

# Economic Risk Capital and Reinsurance: an Extreme Value Theory's Application to Fire Claims of an Insurance Company\*

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## Abstract

The viability of an insurance company depends critically on the size and frequency of large claims. An accurate modelling of the distribution of large claims contributes to correct pricing's and reserving's decisions while maintaining through reinsurance an acceptable level of the unexpected fluctuations in the results. We present an application to the fire claims of an insurance company providing a model for large losses that we evaluate through simulations based on both a traditional and a Peaks over Threshold's approach. Under the first one we estimate separately loss frequency, according to Negative Binomial distribution studying a claims number development triangle, and loss severity, according to Generalized Pareto distribution. A Peaks over Threshold's approach is then developed estimating jointly frequency and severity distribution and considering the time dependence of data. We calculate the economic risk capital as the difference between the expected loss, defined as the expected annual claims amount, and the 99.93<sup>th</sup> quantile of the total cost distribution corresponding to a Standard&Poor's A rating; we then simulate the impact of a quota share and an excess of loss reinsurance structure on the distribution of total cost amount and on economic risk capital. We provide a tool to price alternative programs and investigate how they can affect economic risk capital and explain the rationale of the choice of the optimal reinsurance programmes to smooth economic results.

**Keywords:** Economic Risk Capital; Peaks over Thresholds' Approach; Homogenous and Inhomogenous Poisson Process; Modelling Trends; Negative Binomial Distribution; Simulation.

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*"If things go wrong,  
How wrong can they go?"*

## 1 Introduction

Every portfolio of insurance policies incurs losses of greater and smaller amounts at irregular intervals. The sum of all the losses (paid&reserved) in any one-year period is described as the annual loss burden or as simply total incurred claims amount. The future annual loss burden is estimated on the basis of predicted claims frequency and predicted average individual claim amount, but what actually happens considerably deviates from forecasts on a purely random basis. The economic risk capital represents what needs to be stored as a hedge against adverse developments in the future. There can be relevant and extreme events that will require significant new capital infusions if they occur and this new capital will be additional to what is needed for daily operations or investment. The occurrence of these events also impacts on the credit profile of an insurance company, thus affecting the associated credit spread dynamics. Sudden losses may cause extreme cash-flow stress and investors may require more favorable terms when offering new lines of financing as addressed by Dembo, Deuschel and Duffie [12]. A large class of event-based credit risks cannot be hedged and handled with standard credit instruments but reinsurers have been dealing with event-based risks all along. From this point of view, recent activity in credit-related insurance contracts could be the beginning of a new class of credit derivatives called contingent-capital instruments providing new capital if one or more events occur as outlined by Neftci [22].

An insurance company generally decides to transfer the high cost of contingent capital to a third party, a reinsurance company. The foremost goal of reinsurance is to absorb the impact of unexpected losses on the economic risk capital and from this point of view reinsurance coverage is one of the tool available for capital management. This capability critically depends on the characteristics of the contract and the nature of the underlying risk. By transferring the risk to reinsurer, the insurance company will lower the economic risk capital allocated for the retained risk at the cost of a reinsurance premium to be paid for the transferred risk. A pure comparison of the price of the reinsurance structure versus the cost of capital could lead to wrong conclusions and inappropriate company decisions. It is necessary to take into account not only the spread but also the allocated capital that reinsurance enables to "save" and how this structure affects the economic risk capital whose goal is to cover the volatility of probable annual claims amount not transferred to the reinsurer. The solvency and viability of an insurance company depends on probabilistic calculations of risk and critically on the size and frequency of large claims.

In this paper we are specifically interested in modelling the tail of loss severity distribution according to the extreme value theory (EVT) and the frequency distribution of claims according to both a Negative Binomial and a Poisson distribution. The extreme value theory has found more application in hydrology and climatology [29],[32] than in insurance and finance. This theory is concerned with the modelling of extreme events and recently has begun to play an important methodological role within risk management for insurance, reinsurance and finance [4],[15],[16]. Var-

ious authors have noted that the theory is relevant to the modelling of extreme events for insurance [2],[5],[18],[27],[19],[20],[3],[31], finance through the estimation of Value at Risk [11],[21],[17],[28] and reinsurance and financial engineering of catastrophic risk transfer securization products [14],[25],[33]. The key result in EVT is the Picklands-Blakema-de Haan theorem [24],[1], proving that for a wide class of severity distributions which exceed high enough thresholds the Generalized Pareto distribution (GPD) holds true.

We examined a large data set representing a portion of the fire claims of RAS, an Italian insurance company, analyzing the losses larger than 100 thousand euro from 1990 to 2000. To protect the confidentiality of the source, the data are coded so that the exact time frame and frequency of events covered by data are not revealed. The figures and the comments do not represent the views and/or opinion of the RAS management and risk capital measures are purely indicative.

We calculate the economic risk capital defined as the difference between the expected loss, the expected annual claims amount, and the 99.93<sup>th</sup> quantile of the distribution corresponding to a Standard&Poor's A rating. Then we simulate the impact of a quota share and an excess of loss, analyzing the reinsurance effect on total claims distribution and consecutively on the economic risk capital. We graphically represent the kernel density distributions of total claims. We develop two approaches to derive the total claims amount. A traditional approach is followed assuming that the number of big claims made on a stable portfolio during a fixed premium period is negative binomial distributed. We are concerned with fitting the Generalized Pareto distribution (GPD) to data on exceedances of high thresholds to calculate claim severity. Finally we consider the development of scenarios for loss frequency and for loss severity separately as usual in reinsurance practice [10],[23],[35]. We then develop a second approach according to McNeil and Saladin's work [20], extending the Peak over Thresholds method to obtain a joint description of both the severity and frequency of losses exceeding high thresholds. We also analyze and model the time dependence of our data looking for an effective model for large losses in the past so that we can extrapolate the claims process into the future.

Section 2 discusses why and how we model the probability distribution of the total large claims and how we measure the impact of the reinsurance structures on economic risk capital. Section 3 describes the dataset used and the hypothesis adopted to trend and adjust the data. We provide exploratory graphical methods to detect heavy tail behavior and dependence on time. Section 4 discusses the approaches followed to derive the total claims amount. In subsection 4.1 we introduce a traditional approach, as usual in reinsurance practice, fitting frequency distribution with particular reference to Binomial Negative distribution and separately fitting the severity distribution with Pareto Generalized distribution. In subsection 4.2 we expose the Peak over Thresholds approach: we derive a model for point process of large losses exceeding a high threshold and we obtain a joint description of frequency and severity considering the issue of trend of them. In section 5 we estimate a pure price of an excess of loss treaty adopting the family of fitted Generalized Pareto distributions. Section 6 concentrates on the simulation approach to model the total claims amount and to assess the impact of different reinsurance structures on economic risk capital. Finally, Section 7 presents conclusions and possible future

extensions.

## 2 Measuring Sensitivity of Economic Risk Capital

A portfolio of insurance contracts is considered “exposed” to fluctuations when the actual loss burden is expected to widely deviate from the predicted one. Our methodology provides a tool to measure this volatility. The simulation method can be used to estimate the reinsurance cost and the economic risk capital to cover the probable annual claims amount. We model the loss severity and loss frequency distribution functions to simulate a statistically representative sample of loss experience. When simulating loss experience one should be convinced that the severity and frequency loss distribution used in the simulations reflect the reality to greatest possible extent. A good amount of meticulous work is needed to trend and develop historical individual losses. Loss frequency and loss severity distributions for the projected coverage period are estimated based on the adjusted loss. The times of occurrence provide the information about the frequency and clustering of “rare events”. Independence of widely separated extremes seems reasonable in most applications, but they often display short-range dependence in which clusters of extremes occur together. Large losses may become more or less frequent over time and they may also become more or less severe.

We simulate the impact of reinsurance on total claims amounts. Reinsurance is the transfer of a certain part of portfolio liability (risk) to another carrier. We consider two types of reinsurance structure: a quota share and an excess of loss treaty. The first case is implemented as follows: the primary insurance company transfers a  $q$  share of all premiums and claims; the second case refers to an excess of loss treaty with lower and upper attachment points  $r$  and  $R$  respectively, where  $R > r$ . This means the payout  $y_i$  on a loss  $x_i$  for a reinsurer is given by

$$y_i = \begin{cases} 0 & \text{if } 0 < x_i \leq r \\ x_i - r & \text{if } r < x_i \leq R \\ R - r & \text{if } R < x_i < \infty \end{cases}$$

and the payout of insurance company is

$$c_i = \begin{cases} -x_i & \text{if } 0 < x_i \leq r \\ -r & \text{if } r < x_i \leq R \\ -x_i + (R - r) & \text{if } R < x_i < \infty \end{cases}.$$

As discussed by McNeil [18] there are two related actuarial problems concerning this layer:

- the pricing problem. Given  $r$  and  $R$  what should this insurance layer cost a customer?
- the optimal attachment point problem. If we want payouts greater than a specified amount to occur with at most a specified frequency, how low can we set  $r$ ?

Within this framework, another number of questions arise:

- is there any significant difference in the distributions of total cost amounts associated with different types of reinsurance structures?
- what are the implications of all these issues for the overall risk in terms of economic risk capital on the company?

Given the number of losses in a period  $n$ , the losses are  $x_1, \dots, x_n$  for the reinsurer and the aggregate payout would be  $z = \sum_{i=1}^n y_i$ . The price is the expected payout plus a risk loading factor which is assumed to be  $\gamma$  times the standard deviation of the payout,  $\text{Price} = E[z] + \gamma \sigma[z]$ . The expected payout  $E[z]$  is known as the pure premium and it is defined by  $E[y_i] E[n]$ , where  $E[y_i]$  is the expected value of severity claim and  $E[n]$  is the expected number of claims assuming independence between the frequency and severity of claims. McNeil [18] remarks that the pure premium calculated using the variance principle is a simple price indication and is an unsophisticated tool when the data present a heavy tail behavior, since moments may not exist or may be very large. The attachment point problem is intimately linked with the estimation of a high quantile of the loss severity distribution.

From the insurer point of view there is a particular concern not only about the pricing of the reinsurance structure previously described but also on how this structure affects the economic risk capital whose goal is to cover the volatility of the probable annual claims amount not transferred to reinsurer. The solvency and viability of an insurance company depends on probabilistic calculations of risk and critically on the size and frequency of large claims. We need a good estimate of the loss severity distribution for large  $x_i$  in the tail area and also we need a good estimate of the loss frequency distribution of  $n$ , as we will explain in the next sections.

### 3 Exploratory Data Analysis

We analyse a relevant portion of the fire claims for reported years 1990 through 2000 that exceed 100 thousand euro at 31/5/2001 evaluation date, 1554 observations, and their historical development. To protect the confidentiality of the source, the data are coded so that the exact time frame and the frequency of events covered by data are not revealed.

When trending the historical losses to the prospective experience period claim cost level it is important to select an appropriate severity trend factor. We develop the total incurred losses and the number of claims paid by year to the ultimate one and we calculate an average loss size by report year. Then we adjust the individual cost by an ISTAT inflation factor. Some individual claims in excess of 100 thousand euro are still open at 31/5/2001 and the ultimate values of these claims might be different from their reserved values, carried on the books. Generally, it is not easy to adjust individual claim values for possible development using aggregate development data only. The major complication stems from the fact that aggregate loss development is driven by two different forces: the new claims and the adjustment values for already outstanding claims (reserving review).

We develop some graphical tests to identify the most extreme losses. The QQ-plot is obtained (Figure 1): the quantiles of the empirical distribution function on the  $x$ -axis are plotted against the quantiles of the Exponential distribution on the

$y$ -axis. If the points lie approximately along a straight line, we can infer that the losses come from an Exponential distribution. A concave departure from the ideal shape indicates a heavier tailed distribution whereas convexity indicates a shorter tailed distribution. We observe that our dataset of insurance losses show heavy tailed behavior.

A further useful graphical tool is the plot of the sample mean excess function (Figure 2). Supposing one observes  $x_1, \dots, x_n$  and orders these observations as  $x_{(n)} \geq \dots \geq x_{(1)}$ , the sample mean excess function is defined by

$$e_n(u) = \frac{\sum_{i=1}^n (x_i - u)}{\sum_{i=1}^n 1_{\{x_i > u\}}} \quad (1)$$

and is the sum of the excesses  $(x_1 - u), \dots, (x_n - u)$  over the threshold  $u$  divided by the number of data points which exceeds the threshold  $u$ . The sample mean excess function describes the expected overshoot of a threshold given that exceedance occurs and is an empirical estimate of the mean excess function which is defined as  $e(u) = E[x - u | x > u]$ . If the points show an upward trend, this is a sign of heavy tailed behavior. Exponentially distributed data approximately would give an horizontal line and data from a short tailed distribution would show a downward trend. If the empirical plot seems to follow a reasonably straight line with positive slope above a certain value of  $u$ , then this is an indication that the data follow a generalized Pareto distribution. This is precisely the kind of behavior we observe in our data and a significant positive trend is detected in  $u$  equal to 775 and 1,550 thousand euro.

According to Resnik [27] there are additional techniques and plotting strategies which can be employed to assess the appropriateness of heavy tailed models. We define the Hill estimator as

$$H_{n-s,s} = \frac{1}{s} \sum_{j=1}^s \log x_{n-j+1,n} - \log x_{n-s,n}. \quad (2)$$

The Hill statistic is nothing else than the mean excess estimate of the log-transformed data based on observations which are exceedances over  $\log x_{n-s,n}$  divided by the threshold  $\log x_{n-s,n}$ . The Hill estimator is defined as  $H_{n-s,s} = 1/\alpha_{n,s}$  where  $\alpha_{n,s}$  is the tail index of a semiparametric Pareto type model as  $F(x) = x^{-\alpha} l(x)$  while  $l(x)$  is a nuisance function. We have as many estimates of  $\alpha$  as we have data points; for each value of  $s$  we obtain a new estimate of  $\alpha$ . We plot  $\alpha_{n,s}$  as function of  $s$  (Figure 3) and we observe an increasing trend in our estimates of  $\alpha$  as  $s$  increases revealing again a heavy tailed behavior of our data. One can try to infer the value of  $\alpha$  from a stable region of the graph or choose an optimal  $s$  which minimizes asymptotic mean square error of the estimator  $\alpha$ , but this is difficult and puts a serious restriction on reliability. Close to methods based on high order statistics, such as the Hill estimator, an alternative is offered by the peaks over threshold approach (POT) that will be discussed in the next section.

Resnik [27] and [26] outlines several tests for independence which can help to reassure that a i.i.d. model is adequate and to test that it is not necessary to fit a stationary time series with dependencies to the data. We test the independence hypothesis basing our test on the sample autocorrelation function  $\rho(h)$

$$\rho(h) = \frac{\sum_{t=1}^{n-h} (x_t - \bar{x})(x_{t+h} - \bar{x})}{\sum_{t=1}^n (x_t - \bar{x})^2} \quad (3)$$

where  $h$  is any positive integer and  $\bar{x}$  is the sample mean. We plot  $\rho(h)$  for  $h = 0, \dots, n - h$ , where  $n - h = 30$ , and the horizontal lines denote asymptotic limits of two standard errors (Figure 4). We get values for  $h \neq 0$  giving evidence of possible clustering of our data and this suggests that a more meticulous work should be done.

## 4 Model Approach

### 4.1 "A Traditional Approach"

#### 4.1.1 Fitting Frequency Distribution

For the excess claim frequency distribution we use the Negative Binomial distribution. This discrete distribution has been utilized extensively in actuarial work to represent the number of insurance claims. Since its variance is greater than its mean, the negative binomial distribution is especially useful in situations where the potential claim count can be subject to significant variability. To estimate parameters for the negative binomial distribution we start with the estimate of the expected final number of claims in excess of 100 thousand euro.

Studying a claims number development triangle with a specific “chain ladder” approach, we estimate final claims frequency over the same threshold over ten past years, that will be used in fitting the loss severity distribution, and adjust it on an exposure basis. Estimating on this historical data the empirical expectation and variance, we calculate the parameters for the Negative Binomial distribution.

#### 4.1.2 Fitting Severity Distribution

In order to describe the distribution of the sizes of individual claims made on a portfolio, we assume that insurance losses are denoted by independent, identically distributed random variables and we attempt to find an estimate of the severity distribution function truncated at  $\delta = 100$  thousand euro. Modern methods of extreme value analysis are based on exceedances over high thresholds. Denoting the threshold by  $u$ , the conditional distribution of excesses over  $u$  is modelled by the Generalized Pareto distribution (GPD)

$$\Pr\{x \leq u + y | x > u\} \approx 1 - \left(1 + \frac{\xi(x - u)}{\psi}\right)^{-1/\xi} \text{ for } x \geq 0, \quad (4)$$

where  $\psi > 0$  is a scale parameter,  $\xi$  a shape parameter and  $u$  is the threshold. The three cases  $\xi < 0$ ,  $\xi = 0$  and  $\xi > 0$  correspond to different types of tail behavior. The case  $\xi < 0$  arises in distributions where there is a finite upper bound on the claims, a tendency for claims to cluster near the upper limit. The second case,  $\xi = 0$  arises when an exponentially decreasing tail is observed or also when data are distributed according to a gamma, normal, Weibull or lognormal distribution. However the third case,  $\xi > 0$ , corresponds to a “long-tailed” distribution that is associated to the Pareto tail according to  $\Pr\{X > x\} \sim x^{-\alpha} l(x)$  as  $x \rightarrow \infty$  for a positive constant

$\alpha$ . The relation between  $\xi$  and  $\alpha$  is  $\xi = 1/\alpha$ . When  $0 < \alpha < 2$  the distribution is also tail equivalent to an  $\alpha$ -stable distribution. A critical issue is the selection of an appropriate threshold  $u$  as defined before. We fit the GPD (Generalized Pareto distribution) to data points exceeding a varying high threshold from 300 to 1800 thousand euro with a step of 25 thousand euro. We obtain maximum likelihood estimates for the shape parameter,  $\xi$ , and plot it with respect to the threshold as well as asymptotic confidence intervals for the parameter estimates (Figure 5).

A first conclusion is that the distribution of fire claims is indeed long-tailed as the exploratory analysis suggests. As can be observed from the figures, the choice of the tail index is crucial also for the simulation process explained in the next section. If the threshold is set too high, there will not be enough observations over  $u$  to calculate good estimates of  $\psi$  and  $\xi$  and our estimates will be prone to high standard errors. However we do not want  $u$  to be too low, because we want to analyze “large claims”. The ideal situation would be to have a stable shape’s pattern.

We select the threshold equal to 775 and 1500 thousand euro. We have 141 and 48 exceedance observations respectively and we estimate the Generalized Pareto distributions:  $\xi = 0.492$  for  $u = 775$  and  $\xi = 0.715$  for  $u = 1,500$ . We represent graphically the empirical distribution and the two estimated GPD to make a comparative analysis of fitting (Figure 6). The GPD with  $u = 775$  is a reasonable fit, although its tail is just a little too thin to capture the behavior of the highest observed losses and seems to slightly underestimate the probabilities of medium-to-large losses respect to the GPD with  $u = 1,500$ .

## 4.2 A Peaks Over Threshold Approach

The traditional approach to Extreme Value Theory is based on extreme value limit distributions and on assumption that the original data are independent and identically distributed. A more flexible model is based on a so-called point process characterization and consists in modelling exceedance times in terms of a Poisson process combined with independent excesses over a threshold, permitting separate modelling of frequency and severity without considering time dependence. The excesses over a high threshold  $u$  occur at the times of a Poisson process with intensity

$$\lambda = \left(1 + \frac{\xi(u - \mu)}{\sigma}\right)^{-1/\xi} \quad (5)$$

and the corresponding sizes, occur over  $u$ , are independent and have Generalized Pareto distribution

$$\Pr\{x \leq u + y | x > u\} \approx 1 - \left(1 + \frac{\xi(x - \mu)}{\psi}\right)_+^{-1/\xi} \quad for \quad x \geq 0 \quad (6)$$

where  $\mu$  and  $\sigma$  are location and scaling parameters and  $\psi$  is a positive scaling parameter defined as  $\psi = \sigma + \xi(u - \mu)$ . The parameters  $\xi$  and  $\psi$  correspond to the parameters fitted in subsection 4.1.2. This approach gives us a joint modelling of frequency and severity and focuses on both exceedances of the measurement over some high threshold and the times at which the exceedances occur.



The times of occurrence provide the information about the intensity of the occurrence of “rare events”. Independence of widely separated extremes seems reasonable in most applications, but they often display short-range dependence in which clusters of extremes occur together. Serial dependence generally implies clustering of large values: hot days tend to occur together and sea level maxima often occur during the strongest storm of the year. Data also can be affected by presence of trends of increasing, decreasing or stable loss frequency and severity over the years: large losses may become more or less frequent over time and they may also become more or less severe. These phenomena can be modelled in the Poisson process framework by considering processes with time-dependent intensity and severity measure as investigated in recent literature [31], [19], [9], [8], [34], [6],[7].

It is usual to model non-stationarity of extremes directly through the parameters of the standard models. Suppose we replace the parameters  $\mu$ ,  $\sigma$  and  $\xi$  by  $\mu_t$ ,  $\sigma_t$  and  $\xi_t$  so that the Poisson intensity parameters are functions of the time

$$\lambda_t = \left(1 + \frac{\xi_t(u - \mu_t)}{\sigma_t}\right)^{-1/\xi_t}. \quad (7)$$

and the exceedances are distributed according to a Generalized Pareto distribution whose parameters are time-varying

$$\Pr\{x \leq u + y | x > u\} \approx 1 - \left(1 + \frac{\xi_t(x - \mu_t)}{\psi_t}\right)_+^{-1/\xi_t} \text{ for } x \geq 0, \quad (8)$$

where  $\psi_t$  is a positive scaling parameter, as before, defined as  $\psi_t = \sigma_t + \xi_t(u - \mu_t)$ . For example to allow for a linear trend in the underlying level of extremal behaviour we could use

$$\mu_t = \alpha_1 + \beta_1 t \quad (9)$$

or to allow for a change in the variability of extreme events

$$\sigma_t = \exp(\alpha_2 + \beta_2 t) \quad (10)$$

using an exponential function to retain the positivity of the scale parameter, then it is possible to assume that the shape parameter  $\xi$  is time-dependent according to a linear trend

$$\xi_t = \alpha_3 + \beta_3 t. \quad (11)$$

The full set of model parameters is  $\Theta = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ . According to McNeil and Saladin [20], it is possible to consider all the possible combination of these models’ trends. We denotes Model 0 ( $\xi, \mu, \sigma$ ), the model exposed before, Model 1 ( $\xi_t, \mu, \sigma$ ), Model 2 ( $\xi, \mu_t, \sigma$ ) and Model 3 ( $\xi, \mu, \sigma_t$ ) using respectively the equations (9), (10) and (11). Model 23 denotes a combination of Model 2 and Model 3 ( $\xi, \mu_t, \sigma_t$ ) and Model 123 ( $\xi_t, \mu_t, \sigma_t$ ) denotes a combination of all three trends. We expose the detail of maximum likelihood estimation in Appendix B.

It is necessary to analyze which model offers a significant improvement in the fit with respect to Model 0. We estimate all models from a threshold  $u = 300$  to  $u =$

1600 thousand euro with a step of 25 thousand euro. We plot the shape parameter  $\xi$  across different thresholds  $u$  for the Model 0, the same plotted in Figure 5, and Model 23 and 123 (Figure 7). We can notice that the pattern of the three estimates is similar but the parameter  $\xi$  of the Model 123 is higher than the parameter of Model 0 and Model 23 for values of  $u$  from 300 to 1000 thousand euro, corresponding to a more severe “long-tailed” distribution. The parameter  $\xi$  of Model 23 is lower than the parameter of Model 0 in the same range of  $u$ . The pattern of the shape parameter  $\xi$  of Model 1, 2 and 3 is very similar to the the pattern of Model 0.

We examine the fit of Model 0, 1, 2, 3, 23 and 123 with a  $u = 775$  thousand euro using diagnostics based on residual statistics developed by Smith and Shively [30]. As explained above, we may permit the parameters  $\mu$ ,  $\psi$  and  $\xi$  to be functions of time and the constant intensity  $\lambda$  is replaced by a time-dependent intensity  $\lambda_t$  as in (7). If the observations begin at time  $t_1$  and the successive exceedance times are at  $t_2, t_3, \dots, t_n$  the variables

$$Z_k = \int_{t_{k-1}}^{t_k} \lambda_t dt$$

should be independent exponentially distributed random variables with mean 1 for  $k = 1, \dots, n$ . This may be tested graphically through a QQ plot of observed order statistics versus their expected values under the independent exponential assumption. The assumption is tested for Model 123 (Figure 8) and some departure is observed, but this is not inconsistent with exponential distributions. The same pattern is observed for the other models. A scatterplot is represented in Figure 9 and can be used to search for systematic deviation from unit exponential which may indicate the persistence of trends which are not being adequately modelled. We impose a smooth fit revealing an increasing and then a decreasing trend: losses are becoming more frequent and then less frequent. The scatterplot shows some discrepancy and reveals evidence the POT Model 123 does not adequately model the frequency. The results are not significantly different with respect to the other models.

We provide also the W-statistics to assess our success in modelling the temporal behaviour of the exceedances of the threshold  $u$ . The W-statistics are defined to be

$$W_k = \frac{1}{\xi_{t_k}} \ln \left( 1 + \xi_{t_k} \frac{x_k - u}{\sigma_{t_k} + \xi_{t_k} (u - \mu_{t_k})} \right).$$

If the implied GPD model for the excesses is correct then the  $W_k$  should be also independent exponentially distributed random variables with mean 1. As before we represent the QQ plot and the scatter plot of residuals for Model 123 in Figure 10 and 11 and the unit exponential hypothesis is plausible for residuals of Models 123. This last one, with respect to the others, seems to well model the temporal behaviour suggesting a trend in the size of losses.

A descriptive data analysis by year for the exceedances over a threshold  $u = 775$  allow us to observe an increase in mean size of losses in the last two years. The analysis and the estimates of our models can be significantly affected by the more recent data. As outlined before the ultimate values of these claims might be different from their reserved values, carried on the books. Generally, it is not easy to adjust individual claim values for possible development using aggregate development data

only. The major complication stems from the fact that aggregate loss development is driven by two different forces: the new claims and the adjustment values for already outstanding claims (reserving review).

The pattern of frequency is quite anomalous showing a decreasing trend until a null frequency in the median year and an abrupt increase in the second part of the considered time period. However we plot the intensity  $\lambda_t$  for the models fitted to analyze how the model capture the pattern of frequency observed in our data. As it can be observed in Figure 12, the pattern of estimated intensity is quite different across the models and seems to be captured only by Model 123 and also by Model 23 that is not represented in the figure. Model 0 provides a constant intensity because the parameters are not time-varying. The intensity of Model 2 is monotonically increasing while the Model 3 is monotonically decreasing; both models cannot replicate the pattern of empirical frequency.

Model 123 seems to provide a significant improvement on other models both in capturing pattern of frequency and severity of our data and in possessing sufficient flexibility to model all possible combinations of increasing and decreasing intensities and loss sizes. However it is interesting to consider what the exact consequences of our models are in estimating the economic risk capital and impact of reinsurance.

## 5 Reinsurance

The choice of an appropriate claim size distribution in connection with a rating problem is a fundamental area of discussion, both in the academic as in the practical reinsurance world. In considering the issue of the best choice of threshold we can also investigate how the price of a layer varies with threshold. To give an indication of the prices we get from our model we calculate  $E[y|x > \delta]$  for a layer running from  $r = 2,000$  to  $R = 3,000$  thousand euro

$$E[y|x > \delta] = \int_r^R (x - r)f_{x\delta}(x)dx + (R - r)(1 - F_{x\delta}(R)) \quad (12)$$

where  $F_{x\delta}(x)$  and  $f_{x\delta}(x) = dF_{x\delta}(x)/dx$  respectively denote the cumulative density function and the probability density function for the losses truncated at  $\delta$ , that is 100 thousand euro for the data used in this paper. We estimate the pure price using the cumulative density function and the probability density function derived by model 23 and 123 in order to make a comparative analysis in terms of pure price (Figure 13). As outlined before, Model 123 corresponds to a more severe “long-tailed” distribution and seems to capture the pattern of severity along time. This can significantly affect the pricing of an excess of loss treaty as we observe in Figure 13 where the pure price across varying threshold is significantly higher for the Model 123. The plot shows a behaviour with low thresholds leading to higher prices with  $u$  from 400 to 1400 thousand euro and then an abrupt increase in pure price is detected. The Generalized Pareto distribution models may not fit the data quite so well above this lower threshold as it does above the high threshold  $u = 1,500$ , but it might to use the low threshold to make calculations as outlined in recent literature [18],[19],[2].

## 6 Simulation Method

We concentrate on the estimation of the expected insurer's loss cost and we select the number of simulations necessary to estimate it with an acceptable degree of precision. One simulation is equivalent to the aggregate loss experience for a one-year period. We assume  $N$  is be the number of losses in this period, randomly generated by Negative Binomial distribution or Poisson distributions and the  $N$  losses are  $y_1, \dots, y_N$  and will be such that the excesses are randomly generated by Generalized Pareto. The parameters of this process will be chosen with reference to the fitted models. Thus the loss process will give rise to a total loss volume in a year

$$z = \sum_{i=1}^N y_i.$$

We repeat 100,000 simulations resulting in a sample of frequency and severity loss scenarios. The analysis could also be extended to a multiperiod scenario analysis. We define the economic risk capital as the difference between the expected loss, defined as the expected annual claims amount  $E_{Ctot}$ , and the 99.93<sup>th</sup> quantile of the distribution  $P_{Ctot}$ , corresponding to a Standard&Poor's A rating,  $ERC_{Ctot} = P_{Ctot} - E_{Ctot}$ .

We compare the impact of an excess of loss and a quota share reinsurance structures deriving the respective economic risk capitals  $ERC_{CEoL}$  and  $ERC_{CQS}$ . Each claim value is apportioned to reinsurance layers according to an excess of loss treaty with  $r = 3,500$  and  $R = \infty$  thousand euro obtaining the payout for the insurance company

$$c_i = \begin{cases} -x_i & \text{if } 0 < x_i \leq 3,500 \\ -3,500 & \text{if } 3,500 < x_i < \infty \end{cases}.$$

We impose a ratio  $q$  for the quota share treaty,  $c_i = -qx_i$ , equal to the ratio of mean of total cost netting the excess of loss treaty on total cost

$$q = \frac{E_{CEoL}}{E_{Ctot}} < 1 \rightarrow E_{CQS} = qE_{Ctot} = E_{CEoL}. \quad (13)$$

This allows us to analyze the reinsurance effect on claims distribution, capturing asymmetry and heavy tailedness of total cost.

As explained in the previous sections, we choose a threshold,  $u$ , equal to 775 thousand euro to fit loss severity and frequency distribution. The choice of threshold could be crucial to the results of the simulation but this will be the subject for further investigation in a future extension of this work. The question of which threshold is ultimately the best one depends on which use the results are to be put to as outlined by McNeil [18]. If we are concerned with answering the overall risk problem in terms of economic risk capital on the company, we may want to be conservative and bring answers which are too high rather than too low. On the other hand there may be business reasons for keeping low the attachment point or the premium from the reinsurance point of view.

As usual in reinsurance practice [10],[23],[35], first we generate a random number  $n$  of claims in excess of 100 thousand euro taken from the Negative Binomial

distribution. Second we generate  $n$  claims values; all these values are taken from the Generalized Pareto distribution as specified in section 4.1.2. The model is defined as Model T and the results are reported in Table 1. The distributions present a significant left asymmetry expressed by the skewness coefficient and are characterized by right fat tails expressing relevant risk capital coefficients. The excess of loss treaty “cut” the tail of the empirical distribution as confirmed by the risk capital and skewness coefficient and reduces the volatility of total loss amount. The quota share treaty, as we expect, produces a simple shift of the total loss amount. We estimate a non-parametric empirical probability density function and we graphically represent a normal kernel density distribution of the total cost amount (Figure 14 (a) and (b)).

We then develop the second approach according to McNeil and Saladin’s work [20]. The fitted models are the starting points for developing scenarios. We are interested in examining how the losses that exceed the threshold might develop in the next year beginning in January 2001. Given  $N$  to be the number of losses in the future year, we suppose that  $N$  has a Poisson distribution with mean

$$\int_0^{365} \lambda_t dt \quad ,$$

so that  $\lambda_t$  is the time-dependent intensity of a Poisson process expressed by equation (7). For every developed model we extrapolate under the POT approach a frequency scenario; losses occur as homogenous Poisson process with a constant intensity according to Model 0 and as a non-homogenous Poisson process with an increasing or decreasing intensity according to Model 1, 2, 3, 23 and 123. In addition we also extrapolate trends into the future for severity. We don’t build stress scenarios as combination of stress frequency or severity of different models as it is done in McNeil and Saladin’s approach [20]. We derive, as before, the annual aggregate loss in reinsurance layers; then we use the sample mean as an estimate for the expected insurer’s total cost. The results are reported as before in Table 1. We refer to scenario 0 for the Model 0 and soon. The results derived from developing scenarios with Peaks over Thresholds approach are significantly different with respect to traditional approach’s results in terms of standard deviation and skewness. The skewness index of total cost assumes values significantly higher with respect to the traditional model and at the same time we observe that we get different results, in particular for Model 123, under different scenarios developed with the Peaks over Threshold’s approach. The most salient finding of this simulation is the dramatic effect of Model 123 on total cost distribution gross of reinsurance: a higher standard deviation, economic risk capital and skewness are observed. The excess of loss treaty is more effective with respect to the other models in drastically reducing the values of losses. We observe, graphically, that excess of loss treaty “cut” the tail of the empirical distribution as confirmed by the risk capital and skewness coefficient and reduces the volatility of total loss amount (Figure 14 (a) and (b), Figure 15).

## 7 Conclusion

The viability of an insurance company depends critically on the size and frequency of large claims. An accurate modelling of the distribution of large claims helps an insurance company with pricing and reserving decisions, for example maintaining at an acceptable level the unexpected fluctuations in the results through reinsurance. In this paper we are particularly interested in finding an effective model for large losses in the past so that we can extrapolate through simulation a scenario based on separate estimation of loss frequency and loss severity according to extreme value theory. We present an application to the fire claims of an insurance company. We discuss the severity and frequency distributions of large claims with particular reference to generalized Pareto approximations. A first conclusion is that the distribution of fire claims is long-tailed and an in-depth exploratory data analysis is done to detect heavy tailed behavior and stability of parameters and statistics across different thresholds. Second we consider the development of scenarios for loss frequency according to negative binomial distribution and loss severity separately following standard reinsurance practice. We develop a second approach according to McNeil and Saladin's work [20] extending the Peak over Thresholds method to obtain a joint description of both the severity and frequency of losses exceeding high thresholds. We analyze and model also the time dependence of our data looking for an effective model for large losses in the past so that we can extrapolate the claims process into the future. We simulate the impact of a quota share and an excess of loss reinsurance structure on the distribution of total loss and on economic risk capital. The two approaches give significantly different results in terms of economic risk capital in particular when we detect the effect of an excess of loss treaty. However it is interesting to consider what the exact consequences of our models are in estimating the economic risk capital and the impact of reinsurance.

The analysis highlights the importance of a reinsurance program in term of “capital absorption&release” because “what happens in the tail of the loss distribution” -where things can go very wrong and where the unavailable could sadly happen- has relevant impact on the capital base. A pure comparison of the price of the reinsurance structure versus the cost of capital could lead to wrong conclusions and inappropriate company decision. It is necessary to take into account not only the spread but also the allocated capital that reinsurance enables to “save” and how this structure affects the economic risk capital whose goal is to cover the volatility of probable annual claims amount not transferred to reinsurer.

The rationale behind this simulation is pricing different reinsurance structures and combination of these. The comparison could allow us to choose among different reinsurance programmes the one that is the best (optimal scheme) in the “volatility of results-capital absorption&release” space. We implicitly demonstrate that reinsurance decisions based on costs comparison could lead to wrong conclusions.

**Software** The analyses in this paper were carried out in Matlab using functions implemented by the author. We are very grateful to Kenneth Holmstrom for providing Tomlab Optimization Toolbox. It wouldn't have been possible to achieve these results without it. We are grateful to Stuart Coles and Alexander J. McNeil for providing their S-plus functions.

**Table 1** - Simulation results and comparison of reinsurance structures

		Tot.Cost	Net QS <sup>(1)</sup>	Net EoL <sup>(2)</sup>
Model T	<i>E</i>	28,366	24,307	24,307
	St.dev	24,746	21,205	18,438
	<i>ERC</i>	182,286	156,201	98,766
	Skew	4.32	4.32	1.42
Model 0	<i>E</i>	22,919	19,625	19,625
	St.dev	12,594	10,784	6,193
	<i>ERC</i>	136,690	117,044	23,227
	Skew	10.02	10.02	0.39
Model 1	<i>E</i>	27,734	23,320	23,320
	St.dev	16,502	13,876	6,859
	<i>ERC</i>	149,929	126,067	26,154
	Skew	44.68	44.68	0.37
Model 2	<i>E</i>	23,491	20,158	20,158
	St.dev	12,726	10,920	6,278
	<i>ERC</i>	118,717	101,872	23,354
	Skew	17.3	17.3	0.39
Model 3	<i>E</i>	21,576	18,284	18,284
	St.dev	12,691	10,755	6,009
	<i>ERC</i>	144,915	122,804	22,877
	Skew	13.63	13.63	0.4
Model 23	<i>E</i>	27,964	23,221	23,221
	St.dev	14,401	11,958	7,096
	<i>ERC</i>	122,754	101,934	26,533
	Skew	15.04	15.04	0.37
Model 123	<i>E</i>	33,233	23,230	23,230
	St.dev	37,262	26,046	7,208
	<i>ERC</i>	479,378	335,083	26,848
	Skew	29.21	29.21	0.38

<sup>(1)</sup>Net to insurance company after quota share reinsurance structure<sup>(2)</sup>Net to insurance company after excess of loss reinsurance structure

## Appendix A

We define the conditional excess distribution function  $F_u$  as

$$F_u(x) = \Pr\{x \leq u + y | x > u\} \quad 0 \leq x \leq x_M \quad (\text{A.1})$$

where  $x$  is the random variable,  $u$  is the threshold and  $y = x - u$  are the exceedances and  $x_M < \infty$  is the higher value of  $x$ . The probability associated to  $x$  is  $\Pr\{x \leq u\} = F(u)$  and  $\Pr\{x \leq u + y\} = F(u + y)$ .  $F_u(x)$  expresses the conditional probability so

$$F_u(x) = \Pr\{x \leq u + y | x > u\} = \frac{\Pr\{x \leq u + y \cap x > u\}}{\Pr\{x > u\}} = \frac{F(u + y) - F(u)}{1 - F(u)}. \quad (\text{A.2})$$

We assume that adjusted historical loss data are realizations of independent, identically distributed, truncated random variables of the Generalized Pareto distribution (GPD) according to the Pickands [24] and Balkema and de Haan [1] theorem

$$\widetilde{F}_u(x) \approx G_{\xi\psi}(x) = \begin{cases} 1 - \left(1 + \frac{\widetilde{\xi}(x-u)}{\widetilde{\psi}}\right)_+^{-\frac{1}{\xi}} & \text{if } \xi \neq 0 \\ 1 - e^{-(x-u)/\varphi} & \text{if } \xi = 0 \end{cases} \quad \text{for } x \geq 0. \quad (\text{A.3})$$

Replacing in (A.3) and assuming that  $1 - F(u)$ , the non conditional probability that an observation exceeds  $N_u$ , can be approximated as  $1 - F(u) \cong \frac{N_u}{N}$ , where  $N_u$  is the number of the extreme observations and  $N$  is the total number of observations, we obtain

$$F(x) = F(u) + \widetilde{F}_u(x) [1 - F(u)] = 1 - \frac{N_u}{N} \left(1 + \frac{\widetilde{\xi}(x-u)}{\widetilde{\psi}}\right)^{-\frac{1}{\xi}}. \quad (\text{A.4})$$

Inverting the expression (A.4), we obtain for a given probability  $p$  the VaR (Value at Risk)

$$VaR_p = u + \frac{\widetilde{\psi}}{\widetilde{\xi}} \left[ \left( \frac{pN}{N_u} \right)^{\widetilde{\xi}} - 1 \right] \quad (\text{A.5})$$

and so we can derive the expected shortfall as

$$ES_p = VaR_p + E \left[ \underbrace{x - VaR_p}_y | x > VaR_p \right]. \quad (\text{A.6})$$

The second addend when  $\xi < 1$  is

$$e(u) = E(x - u | x > u) = \frac{\psi + \xi u}{1 - \xi} \quad (\text{A.7})$$

and then we derive



$$ES_p = VaR_p + \frac{\psi + \xi (VaR_p - u)}{1 - \xi}. \quad (\text{A.8})$$

The associated density function is

$$\frac{\partial F_u(x)}{\partial x} = g(x, \xi, \psi) = \begin{cases} \frac{1}{\psi} \left(1 + \frac{\xi}{\psi} (x - u)\right)^{-\frac{1}{\xi}-1} & \text{if } \xi \neq 0 \\ \frac{1}{\psi} e^{-(x-u)/\varphi} & \text{if } \xi = 0 \end{cases} \quad (\text{A.9})$$

and the maximum likelihood function associated and maximizing respect to the two parameters  $\xi$  and  $\psi$  is

$$\max_{\xi, \psi} L(\xi, \psi) = \begin{cases} -N_u \ln \psi - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{N_u} \ln \left(1 + \frac{\xi}{\psi} (x_i - u)\right) & \text{if } \xi \neq 0 \\ -N_u \ln \psi - \frac{1}{\psi} \left(\sum_{i=1}^{N_u} x_i + N_u u\right) & \text{if } \xi = 0 \end{cases}. \quad (\text{A.10})$$

## Appendix B

We consider jointly the exceedances and exceeding times according to a two dimensional approach Poisson process proposed by Smith [29]. We assume that the baseline time interval is  $T = 365$  which is a year. Let  $t$  be the time interval of the data points, for example daily, and denote the data span by  $t = 1, 2, \dots, N$  where  $N$  is the total number of data points. For a given threshold  $u$ , the exceeding times over the threshold are denoted by  $\{t_i, i = 1, \dots, N_u\}$  and the observed severity claims at  $t_i$  is  $x_i$ . We focus on modelling  $\{t_i, x_i\}$  for  $i = 1, \dots, N_u$ , where  $N_u$  depends on threshold  $u$ .

We analyze the assumption that the Poisson process is homogenous. The exceeding times and the exceedances  $\{t_i, x_i\}$  form a two-dimensional Poisson Process with intensity measure given by

$$\Lambda[(T_2, T_1) \cdot (x, \infty)] = \frac{T_2 - T_1}{T} \left(1 + \frac{\xi(x - \mu)}{\sigma}\right)^{-1/\xi} \quad (\text{B.1})$$

where  $0 \leq T_1 \leq T_2 \leq N$ ,  $x > u$ , and  $\mu, \sigma$  and  $\xi$  are the parameters. This intensity measure states that the occurrence of exceeding the threshold is proportional to the length of the time interval  $[T_1, T_2]$ . We consider the implied conditional probability of  $x = y + u$  given  $x > u$  over the time interval  $[0, T]$  and where  $x > 0$ ,

$$\frac{\Lambda[(0, T) \cdot (y + u, \infty)]}{\Lambda[(0, T) \cdot (u, \infty)]} = \left[\frac{1 + \xi(y + u - \mu)/\sigma}{1 + \xi(u - \mu)/\sigma}\right]^{-1/\xi} = \left[1 + \frac{\xi y}{\sigma - \xi(u - \mu)}\right]^{-1/\xi} \quad (\text{B.2})$$

which is the survival function of the conditional distribution. The intensity measure in (B.1) can be written as an integral of an intensity function

$$\Lambda[(T_2, T_1) \cdot (x, \infty)] = (T_2 - T_1) \int_x^\infty \lambda(z; \mu, \sigma, \xi) dz \quad (\text{B.3})$$

where the intensity function  $\lambda(x; \mu, \sigma, \xi)$  is defined as

$$\lambda(t, x; \mu, \sigma, \xi) = \frac{1}{T} g(x; \mu, \sigma, \xi) = \begin{cases} \frac{1}{T} \frac{1}{\sigma} \left(1 + \frac{\xi}{\sigma} (x - u)\right)^{-\frac{1}{\xi}-1} & \text{if } \xi \neq 0 \\ \frac{1}{T} \frac{1}{\sigma} e^{-(x-u)/\sigma} & \text{if } \xi = 0 \end{cases} \quad (\text{B.4})$$

The likelihood function for the observed exceeding times and their corresponding claims  $\{t_i, x_i\}$  over the two-dimensional space  $(0, N) \cdot (u, \infty)$  is given by

$$L(\mu, \sigma, \xi) = \left( \prod_{i=1}^{N_u} \frac{1}{T} g(x_i; \mu, \sigma, \xi) \right) \cdot \exp \left( -\frac{N_u}{T} \left(1 + \frac{\xi(u-\mu)}{\sigma}\right)^{-1/\xi} \right) \quad (\text{B.5})$$

maximizing respect to the three parameters  $\mu$ ,  $\sigma$  and  $\xi$  is reduced to

$$\max_{\mu, \sigma, \xi} L(\mu, \sigma, \xi) = \begin{cases} -\frac{N_u}{T} \ln \sigma - \frac{1}{T} \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{N_u} \ln \left(1 + \frac{\xi}{\sigma} (x_i - u)\right) & \text{if } \xi \neq 0 \\ -\frac{N_u}{T} \frac{1}{\xi} \ln \left(1 + \frac{\xi(u-\mu)}{\sigma}\right) & \\ -\frac{N_u}{T} \ln \sigma - \frac{1}{T} \sum_{i=1}^{N_u} \frac{(x_i - u)}{\sigma} - \frac{N_u}{T} \frac{(u-\mu)}{\sigma} & \text{if } \xi = 0 \end{cases} \quad (\text{B.6})$$

The two-dimensional Poisson process model discussed above is called a homogeneous Poisson model because the three parameters  $\mu$ ,  $\sigma$  and  $\xi$  are constant over time. When the parameters are time-varying  $(\xi_t, \mu_t, \sigma_t)$ , we have an inhomogeneous Poisson process. The intensity measure becomes

$$\Lambda[(T_2, T_1) \cdot (x, \infty)] = \frac{T_2 - T_1}{T} \left(1 + \frac{\xi_t(x - \mu_t)}{\sigma_t}\right)^{-1/\xi_t} \quad (\text{B.7})$$

and can be written as an integral of an intensity function

$$\Lambda[(T_2, T_1) \cdot (x, \infty)] = \int_{T_1}^{T_2} \int_x^\infty \lambda(z; \mu_t, \sigma_t, \xi_t) dt dz \quad (\text{B.8})$$

where the intensity function  $\lambda(t, x; \mu_t, \sigma_t, \xi_t)$  is defined as

$$\lambda(t, x; \mu_t, \sigma_t, \xi_t) = \frac{1}{T} g(x; \mu_t, \sigma_t, \xi_t) = \begin{cases} \frac{1}{T} \frac{1}{\sigma_t} \left(1 + \frac{\xi_t}{\sigma_t} (x - u)\right)^{-\frac{1}{\xi_t}-1} & \text{if } \xi_t \neq 0 \\ \frac{1}{T} \frac{1}{\sigma_t} e^{-(x-u)/\sigma_t} & \text{if } \xi_t = 0 \end{cases} \quad (\text{B.9})$$

The likelihood function of the exceeding times and exceedances  $\{t_i, x_i\}$  is given by

$$L(\mu_t, \sigma_t, \xi_t) = \left( \prod_{i=1}^{N_u} \frac{1}{T} g(x_i; \mu_{t_i}, \sigma_{t_i}, \xi_{t_i}) \right) \cdot \exp \left( -\frac{1}{T} \sum_{t=1}^{N_u} \left(1 + \frac{\xi_{t_i}(u - \mu_{t_i})}{\sigma_{t_i}}\right)^{-1/\xi_{t_i}} \right). \quad (\text{B.10})$$

We assume in our analysis that  $\mu_t$ ,  $\sigma_t$  and  $\xi_t$  are functions of a time index  $t$

$$\mu_t = \alpha_1 + \beta_1 t$$

$$\sigma_t = \exp(\alpha_2 + \beta_2 t)$$

$$\xi_t = \alpha_3 + \beta_3 t.$$

The days in our data vary from 01/01/1990 to 31/12/2000 and for numerical purposes it is preferable to re-scale the day variable so it is centered and over a simple range according to this transformation

$$t = \frac{(day - me)}{(me - day_1)}$$

where  $me$  is the median of days and  $day_1$  is the first date. We maximize respect to the set of parameters  $\Theta = \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ , and the likelihood functions is reduced to

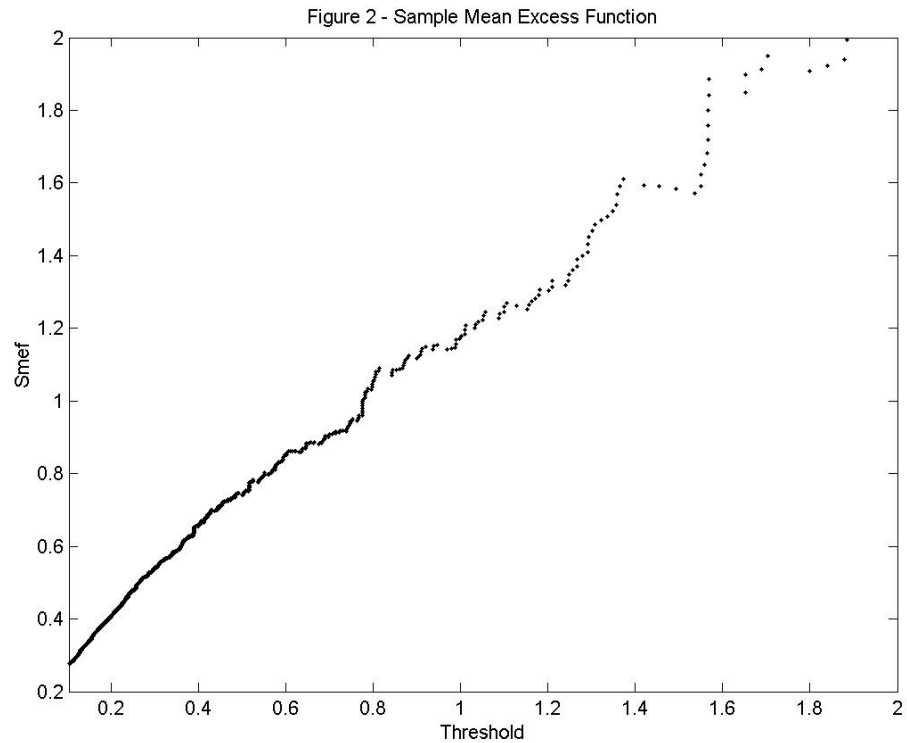
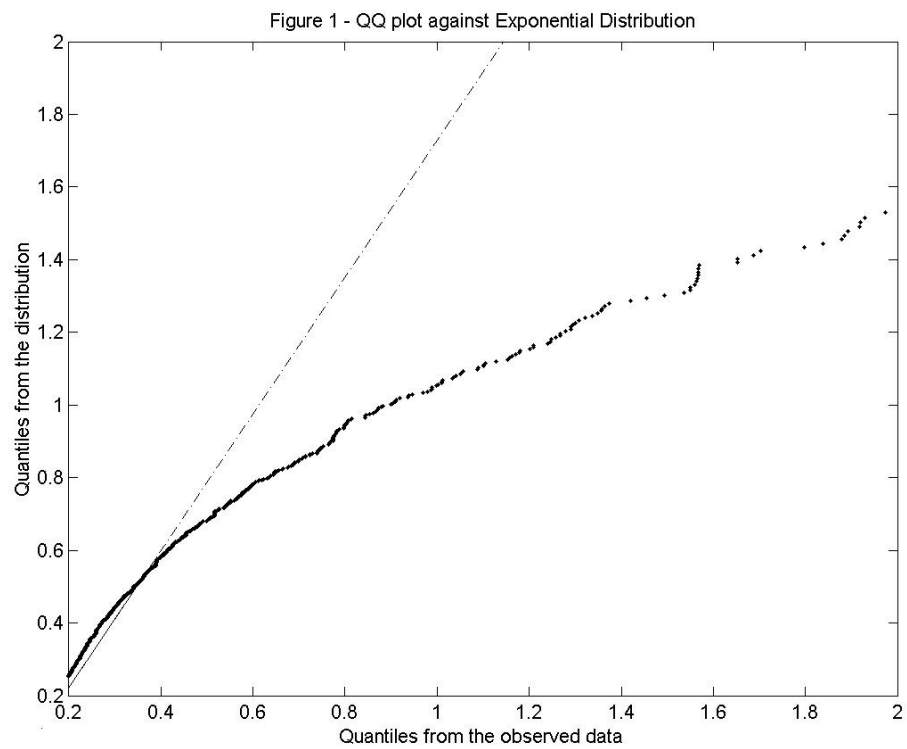
$$\max_{\Theta} L(\Theta) = \begin{cases} -\sum_{i=1}^{N_u} \ln \sigma_{t_i} - \sum_{i=1}^{N_u} \ln \left( 1 + \frac{\xi_{t_i}}{\sigma_{t_i}} (x_i - u) \right)^{\left( -\frac{1}{\xi_{t_i}} - 1 \right)} & \text{if } \xi_t \neq 0 \\ -\frac{1}{T} \sum_{i=1}^{N_u} \ln \left( 1 + \frac{\xi_{t_i} (u - \mu_{t_i})}{\sigma_{t_i}} \right)^{\frac{1}{\xi_{t_i}}} & \\ -\frac{1}{T} \sum_{i=1}^{N_u} \ln \sigma_{t_i} - \frac{1}{T} \sum_{i=1}^{N_u} \frac{(x_i - u)}{\sigma_{t_i}} - \frac{1}{T} \sum_{i=1}^{N_u} \frac{(u - \mu_{t_i})}{\sigma_{t_i}} & \text{if } \xi_t = 0 \end{cases} . \quad (\text{B.11})$$

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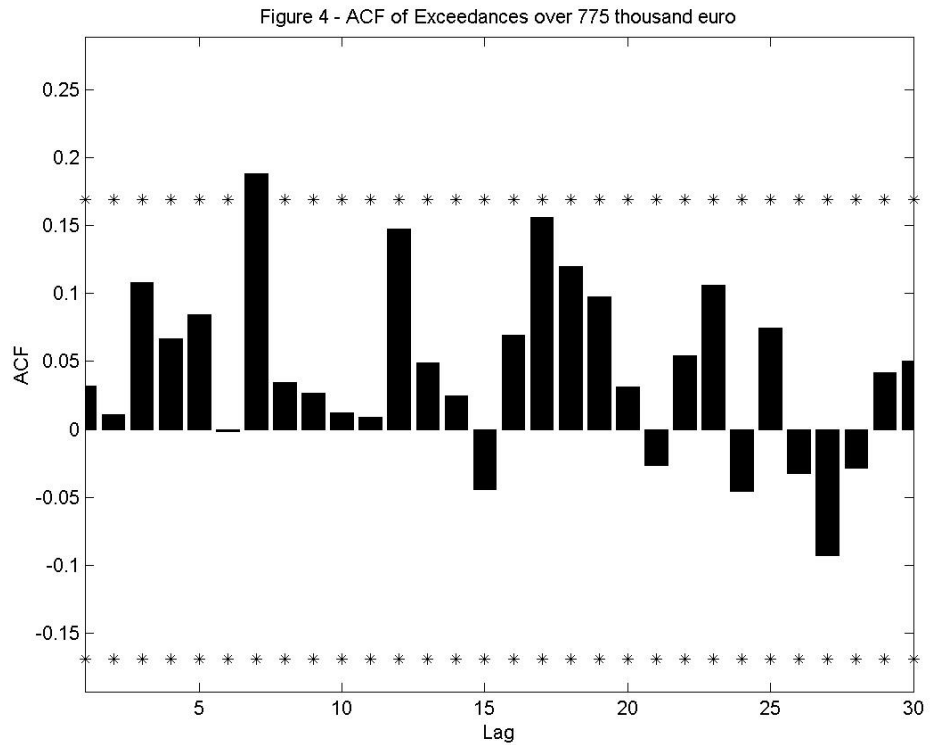
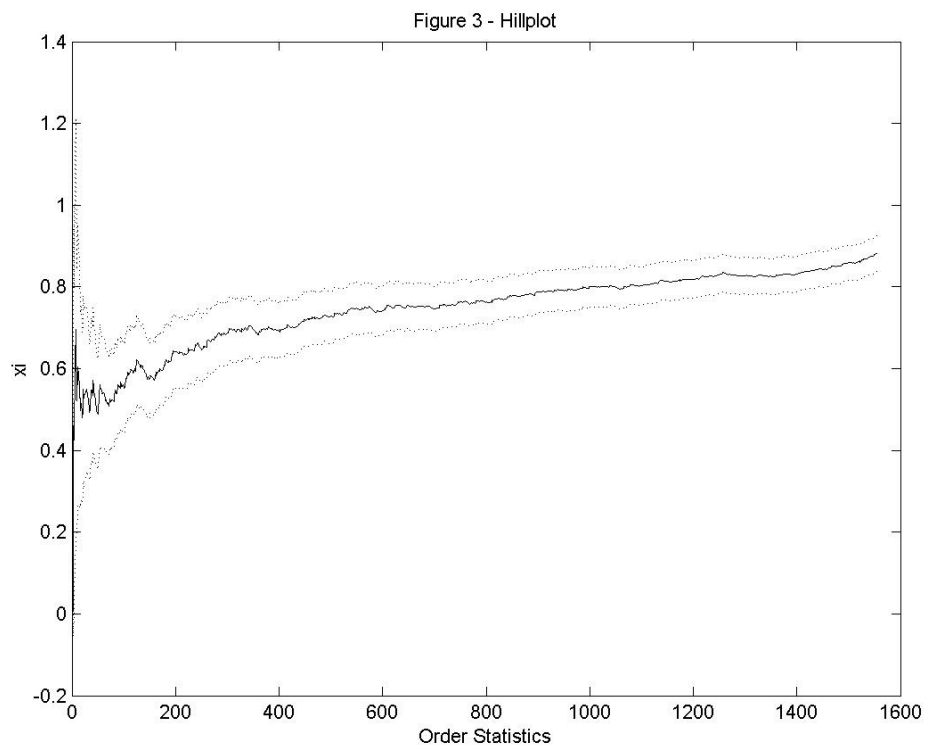




Figure 5 - Estimates of Shape and Threshold

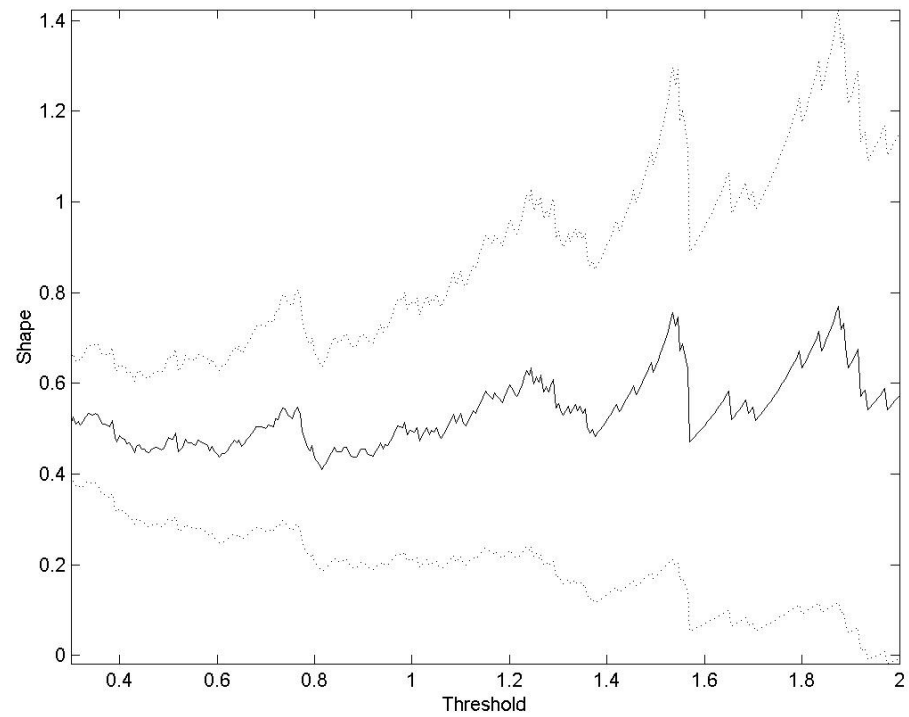


Figure 6 - Empirical Distribution, GPD ( $u = 775$ ) and GPD ( $u = 1500$ )

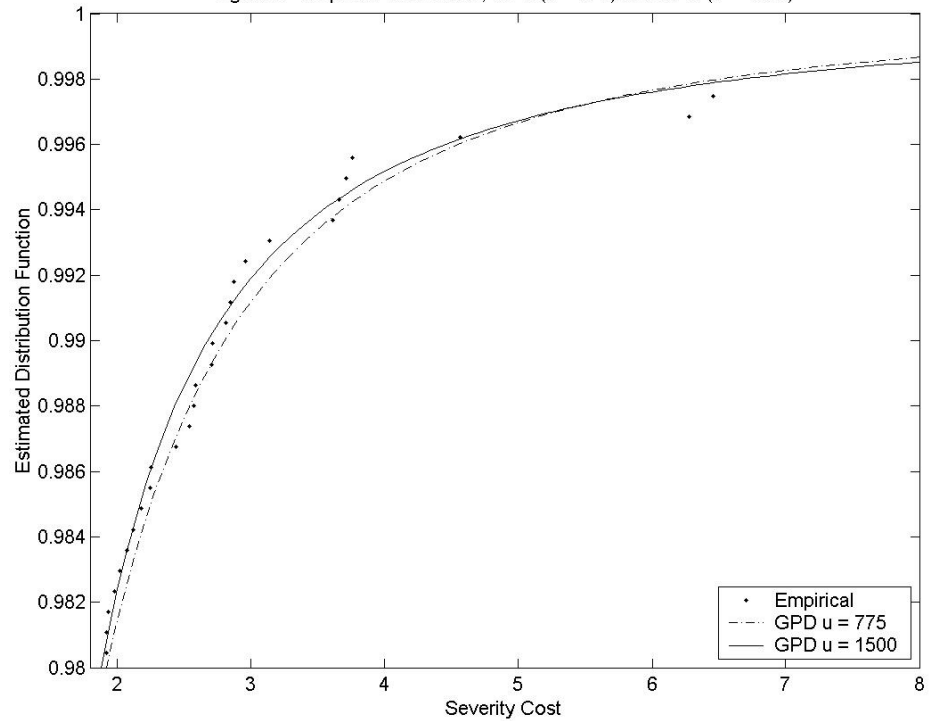


Figure 7 - Estimates of Shape for Model 0, Model 123 and Model 23

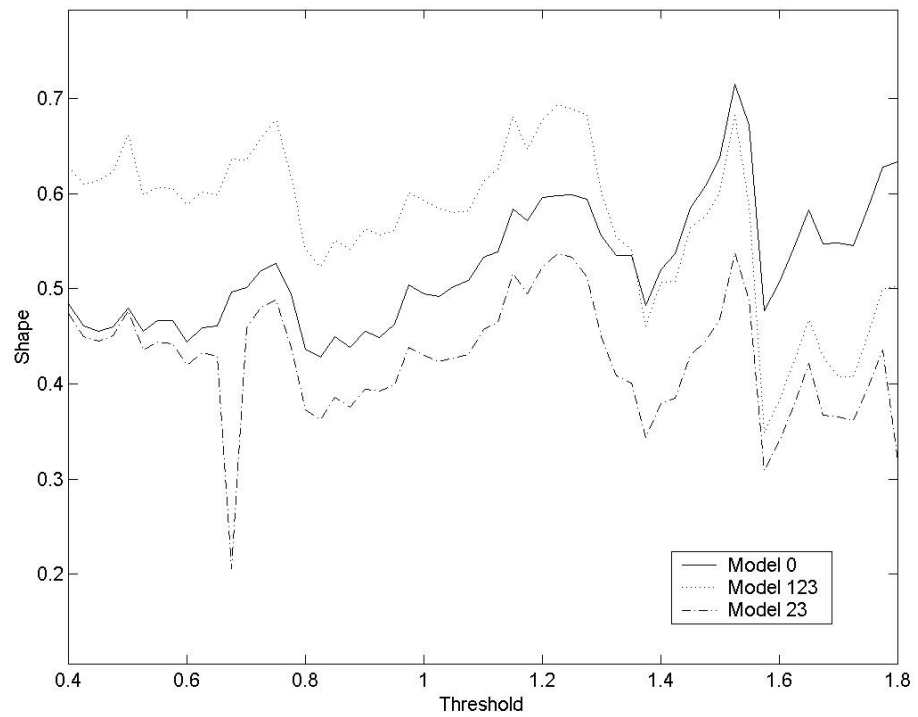


Figure 8 - QQ plot of Z statistics over  $u=775$  Model 123

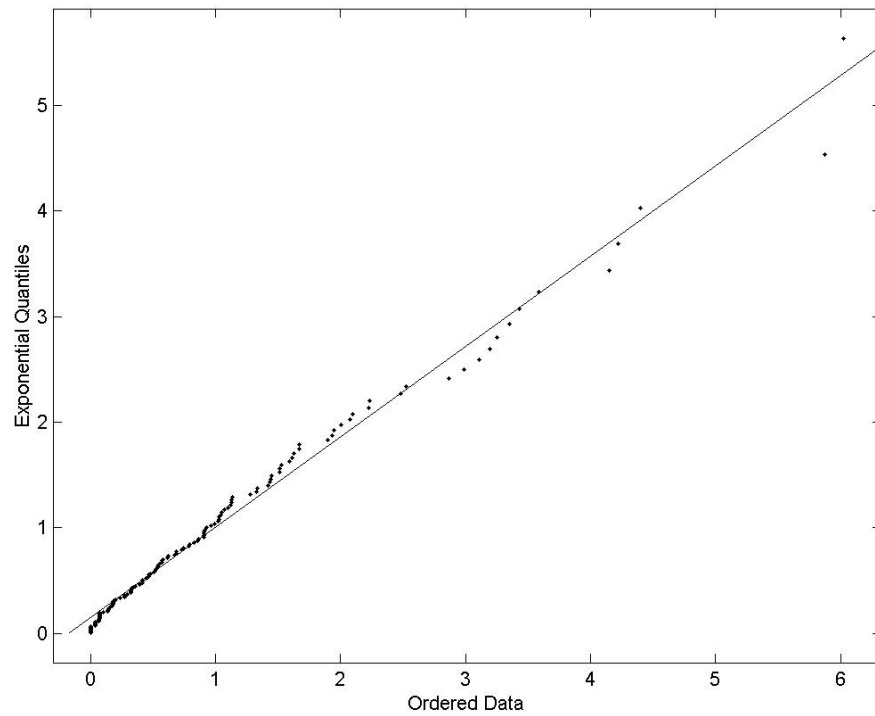


Figure 9 - Scatter plot of Z statistics over u=775 Model 123

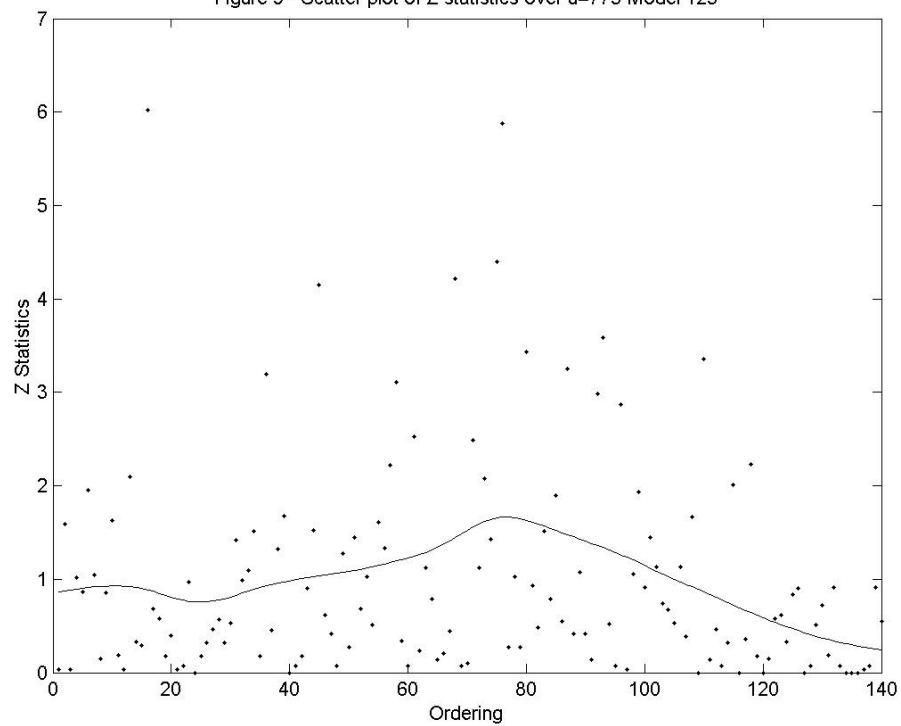


Figure 10 - QQ plot of W statistics over u=775 Model 123

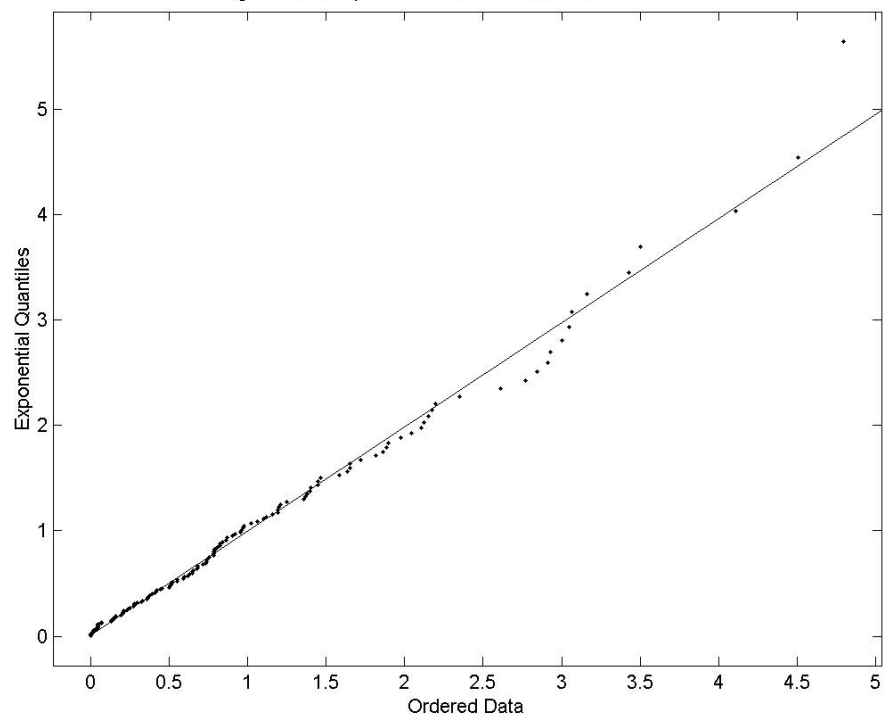


Figure 11 - Scatter plot of W statistics over u=775 Model 123

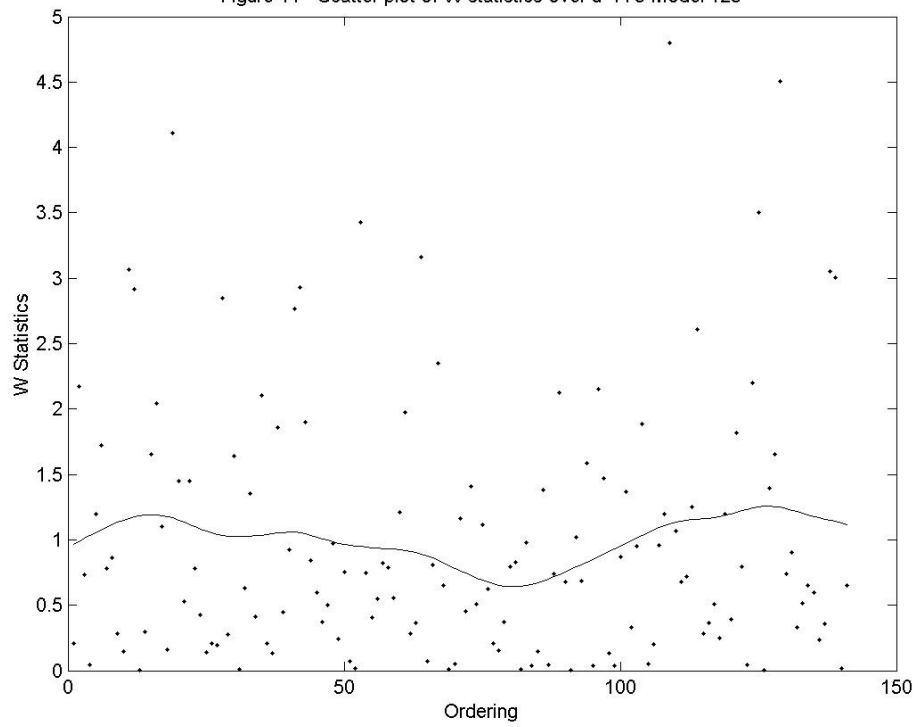


Figure 12 - Intensity of Exceedances

