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Insurance pricing and increased limits ratemaking by proportional hazards transforms

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Abstract

This paper proposes a new premium principle, where risk loadings are imposed by a proportional decrease in the hazard rates. This premium principle is scale invariant and additive for layers. It is shown that this principle will generate stop-loss contracts as optimal reinsurance arrangements in a competitive market when the reinsurer is less risk-averse than the direct insurer. Finally, increased limits factors are calculated based on this principle.

Keywords: Risk-averse; Premium principle; Proportional hazards transform; Optimal reinsurance; Increased limits factors

1. Introduction

Insurance is a practice of exchanging a contingent claim for a fixed payment called premium. The principle of assigning premiums according to the underlying risk is an essential element of actuarial science.

According to Freifelder (1976), an individual's assessment of a risk reflexes the subjective attitude of the individual toward its uncertain outcome. Firstly, it is believed that policy-holders are more risk averse than insurers; this difference provides a mechanism for insurance business. Secondly, for a fixed insurer, a risk which is more uncertain than another is assigned a higher premium. By attaching a premium to each risk, an insurer has implicitly ordered the risks.

Based on the proportional hazards (PH) transform, this paper proposes a risk-adjusted premium for pricing risks. For the same underlying risk, the risk-adjusted premium is larger for a party which is more risk-averse. For an insurer, the risk-adjusted premium automatically and consistently adjusts the risk loading relative to the expected loss for different risks.

The risk-adjusted premium based on the PH transform is additive when a risk is split into layers, which makes it very appealing to insurance layer pricing. Based on a preference argument, an optimal reinsurance cooperation is identified. The increased limits factors, which are widely used in casualty insurance practice, can also be

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calculated by this method. The Panjer recursion can be used to evaluate the risk-adjusted stop-loss premium for a compound risk.

2. PH transforms and risk-adjusted premiums

In casualty insurance, a risk is a non-negative random loss X defined by its distribution function $F_X(t) = \Pr\{X \le t\}$ or survivor function $S_X(t) = 1 - F_X(t)$. Thinking of claim size as 'lifetime', we can define a hazard rate for a risk X provided that $F_X(t)$ is absolutely continuous:

$$\mu_X(t) = \frac{F_X'(t)}{1 - F_X(t)} = -\frac{\mathrm{d}}{\mathrm{d}t} \log S_X(t).$$

An adverse claims experience means that the realized loss is larger than expected, or the 'lifetime' is longer than the mean 'lifetime' and the hazard rate is lower (which is opposite to mortality risk for life insurers). To impose a safety margin, one can deflate the hazard rate by a multiple:

$$\mu_Y(t) = \frac{1}{\rho}\mu_X(t), \quad \rho > 1 \quad (t \ge 0), \tag{1}$$

which defines another random variable Y with [1]

$$S_Y(t) = S_X(t)^{1/\rho}.$$

Definition 1. Given any random variable X with survivor function $S_X(x)$, the equation

$$S_Y(t) = S_X(t)^{1/\rho}, \quad \rho > 0, \tag{2}$$

defines another random variable Y with survivor function $S_Y(t)$.

The mapping: $\Pi_\rho : X \mapsto Y$ is called the *proportional hazards (PH)* transform.

Remark 1. It is noted that relation (2) is *more general* than relation (1), since in relation (2) X can be a discrete variable or a continuous variable with spikes.

Remark 2. If X is continuous with density function $f_X(x)$, $x \in I$, then $Y = \Pi_\rho(X)$ is also continuous with density function

$$f_Y(t) = \left[\frac{1}{\rho} S_X(t)^{(1/\rho)-1}\right] f_X(t), \quad t \in I.$$

The weight function $1/\rho S_X(t)^{1/\rho - 1}$ increases with the loss size t, thus gives more weight to the unfavorable events.

Definition 2. For a risk X with survivor function $S_X(t)$, the *risk-adjusted premium* is defined as

$$\pi_\rho(X) = \mathrm{E}[\Pi_\rho(X)] = \int_0^\infty S_X(t)^{1/\rho}\, \mathrm{d}t, \quad \rho \ge 1, \tag{3}$$

where ρ is called the *(risk-averse) index*.

When $\rho = 1$, $\pi_1(X) = \int_0^\infty S_X(t)\, \mathrm{d}t = \mathrm{E}(X)$, which is the net expected loss.

Remark 3. To life insurers, an adverse mortality experience means that insureds die earlier than expected. When calculating insurance premiums, it is common practice to add a safety margin (by some percentage) to the expected mortality rates q_x. Since $q_x \approx \mu(x)$, it can be interpreted as inflating the hazard rate by a multiple:

$$\tilde{\mu}(x) = \alpha\mu(x), \quad \alpha > 1 \quad (x \ge 0).$$

Therefore, the PH transform method is consistent with the practice of adding a safety margin to the mortality rates q_x in life insurance.

3. The PH transform of some loss distributions

(1) If X has an exponential distribution with mean b:

$$S_X(t) = \mathrm{e}^{-t/b}, \quad \mu_X(t) = 1/b, \tag{4}$$

[1] Strictly speaking, (1) defines another distribution function $F_Y(t) = 1 - \exp(-\int_0^t \mu_Y(s)\, \mathrm{d}s)$, not a random variable. For notational convenience, we shall not distinguish between a random variable and its distribution function.

the PH transform $\Pi_\rho(X)$ also has an exponential distribution with mean ρb:

$$\mathrm{E}(X) = b, \quad \pi_\rho(X) = \rho b.$$

The risk-adjusted premium increases linearly with respect to ρ.

(2) If X has a uniform distribution on $[0,a]$, then

$$\mathrm{E}(X) = a/2, \quad \pi_\rho(X) = \frac{\rho a}{\rho+1}.$$

There is a decelerated increase in $\pi_\rho(X)$ as ρ increases $(\mathrm{d}^2/(\mathrm{d}\rho^2)\pi_\rho(X) < 0)$.

(3) If X has a Pareto distribution with parameters (α,λ):

$$S_X(x) = \left(\frac{\lambda}{\lambda+t}\right)^\alpha, \quad \mu_X(t) = \frac{\alpha}{\lambda+t}, \tag{5}$$

the PH transform $\Pi_\rho(X)$ also has a Pareto distribution with parameters $(\alpha/\rho,\lambda)$. When $\alpha > 1$,

$$\mathrm{E}(X) = \frac{\lambda}{\alpha-1}, \quad \pi_\rho(X) = \begin{cases} \dfrac{\lambda}{\dfrac{\alpha}{\rho}-1}, & \rho < \alpha; \\ \infty, & \rho \geq \alpha. \end{cases}$$

There is an accelerated increase in premium as ρ increases $(d^2/(d\rho^2)\pi_\rho(X) > 0)$. If we impose an upper limit ω on the Pareto loss distribution in (5), then the risk-adjusted premium is the limited expected value:

$$\mathrm{E}[X;\omega]_{(\alpha/\rho,\lambda)} = \begin{cases} \dfrac{\lambda}{\dfrac{\alpha}{\rho}-1}\left[1-\left(\dfrac{\lambda}{\lambda+\omega}\right)^{\alpha/\rho-1}\right], & \rho \neq \alpha, \\ \lambda \ln\left(1+\dfrac{\omega}{\lambda}\right), & \rho = \alpha. \end{cases}$$

(4) If X has a Weibull distribution with parameters (c,τ) (Hogg and Klugman, 1984):

$$S_X(t) = \mathrm{e}^{-ct^\tau}, \quad \mu_X(t) = c\tau t^{\tau-1},$$

the PH transform $Y = \Pi_\rho(X)$ also has a Weibull distribution with parameters $(c/\rho,\tau)$.

(5) If X has a Burr distribution with parameters (α,λ,τ) (Hogg and Klugman, 1984):

$$S_X(t) = \left(\frac{\lambda}{\lambda+t^\tau}\right)^\alpha, \quad \mu_X(t) = \frac{\alpha\tau t^{\tau-1}}{\lambda+t^\tau},$$

the PH transform $Y = \Pi_\rho(X)$ also has a Burr distribution with parameters $(\alpha/\rho,\lambda,\tau)$.

(6) If X has a discrete or non-parametric loss distribution with support $\{t_0 < t_1 < t_2 < \cdots\}$, then the PH transform $Y = \Pi_\rho(X)$ has the same support but with modified probabilities.

4. Properties of the risk-adjusted premium

In the analysis that follows, we will always assume that $\rho \geq 1$.

4.1. Positive loading and no ripoff

One can verify that

$$\mathrm{E}(X) \leq \pi_\rho(X) \leq \max(X),$$

and

$$\lim_{\rho\to 1} \pi_\rho(X) = \mathrm{E}(X), \quad \lim_{\rho\to\infty} \pi_\rho(X) = \max(X).$$

By choosing a value of ρ, $\pi_\rho(X)$ can assume any value between $\mathrm{E}(X)$ and $\max(X)$.

4.2. No unjustified risk-loading

A risk X is called *degenerate*, if there exists a constant b such that $\Pr(X = b) = 1$. When X is degenerate,

$$S_X(x) = \begin{cases} 1, & x < b; \\ 0, & b \leq x. \end{cases}$$

Therefore

$$\pi_\rho(X) = \int_0^\infty S_X(t)^{1/\rho}\,\mathrm{d}t = \int_0^b 1\,\mathrm{d}t = b.$$

4.3. Scale-invariant

For any constant $a > 0$, $\pi_\rho(aX) = a\pi_\rho(X)$.

Proof. Let $U = aX$, then $S_U(u) = S_X(u/a)$ and thus

$$\pi_\rho(U) = \int_0^\infty S_X\left(\frac{u}{a}\right)^{1/\rho} \mathrm{d}u = a\int_0^\infty S_X(t)^{1/\rho} \mathrm{d}t = a\pi_\rho(X).$$

The risk-adjusted premium is invariant under a scale change (e.g., a translation of currency). It also follows from the scale-invariant property that

$$\pi_\rho(X) = \pi_\rho(aX) + \pi_\rho((1-a)X).$$

The risk-adjusted premium is additive for quota share insurance.

4.4. Translativity

$$\pi_\rho(X+b) = \pi_\rho(X) + b.$$

Proof. Let $U = X + b$, then

$$S_U(u) = \begin{cases} 1, & u < b; \\ S_X(u-b), & b \le u. \end{cases}$$

Therefore

$$\pi_\rho(U) = \int_0^b 1 \,\mathrm{d}u + \int_b^\infty S_X(u-b)^{1/\rho} \mathrm{d}u = b + \pi_\rho(X).$$

Properties 4.2, 4.3 and 4.4 together are called linearity: $\pi_\rho(aX+b) = a\pi_\rho(X) + b$ for $a,b \ge 0$.

4.5. Sub-additivity

Theorem 1 (sub-additivity). *For any two non-negative random variables U and V without assuming independence, the following inequality holds*

$$\pi_\rho(U+V) \le \pi_\rho(U) + \pi_\rho(V), \quad \rho \ge 1.$$

A complete proof is given in the Appendix.

The sub-additivity is of importance since it implies that there is no advantage for policyholders to split a risk into pieces. In contrast, the variance principle is not sub-additive for positively correlated risks.

4.6. Extra loading for parameter uncertainty

Applying Jensen's inequality to the concave function $h(x) = x^{1/\rho}$, we have

$$\mathrm{E}_\theta h(S_X(t\,|\,\theta)) \le h(\mathrm{E}_\theta S_X(t\,|\,\theta)) = h(S_X(t)).$$

Therefore,

$$\pi_\rho(X) = \int_0^\infty h(S_X(t))\,\mathrm{d}t \ge \int_0^\infty \mathrm{E}_\theta h(S_X(t\,|\,\theta))\,\mathrm{d}t = \mathrm{E}_\theta \pi_\rho(X\,|\,\theta),$$

with strict inequality unless θ is degenerate.

As a special case, if $U \ne V$ and X is a mixture:

$$X = \begin{cases} U, & \text{with probability } a > 0; \\ V, & \text{with probability } 1-a > 0; \end{cases}$$

then for $\rho > 1$, $\pi_\rho(X) > a\pi_\rho(U) + (1-a)\pi_\rho(V)$.

Thus parameter risks can be quantified via the risk-adjusted premium.

4.7. Preserving of stochastic order

Two risks U and V are *stochastically ordered* (notation: $U \le_{st} V$) when $S_U(t) \le S_V(t)$ for all $t \ge 0$. It is straight-forward to show that the risk-adjusted premium preserves the stochastic ordering:

$$U \le_{st} V \Rightarrow \pi_\rho(U) \le \pi_\rho(V).$$

4.8. The mechanism of insurance without using utility functions

The risk-adjusted premium $\pi_\rho(X)$ is an increasing function of ρ:

$$\rho_1 > \rho_2 \ge 1 \Rightarrow \pi_{\rho_1}(X) > \pi_{\rho_2}(X).$$

Proof.

$$\rho_1 > \rho_2 \Rightarrow S_X(t)^{1/\rho_1} > S_X(t)^{1/\rho_2} \Rightarrow \pi_{\rho_1}(X) > \pi_{\rho_2}(X).$$

Table 1
Some numerical values of risk-adjusted premium $\pi_\rho(.)$

	U	V	W
$\rho_2 = 1.2$	$1.09b$	$1.2b$	$1.5b$
$\rho_1 = 1.5$	$1.2b$	$1.5b$	$3.0b$
$\rho_0 = 1.8$	$1.29b$	$1.8b$	$9.0b$

The risk-adjusted premium $\pi_\rho(X)$ reflects both the relative riskiness of the underlying risk and the risk-averse attitude of the decision-maker toward its uncertain outcome.

It is generally assumed that

$$\rho_0[\text{of policy-holder}] > \rho_1[\text{of insurer}] > \rho_2[\text{of reinsurer}].$$

Therefore, for a given risk X, $\pi_{\rho 0}(X) > \pi_{\rho 1}(X) > \pi_{\rho 2}(X)$, which explains the mechanism of the insurance business without using utility functions. The risk-adjusted premium is relatively easier to implement than the traditional utility theory.

Example 1. Consider three different loss distributions

$$S_U(t) = 1 - \frac{1}{2b}t, \quad 0 \le t \le 2b \quad \text{(uniform)}$$

$$S_V(t) = e^{-t/b} \quad \text{(exponential)}$$

$$S_W(t) = \left(\frac{b}{b+t}\right)^2 \quad \text{(Pareto)}$$

which all have an expected loss of b.

From Section 3 we have

$$\pi_\rho(U) = \frac{2\rho}{\rho+1}b, \quad \pi_\rho(V) = \rho b,$$

$$\pi_\rho(W) = \begin{cases} [\rho/(2-\rho)]b, & \rho < 2; \\ \infty, & \rho \ge 2. \end{cases}$$

Some values of risk-adjusted premium are given in Table 1 for different ρ.

5. Premium allocation among layers

Definition 3. A *layer* $(a,b]$ of a given risk X is defined by a stop-loss cover:

$$I_{(a,b]} = \begin{cases} 0, & 0 \le X < a; \\ (X-a), & a \le X < b; \\ (b-a), & b < X. \end{cases}$$

which has a survivor function

$$S_{I_{(a,b]}}(t) = \begin{cases} S_X(a+t), & 0 \le t < b-a, \\ 0, & b-a \le t. \end{cases} \tag{6}$$

Theorem 2. *When a risk X is divided into layers $\{(x_i, x_{i+1}], i = 0, 1, \cdots\}$:*

$$X = I_{(0,x_1]} + I_{(x_1,x_2]} + \cdots,$$

$$0 = x_0 < x_1 < x_2 < \cdots$$

its risk-adjusted premium is the summation of the risk-adjusted premiums of all layers:

$$\pi_\rho(X) = \sum_{i=0}^{\infty} \pi_\rho(I_{(x_i,x_{i+1}]}).$$

Proof. For a layer $(x_i, x_{i+1}]$, from (6) we have

$$\pi_\rho(I_{(x_i,x_{i+1}]}) = \int_{x_i}^{x_{i+1}} S_X(t)^{1/\rho}\,\mathrm{d}t, \quad \rho > 1.$$

Adding up the risk-adjusted premium of all the layers, we have

$$\sum_{i=0}^{\infty} \pi_\rho(I_{(x_i,x_{i+1}]}) = \int_0^\infty S_X(t)^{1/\rho}\,\mathrm{d}t = \pi_\rho(X).$$

Remark 1. Venter (1991, pp. 227–228) has shown that the only premium principles that preserve layer additivity [2] are those that can be generated from transformed distributions, where the price for any layer is the expected loss for that layer under the transformed distribution. To the contrast, risk loading based on variance or standard deviation does not satisfy layer-additivity due to the positive correlation between layers. Assume that the risk loading for X is α Var(X). By dividing X into many small layers of length ε, one can totally eliminate the risk loading:

$$\Sigma\alpha \operatorname{Var}(I_{(x,x+\varepsilon]}) \approx \Sigma\alpha S_X(x) F_X(x)\varepsilon^2 \approx 0.$$

[2] Ole Hesselager communicated to the author a more precise statement: a premium principle which is layer-additive *and* 'gives no risk load for a degenerate risk' comes from a transformed distribution.

Remark 2. Based on considerations of layer additivity, Venter (1991) advocated calculation of layer prices using the expected value of a transformed distribution. He proposed using a scale transform $Y = cX$ or a power transform $Y = X^{\eta}$. Under a scale transform, the price for a layer $(a,b]$ is the expected loss for that layer under the transformed distribution:

$$E\left[I_{(a,b]}(cX)\right] = \int_a^b S_{cX}(t)\,\mathrm{d}t = \int_a^b S_X(t/c)\,\mathrm{d}t$$

$$= \int_{a/c}^{b/c} S_X(x)\,\mathrm{d}x,$$

where the relative loading increases for higher layers under most practical circumstances. For a Pareto distribution in (5), a scale transform changes the scale parameter λ while the PH transform modifies the shape parameter α. The power transform is not recommended since it depends on the choice of monetary unit: $(aX)^{\eta} = a^{\eta}X^{\eta}$.

We have the following result regarding layer pricing using risk-adjusted premiums.

Theorem 3. (1) *Given risk X, for any $h > 0$ and $\rho > 1$,*

$$x < y \Rightarrow \pi_\rho\left(I_{(x,x+h]}\right) \geq \pi_\rho\left(I_{(y,y+h]}\right).$$

(2) *For a given risk X and $\rho_1 > \rho_2 \geq 1$, the following ratios:*

$$\phi(x;\,\rho_1,\,\rho_2) = \frac{\pi_{\rho_1}\left(I_{(x,x+\varepsilon]}\right)}{\pi_{\rho_2}\left(I_{(x,x+\varepsilon]}\right)} \quad \textit{and}$$

$$\Phi(x;\,\rho_1,\,\rho_2) = \frac{\pi_{\rho_1}\left(I_{(0,x]}\right)}{\pi_{\rho_2}\left(I_{(0,x]}\right)} \tag{7}$$

are increasing functions of x.

Proof. (1) For any two layers $(x,x+h]$ and $(y,x+h]$,

$$\pi_\rho\left(I_{(x,x+h]}\right) = \int_0^h S_X(x+t)^{1/\rho}\,\mathrm{d}t,$$

$$\pi_\rho\left(I_{(y,y+h]}\right) = \int_0^h S_X(y+t)^{1/\rho}\,\mathrm{d}t.$$

If $x < y$, then $S_X(x+t) \geq S_X(y+t)$ and thus $\pi_\rho(I_{(x,x+h]}) \geq \pi_\rho(I_{(y,y+h]})$.

(2) When $\rho_1 > \rho_2 \geq 1$, the ratio

$$\phi(x;\,\rho_1,\,\rho_2) = \frac{\pi_{\rho_1}\left(I_{(x,x+\varepsilon]}\right)}{\pi_{\rho_2}\left(I_{(x,x+\varepsilon]}\right)} = \frac{S_X(x)^{1/\rho_1}\varepsilon}{S_X(x)^{1/\rho_2}\varepsilon}$$

$$= S_X(x)^{(1/\rho_1)-(1/\rho_2)} \tag{8}$$

is an increasing function of x.

Since $\Phi(x;\rho_1,\rho_2) \leq \phi(x;\rho_1,\rho_2)$ and

$$\Phi(x+\varepsilon;\,\rho_1,\,\rho_2) = \frac{\pi_{\rho_1}\left(I_{(0,x]}\right) + \pi_{\rho_1}\left(I_{(x,x+\varepsilon]}\right)}{\pi_{\rho_2}\left(I_{(0,x]}\right) + \pi_{\rho_2}\left(I_{(x,x+\varepsilon]}\right)},$$

we infer that $\Phi(x;\rho_1,\rho_2)$ is also an increasing function of x.

Remark 3. It is noted that when $x < y$, the loss for the layer $(y,y+h]$ cannot exceed that for the layer $(x,x+h]$. The risk-adjusted premium preserves this stochastic ordering. By contrast, the variance and standard deviation principles do not always preserve this stochastic order (e.g. Kaas et al., 1994, p. 17).

Remark 4. Using risk-adjusted premium with index ρ, the relative risk loadings with respect to the expected loss

$$\phi(x;\,\rho,\,1) = \frac{\pi_\rho\left(I_{(x,x+\varepsilon]}\right)}{E\left(I_{(x,x+\varepsilon]}\right)}$$

and

$$\Phi(x;\,\rho,\,1) = \frac{\pi_\rho\left(I_{(0,x]}\right)}{E\left(I_{(0,x]}\right)}$$

are increasing functions of x. This satisfies the empirical observed constraint (Venter, 1991, p. 224): '*a premium calculation principle should produce a higher loading, relative to expected losses, for an excess of loss cover than for a primary cover on the same risk.*'

6. Risk-reward trade-off and insurer's preference

An insurer's index ρ is determined by its aversion to risks. An insurer should apply the same index ρ to all the insured risks. Using one index

ρ, the risk-adjusted premium automatically adjusts the risk loading for different loss distributions. It is superior to the risk-loading method by using variance or higher moments (e.g. Ramsay, 1994).

Example 2. Consider the following two loss distributions:

$$S_U(t) = \begin{cases} \frac{1}{4}, & 0 \le t < 4, \\ 0, & 4 < t. \end{cases} \qquad \text{(two-point)}$$

$$S_V(t) = \left(\frac{2}{2+t}\right)^3 \quad \text{(Pareto)}$$

where both have the same mean(= 1) and same variance(= 3).

One can easily verify that

$$\pi_\rho(U) = 4^{1-1/\rho}, \quad \pi_\rho(V) = \begin{cases} \dfrac{2\rho}{3-\rho}, & \rho < 3; \\ \infty, & \rho \ge 3. \end{cases}$$

For any $\rho > 1$, the risk-adjusted premium always assigns a higher premium for the Pareto risk (see Fig. 1).

It is argued that, for any two risks U and V, the insurer has no preference between insuring any one of them provided that U and V are rewarded at the insurer's risk-adjusted premiums, $\pi_\rho(U)$ and $\pi_\rho(V)$, respectively. Sometimes, due to market conditions, risks may not be rewarded exactly at their risk-adjusted premiums. In which case, an insurer's preference can be described by the following risk-reward trade-off relationship:

Definition 4. Assume that two risks U and V can earn premiums $P(U)$ and $P(V)$, respectively. An insurer with index ρ is *consistent* if the following conditions are met:

- When

$$\frac{P(U)}{P(V)} = \frac{\pi_\rho(U)}{\pi_\rho(V)},$$

the insurer has no preference between insuring U or V.

- When

$$\frac{P(U)}{P(V)} > \frac{\pi_\rho(U)}{\pi_\rho(V)},$$

the insurer prefers insuring U to insuring V.

Given a risk X associated with a premium $P(X)$, to an insurer with index ρ, a *fair* premium allocation of $P(X)$ to the layer $(x, x+\varepsilon]$ should be

$$a\pi_\rho(I_{(x,x+\varepsilon]}), \quad \text{where} \quad a = \frac{P(X)}{\pi_\rho(X)}.$$

Under this allocation, the insurer has no preference to having any particular layer.

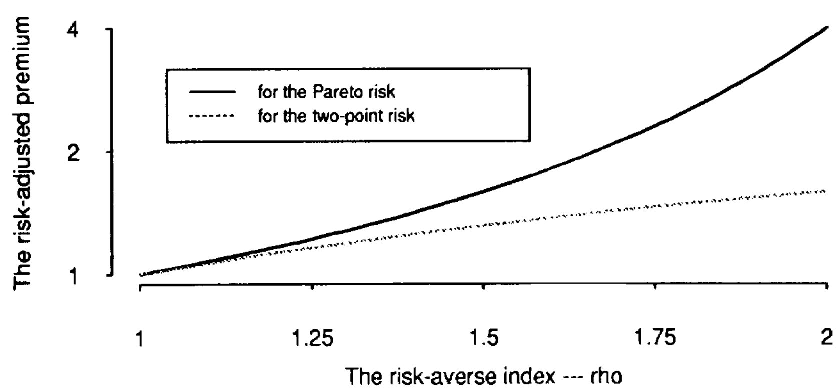

Fig. 1. Risk-adjusted premium for two risks with the same mean and variance.

7. Optimal reinsurance

Now consider the case that two parties share risk X as well as its associated premium $P(X)$. One party is the primary (or ceding) company who has an index ρ_1. The other is the reinsurer who has an index ρ_2. It is assumed that $\rho_1 > \rho_2 \geq 1$.

Assume that

$$P(X) = (1+\theta)\pi_{\rho_1}(X),$$

where θ is the expense loading factor of the primary company. We assume that the expense loading of the primary company is unaffected by obtaining reinsurance, and thus excluded in later discussions. Excluding expenses, the primary company requires a risk-adjusted premium $\pi_{\rho 1}(X)$ for insuring risk X.

Since $\pi_{\rho 1}(X) - \pi_{\rho 2}(X) > 0$, the reinsurer eyes this difference as extra profit and wants to have a share of this extra profit by providing reinsurance. There are also additional expenses associated with the reinsurance service. Thus we assume that, for any layer $(d,\omega]$, the reinsurer charges a (gross) premium at

$$C\pi_{\rho_2}(I_{(d,\omega]}), \quad C > 1,$$

where C is called the *pricing factor* of the reinsurer.

For each layer $(x,x+\varepsilon]$, the primary company and the reinsurer require a premium at $\pi_{\rho_1}(I_{(x,x+\varepsilon]})$ and $C\,\pi_{\rho 2}(I_{(x,x+\varepsilon]})$, respectively. From Theorem 3, their ratio

$$\frac{\pi_{\rho_1}(I_{(x,x+\varepsilon]})}{C\pi_{\rho_2}(I_{(x,x+\varepsilon]})} = \frac{\phi(x;\,\rho_1,\,\rho_2)}{C}$$

is an increasing function of x, which implies the following:

(i) under the ceding company's fair premium allocation where $\pi_{\rho_1}(I_{(x,x+\varepsilon]})$ is assigned to each layer $(x,x+\varepsilon]$, the reinsurer always prefers upper layers to lower layers;

(ii) under the reinsurer's fair premium allocation where $C\,\pi_{\rho 2}(I_{(x,x+\varepsilon]})$ is assigned to each layer $(x,x+\varepsilon]$, the ceding company always prefers lower layers to upper layers.

Therefore, an optimal cooperation is naturally a stop-loss reinsurance: the ceding company retain a lower layer $(0,d]$ and cede the upper layer $(d,\omega]$ to the reinsurer, where the dividing point d is called the *retention*.

Theorem 4. *Under the optimal cooperation, the retention d and the reinsurer's pricing factor C satisfy the following condition*:

$$\phi(d;\,\rho_1,\,\rho_2) = C. \tag{9}$$

(1) *If the reinsurer has a pricing factor C, the optimal retention for the ceding company is $d = \phi^{-1}(C;\rho_1,\rho_2)$.*

(2.) *If the ceding company proposes a retention d, the optimal pricing factor for the reinsurer is $C = \phi(d;\rho_1,\rho_2)$.*

Proof. Case 1: If the reinsurer has a pricing factor C and $d = \phi^{-1}(C;\rho_1,\rho_2)$, for any $\delta > 0$,

$$\pi_{\rho_1}(I_{(d-\delta,d]}) < C\pi_{\rho_2}(I_{(d-\delta,d]}),$$

$$\pi_{\rho_1}(I_{(d,d+\delta]}) > C\pi_{\rho_2}(I_{(d,d+\delta]}).$$

It would require a higher premium for the ceding company to cede the layer $(d-\delta,d]$ or to retain the layer $(d,d+\delta]$. By choosing a retention of $d = \phi^{-1}(C;\rho_1,\rho_2)$, the total required premium for risk X is minimal.

Case 2: Assume that the ceding company proposes a retention d based on its needs. If $C > \phi(d;\rho_1,\rho_2)$, there exists $\delta > 0$ such that $\phi(d+\delta;\rho_1,\rho_2) = C$. For the layer $(d,d+\delta]$,

$$\pi_{\rho_1}(I_{(d,d+\delta]}) < C\pi_{\rho_2}(I_{(d,d+\delta]}),$$

the reinsurer asks a premium which is higher than what the ceding company earned from the policy-holder. The ceding company would be unhappy about the reinsurance price for the layer $(d,d+\delta]$, and may try to raise the retention to $d+\delta$, which would result in a reduction of business (layer size) for the reinsurer. Therefore, given a retention d, the optimal pricing factor C for the reinsurer is $\phi(d;\rho_1,\rho_2)$, which gives the highest reinsurance price to keep the ceding company happy and not raising the retention.

Taking reinsurance into account, the ceding company can give the policy-holder a premium reduction to promote market competition. Under the optimal reinsurance cooperation, a premium reduction can be achieved at $\pi_{\rho 1}(I_{(d,\omega]}) - \phi(d;\rho_1,\rho_2)\pi_{\rho 2}(I_{(d,\omega]})$.

8. Examples

Example 3. Consider a commercial liability insurance case where the underlying risk has a Pareto loss distribution with $\alpha = 2$ and $\lambda = 1{,}000$ with an upper limit of one million ($\omega = 1{,}000{,}000$). The primary company has an index $\rho_1 = 1.8$. The reinsurer has an index $\rho_2 = 1.65$ and a pricing factor $C = 1.36$.

(i) The total net expected loss is $\mathrm{E}[X;\omega] = 999$.

(ii) Without reinsurance, the primary company would charge a premium at

$$P = \pi_{\rho_1}(X) = \mathrm{E}[X;\ \omega]_{(\alpha/\rho_1,\lambda]} = 4822.$$

(iii) From Eq. (9), one gets

$$\phi(d;\ \rho_1,\ \rho_2) = 1.36 = \left(\frac{1000}{1000+d}\right)^{1/1.8-1/1.65},$$

which gives an optimal retention $d = 20{,}000$.

(iv) For the retained layer $(0,d]$, the ceding company requires a premium at:

$$\pi_{\rho_1}(I_{(0,d]}) = \mathrm{E}[X;\ d]_{(\alpha/\rho_1,\lambda)} = 2515,$$

which is 265% of the net expected loss ($= 947.8$).

(v) For the reinsurance layer $[d,\omega)$, $\pi_{\rho 2}(I_{(d,\omega]}) = 1432$, the reinsurance price is

$$C\pi_{\rho_2}(I_{(d,\omega]}) = 1948,$$

which is 38 times of expected loss ($= 51$).

(vi) With reinsurance, the primary company can give its policy-holder a premium reduction of $(4822 - 2515 - 1948) = 359$ from the original premium 4822.

By using the Panjer recursion, the risk-adjusted premium and the optimal reinsurance theory can be applied effectively to a collective risk model.

Example 4. Consider a group dental plan for a firm. The firm has set-up a self-insurance program by contributing half the cost and letting the employees pay the other half.

Suppose that the claim frequency has a negative binomial distribution:

$$p_n = \frac{(9+n)!}{9!n!}(0.1)^{10}(0.9)^n, \quad n = 0, 1, 2, \cdots$$

The claim severity distribution $f(x)$ is given below (in units of 25 dollars):

x	1	2	3	4	5
$f(x)$	0.15	0.20	0.25	0.125	0.075

x	6	7	8	9	10
$f(x)$	0.05	0.05	0.05	0.025	0.025

The total collected premium is set to be the risk-adjusted premium ($\rho_1 = 1.8$) of the aggregate claim distribution. The firm is seeking reinsurance for the layer $[400, \infty)$ from a reinsurer who has an index $\rho_2 = 1.5$.

(i) The total claim distribution can be evaluated using the Panjer recursion (e.g. Sundt and Jewell, 1981):

$$g(x) = \sum_{y=0}^{x}\left(a + b\frac{y}{x}\right)f(y)g(x-y),$$

with $a = 0.9$ and $b = 8.1$ and $g(0) = 10^{-8}$.

The distribution function $G(x)$ can also be calculated recursively

$$G(x) = G(x-1) + g(x), \quad x = 1, 2, \cdots$$

with $G(0) = g(0)$.

(ii) The total expected loss is $\mathrm{E}(X) = 333.0$.

(iii) The firm collected a total premium at

$$\pi_{1.8}(X) = \sum_{i=0}^{\infty}[1 - G(i)]^{1/1.8} = 408.36.$$

(iv) For the retained layer $(0, 400]$, the required premium by the firm is 343.49, which implies a loading of 10% over the net expected loss 311.94.

(v) For the ceded layer (400, ∞), the optimal pricing factor is

$$C = [1 - G(400)]^{1/1.8 - 1/1.5} = 1.165,$$

which gives the maximum reinsurance pricing at

$$1.165\pi_{1.5}(I_{(400,\infty]}) = 49.12,$$

and implies a loading of 133% over the net expected loss 21.06.

(vi) Through optimal reinsurance cooperation, the firm can give a reduction in the total premium contribution by (408.36 − 343.49 − 49.12) = 15.75.

9. Competitive market premium with reinsurance

For catastrophic insurance, if the saving through reinsurance is transferred back to the policy-holders in a competitive market, one can modify the risk-adjusted premium according to the optimal reinsurance cooperation.

Assume that, in a competitive market, the insurer has an index ρ_1, while the reinsurer has an index ρ_2. Under the optimal cooperation for a given risk X, the retention d and the reinsurer's pricing factor C satisfy the equation:

$$\phi(d;\, \rho_1,\, \rho_2) = S_X(d)^{1/\rho_1 - 1/\rho_2} = C.$$

Definition 5. The *competitive market* premium for risk X is defined as

$$\pi_{cm}(X) = \pi_{\rho_1}(I_{(0,d]}) + C\pi_{\rho_2}(I_{(d,\omega]}).$$

For an infinitesimal layer $(x, x+\varepsilon]$,

$$\pi_{cm}(I_{(x,x+\varepsilon]}) = \begin{cases} \pi_{\rho_1}(I_{(x,x+\varepsilon]}), & x < d; \\ C\pi_{\rho_2}(I_{(x,x+\varepsilon]}), & d \le x. \end{cases}$$

One can show that

(1) $\pi_{cm}(aX) = a\pi_{cm}(X)$, for $a > 0$;

(2) $\pi_{cm}(I_{(x,x+\delta]}) \ge \pi_{cm}(I_{(y,y+\delta]})$, for $x < y$, $\delta > 0$;

(3) $\dfrac{\pi_{cm}(I_{(x,x+\delta]})}{\mathrm{E}(I_{(x,x+\delta]})} \le \dfrac{\pi_{cm}(I_{(y,y+\delta]})}{\mathrm{E}(I_{(y,y+\delta]})}$, for $x < y$, $\delta > 0$.

10. Increased limits ratemaking

In commercial liability insurance, a policy generally covers a loss up to a specified maximum dollar amount that will be paid to any one loss.

It is general practice to publish rates for some standard limit called the basic limit (usually \$25,000). Increased limits rates are calculated using a multiple factor, called the increased limits factors (ILFs). Without risk loading, the increased limits factor is the expected loss at the increased limit divided by the expected loss at the basic limit. The increased limits factor with risk loading is the summation of expected loss and risk loading at the increased limit divided by the summation of expected loss and risk loading at the basic limit:

$$ILF(\omega) = \frac{\mathrm{E}[X;\omega] + RL_{(0,\omega]}}{\mathrm{E}[X;25000] + RL_{(0,25000]}}.$$

The ILFs should satisfy the following conditions (Meyers, 1991; Robbin, 1992):

(1) The relative loading with respect to the expected loss is higher for increased limits;
(2) ILFs should produce the same price under any arbitrary division of layers;
(3) The ILFs should exhibit a pattern of declining marginal increases as the limit of coverage is raised. In other words, when $x < y$,

$$ILF(x+h) - ILF(x) \ge ILF(y+h) - ILF(y).$$

In the U.S., most companies use the Insurance Service Office (ISO) published ILFs. Traditionally, when producing ILFs, ISO assumed a Pareto loss distribution and used variance-based risk loading method. Some authors raised serious questions about the variance-based risk loading method for the calculation of ILFs (Robbin, 1992). Using a variance-based risk loading, a customer can eliminate the risk loading by dividing a layer into a package of sub-layers, which violates condition (2). Since variance-based risk loading does not satisfy condition (3), consistency tests were needed for the produced ILFs.

Here we use the risk-adjusted premium of this paper to produce ILFs.

Assume that an 'average' insurer has an index ρ_1. Each layer $(d,\omega]$ is priced at the risk-adjusted premium $\pi_{\rho 1}(I_{(d,\omega]})$. The risk loading for the layer $(d,\omega]$ is

$$RL_{(d,\omega]} = \pi_{\rho_1}(I_{(d,\omega]}) - \mathrm{E}(I_{(d,\omega]}).$$

The ILFs with risk loading is then the ratio

$$ILF(\omega) = \frac{\pi_{\rho_1}(I_{(0,\omega]})}{\pi_{\rho_1}(I_{(0,25000]})}.$$

For very high limits, to take account of reinsurance, the competitive market premium can be used:

$$RL_{(d,\omega]} = \pi_{cm}(I_{(d,\omega]}) - \mathrm{E}(I_{(d,\omega]}),$$

$$ILF(\omega) = \frac{\pi_{cm}(I_{(0,\omega]})}{\pi_{cm}(I_{(0,25000]})}.$$

One can verify that the ILFs produced by using the risk-adjusted premium or the competitive market premium always satisfy the consistency conditions (1–3).

To illustrate, assuming a Pareto loss X with survivor function:

$$S_X(t) = \left(\frac{5000}{5000+t}\right)^{1.2},$$

and an index $\rho_1 = 1.5$, one can easily produce ILFs as in Table 2.

Table 2
Risk-loadings and ILFs produced by risk-adjusted premiums (the numbers between parenthesis are based on the reinsurance prices for the layer between 100,000 and 1,000,000)

Policy limit ω	Expected loss $\mathrm{E}[X;\omega]$	ILF without RL	Risk load	ILF with RL
25,000	6,364	1.00	4,410	1.00
50,000	8,961	1.41	6,422	1.43
75,000	10,282	1.62	8,245	1.72
100,000	11,142	1.74	9,820	1.95
250,000	13,523	2.12	16,364	2.78
			(15,880)	(2.73)
500,000	15,028	2.36	22,895	3.52
			(21,280)	(3.37)
750,000	15,811	2.48	27,384	4.01
			(24,734)	(3.76)
1,000,000	16,327	2.56	30,885	4.38
			(27,312)	(4.05)

If the reinsurance market has an index $\rho_2 = 1.3$ and a pricing factor $C = 1.455$, the optimal retention d can be solved from (9):

$$\left[\left(\frac{5000}{5000+d}\right)^{1.2}\right]^{(1/1.5-1/1.3)} = 1.455 \Rightarrow d = 100{,}000.$$

For any layer $(x,x+\delta]$ above 100,000, the competitive market premium is the reinsurance price:

$$\pi_{cm}(I_{(x,x+\delta]}) = 1.455\pi_{1.3}(I_{(x,x+\delta]}).$$

In Table 2, the revised ILFs by using the competitive market premium are given between parenthesis.

The risk-adjusted premium provides a simple way to produce consistent ILFs. In practice, an individual insurer can easily adapt the published ILFs to its own use.

11. Conclusion

Based on the PH transform, this paper proposes a method to calculate risk-adjusted premiums, which provides an alternative to many existing premium principles (Goovaerts et al., 1984; van Heerwaarden and Kaas, 1992). It can be used to equalize the risk loading for different risk classes or different lines of insurance.

The risk-adjusted premium is applied to premium allocation among insurance layers. Based on a preference argument, an optimal reinsurance cooperation is identified. It may provide a theoretical basis for the negotiation process between the reinsurer and the ceding company. The method presented in this paper can be extended to the case where more than two companies split the risk.

The risk-adjusted premium based on the PH transform provides the casualty actuaries a simple and consistent method to calculate increased limits factors.

Further research is needed to investigate the properties of the PH transform and the risk-adjusted premium. Empirical studies of the consistency of insurers in charging premiums is also an area to work on.

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Appendix: A proof of Theorem 1

Lemma 1. *If* $0 < a < b$ *and* $\rho > 1$, *then for any* $x > 0$,

$$(b+x)^{1/\rho} - (a+x)^{1/\rho} < b^{1/\rho} - a^{1/\rho}.$$

Proof. Let $g(x) = (b+x)^{1/\rho} - (a+x)^{1/\rho}$. For $x > 0$,

$$g'(x) = \frac{1}{\rho}\left[(b+x)^{1/\rho-1} - (a+x)^{1/\rho-1}\right] < 0.$$

In an earlier draft the author proved Theorem 1 for independent variables. The following simplified and generalized proof for non-independent risks is attributed to Ole Hesselager.

Proof of Theorem 1. We prove the result for arbitrary V, and U a discrete variable taking values in $\{0, \dots, n\}$. By the translation invariance it then also holds for $U \in \{k, \dots, n+k\}$ and by the scale invariance for $U \in \{kh, \dots, (n+k)h\}$, $h > 0$. Any random variable can be approximated arbitrarily close by a discrete variable with small span h.

Mathematical induction:

(i) $n = 0$: $U = 0$ a.s. Trivial.

(ii) $n \to n+1$. For (U,V) with $U \in \{0, 1, \dots, n+1\}$, let (U^*, V^*) be distributed as $(U,V \mid U > 0)$. Since $U^* \in \{1, \cdots, n+1\}$, the induction hypothesis states that $\pi_\rho(U^* + V^*) \le \pi_\rho(U^*) + \pi_\rho(V^*)$.

With $w_0 = \Pr(U = 0)$ and $S_{V|0}(t) = \Pr(V > t \mid U = 0)$, we then have for $t > 0$ that

$$S_U(t) = (1 - w_0) S_{U^*}(t),$$

$$S_V(t) = w_0 S_{V|0}(t) + (1 - w_0) S_{V^*}(t),$$

$$S_{U+V}(t) = w_0 S_{V|0}(t) + (1 - w_0) S_{U^*+V^*}(t).$$

This yields (according to Lemma 1) for $t > 0$ that

$$\begin{aligned}
&S_{U+V}(t)^{1/\rho} - S_U(t)^{1/\rho} - S_V(t)^{1/\rho} \\
&\quad = \left[w_0 S_{V|0}(t) + (1 - w_0) S_{U^*+V^*}(t)\right]^{1/\rho} \\
&\qquad - \left[(1 - w_0) S_{U^*}(t)\right]^{1/\rho} \\
&\qquad - \left[w_0 S_{V|0}(t) + (1 - w_0) S_{V^*}(t)\right]^{1/\rho} \\
&\quad \le (1 - w_0)^{1/\rho} \\
&\qquad \times \left[S_{U^*+V^*}(t)^{1/\rho} - S_{U^*}(t)^{1/\rho} - S_{V^*}(t)^{1/\rho}\right].
\end{aligned}$$

Integration over t on both sides of this inequality and using the induction assumption yields that $\pi_\rho(W + V) \le \pi_\rho(W) + \pi_\rho(V)$.

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