Rank pt. 1

John Hohman

Table of Contents

Outer Product

Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Then the outer product is:

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \\ u_3 v_1 & u_3 v_2 \end{bmatrix} = A$$

The outer product captures pairwise multiplicative interactions

Kronecker Product

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

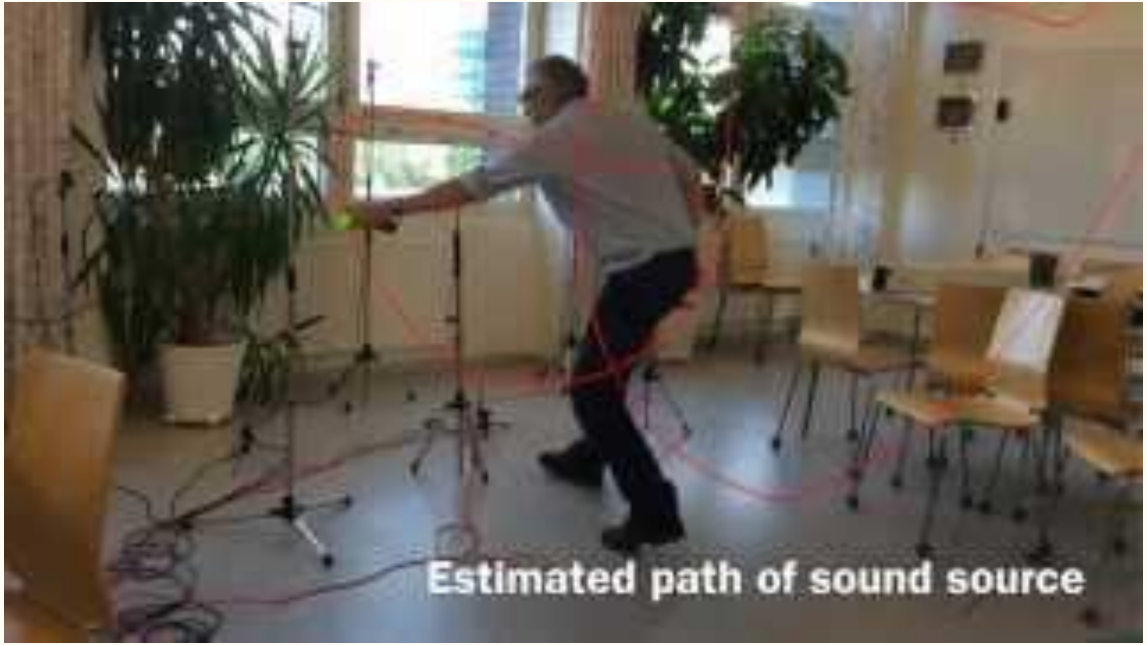
Then the Kronecker product is:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix} = \left[\begin{array}{cc:cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ \hdashline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right]$$

Indepth

Feature	Outer Product $\mathbf{uv}^T$	Kronecker Product $\mathbf{A} \otimes \mathbf{B}$
Inputs	Two vectors $\mathbf{u}, \mathbf{v}$	Two matrices $\mathbf{A}, \mathbf{B}$
Output	Matrix capturing pairwise **scalar** interactions	Larger matrix capturing **block-wise** interactions
Size	$m \times n$ (if $\mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n$)	$(m \cdot p) \times (n \cdot q)$ (if $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{B} \in \mathbb{R}^{p \times q}$)
Interpretation	Forms rank-1 matrix; captures scalar-level pairings	Tensor-like product; encodes interactions between matrix structures
Applications	Rank-1 approximations, covariance matrices, basic interactions	Tensor algebra, system modeling, multi-task learning, structured matrix equations
Notation	$\mathbf{u} \otimes \mathbf{v} = \mathbf{uv}^T$ (vector case)	$\mathbf{A} \otimes \mathbf{B}$

Use of rank



Linear Subspace

Let V be a linear vector space over the scalars $\mathbb{R}$. $W \subset V$ is a a subspace if and only if W satisfies the following conditions:

1. The zero vector is in the subspace:

$$\mathbf{0} \in W$$

2. The subspace is closed under vector addition:

$$\text{If } \mathbf{v}_1 \in W \text{ and } \mathbf{v}_2 \in W, \text{ then } \mathbf{v}_1 + \mathbf{v}_2 \in W$$

3. The subspace is closed under scalar multiplication:

$$\text{If } \mathbf{v} \in W \text{ and } \lambda \in \mathbb{R}, \text{ then } \lambda \mathbf{v} \in W$$

Let solve a few!

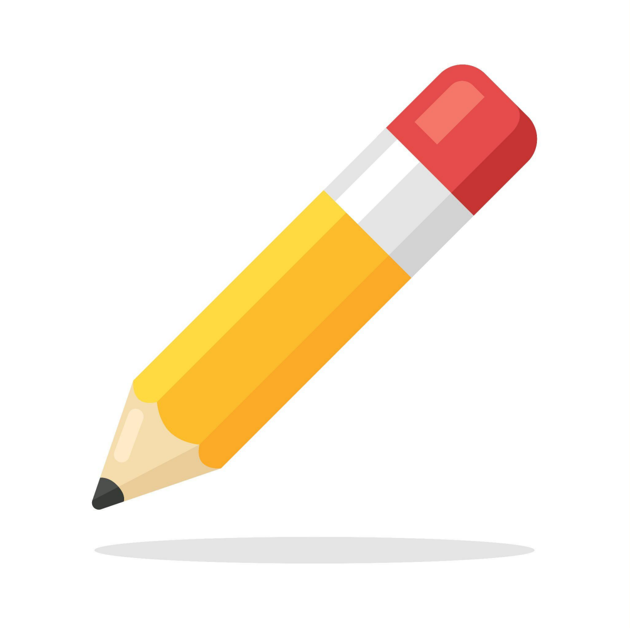


Image by juicy_fish on Freepik

Questions?

Onward to practicality

https://github.com/fastai/numerical-linear-algebra/blob/master/nbs/2.%20Topic%20Modeling%20with%20NMF%20and%20SVD.ipynb

Next Week - Rank

Reinforcement Learning

Goal: Continue Notebook 3 in https://github.com/fastai/numerical-linear-algebra/blob/master/README.md