Espérances tronquées et mesure VaR et TVaR
Développements

Jérémie Barde

14 août 2024

Résumé

Ce document contient les développements pour les expressions des espérances tronquées et les mesures VaR et TVaR pour différentes lois.

Table des matières

1 Loi uniforme

1.1 Espérance tronquée

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_a^d x \frac{1}{b-a} \times 1_{\{x \in [a,b]\}} \mathrm{d}x \\
&= \frac{1}{b-a} \frac{x^2}{2} \Big|_a^d \\
&= \frac{1}{b-a} \left[\frac{d^2}{2} - \frac{a^2}{2} \right] \\
&= \frac{d^2 - a^2}{2(b-a)}
\end{aligned}$$

□

1.2 TVaR

Démonstration.

$$\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa} \int_\kappa^1 VaR_u(X) \mathrm{d}u \\
&= \frac{1}{1-\kappa} \int_\kappa^1 a + (b-a)\kappa \ \mathrm{d}u \\
&= \frac{1}{1-\kappa} \left[a(1-\kappa) + (b-a) \frac{u^2}{2} \Big|_\kappa^1 \right] \\
&= \frac{1}{1-\kappa} \left[a(1-\kappa) + \frac{b-a}{2} \left(1-\kappa^2\right) \right] \\
&= \frac{1}{1-\kappa} \left[a(1-\kappa) + \frac{b-a}{2} \left((1-\kappa)(1+\kappa)\right) \right] \\
&= a + \frac{b-a}{2}(1+\kappa)
\end{aligned}$$

□

2 Loi normale

2.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_{\infty}^{d} (\mu + x - \mu) f(x) \mathrm{d}x \\
&= \int_{-\infty}^{d} \mu f(x) \mathrm{d}x + \int_{-\infty}^{d} (x - \mu) \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \mathrm{d}x, \quad \text{(C.V.)} \\
&= \mu \Phi \left(\frac{d - \mu}{\sigma} \right) + \frac{\sigma}{\sqrt{2\pi}} \int_{\infty}^{\frac{1}{2}\left(\frac{d-\mu}{\sigma}\right)^2} \mathrm{e}^{-u} \mathrm{d}u \\
&= \mu \Phi \left(\frac{d - \mu}{\sigma} \right) - \frac{\sigma}{\sqrt{2\pi}} \int_{\frac{1}{2}\left(\frac{d-\mu}{\sigma}\right)^2}^{\infty} \mathrm{e}^{-u} \mathrm{d}u \\
&= \mu \Phi \left(\frac{d - \mu}{\sigma} \right) - \frac{\sigma}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{d-\mu}{\sigma}\right)^2}
\end{aligned}
$$

□

Changement de variable :

$$
\begin{aligned}
u &= \frac{1}{2} \left(\frac{x - \mu}{\sigma} \right)^2 \\
\sigma \mathrm{d}u &= \frac{x - u}{\sigma} \mathrm{d}x
\end{aligned}
$$

2.2 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1 - \kappa} \left(E[X] - E[X \times 1_{\{X \leq VaR_\kappa(X)\}}] \right) \\
&= \frac{1}{1 - \kappa} \left(\mu - \mu \Phi \left(\frac{VaR_\kappa(X) - \mu}{\sigma} \right) + \frac{\sigma}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{VaR_\kappa(X)-\mu}{\sigma}\right)^2} \right) \\
&= \frac{1}{1 - \kappa} \left(\mu - \mu \Phi \left(\Phi^{-1}(\kappa) \right) + \frac{\sigma}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \right) \\
&= \mu + \frac{1}{1 - \kappa} \frac{\sigma}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\Phi^{-1}(\kappa)\right)^2} \\
&= \mu + \sigma TVaR_\kappa(Z)
\end{aligned}
$$

□

3 Loi lognormale

3.1 Espérance tronquée

On utilise le fait que $X \sim LN(\mu, \sigma^2)$ quand $X = \mathrm{e}^Y$ où $Y \sim N(\mu, \sigma^2)$.

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= E[\mathrm{e}^Y \times 1_{\{Y \leq \ln(d)\}}] \\
&= \int_{-\infty}^{\ln(d)} \mathrm{e}^y \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(y-\mu)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \int_{-\infty}^{\ln(d)} c\mathrm{e}^{y-\frac{y^2-2\mu y+\mu^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \int_{-\infty}^{\ln(d)} c\mathrm{e}^{\frac{2\sigma^2 y+y^2-2\mu y+\mu^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \int_{-\infty}^{\ln(d)} c\mathrm{e}^{\frac{-[y^2-2y(\mu+\sigma^2)+(\mu+\sigma^2)^2-(\mu+\sigma^2)^2+\mu^2]}{2\sigma^2}} \,\mathrm{d}x \\
&= \int_{-\infty}^{\ln(d)} c\mathrm{e}^{\frac{-\left[\left(y-(\mu+\sigma^2)\right)^2-\mu^2-2\sigma^2\mu-\sigma^4+\mu^2\right]}{2\sigma^2}} \,\mathrm{d}x \\
&= \int_{-\infty}^{\ln(d)} c\mathrm{e}^{\frac{-\left(y-(\mu+\sigma^2)\right)^2}{2\sigma^2}+\mu+\frac{\sigma^2}{2}} \,\mathrm{d}x \\
&= \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \Phi\left(\frac{\ln(d)-\mu-\sigma^2}{\sigma}\right)
\end{aligned}$$

□

3.2 VaR

On utilise la propriété suivante de la $VaR_\kappa(X)$ pour une fonction strictement croissante :

$$VaR_p(\varphi(X)) = \varphi(VaR_p(X))$$

Démonstration.

$$VaR_\kappa(X) = VaR_\kappa(\mathrm{e}^Y) = \mathrm{e}^{VaR_\kappa(Y)} = e^{\mu+\sigma VaR_\kappa(Z)} \quad \square$$

□

3.3 TVaR

Démonstration.

$$\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa}\left(\mathrm{e}^{\mu+\frac{\sigma^2}{2}} - \mathrm{e}^{\mu+\frac{\sigma^2}{2}} \Phi\left(\frac{\ln(VaR_\kappa(X))-\mu-\sigma^2}{\sigma}\right)\right) \\
&= \frac{\mathrm{e}^{\mu+\frac{\sigma^2}{2}}}{1-\kappa}\left(1-\Phi\left(VaR_\kappa(Z)-\sigma\right)\right)
\end{aligned}$$

□

4 Loi exponentielle

4.1 Espérance tronquée

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x\beta \mathrm{e}^{-\beta x} \mathrm{d}x \\
&= -x\mathrm{e}^{-\beta x}\Big|_0^d + \int_0^d \mathrm{e}^{-\beta x}\mathrm{d}x \\
&= \frac{1}{\beta}\int_0^d \beta \mathrm{e}^{-\beta x}\mathrm{d}x - d\mathrm{e}^{-\beta d} \\
&= \frac{1}{\beta}\left(1 - \mathrm{e}^{-\beta d}\right) - d\mathrm{e}^{-\beta d}
\end{aligned}$$

□

4.2 VaR

La fonction de répartition s'inverse facilement

4.3 TVaR

Démonstration.

$$\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa} E[X \times 1_{\{X > VaR_\kappa(X)\}}] \\
&= \frac{1}{1-\kappa}\left(E[X] - E[X \times 1_{\{X \leq VaR_\kappa(X)\}}]\right) \\
&= \frac{1}{1-\kappa}\left(\frac{1}{\beta} - \frac{1}{\beta}\left(1 - \mathrm{e}^{-\beta VaR_\kappa(X)}\right) + VaR_\kappa(X)\mathrm{e}^{-\beta VaR_\kappa(X)}\right) \\
&= \frac{1}{1-\kappa}\frac{1}{\beta}\left(1 - \left(1 - \mathrm{e}^{-\beta VaR_\kappa(X)}\right) + \beta VaR_\kappa(X)\mathrm{e}^{-\beta VaR_\kappa(X)}\right) \\
&= \frac{1}{1-\kappa}\frac{1}{\beta}\left(\mathrm{e}^{-\beta VaR_\kappa(X)} + \beta VaR_\kappa(X)\mathrm{e}^{-\beta VaR_\kappa(X)}\right) \\
&= \frac{1}{1-\kappa}\frac{1}{\beta}\left(\mathrm{e}^{-\beta(-\frac{1}{\beta}\ln(1-\kappa))} + \beta VaR_\kappa(X)\mathrm{e}^{-\beta(-\frac{1}{\beta}\ln(1-\kappa))}\right) \\
&= \frac{1}{1-\kappa}\frac{1}{\beta}\left((1-\kappa) + \beta VaR_\kappa(X)\right) \\
&= VaR_\kappa(X) + \frac{1}{\beta} \\
&= VaR_\kappa(X) + E[X]
\end{aligned}$$

□

5 Loi Gamma

5.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-\beta x} \mathrm{d}x \\
&= \int_0^d \frac{\beta^\alpha}{\Gamma(\alpha)} x^{(\alpha+1)-1} \mathrm{e}^{-\beta x} \mathrm{d}x \\
&= \frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha)} \int_0^d \frac{\beta^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} \mathrm{e}^{-\beta x} \mathrm{d}x \\
&= \frac{\alpha}{\beta} H(d; \alpha+1, \beta)
\end{aligned}
$$

□

5.2 VaR

Outil d'optimisation

5.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa} \left(E[X] - E[X \times 1_{\{X \leq VaR_\kappa(X)\}}]\right) \\
&= \frac{1}{1-\kappa} \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta} H(VaR_\kappa(X); \alpha+1, \beta)\right) \\
&= \frac{1}{1-\kappa} \frac{\alpha}{\beta} \bar{H}(VaR_\kappa(X); \alpha+1, \beta)
\end{aligned}
$$

□

6 Loi Pareto

6.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \frac{\alpha \lambda^\alpha}{(\lambda + x)^{\alpha+1}} \mathrm{d}x \\
&= \int_0^d x \alpha \lambda^\alpha (\lambda + x)^{-\alpha-1} \mathrm{d}x \\
&= -x \left(\frac{\lambda}{\lambda + x} \right)^\alpha \Bigg|_0^d + \int_0^d \lambda^\alpha (\lambda + x)^{-\alpha} \mathrm{d}x \\
&= -d \left(\frac{\lambda}{\lambda + d} \right)^\alpha + \frac{\lambda^\alpha}{-\alpha + 1} (\lambda + x)^{-\alpha+1} \Bigg|_0^d \\
&= -d \left(\frac{\lambda}{\lambda + d} \right)^\alpha - \frac{1}{\alpha - 1} \left[\lambda^\alpha (\lambda + d)^{-\alpha+1} - \lambda \right] \\
&= -d \left(\frac{\lambda}{\lambda + d} \right)^\alpha - \frac{\lambda}{\alpha - 1} \left[\left(\frac{\lambda}{\lambda + d} \right)^{\alpha-1} - 1 \right] \\
&= \frac{\lambda}{\alpha - 1} \left[1 - \left(\frac{\lambda}{\lambda + d} \right)^{\alpha-1} \right] - d \left(\frac{\lambda}{\lambda + d} \right)^\alpha
\end{aligned}
$$

□

Intégration par partie :

$$
\begin{aligned}
u = x, &\quad \mathrm{d}v = \alpha \lambda (\lambda + x)^{-\alpha-1} \\
\mathrm{d}u = \mathrm{d}x, &\quad v = -\lambda^\alpha (\lambda + x)^{-\alpha}
\end{aligned}
$$

6.2 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1 - \kappa} \int_\kappa^1 VaR_u(X) \mathrm{d}u \\
&= \frac{1}{1 - \kappa} \int_\kappa^1 \lambda \left((1 - \kappa)^{-\frac{1}{\alpha}} - 1 \right) \mathrm{d}u \\
&= \frac{\lambda}{1 - \kappa} \left[\int_\kappa^1 (1 - \kappa)^{-\frac{1}{\alpha}} \mathrm{d}u - \int_\kappa^1 \mathrm{d}u \right] \\
&= \frac{\lambda}{1 - \kappa} \left[- \frac{(1 - u)^{-\frac{1}{\alpha}+1}}{-\frac{1}{\alpha} + 1} - (1 - \kappa) \right] \\
&= \lambda \left[\frac{\alpha}{\alpha - 1} (1 - \kappa)^{-\frac{1}{\alpha}} - 1 \right]
\end{aligned}
$$

□

7 Loi weibull

7.1 Espérance tronquée

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x\beta\tau(\beta x)^{\tau-1}\mathrm{e}^{-(\beta x)^\tau}\,\mathrm{d}x \\
&= \int_0^d \tau(\beta x)^\tau \mathrm{e}^{-(\beta x)^\tau}\,\mathrm{d}x \\
&= \int_0^{(\beta d)^\tau} u\mathrm{e}^{-u}\frac{u^{\frac{1}{\tau}}}{\beta u}du, \qquad \text{(C.V.)} \\
&= \int_0^{(\beta d)^\tau} \frac{u^{\frac{1}{\tau}}}{\beta}\mathrm{e}^{-u}du \\
&= \frac{\Gamma(\frac{1}{\tau}+1)}{\beta}\int_0^{(\beta d)^\tau}\frac{1}{\Gamma(\frac{1}{\tau}+1)}u^{\frac{1}{\tau}+1-1}\mathrm{e}^{-u}du \\
&= \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \times H\left(u, \frac{1}{\tau}+1, 1)\right) \\
&= \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \times H\left(\beta^\tau d^\tau, \frac{1}{\tau}+1, 1)\right) \\
&= \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \times H\left(d^\tau, \frac{1}{\tau}+1, \beta^\tau)\right)
\end{aligned}$$

□

Changement de variable :

$$\begin{aligned}
u = (\beta x)^\tau &\Rightarrow x = \frac{u^{\frac{1}{\tau}}}{\beta} \\
du &= \beta\tau(\beta x)^{\tau-1}\mathrm{d}x \\
\tau\mathrm{d}x &= \frac{du}{\beta^\tau x^{\tau-1}} \\
\tau\mathrm{d}x &= \frac{u^{\frac{1}{\tau}}du}{\beta u}
\end{aligned}$$

7.2 VaR

Démonstration.

$$\begin{aligned}
\kappa &= 1 - \mathrm{e}^{-(\beta x)^\tau} \\
1 - k &= \mathrm{e}^{-(\beta x)^\tau} \\
-\ln(1-k) &= -(\beta x)^\tau \\
\frac{-\ln(1-k)}{\beta^\tau} &= x^\tau \\
\frac{[-\ln(1-k)]^{\frac{1}{\tau}}}{\beta} &= x
\end{aligned}$$

Donc,

$$VaR_\kappa(X) = \frac{1}{\beta}[-\ln(1-k)]^{\frac{1}{\tau}}$$

□

7.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(x) &= \frac{1}{1-\kappa} E[X \times 1_{\{X > VaR_\kappa(X)\}}] \\
&= \frac{1}{1+\kappa} \left(E[X] - E[X \times 1_{\{X \leq VaR_\kappa(X)\}}] \right) \\
&= \frac{1}{1+\kappa} \left(\frac{\Gamma(\frac{1}{\tau}+1)}{\beta} - \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \times H\left((VaR_\kappa(X))^\tau, \frac{1}{\tau}+1, \beta^\tau) \right) \right) \\
&= \frac{1}{1+\kappa} \times \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \left(1 - H\left((VaR_\kappa(X))^\tau, \frac{1}{\tau}+1, \beta^\tau) \right) \right) \\
&= \frac{1}{1+\kappa} \times \frac{\Gamma(\frac{1}{\tau}+1)}{\beta} \left(\bar{H}\left(\frac{1}{\beta^\tau}[-\ln(1-k)], \frac{1}{\tau}+1, \beta^\tau) \right) \right) \\
&= \frac{\Gamma(\frac{1}{\tau}+1)}{\beta(1-\kappa)} \left(\bar{H}\left(-\ln(1-k), \frac{1}{\tau}+1, 1) \right) \right)
\end{aligned}
$$

□

8 Loi burr

8.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \frac{\alpha\tau\lambda^\alpha x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+1}} \mathrm{d}x \\
&= \int_0^d \frac{x^\tau \alpha\tau}{(\lambda + x^\tau)} \left(\frac{\lambda}{\lambda + x^\tau}\right)^\alpha \mathrm{d}x \\
&= \int_1^{\frac{\lambda}{\lambda+d^\tau}} \frac{u^\alpha \alpha \left(\frac{\lambda(1-u)}{u}\right)^{\frac{\tau}{\tau}}}{\frac{\lambda}{u}} \times - \frac{\lambda^{\frac{1}{\tau}+1-1}(1-u)^{\frac{1}{\tau}-1}}{u^{2+\frac{1}{\tau}-1}} du \qquad \text{(C.V. 1)} \\
&= \int_{\frac{\lambda}{\lambda+d^\tau}}^1 u^{\alpha-\frac{1}{\tau}-1} \lambda^{\frac{1}{\tau}} (1-u)^{\frac{1}{\tau}} du \\
&= \lambda^{\frac{1}{\tau}} \alpha(-1) \int_{\frac{d^\tau}{\lambda+d^\tau}}^0 (1-v)^{\alpha-\frac{1}{\tau}-1} v^{\frac{1}{\tau}+1-1} dv \qquad \text{(C.V. 2)} \\
&= \lambda^{\frac{1}{\tau}} \alpha I(\tau+1, \alpha - \frac{1}{\tau}) \int_0^{\frac{d^\tau}{\lambda+d^\tau}} \frac{1}{I(\tau+1, \alpha-\frac{1}{\tau})} v^{\frac{1}{\tau}+1-1} (1-v)^{\alpha-\frac{1}{\tau}-1} dv \\
&= \frac{\lambda^{\frac{1}{\tau}} \Gamma(\frac{1}{\tau}+1)\Gamma(\alpha-\frac{1}{\tau})}{\Gamma(\alpha)} B\left(\frac{d^\tau}{\lambda+d^\tau}, 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)
\end{aligned}
$$

□

Changement de variable :

(1)

$$
\begin{aligned}
u = \frac{\lambda}{\lambda + x^\tau} &\Rightarrow x = \left(\frac{\lambda(1-u)}{u}\right)^{\frac{1}{\tau}} \\
tdx &= \frac{-\lambda^{\frac{1}{\tau}+1}}{u^2} \times \frac{(1-u)^{\frac{1}{\tau}-1}}{u^{\frac{1}{\tau}}-1} du
\end{aligned}
$$

(2)

$$
\begin{aligned}
v = 1 - u &\Rightarrow u = 1 - v \\
du &= -dv
\end{aligned}
$$

8.2 VaR

La fonction de répartition s'inverse facilement.

8.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-k} \left(\frac{\lambda^{\frac{1}{\tau}} \Gamma(\frac{1}{\tau}+1)\Gamma(\alpha-\frac{1}{\tau})}{\Gamma(\alpha)} - \frac{\lambda^{\frac{1}{\tau}} \Gamma(\frac{1}{\tau}+1)\Gamma(\alpha-\frac{1}{\tau})}{\Gamma(\alpha)} B\left(\frac{VaR_\kappa(X)^\tau}{\lambda + VaR_\kappa(X)^\tau}, 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)\right) \\
&= \frac{\lambda^{\frac{1}{\tau}} \Gamma(\frac{1}{\tau}+1)\Gamma(\alpha-\frac{1}{\tau})}{\Gamma(\alpha)(1-\kappa)} \left(1 - B\left(\frac{VaR_\kappa(X)^\tau}{\lambda + VaR_\kappa(X)^\tau}, 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)\right) \\
&= \frac{\lambda^{\frac{1}{\tau}} \Gamma(\frac{1}{\tau}+1)\Gamma(\alpha-\frac{1}{\tau})}{\Gamma(\alpha)(1-\kappa)} \bar{B}\left(\frac{VaR_\kappa(X)^\tau}{\lambda + VaR_\kappa(X)^\tau}, 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)
\end{aligned}
$$

□

9 Loi log-logistique

9.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \frac{\tau x^{\tau-1}\lambda^\tau}{(\lambda^\tau + x^\tau)^2} \mathrm{d}x \\
&= \int_0^d \frac{\tau\lambda^\tau}{\lambda^\tau + x^\tau} \times \frac{x^\tau}{\lambda^\tau + x^\tau} \mathrm{d}x \\
&= \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \frac{u\tau\lambda^{\tau+1}}{\lambda^\tau(1+\frac{u}{1-u})} \times \frac{\lambda}{\tau}\left(\frac{u}{1-u}\right)^{\frac{1}{\tau}-1} \times \frac{1}{(1-u)^2}\,\mathrm{d}u \qquad \text{(C.V.)} \\
&= \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \frac{\lambda(1-u)u^{\frac{1}{\tau}}}{(1-u)^{\frac{1}{\tau}-1}(1-u)^2} du \\
&= \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \frac{\lambda u^{\frac{1}{\tau}+1-1}}{(1-u)^{\frac{1}{\tau}-1}(1-u)} du \\
&= \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \frac{\lambda u^{\frac{1}{\tau}+1-1}}{(1-u)^{\frac{1}{\tau}}} du \\
&= \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \lambda u^{\frac{1}{\tau}+1-1}(1-u)^{1-\frac{1}{\tau}-1} du \\
&= \frac{\lambda\Gamma(1+\frac{1}{\tau})\Gamma(1-\frac{1}{\tau})}{\Gamma(1+\frac{1}{\tau}+1-\frac{1}{\tau})} \int_0^{\frac{d^\tau}{d^\tau+\lambda^\tau}} \lambda u^{\frac{1}{\tau}+1-1}(1-u)^{1-\frac{1}{\tau}-1} \frac{\Gamma(1+\frac{1}{\tau}+1-\frac{1}{\tau})}{\Gamma(1+\frac{1}{\tau})\Gamma(1-\frac{1}{\tau})} du \\
&= \lambda\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(1-\frac{1}{\tau}\right)B\left(\frac{d^\tau}{d^\tau+\lambda^\tau}, 1+\frac{1}{\tau}, 1-\frac{1}{\tau}\right)
\end{aligned}
$$

□

Changement de variable :

$$
\begin{aligned}
u = \frac{x^\tau}{\lambda^\tau + x^\tau} &\Rightarrow x = \lambda\left(\frac{u}{1-u}\right)^{\frac{1}{\tau}} \\
\mathrm{d}x = \frac{\lambda}{\tau}&\left(\frac{u}{1-u}\right)^{\frac{1}{\tau}-1} \frac{1}{(1-u)^2} du
\end{aligned}
$$

9.2 VaR

La fonction de répartition s'inverse facilement.

9.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-k}\left(\lambda\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(1-\frac{1}{\tau}\right) - \lambda\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(1-\frac{1}{\tau}\right)B\left(\frac{VaR_\kappa(X)^\tau}{VaR_\kappa(X)^\tau+\lambda^\tau}, 1+\frac{1}{\tau}, 1-\frac{1}{\tau}\right)\right) \\
&= \frac{\lambda\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(1-\frac{1}{\tau}\right)}{1-k}\bar{B}\left(\frac{VaR_\kappa(X)^\tau}{VaR_\kappa(X)^\tau+\lambda^\tau}, 1+\frac{1}{\tau}, 1-\frac{1}{\tau}\right) \\
&= \frac{\lambda\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(1-\frac{1}{\tau}\right)}{1-k}\bar{B}\left(\kappa, 1+\frac{1}{\tau}, 1-\frac{1}{\tau}\right)
\end{aligned}
$$

□

10 Loi bêta généralisée de type 2

10.1 Espérance tronquée

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \frac{ax^{ap-1}}{b^{ap} I(p,\ q)(1+(\frac{x}{b})^a)^{p+q}}\, \mathrm{d}x \\
&= \int_0^d \frac{ax^{ap}}{b^{ap} I(p,\ q)(1+(\frac{x}{b})^a)^{p+q}}\, \mathrm{d}x \\
&= \int_0^d \frac{ax^{(ap+1)-1}}{b^{ap} I(p,\ q)(1+(\frac{x}{b})^a)^{p+q}}\, \mathrm{d}x \\
&= \int_0^d \frac{ax^{a(p+\frac{1}{a})-1}}{b^{ap} I(p,\ q)(1+(\frac{x}{b})^a)^{(p+\frac{1}{a})+(q-\frac{1}{a})}}\, \mathrm{d}x \\
&= \frac{b^{a(p+\frac{1}{a})} I(p+\frac{1}{a}, q-\frac{1}{a})}{b^{ap} I(p,q)} \int_0^d \frac{ax^{a(p+\frac{1}{a})-1}}{b^{a(p+\frac{1}{a})} I(p+\frac{1}{a}, q-\frac{1}{a})(1+(\frac{x}{b})^a)^{(p+\frac{1}{a})+(q-\frac{1}{a})}}\, \mathrm{d}x \\
&= \frac{b^{ap+1} I(p+\frac{1}{a}, q-\frac{1}{a})}{b^{ap} I(p,q)} B\left(\frac{(\frac{d}{b})^a}{1+(\frac{d}{b})^a}, p+\frac{1}{a}, q-\frac{1}{a}\right) \\
&= \frac{bI(p+\frac{1}{a}, q-\frac{1}{a})}{I(p,q)} B\left(\frac{(\frac{d}{b})^a}{1+(\frac{d}{b})^a}, p+\frac{1}{a}, q-\frac{1}{a}\right)
\end{aligned}$$

□

Avec

$$F_X(x) = B\left(\frac{(\frac{d}{b})^a}{1+(\frac{d}{b})^a}, p, q\right).$$

10.2 VaR

Outil d'optimisation

10.3 TVaR

Démonstration.

$$\begin{aligned}
TVaR_\kappa(X) &= \frac{E[X \times 1_{\{X > VaR_\kappa(X)\}}]}{1-\kappa} \\
&= \frac{1}{1-\kappa}(E[X] - E[X \times 1_{\{X \leq VaR_\kappa(X)\}}]) \\
&= \frac{1}{1-\kappa}\left(\frac{bI(p+\frac{1}{a}, q-\frac{1}{a})}{I(p,q)} - \frac{bI(p+\frac{1}{a}, q-\frac{1}{a})}{I(p,q)} B\left(\frac{\left(\frac{VaR_\kappa(X)}{b}\right)^a}{1+\left(\frac{VaR_\kappa(X)}{b}\right)^a}, p+\frac{1}{a}, q-\frac{1}{a}\right)\right) \\
&= \frac{1}{1-\kappa} \frac{bI(p+\frac{1}{a}, q-\frac{1}{a})}{I(p,q)} \bar{B}\left(\frac{\left(\frac{VaR_\kappa(X)}{b}\right)^a}{1+\left(\frac{VaR_\kappa(X)}{b}\right)^a}; p+\frac{1}{a}, q-\frac{1}{a}\right)
\end{aligned}$$

□

11 Pareto généralisée

11.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_u^d x \frac{1}{\sigma} \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}-1} \mathrm{d}x \\
&= -x\sigma \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}} \Bigg|_u^d + \int_u^d -\sigma \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}} \mathrm{d}x \\
&= -d\sigma \left(1 + \frac{\xi}{\sigma}(d-u)\right)^{\frac{1}{\xi}} + u - \frac{\sigma}{1-\xi} \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{1-\frac{1}{\xi}} \Bigg|_u^d \\
&= u - d(1 + \frac{\xi}{\sigma}(d-u))^{-\frac{1}{\xi}} - \frac{\sigma}{1-\xi} \left((1 + \frac{\xi}{\sigma}(d-u))^{1-\frac{1}{\xi}} - 1\right)
\end{aligned}
$$

□

Intégration par partie :

$$
\begin{aligned}
u = x, \quad \mathrm{d}v &= \frac{1}{\sigma} \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}-1} \mathrm{d}x \\
\mathrm{d}u = \mathrm{d}x, \quad v &= -\sigma \left(1 + \frac{\xi}{\sigma}(x-u)\right)^{-\frac{1}{\xi}}
\end{aligned}
$$

11.2 VaR

La fonction de répartition s'inverse facilement

11.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa} \int_\kappa^1 VaR_p(X) \ \mathrm{d}p \\
&= \frac{1}{1-\kappa} \int_\kappa^1 \frac{\sigma}{\xi} \left((1-p)^{-\xi} - 1\right) + u \ \mathrm{d}p \\
&= \frac{1}{1-\kappa} \left[\int_\kappa^1 \frac{\sigma}{\xi}(1-p)^{-\xi} \ \mathrm{d}p - \int_\kappa^1 \frac{\sigma}{\xi} \ \mathrm{d}p + \int_\kappa^1 u \ \mathrm{d}p\right] \\
&= \frac{1}{1-\kappa} \left[-\frac{\sigma}{\xi} \frac{(1-p)^{1-\xi}}{1-\xi} \Bigg|_\kappa^1 - \frac{\sigma}{\xi}(1-\kappa) + u(1-\kappa)\right] \\
&= \frac{1}{1-\kappa} \frac{\sigma}{\xi} \frac{(1-\kappa)^{1-\xi}}{1-\xi} + \frac{\sigma}{\xi} + u \\
&= \frac{\sigma}{\xi} \left(\frac{1}{1-\xi}(1-\kappa)^{-\xi} - 1\right) + u
\end{aligned}
$$

□

12 Mélange d'Erlang

12.1 Espérance tronquée

Démonstration.

$$\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \sum_{k=0}^{\infty} \gamma_k \frac{\beta^k}{\Gamma(k)} x^{k-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \int_0^d x \frac{\beta^k}{\Gamma(k)} x^{k-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{\Gamma(k+1)}{\Gamma(k)\beta} \int_0^d \frac{\beta^{k+1}}{\Gamma(k+1)} x^{(k+1)-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{k!}{(k-1)!\beta} H(d; k+1, \beta) \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{k}{\beta} H(d; k+1, \beta)
\end{aligned}$$

□

12.2 VaR

Outil d'optimisation

12.3 TVaR

Démonstration.

$$\begin{aligned}
TVaR_k(X) &= \frac{1}{1-\kappa} \int_{VaR_\kappa(X)}^{\infty} x \sum_{k=0}^{\infty} \gamma_k \frac{\beta^k}{\Gamma(k)} x^{k-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \int_{VaR_\kappa(X)}^{\infty} x \frac{\beta^k}{\Gamma(k)} x^{k-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{\Gamma(k+1)}{\Gamma(k)\beta} \int_{VaR_\kappa(X)}^{\infty} \frac{\beta^{k+1}}{\Gamma(k+1)} x^{(k+1)-1} \mathrm{e}^{-\beta x} \, \mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{k!}{(k-1)!\beta} \bar{H}(VaR_\kappa(X); k+1, \beta) \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{k}{\beta} \bar{H}(VaR_\kappa(X); k+1, \beta)
\end{aligned}$$

□

13 Mélange Erlang 2 (Gamma)

13.1 Espérance tronquée

Démonstration.

$$
\begin{aligned}
E[X \times 1_{\{X \leq d\}}] &= \int_0^d x \sum_{k=0}^{\infty} \gamma_k \frac{\beta^{\alpha k}}{\Gamma(\alpha k)} x^{\alpha k - 1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \int_0^d x \frac{\beta^{\alpha k}}{\Gamma(\alpha k)} x^{k-1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k)\beta} \int_0^d \frac{\beta^{k+1}}{\Gamma(\alpha k + 1)} x^{(\alpha k + 1) - 1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{(\alpha k)!}{(\alpha k - 1)!\beta} H(x; \alpha k + 1, \beta) \\
&= \sum_{k=0}^{\infty} \gamma_k \frac{\alpha k}{\beta} H(x; \alpha k + 1, \beta)
\end{aligned}
$$

□

13.2 VaR

Outil d'optimisation

13.3 TVaR

Démonstration.

$$
\begin{aligned}
TVaR_\kappa(X) &= \frac{1}{1-\kappa} \int_{VaR_\kappa(X)}^{\infty} x \sum_{k=0}^{\infty} \gamma_k \frac{\beta^{\alpha k}}{\Gamma(\alpha k)} x^{\alpha k - 1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \int_{VaR_\kappa(X)}^{\infty} x \frac{\beta^{\alpha k}}{\Gamma(\alpha k)} x^{\alpha k - 1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k)\beta} \int_{VaR_\kappa(X)}^{\infty} \frac{\beta^{\alpha k+1}}{\Gamma(\alpha k + 1)} x^{(\alpha k + 1) - 1} \mathrm{e}^{-\beta x} \,\mathrm{d}x \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{\alpha k!}{(\alpha k - 1)!\beta} \bar{H}(VaR_\kappa(X); \alpha k + 1, \beta) \\
&= \frac{1}{1-\kappa} \sum_{k=0}^{\infty} \gamma_k \frac{\alpha k}{\beta} \bar{H}(VaR_\kappa(X); \alpha k + 1, \beta)
\end{aligned}
$$

□