ON FIRE EXPOSURE RATING AND THE IMPACT OF THE RISK PROFILE TYPE

BY

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ABSTRACT

We analyze fire exposure rating for three types of risk profiles: policy profiles, top location profiles and location profiles. Location profiles offer more detailed information than top location profiles, which in turn are better than policy profiles. We prove criteria to ensure that a better quality of risk profile leads to a lower price. These criteria are discussed with respect to standard exposure rating and an alternative method called 'burning cost-adjusted exposure rating', where the loss ratio is adjusted by means of the burning cost of a low reference layer. Further, we introduce a family of analytic exposure curves, the so-called EP exposure curves. These curves are useful for practical application, since the criteria given in this paper can be checked easily for exposure curves that can be approximated by EP exposure curves. This is of particular interest for discrete exposure curves. Finally, we apply the results to the MBBEFD exposure curves.

KEYWORDS

Exposure rating, exposure curve, fire insurance, risk profile, Pareto distribution, excess of loss reinsurance, MBBEFD exposure curve, EP exposure curve

1 INTRODUCTION

In reinsurance, fire exposure rating is one of the standard pricing methods for non-proportional property per risk covers. For this purpose, ceding companies usually provide risk profiles to the reinsurer which contain the sum insured (SI) and the gross premium for every risk in the portfolio. Exposure rating for such a profile can be divided into two steps (see Bernegger (1997)):

Step 1: Estimate the risk premium of each risk by applying an appropriate loss ratio to its gross premium.

Step 2: Divide each risk premium into a risk premium for the retention of the ceding company and a risk premium for the cession to the reinsurer.

The second step could be carried out easily if the severity distributions of all risks in the portfolio were available. However, the correct distribution function of an individual risk is hardly known in practice. Therefore, distributions derived from large portfolios of similar risks are used. Such distributions are available in form of so-called *exposure curves.* These curves describe the functional dependance between the deductible of an excess layer (in % of SI) and the risk premium contained below the deductible (in % of total), i.e. an exposure curve provides the loss elimination ratio for deductibles in % of SI. Details can be found in Ludwig (1991) or Mack and Fackler (2003).

Fire exposure rating has a long tradition in reinsurance. For example Riebesell (1936) presents an exposure curve which was derived from statistics of fire insurance for residential buildings in the United States (see Table 1).

Deductible of an excess layer (in % of the SI)	Risk premium contained below the deductible (in % of the total risk premium)
10%	57%
20%	67%
30%	75%
40%	81%
50%	86%
60%	91%
70%	94%
80%	97%
90%	99%

TABLE 1: Riebesell's exposure curve

Over the years, many exposure curves for various areas of application have been developed. Well known are for example the 'Gasser curves' which were derived from the 'Brandstatistik der Vereinigung kantonaler Feuerversicherungsanstalten' (fire statistics of the Swiss Association of Cantonal Fire Insurance Institutions) for the years 1959–1967 (see Gasser (1970)). Today, analytical exposure curves are often used, e.g. the MBBEFD exposure curves introduced by Bernegger (1997). Analytical curves derived from Pareto densities can be found in Mack (1980).

Example 1.1 (Standard exposure rating). In risk profiles, risks of similar size and the same risk category are often summarized in risk bands. Table 2 shows an example of exposure rating for such a profile and a layer 1 000 xs 1 000. We have assumed a loss ratio of 60% and the exposure curve from Table 1.

Risk band	# of risks	Avg. SI	Gross premium	Total risk premium	Ded. in % of the SI	Exit pt. in % of the SI	Risk premium of XL in %	Risk premium of XL in amount
1	2 300	1 250	2 000	1 200	80%	>100%	3%	36
2	1 300	1 667	1 500	900	60%	>100%	9%	81
3	600	2 000	1 000	600	50%	100%	14%	84
4	300	2 500	600	360	40%	80%	16%	58
5	150	3 333	400	240	30%	60%	16%	38
6	80	5 000	300	180	20%	40%	14%	25
7	20	10 000	100	60	10%	20%	10%	6
Total			5 900	3 540			9%	328

TABLE 2: Standard exposure rating

By way of example, for risk band 5, the risk premium of the XL in percent of the total risk premium is calculated as $G(60\%) - G(30\%)$, where G denotes the assumed exposure curve. □

At first sight the choice of the loss ratio in step 1 seems to be straightforward. In practice, however, the estimation of the loss ratio turns out to be difficult:

- The expected loss ratio of the exposed segment of risks can be quite different to the loss ratio of the overall portfolio.

- Property policies often cover sublines of business which do not have much potential for large losses or which are strictly sublimited (e.g. coverage for burglary or natural catastrophes). Nevertheless, in some markets pure fire exposure curves are used. In this case, only the fire part of the premium should be taken into account for exposure rating. However, the reinsurer often does not have access to data per subline of business.
- Exposure curves typically only fit well if the original deductibles of the portfolio are close to the market average. If this is not the case, the curves can still be used (with a small approximation error) but the loss ratio has to be adjusted accordingly. For details see Appendix A.

Due to these difficulties, sometimes the following approach called *burning cost-adjusted exposure rating* is preferred: Choose a reference layer below or equal to the first layer of the non-proportional program and adjust the loss ratio in step 1 in such a way that the expected loss of the exposure model coincides with the burning cost (BC) for this reference layer. In other words, the severity distribution of the exposure model is used to extrapolate the burning cost for the reference layer to the higher layers of the reinsurance program.

Example 1.2 (Burning cost-adjusted exposure rating). Consider the profile from Example 1.1. If we use the same loss ratio and exposure curve to calculate two additional layers 2 000 xs 2 000 and 6 000 xs 4 000, we obtain the following risk premiums:

Layer	Standard exposure rating
1 000 xs 1 000	328
2 000 xs 2 000	70
6 000 xs 4 000	17

Assume that we have a reliable burning cost of 290 for the layer 1 000 xs 1 000 and that the loss history of the portfolio is not sufficient for experience rating of the higher layers. If we are not sure that the loss ratio used for standard exposure rating is correct, it makes sense to rely on the burning cost for the first layer and to adjust the whole exposure model by the factor 290/328, which is equivalent to choosing a loss ratio of 53% instead of 60%:

Layer	BC-adjusted exposure rating
1 000 xs 1 000	290
2 000 xs 2 000	62
6 000 xs 4 000	15

□

So far we have considered risk profiles without discussing an important detail: What is a risk? This is not self-evident since industrial fire policies typically cover multiple locations. There are mainly three different types of profiles:

- *Policy profile:* Each policy is understood as one risk. The risk profile contains the cumulated sum insured of all locations and the total premium of the policy.
- *Top location profile:* Each policy is understood as one risk, but the risk profile contains the sum insured of the largest location and the total premium of the policy.
- *Location profile:* Each location covered by a policy is understood as a risk and is contained in the profile with a separate sum insured and the part of the gross premium which is allocated to the location.

Policy profiles are not very useful for exposure rating since a fire will not affect more than one location of a policy, i.e. the loss amount per event is limited by the sum insured of the largest location. Top location profiles are much better since the reported sum insured corresponds to the largest possible loss amount. From an underwriter's perspective, location profiles offer the best information because they contain more details than top location profiles.

Example 1.3. Consider an unlimited excess layer with a deductible of 2 400 which covers a portfolio consisting of two multi location policies. Table 3 shows the exposure calculation for the location profile. We have assumed a loss ratio of 60% and the exposure curve from Table 1. Tables 4 and 5 show the calculations for the corresponding top location profile and policy profile.

			Gross	Total risk	Ded. in %	Risk premium of XL	
Policy	Location	SI	premium	premium	of the SI	in %	in amount
A	Munich	6 000	6.0	3.6	40%	19%	0.68
	Rome	4 000	4.0	2.4	60%	9%	0.22
	Madrid	2 000	2.0	1.2	> 100%	0%	0.00
B	Zurich	4 000	6.0	3.6	60%	9%	0.32
	Paris	2 000	3.0	1.8	> 100%	0%	0.00
Total		18 000	21.0	12.6			1.22

Table 3: Standard exposure rating for the location profile

		Gross	Risk	Ded. in %	Risk premium of XL	
Policy	SI	premium	premium	of the SI	in %	in amount
A	6 000	12.0	7.2	40%	19%	1.37
B	4 000	9.0	5.4	60%	9%	0.49
Total	10 000	21.0	12.6			1.85

Table 4: Standard exposure rating for the top location profile

		Gross	Risk	Ded. in %	Risk premium of the XL	
Policy	SI	premium	premium	of the SI	in %	in amount
A	12 000	12.0	7.2	20%	33%	2.38
B	6 000	9.0	5.4	40%	19%	1.03
Total	18 000	21.0	12.6			3.40

Table 5: Standard exposure rating for the policy profile

In the location profile, there are locations which do not expose the layer. This is not reflected by the top location profile which therefore leads to a higher price. The policy profile leads to the highest price because it contains larger sums insured than the top location profile. □

Example 1.3 is quite typical: In this paper we show that under certain conditions, which often hold in practice, a better quality of risk profile always leads to a lower price.

We start with formal definitions of exposure curves and exposure rating. In Section 3 we introduce the class of Pareto increasing exposure curves which turns out to be closely related to our analysis in respect of burning cost-adjusted exposure rating. The influence of the profile type is analyzed in Section 4. For standard exposure rating we show that exposure curves with local Pareto alphas greater than 1 always ensure a lower price for the better profile type (see Theorem 4.5). For burning cost-adjusted exposure rating, the same holds true for Pareto increasing exposure curves if the underlying portfolio is ‘homogeneous’ and approximately Pareto distributed (see Theorem 4.12 and Corollary 4.14). In Section 5 we introduce a large and flexible family of Pareto increasing exposure curves. These curves are useful when the results shall be applied to discrete exposure curves. Moreover, they are used in Section 6 where we analyze the MBBEFD exposure curves. Furthermore, Section 7 provides an example which shows that the results are useful for the evaluation of loss models if data for exposure rating is not available in the required granularity. In order to improve readability, all the mathematical proofs are found in the appendix.

Throughout this paper the results are presented for profiles based on sums insured. For large industrial risks, however, PMLs rather than sums insured are typically used for exposure rating. The PML (probable maximum loss) is an estimate of the largest loss that could result from a single fire, considering the existing mitigation measures (like firewalls or sprinklers). Like sums insured, PMLs can be considered per location, per top location or per policy. The policy PML is typically equal to the top location PML (since a single fire will not affect more than one location of the policy), i.e. the policy profile coincides with the top location profile. The findings regarding location and top location profiles obviously remain valid in the PML case.

2 Exposure Rating

Fire exposure rating of non-proportional reinsurance covers is based on the assumption that (for homogeneous segments of risks) the claims sizes scale with the sums insured, i.e. that the degree of loss (in % of SI) is identically distributed for all risks. Adequate distributions are available in form of exposure curves. These can be defined in a mathematically rigorous manner (see Riegel (2008) for technical details):

Definition 2.1. An *exposure curve* is a concave function $G\colon [0,1] \to [0,1]$ satisfying $G(0) = 0$, $G^{-1}(1) = \{1\}$ and $\lim_{x \searrow 0} G(x)/x < \infty$.

If x is the deductible of an unlimited excess layer in % of the SI of a risk, then $G(x)$ is the corresponding loss elimination ratio, i.e. the ratio of the risk premium contained below the deductible to the total risk premium of the risk. The reinsurer's risk premium is $1 - G(x)$ times the total risk premium.

It is easy to see that exposure curves are continuous and right differentiable. We will use $G'(x)$ and $\frac{d}{dx}G(x)$ for the right derivative throughout this paper.

Let X be the degree of loss and let F_X be the distribution function of X. If the largest possible loss is equal to the sum insured, we have $\inf(F^{-1}(1)) = 1$, and the corresponding exposure curve $G_X\colon [0,1] \to [0,1]$ is given by

$$G_X(x) := \frac{E(\min(X,x))}{E(X)} = \frac{\int_0^x (1 - F_X(t))\,dt}{\int_0^1 (1 - F_X(t))\,dt}.$$

Conversely, given an exposure curve G, the corresponding distribution function F_G of the degree of loss can be reconstructed via

$$F_G(x) := 1 - \frac{G'(x)}{G'(0)}$$

for $x \in [0,1)$ and $F_G(x) := 1$ for $x \geq 1$.

In practice, sometimes curves with a vertical tangent at zero are used for exposure rating (e.g. the curve $G(x) = x^{\log_2(1+z)}$ with a constant parameter $z \in (0,1)$ which is often used in liability insurance, see Riegel (2008)). Therefore, we will analyze the following, slightly more general class of curves:

Definition 2.2. A *quasi exposure curve* is a continuous concave function $G\colon [0,1] \to [0,1]$ satisfying $G(0) = 0$ and $G^{-1}(1) = \{1\}$.

Note that quasi exposure curves are right differentiable in $(0, 1)$ but not necessarily at 0. One can show that every quasi exposure curve is strictly increasing and surjective, thus bijective. The inverse of a quasi exposure curve is right differentiable in $[0, 1)$.

For a quasi exposure curve $G\colon [0,1] \to [0,1]$ we denote with $G_+\colon [0,+\infty) \to [0,1]$ the extension of G with $G_+(x) := 1$ for all $x > 1$.

In practice, portfolio information for exposure rating is often provided in risk bands. For our analysis, however, it is advantageous to assume that a risk profile contains sum insured and premium per risk. This is not a restriction, since risk bands are just a condensed representation of the information per risk. Mathematically, we define a risk profile as follows:

Definition 2.3. A *risk profile* is a finite family $\mathcal{P} = \{(s_i, p_i)\}_{i\in I}$ of pairs with $s_i > 0$ and $p_i > 0$ for all $i \in I$. The index set I should be interpreted as the set of risks, s_i as the sum insured and p_i as the premium of the risk $i \in I$. For a risk profile $\mathcal{P} = \{(s_i, p_i)\}_{i\in I}$ we denote with $\max(\mathcal{P}) := \max\{s_i \,|\, i \in I\}$ the largest sum insured.

With this notion of a risk profile, we can now formally define the concepts of 'standard exposure rating' and 'burning cost-adjusted exposure rating'.

Standard Fire Exposure Rating:
Let G be a quasi exposure curve and let $l > 0$ be the selected loss ratio. Given a risk profile $\mathcal{P} = \{(s_i, p_i)\}_{i\in I}$ the expected loss of the layer C xs D $(C, D > 0)$ is calculated as follows:

$$\mathrm{EL}_{C,D}(\mathcal{P}, l, G) := l \cdot \sum_{i\in I} \left[G_+\left(\frac{C+D}{s_i}\right) - G_+\left(\frac{D}{s_i}\right)\right] \cdot p_i.$$

Burning Cost-Adjusted Fire Exposure Rating:
Let $\mathcal{P} = \{(s_i, p_i)\}_{i\in I}$ be a risk profile and let G be a quasi exposure curve. Assume that the burning cost BC_0 of a reference layer C_0 xs D_0 with $D_0 < \max(\mathcal{P})$ is known. Let $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$. Then the expected loss of the layer C xs D is calculated by

$$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}, \rho_0, G) := \frac{\mathrm{BC}_0}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}, 1, G)} \mathrm{EL}_{C,D}(\mathcal{P}, 1, G) = \mathrm{EL}_{C,D}(\mathcal{P}, l_0, G),$$

where $l_0 := \frac{\mathrm{BC}_0}{\mathrm{EL}_{C_0,D_0}(\mathcal{P},1,G)}$. Since C_0 xs D_0 is supposed to be a 'low' reference layer, we will always presume $C + D \geq C_0 + D_0$ and $D > D_0$.

Note that the technical condition $D_0 < \max(\mathcal{P})$ is needed because otherwise the profile $\mathcal{P}$ would not expose the reference layer and consequently $\mathrm{EL}_{C_0,D_0}(\mathcal{P}, 1, G) = 0$. In practice, this is not a restriction since actually $\max(\mathcal{P}) > D > D_0$ in every realistic case.

Remark 2.4. According to our definition, a quasi exposure curve G satisfies $G^{-1}(1) = \{1\}$. This is certainly the normal case and corresponds to the assumption that the largest possible loss for a risk is equal to its sum insured. However, sometimes curves $H\colon [0, b] \to [0, 1]$ with $H^{-1}(1) = \{b\}$ are used (cf. e.g. Ludwig (1991)). Then $G\colon [0, 1] \to [0, 1]$, $x \mapsto H(bx)$ is a quasi exposure curve in the sense of this paper. If all sums insured of a profile are scaled by b then G yields the same results as H and the original profile. Using this fact, all results of this paper can be applied to these slightly more general curves.

3 Pareto Increasing Quasi Exposure Curves

In this section the class of Pareto increasing quasi exposure curves is introduced, which is useful for the analysis of burning cost-adjusted exposure rating. The definition of these curves uses local properties of the Pareto distribution. This paper refers to the 'European' version of the Pareto distribution as defined in Rytgaard (1990), also known as 'Single-parameter Pareto' (cf. Klugman et al. (2004)).

The Pareto distribution with threshold $x_0 > 0$ and shape parameter $\alpha > 0$ (the so-called 'Pareto alpha') is defined by the distribution function

$$F_\alpha(x) := 1 - \left(\frac{x_0}{x}\right)^\alpha$$

for $x \geq x_0$. The survival function $1 - F_\alpha(x)$ satisfies

$$\ln(1 - F_\alpha(x)) = -\alpha \cdot \ln(x) + \alpha \cdot \ln(x_0)$$

for $x \geq x_0$, i.e. on double logarithmic paper the graph of $1 - F_\alpha$ is a straight line with slope $-\alpha$. In other words, the function

$$\gamma_\alpha(t) := -\ln(1 - F_\alpha(e^t)) = \alpha \cdot t - \alpha \cdot \ln(x_0)$$

has constant derivative $\gamma'_\alpha(t) = \alpha$ for $t \geq \ln(x_0)$. This fact can be used to define 'local Pareto alphas' for right differentiable distribution functions.

Definition 3.1. Let F be a distribution function which is right differentiable at $x > 0$. Let $F(x) < 1$. Then

$$\alpha(x) := -\frac{d}{dt}\bigg|_{t=\ln(x)} \ln(1 - F(e^t)) = \frac{xF'(x)}{1 - F(x)}$$

is called the *local Pareto alpha of F at x.*

Remark 3.2. If F is differentiable at x then $\alpha(x) = xh(x)$ with the *hazard rate*

$$h(x) := \frac{F'(x)}{1 - F(x)}.$$

The hazard rate $h(x)$ may be interpreted as the probability density at x given that the argument will be at least x (see Klugman et al. (2004)).

Let G be an exposure curve which is twice right differentiable and let F be the corresponding distribution function, i.e. $F(x) := 1 - G'(x)/G'(0)$ for $x \in [0, 1)$. Then the local Pareto alphas are calculated by

$$\alpha(x) = \frac{xF'(x)}{1 - F(x)} = -\frac{xG''(x)}{G'(x)}$$

for $x \in (0, 1)$. The same formula can be used to define local Pareto alphas slightly more generally for quasi exposure curves:

Definition 3.3. Let G be a quasi exposure curve which is twice right differentiable in $(0, 1)$. Then for $x \in (0, 1)$

$$\alpha(x) := -\frac{xG''(x)}{G'(x)}$$

is called the *local Pareto alpha of G at x.*

Definition 3.4. We say that a quasi exposure curve G is *Pareto increasing* if

1. G is differentiable and twice right differentiable in $(0,1)$ and
2. $\alpha\colon (0,1) \to \mathbb{R}$, $x \mapsto -\frac{xG''(x)}{G'(x)}$ is increasing (i.e. G has increasing local Pareto alphas).

Example 3.5. Exposure rating in liability reinsurance is often based on the assumption that there is a constant percentage surcharge $z \in (0,1)$ for doubling the policy limit (the *doubled limits surcharge*, see Riegel (2008)). This assumption leads to the same results as fire exposure rating with the quasi exposure curve

$$G\colon [0,1] \to [0,1], \quad x \mapsto x^{\log_2(1+z)}.$$

G is twice differentiable in $(0,1)$ and

$$\alpha(x) = -\frac{x \cdot G''(x)}{G'(x)} = -\frac{x \cdot \log_2(1+z)(\log_2(1+z)-1)x^{\log_2(1+z)-2}}{\log_2(1+z)x^{\log_2(1+z)-1}} = 1 - \log_2(1+z)$$

for $x \in (0,1)$, i.e. G is Pareto increasing. □

The following proposition, which provides the link between Pareto increasing exposure curves and BC-adjusted exposure rating, will be key to prove the results of the next section.

Proposition 3.6. *A quasi exposure curve $G\colon [0,1] \to [0,1]$ is Pareto increasing if and only if the function*

$$(0, 1/x_1) \to \mathbb{R}, \quad t \mapsto \frac{G_+(ty_2) - G_+(ty_1)}{G_+(tx_2) - G_+(tx_1)}$$

is decreasing for all $0 \le x_1 < x_2$ and $y_1 < y_2$ with $x_1 < y_1$ and $x_2 \le y_2$ (read $1/x_1 = +\infty$ in the case $x_1 = 0$).

A proof of this proposition is found in Appendix B.

4 Exposure Rating for Different Types of Risk Profiles

We will now introduce the mathematical notation for the three risk profile types which have already been mentioned in the introduction: location profiles, top location profiles and policy profiles. Recall that location profiles offer the best information for exposure rating, followed by top location profiles. We will then focus on criteria which ensure that a higher quality of the risk profile type results in a lower price. The proofs are found in Appendix C.

Notation 4.1. Let I be a finite set and let $j\colon I \to \mathbb{N}$ be a map. We write

$$J(I,j) := \{(i,n) \in I \times \mathbb{N} \mid n \le j(i)\} \subset I \times \mathbb{N}.$$

Definition 4.2. A risk profile with an index set of the form $J(I,j)$ is called a *location profile*. The elements $i \in I$ are then called the *policies* of the profile and (i,n) is the nth *location* of policy i. In the case $|I| = 1$ we speak of a *one policy location profile*. Given a location profile $\mathcal{L} = \{(s_{(i,n)}, p_{(i,n)}\}_{(i,n)\in J(I,j)}$, the risk profile

$$\mathcal{T}(\mathcal{L}) := \{(s_i^T, p_i^T)\}_{i\in I} \quad \text{with} \quad s_i^T = \max_{1\le n\le j(i)} s_{(i,n)},\ p_i^T = \sum_{n=1}^{j(i)} p_{(i,n)}$$

is called the *corresponding top location profile*. The risk profile

$$\mathcal{P}(\mathcal{L}) := \{(s_i^P, p_i^P)\}_{i \in I} \quad \text{with} \quad s_i^P = \sum_{n=1}^{j(i)} s_{(i,n)}, \ p_i^P = \sum_{n=1}^{j(i)} p_{(i,n)}$$

is called the *corresponding policy profile*.

Example 4.3. Consider a portfolio consisting of three policies A, B, and C which have 3, 1, and 2 locations respectively:

Policy	Location	Sum insured	Premium
Policy A	1	10 000	7
	2	6 000	3
	3	2 000	1
Policy B	1	8 000	4
Policy C	1	3 000	2
	2	7 000	4

Let $I := \{A, B, C\}$ and define $j\colon I \to \mathbb{N}$ by $j(A) := 3$, $j(B) := 1$ and $j(C) := 2$, i.e.

$$J(I, j) = \{(A, 1), (A, 2), (A, 3), (B, 1), (C, 1), (C, 2)\}.$$

Then we have the location profile $\mathcal{L} := \{(s_{(i,n)}, p_{(i,n)}\}_{(i,n) \in J(I,j)}$ with

$$\begin{aligned}
s_{(A,1)} &:= 10\,000, & p_{(A,1)} &:= 7, & s_{(A,2)} &:= 6\,000, & p_{(A,2)} &:= 3,\\
s_{(A,3)} &:= 2\,000, & p_{(A,3)} &:= 1, & s_{(B,1)} &:= 8\,000, & p_{(B,1)} &:= 4,\\
s_{(C,1)} &:= 3\,000, & p_{(C,1)} &:= 2, & s_{(C,2)} &:= 7\,000, & p_{(C,2)} &:= 4,
\end{aligned}$$

the corresponding top location profile $\mathcal{T}(\mathcal{L}) := \{(s_A^T, p_A^T), (s_B^T, p_B^T), (s_C^T, p_C^T)\}$ with

$$s_A^T := 10\,000, \quad p_A^T := 11, \quad s_B^T := 8\,000, \quad p_B^T := 4, \quad s_C^T := 7\,000, \quad p_C^T := 6$$

and the corresponding policy profile $\mathcal{P}(\mathcal{L}) := \{(s_A^P, p_A^P), (s_B^P, p_B^P), (s_C^P, p_C^P)\}$ with

$$s_A^P := 18\,000, \quad p_A^T := 11, \quad s_B^P := 8\,000, \quad p_B^T := 4, \quad s_C^P := 10\,000, \quad p_C^T := 6.$$

□

Naturally, the question arises: Does a better profile type always lead to a lower price? The following proposition shows that the answer is yes, if we consider standard exposure rating for unlimited layers (or layers with sufficiently high exit points).

Proposition 4.4. *Let $\mathcal{L}$ be a location profile, C xs D a reinsurance layer, $l > 0$ and let G be an arbitrary quasi exposure curve. If* $\max(\mathcal{L}) \leq C + D$ *then*

$$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G).$$

If $\max(\mathcal{P}(\mathcal{L})) \leq C + D$ *then*

$$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G).$$

Note that $\max(\mathcal{L}) = \max(\mathcal{T}(\mathcal{L}))$. This proposition is an immediate consequence of the fact that quasi exposure curves are increasing functions. For layers with lower exit points, the situation is more complex:

Theorem 4.5. *Let G be a quasi exposure curve and let $l > 0$, $\mu \in (0,1)$. The following assertions are equivalent:*

1. *The function $x \mapsto xG'(x)$ is decreasing in the interval $[\mu, 1)$.*
2. *For every choice of $C, D > 0$ and every location profile $\mathcal{L}$ with $\mu \cdot \max(\mathcal{T}(\mathcal{L})) \leq D$ we have*
$$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G),$$
i.e. standard exposure rating with a location profile is always cheaper than with the corresponding top location profile.
3. *For every choice of $C, D > 0$ and every location profile $\mathcal{L}$ with $\mu \cdot \max(\mathcal{P}(\mathcal{L})) \leq D$ we have*
$$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G),$$
i.e. standard exposure rating with a top location profile is always cheaper than with the corresponding policy profile.

In the case that G is twice right differentiable in $[\mu, 1)$, this is equivalent to

4. *The local Pareto alphas $\alpha(x) = -\frac{xG''(x)}{G'(x)}$ are larger or equal to 1 for $x \in [\mu, 1)$.*

An example illustrating this theorem with MBBEFD exposure curves can be found in Section 6 (see Example 6.4). In practice, the type of the risk profile at hand is not always known. In this case the following corollary is useful which is a direct consequence of Theorem 4.5.

Corollary 4.6. *Let $\mathcal{Q}$ be a risk profile, $C, D > 0$, $l > 0$ and $\mu := D/\max(\mathcal{Q})$. Let G be a quasi exposure curve such that $x \mapsto xG'(x)$ is decreasing in the interval $[\mu, 1)$.*

1. *If $\mathcal{Q}$ is a policy profile, i.e. $\mathcal{Q} = \mathcal{P}(\mathcal{L})$ for a location profile $\mathcal{L}$ then*
$$\mathrm{EL}_{C,D}(\mathcal{Q}, l, G) = \mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G).$$
2. *If $\mathcal{Q}$ is a top location profile, i.e. $\mathcal{Q} = \mathcal{T}(\mathcal{L})$ for a location profile $\mathcal{L}$ then*
$$\mathrm{EL}_{C,D}(\mathcal{Q}, l, G) = \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G).$$

Example 4.7. Consider the risk profile from Example 1.1:

Risk band	# of risks	Avg. SI	Gross premium
1	2 300	1 250	2 000
2	1 300	1 667	1 500
3	600	2 000	1 000
4	300	2 500	600
5	150	3 333	400
6	80	5 000	300
7	20	10 000	100

A pricing actuary wants to use standard exposure rating to estimate the loss cost of a layer 2 000 xs 2 000. He does not know the type of the risk profile but he assumes that it is a location profile. Therefore, he uses an exposure curve which is suitable for location profiles. If his assumption on the risk profile type is wrong, this will never lead to underestimation of the loss cost if the exposure curve has local Pareto alphas $\alpha(x) \geq 1$ for $x \geq 0.2$. □

Remark 4.8. If the condition $\alpha(x) \geq 1$ is not fulfilled, Theorem 4.5 states that there are portfolios and layers where a better quality of risk profile leads to a higher price. However, this can only occur if a good portion of the locations have sums insured which are larger than the exit point of the considered layer. In many realistic situations a better risk profile type leads to a lower price, even if the layer is limited and the local Pareto alphas of the exposure curve are below 1.

For burning cost-adjusted exposure rating Pareto increasing exposure curves ensure a lower price for a better risk profile type if we restrict our attention to portfolios consisting of only one policy:

Theorem 4.9. *Let G be a quasi exposure curve. The following assertions are equivalent:*

1. *G is Pareto increasing.*
2. *For every choice of $\rho_0 = (C_0, D_0, \mathrm{BC}_0) \in \mathbb{R}_+^3$ and $C, D > 0$ with $C + D \geq C_0 + D_0$, $D > D_0$ and every one policy location profile $\mathcal{L}$ with $\max(\mathcal{L}) > D_0$ we have*

   $$\mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}), \rho_0, G) \geq \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{L}, \rho_0, G),$$

   i.e. burning cost-adjusted exposure rating with a one policy location profile is always cheaper than with the corresponding top location profile.
3. *For every choice of $\rho_0 = (C_0, D_0, \mathrm{BC}_0) \in \mathbb{R}_+^3$ and $C, D > 0$ with $C + D \geq C_0 + D_0$, $D > D_0$ and every one policy location profile $\mathcal{L}$ with $\max(\mathcal{L}) > D_0$ we have*

   $$\mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{P}(\mathcal{L}), \rho_0, G) \geq \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}), \rho_0, G),$$

   i.e. burning cost-adjusted exposure rating with a one policy top location profile is always cheaper than with the corresponding policy profile.

For an illustration see Example 6.6. Theorem 4.9 indicates that Pareto increasing exposure curves are important for the analysis of burning cost-adjusted exposure rating. However, given an exposure curve and a policy profile or a top location profile, one can never be absolutely certain that a better risk profile type would lead to a lower price:

Proposition 4.10. *Let G be a quasi exposure curve, $C_0, D_0, C, D > 0$ with $C + D \geq C_0 + D_0$ and $D > D_0$. Let $\mathrm{BC}_0 > 0$ and $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$. Let $\mathcal{Q} = \{(s_i, p_i)\}_{i \in I}$ be a profile such that $s_\mu, s_\nu > D_0$ and*

$$\frac{G_+\left(\frac{C+D}{s_\mu}\right) - G_+\left(\frac{D}{s_\mu}\right)}{G_+\left(\frac{C_0+D_0}{s_\mu}\right) - G_+\left(\frac{D_0}{s_\mu}\right)} \neq \frac{G_+\left(\frac{C+D}{s_\nu}\right) - G_+\left(\frac{D}{s_\nu}\right)}{G_+\left(\frac{C_0+D_0}{s_\nu}\right) - G_+\left(\frac{D_0}{s_\nu}\right)} \tag{1}$$

for at least two policies $\mu, \nu \in I$. (Note that (1) is a very weak assumption which is fulfilled in every realistic case.)

1. *There is a location profile $\mathcal{L}$ such that $\mathcal{Q} = \mathcal{P}(\mathcal{L})$ and*

   $$\mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{Q}, \rho_0, G) = \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{P}(\mathcal{L}), \rho_0, G) < \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}), \rho_0, G).$$

2. *There is a location profile $\mathcal{L}'$ such that $\mathcal{Q} = \mathcal{T}(\mathcal{L}')$ and*

   $$\mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{Q}, \rho_0, G) = \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}'), \rho_0, G) < \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{L}', \rho_0, G).$$

Example 4.11. Assume that the risk profile of Example 1.1 is a policy profile. We consider burning cost-adjusted exposure rating for the layers 2 000 xs 2 000 and 6 000 xs 4 000, where 1 000 xs 1 000 is used as reference layer. Given Proposition 4.10, we can not be certain that the corresponding top location profile leads to a lower price.

For example, assume that policies in the risk bands 2–7 have only one location each and that all policies in the first risk band have two equally sized locations. Then we have the following profiles:

Policy profile

Risk band	# of risks	Avg. SI	Gross premium
1	2 300	1 250	2 000
2	1 300	1 667	1 500
3	600	2 000	1 000
4	300	2 500	600
5	150	3 333	400
6	80	5 000	300
7	20	10 000	100

Top location profile

Risk band	# of risks	Avg. SI	Gross premium
1	2 300	625	2 000
2	1 300	1 667	1 500
3	600	2 000	1 000
4	300	2 500	600
5	150	3 333	400
6	80	5 000	300
7	20	10 000	100

For standard exposure rating with an arbitrary loss ratio, using the top location profile instead of the policy profile obviously reduces the price for the reference layer 1 000 xs 1 000 but leaves the prices for the layers 2 000 xs 2 000 and 6 000 xs 4 000 unchanged. This, in turn, implies that burning cost-adjusted exposure rating with the policy profile is cheaper than with the top location profile. □

Proposition 4.10 shows that for burning cost-adjusted exposure rating, we have to take the portfolio structure into account. We will show that for 'homogeneous' portfolios with approximately Pareto distributed sums insured a better type of risk profile is rewarded by a lower price if a Pareto increasing exposure curve is used. For this purpose, we consider the following situation:

Let $(\mathcal{L}_\nu)_{\nu\geq 1}$ be a sequence of location profiles, $\mathcal{L}_\nu =: \{(s_{\nu,(i,n)}, p_{\nu,(i,n)}\}_{(i,n)\in J(I_\nu,j_\nu)}$, such that

1. $j_\nu \equiv N \in \mathbb{N}$ for all $\nu \geq 1$ (i.e. each policy has N locations)
2. all locations have the same premium rate, i.e. there is a $\pi > 0$ such that $p_{\nu,(i,n)} = \pi s_{\nu,(i,n)}$ for $\nu \geq 1$ and $(i,n) \in J(I_\nu, j_\nu) = I_\nu \times \{1,\dots,N\}$
3. there are $\gamma_1 \geq \dots \geq \gamma_N > 0$, $\sum_{n=1}^N \gamma_n = 1$ with $s_{\nu,(i,n)} = \gamma_n s^P_{\nu,i}$ for all $\nu \geq 1$, $i \in I_\nu$, i.e. the percental distribution of the total sum insured to the locations is the same for all policies
4. the distribution functions
$$F_\nu(x) := \frac{\#\{i \in I_\nu \,|\, s^P_{\nu,i} \leq x\}}{\#I_\nu}$$
of the total sums insured per policy converge to a truncated Pareto distribution, i.e. there are $U > t > 0$ and $\alpha > 0$ such that
$$\lim_{\nu\to\infty} F_\nu(x) = F(x)$$
for all x, where
$$F(x) := \begin{cases} 0 & \text{for } x < t \\ \left(1 - \left(\frac{t}{x}\right)^\alpha\right) / \left(1 - \left(\frac{t}{U}\right)^\alpha\right) & \text{for } t \leq x < U \\ 1 & \text{for } x \geq U. \end{cases}$$

Theorem 4.12. *Let G be a Pareto increasing quasi exposure curve, $C_0, D_0, C, D > 0$ with $C+D \geq C_0 + D_0$, $D > D_0$ and $t \leq D_0 < \gamma_1 U$. Let $\mathrm{BC}_0 > 0$ and $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$. Then*

$$\lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) \geq \lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G) \geq \lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{L}_\nu, \rho_0, G).$$

The technical condition $t \leq D_0 < \gamma_1 U$ ensures that the approximation by the Pareto distribution starts below the deductible of the reference layer and that the condition $\max(\mathcal{L}_\nu) > D_0$ (which is necessary for burning cost-adjusted exposure rating with $\mathcal{L}_\nu$ and $\mathcal{T}(\mathcal{L}_\nu)$) is satisfied at least for large ν.

Remark 4.13. In the nontrivial case $\gamma_1 \neq \gamma_N$, $U > D/\gamma_1$ even the strict inequality

$$\lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) > \lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G) > \lim_{\nu\to\infty} \mathrm{EL}_{C,D}^{\mathrm{BC}}(\mathcal{L}_\nu, \rho_0, G).$$

holds (see Remark C.2). These inequalities are rather robust, i.e. they often hold true even if the assumptions of Theorem 4.12 are not strictly fulfilled. However, the example in Appendix D shows that the inequalities do not hold in general if the used quasi exposure curve is 'far away' from being Pareto increasing.

Theorem 4.12 is rather technical. However, if we take Remark 4.13 into account we can deduce the following corollary, which shows how this result can be applied in practice.

Corollary 4.14. *Let $\mathcal{L}$ be a location profile and G a Pareto increasing quasi exposure curve. Let $C_0, D_0, C, D > 0$ with $C + D \geq C_0 + D_0$, $D > D_0$ and $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$ with $\mathrm{BC}_0 > 0$. If*

a) the sums insured per policy (or top location) can be approximated by a (truncated) Pareto distribution in the relevant range,

b) the number of locations of a policy and the distribution of its sum insured to the locations does not systematically and materially depend on the total sum insured of the policy and

c) the premium rate of a location does not systematically and substantially depend on the size of the location and/or the corresponding policy

then

$$\mathrm{EL}_{C,D}^{BC}(\mathcal{P}(\mathcal{L}), \rho_0, G) \gtrsim \mathrm{EL}_{C,D}^{BC}(\mathcal{T}(\mathcal{L}), \rho_0, G) \gtrsim \mathrm{EL}_{C,D}^{BC}(\mathcal{L}, \rho_0, G),$$

where $a \gtrsim b$ stands for $a \geq (1-\varepsilon)\cdot b$ with a 'small' $\varepsilon \geq 0$.

In practice, condition a) often holds (and can be checked with a given top location or policy profile). For b) and c) it might make sense to split a given portfolio into homogeneous segments of risks. For such homogeneous segments b) and c) are realistic with the needed accuracy.

5 A Family of Pareto Increasing Exposure Curves

As regards burning cost-adjusted exposure rating, we have seen in Section 4 that for a 'homogeneous Pareto portfolio' a better type of risk profile always leads to a lower price if a Pareto increasing exposure curve is used (see Corollary 4.14). This remains true even if the used exposure curve is only approximately Pareto increasing (see Remark C.2). We will now introduce a large and flexible family of Pareto increasing exposure curves that can be used to approximate many exposure curves that are used in practice. This is of particular interest for discrete exposure curves, for which there may be practical challenges with estimating local Pareto alphas.

By definition, an exposure curve G which is twice right differentiable has local Pareto alphas $\alpha(x)$ if and only if it satisfies the differential equation

$$\alpha(x)y' = -xy'' \tag{2}$$

(read y' and y'' as right derivatives of y). This differential equation can be used to construct exposure curves with prescribed local Pareto alphas: Let $\alpha\colon (0,1) \to [0,\infty)$ be a right continuous function such that $x \mapsto \alpha(x)/x$ is Lebesgue integrable on $(0,1)$. Since (2) implies

$$\frac{d}{dx}\ln(y') = \frac{y''}{y'} = -\frac{\alpha(x)}{x}$$

we consider a primitive of $-\alpha(x)/x$:

$$S(x) := -\int_0^x \frac{\alpha(\tau)}{\tau}\, d\tau.$$

If $x \mapsto \exp(S(x))$ is Lebesgue integrable on $(0,1)$ then $x \mapsto c \cdot \int_0^x \exp(S(\tau))\, d\tau$ is a solution of the differential equation (2). With an appropriate choice of the constant c we obtain an exposure curve

$$G(x) := \frac{\int_0^x \exp(S(\tau))\, d\tau}{\int_0^1 \exp(S(\tau))\, d\tau}$$

which is differentiable, twice right differentiable and has local Pareto alphas $\alpha(x)$.

If α is continuous on the complement of a finite set $\{x_1, \ldots, x_n\} \subset (0,1)$, it follows from standard theory of linear differential equations that G is the only *differentiable* exposure curve with local Pareto alphas $\alpha(x)$.

Definition 5.1. For $t \in (0,1)$ and $\beta \geq 0$ the differentiable exposure curve $G_{t,\beta}^{EP}$ with local Pareto alphas

$$\alpha_{t,\beta}\colon (0,1) \to [0,\infty), \quad x \mapsto \begin{cases} \frac{\beta}{t}x & \text{for } x < t \\ \beta & \text{for } x \geq t \end{cases}$$

is called the *Exponential & Pareto (EP) exposure curve* with parameters (t,β).

It is easy to see that the corresponding degree of loss is exponentially distributed between 0 and t and Pareto distributed between t and 1. By simple integration $G_{t,\beta}^{EP}(x)$ can be calculated explicitly.

For $\beta > 0$, $\beta \neq 1$:

$$G_{t,\beta}^{EP}(x) = \begin{cases} \dfrac{\frac{t}{\beta}(1-e^{-\frac{\beta}{t}x})}{\frac{t}{\beta}(1-e^{-\beta})+\frac{1-t^{1-\beta}}{1-\beta}e^{\beta(\ln(t)-1)}} & \text{for } x < t \\[2ex] \dfrac{\frac{t}{\beta}(1-e^{-\beta})+\frac{x^{1-\beta}-t^{1-\beta}}{1-\beta}e^{\beta(\ln(t)-1)}}{\frac{t}{\beta}(1-e^{-\beta})+\frac{1-t^{1-\beta}}{1-\beta}e^{\beta(\ln(t)-1)}} & \text{for } x \geq t \end{cases}$$

For $\beta = 1$:

$$G_{t,\beta}^{EP}(x) = \begin{cases} \dfrac{\frac{t}{\beta}(1-e^{-\frac{\beta}{t}x})}{\frac{t}{\beta}(1-e^{-\beta})-\ln(t)e^{\beta(\ln(t)-1)}} & \text{for } x < t \\[2ex] \dfrac{\frac{t}{\beta}(1-e^{-\beta})+(\ln(x)-\ln(t))e^{\beta(\ln(t)-1)}}{\frac{t}{\beta}(1-e^{-\beta})-\ln(t)e^{\beta(\ln(t)-1)}} & \text{for } x \geq t \end{cases}$$

For $\beta = 0$ we get the diagonal $G_{t,0}^{EP}(x) = x$. Figure 1 provides plots of some EP exposure curves.

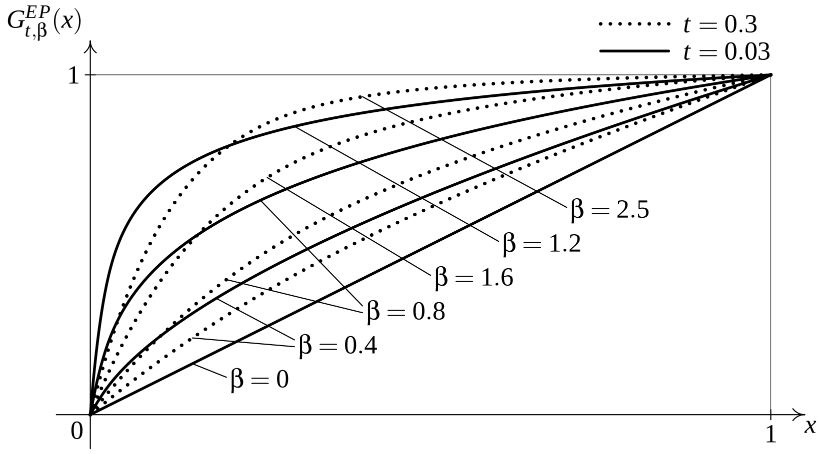

FIGURE 1: EP curves for various parameters

Example 5.2. Figure 2 shows two prominent empirical exposure curves, the Salzmann curve and the Hartford Commercial Property curve (see Ludwig (1991)). These curves can be approximated by the EP exposure curves $E^{EP}_{0.03,0.9}$ and $E^{EP}_{0.10,1.6}$ respectively. The fits are not perfect, but the accuracy is sufficient to conclude that the assertion of Corollary 4.14 holds true for the Salzmann curve and the Hartford Commercial Property curve. $\square$

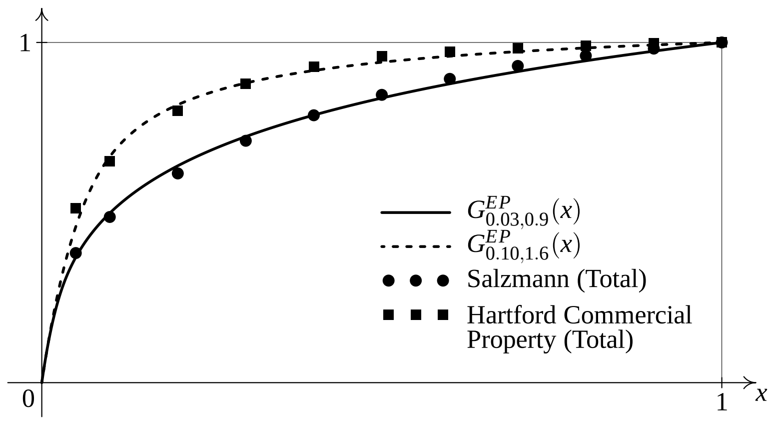

FIGURE 2: Approximation of the Salzmann and the Hartford curve by EP curves

6 Application to the MBBEFD Exposure Curves

The MBBEFD class of exposure curves has been introduced by Bernegger (1997) in order to replace commonly used discrete exposure curves by similar analytical ones. The MBBEFD class has two parameters and is very flexible. These parameters, however, are not very intuitive. Therefore, in practice, typically the following one-parameter subclass is used, which has also been defined by Bernegger (1997).

Definition 6.1. Let $g_c := g(c) := e^{(0.78+0.12c)c}$, $b_c := b(c) := e^{3.1-0.15(1+c)c}$, $a_c := a(c) := \frac{(g_c-1)b_c}{1-g_c b_c}$ and $c_0 := -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{62}{3}} \approx 4.0735$. The *(one-parameter) MBBEFD exposure curve* G_c with parameter $c \in [0, 10]$ is defined by

$$G_c(x) := \begin{cases} \dfrac{\ln\left(\frac{a_c + b_c^x}{a_c + 1}\right)}{\ln\left(\frac{a_c + b_c}{a_c + 1}\right)} & \text{if } c \neq c_0 \\ \dfrac{\ln\left(1 + x\frac{b'(c_0)}{a'(c_0)}\right)}{\ln\left(1 + \frac{b'(c_0)}{a'(c_0)}\right)} & \text{if } c = c_0. \end{cases}$$

Using l'Hospital's rule one can show that the functions $[0, 10] \times [0, 1] \to [0, 1]$, $(c, x) \mapsto G_c(x)$ and $[0, 10] \times [0, 1] \to [0, 1]$, $(c, x) \mapsto G_c'(x)$ are continuous.

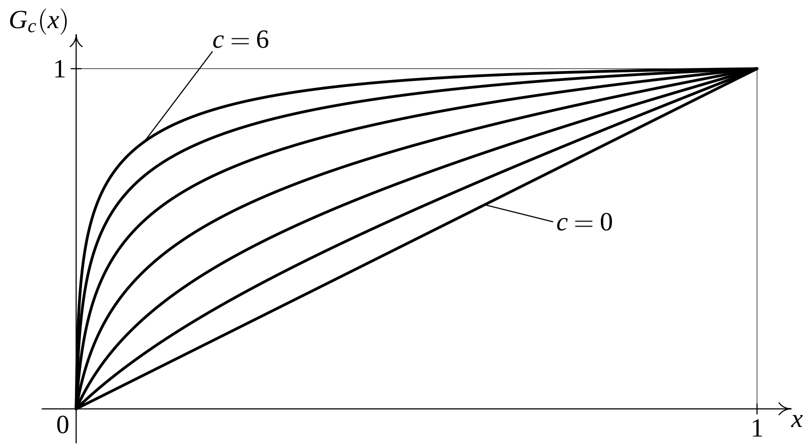

Figure 3: One-parameter family of MBBEFD exposure curves

Figure 3 shows the graphs of G_c for $c = 0$, 1, 2, 3, 4, 5, and 6. Table 6 summarizes the relation to some empirical exposure curves and gives an overview to the scope of application of the MBBEFD curves (for details see Bernegger (1997) and Guggisberg (2004)).

MBBEFD parameter c	Empirical exposure curve	Scope of application	Basis
0		Total losses only	
1.5	Gasser Y1	Personal lines	SI
1.7	OPC MD&BI	Oil & Petro business	PML
2.0	Gasser Y2	Commercial lines (small scale)	SI
3.0	Gasser Y3	Commercial lines (medium scale)	SI
3.4	Hopewell MD&BI	Industrial and large commercial	PML
4.0	Gasser Y4	Industrial and large commercial	PML
5.0	Lloyd's curve (Y5)	Industrial	Top location
up to 8		Large-scale industry	PML

Table 6: Relation to empirical exposure curves and scope of application

With regard to Theorem 4.5 and Corollary 4.6 we will now define a function $\mu(c)$ such that $x \mapsto xG_c'(x)$ is decreasing on the interval $[\mu(c), 1)$.

Let $W\colon [-\frac{1}{e},\infty) \to [-1,\infty)$ denote the Lambert W function, i.e. the inverse of the homeomorphism $[-1,\infty) \to [-\frac{1}{e},\infty)$, $x \mapsto xe^x$ (see Corless et al. (1996)) and let

$$\mu\colon (c_0, 10] \to (0,\infty), \quad c \mapsto -\frac{1}{\ln(b_c)}\left(1 + W\left(\frac{1}{ea_c}\right)\right).$$

Figure 4 shows the graph of μ.

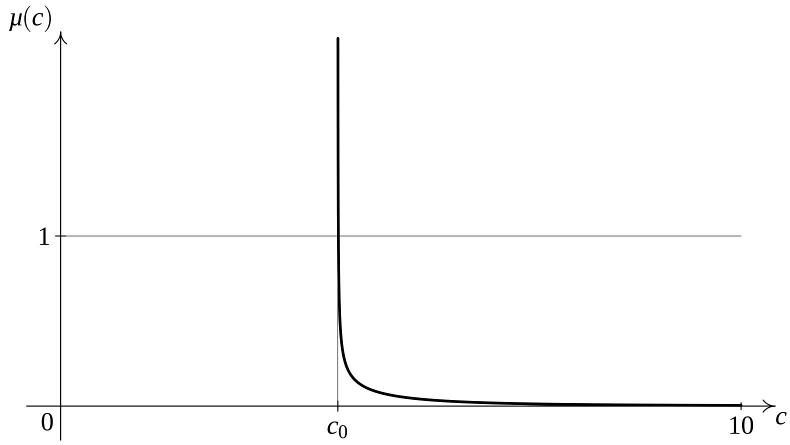

FIGURE 4: Graph of μ

Proposition 6.2. *If $\mu(c) < 1$ for a parameter $c \in (c_0, 10]$ then the local Pareto alphas $\alpha_c(x)$ of the MBBEFD exposure curve G_c are larger or equal to 1 for $x \in [\mu(c), 1)$. Consequently, the function $x \mapsto xG_c'(x)$ is decreasing on the interval $[\mu(c), 1)$. For $c \leq c_0$ we have $\lim_{x \nearrow 1} \alpha_c(x) < 1$, i.e. there is no $\nu \in (0,1)$ such that $\alpha_c(x) \geq 1$ on $[\nu, 1)$.*

The proofs of this section are presented in Appendix E. We use this proposition to apply Theorem 4.5 to the special case of the MBBEFD exposure curves.

Corollary 6.3. *Let $c > c_0$ and let $C, D, l > 0$. Let $\mathcal{L}$ be a location profile which satisfies $\mu(c) \cdot \max(\mathcal{T}(\mathcal{L})) \leq D$. Then*

$$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G_c) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G_c).$$

If $\mathcal{L}$ satisfies $\mu(c) \cdot \max(\mathcal{P}(\mathcal{L})) \leq D$ then

$$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G_c) \geq \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G_c).$$

In the case $\mu(c) < 1$ this is a direct consequence of Theorem 4.5 and Proposition 6.2. If $\mu(c) \geq 1$ the inequalities are true since both sides are zero.

Example 6.4. Consider a location profile $\mathcal{L}$ consisting of two policies:

Policy	Location	Sum insured	Premium
Policy A	1	10 000	7
	2	6 000	3
	3	2 000	1
Policy B	1	6 000	4
	2	3 000	2

The corresponding top location profile $\mathcal{T}(\mathcal{L})$ consists of two risks: risk A with a sum insured of 10 000 and a premium of 11 and risk B with a sum insured of 6 000 and a premium of 6. The corresponding policy profile $\mathcal{P}(\mathcal{L})$ consists of two risks: risk A with a sum insured of 18 000 and a premium of 11 and risk B with a sum insured of 9 000 and a premium of 6.

We pick a loss ratio of $l = 60\%$ and use standard exposure rating to calculate the expected loss for the layer 1 000 xs 1 000 (= C xs D). For $c = 5$, we have $\mu(5) \approx 0.055$ and consequently $\mu(5) \cdot \max(\mathcal{P}(\mathcal{L})) \approx 990 < 1\,000 = D$. Consequently, Corollary 6.3 ensures that

$$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G_5) \geq \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G_5) \geq \mathrm{EL}_{C,D}(\mathcal{L}, l, G_5). \tag{3}$$

If we explicitly calculate the expected losses with $c = 3$ and $c = 5$ we obtain

	$c = 3$	$c = 5$
$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G_c)$	1.362	1.161
$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G_c)$	1.525	1.115
$\mathrm{EL}_{C,D}(\mathcal{L}, l, G_c)$	1.641	1.059

As expected, inequality (3) holds for $c = 5$. But we see that (3) is not fulfilled if we choose $c = 3$. This example is artificial though: all locations have sum insureds which are larger or equal to the exit point of the considered layer. In practice, inequality (3) will be satisfied in most cases for $c = 3$ as well. □

In view of Theorems 4.9 and 4.12 and Corollary 4.14 we will now analyze for which parameters c the MBBEFD exposure curve G_c is Pareto increasing. For this purpose, we consider the function

$$\varphi\colon [0, 10] \to \mathbb{R}, \quad c \mapsto a_c + (1 - \ln(b_c))b_c.$$

Proposition 6.5. *Let c_1 be the smallest zero of φ. Then $c_1 \in (4.0651, 4.0652)$. The MBBEFD exposure curve G_c is Pareto increasing if and only if $c = 0$ or $c \geq c_1$.*

Figure 5 shows the local Pareto alphas $\alpha_c(x) = -\frac{x \cdot G_c''(x)}{G_c'(x)}$ for $c = 1$, 2, 3, 4, 5, and 6.

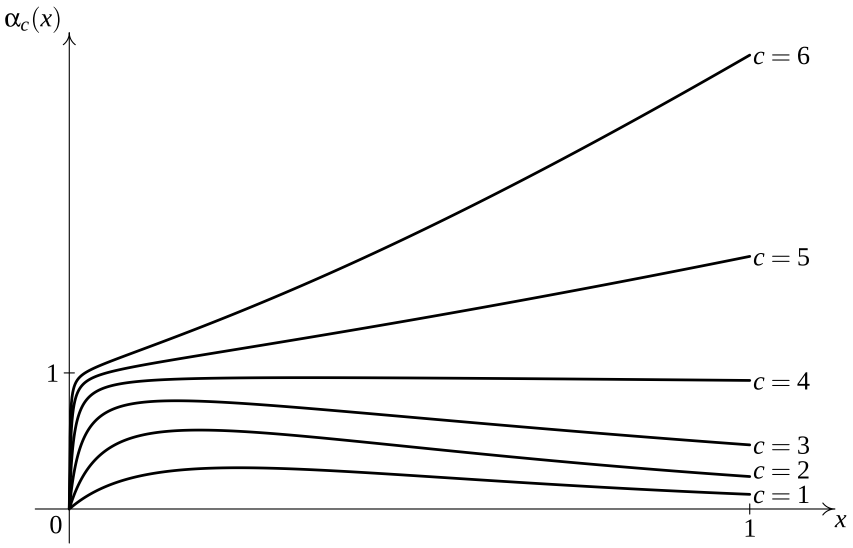

FIGURE 5: Local Pareto alphas of the MBBEFD curves with $c = 1$, 2, 3, 4, 5, and 6

Theorem 4.9 thus states that for the MBBEFD curves and *one policy profiles* a better type of risk profile always leads to a lower price if and only if $c = 0$ or $c \geq c_1$.

Example 6.6. Consider the following one policy location profile $\mathcal{L}$:

Location	Sum insured	Premium
1	7 000	7
2	6 000	3
3	5 000	2

Let $C_0 = D_0 = 1\,000$ and $D = C = 2\,000$, i.e. we want to use the burning cost of the reference layer 1 000 xs 1 000 to calculate the expected loss of the layer 2 000 xs 2 000. Assume that the burning cost of C_0 xs D_0 is $\mathrm{BC}_0 = 0.6$. Standard exposure rating with a loss ratio $l = 100\%$ yields the following results for the parameters $c = 3$ and $c = 5$:

	$c = 3$	$c = 5$
$\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G_c)$	1.514	1.387
$\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G_c)$	1.859	1.288
$\mathrm{EL}_{C_0,D_0}(\mathcal{L}, 1, G_c)$	1.899	1.269
$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G_c)$	1.763	1.328
$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G_c)$	2.201	1.111
$\mathrm{EL}_{C,D}(\mathcal{L}, 1, G_c)$	2.269	1.075

	$c = 3$	$c = 5$
$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G_c)/\,\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G_c)$	116.5%	95.7%
$\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G_c)/\,\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G_c)$	118.4%	86.2%
$\mathrm{EL}_{C,D}(\mathcal{L}, 1, G_c)/\,\mathrm{EL}_{C_0,D_0}(\mathcal{L}, 1, G_c)$	119.5%	84.7%

With $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$ we get the following results:

	$c = 3$	$c = 5$
$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G_c)$	0.699	0.574
$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G_c)$	0.710	0.517
$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}, \rho_0, G_c)$	0.717	0.508

As expected, for $c = 5$ we have

$$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G_5) \geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G_5) \geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}, \rho_0, G_5).$$

But for $c = 3$ we see that

$$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G_3) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G_3) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}, \rho_0, G_3).$$

□

Note that this example is only characteristic of one policy profiles. In the next corollary, we will see that all MBBEFD curves typically reward a better quality of risk profile by a lower price if we take the portfolio structure into account.

Proposition 6.5 states that the MBBEFD exposure curves G_c are not Pareto increasing for $0 < c < c_1 \approx 4.065$. However, in Appendix F we show that G_c can be approximated very well by EP exposure curves for $0 < c \leq 4.1$. Since EP exposure curves are Pareto increasing the following statement is a direct consequence of Corollary 4.14.

Corollary 6.7. *Let $\mathcal{L}$ be the location profile of a 'homogeneous' portfolio with approximately Pareto distributed insured sums. More precisely, assume that $\mathcal{L}$ satisfies conditions a), b) and c) of Corollary 4.14. Let $C_0, D_0, C, D > 0$ with $C + D \geq C_0 + D_0$, $D > D_0$ and $\rho_0 := (C_0, D_0, \mathrm{BC}_0)$ with $\mathrm{BC}_0 > 0$. Then*

$$\mathrm{EL}^{BC}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G_c) \gtrsim \mathrm{EL}^{BC}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G_c) \gtrsim \mathrm{EL}^{BC}_{C,D}(\mathcal{L}, \rho_0, G_c)$$

for all MBBEFD exposure curves G_c with $c \in [0, 10]$.

7 A Practical Example

The following example shows that the presented results do not only provide an incentive for underwriters and actuaries to push for better information but are also useful for the selection of loss models in certain situations.

Assume that we want to estimate the expected loss cost of a property per risk program consisting of three layers 1 000 xs 1 000, 3 000 xs 2 000 and 5 000 xs 5 000. The program covers an industrial fire portfolio. The loss history is sufficient for a reliable burning cost calculation for the first layer but not for the higher layers. The burning cost of the first layer is 507. In practice, it is common to use a Pareto model with a 'typical' Pareto alpha, e.g. $\alpha = 1.6$, to extrapolate the burning cost of the first layer to the higher layers.

In addition to the loss history, the cedent has provided a risk profile, which is displayed in Table 7.

Risk band	# of risks	Average SI	Gross premium	Risk band	# of risks	Average SI	Gross premium
1	7 128	1 420	10 121	8	45	8 486	382
2	1 488	2 453	3 650	9	33	9 488	313
3	559	3 466	1 938	10	25	10 489	262
4	273	4 474	1 221	11	19	11 490	218
5	154	5 479	844	12	15	12 491	187
6	96	6 482	622	13	12	13 491	162
7	64	7 484	479	14	10	14 492	145

Table 7: Policy profile of the covered portfolio

The profile is a policy profile and therefore suboptimal for exposure rating. Nevertheless, we use the MBBEFD exposure curve with $c = 5$ to obtain two additional models: Standard exposure rating with a loss ratio of 60% and BC-adjusted exposure rating, where the first layer is used as reference layer. The following table shows the expected loss cost of the three layers and the considered models.

Layer	BC and Pareto extrapolation with $\alpha = 1.6$	Standard exposure rating	BC-adjusted exposure rating
1 000 xs 1 000	507	795	507
3 000 xs 2 000	416	432	276
5 000 xs 5 000	193	96	61

We see that burning cost-adjusted exposure rating is by far the cheapest option. It corresponds to standard exposure rating with a loss ratio of 38%, which seems to be unrealistic. Is it justified to choose this option? The models can be evaluated as follows.

Pareto Model: Since the loss history seems to be sufficient, it is reasonable to use the burning cost for the first layer. The Pareto extrapolation with $\alpha = 1.6$ might be appropriate for the market average but doesn't reflect the individual portfolio structure and can thus be challenged.

Standard Exposure Rating: The MBBEFD exposure with $c = 5$ is suitable for top location profiles (see Table 6). Since $\mu(5) \cdot 15\,000 \approx 825 < 1\,000$ Corollary 6.3 states that standard exposure rating with the policy profile overestimates the loss cost of all layers. This explains why standard exposure rating for the first layer is much more expensive than the burning cost.

BC-Adjusted Exposure Rating: The distribution of the sums insured in the profile is close to a truncated Pareto distribution with $\alpha = 1.8$. If the portfolio is sufficiently homogeneous (which is arguably true), BC-adjusted exposure rating with the available policy profile is more expensive than with the corresponding top location profile (Corollary 6.7). Since the MBBEFD exposure curve is designed for top location profiles, we can expect that BC-adjusted exposure rating with the policy profile is a conservative estimate of the loss cost.

We see that there is a rationale to use BC-adjusted exposure rating, which is the cheapest of the three models. It is remarkable that the findings of this paper enable us to make a statement on exposure rates even if the data on risk exposures is not available in the required granularity. However, the proven facts about the influence of the risk profile type should not obscure that in practice it will still often be a challenge to find adequate exposure curves for certain portfolios.

8 Summary and Conclusions

We have analyzed the influence of the risk profile type for standard exposure rating and burning cost-adjusted exposure rating. Three types of risk profiles have been considered: Location profiles, top location profiles and policy profiles.

We have demonstrated that for standard exposure rating a better type of risk profile always leads to a lower price if and only if the used exposure curve has local Pareto alphas larger or equal to 1 in the relevant range. However, for realistic portfolios and layers, we mostly get a lower price for the better profile type even if the local Pareto alphas are below 1.

For burning cost-adjusted exposure rating there is no exposure curve which always ensures a lower price for a better type of risk profile. But we have shown that for homogeneous portfolios with Pareto distributed sums insured it is sufficient that the exposure curve is Pareto increasing, i.e. that the curve is differentiable and the local Pareto alphas are increasing. For practical application, we have provided a large and flexible family of Pareto increasing exposure curves (the EP exposure curves).

Further, we have discussed the one-parameter family of MBBEFD exposure curves which are very often used in practice. With regard to standard exposure rating, we have determined the arguments with local Pareto alphas larger or equal to 1. As for burning cost-adjusted exposure rating, we have seen that all MBBEFD exposure curves are either Pareto increasing or can be approximated very well by Pareto increasing exposure curves. For realistic portfolios, a better type of risk profile therefore typically leads to a lower price if a MBBEFD exposure curve is used.

Finally, we have discussed a practical example which shows that in certain situations the results of this paper are useful for the selection of loss models.

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Appendix A

Exposure Rating and Original Deductibles

Theoretically, exposure curves contain the complete information on the severity distribution (in percent of the SI). In reality, however, the curves are often derived from statistics of large losses. In these statistics, detailed information on small losses and original deductibles is missing. Therefore, the curves are not suitable to evaluate the loss elimination ratio of original deductibles being small compared to the sum insured. In a way, an exposure curve contains the information on the distribution of large losses and the ratio between expected large and attritional losses. Original deductibles have almost no impact on the distribution of large losses but they can heavily influence the ratio between large and attritional losses. In the following, we will see that a change of original deductibles can be taken into account by an adjustment of the loss ratio that is used for standard exposure rating.

Consider a reinsurance layer C xs D and a risk with sum insured $v > D$ which does not have an original deductible. Let X be the degree of loss, i.e. the claim size of the risk in percent of the sum insured and let F_X be the distribution function of X. An exposure curve G is appropriate to rate this risk if

$$1 - G(x) \approx \frac{E(\max(X - x, 0))}{E(X)} = \frac{\int_x^1 (1 - F_X(t))\, dt}{\int_0^1 (1 - F_X(t))\, dt}$$

for all $x \geq D/v$. Let Y denote the degree of loss of the risk after introduction of an ordinary deductible of the policy holder, and let d denote this original deductible in percent of the sum insured. Then $Y = (X - d)|(X > d)$ and

$$1 - F_Y(x) = \frac{1 - F_X(x + d)}{1 - F_X(d)}$$

for $x \geq 0$. Let

$$a := \frac{E(\min(X, d))}{E(X)} = \frac{\int_0^d (1 - F_X(t))\, dt}{\int_0^1 (1 - F_X(t))\, dt} = 1 - \frac{\int_d^1 (1 - F_X(t))\, dt}{\int_0^1 (1 - F_X(t))\, dt}$$

denote the loss elimination ratio of the deductible d (see Klugman et al. (2004)). Since typically $d \ll 1$, it follows that

$$\begin{aligned}
\frac{\int_x^1 (1-F_Y(t))\,dt}{\int_0^1 (1-F_Y(t))\,dt} &= \frac{\int_{x+d}^1 (1-F_X(t))\,dt}{\int_d^1 (1-F_X(t))\,dt} = \frac{1}{1-a}\cdot\frac{\int_{x+d}^1 (1-F_X(t))\,dt}{\int_0^1 (1-F_X(t))\,dt} \\
&\approx \frac{1}{1-a}\cdot(1-G(x+d)) \approx \frac{1}{1-a}\cdot(1-G(x))
\end{aligned}$$

for $x \geq D/v$, i.e. for the risk with original deductible d a realistic loss ratio l and an exposure curve H with $1 - H(x) = \frac{1}{1-a}(1-G(x))$ for $x \geq D/v$ should be used for standard exposure rating. Alternatively, one can use the exposure curve G and adjust the loss ratio l by the factor $1/(1-a)$.

Assume now that we want to rate a portfolio with an average percental deductible d_1 and that the available exposure curve has been derived from a segment of risks with an average deductible d_0. Then the exposure curve can be used if the loss ratio is adjusted by the factor $(1-a_0)/(1-a_1)$, where a_i denotes the loss elimination ratio of the deductible d_i.

Of course, it is almost impossible to determine this adjustment factor $(1-a_0)/(1-a_1)$. But note that this factor is independent of the considered reinsurance layer. Therefore, burning cost-adjusted exposure rating is a good way to deal with the issues caused by original deductibles.

Appendix B
Proof of Proposition 3.6

The following propositions provide technical facts that are used in the proof of Proposition 3.6.

Proposition B.1. *A quasi exposure curve G is Pareto increasing if and only if*

$$\frac{G^{-1}}{(G^{-1})'} \colon (0,1) \to \mathbb{R}$$

is concave.

Proof. Let G be differentiable and twice right differentiable in $(0,1)$. Then G^{-1} is differentiable and twice right differentiable in $(0,1)$ as well. For $x \in (0,1)$ we have

$$\begin{aligned}
1 &= \frac{d}{dx} G^{-1}(G(x)) = (G^{-1})'(G(x)) \cdot G'(x) \quad \text{and} \\
0 &= \frac{d}{dx}\left[(G^{-1})'(G(x)) \cdot G'(x)\right] = \\
&= (G^{-1})''(G(x)) \cdot G'(x)^2 + (G^{-1})'(G(x)) \cdot G''(x),
\end{aligned}$$

i.e.

$$\frac{(G^{-1})''(G(x))}{(G^{-1})'(G(x))} = -\frac{G''(x)}{G'(x)^2}.$$

Thus

$$\begin{aligned}
\left.\frac{d}{dy}\right|_{y=G(x)} \frac{G^{-1}(y)}{(G^{-1})'(y)} &= \left.\frac{(G^{-1})'(y)^2 - G^{-1}(y)\cdot (G^{-1})''(y)}{(G^{-1})'(y)^2}\right|_{y=G(x)} \\
&= 1 + \frac{x \cdot G''(x)}{(G^{-1})'(G(x)) \cdot G'(x)^2} \\
&= 1 + \frac{x \cdot G''(x)}{G'(x)}.
\end{aligned}$$

Since G is strictly increasing, $\frac{G^{-1}}{(G^{-1})'}$ is concave if and only if $(0,1) \to \mathbb{R}$, $x \mapsto \frac{xG''(x)}{G'(x)}$ is decreasing, i.e. if and only if G is Pareto increasing.

It remains to show that concavity of $\gamma := \frac{G^{-1}}{(G^{-1})'} : (0,1) \to \mathbb{R}$ implies that G is differentiable and twice right differentiable. If γ is concave then γ is continuous and right differentiable. Since $G^{-1}|(0,1)$ is a positive and convex function, $\frac{1}{G^{-1}} : (0,1) \to \mathbb{R}$ is continuous and right differentiable. Consequently, $\frac{\gamma}{G^{-1}} = \frac{1}{(G^{-1})'}$ is continuous and right differentiable which finally implies that $(G^{-1})'$ and thus G' are continuous and right differentiable in $(0,1)$. $\square$

Proposition B.2. *A quasi exposure curve G is Pareto increasing if and only if*

$$\frac{d}{dt}\bigg|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0$$

for every choice of $0 < x_1 < x_2 < x_3 < 1$.

Proof. Let G be a quasi exposure curve and let $0 < x_1 < x_2 < x_3 < 1$. Then

$$\begin{aligned}
&\frac{d}{dt}\bigg|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2) - G(tx_1)} = \\
&\quad = \frac{(x_3G'(x_3) - x_2G'(x_2))(G(x_2) - G(x_1)) - (G(x_3) - G(x_2))(x_2G'(x_2) - x_1G'(x_1))}{(G(x_2) - G(x_1))^2} \\
&\quad = \frac{x_3G'(x_3)(G(x_2) - G(x_1)) - x_2G'(x_2)(G(x_3) - G(x_1)) + x_1G'(x_1)(G(x_3) - G(x_2))}{(G(x_2) - G(x_1))^2}.
\end{aligned}$$

Thus,

$$\frac{d}{dt}\bigg|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0$$

is equivalent to

$$x_2G'(x_2) \geq \frac{G(x_3) - G(x_2)}{G(x_3) - G(x_1)} x_1G'(x_1) + \frac{G(x_2) - G(x_1)}{G(x_3) - G(x_1)} x_3G'(x_3). \tag{4}$$

Let $y_i := G(x_i)$. Using

$$x_iG'(x_i) = G^{-1}(y_i) \cdot G'(G^{-1}(y_i)) = \frac{G^{-1}(y_i)}{(G^{-1})'(y_i)}$$

we see that (4) is equivalent to

$$\frac{G^{-1}(y_2)}{(G^{-1})'(y_2)} \geq \frac{y_3 - y_2}{y_3 - y_1} \cdot \frac{G^{-1}(y_1)}{(G^{-1})'(y_1)} + \frac{y_2 - y_1}{y_3 - y_1} \cdot \frac{G^{-1}(y_3)}{(G^{-1})'(y_3)}.$$

Since G is increasing and bijective, we conclude that

$$\frac{d}{dt}\bigg|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0$$

for all $0 < x_1 < x_2 < x_3 < 0$ if and only if $G^{-1}/(G^{-1})'$ is concave in the interval $(0,1)$. $\square$

Proposition B.3. *Let G be Pareto increasing and let $0 < x_1 < x_2 < x_3 < 1$. Then G satisfies*

$$\left.\frac{d}{dt}\right|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2)} \leq 0 \quad \textit{and} \tag{5}$$

$$\left.\frac{d}{dt}\right|_{t=1} \frac{1 - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0 \quad \textit{and} \tag{6}$$

$$\left.\frac{d}{dt}\right|_{t=1} \frac{1 - G(tx_2)}{G(tx_2)} \leq 0. \tag{7}$$

Proof. Inequality (5): We have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2)} &= \frac{(x_3G'(x_3) - x_2G'(x_2))G(x_2) - (G(x_3) - G(x_2))x_2G'(x_2)}{G(x_2)^2} = \\ &= \frac{x_3G'(x_3)G(x_2) - x_2G'(x_2)G(x_3)}{G(x_2)^2} \leq 0\end{aligned}$$

if and only if

$$x_2G'(x_2) \geq \frac{G(x_2)}{G(x_3)} x_3G'(x_3).$$

With $y_i := G(x_i)$ this is equivalent to

$$\frac{G^{-1}(y_2)}{(G^{-1})'(y_2)} \geq \frac{y_2}{y_3} \frac{G^{-1}(y_3)}{(G^{-1})'(y_3)}.$$

This inequality holds true since $G^{-1}/(G^{-1})' \colon (0,1) \to \mathbb{R}$ is concave and positive.
Inequality (6): Moreover, we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=1} \frac{1 - G(tx_2)}{G(tx_2) - G(tx_1)} &= \frac{-x_2G'(x_2)(G(x_2) - G(x_1)) - (1 - G(x_2))(x_2G'(x_2) - x_1G'(x_1))}{(G(x_2) - G(x_1))^2} \\ &= \lim_{\gamma \nearrow 1} \frac{-x_2G'(x_2)(G(x_2) - G(x_1)) - (G(\gamma) - G(x_2))(x_2G'(x_2) - x_1G'(x_1))}{(G(x_2) - G(x_1))^2} \\ &\leq \lim_{\gamma \nearrow 1} \frac{(\gamma G'(\gamma) - x_2G'(x_2))(G(x_2) - G(x_1)) - (G(\gamma) - G(x_2))(x_2G'(x_2) - x_1G'(x_1))}{(G(x_2) - G(x_1))^2} \\ &= \lim_{\gamma \nearrow 1} \left.\frac{d}{dt}\right|_{t=1} \frac{G(t\gamma) - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0.\end{aligned}$$

Inequality (7): Obviously true, since G is increasing. $\square$

Proof of Proposition 3.6. Let G be Pareto increasing. Let $0 \leq x_1 < x_2$ and $y_1 < y_2$ with $x_1 < y_1$ and $x_2 \leq y_2$. We have to show that

$$f \colon (0, 1/x_1) \to \mathbb{R}, t \mapsto \frac{G_+(ty_2) - G_+(ty_1)}{G_+(tx_2) - G_+(tx_1)}$$

is decreasing. (The opposite direction follows directly from Proposition B.2.)

Case 1: $y_1 = x_2$
For $t \in (0, 1/y_2)$ we have

$$f'(t) = \left.\frac{d}{d\tau}\right|_{\tau=1} \frac{f(\tau t)}{t} = \frac{1}{t} \left.\frac{d}{d\tau}\right|_{\tau=1} \frac{G(\tau t y_2) - G(\tau t x_2)}{G(\tau t x_2) - G(\tau t x_1)} \leq 0.$$

In the case $x_1 > 0$ this follows directly from Proposition B.2, in the case $x_1 = 0$ this is a consequence of Proposition B.3 (5).

For $t \in [1/y_2, 1/x_2)$ we have

$$f'(t) = \frac{d}{d\tau}\bigg|_{\tau=1} \frac{f(\tau t)}{t} = \frac{1}{t}\frac{d}{d\tau}\bigg|_{\tau=1} \frac{1 - G(\tau t x_2)}{G(\tau t x_2) - G(\tau t x_1)} \leq 0.$$

In the case $x_1 > 0$ this follows from Proposition B.3 (6) and in the case $x_1 = 0$ from Proposition B.3 (7). For $t \in [1/x_2, 1/x_1)$ we have $f(t) = 0$.

Case 2: $y_1 > x_2$
For $t \in (0, 1/y_1)$

$$f(t) = \frac{G_+(ty_2) - G_+(ty_1)}{G_+(tx_2) - G_+(tx_1)} = \frac{G_+(ty_2) - G_+(ty_1)}{G_+(ty_1) - G_+(tx_2)} \cdot \frac{G_+(ty_1) - G_+(tx_2)}{G_+(tx_2) - G_+(tx_1)}.$$

is decreasing since both factors are decreasing and positive (apply Case 1). For $t \in [1/y_1, 1/x_1)$ we have $f(t) = 0$.

Case 3: $y_1 < x_2$, $y_2 > x_2$
For $t \in (0, 1/y_1)$ we have

$$f(t) = \frac{G_+(ty_2) - G_+(ty_1)}{G_+(tx_2) - G_+(tx_1)} = \frac{\frac{G_+(ty_2) - G_+(tx_2)}{G_+(tx_2) - G_+(ty_1)} + 1}{\frac{G_+(ty_1) - G_+(tx_1)}{G_+(tx_2) - G_+(ty_1)} + 1}.$$

From Case 1 we know that the numerator is decreasing and the denominator is increasing. Since numerator and denominator are positive, f is decreasing. Moreover, $f(t) = 0$ for $t \in [1/y_1, 1/x_1)$.

Case 4: $y_1 < x_2$, $y_2 = x_2$
We have

$$f(t) = \frac{G_+(ty_2) - G_+(ty_1)}{G_+(tx_2) - G_+(tx_1)} = \frac{1}{\frac{G_+(ty_1) - G_+(tx_1)}{G_+(tx_2) - G_+(ty_1)} + 1}$$

for $t \in (0, 1/y_1)$. Like in Case 3 the denominator is positive and increasing. For $t \in [1/y_1, 1/x_1)$ we have $f(t) = 0$. □

Appendix C
Proofs of Section 4

Proof of Proposition 4.4. Let $\mathcal{L} =: \{(s_{(i,n)}, p_{(i,n)}\}_{(i,n) \in J(I,j)}$ with $\max(\mathcal{L}) \leq C + D$. Since G_+ is an increasing function, we have

$$\begin{aligned}
\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) &= l \sum_{i \in I} \left[1 - G_+\left(\frac{D}{s_i^T}\right)\right] \cdot p_i^T \\
&= l \sum_{i \in I} \sum_{n=1}^{j(i)} \left[1 - G_+\left(\frac{s_{(i,n)}}{s_i^T} \cdot \frac{D}{s_{(i,n)}}\right)\right] \cdot p_{(i,n)} \\
&\geq l \sum_{i \in I} \sum_{n=1}^{j(i)} \left[1 - G_+\left(\frac{D}{s_{(i,n)}}\right)\right] \cdot p_{(i,n)} \\
&= \mathrm{EL}_{C,D}(\mathcal{L}, l, G).
\end{aligned}$$

Moreover, if $\max(\mathcal{P}(\mathcal{L})) \le C + D$ then

$$\begin{aligned}\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) &= l \sum_{i\in I} \left[1 - G_+\left(\frac{D}{s_i^P}\right)\right] \cdot p_i^P \\ &\ge l \sum_{i\in I} \left[1 - G_+\left(\frac{D}{s_i^T}\right)\right] \cdot p_i^T = \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G)\end{aligned}$$

since $s_i^P \ge s_i^T$ and $p_i^P = p_i^T$ for all $i \in I$. □

Proof of Theorem 4.5. 1. ⇒ 2.: Let $[\mu, 1) \to \mathbb{R}$, $x \mapsto xG'(x)$ be decreasing. Then $[\mu, \infty) \to \mathbb{R}$, $x \mapsto xG'_+(x)$ is decreasing and we have

$$\left.\frac{d}{dt}\right|_{t=1} \big[G_+(tx_2) - G_+(tx_1)\big] = x_2 G'_+(x_2) - x_1 G'_+(x_1) \le 0$$

for all $x_2 > x_1 \ge \mu$, i.e.

$$[\mu/x_1, \infty) \to \mathbb{R}, \quad t \mapsto G_+(tx_2) - G_+(tx_1)$$

is decreasing for all $x_2 > x_1 > 0$. Let $\mathcal{L} = \{(s_{(i,n)}, p_{(i,n)}\}_{(i,n)\in J(I,j)}$ be a location profile with $\mu \cdot \max(\mathcal{T}(\mathcal{L})) \le D$. Then

$$\begin{aligned}\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G) &= l \sum_{i\in I} \left[G_+\left(\frac{D+C}{s_i^T}\right) - G_+\left(\frac{D}{s_i^T}\right)\right] \cdot p_i^T \\ &= l \sum_{i\in I} \sum_{n=1}^{j(i)} \left[G_+\left(\frac{s_{(i,n)}}{s_i^T} \cdot \frac{D+C}{s_{(i,n)}}\right) - G_+\left(\frac{s_{(i,n)}}{s_i^T} \cdot \frac{D}{s_{(i,n)}}\right)\right] \cdot p_{(i,n)} \\ &\ge l \sum_{i\in I} \sum_{n=1}^{j(i)} \left[G_+\left(\frac{D+C}{s_{(i,n)}}\right) - G_+\left(\frac{D}{s_{(i,n)}}\right)\right] \cdot p_{(i,n)} \\ &= \mathrm{EL}_{C,D}(\mathcal{L}, l, G).\end{aligned}$$

2. ⇒ 3.: Let $\mathcal{L} = \{(s_{(i,n)}, p_{(i,n)}\}_{(i,n)\in J(I,j)}$ be a location profile with $\mu \cdot \max(\mathcal{P}(\mathcal{L})) \le D$. We define an artificial location profile $\mathcal{L}' = \{(s'_{(i,n)}, p'_{(i,n)}\}_{(i,n)\in I\times\{1,2\}}$ by

$$s'_{(i,1)} := s_i^P,\ p'_{(i,1)} := p_i^P,\ s'_{(i,2)} := s_i^T,\ p'_{(i,2)} := p_i^T = p_i^P$$

for $i \in I$. Since $\mu \cdot \max(\mathcal{T}(\mathcal{L}')) = \mu \cdot \max(\mathcal{P}(\mathcal{L})) \le D$ we have

$$\begin{aligned}2 \cdot \mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) &= \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}'), l, G) \\ &\ge \mathrm{EL}_{C,D}(\mathcal{L}', l, G) \\ &= \mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) + \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G)\end{aligned}$$

and thus

$$\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) \ge \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G).$$

3. ⇒ 1.: Assume that $[\mu, 1) \to \mathbb{R}$, $x \mapsto xG'(x)$ is not decreasing. Choose $1 > x_2 > x_1 \ge \mu$ with $x_2 G'(x_2) > x_1 G'(x_1)$. Since

$$\left.\frac{d}{dt}\right|_{t=1} \big[G(tx_2) - G(tx_1)\big] = x_2 G'(x_2) - x_1 G'(x_1) > 0,$$

we can choose a $\xi \in (1, 1/x_2)$, $\xi < 2$ with $G(\xi x_2) - G(\xi x_1) > G(x_2) - G(x_1)$. Let $D := x_1$, $C := x_2 - x_1$ and consider the one policy location profile $\{(s_1, p_1), (s_2, p_2)\}$ with $s_1 = 1/\xi$, $s_2 = 1 - 1/\xi$ and $p_1 = p_2 = 1/2$. Then $s^T = \max(s_1, s_2) = s_1 = 1/\xi$, $s^P = s_1 + s_2 = 1$ $p^T = p^P = p_1 + p_2 = 1$ and thus

$$\begin{aligned}
\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), l, G) &= l\left[G_+\left(\frac{D+C}{s^P}\right) - G_+\left(\frac{D}{s^P}\right)\right] \cdot p^P \\
&= l\left[G(x_2) - G(x_1)\right] \\
&< l\left[G(\xi x_2) - G(\xi x_1)\right] \\
&= l\left[G_+\left(\frac{D+C}{s_1}\right) - G_+\left(\frac{D}{s_1}\right)\right] \cdot p_T \\
&= \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), l, G).
\end{aligned}$$

1. $\Leftrightarrow$ 4.: Let G be twice right differentiable in $[\mu, 1)$. Then

$$\begin{aligned}
x \mapsto xG'(x) \text{ decreasing in } [\mu, 1) &\underset{(8)}{\Leftrightarrow} 0 \geq \frac{d}{dx} xG'(x) = G'(x) + xG''(x) \text{ for all } x \in [\mu, 1) \\
&\Leftrightarrow \alpha(x) = -\frac{xG''(x)}{G'(x)} \geq 1 \text{ for all } x \in [\mu, 1).
\end{aligned}$$

Using the fact that G' is right differentiable and decreasing, the proof of (8) is similar to the proof of the mean value theorem. $\square$

Although the following remark is obvious, it is very useful for the following proofs.

Remark C.1. Let $a_1, \ldots, a_n$ be real numbers and let $b_1, \ldots, b_n > 0$. Then

$$\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} = \frac{b_1}{\sum_{i=1}^n b_i} \cdot \frac{a_1}{b_1} + \cdots + \frac{b_n}{\sum_{i=1}^n b_i} \cdot \frac{a_n}{b_n},$$

i.e. $\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n}$ is a convex combination of $\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}$. In particular, we have

$$\min\left\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\right\} \leq \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max\left\{\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\right\}.$$

Proof of Theorem 4.9. 1. $\Rightarrow$ 2.: Let $\mathcal{L} = \{(s_n, p_n)\}_{n \in \{1,\ldots,j\}}$ be a one policy location profile which satisfies $\max(\mathcal{L}) > D_0$. W.l.o.g. we can assume $s_1 \geq \cdots \geq s_j$. Let $j_0 := \max\{n \in \{1, \ldots, j\} \mid s_n > D_0\}$. Since G is Pareto increasing, we have

$$\frac{G_+\left(\frac{C+D}{s^T}\right) - G_+\left(\frac{D}{s^T}\right)}{G_+\left(\frac{C_0+D_0}{s^T}\right) - G_+\left(\frac{D_0}{s^T}\right)} \geq \frac{G_+\left(\frac{C+D}{s_n}\right) - G_+\left(\frac{D}{s_n}\right)}{G_+\left(\frac{C_0+D_0}{s_n}\right) - G_+\left(\frac{D_0}{s_n}\right)}$$

for $n = 1, \dots, j_0$ (Proposition 3.6). Thus, we have

$$\begin{aligned}
\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G) &= \mathrm{BC}_0 \frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G)} \\
&= \mathrm{BC}_0 \frac{G_+\left(\frac{C+D}{s^T}\right) - G_+\left(\frac{D}{s^T}\right)}{G_+\left(\frac{C_0+D_0}{s^T}\right) - G_+\left(\frac{D_0}{s^T}\right)} \\
&\underset{(9)}{\geq} \mathrm{BC}_0 \frac{\sum_{n=1}^{j_0} \left[G_+\left(\frac{C+D}{s_n}\right) - G_+\left(\frac{D}{s_n}\right)\right] p_n}{\sum_{n=1}^{j_0} \left[G_+\left(\frac{C_0+D_0}{s_n}\right) - G_+\left(\frac{D_0}{s_n}\right)\right] p_n} \\
&= \mathrm{BC}_0 \frac{\mathrm{EL}_{C,D}(\mathcal{L}, 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{L}, 1, G)} \\
&= \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}, \rho_0, G).
\end{aligned}$$

For inequality (9) we have used Remark C.1.

2. $\Rightarrow$ 3.: Let $\mathcal{L} = \{(s_{(i,n)}, p_{(i,n)})\}_{(i,n)\in J(I,j)}$ be a location profile with $\max(\mathcal{L}) \geq D_0$. Like in the proof of Theorem 4.5 we consider the artificial location profile $\mathcal{L}' = \{(s'_{(i,n)}, p'_{(i,n)})\}_{(i,n)\in I\times\{1,2\}}$ defined by

$$s'_{(i,1)} := s_i^P,\ p'_{(i,1)} := p_i^P,\ s'_{(i,2)} := s_i^T,\ p'_{(i,2)} := p_i^T = p_i^P$$

for $i \in I$. Then $\max(\mathcal{L}') = \max(\mathcal{P}(\mathcal{L})) \geq D_0$ and we have

$$\begin{aligned}
\mathrm{BC}_0 \frac{2\cdot \mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G)}{2\cdot \mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G)} &= \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}'), \rho_0, G) \\
&\geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}', \rho_0, G) \\
&= \mathrm{BC}_0 \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G) + \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G) + \mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G)}.
\end{aligned}$$

Using Remark C.1 we obtain

$$\begin{aligned}
\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G) &= \mathrm{BC}_0 \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G)} \\
&\geq \mathrm{BC}_0 \frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G)} = \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G).
\end{aligned}$$

3. $\Rightarrow$ 1.: Let $0 < x_1 < x_2 < x_3 < 1$. Given Proposition B.2 we have to show that

$$\left.\frac{d}{dt}\right|_{t=1} \frac{G(tx_3) - G(tx_2)}{G(tx_2) - G(tx_1)} \leq 0.$$

Let $\xi \in (1, 1/x_3)$, $\xi < 2$. It is sufficient to show

$$\frac{G(x_3) - G(x_2)}{G(x_2) - G(x_1)} \geq \frac{G(\xi x_3) - G(\xi x_2)}{G(\xi x_2) - G(\xi x_1)}.$$

Let $D_0 := x_1$, $C_0 := x_2 - x_1$, $D := x_2$, $C := x_3 - x_2$ and consider the one policy location profile $\{(s_1, p_1), (s_2, p_2)\}$ with $s_1 = 1/\xi$, $s_2 = 1 - 1/\xi$ and $p_1 = p_2 = 1/2$. Then $s^T = \max(s_1, s_2) = s_1 = 1/\xi$, $s^P = s_1 + s_2 = 1$ and $p^T = p^P = p_1 + p_2 = 1$. With $\rho_0 := (C_0, D_0, 1)$ we get

$$\begin{aligned}
\frac{G(x_3) - G(x_2)}{G(x_2) - G(x_1)} &= \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G) \\
&\geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G) = \frac{G(\xi x_3) - G(\xi x_2)}{G(\xi x_2) - G(\xi x_1)}.
\end{aligned}$$

$\square$

Proof of Proposition 4.10. Let $I_0 := \{i \in I \mid s_i > D_0\}$. Choose $\mu \in I_0$ with

$$\frac{G_+\left(\frac{C+D}{s_\mu}\right) - G_+\left(\frac{D}{s_\mu}\right)}{G_+\left(\frac{C_0+D_0}{s_\mu}\right) - G_+\left(\frac{D_0}{s_\mu}\right)} \le \frac{G_+\left(\frac{C+D}{s_i}\right) - G_+\left(\frac{D}{s_i}\right)}{G_+\left(\frac{C_0+D_0}{s_i}\right) - G_+\left(\frac{D_0}{s_i}\right)} \tag{10}$$

for all $i \in I_0$. Choose a $N \in \mathbb{N}$ with $s_\mu/N < D_0$ and define $j\colon I \to \mathbb{N}$ by $j(\mu) := N$ and $j(i) := 1$ for $i \in I \setminus \{\mu\}$.

Let $\sigma_{(\mu,n)} := s_\mu/N$, $\pi_{(\mu,n)} := p_\mu/N$ for $n = 1, \dots, N$ and $\sigma_{(i,1)} := s_i$, $\pi_{(i,1)} := p_i$ for $i \in I \setminus \{\mu\}$. Then $\mathcal{L} := \{\sigma_{(i,n)}, \pi_{(i,n)}\}_{(i,n)\in J(I,j)}$ is a location profile with $\mathcal{P}(\mathcal{L}) = \mathcal{Q}$. Using (1) and (10) it is easy to see that $\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G)$.

Let $\sigma'_{(\mu,1)} := s_\mu$, $\sigma'_{(\mu,n)} := s_\mu/N$ for $n = 2, \dots, N$ and $\pi'_{(\mu,n)} := p_\mu/N$ for $n = 1, \dots, N$. Moreover, let $\sigma'_{(i,1)} := s_i$, $\pi'_{(i,1)} := p_i$ for $i \in I \setminus \{\mu\}$. Then $\mathcal{L}' := \{\sigma'_{(i,n)}, \pi'_{(i,n)}\}_{(i,n)\in J(I,j)}$ is a location profile with $\mathcal{T}(\mathcal{L}') = \mathcal{Q}$. Using (1) and (10) it is easy to see that $\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}'), \rho_0, G) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}', \rho_0, G)$. $\square$

Proof of Theorem 4.12. The inequality

$$\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) \ge \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G)$$

follows from

$$\begin{aligned}
&\lim_{\nu\to\infty} \frac{\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G)}{\mathrm{BC}_0} = \lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}_\nu), 1, G)} \\
&= \lim_{\nu\to\infty} \frac{\sum_{i\in I_\nu} \pi s^P_{\nu,i}(G_+(\frac{C+D}{s^P_{\nu,i}}) - G_+(\frac{D}{s^P_{\nu,i}}))}{\sum_{i\in I_\nu} \pi s^P_{\nu,i}(G_+(\frac{C_0+D_0}{s^P_{\nu,i}}) - G_+(\frac{D_0}{s^P_{\nu,i}}))} = \lim_{\nu\to\infty} \frac{\int_t^U \pi x(G_+(\frac{C+D}{x}) - G_+(\frac{D}{x}))\, dF_\nu(x)}{\int_t^U \pi x(G_+(\frac{C_0+D_0}{x}) - G_+(\frac{D_0}{x}))\, dF_\nu(x)} \\
&= \frac{\int_t^U \pi x(G_+(\frac{C+D}{x}) - G_+(\frac{D}{x}))\, dF(x)}{\int_t^U \pi x(G_+(\frac{C_0+D_0}{x}) - G_+(\frac{D_0}{x}))\, dF(x)} = \frac{\int_{D_0}^U x(G_+(\frac{C+D}{x}) - G_+(\frac{D}{x}))x^{-\alpha-1}\, dx}{\int_{D_0}^U x(G_+(\frac{C_0+D_0}{x}) - G_+(\frac{D_0}{x}))x^{-\alpha-1}\, dx} \\
&= \frac{\int_{D_0/\gamma_1}^{U/\gamma_1} x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))x^{-\alpha-1}\, dx}{\int_{D_0/\gamma_1}^{U/\gamma_1} x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))x^{-\alpha-1}\, dx} \underset{(11)}{\ge} \frac{\int_{D_0/\gamma_1}^{U} x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))x^{-\alpha-1}\, dx}{\int_{D_0/\gamma_1}^{U} x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))x^{-\alpha-1}\, dx} \\
&= \frac{\int_t^U \pi\gamma_1 x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))\, dF(x)}{\int_t^U \pi\gamma_1 x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))\, dF(x)} = \lim_{\nu\to\infty} \frac{\int_t^U \pi\gamma_1 x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))\, dF_\nu(x)}{\int_t^U \pi\gamma_1 x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))\, dF_\nu(x)} \\
&= \lim_{\nu\to\infty} \frac{\sum_{i\in I_\nu} \pi\gamma_1 s^P_{\nu,i}(G_+(\frac{C+D}{\gamma_1 s^P_{\nu,i}}) - G_+(\frac{D}{\gamma_1 s^P_{\nu,i}}))}{\sum_{i\in I_\nu} \pi\gamma_1 s^P_{\nu,i}(G_+(\frac{C_0+D_0}{\gamma_1 s^P_{\nu,i}}) - G_+(\frac{D_0}{\gamma_1 s^P_{\nu,i}}))} = \lim_{\nu\to\infty} \frac{\sum_{i\in I_\nu} p^T_{\nu,i}(G_+(\frac{C+D}{s^T_{\nu,i}}) - G_+(\frac{D}{s^T_{\nu,i}}))}{\sum_{i\in I_\nu} p^T_{\nu,i}(G_+(\frac{C_0+D_0}{s^T_{\nu,i}}) - G_+(\frac{D_0}{s^T_{\nu,i}}))} \\
&= \lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}_\nu), 1, G)} = \lim_{\nu\to\infty} \frac{\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G)}{\mathrm{BC}_0}.
\end{aligned}$$

In order to see that the inequality (11) really holds, we define

$$\psi(x) := \frac{G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x})}{G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x})} = \frac{x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))x^{-\alpha-1}}{x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))x^{-\alpha-1}}$$

and

$$\Psi(a,b) := \frac{\int_a^b x(G_+(\frac{C+D}{\gamma_1 x}) - G_+(\frac{D}{\gamma_1 x}))x^{-\alpha-1}\,dx}{\int_a^b x(G_+(\frac{C_0+D_0}{\gamma_1 x}) - G_+(\frac{D_0}{\gamma_1 x}))x^{-\alpha-1}\,dx}.$$

Then (11) can be rewritten as

$$\Psi(D_0/\gamma_1, U/\gamma_1) \geq \Psi(D_0/\gamma_1, U). \tag{12}$$

Since the case $\gamma_1 = 1$ (i.e. $N = 1$) is trivial, we assume $\gamma_1 < 1$. Since G is Pareto increasing, $x \mapsto \psi(x)$ is increasing (Proposition 3.6). It follows that

$$\psi(x) \leq \psi(U) \leq \psi(y) \quad \text{for all } x \in (D_0/\gamma_1, U),\ y \in (U, U/\gamma_1)$$

and consequently $\Psi(D_0/\gamma_1, U) \leq \psi(U) \leq \Psi(U, U/\gamma_1)$. Since $\Psi(D_0/\gamma_1, U/\gamma_1)$ is a weighted average of $\Psi(D_0/\gamma_1, U)$ and $\Psi(U, U/\gamma_1)$ (see Remark C.1), the inequality (12) holds.

For the proof of the inequality $\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G) \geq \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}_\nu, \rho_0, G)$ we can w.l.o.g. assume $N > 1$. Choose $M \in \mathbb{N}$ with $(\gamma_1 - \gamma_N)/M \leq \gamma_N$. For $\mu = 1, \ldots, N$ and $\nu \geq 1$ we consider the location profiles

$$\mathcal{L}_{\nu,\mu} := \{(s_{\nu,\mu,(i,m)}, p_{\nu,\mu,(i,m)})\}_{(i,m)\in I_\nu \times \{1,\ldots,M+1\}}$$

where $s_{\nu,\mu,(i,1)} := s_{\nu,(i,\mu)}$, $p_{\nu,\mu,(i,1)} := p_{\nu,(i,\mu)}/2$, $s_{\nu,\mu,(i,m)} := (s_{\nu,(i,1)} - s_{\nu,(i,\mu)})/M$ and $p_{\nu,\mu,(i,m)} := p_{\nu,(i,\mu)}/(2M)$ for $m = 2, \ldots, M+1$. For every $\mu = 1, \ldots, N$ the sequence $(\mathcal{L}_{\nu,\mu})_{\nu\geq 1}$ fulfills the assumptions 1. to 4. of Theorem 4.12. Using the first part of this proof, we have

$$\lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)} \geq \lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}_{\nu,\mu}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}_{\nu,\mu}), 1, G)}$$

for $\mu = 1, \ldots, N_0$, where $N_0 := \max\{n \in \{1, \ldots, N\} \mid \gamma_n U > D_0\}$. Moreover,

$$\frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}_\nu), 1, G)} = \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)}$$

for $\nu \geq 1$ and $\mu = 1, \ldots, N_0$. Using Remark C.1 we obtain

$$\begin{aligned}
\lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}_\nu), 1, G)} &= \lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}_{\nu,\mu}), 1, G)} \\
&\geq \lim_{\nu\to\infty} \frac{\sum_{j=1}^{N_0} \mathrm{EL}_{C,D}(\mathcal{T}(\mathcal{L}_{\nu,j}), 1, G)}{\sum_{j=1}^{N_0} \mathrm{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}_{\nu,j}), 1, G)} \\
&= \lim_{\nu\to\infty} \frac{\mathrm{EL}_{C,D}(\mathcal{L}_\nu, 1, G)}{\mathrm{EL}_{C_0,D_0}(\mathcal{L}_\nu, 1, G)},
\end{aligned}$$

which implies the second inequality. □

Remark C.2. Let us take a closer look at the inequalities of Theorem 4.12. In the trivial cases, i.e. $\gamma_1 = 1$ or $U \leq D$, we obviously have

$$\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) = \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G).$$

If $\gamma_1 < 1$ and $U > D/\gamma_1$, we have (with the functions ψ and Ψ from the proof of Theorem 4.12)

$$\psi(x) = 0 < \psi(y) \leq \psi(z) \quad \text{for all } x \in (D_0/\gamma_1, D/\gamma_1),\ y \in (D/\gamma_1, U),\ z \in (U, U/\gamma_1)$$

and consequently

$$\Psi(D_0/\gamma_1, D/\gamma_1) = 0 < \Psi(D/\gamma_1, U) \leq \Psi(U, U/\gamma_1).$$

Since $\Psi(D_0/\gamma_1, U)$ is a weighted average of $\Psi(D_0/\gamma_1, D/\gamma_1)$ and $\Psi(D/\gamma_1, U)$ with positive weights (see Remark C.1), we have $\Psi(D_0/\gamma_1, U) < \Psi(U, U/\gamma_1)$. Since $\Psi(D_0/\gamma_1, U/\gamma_1)$ is a weighted average of $\Psi(D_0/\gamma_1, U)$ and $\Psi(U, U/\gamma_1)$ with positive weights, we can replace inequality (12) in the proof of Theorem 4.12 by the strict inequality

$$\Psi(D_0/\gamma_1, U/\gamma_1) > \Psi(D_0/\gamma_1, U).$$

If $\gamma_1 < 1$ and $D < U \leq D/\gamma_1$, we obviously have

$$\Psi(D_0/\gamma_1, U/\gamma_1) > 0 = \Psi(D_0/\gamma_1, U).$$

Thus, for $\gamma_1 < 1$ and $U > D$ (which is the nontrivial case) we have proven

$$\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) > \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G).$$

For $N > 1$, $\gamma_1 > \gamma_N$ and $U > D/\gamma_1$ one can analogously prove

$$\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G) > \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}_\nu, \rho_0, G).$$

For realistic choices of the parameters of the Pareto distribution and of the layers, these inequalities are ‘far away’ from equality. This makes the inequalities

$$\lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}_\nu), \rho_0, G) \geq \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}_\nu), \rho_0, G) \geq \lim_{\nu\to\infty} \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L}_\nu, \rho_0, G)$$

rather robust, i.e. the inequalities often hold even if the assumptions of Theorem 4.12 are not strictly fulfilled.

Appendix D
Example of a Pareto Distributed Portfolio

The following example shows that the assertion of Theorem 4.12 does not hold in general if G is not Pareto increasing.

For $t = 1\,000$, $U = 10\,000$ and $\alpha = 0.8$ we consider the truncated Pareto distribution

$$F(x) := \begin{cases} 0 & \text{for } x < t \\ \left(1 - \left(\frac{t}{x}\right)^\alpha\right) / \left(1 - \left(\frac{t}{U}\right)^\alpha\right) & \text{for } t \leq x < U \\ 1 & \text{for } x \geq U. \end{cases}$$

Let

$$F^-(x) := (F|[t, U])^{-1}(x) = \frac{t}{(1 - (1 - (\frac{t}{U})^\alpha)x)^{1/\alpha}},$$

$I = \{1, \ldots, 10\,000\}$, $N = 2$, $\gamma_1 = 0.6$, $\gamma_2 = 0.4$, $\pi = 0.05\%$ and

$$s_i^P := F^-\left(\tfrac{i}{10\,000}\right), \; p_i^P := \pi s_i^P, \; s_{(i,n)} := \gamma_n s_i^P, \; p_{(i,n)} := \gamma_n p_i^P, \; s_i^T := \gamma_1 s_i^P, \; p_i^T := p_i^P$$

for $i \in I$ and $n \in \{1, 2\}$. Then $\mathcal{L} = \{(s_{(i,n)}, p_{(i,n)})\}_{(i,n)\in I\times\{1,2\}}$ is a single location profile with corresponding top location profile $\mathcal{T}(\mathcal{L}) = \{(s_i^T, p_i^T)\}_{i\in I}$ and policy profile $\mathcal{P}(\mathcal{L}) = \{(s_i^P, p_i^P)\}_{i\in I}$. The distribution function

$$F_{\mathcal{P}(\mathcal{L})}(x) := \frac{\#\{i \in I \mid s_i^P \leq x\}}{\#I}$$

is very close to F. Figure 6 shows the graphs of $F_{\mathcal{L}}$, $F_{\mathcal{T}(\mathcal{L})}$ and $F_{\mathcal{P}(\mathcal{L})}$ where

$$F_{\mathcal{L}}(x) := \frac{\#\{(i,n) \in I \times \{1,2\} \,|\, s_{(i,n)} \le x\}}{\#(I \times \{1,2\})} \quad \text{and} \quad F_{\mathcal{T}(\mathcal{L})}(x) := \frac{\#\{i \in I \,|\, s_i^T \le x\}}{\#I}.$$

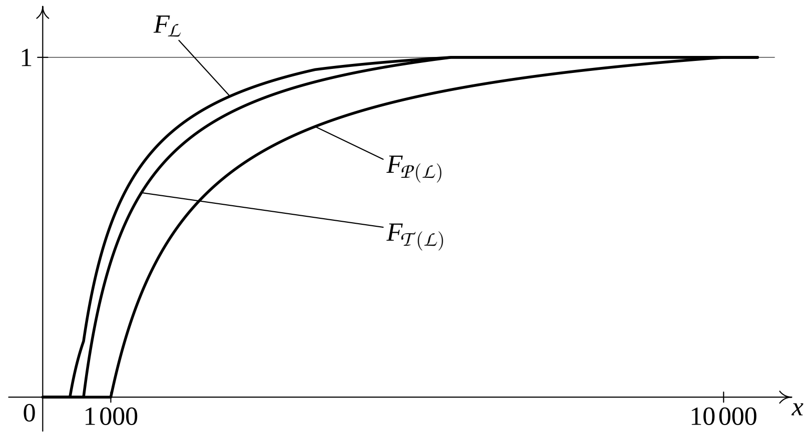

FIGURE 6: Distributions of the sums insured of $\mathcal{L}$, $\mathcal{T}(\mathcal{L})$ and $\mathcal{P}(\mathcal{L})$

For $0 < t_1 < t_2 < 1$ and $0 < a \neq 1$ we define

$$g_{t_1,a,t_2}(x) := \begin{cases} x & \text{if } 0 \le x \le t_1 \\ t_1 + \frac{t_1^a}{1-a}(x^{1-a} - t_1^{1-a}) & \text{if } t_1 < x \le t_2 \\ t_1 + \frac{t_1^a}{1-a}(t_2^{1-a} - t_1^{1-a}) + \frac{t_1^a}{t_2^a}(x - t_2) & \text{if } t_1 < x \le 1 \end{cases}$$

and

$$G^{\text{step}}_{t_1,a,t_2}(x) := \frac{g_{t_1,a,t_2}(x)}{g_{t_1,a,t_2}(1)}.$$

Then $G^{\text{step}}_{t_1,a,t_2}$ is the differentiable and twice right differentiable exposure curve with local Pareto alphas

$$\alpha^{\text{step}}_{t_1,a,t_2}(x) = \begin{cases} a & \text{for } x \in [t_1, t_2) \\ 0 & \text{for } x \in [0,1] \setminus [t_1, t_2) \end{cases}$$

(cf. Section 5). In particular, $G^{\text{step}}_{t_1,\alpha,t_2}$ is not Pareto increasing. Consider the exposure curve

$$G_{\text{step}}(x) := G^{\text{step}}_{0.2,4,0.5}(x)$$

and the Pareto increasing quasi exposure curve

$$G_{\text{DLS}}(x) := x^{\log_2(1.1)}$$

which corresponds to a doubled limits surcharge of $z = 10\%$ (see example 3.5). Figures 7 and 8 show the graphs of the exposure curves and the corresponding local Pareto alphas.

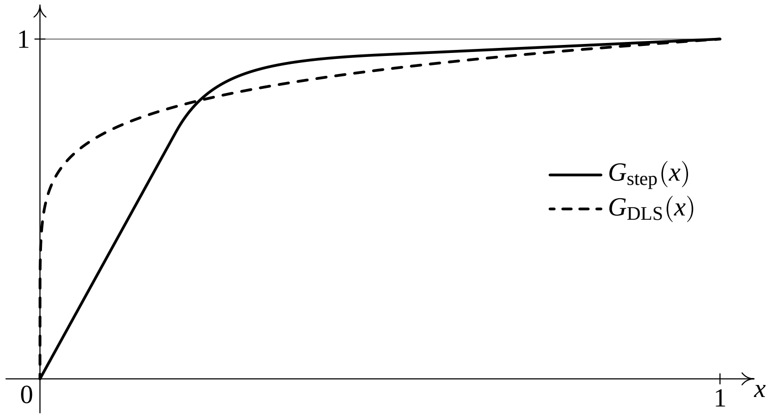

FIGURE 7: Graphs of the exposure curves G_{step} and G_{DLS}

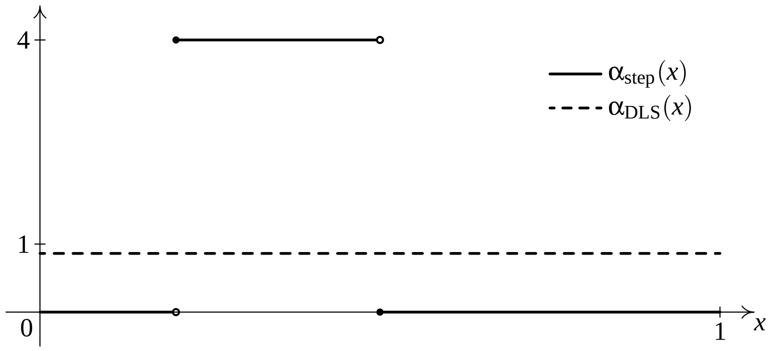

FIGURE 8: Local Pareto alphas of G_{step} and G_{DLS}

Let $C_0 = C = 500$, $D_0 = 1\,500$ and $C = 2\,000$, i.e. we want to use the burning cost of the reference layer 500 xs 1 500 to calculate the expected loss of the layer 500 xs 2 000. Assume that the burning cost of C_0 xs D_0 is $\text{BC}_0 = 200$. Standard exposure rating with a loss ratio $l = 100\%$ yields the following results for the curves G_{DLS} and G_{step}:

	$G = G_{\text{DLS}}$	$G = G_{\text{step}}$
$\text{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G)$	388.73	731.23
$\text{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G)$	300.14	224.46
$\text{EL}_{C_0,D_0}(\mathcal{L}, 1, G)$	266.64	178.76
$\text{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G)$	268.95	319.51
$\text{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G)$	193.12	105.44
$\text{EL}_{C,D}(\mathcal{L}, 1, G)$	164.60	89.09

	$G = G_{\text{DLS}}$	$G = G_{\text{step}}$
$\text{EL}_{C,D}(\mathcal{P}(\mathcal{L}), 1, G) / \text{EL}_{C_0,D_0}(\mathcal{P}(\mathcal{L}), 1, G)$	69.19%	43.69%
$\text{EL}_{C,D}(\mathcal{T}(\mathcal{L}), 1, G) / \text{EL}_{C_0,D_0}(\mathcal{T}(\mathcal{L}), 1, G)$	64.35%	46.98%
$\text{EL}_{C,D}(\mathcal{L}, 1, G) / \text{EL}_{C_0,D_0}(\mathcal{L}, 1, G)$	61.73%	49.84%

With $\rho_0 := (C_0, D_0, \text{BC}_0)$ we get the following results:

	$G = G_{\text{DLS}}$	$G = G_{\text{step}}$
$\text{EL}^{\text{BC}}_{C,D}(\mathcal{P}(\mathcal{L}), \rho_0, G)$	138.38	87.39
$\text{EL}^{\text{BC}}_{C,D}(\mathcal{T}(\mathcal{L}), \rho_0, G)$	128.69	93.96
$\text{EL}^{\text{BC}}_{C,D}(\mathcal{L}, \rho_0, G)$	123.46	99.67

As expected from theorem 4.12, we obtain

$$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}),\rho_0,G_{\mathrm{DLS}}) \geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}),\rho_0,G_{\mathrm{DLS}}) \geq \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L},\rho_0,G_{\mathrm{DLS}})$$

for the Pareto increasing quasi exposure curve G_{DLS}. For the exposure curve G_{step}, which is far away from being Pareto increasing, we see that

$$\mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{P}(\mathcal{L}),\rho_0,G_{\mathrm{step}}) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{T}(\mathcal{L}),\rho_0,G_{\mathrm{step}}) < \mathrm{EL}^{\mathrm{BC}}_{C,D}(\mathcal{L},\rho_0,G_{\mathrm{step}}).$$

Appendix E
Proofs of Section 6

Since Proposition 6.5 is used in the proof of Proposition 6.2, we prove the propositions in reverse order. Like in Section 6 let

$$\varphi\colon [0,10] \to \mathbb{R}, \quad c \mapsto a_c + (1-\ln(b_c))b_c.$$

Since $c_0 = -\frac{1}{2} + \sqrt{\frac{1}{4}+\frac{62}{3}} \approx 4.0735$ is a zero of $\ln(b_c) = 3.1 - 0.15(1+c)c$, we have $b(c_0) = 1$ and $a(c_0) = \frac{(g(c_0)-1)b(c_0)}{1-g(c_0)b(c_0)} = -1$. Consequently $\varphi(c_0) = 0$. In the proof of Proposition 6.5 we will use the following fact:

Proposition E.1. *Apart from c_0, φ has only one additional zero $c_1 \in (4.0651, 4.0652)$. In particular $c_1 < c_0$. Regarding a_c, $\ln(b_c)$ and $\varphi(c)$ we have*

	$c=0$	$c \in (0,c_1)$	$c=c_1$	$c \in (c_1,c_0)$	$c=c_0$	$c \in (c_0,10]$
a_c	$=0$	<0	<0	<0	$=-1$	<-1
$\ln(b_c)$	>0	>0	>0	>0	$=0$	<0
$\varphi(c)$	<0	<0	$=0$	>0	$=0$	<0

Moreover, we have $a'(c_0) < 0$ and $b'(c_0) < 0$.

Proof. The assertion about the signs of $\ln(b_c) = 3.1 - 0.15(1+c)c$ is obvious. Solving the corresponding quadratic equations, we easily see that $\ln(g_c) = (0.78+0.12c)c > 0$ and $\ln(g_c b_c) = (0.78+0.12c)c + 3.1 - 0.15(1+c)c > 0$ for all $c \in (0,10]$. Thus, we have $g_c > 1$ and $g_c b_c > 1$ for $c \in (0,10]$. Since $b_c > 0$ we see that $a_c = \frac{(g_c-1)b_c}{1-g_c b_c} < 0$ for all $c \in (0,10]$. Because of $g_0 = 1$ and $b_{c_0} = 1$ we have $a_0 = 0$ and $a_{c_0} = \frac{g_{c_0}-1}{1-g_{c_0}} = -1$. For $c > c_0$ we have $0 < b_c < 1$ and thus $a_c = -\frac{g_c b_c - b_c}{g_c b_c - 1} < -1$.

With $\frac{d}{dc}\big|_{c=c_0} \ln(b(c)) = -0.15(1+2c_0) < 0$ we get $b'(c_0) < 0$. Using $b(c_0) = 1$, a short calculation shows

$$a'(c_0) = \frac{b'(c_0)}{g(c_0)-1}.$$

With $g(c_0) > 1$ and $b'(c_0) < 0$ we see that $a'(c_0) < 0$.

Since $\varphi(4.0651) \approx -7.22 \cdot 10^{-8} < 0$ and $\varphi(4.0652) \approx 7.10 \cdot 10^{-7} > 0$, φ has a zero c_1 in the interval $(4.0651, 4.0652)$.

Zooming into the graph of φ, it is plausible that c_0 and c_1 are the only zeros of φ and that the assertion about the signs of φ is true (see Figure 9). For a rigorous proof (which is omitted here) one can proceed as follows:

- Show that $\gamma(c) := (1-\ln(b_c))b_c$ is strictly decreasing on $[c_0, 10]$ (easy). With $a(c) < a(c_0)$ for $c \in (c_0, 10]$ we see that $\varphi(c) < \varphi(c_0) = 0$ for $c \in (c_0, 10] \supset [4.1, 10]$.

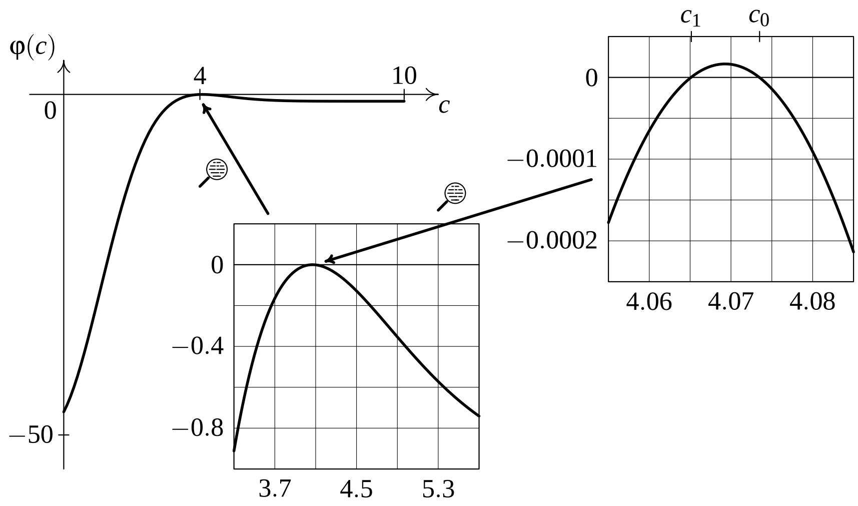

FIGURE 9: Graph of φ

- Show that $\gamma(c)$ is strictly increasing and $a(c)$ is strictly decreasing on $[0, c_0]$ (easy). Consequently, $\varphi(c) \le a(\frac{i-1}{10}) + \gamma(\frac{i}{10})$ for $i = 1, \ldots, 40$ and $c \in [\frac{i-1}{10}, \frac{i}{10}]$. Since $a(\frac{i-1}{10}) + \gamma(\frac{i}{10}) < 0$ for $i = 1, \ldots, 40$ (use e.g. Excel) we obtain $\varphi(c) < 0$ for $c \in [0, 4]$.

- Show that $\gamma''(c) < -1$ and $a''(c) < 0.5$ for $c \in (4, 4.1)$ (lengthy). It follows that $\varphi''(c) < 0$ for $c \in (4, 4.1)$. Consequently, $\varphi(c) > 0$ for $c_1 < c < c_0$ and $\varphi(c) < 0$ for $c \notin [c_1, c_0]$. ☐

Proof of Proposition 6.5. The zero $c_1 \in (4.0651, 4.0652)$ from Proposition E.1 is smaller than c_0. Thus, c_1 is the smallest zero of φ.

The case $c = 0$ is trivial. Let $\alpha_c(x)$ denote the local Pareto alphas of G_c. For $c = c_0$ a short calculation shows

$$\alpha_{c_0}(x) = -\frac{xG''_{c_0}(x)}{G'_{c_0}(x)} = \frac{xb'(c_0)}{a'(c_0) + xb'(c_0)}$$

and

$$\alpha'_{c_0}(x) = -\frac{d}{dx}\frac{xG''_{c_0}(x)}{G'_{c_0}(x)} = \frac{a'(c_0)b'(c_0)}{(a'(c_0) + xb'(c_0))^2} > 0$$

(use $a'(c_0) < 0$ and $b'(c_0) < 0$), i.e. G_{c_0} is Pareto increasing.

We will now look at the cases $c \in (0, 10] \setminus \{c_0\}$. A short calculation yields

$$\alpha_c(x) = -\frac{xG''_c(x)}{G'_c(x)} = -\frac{xa_c \ln(b_c)}{a_c + b_c^x}$$

and

$$\alpha'_c(x) = -\frac{d}{dx}\frac{xG''_c(x)}{G'_c(x)} = -\frac{a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x]}{(a_c + b_c^x)^2}.$$

Thus, G_c is Pareto increasing if and only if

$$a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x] \le 0$$

for all $x \in (0, 1)$. Since

$$\frac{d}{dx}[a_c + (1 - \ln(b_c)x)b_c^x] = -\ln(b_c)^2 x b_c^x \le 0$$

for $x \in [0,1]$ we get

$$a_c + 1 \geq a_c + (1 - \ln(b_c)x)b_c^x \geq a_c + (1 - \ln(b_c))b_c = \varphi(c)$$

for $x \in [0,1]$.

In the case $c_0 < c \leq 10$ we have $a_c \ln(b_c) > 0$ and $a_c + 1 < 0$. Thus

$$a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x] \leq a_c \ln(b_c)(a_c + 1) < 0,$$

i.e. G_c is Pareto increasing.

In the case $c_1 \leq c < c_0$ we have $a_c \ln(b_c) < 0$ and $\varphi(c) = a_c + (1 - \ln(b_c))b_c \geq 0$. Thus

$$a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x] \leq a_c \ln(b_c)[a_c + (1 - \ln(b_c))b_c] \leq 0,$$

i.e. G_c is Pareto increasing.

In the case $0 < c < c_1$ we have $a_c \ln(b_c) < 0$ and $\varphi(c) = a_c + (1 - \ln(b_c))b_c < 0$. Thus

$$\lim_{x \nearrow 1} a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x] = a_c \ln(b_c)[a_c + (1 - \ln(b_c))b_c] > 0,$$

i.e. $a_c \ln(b_c)[a_c + (1 - \ln(b_c)x)b_c^x] > 0$ for some $x \in (0,1)$, i.e. G_c is not Pareto increasing. $\square$

Proof of Proposition 6.2. Let $c \in (c_0, 10]$ with $\mu(c) < 1$. Using $\alpha_c(x) = -\frac{xG_c''(x)}{G_c'(x)} = -\frac{xa_c \ln(b_c)}{a_c + b_c^x}$ and $W(x)e^{W(x)} = x$ we obtain

$$\alpha_c(\mu(c)) = -\frac{\mu(c)a_c \ln(b_c)}{a_c + b_c^{\mu(c)}} = \frac{\left(1 + W\left(\frac{1}{ea_c}\right)\right) a_c}{a_c + \frac{1}{\exp\left(1 + W\left(\frac{1}{ea_c}\right)\right)}}$$

$$= \frac{\frac{a_c}{W\left(\frac{1}{ea_c}\right)} + a_c}{\frac{a_c}{W\left(\frac{1}{ea_c}\right)} + \frac{1}{eW\left(\frac{1}{ea_c}\right)\exp\left(W\left(\frac{1}{ea_c}\right)\right)}} = \frac{\frac{a_c}{W\left(\frac{1}{ea_c}\right)} + a_c}{\frac{a_c}{W\left(\frac{1}{ea_c}\right)} + \frac{1}{e\frac{1}{ea_c}}} = 1.$$

Since G_c is Pareto increasing (Proposition 6.5), it follows that $\alpha_c(x) \geq 1$ for all $x \in [\mu(c), 1)$.

It remains to show that $\lim_{x \nearrow 1} \alpha_c(x) < 1$ for all $c \leq c_0$. Recall that for $x \in (0,1)$ we have

$$\alpha_c(x) = -\frac{xa_c \ln(b_c)}{a_c + b_c^x} \text{ for } c \neq c_0 \quad \text{and} \quad \alpha_{c_0}(x) = \frac{xb'(c_0)}{a'(c_0) + xb'(c_0)}.$$

Consequently

$$\lim_{x \nearrow 1} \alpha_c(x) = -\frac{a_c \ln(b_c)}{a_c + b_c} \text{ for } c \neq c_0 \quad \text{and} \quad \lim_{x \nearrow 1} \alpha_{c_0}(x) = \frac{b'(c_0)}{a'(c_0) + b'(c_0)}.$$

Note that $\lim_{x \nearrow 1} \alpha_{c_0}(x) < 1$ because of $a'(c_0) < 0$ and $b'(c_0) < 0$. With l'Hospital's rule it is easy to see that

$$\lim_{c \to c_0} \lim_{x \nearrow 1} \alpha_c(x) = \lim_{x \nearrow 1} \alpha_{c_0}(x) < 1.$$

Since $\lim_{x \nearrow 1} \alpha_0(x) = 0$, it is sufficient to show that $(0, c_0) \to (0, \infty)$, $c \mapsto \lim_{x \nearrow 1} \alpha_c(x)$ is increasing. For $c \in (0, c_0)$ a short calculation yields

$$\lim_{x \nearrow 1} \alpha_c(x) = -\frac{a_c \ln(b_c)}{a_c + b_c} = \left(1 - \frac{1}{g_c}\right) \cdot \frac{\ln(b_c)}{b_c - 1} = \left(1 - \frac{1}{g_c}\right) \cdot \gamma(b_c),$$

where $\gamma\colon (1,\infty) \to (0,\infty)$, $x \mapsto \frac{\ln(x)}{x-1}$. It is easy to check that $(0,c_0) \to (0,\infty)$, $c \mapsto 1-g_c^{-1}$ is increasing and that $(0,c_0) \to (1,\infty)$, $c \mapsto b_c$ is decreasing. Since γ is decreasing, we can conclude that $(0,c_0) \to (0,\infty)$, $c \mapsto \lim_{x \nearrow 1} \alpha_c(x)$ is increasing. □

Appendix F

Approximation of MBBEFD Exposure Curves by EP Exposure Curves

The EP exposure curves are specified by two parameters t and β. We define a subclass of one-parameter EP exposure curves by $G_c^{EP} := G_{t(c),\beta(c)}^{EP}$ $(0 \le c \le 4.1)$ with

$$\begin{aligned} t(c) &= 0.0001008554c^6 - 0.0009943289c^5 + 0.002317257c^4 + 0.0043010816c^3 \\ &\quad - 0.0097699698c^2 - 0.0516128438c + 0.1200082633 \\ \beta(c) &= -0.0000421493c^6 + 0.0003050501c^5 + 0.001466131c^4 \\ &\quad - 0.010656745c^3 + 0.0057139018c^2 + 0.2554579398c. \end{aligned}$$

For $0 \le c \le 4.1$ the curves G_c^{EP} approximate the MBBEFD exposure curves G_c very well. Figures 10 and 11 compare the MBBEFD and EP curves for $c = 0$, 1, 2, 3, and 4. The approximation errors are plotted in Figure 12.

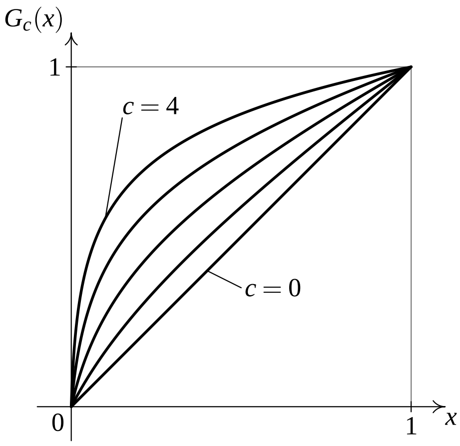

Figure 10: MBBEFD and EP curves for $c = 0$, 1, 2, 3, and 4

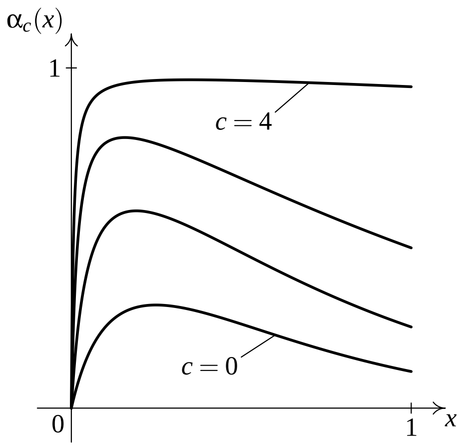

Figure 11: Local Pareto alphas of MBBEFD and EP curves for $c = 0$, 1, 2, 3, and 4

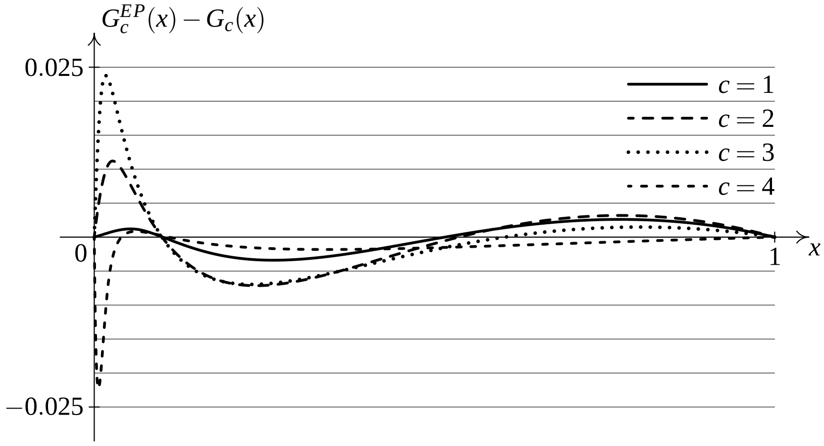

FIGURE 12: Approximation errors for $c = 0$, 1, 2, 3, and 4

For $x \geq 0.1$ (which is the range in which exposure curves are mostly used), the deviation between $G_c^{EP}(x)$ and $G_c(x)$ is below 0.008, which is sufficient for practical purposes.

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