Series

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Table of Contents

What is a Series?

A series is an infinite sum of terms of a sequence. Given a sequence $\{a_n\}$, the corresponding series is:

$$S = \sum_{n=1}^{\infty} a_n. \tag{1}$$

We classify series as either convergent or divergent based on their behavior as the number of terms increases.

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

←STOP: Think about why this expression is important

Convergent Series

A series $\sum a_n$ is said to be **convergent** if the sequence of its partial sums:

$$S_N = \sum_{n=1}^{N} a_n \tag{2}$$

converges to a finite limit S as $N \to \infty$. That is,

$$\lim_{N\to\infty} S_N = S < \infty. \tag{3}$$

Example: A telescoping series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \tag{4}$$

converges to a finite value as all intermediate terms cancel out.

Divergent Series

A series $\sum a_n$ is **divergent** if the sequence of partial sums S_N does not converge to a finite limit. This may happen in several ways:

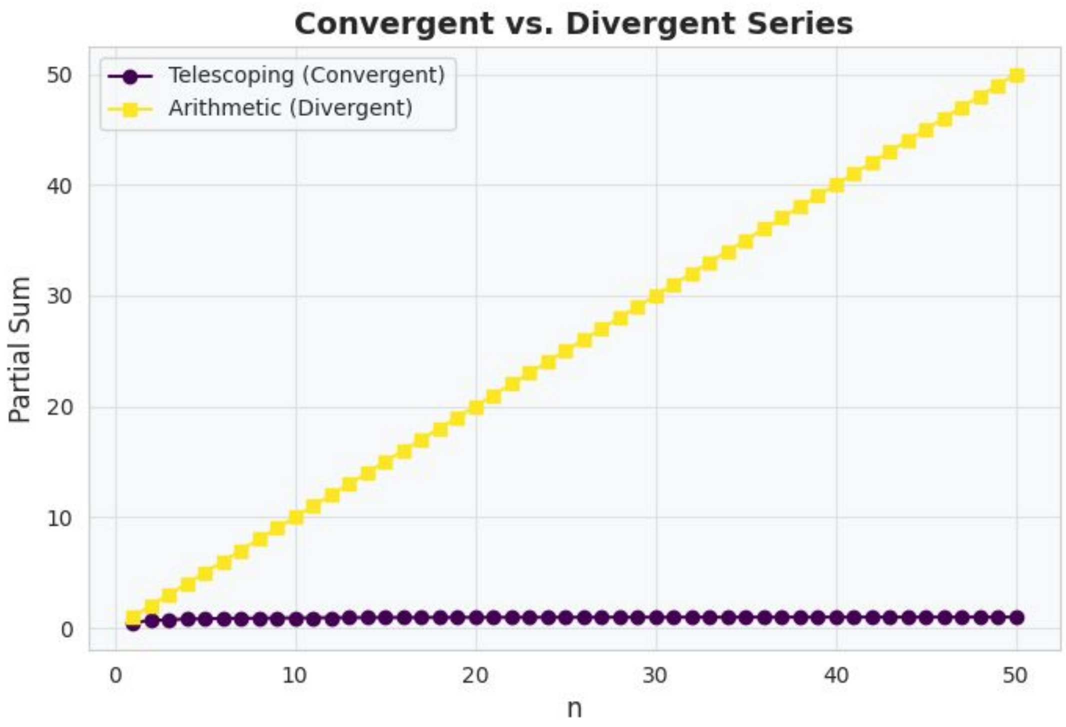
- The partial sums tend to $\pm\infty$.
- The partial sums oscillate without approaching a single value.

Example: An arithmetic series

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \tag{5}$$

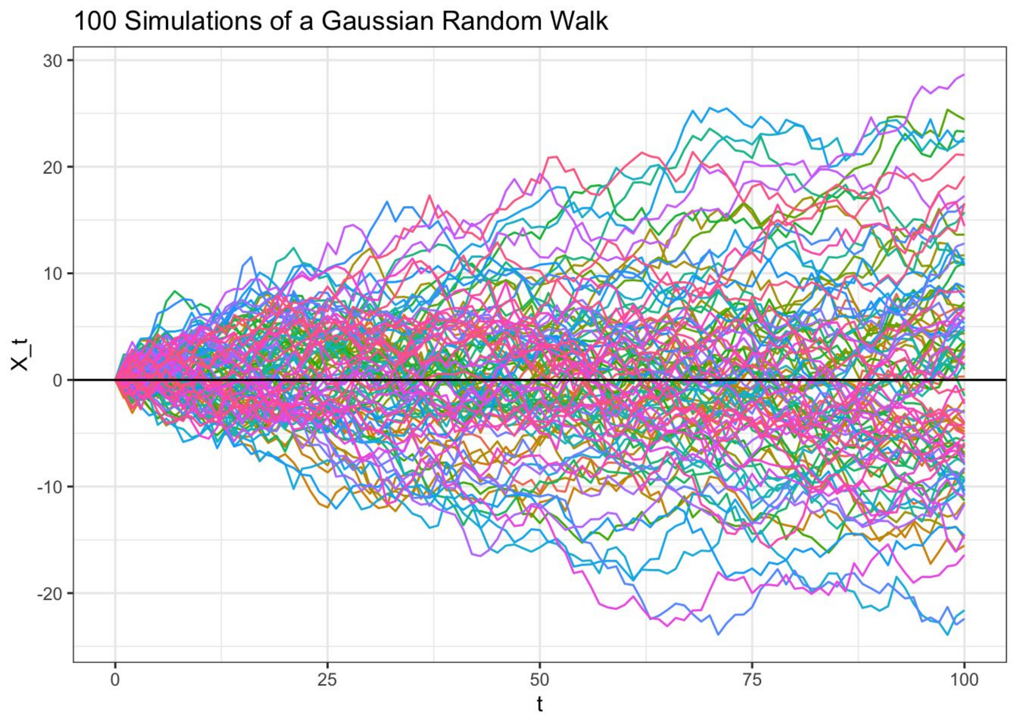
is divergent if a_i is a constant nonzero value, as the sum grows indefinitely.

Example:


Convergent vs. Divergent Series
Telescoping (Convergent)
Arithmetic (Divergent)
Partial Sum
n
0
10
20
30
40
50
0
10
20
30
40
50

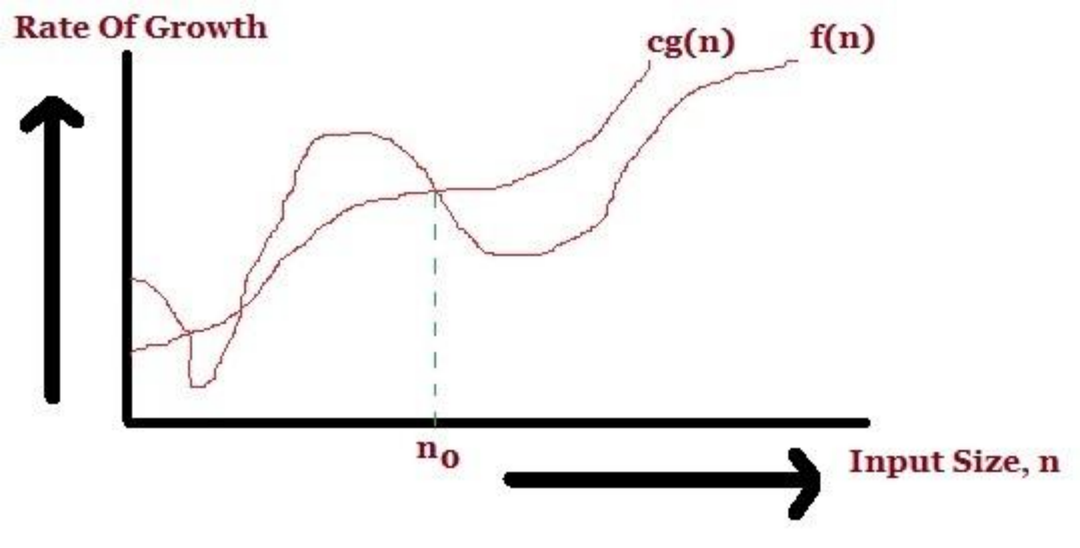
The Importance

1. Computer Science
2. Numerical analysis
3. Probability
4. Physics



https://bookdown.org/probability/bookdown-demo/random-walks.html

Importance in Algorithms

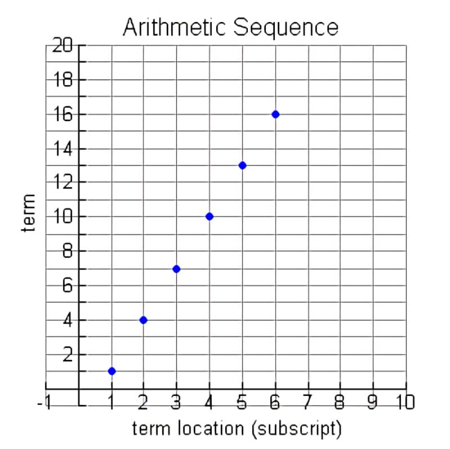


https://javatbrains.blogspot.com/2017/06/big-o-omega-and-theta-notations-in.html

Arithmetic Series

A sum of sequence where each number is equidistant from the others

Example: 1 + 2 + 3 + 4 + 5 + ... + n-2 + n-1 + n



https://mathbitsnotebook.com/Algebra2/Sequences/SSArithmetic.html

Zeno's Paradox of Movement

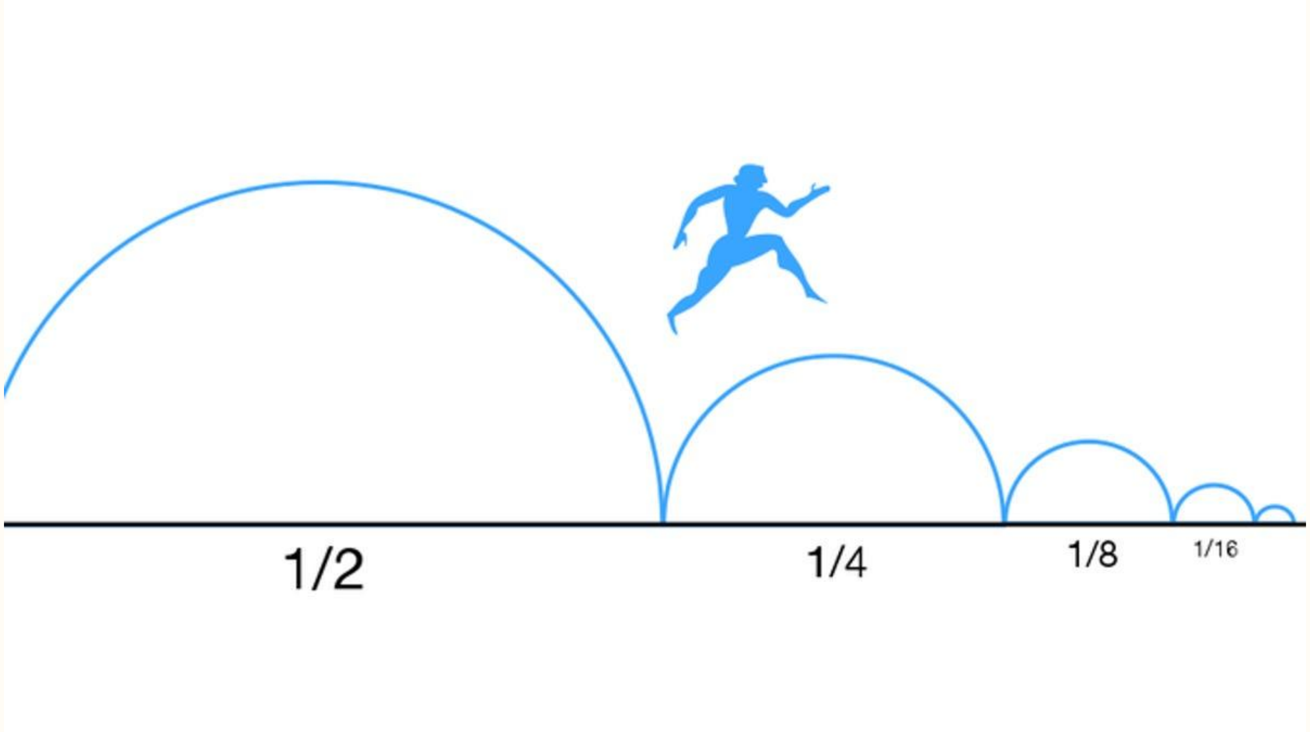


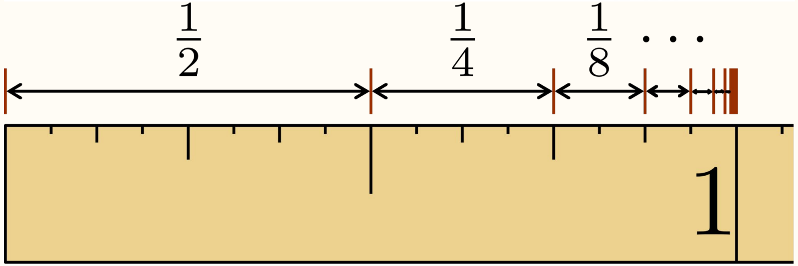
Image: Martin Grandjean

https://qz.com/emails/quartz-obsession/1804821/zenos-paradoxes

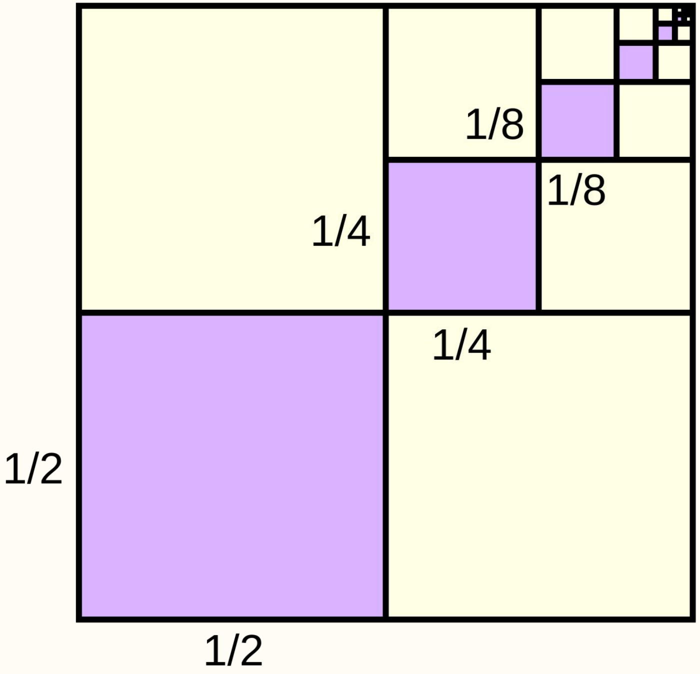
Geometric series

A series in which the ratio of consecutive terms is constant

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k.$$



https://en.wikipedia.org/wiki/1/2_%2B_1/4_%2B_1/8_%2B_1/16_%2B_%E2%8B%AF#/media/File:Geometric_Segment.svg



https://en.wikipedia.org/wiki/Geometric_series

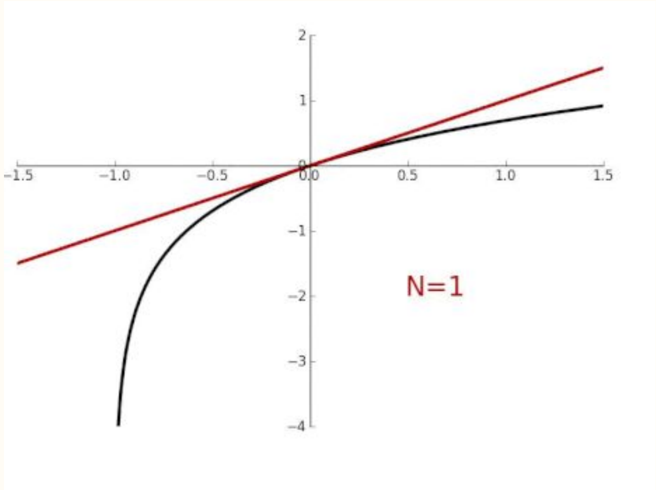
Uses of the Geometric Series

- Recurrence relations in divide and conquer algorithms
- Transformations and rendering techniques
- Signal processing and digital filters

Taylor Series

An infinitely differentiable series expansion of a function

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$



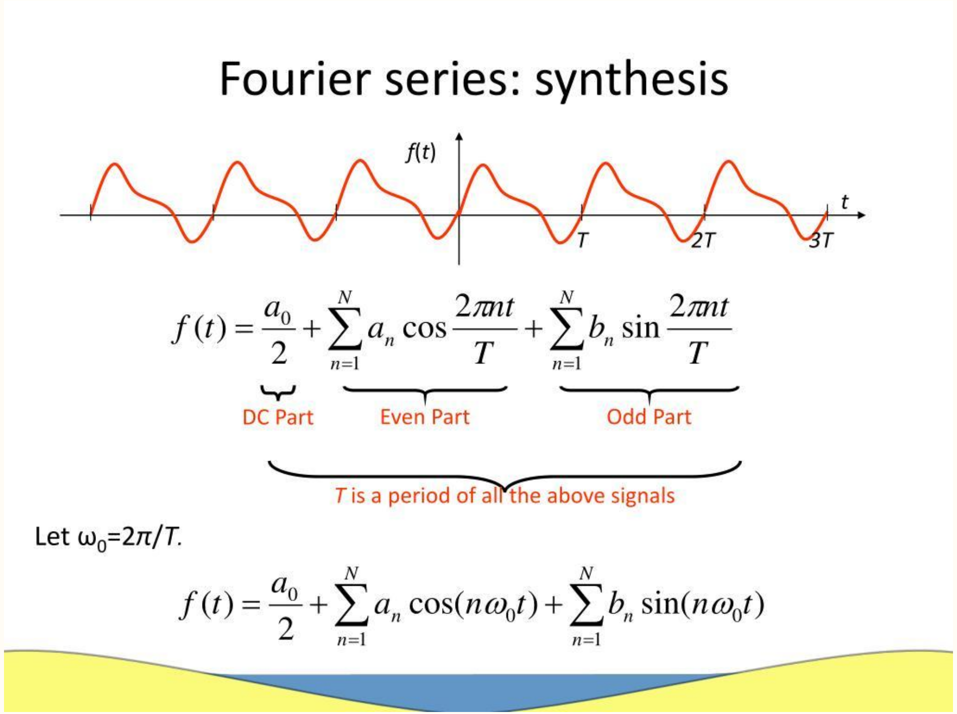
https://en.wikipedia.org/wiki/Taylor_series

Uses of the Taylor Series

- Approximating functions (e.g., transcendental functions like)
- Error analysis and truncation
- Computational efficiency in machine learning and optimization

Fourier Series

- A convergent series of trig functions
- Therefore a series of series



https://www.slideserve.com/lara-randolph/fourier-series-discrete-time-fourier-transform-and-characteristic-functions

Zeno's Paradox Proof

Let's try together!

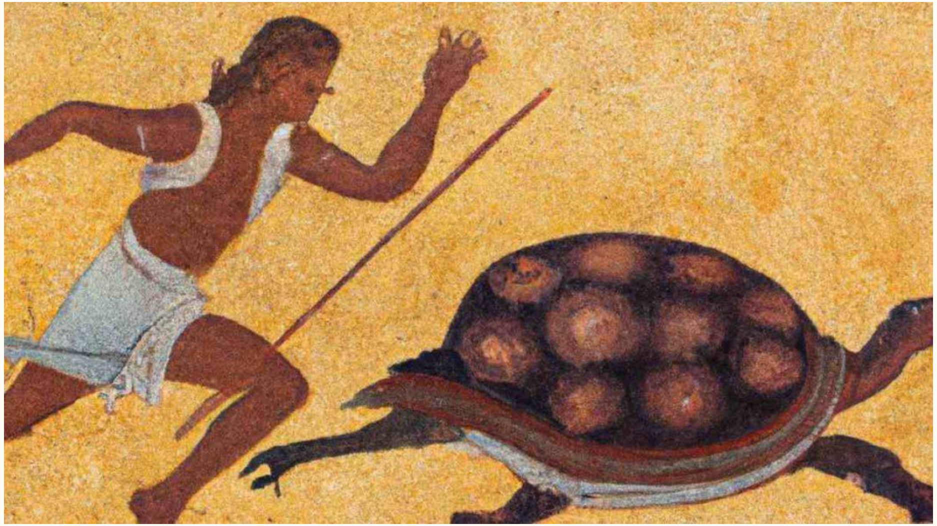


Photo Credit: Twitter/raphaelmilliere

Questions?

Next Week -

More Review or Linear Algebra Begin you vote

Good Reading:

https://math.libretexts.org/Bookshelves/Precalculus/Precalculus_1e_(Open Stax)/11%3A_Sequences_Probability_and_Counting_Theory

Reinforcement Learning

`Goal: Extend project 3 with Monte Carlo Error Reduction with Series Expansions`

`Collaborate on the following file project_6.cpp`