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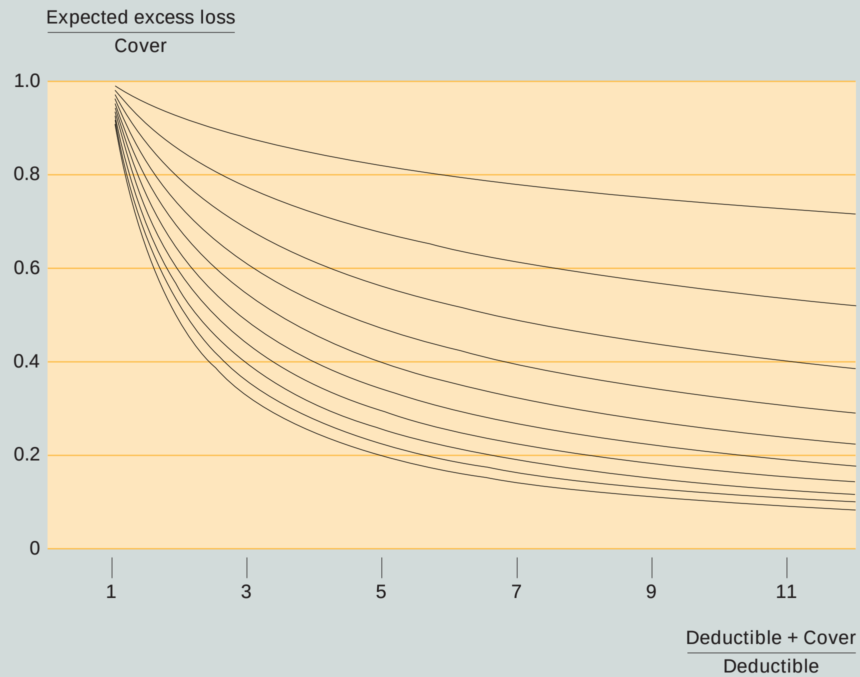


The Pareto model in property reinsurance

Formulas and applications

The Pareto model in property reinsurance

Formulas and applications

with reference to other loss distribution functions

Markus Schmutz/Richard R. Doerr

Contents

Foreword

Towards the end of the seventies, Swiss Re first issued two publications on the Pareto distribution. In the first of these brochures,[1] Hans Schmitter introduced the reader to the Pareto model in a clear, graphic manner. Here, the emphasis was not on formulas, but on a simple, easily comprehensible derivation of the model. The second brochure,[2] by Richard Doerr, was a sequel to the first, and consisted mainly of a collection of useful formulas and applied examples. This brochure was helpful mainly to those seeking to use the Pareto model with the aid of a pocket calculator or personal computer. Recently, the first brochure was revised and re-issued under the title "Estimating property excess of loss risk premiums by means of the Pareto model."

Analogously, the present publication is a revision of the second Pareto brochure. It is in large measure based on the original, but several things have been added (eg additional practical examples). The abbreviations and definitions have been harmonized with the other publication. A new feature is the electronic spreadsheets on diskette (or available from within Swiss Re via Lotus Notes) that will simplify the reader's experiments with the most important formulas.

Markus Schmutz, April 1998

1 "Property Excess Loss Rating by means of the Pareto Model", Swiss Re 1978
2 "Property Excess of Loss: Pareto Rating", Swiss Re 1980

1 Index of symbols and abbreviations

Functions	$F\ (.)$	Distribution function
	$f\ (.)$	Density function
	$E\ (.)$	Expected value
	$Var\ (.)$	Variance
	$P\ (.)$	Probability
Random variables	X	Size of losses
	N	Number of losses
	Z	Loss burden
	α	Pareto parameter (alpha)
	$\hat{\alpha}$	Estimation of α
Key figures of the cover	CO	Cover
	DE	Deductible
	OP	Observation point
	FQ	Frequency
	EL	Expected excess loss
	RP	Risk premium
	PR	Premium
Derived quantities	$EP = CO + DE$	Cover plus deductible (exit point)
	$RL = \frac{CO + DE}{DE}$	Relative length of the layer
	$GLM = \sqrt{DE \cdot (DE + CO)}$	Geometrical layer mid-point
	$ROL = \frac{PR}{CO}$	Rate on line
	$RROL = \frac{RP}{CO}$	Risk rate on line
Other	ln	Natural logarithm (available from scientific pocket calculators and in Lotus or Excel)
	XL	Excess of loss
	CatXL	Catastrophe XL
	WXL	Working XL: excess of loss exposed by single risks (loss occurrence can be defined per risk or per event)
	Aggregates	Total of the insured values

2 Mathematical basis

Let us consider the single losses in a given portfolio during a given period (the period of a treaty is usually one year). As we want to calculate premiums for XL treaties, we may limit our attention to the losses above a certain amount, the “observation point” (OP). Of course, the OP must be lower than the deductible of the layer for which we wish to calculate the premium. The amount of these single losses is random, or, as the mathematicians put it, a random variable.

Losses above this OP are Pareto-distributed if:

The probability that a loss X will fall in the very small interval $x \leq X < x + dx$ is equal to $\alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1} dx$.

2.1 Density function

The density function of the loss distribution is then:

$$f(x) = \alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1}$$

The sum of the probabilities over all possible loss amounts is of course 1. As a formula, this can be written as follows:

$$\int_{OP}^{\infty} \alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1} dx = 1$$ (The integral of the density function is equal to 1.)

2.2 Pareto parameter

α (alpha) is what we call the “Pareto parameter”. This parameter characterizes the portfolio’s loss pattern. In 4.1, you will find typical values for α for different lines of business.

2.3 Distribution function

The distribution function of the Pareto distribution is:

$$F(x) = \int_{OP}^{x} \alpha \cdot OP^{\alpha} \cdot y^{-\alpha-1} dy = 1 - \left(\frac{OP}{x}\right)^{\alpha}$$

This formula gives the probability that the loss is smaller than or equal to x. As a formula, we can write: $F(x) = P(X \leq x)$.

2.4 Expected value, variance

The expected value of the Pareto distribution is the average expected loss. The formula for this is:

$$E(X) = \int_{OP}^{\infty} x \cdot \alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1} dx = OP \cdot \frac{\alpha}{\alpha - 1} \quad \text{for } \alpha > 1$$

For $\alpha \leq 1$, the Pareto distribution has an infinite expected value.

The variance is an indication of how strongly the losses vary from the expected value. The variance of the Pareto distribution can be calculated as follows:

$$\mathrm{Var}(X) = OP^2 \cdot \frac{\alpha}{(\alpha-1)^2 \cdot (\alpha-2)} \quad \text{for } \alpha > 2$$

For $\alpha \leq 2$, the Pareto distribution has an infinite variance.

2.5 Calculating the risk premium for an XL layer

In order to calculate the premium of an XL layer CO xs DE, we proceed as follows: First, we calculate the expected value of a single loss in the layer (ie the expected excess loss). This is then multiplied by the expected number of losses per year (ie the frequency) in that layer. The result is the total expected annual loss burden for the layer; in other words, the risk premium. As we did not limit the frequency, the risk premium calculated in this manner applies to an infinite number of reinstatements.

Expected excess loss

For losses x between DE and EP, the excess loss is equal to $x-DE$. For losses greater than EP, it is equal to the cover CO. Thus, we may calculate the expected excess loss as:

$$EL = \int_{DE}^{EP} (x-DE) \cdot \alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1}\,dx + \int_{EP}^{\infty} CO \cdot \alpha \cdot OP^{\alpha} \cdot x^{-\alpha-1}\,dx$$

This gives us

$$\frac{DE}{1-\alpha} \cdot (RL^{1-\alpha} - 1) \quad \text{for } \alpha \neq 1\text{, or}$$

$$DE \cdot \ln(RL) \quad \text{for } \alpha = 1$$

where RL is the relative length of the layer, defined as

$$\frac{\text{cover + deductible}}{\text{cover}}$$

ln is the natural logarithm. All scientific calculators and spreadsheet programs such as Lotus or Excel have this function.

Frequency

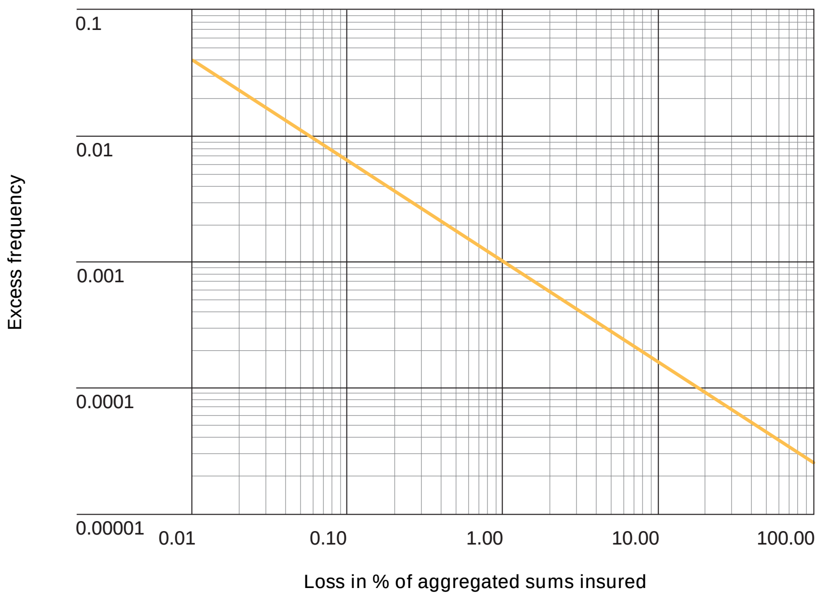
We can assume that the frequency is known at OP. For a given portfolio with loss experience, we should set OP low enough to have a sufficient number of losses to give a reasonable estimation of this frequency.

For the frequency at an arbitrary point x (where $x \geq OP$), the following applies:

$$FQ(x) = FQ(OP) \cdot P(X > x) = FQ(OP) \cdot (1-F(x)) = FQ(OP) \cdot \left(\frac{OP}{x}\right)^{\alpha}$$

Loss-frequency curve

The loss-frequency curve is a commonly used concept in catastrophe reinsurance. This curve shows, for any loss level, the average frequency with which that level will be exceeded. For the Pareto distribution, the loss-frequency curve can be computed directly with the aid of the formula in the previous section.



Example: Loss-frequency curve of a Pareto distribution where OP = 0.01%, FQ(OP) = 0.04, and $\alpha = 0.8$. Note the log-log scale, chosen to make it easier to read the graph. Plotted this way, the loss-frequency curve of a Pareto distribution is always a straight line.

In setting up a CatXL programme, the loss value that is equalled or exceeded once a century on the average, or "hundred-year loss", is frequently used as a reference. To read the hundred-year loss in the loss-frequency curve, we start on the Y axis at 0.01 (1 divided by 100). The corresponding value on the X axis is approximately 0.06%. In other words, in this example, the hundred-year loss will be 0.06% of the aggregated sums insured.

Risk premium

The risk premium for our layer can now be calculated as follows:

$$RP = FQ(DE) \cdot EL$$

$$= FQ(OP) \cdot \left(\frac{OP}{DE}\right)^{\alpha} \cdot \frac{DE}{(1-\alpha)} \cdot (RL^{1-\alpha} - 1) \quad \text{for } \alpha \neq 1$$

$$= FQ(OP) \cdot \frac{OP}{DE} \cdot DE \cdot \ln(RL) = FQ(OP) \cdot OP \cdot \ln(RL) \quad \text{for } \alpha = 1$$

3 Useful formulas

This chapter contains a collection of the formulas that are most commonly used to solve problems in practice. Some of the formulas included are taken from Chapter 2, while others are transformations of those formulas, and some are new. We will dispense with the derivations for reasons of simplicity.

For ease of use, the most important formulas have been programmed in Lotus and Excel spreadsheets. These spreadsheets are included on the accompanying diskette (brochure for distribution outside Swiss Re) and in Lotus Notes[3] for internal distribution. The corresponding file name is given for each of the formulas below.

3.1 Solving for frequency

3.1.1 Given: frequency at OP, OP, α

Solving for: Frequency at an arbitrary point $x \geq OP$

Formula: $FQ(x) = FQ(OP) \cdot \left(\frac{OP}{x}\right)^{\alpha}$

Worksheet: FQ2_FQ1

3.1.2 Given: frequency at another deductible DE_1, DE_1, α

Solving for: Frequency at deductible DE_2

Formula: $FQ(DE_2) = \left(\frac{DE_1}{DE_2}\right)^{\alpha} \cdot FQ(DE_1) = \frac{1}{r^{\alpha}} \cdot FQ(DE_1)$

where $r = \frac{DE_2}{DE_1}$

Worksheet: FQ2_FQ1

This formula applies for any observation point. It also makes no difference whether DE_1 is smaller than DE_2 or the inverse.

3.1.3 Given: risk premium, cover, deductible, α

Solving for: Penetration frequency FQ(DE) for a layer

Formula for $\alpha \neq 1$: $FQ(DE) = \frac{RP \cdot (1-\alpha)}{DE \cdot (RL^{1-\alpha} - 1)}$

Formula for $\alpha = 1$: $FQ(DE) = \frac{RP}{DE \cdot \ln(RL)}$

Worksheet: FQ_RP

3 Under “Groupwide Interest Discussions/Resources/By Audience/Non-life/property insurance/The Pareto Model in Property Reinsurance: Worksheets”

3.1.4 Given: risk premium, cover, deductible, α

Solving for	Exit frequency FQ(EP) for a layer
Formula for $\alpha \neq 1$	$FQ(EP) = \frac{RP \cdot (1-\alpha) \cdot RL^{1-\alpha}}{EP \cdot (RL^{1-\alpha}-1)}$
Formula for $\alpha=1$	$FQ(EP) = \frac{RP}{EP \cdot \ln(RL)}$
Worksheet	FQ_RP

3.2 Solving for expected excess loss

3.2.1 Given: cover, deductible, α

Solving for	Expected excess loss EL
Formula for $\alpha \neq 1$	$EL = \frac{DE}{1-\alpha} \cdot (RL^{1-\alpha}-1)$
Formula for $\alpha=1$	$EL = DE \cdot \ln(RL)$
Worksheet	EL_LAY

3.3 Solving for risk premium

3.3.1 General formula

The risk premium is calculated as the product of the expected excess loss and the frequency at the deductible.

$$RP = FQ(DE) \cdot EL$$

3.3.2 Given: cover, deductible, frequency at OP, OP, α

Solving for	Risk premium RP
Formula for $\alpha \neq 1$	$RP = FQ(OP) \cdot OP^{\alpha} \cdot \frac{DE^{1-\alpha}}{1-\alpha} \cdot (RL^{1-\alpha}-1)$
Formula for $\alpha=1$	$RP = FQ(OP) \cdot OP \cdot \ln(RL)$
Worksheet	RP_FQLAY

3.3.3 Given: cover, deductible, frequency at DE, α

Solving for	Risk premium RP
Formula for $\alpha \neq 1$	$RP = FQ(DE) \cdot \frac{DE}{1-\alpha} \cdot (RL^{1-\alpha}-1)$
Formula for $\alpha=1$	$RP = FQ(DE) \cdot DE \cdot \ln(RL)$
Worksheet	RP_FQLAY

3.3.4 Given: risk premium RP_1 of a layer (1) in a programme, CO_1, DE_1, CO_2, DE_2, α

Solving for: Risk premium RP_2 of a layer (2) in the same programme

Formula for $\alpha \neq 1$: $RP_2 = r^{1-\alpha} \cdot \frac{RL_2^{1-\alpha}-1}{RL_1^{1-\alpha}-1} \cdot RP_1$ where $r = \frac{DE_2}{DE_1}$

Formula for $\alpha=1$: $RP_2 = \frac{\ln(RL_2)}{\ln(RL_1)} \cdot RP_1$

Worksheet: RP2_RP1

3.4 Solving for risk rate on line (RROL)

The risk rate on line is calculated as the risk premium divided by the length of the layer (CO). It can be written as [penetration frequency FQ(DE)] times [expected excess loss EL divided by cover CO]. The term “expected excess loss divided by cover” is equivalent to the risk rate on line when the penetration frequency is equal to 1. This value is designated as RROL*.

3.4.1 Given: cover, deductible, FQ(OP), OP, α

Solving for: Risk rate on line RROL

Formula for $\alpha \neq 1$: $RROL = \frac{FQ(OP) \cdot OP^{\alpha} \cdot (RL^{1-\alpha}-1)}{DE^{\alpha} \cdot (RL-1) \cdot (1-\alpha)}$

Formula for $\alpha=1$: $RROL = \frac{FQ(OP) \cdot OP \cdot \ln(RL)}{DE \cdot (RL-1)}$

Worksheet: RP_FQLAY

3.4.2 Given: cover, deductible, FQ(DE), α

Solving for: Risk rate on line RROL

Formula for $\alpha \neq 1$: $RROL = FQ(DE) \cdot \frac{RL^{1-\alpha}-1}{(RL-1) \cdot (1-\alpha)} = FQ(DE) \cdot RROL^*$

where $RROL^* = \frac{RL^{1-\alpha}-1}{(RL-1) \cdot (1-\alpha)}$

Formula for $\alpha=1$: $RROL = FQ(DE) \cdot \frac{\ln(RL)}{RL-1} = FQ(DE) \cdot RROL^*$

where $RROL^* = \frac{\ln(RL)}{RL-1}$

Worksheet: RP_FQLAY

As RROL* is dependent only on RL and α (and not on the frequency), this value is well suited for graphical representations of Pareto rating (see Annex). RROL* is a suitable base value for quick calculations of risk rates on line at different penetration frequencies (for example in an Excel or Lotus worksheet).

3.4.3 Given: Risk rate on line $RROL_1$ of a layer (1) in a programme, CO_1, DE_1, CO_2, DE_2, α

Solving for: Risk rate on line $RROL_2$ of a layer (2) in the same programme

Formula for $\alpha \neq 1$: $$RROL_2 = r^{1-\alpha} \cdot \frac{RL_2^{1-\alpha} - 1}{RL_1^{1-\alpha} - 1} \cdot \frac{RROL_1 \cdot CO_1}{CO_2} \quad \text{where } r = \frac{DE_2}{DE_1}$$

Formula for $\alpha = 1$: $$RROL_2 = \frac{\ln(RL_2)}{\ln(RL_1)} \cdot \frac{RROL_1 \cdot CO_1}{CO_2}$$

Worksheet: RR2_RR1

This formula (or the corresponding worksheet) has the widest application of any of the Pareto formulas presented so far. By defining the parameters appropriately, all the problems from 3.1 to 3.3 can be solved. Risk premiums for layers are entered and output as rates on line. If the frequency at a defined point x is given or to be solved for, this point can be defined as a very small layer (0.00001 xs x, for example). The corresponding frequency is then the ROL of this small layer (cf "The geometrical layer mid-point" in section 4.7). The expected excess loss is equivalent to the risk premium for a penetration frequency of 1.

3.5 Solving for alpha

3.5.1 Given: all indexed as-if losses (X_1, X_2, X_3, ...X_n) larger than OP, OP

Solving for: α

Formula: $$\hat{\alpha} = \frac{n}{\ln \frac{X_1}{OP} + \ln \frac{X_2}{OP} + \ldots + \ln \frac{X_n}{OP}} = \frac{n}{\sum_{i=1}^{n} \ln \frac{X_i}{OP}}$$

Worksheet: A_LOSSES

This is the so-called "maximum likelihood" estimation for α. In practice, only the indexed as-if losses are given. One must select an appropriate OP.[4]

3.5.2 Given: two layers (CO_1, DE_1, CO_2, DE_2) with the corresponding risk rates on line ($RROL_1$, $RROL_2$)

Solving for: α

Formula: $$\alpha = \frac{\ln\left(\frac{RROL_2}{RROL_1}\right)}{\ln \frac{\sqrt{DE_1 \cdot (DE_1 + CO_1)}}{\sqrt{DE_2 \cdot (DE_2 + CO_2)}}}$$

Worksheet: A_RR1RR2

Note: The term $\sqrt{DE \cdot (DE + CO)}$ is known as the "geometrical layer mid-point". See Example 4.7 for details on the derivation of Formula 3.5.2 and its applications.

4 Selecting an OP: cf the Swiss Re publication "Burning Cost and Pareto in Property Insurance", 1988, Part B.

4.1 Rule-of-thumb values for alpha in property reinsurance

The graph in section 6.1.1 shows that the frequency of large losses in comparison to small losses increases as the value of α decreases. This means that α is low for large-loss orientated portfolios, and high for portfolios with few large and many small losses.

Fire (per risk)

In the range of 1.0 to 2.5. For industrial business, α is in the lower part of this range: around 1.2. For simple business, α is high, in the order of 1.8 to 2.5. In general, it can be said that α is smaller for a WXL on gross than it is for a WXL on the retention of a surplus treaty. This is because the large losses are cut off by the surplus.

Cat perils

In general, 1.0. Earthquake is more in the neighbourhood of 0.8, while European windstorm is around 1.3.

4.2 Calculating all important values for a cover

Solving for	Penetration frequency FQ(DE), expected excess loss EL and risk premium RP
Given	OP, α (rule-of-thumb value or calculated from the as-if losses using Formula 3.5.1), FQ(OP) (from loss experience)

Formulas needed

Penetration frequency (Formula 3.1.1):

$$FQ(DE) = FQ(OP) \cdot \left(\frac{OP}{DE}\right)^{\alpha}$$

Expected excess loss (Formula 3.2.1):

$$EL = \frac{DE}{1-\alpha} \cdot (RL^{1-\alpha} - 1) \quad \text{for } \alpha \neq 1$$

$$EL = DE \cdot \ln(RL) \quad \text{for } \alpha = 1$$

According to Formula 3.3.1, the risk premium is then:

$$RP = FQ(DE) \cdot EL$$

Numerical example

OP = 1 250 000
α = 0.75
FQ(OP) = 0.67
CO xs DE = 5 000 000 xs 2 000 000

Step 1: Calculate RL

$$RL = \frac{CO+DE}{DE} = \frac{5\,000\,000 + 2\,000\,000}{2\,000\,000} = \underline{3.5}$$

Step 2:
If $\alpha = 1$ calculate $\ln(RL)$
If $\alpha \neq 1$ calculate $RL^{1-\alpha}$

$RL^{1-\alpha} = 3.5^{0.25} = \underline{1.3678}$

Step 3: Calculate $\left(\frac{OP}{DE}\right)^{\alpha}$

$\left(\frac{OP}{DE}\right)^{\alpha} = \left(\frac{1\,250\,000}{2\,000\,000}\right)^{0.75} = 0.625^{0.75} = \underline{0.7029}$

Results:

$FQ(DE) = FQ(OP) \cdot \left(\frac{OP}{DE}\right)^{\alpha} = 0.67 \cdot 0.7029 = \underline{\underline{0.4710}}$

$EL = \frac{DE}{1-\alpha} \cdot (RL^{1-\alpha} - 1) = \frac{2\,000\,000}{1-0.75} \cdot (1.3678 - 1) = \underline{\underline{2\,942\,400}}$

$RP = FQ(DE) \cdot EL = 0.4710 \cdot 2\,942\,400 = \underline{\underline{1\,385\,870}}$

4.3 Calculating the penetration and exit frequencies from the risk premium

Solving for: Penetration frequency FQ(DE), exit frequency FQ(EP)

Given: RP, α (rule-of-thumb value or calculated from the as-if losses using Formula 3.5.1)

Formulas needed

Penetration frequency (Formula 3.1.3)

$$FQ(DE) = \frac{RP \cdot (1-\alpha)}{DE \cdot (RL^{1-\alpha} - 1)} \quad \text{for } \alpha \neq 1$$

$$FQ(DE) = \frac{RP}{DE \cdot \ln(RL)} \quad \text{for } \alpha = 1$$

Exit frequency (Formula 3.1.4)

$$FQ(EP) = \frac{RP \cdot (1-\alpha) \cdot RL^{1-\alpha}}{EP \cdot (RL^{1-\alpha} - 1)} \quad \text{for } \alpha \neq 1$$

$$FQ(EP) = \frac{RP}{EP \cdot \ln(RL)} \quad \text{for } \alpha = 1$$

Numerical example

$\alpha = 0.8$
RP = 500 000
CO xs DE = 9 000 000 xs 6 000 000

Step 1: Calculate RL

$RL = \frac{CO + DE}{DE} = \frac{9\,000\,000 + 6\,000\,000}{6\,000\,000} = \underline{2.5}$

Step 2:
If $\alpha = 1$ calculate $\ln(RL)$
If $\alpha \neq 1$ calculate $RL^{1-\alpha}$

$RL^{1-\alpha} = 2.5^{0.2} = \underline{1.2011}$

Results:

$$FQ(DE) = \frac{RP \cdot (1-\alpha)}{DE \cdot (RL^{1-\alpha}-1)} = \frac{500\,000 \cdot 0.2}{6\,000\,000 \cdot 0.2011} = \frac{100\,000}{1\,206\,600} = \underline{\underline{0.0829}}$$

$$FQ(EP) = \frac{RP \cdot (1-\alpha) \cdot RL^{1-\alpha}}{EP \cdot (RL^{1-\alpha}-1)} = \frac{500\,000 \cdot 0.2 \cdot 1.2011}{15\,000\,000 \cdot 0.2011} = \underline{\underline{0.0398}}$$

This example shows that rating considerations such as the following will lead to incorrect results:

"I estimate that a loss of 15 000 000 will occur about once every 25 years. For a 9 000 000 xs 6 000 000 cover, I will thus need $0.04 \cdot 9\,000\,000$ or 360 000."

The important parameters are the penetration frequency (not the exit frequency) and the expected excess loss (which, exceptions aside, is not the same as the whole cover).

The correct risk premium is:

$$EL = \frac{DE}{1-\alpha} \cdot (RL^{1-\alpha}-1) = \frac{6\,000\,000}{0.2} \cdot 0.2011 = \underline{\underline{6\,033\,000}}$$

$$RP = FQ(DE) \cdot EL = 0.0829 \cdot 6\,033\,000 = \underline{\underline{500\,135}}$$

(The difference to the 500 000 in the example is caused by rounding.)

4.4 Rating CatXL treaties with the aid of a frequency extrapolation

Rating software has been developed for many catastrophic perils (earthquakes, for example). This software is based on detailed data on the particular peril, and is able to calculate a very reliable risk premium for a given layer.

There are also perils for which there is no detailed model, either because development is not worthwhile given the amount of business, or because the necessary data is not available.

In such cases, the Pareto approach can provide assistance:

We first estimate the frequency at a certain point. For example, for floods in country XY, the frequency at 0.05% of the aggregated sums insured is 0.1. This estimate should "feel right" to the underwriter. The ideal case is when it can be supported by loss experience: "We know about how many events of this magnitude have occurred in the last 20 years." The observation point is expressed as a percentage of the total sums insured (aggregates) so that it can be adjusted to a portfolio of any size. A ten-year event will cause a loss of at least 0.05% of the aggregates in an average portfolio. For a cedent with aggregates of 2 000 000 000, a ten-year loss will therefore be at around 1 000 000, while it will amount to only 500 000 for a 1 000 000 000 portfolio.

We estimate a Pareto-α. This estimate will in most cases be based on general knowledge about the peril in question. For example, "Flood in country AB has an α of 1.1, so we will use this same value for country XY." If no information at all is available, we can use $\alpha = 1$ for CatXL treaties.

Example

Market approach: FQ(0.05% of aggregates) = 0.1, α=1
Cedent: aggregates of 2 000 000 000, cover of 1 500 000 xs 1 500 000

Calculation
OP = 0.0005 · 2 000 000 000 = 1 000 000
FQ(OP) = 0.1

Penetration frequency (Formula 3.1.1)

$$FQ(DE) = FQ(OP) \cdot \left(\frac{OP}{DE}\right)^{\alpha}$$

Expected excess loss (Formula 3.2.1)
$EL = DE \cdot \ln(RL)$ for α=1

Risk premium (Formula 3.3.1)
$RP = FQ(DE) \cdot EL$

Step 1: Calculate RL

$$RL = \frac{CO + DE}{DE} = \frac{1\,500\,000 + 1\,500\,000}{1\,500\,000} = \underline{2}$$

Step 2: Calculate ln(RL)

ln(RL) = ln(2) = <u>0.6931</u> (with the aid of a calculator)

Step 3: Calculate $\left(\frac{OP}{DE}\right)^{\alpha}$

$$\left(\frac{OP}{DE}\right)^{\alpha} = \left(\frac{1\,000\,000}{1\,500\,000}\right)^{1} = \underline{0.6667}$$

Results:

$$FQ(DE) = FQ(OP) \cdot \left(\frac{OP}{DE}\right)^{\alpha} = 0.1 \cdot 0.6667 = \underline{0.0667}$$

$$EL = DE \cdot \ln(RL) = 1\,500\,000 \cdot 0.6931 = \underline{1\,039\,650}$$

$$RP = FQ(DE) \cdot EL = 0.0667 \cdot 1\,039\,650 = \underline{\underline{69\,345}}$$

4.5 Rating with the aid of a reference layer

General

With Formula 3.3.4, it is possible to calculate the price of a layer based on a reference layer for which the price is known. This is predicated on the losses roughly following a Pareto distribution, and on our knowing the corresponding value for α. If enough loss data is available, we can determine α with Formula 3.5.1 (Example 4.6). Another alternative would be to use a rule-of-thumb value for α (Example 4.4); or we could try to find out the α contained in the price structure of the reference programme (Example 4.7).

This method, for example, can be used for a reinsurance programme for which we have enough loss data to determine the prices for the lower layers. For the upper layers, where only insufficient loss experience is available (if any at all) we can extrapolate the premiums. This method is also suitable for rating CatXLs with the help of a reference layer (see the numerical example in this section).

In fire business, extrapolations from lower layers to higher ones provide better results than the other way around. This is due to the fact that the loss experience is more extensive in the lower layers (and thus more reliable) than in the higher ones.

With CatXL treaties, extrapolations from lower layers to higher ones can lead to very low prices for the higher layers. Still, we must keep in mind that we are calculating only the risk premium here. The fluctuation loading, which of course accounts for a substantial part of the total price for high layers with a low ROL, is not considered.

Comparing or extrapolating between different layers is only meaningful when the same kind of business is involved: the same perils, WXL or CatXL, per risk or per event, similar portfolios, similar geographical spread, and similar underwriting limits.

Rating a CatXL with the aid of a market layer

The initial situation for this example is simliar to that in the previous section: a country or a peril for which no detailed data on loss frequencies is available.

Instead of a frequency at a given observation point, we can also use a market layer as a reference. For example, this may be a layer taken from the programme of the largest insurer in the market. We assume that the price for this layer is known. (In some cases, we can assume that the market will give a fair price for that layer.) Our goal is to rate the other layers in the market so as to be consistent with the reference layer. To do this, we express the reference layer as a percentage of the total sums insured (aggregates), so that it can be adjusted to the size of other companies. For the same reason, we must define the price for the reference layer as a rate on line.

Example

Reference company XY
Aggregates: 2 000 000 000
Layer: 2 000 000 xs 1 000 000
Risk premium 200 000 or 10% ROL

Thus, the reference layer is:
0.1% xs 0.05% of the aggregates cost 10% ROL
$\alpha = 1.1$ (assumption)

The layer to be rated:
Company AB
Aggregates: 1 250 000 000
Layer: 500 000 xs 500 000

Reference layer transferred to company AB:
1 250 000 xs 625 000 at a cost of 10% ROL or 125 000

The conversion of prices from one layer to another (Formula 3.3.4):

$$RP_2 = r^{1-\alpha} \cdot \frac{RL_2^{1-\alpha}-1}{RL_1^{1-\alpha}-1} \cdot RP_1 \quad \text{for values of } \alpha \neq 1 \text{ where } r = \frac{DE_2}{DE_1}$$

$$RP_2 = \frac{\ln(RL_2)}{\ln(RL_1)} \cdot RP_1 \quad \text{for } \alpha = 1$$

Step 1: Calculate RL

$$RL_1 = \frac{CO_1 + DE_1}{DE_1} = \frac{1\,250\,000 + 625\,000}{625\,000} = \underline{3}$$

$$RL_2 = \frac{CO_2 + DE_2}{DE_2} = \frac{500\,000 + 500\,000}{500\,000} = \underline{2}$$

Step 2: Calculate r

$$r = \frac{DE_2}{DE_1} = \frac{500\,000}{625\,000} = \underline{0.8}$$

Results:

$$RP_2 = r^{1-\alpha} \cdot \frac{RL_2^{1-\alpha}-1}{RL_1^{1-\alpha}-1} \cdot RP_1 = 0.8^{-0.1} \cdot \frac{2^{-0.1}-1}{3^{-0.1}-1} \cdot 200\,000 = \underline{\underline{131\,636}} \text{ or } 26\% \text{ ROL}$$

4.6 Computing an estimation for alpha using loss statistics

This is a numerical example for the application of Formula 3.5.1.

Solving for α

Given: All losses X_i above observation point OP. Of course, these losses must first be indexed and corrected to an "as-if" basis so as to reflect the conditions in the year for which the rating is required (inflation, portfolio growth, underlying reinsurance, etc[5]).

$$\text{Formula: } \hat{\alpha} = \frac{n}{\ln \frac{X_1}{OP} + \ln \frac{X_2}{OP} + \ldots + \ln \frac{X_n}{OP}} = \frac{n}{\sum_{i=1}^{n} \ln \frac{X_i}{OP}}$$

Numerical example

OP = 98 000
n = 12

X_i	$\frac{X_i}{OP}$	$\ln \frac{X_i}{OP}$
101 000	1.0306	0.03014
104 000	1.0612	0.05940
110 000	1.1224	0.11547
122 000	1.2449	0.21906
135 000	1.3776	0.32034
175 000	1.7857	0.57981
177 000	1.8061	0.59117
186 000	1.8980	0.64080
241 000	2.4592	0.89984
353 000	3.6020	1.28149
378 000	3.8571	1.34992
554 000	5.6531	1.73220
	Total	**7.81964**

$$\alpha = \frac{12}{7.81964} = \underline{\underline{1.5346}}$$

5 cf the Swiss Re publication "Burning Cost and Pareto in Property Insurance"

4.7 Determining alpha in a price structure

This section explains Formula 3.5.2.

Sometimes we receive from one source or another a reinsurance programme complete with premiums, and we would like to know whether these premiums are based on a Pareto distribution, and if so, which value was used for α. Such situations can come up as the result of:
– a client offer;
– a calculation with a rating tool.

Using a Pareto distribution to model a fairly accurate reproduction of these prices can bring the following advantages:
– quick calculation of programme variants (with Formula 3.3.4);
– application of the programme as reference for a market or a peril (see preceding section).

Aid: the geometrical layer mid-point

For every layer, the following applies: FQ(EP) < RROL < FQ(DE). Put into words, the layer's RROL lies between its penetration frequency and its exit frequency. As the frequencies decrease at increasing levels, there has to be a point x somewhere between DE and EP for which FQ(x) = RROL. This point can be approximated as

$$\mathrm{GLM} = \sqrt{\mathrm{DE} \cdot \mathrm{EP}} = \sqrt{\mathrm{DE} \cdot (\mathrm{CO} + \mathrm{DE})}$$

where GLM is the "geometrical layer mid-point". Thus,

$$\mathrm{FQ}(\mathrm{GLM}) \approx \mathrm{RROL}$$

If the premium structure is based on a Pareto distribution, the relationship applies exactly for $\alpha = 2$, and for other values of α it provides a fairly good approximation.

Derivation of the formula

Solving for α

Given: two layers (CO_1, DE_1, CO_2, DE_2) with the related risk rates on line (RROL_1, RROL_2).

The basis for the calculation is Formula 3.1.2.

$$\mathrm{FQ}(\mathrm{DE}_2) = \left(\frac{\mathrm{DE}_1}{\mathrm{DE}_2}\right)^{\alpha} \cdot \mathrm{FQ}(\mathrm{DE}_1)$$

From our considerations on GLM, we can rewrite this formula as:

$$\mathrm{RROL}_2 = \left(\frac{\mathrm{GLM}_1}{\mathrm{GLM}_2}\right)^{\alpha} \cdot \mathrm{RROL}_1$$

After several transformations, we arrive at Formula 3.5.2:

$$\alpha = \frac{\ln\left(\frac{RROL_2}{RROL_1}\right)}{\ln\left(\frac{GLM_1}{GLM_2}\right)} = \frac{\ln\left(\frac{RROL_2}{RROL_1}\right)}{\ln \frac{\sqrt{DE_1 \cdot (DE_1 + CO_1)}}{\sqrt{DE_2 \cdot (DE_2 + CO_2)}}}$$

Numerical example

First layer: 2 000 000 xs 1 000 000; RROL 20%
Second layer: 2 000 000 xs 3 000 000; RROL 8%

What is the premium for the third layer (4 000 000 xs 5 000 000)?

Step 1: Calculate the GLM

$$GLM_1 = \sqrt{1\,000\,000 \cdot (2\,000\,000 + 1\,000\,000)} = \sqrt{1\,000\,000 \cdot 3\,000\,000} = \underline{1\,732\,051}$$

$$GLM_2 = \sqrt{3\,000\,000 \cdot (2\,000\,000 + 3\,000\,000)} = \sqrt{3\,000\,000 \cdot 5\,000\,000} = \underline{3\,872\,983}$$

Step 2: Calculate α

$$\alpha = \frac{\ln\left(\frac{RROL_2}{RROL_1}\right)}{\ln\left(\frac{GLM_1}{GLM_2}\right)} = \frac{\ln\left(\frac{0.08}{0.2}\right)}{\ln\left(\frac{1\,732\,051}{3\,872\,983}\right)} = \underline{\underline{1.14}}$$

If more than two layers are given with their prices, different values for α should be calculated: first layer – second layer; first layer – third layer; first layer – fourth layer, and so on. If the resulting values for α vary greatly, this is an indication that the price structure is not based on a Pareto distribution. Provisionally, however, one can calculate a further layer (Example 4.5) using the value for α which results from the two layers nearest the layer for which one is solving. As reference layer for the price conversion, one should use the layer closest to the one being solved for.

4.8 Modelling technical premiums using the Pareto distribution

So far, we have used the Pareto distribution to model risk premiums. However, the technical premium also contains loadings for administration costs, fluctuation and uncertainty, as well as discounts for loss-dependent premium systems. Although the technical premium is no longer derived directly from a loss distribution, it is sometimes helpful to approach the problem as if the technical premium were based directly on a Pareto distribution. This is particularly useful in cases where a large number of variants must be calculated for one programme, and the risk premium contains components for many different perils. If we can find an α that will reproduce the structure of the technical premiums, we can simply rate any new variants with Formula 3.3.4 (Example 4.5).

Procedure

One variant is first rated in the conventional manner (at least two to three layers). That is, we determine the risk premium with the usual tools, add all the loadings, deduct all the discounts, and arrive at the technical premium. With Formula 3.5.2, we then solve for the values of α that correspond to these technical premiums. These will be somewhat lower than the α values that would result from the risk premiums. The reason for this is the fluctuation loading, which is a higher percentage of the premium in the upper layers than it is in the lower ones. (Typical values of α for technical premiums in CatXL treaties range between 0.5 and 0.8.) As in Example 4.5, we can then calculate the prices for additional variants on the basis of the prices for the first variant by using the value determined for α and Formula 3.3.4. The entire process can be automated with a spreadsheet.

5 Other distribution functions

5.1 Advantages and disadvantages of the Pareto model

The Pareto distribution is very serviceable in many situations. Its main advantages are the following:

- It can be used in most situations (WXL, CatXL).
- The distribution is characterized with only one parameter. This makes it easier to get a “feeling” for its influence.
- Mathematically, it is simple to handle. All problems can be calculated with the aid of relatively simple formulas.
- It is widely used, and there is a good amount of knowledge on typical parameter values for certain perils.

However, the Pareto distribution also has certain disadvantages. These include:

- The loss burden can only be modelled from a certain point $OP > 0$. In some cases, however, particularly proportional, stop-loss and multiline treaties, the distribution of losses from 0 is important.
- The Pareto distribution has only one parameter. Thus it is not very flexible when approximating as closely as possible the distribution of losses that actually occurred in a portfolio. Frequently, for example, there are real-world loss distributions in which medium-sized losses are more probable than large or small losses. This means that the distribution function of such distributions contains an inflection or turning point. The Pareto distribution cannot fulfil this requirement because the probability decreases continually as the size of the loss increases.

5.2 Other distributions

We will consider below several further distributions that, along with the Pareto distribution, play an important role in property reinsurance. They can compensate in part for the shortcomings of the Pareto distribution. For example, they have two parameters (α and γ), meaning that they are much more flexible in their application. Some are also capable of modelling the loss distribution from 0. All have one fundamental disadvantage, however: they are rather difficult to handle mathematically. Though the density functions for all distributions can be expressed by formulas, this is not possible for distribution functions and the prices for XL layers, which can only be calculated by means of numerical methods.[6] At Swiss Re, we use the rating tool “Win-NP”, a comprehensive package which, in addition to risk premiums, can calculate many other functions that are relevant for rating (fluctuation loading, discounts for cover limitations, loss-dependent premium systems). Contact us if you would like further information.

6 For further information, see the following:
- “Schadenversicherungsmathematik”, Thomas Mack, Verlag Versicherungswirtschaft E.V., Karlsruhe 1997 (*Schriftenreihe angewandte Versicherungsmathematik*, Heft 28)
- “Loss Distributions”, Robert V. Hogg, Stuart A. Klugmann, Wiley, New York 1984 (*Wiley Series in Probability and Mathematical Statistics*)

Generalized Pareto distribution

- Very flexible. Depending on the choice of parameters, it can even be a Pareto or a gamma distribution.
- No inflection point.
- Only losses above an observation point OP can be modelled.
- Well suited for modelling single risk fire losses (WXL per risk).

Gamma distribution

- With the gamma distribution, the probability of large losses is very small. This means that when extrapolating from low layers (with loss experience) to high layers (with no loss experience), we will become very cheap at higher levels. Thus, great care should be taken when using the gamma distribution for extrapolation purposes.
- However, the gamma distribution is very well suited for fairly exact approximations of the loss experience in a defined range throughout which losses have occurred (interpolation).
- Inflection point.
- Losses can be modelled from 0.
- Well suited for modelling the overall loss distribution in large portfolios (stop loss).

Log-gamma distribution

- The log-gamma distribution is a Pareto distribution of sorts with "variable" α. For values of $\gamma > 1$, the "Pareto-α" increases with increasing loss size. For values of $\gamma < 1$, it decreases with increasing loss size. For $\gamma = 1$, the log-gamma distribution is a Pareto distribution.
- Inflection point.
- Only the losses above the OP can be modelled.
- More flexible than the Pareto distribution in the lower loss range (two parameters), but very similar to Pareto in the upper loss ranges. For this reason it is also suited for modelling large losses (WXL, CatXL, stop loss).

Log-normal distribution

- The log-normal distribution returns a higher probability for large losses than the gamma distribution. For this range of losses, it lies between the gamma distribution and the log-gamma distribution.
- Inflection point.
- Losses can be modelled from 0.
- Well-suited for modelling the overall loss distribution in a large portfolio (stop loss) and for single-risk fire losses (WXL per risk).

Discrete distributions

Discrete distributions are defined by a finite set of possible losses, each of which can occur with a certain probability. They differ from the continuous distributions considered above, in which, in principle, any size of loss is possible above the observation point.

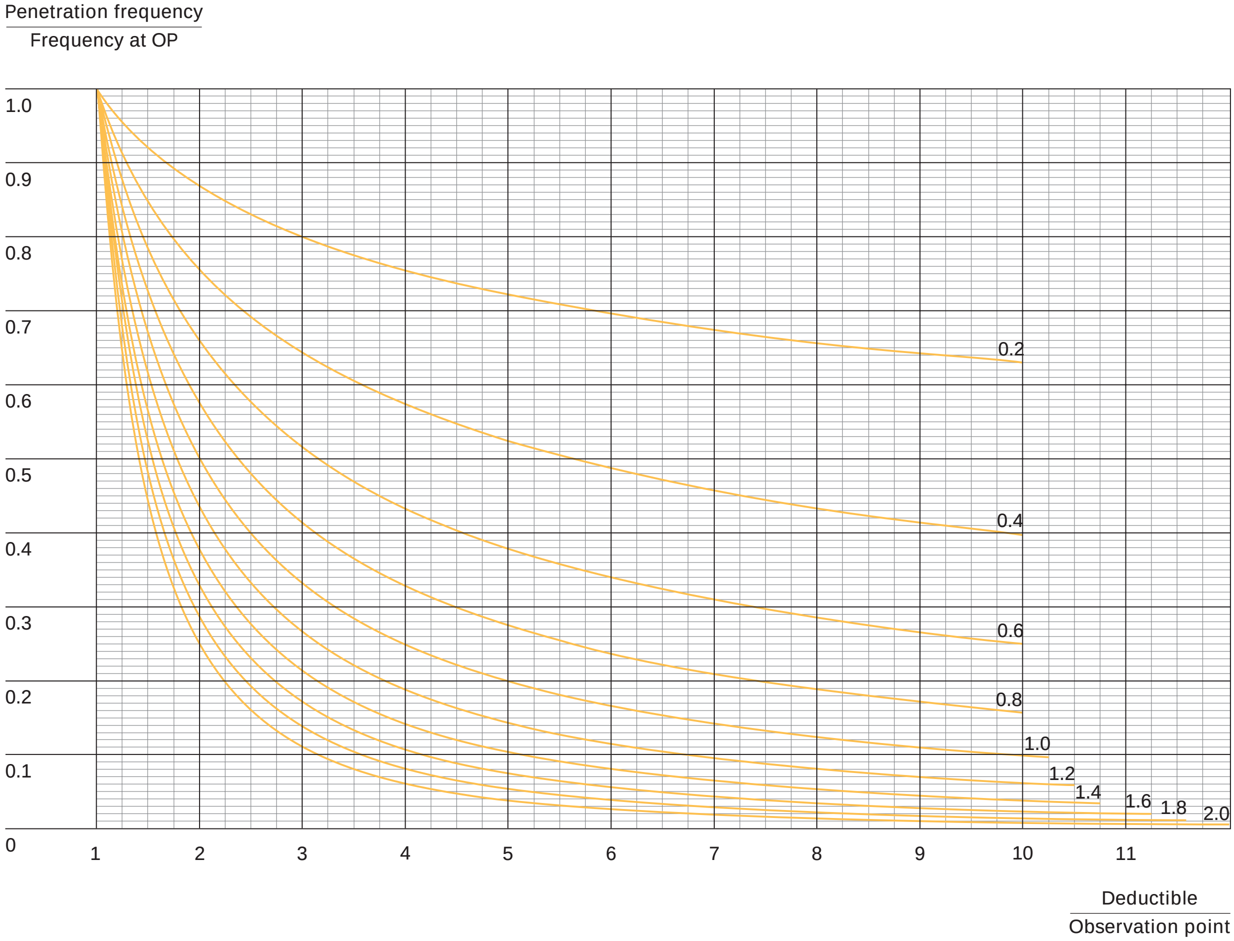
One advantage of discrete distributions lies in the fact that they are easy to use in computer programs due to the limited number of possible losses. (Many of the numerical procedures used in connection with the continuous distributions considered above are actually based on ”discretized” versions of these distributions.)

6 Annex

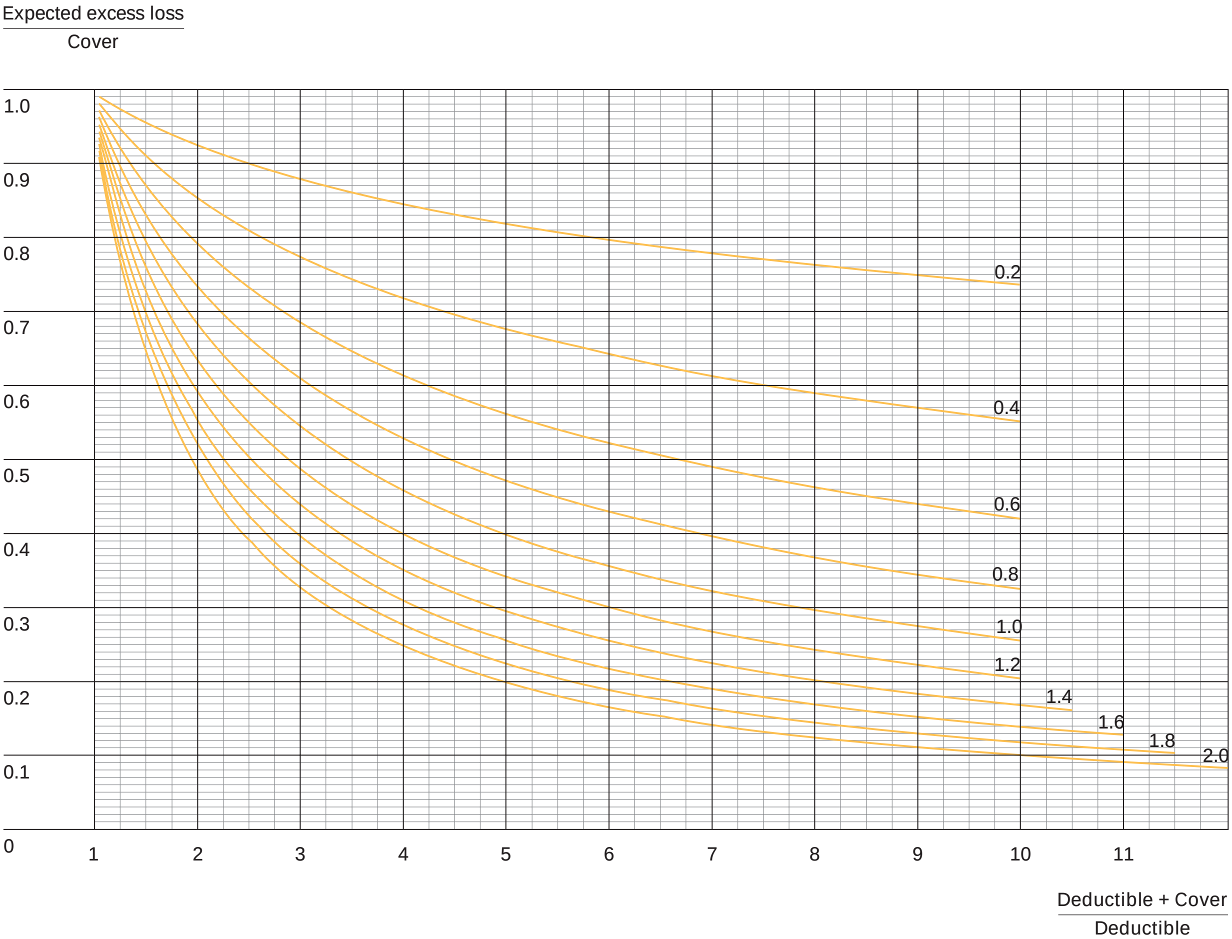
6.1 Graphs

The two graphs below make it possible to calculate risk premiums (similar to Example 4.2). In doing this, FQ(DE) and EL are read from the graphs rather than calculated. It is left to the user to discover which method he prefers. However, once the formula has been programmed, the "electronic" method is definitely faster.

6.1.1 Penetration frequency FQ(DE) (as a multiple of the frequency at the observation point FQ(OP))



6.1.2 Expected excess loss EL
(as a multiple of the cover CO)
relative to layer length RL



6.2 Example for applying the graphs

The following example is analogous to Example 4.2.

Solving for	Penetration frequency FQ(DE), expected excess loss EL and risk premium RP
Given	OP, α (rule-of-thumb value, or calculated from the as-if losses using Formula 3.5.1), FQ(OP) (from loss experience)

OP = 1 250 000
α = 0.75
FQ(OP) = 0.67
CO xs DE = 5 000 000 xs 2 000 000

Step 1: Calculate $\frac{DE}{OP}$

$$\frac{DE}{OP} = \frac{2\,000\,000}{1\,250\,000} = \underline{1.6}$$

Step 2: Calculate RL

$$RL = \frac{CO+DE}{OP} = \frac{5\,000\,000+2\,000\,000}{2\,000\,000} = \underline{3.5}$$

Step 3: Calculate FQ(DE)

Using Graph 6.1.1, we enter the result from Step 1 on the horizontal axis. The vertical extended from this cuts the curve for $\alpha = 0.75$ at 0.705. Thus 0.705 is FQ(DE) expressed as a multiple of FQ(OP), that is, in order to determine FQ(DE), we multiply the curve value by FQ(OP):

$$FQ(DE) = FQ(OP) \cdot 0.705 = 0.67 \cdot 0.705 = \underline{\underline{0.4724}}$$

(The value calculated in 4.2 was 0.4710.)

Step 4: Calculate the expected excess loss EL

Using Graph 6.1.2, we enter the relative layer length RL from Step 2 on the horizontal axis. The vertical extended from this cuts the curve for $\alpha = 0.75$ at 0.59. Thus, 0.59 is the EL expressed as a multiple of the cover CO; that is, in order to determine EL, we multiply the curve value by CO:

$$EL = CO \cdot 0.59 = 5\,000\,000 \cdot 0.59 = \underline{\underline{2\,950\,000}}$$

(The value calculated in 4.2 was 2 942 400.)

Step 5: Calculate the risk premium RP

$$RP = FQ(DE) \cdot EL = 0.4724 \cdot 2\,950\,000 = \underline{\underline{1\,393\,580}}$$

(The value calculated in 4.2 was 1 385 870.)

7 Worksheets

A diskette with the worksheets mentioned in Chapter 3 is included in the external edition of this brochure. Interested readers within Swiss Re will find the worksheets in Lotus Notes under: "Groupwide Interest Discussions / Resources / By Audience / Non-life / property insurance / The Pareto Model in Property Reinsurance: Worksheets".

The worksheets are presented in two versions: *.wk1 for Lotus 123 and *.xls for Excel. More detailed information on applications are in the worksheets themselves and in Chapters 3 and 4.

Notes

Notes

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