Loi de Poisson bivariée Teicher

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Résumé

Ce document contient les preuves de la fonction de densité et de l'espérance conditionnelle de la loi Poisson Teicher et celle de l'espérance tronquée de la loi exponentielle EFGM.

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1 Poisson Teicher

1.1 Définition

On a

$$M_1 = K_1 + K_0, \qquad M_2 = K_2 + K_0$$

où $M_i \sim \text{Pois}(\lambda_i),\ i = 1, 2$ et $K_i = \text{Pois}(\alpha_i),\ i = 0, 1, 2$ avec $K_0 \perp\!\!\!\perp K_1 \perp\!\!\!\perp K_2$.

1.2 Fonction de densité

Démonstration.

$$\begin{aligned}
f_{M_1,M_2}(m_1, m_2) &= \Pr(M_1 = m_1, M_2 = m_2) \\
&= \sum_{j=0}^{\min(m_1,m_2)} \Pr(M_1 = m_1, M_2 = m_2 | J_0 = j) \times \Pr(J_0 = j) \\
&= \sum_{j=0}^{\min(m_1,m_2)} \Pr(K_1 - j = m_1, K_2 - j = m_2) \times \Pr(J_0 = j) \\
&= \sum_{j=0}^{\min(m_1,m_2)} \Pr(K_1 = m_1 - j) \times \Pr(K_2 = m_2 - j) \times \Pr(J_0 = j) \\
&= \sum_{j=0}^{\min(m_1,m_2)} f_{K_1}(m_1 - j) f_{J_2}(m_2 - j) f_{J_0}(j)
\end{aligned}$$

$\square$

1.3 Espérance conditionnelle

Démonstration.

$$\begin{aligned} E[M_1|M_2 = m_2] &= E[K_1 + K_0 = m_1|K_2 + K_0 = m_2] \\ &= \underbrace{E[K_1|K_2 + K_0 = m_2]}_{A} + \underbrace{E[K_0|K_2 + K_0 = m_2]}_{B} \end{aligned}$$

On commence part la partie A :

$$E[K_1|K_2 + K_0 = m_2] \overset{\perp\!\!\!\perp}{=} E[K_1] = \alpha_1 = \lambda_1 - \alpha_0$$

On passe à la partie B :

$$\begin{aligned} E[K_0|K_2 + K_0 = m_2] &= \sum_{j=1}^{m_2} j \frac{\Pr(K_0 = j)\Pr(K_2 = m_2 - j)}{\Pr(K_2 + K_0 = m_2)} \\ &= \sum_{j=1}^{m_2} j \frac{\frac{\alpha_0^j \mathrm{e}^{-\alpha_0}}{j!} \frac{\alpha_2^{m_2-j}\mathrm{e}^{-\alpha_2}}{(m_2-j)!}}{\frac{(\alpha_0+\alpha_2)^{m_2}\mathrm{e}^{-(\alpha_0+\alpha_2)}}{m_2!}} \\ &= \sum_{j=1}^{m_2} j \frac{m_2!}{j!(m_2-j)!} \left(\frac{\alpha_0}{\alpha_2}\right)^j \left(\frac{\alpha_2}{\alpha_0+\alpha_2}\right)^{m_2} \\ &= \sum_{j=1}^{m_2} j \frac{m_2!}{j!(m_2-j)!} \left(\frac{\alpha_0}{\alpha_2}\right)^j \left(\frac{\alpha_2}{\alpha_0+\alpha_2}\right)^{m_2} \left(\frac{\alpha_2}{\alpha_0+\alpha_2}\right)^{-j} \left(\frac{\alpha_2}{\alpha_0+\alpha_2}\right)^{j} \\ &= \sum_{j=1}^{m_2} j \frac{m_2!}{j!(m_2-j)!} \left(\frac{\alpha_0}{\alpha_0+\alpha_2}\right)^j \left(\frac{\alpha_2}{\alpha_0+\alpha_2}\right)^{m_2-j} \\ &= \sum_{j=1}^{m_2} j \frac{m_2!}{j!(m_2-j)!} \left(\frac{\alpha_0}{\alpha_0+\alpha_2}\right)^j \left(1-\frac{\alpha_0}{\alpha_0+\alpha_2}\right)^{m_2-j} \\ &= \sum_{j=1}^{m_2} j \binom{m_2}{j} \left(\frac{\alpha_0}{\alpha_0+\alpha_2}\right)^j \left(1-\frac{\alpha_0}{\alpha_0+\alpha_2}\right)^{m_2-j} \end{aligned}$$

On reconnais une loi binomial $n = m_2$ et $q = \frac{\alpha_0}{\alpha_0+\alpha_2}$. Donc,

$$E[K_0|K_2 + K_0 = m_2] = m_2 \frac{\alpha_0}{\alpha_0+\alpha_2} = m_2 \frac{\alpha_0}{\lambda_2}$$

En recombinant A avec B on obtient

$$E[M_1|M_2 = m_2] = \lambda_1 - \alpha_0 + m_2 \frac{\alpha_0}{\lambda_2}$$

□

2 Expo EFGM

On a

$$
\begin{aligned}
f_{X_2|X_1=x_1}(x_2) &= \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} \\
&= (1+\theta)\beta_2 \mathrm{e}^{-\beta_2 x_2} + 4\theta \mathrm{e}^{-\beta_1 x_1}\beta_2 \mathrm{e}^{-2\beta_2 x_2} - 2\theta \mathrm{e}^{-\beta_1 x_1}\beta_2 \mathrm{e}^{-\beta_2 x_2} - 2\theta\beta_2 \mathrm{e}^{-\beta_2 x_2}.
\end{aligned}
$$

Alors,

$$
\begin{aligned}
\mathrm{E}[X_2|X_1=x_1] &= \int_0^\infty x_2 f_{X_2|X_1=x_1}(x_2)\,\mathrm{d}x_2 \\
&= \int_0^\infty x_2((1+\theta)\beta_2 \mathrm{e}^{-\beta_2 x_2} + 4\theta \mathrm{e}^{-\beta_1 x_1}\beta_2 \mathrm{e}^{-2\beta_2 x_2} - 2\theta \mathrm{e}^{-\beta_1 x_1}\beta_2 \mathrm{e}^{-\beta_2 x_2} - 2\theta\beta_2 \mathrm{e}^{-\beta_2 x_2})\,\mathrm{d}x_2 \\
&= (1+\theta)\frac{1}{\beta_2} + 2\theta \mathrm{e}^{-\beta_1 x_1}\frac{1}{2\beta_2} - \frac{2\theta \mathrm{e}^{-\beta_1 x_1}}{\beta_2} - \frac{\theta}{2\beta_2} \\
&= \frac{1}{\beta_2} + \frac{\theta}{2\beta_2} - \frac{\theta \mathrm{e}^{-\beta_1 x_1}}{\beta_2} \\
&= \mathrm{E}[X_2] + \theta\frac{1}{\beta_2}\left(\frac{1}{2} - \mathrm{e}^{-\beta_1 x_1}\right).
\end{aligned}
$$