1}{x+2}\,dx = -\ln\lvert x+2\rvert + C_2.$$
Final Answer	Combine the integrals: $$\int \frac{2x+3}{x^2-x-6}\,dx = 3\ln\lvert x-3\rvert - \ln\lvert x+2\rvert + C.$$

Integral Rules

$\int a dx = ax + C$	$\int x dx = \frac{x^2}{2} + C$
$\int x^2 dx = \frac{x^3}{3} + C$	$\int x^p dx = \frac{x^{p+1}}{p+1} + C$
$\int \frac{dx}{x} = \ln\lvert x\rvert + C$	$\int e^x dx = e^x + C$
$\int b^x dx = \frac{b^x}{\ln b} + C$	$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$	$\int \tan x dx = -\ln\lvert\cos x\rvert + C$
$\int \cot x dx = \ln\lvert\sin x\rvert + C$	$\int \sec x dx = \ln\left\lvert\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\rvert + C$ $= \ln\lvert\sec x + \tan x\rvert + C$
$\int \csc x dx = \ln\left\lvert\tan\frac{x}{2}\right\rvert + C$ $= -\ln\lvert\csc x + \cot x\rvert + C$	$\int \sec^2 x dx = \tan x + C$
$\int \csc^2 x dx = -\cot x + C$	$\int \sec x \tan x dx = \sec x + C$
$\int \csc x \cot x dx = -\csc x + C$	$\int \frac{dx}{1+x^2} = \arctan x + C$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\frac{x}{a} + C$	$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left\lvert\frac{1+x}{1-x}\right\rvert + C$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert + C$	$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$	$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left\lvert x + \sqrt{x^2 \pm a^2}\right\rvert + C$
$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}\lvert x\rvert + C$	$\int \sinh x dx = \cosh x + C$

Source: https://trigidentities.net/integration-formula/

This website has a highly comprehensive list of integrals identities

Specialized Techniques

1. Improper Integration
2. Reduction Formula
3. Leibniz's rule

https://www.reddit.com/r/notinteresting/comments/pj411k/this_square_peg_wont_fit_in_this_circular_hole/

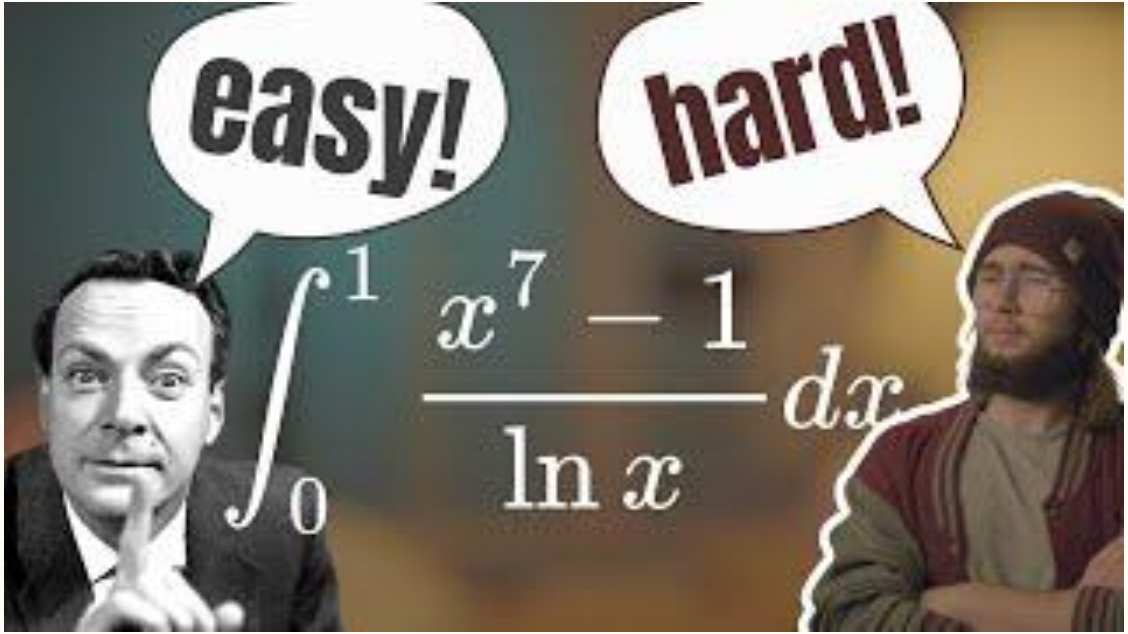
Improper Integration

Step	Explanation
Theoretical Framework	Improper integrals occur when: - The limits of integration are infinite, or - The integrand has an infinite discontinuity within the integration limits. To evaluate improper integrals, we take the limit of the integral as the boundary approaches infinity (or the point of discontinuity).
Given Integral	$\int_1^\infty \frac{1}{x^2}\,dx$
Step 1: Recognize the Type	This is an improper integral because the upper limit is infinite. We need to evaluate the limit of the integral as the upper bound approaches infinity.
Step 2: Set up the Limit	Rewrite the integral as a limit: $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx.$$
Step 3: Compute the Integral	The integral of $\frac{1}{x^2}$ is: $$\int \frac{1}{x^2}\,dx = -\frac{1}{x}.$$ So, we now compute: $$\int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = -\frac{1}{b} + \frac{1}{1} = 1 - \frac{1}{b}.$$
Step 4: Take the Limit	Now, take the limit as $b \to \infty$: $$\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1 - 0 = 1.$$
Final Answer	The value of the improper integral is: $$\int_1^\infty \frac{1}{x^2}\,dx = 1.$$

Reduction Formula

Step	Explanation
Theoretical Framework	A reduction formula is a recursive method to simplify the evaluation of integrals, especially for higher powers of functions. By expressing the integral in terms of simpler integrals, we can reduce the power of the function step by step. For example, consider the integral $\int x^n\,dx$.
Given Integral	$\int x^n\,dx$ where $n \neq -1$.
Step 1: Use Integration by Parts	We can apply the integration by parts formula: $$\int u\,dv = uv - \int v\,du.$$ Let $u = x^n$ and $dv = dx$. Thus, $du = nx^{n-1}\,dx$ and $v = x$.
Step 2: Apply the Formula	Applying the integration by parts formula: $$\int x^n\,dx = x^n \cdot x - \int x \cdot nx^{n-1}\,dx$$ Simplifying this gives: $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$
Step 3: Derive the Recursive Formula	To reduce the power, use the recursive step. For any n: $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$ For higher powers, continue applying the formula until the power becomes simpler or manageable. For example, applying to $\int x^{n-1}\,dx$, we would get the next step of the formula recursively.
Final Answer	The reduction formula for $\int x^n\,dx$ is: $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for} \quad n \neq -1.$$

Integration of Parametric Functions (Leibniz's rule)



Video:
https://www.youtube.com/watch?v=hfLFO-X6Cug
Reading:
https://zackyzz.github.io/feynman.html#:~:text=Feynman%27s%20trick%20aims%20to%20get%20rid%20of%20this,obtain%20an%20integral%20that%20is%20easier%20to%20evaluate.

Questions?

Next Week -

Series or Review you vote

Reinforcement Learning

`Goal: A variety of smaller problems posted to the github/discord`

`Push your work as [Your_name]_project_4.[file_extention]`