Derivatives

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What is a Derivative?

A derivative is an instantaneous rate of change relative to the infinitesimal values around a point.

$$\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]=\lim_{dx\to 0}\left[\frac{f(x+dx)-f(x)}{dx}\right]=\frac{dy}{dx}=\frac{\Delta y}{\Delta x}=f'(x)=Df(x)=D_x f$$

Pierre de Fermat

Derivative Rules

Power Rule: $$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \text{where } n \in \mathbb{R}$$

Constant Rule: $$\frac{d}{dx}[c] = 0, \quad \text{where } c \text{ is a constant}$$

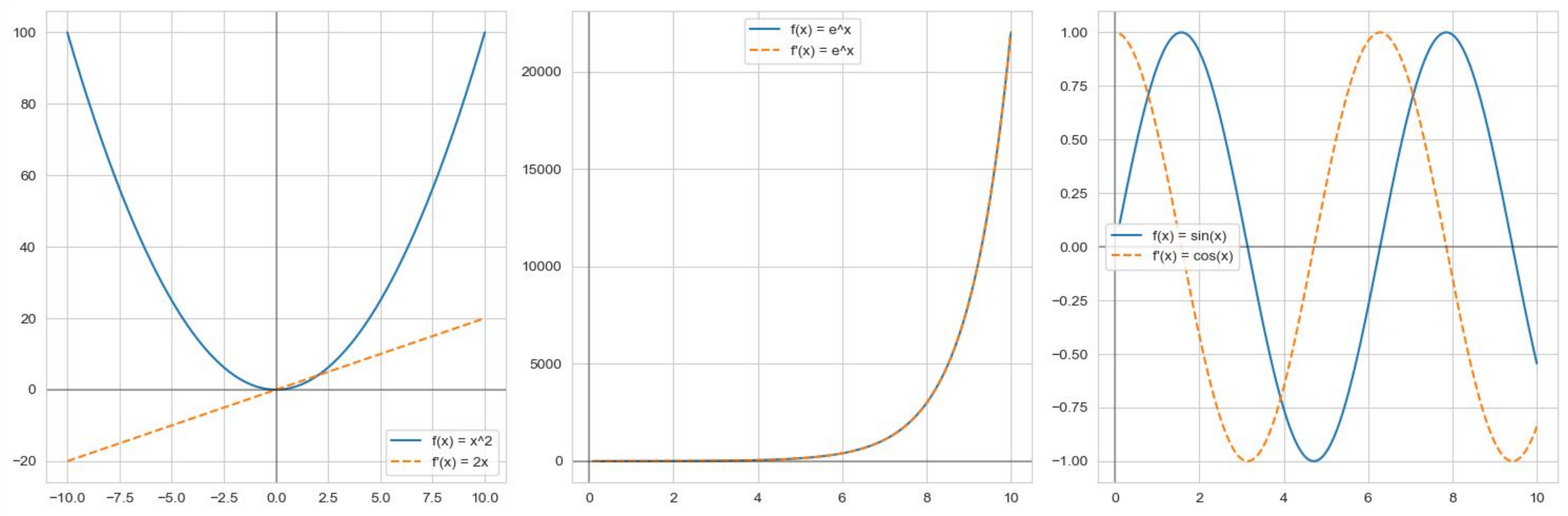
Sum Rule: $$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Product Rule: $$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad g(x) \neq 0$$

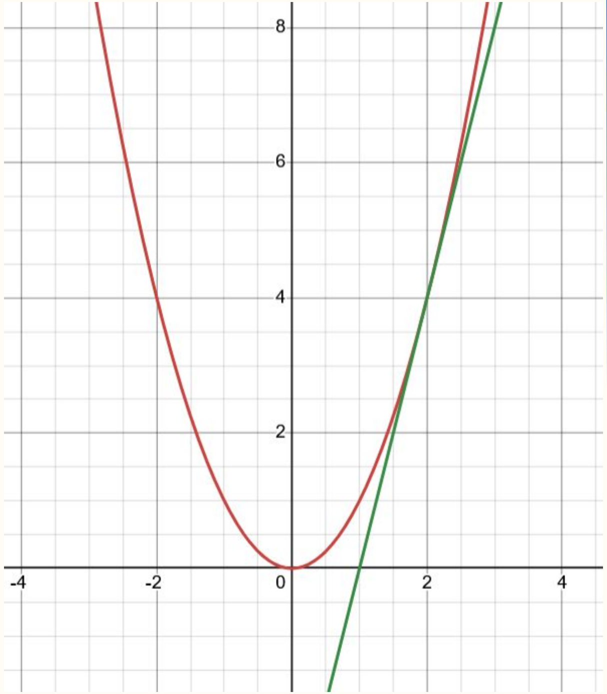
Chain Rule: $$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Derivatives

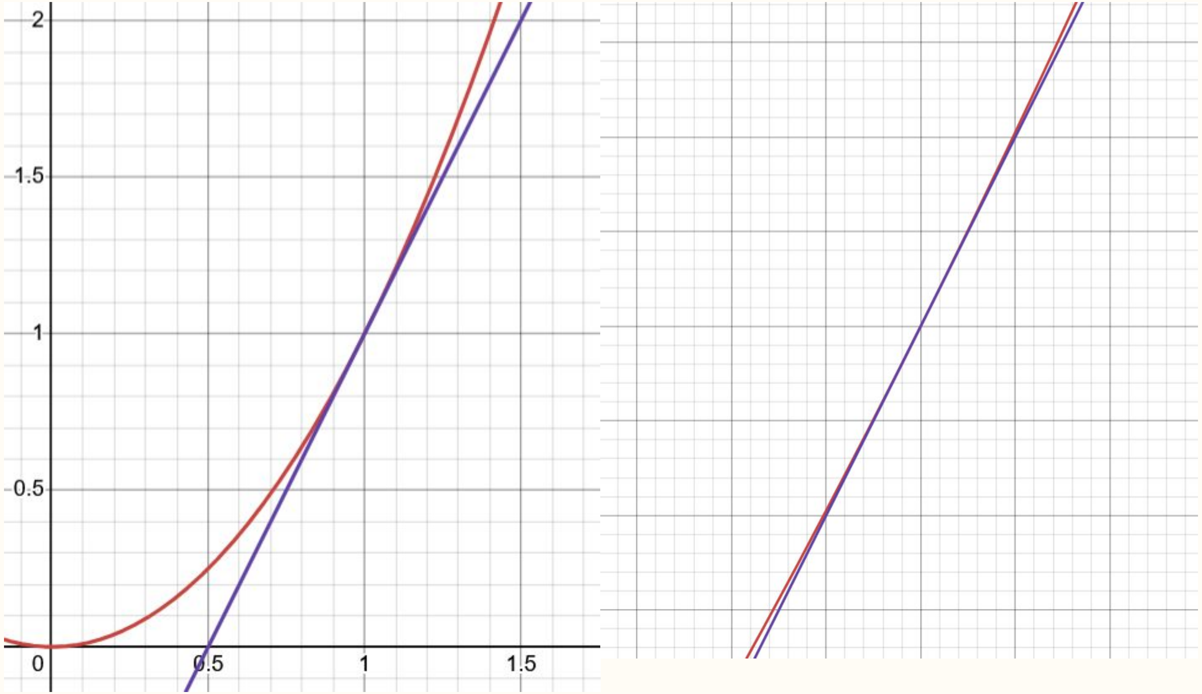


Rates Displayed

4x-4



2x-1



Numerical Methods

Euler's Method

h is the step size, for our purposes use h = 1/(num_time_points)
t is the index of y_n value

Aspect	Forward Euler	Backward Euler	Trapezoid Method
Type	Explicit (direct computation)	Implicit (requires solving equations)	Implicit
Stability	Less stable for stiff problems	More stable, suitable for stiff ODEs	More stable, suitable for stiff ODEs
Accuracy	First-order accurate	First-order accurate	Second-order accurate
Ease of Implementation	Simple	More computationally demanding	Requires solving equations, but more accurate
Formula	$y_{n+1} = y_n + h \cdot f(t_n, y_n)$	$y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1})$	$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big)$

I will cover Runge-Kutta Methods at a later date, during stability region discussions

Special Cases

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[a^x] = a^x \ln a$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\log_a x] = \frac{1}{x \ln a}$$

$$\frac{d}{dx}[|x|] = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\sinh x] = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \operatorname{sech}^2 x$$

$$\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}[\operatorname{arsinh} x] = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}[\operatorname{arcosh} x] = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{artanh} x] = \frac{1}{1-x^2}$$

$$\frac{d}{dx}[\operatorname{arsech} x] = -\frac{1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\operatorname{arcsch} x] = -\frac{1}{|x|\sqrt{1+x^2}}$$

$$\frac{d}{dx}[\operatorname{arcoth} x] = \frac{1}{1-x^2}$$

L'Hôpital's rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad \text{provided } \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ exists.}$$

Conditions for L'Hôpital's Rule:

- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ (indeterminate form $\frac{0}{0}$) or $\pm\infty$.
- $g'(x) \neq 0$ near a.
- $f'(x)$ and $g'(x)$ are continuous near a.

Derivative Examples

1.

$$\frac{d}{dx} 2^x$$

2.

$$\frac{d}{dx} \log(42^3)$$

3.

$$\frac{d}{dx} \ln(\sin(x)) + 5$$

4.

$$\lim_{x \to 1} \frac{\sin(\pi x)}{1 - x^2}$$

5.

$$\frac{d}{dx} \frac{\sin(\pi x)}{1 - x^2}$$

6.

$$\frac{d}{dx} 2x \tan(x)$$

7.

$$\frac{d}{dx} \sec(x^2 - e^x)$$

8.

$$\lim_{x \to 1} \frac{1}{1 + x^2}$$

Proof Time!

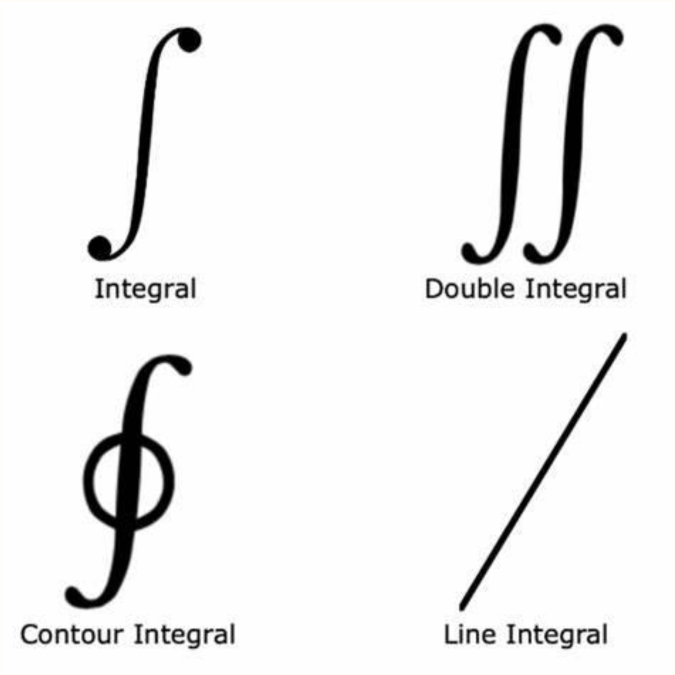
Let us prove the chain rule!

https://math.stackexchange.com/questions/1132510/chain-rule-proof-doubt/1132523#1132523

Questions?

Next Week -

We're going to find the area of Integrals, and sum up new knowledge!



Code Project -

```
Goal 1: Implement the Trapezoid Method for solving ordinary differential equations
Goal 2: Approximate the true functions
Data is provided in the github under topic/lesson_2/projects/
Push your code as [Your_name]_project_2.[file_extention]
```