Economic Risk Capital and Reinsurance: an Application to Fire Claims of an Insurance Company

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Abstract

The viability of an insurance company depends critically on the size and frequency of large claims. An accurate modelling of the distribution of large claims contributes to correct pricing's and reserving's decisions while maintaining, at an acceptable level, the unexpected fluctuations in the results through reinsurance. We provide a model for large losses and we extrapolate through a simulation a scenario based on separate estimation of loss frequency and loss severity according to extreme value theory, with particular reference to generalized Pareto approximations. We present an application to the fire claims of an insurance company. One conclusion is that the distribution of fire claims is long-tailed and an accurate exploratory data analysis is done to detect heavy tailed behavior and stability of parameters and statistics across different thresholds. We simulate the impact of a quota share and an excess of loss reinsurance structure on the distribution of total loss and on economic risk capital. We provide also a tool to price and investigate how different reinsurance programs can affect economic risk capital and explain the rationale of the choice of the optimal reinsurance programmes to smooth economic results.

Keywords: Economic risk capital; reinsurance; high excess layers; total large claims amount distribution; loss frequency distribution; loss severity distribution; extreme value theory; peaks over thresholds method; Binomial Negative distribution; Generalized Pareto distribution; scenarios.

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"If things go wrong,
How wrong can they go?"

1 Introduction

The foremost goal of reinsurance is to maintain at an acceptable level the unexpected fluctuations in the results of primary insurers. How can the unexpected fluctuations in the results of primary insurers be effectively maintained at an acceptable level with the help of reinsurance? In this context "effectively" means at the price one is prepared to pay for: as a matter of fact reinsurance function is to minimize fluctuations as much by means of reduced volatility and smoothed results, as through price negotiation and this can be achieved through the skilful use of retentions. From this point of view reinsurance can be considered an efficient way to manage the company capital base. Reinsurance coverage is therefore a viable alternative to capital management because it serves the purpose of absorbing the impact of unexpected losses. But a pure comparison of the price of reinsurance versus the cost of capital (WACC–Weighted Average Cost of Capital) could lead to wrong conclusions and inappropriate company decision: it is necessary to take into account not only the spread but also the allocated capital that reinsurance enables to "save". Furthermore reinsurance produces value by producing stability. This can translate into higher earnings through reduced financing costs, improved access to markets and stronger product pricing. It can lead to a higher earnings multiple through reduction in the market price of possible bankruptcy and fewer misreading of downwards earnings hits. Measuring these earnings and valuations effects is a still developing science even if the practice is very well known.

These issues depend critically on the size and frequency of large claims that are usually covered by excess of loss protection. We are specifically interested in modelling separately the tail of loss severity distribution according to the extreme value theory (EVT) and the frequency distribution of claims. Then trough a simulation we estimate the total large claims amount distribution. The extreme value theory has found more application in hydrology and climatology [20],[22] than in insurance and finance. This theory is concerned with the modelling of extreme events and recently has begun to play an important methodological role within risk management for insurance, reinsurance and finance [4],[8],[9]. Various authors have noted that the theory is relevant to the modelling of extreme events for insurance [2],[11],[18],[12],[13],[3],[21], finance through the estimation of Value at Risk [6],[14],[10],[19] and reinsurance and financial engineering of catastrophic risk transfer securization products [7],[17],[23]. The key result in EVT is the Picklands-Blakema-de Haan theorem [16],[1] that says that for a wide class of distributions losses which exceed high enough thresholds the Generalized Pareto distribution (GPD) holds true.

In this paper we are concerned with fitting the GPD to data on exceedances of high thresholds and we model the claims frequency distribution separately. We assume that the number of big claims made on a stable portfolio during a fixed premium period is negative binomial distributed. Finally we consider the development of scenarios for loss frequency and loss severity as in reinsurance practice is done

[5],[15],[24]. We generate randomly a number of claims per year and we calculate each claim severity through the Generalized Pareto distribution. We simulate the impact of a quota share and an excess of loss to analyze the reinsurance effect on claims distribution. The rationale behind this simulation is one that leads to price an excess of loss layer and/or to assess the capital released by ceded reinsurance. We investigate how the reinsurance treaty from the insurer point of view reduces risk exposure measuring the changes in total cost distribution and we capture asymmetry and heavy taildness of total cost. We represent graphically the kernel density distributions.

In this paper the severity and frequency of large claims are examined with reference to a large data set representing a relevant portion of the fire claims of RAS, an italian insurance company, analyzing the losses over 200 million lire from 1990 to 2000. To protect the confidentiality of the source, the data are coded so that the exact time frame and frequency of events covered by data are not revealed. The figures and the comments do not represent the views and/or opinion of the RAS management and risk capital measures are purely indicative.

Section 2 discusses why and how we model the probability distribution of the total large claims and how we measure the impact of the reinsurance structures on economic risk capital. Section 3 describes the dataset used and the hypothesis adopted to trend and adjust data. Section 4 discusses the fitting of frequency distribution of large claims with particular reference to Binomial Negative distribution. Section 5 extends the analysis to the extreme value theory. We provide explanatory graphical methods to detect heavy tail behavior and preliminary information about the data (Subsection 5.1). The severity distribution is fitted with Pareto Generalized distribution in Subsection 5.2 and we analyse the stability of parameters, quantiles and expected shortfall across different thresholds. We provide a pricing of a high-excess layer across lower attachment points and different thresholds. Section 6 concentrates on the simulation approach to model the total claims amount and to detect the impact of different reinsurance structures on economic risk capital. Finally, Section 7 presents conclusions and possible future extensions.

2 Measuring sensitivity of economic risk capital

Every portfolio of risk policies incurs losses of greater and lower amounts at irregular intervals. The sum of all the losses (paid&reserved) in any one-year period is described as the annual loss burden or simply total incurred claims amounts. The future annual loss burden is estimated on the basis of the predicted claims frequency and predicted average individual claim amount, but what actually happens usually deviates considerably from forecasts on a purely random basis. A portfolio of insurance contracts is considered "exposed" to fluctuations when the actual loss burden is expected to deviate widely from the predicted one.

Our methodology provides a tool to measure this volatility. The simulation method can be used to estimate the reinsurance loss cost and the economic risk capital to cover the probable annual claims amount. We model the loss severity and loss frequency distribution functions to simulate a statistically representative sample of loss experience reinsurance treaty. When simulating loss experience one should be convinced that the severity and frequency loss distribution used in the simulations reflect the reality to the greatest extent possible. To assure that, a good amount of meticulous work is done. Historical individual losses are trended and developed. Loss frequency and loss severity distribution for the projected coverage period are estimated based on the adjusted loss.

We simulate the impact of reinsurance on total claims amounts. Reinsurance is the transfer of a certain part of portfolio liability (risk) to another carrier. We consider two types of reinsurance structure: a quota share and an excess of loss treaty. The first case is implemented as follows: the primary insurance company cedes a q share of all premiums and claims; the second type refers to an excess of loss treaty with lower and upper attachment points r and R respectively, where $R > r$. This means the payout y_i on a loss x_i for a reinsurer is given by

$$y_i = \begin{cases} 0 & if \quad 0 < x_i \leq r \\ x_i - r & if \quad r < x_i \leq R \\ R - r & if \quad R < x_i < \infty \end{cases}$$

and the payout of insurance company is

$$c_i = \begin{cases} -x_i & if \quad 0 < x_i \leq r \\ -r & if \quad r < x_i \leq R \\ -x_i + (R - r) & if \quad R < x_i < \infty \end{cases} .$$

As discussed by McNeil [11] there are two related actuarial problems concerning this layer

- the pricing problem. Given r and R what should this insurance layer cost a customer?

- the optimal attachment point problem. If we want payouts greater than a specified amount to occur with at most a specified frequency, how low can we set r?

  Within this framework, another number of questions arise:

- Is there any significant difference in the distributions of total cost amounts associated with different types of reinsurance structures?

- What are the implications of all these issues for the overall risk in terms of economic risk capital on the company?

Given the number of losses in a period n, the losses are $x_1, ..., x_n$ for the reinsurer and the aggregate payout would be $z = \sum_{i=1}^{n} y_i$. The price is the expected payout plus a risk loading factor which is assumed to be k times the standard deviation of the payout, Price $= E[z] + k\sqrt{Var[z]}$. The expected payout $E[z]$ is known as the pure premium and it is defined by $E[y_i] E[n]$, where $E[y_i]$ is the expected value of severity claim and $E[n]$ is the expected number of claims. McNeil [11] remarks that the pure premium calculated using the variance principle is a simple price indication and is an unsophisticated tool when the data present a heavy tail behavior, since moments may not exist or may be very large. The attachment point problem essentially is reassumed by the estimation of a high quantile of the loss severity distribution.

From the insurer point of view there is a particular concern not only about the pricing of the reinsurance structure described previously but also on how this structure affects the economic risk capital whose goal is to cover the volatility of probable annual claims amount not transferred to reinsurer. The solvency and viability of an insurance company depends on probabilistic calculations of risk and critically on the size and frequency of large claims. We need a good estimate of the loss severity distribution for large x_i in the tail area and also we need a good estimate of the loss frequency distribution of n, as we will explain in the next sections.

3 Data

We use a relevant portion of the fire claims for reported years 1990 through 2000 that exceed 200 million lire at 31/5/2001 evaluation date, 1554 observations, and their historical development. To protect the confidentiality of the source, the data are coded so that the exact time frame and the frequency of events covered by data are not revealed.

When trending the historical losses to the prospective experience period claim cost level it is important to select an appropriate severity trend factor. We develop the total incurred losses and the number of claims paid by year to the ultimate one and we calculate an average loss size by report year. Then we adjust the individual cost by an ISTAT inflation factor. Some individual claims in excess of 200 million lire are still open at 31/5/2001. The ultimate values of these claims might be different from their reserved values, carried on the books. Generally, it is not easy to adjust individual claim values for possible development using aggregate development data only. The major complication stems from the fact that aggregate loss development is driven by two different forces: new claims and the adjustment values for already outstanding claims (reserving review).

4 Fitting Frequency Distribution

For the excess claim frequency distribution we use the Negative Binomial distribution. This discrete distribution has been utilized extensively in actuarial work to represent the number of insurance claims. Since its variance is greater than its mean, the negative binomial distribution is especially useful in situations where the potential claim count can be subject to significant variability. To estimate parameters for the negative binomial distribution we start with the estimate of the expected final number of claims in excess of 200 million lire.

Studying a claims number development triangle with a kind of "chain ladder" approach, we estimate final claims frequency over the same threshold over ten past years, that will be used in fitting the loss severity distribution, and adjust it on an exposure basis. Estimating on this historical data the empirical expectation and variance, we calculate the parameters for the Negative Binomial distribution.

5 Fitting Severity Distribution

In order to describe the distribution of the sizes of individual claims made on a portfolio, we assume that insurance losses are denoted by independent, identically distributed random variables and we attempt to find an estimate of the severity distribution function truncated at 200 million lire.

Generally three types of methods are distinguished to estimate extreme quantiles from heavy tailed data in literature:

- the method of block of maxima;
- the peaks over-the-thresholds;
- quantile based methods.

In this paper we use the second method (POT) and we fit the generalized Pareto distribution on our dataset. A first step before model fitting is undertaken, a number of exploratory graphical methods is provided to make a preliminary analysis about the data and in particular their tail.

5.1 Exploratory Data Analysis

We develope some graphical tests to identify the most extreme losses. The QQ-plot is obtained (Figure 1): the quantiles of the empirical distribution function on the x-axis are plotted against the quantiles of the Exponential distribution on the y-axis. It examines graphically the hypothesis that the losses come from an Exponential distribution and this is confirmed if the points should lie approximately along a straight line. A concave departure from the ideal shape indicates a heavier tailed distribution whereas convexity indicates a shorter tailed distribution. We observe that our dataset of insurance losses show heavy tailed behavior.

A further useful graphical tool is the plot of the sample mean excess function (Figure 2). The sample mean excess function is defined by

$$e_n(u) = \frac{\sum_{i=k}^{n} (x_i - u)^+}{n-k+1} \tag{1}$$

and is the sum of the excesses over the threshold u divided by the number of data points, $n-k+1$, which exceed the threshold u. The sample mean excess function describes the expected overshoot of a threshold given that exceedance occurs and is an empirical estimate of the mean excess function which is defined as $e(u) = E[x-u|x>u]$. If the points show an upward trend, this is a sign of heavy tailed behavior. Exponentially distributed data would give an approximately horizontal line and data from a short tailed distribution would show a downward trend. If the empirical plot seems to follow a reasonably straight line with positive slope above a certain value of u, then this is an indication that the data follow a generalized Pareto distribution. This is a precisely the kind of behavior we observe in our data and a significant positive trend is detected in u equal to 1,500 and 3,000 million lire.

According to Resnik [18] there are additional techniques and plotting strategies which can be employed to assess the appropriateness of heavy tailed models. We define the Hill estimator as

$$H_{n-k,k} = \frac{1}{k}\sum_{j=1}^{k} \log x_{n-j+1,n} - \log x_{n-k,n}, \tag{2}$$

supposing one observes $x_1, ..., x_n$ and orders these observations as $x_{(1)} \geq ... \geq x_{(n)}$. The Hill statistic is nothing else than the mean excess estimate of the log-transformed data based on observations which are exceedances over $\log x_{n-k,n}$ divided by the threshold $\log x_{n-k,n}$. The hill estimator is defined as $H_{n-k,k} = 1/\alpha_{n,k}$ where $\alpha_{n,k}$ is the tail index of a semiparametric Pareto type model as $F(x) = x^{-\alpha} l(x)$ while $l(x)$ is a nuisance function. We have as many estimates of α as we have data points; for each value of k we obtain a new estimate of α. We plot $\alpha_{n,k}$ as function of k (Figure 3) and we observe an increasing trend in our estimates of α as k increases revealing again a heavy tailed behavior of our data. One can try to infer the value of α from a stable region of graph or choose an optimal k which minimizes asymptotic mean square error of the estimator α, but this is difficult and puts a serious restriction on reliability. Next to methods based on high order statistics, such as the Hill estimator, an alternative is offered by the peaks over threshold approach (POT) that will be discussed in the next section.

5.2 Generalized Pareto Distribution

Modern method of extreme value analysis are based on exceedances over high thresholds. Denoting the threshold by u, the conditional distribution of excesses over u is modelled by the Generalized Pareto distribution (GPD)

$$\Pr\{x \leq u + y | x > u\} \approx 1 - \left(1 + \frac{\xi(x-u)}{\psi}\right)_{+}^{-1/\xi} \quad for \quad x \geq 0 \tag{3}$$

and $(x_+ = \max(x, 0))$, where $\psi > 0$ is a scale parameter, ξ a shape parameter and u is the threshold. The three cases $\xi < 0$, $\xi = 0$ and $\xi > 0$ correspond to different types of tail behavior. The case $\xi < 0$ arises in distributions where there is a finite upper bound on the claims, a tendency for claims to cluster near the upper limit. The second case, $\xi = 0$ arises in case with an exponentially decreasing tail or also when data are distributed according to a gamma, normal, Weibull, lognormal distribution. However the third case, $\xi > 0$, corresponds to a "long-tailed" distribution that is associated to the Pareto tail according to $\Pr\{X > x\} \sim x^{-\alpha} l(x)$ as $x \to \infty$ for a positive constant α. The relation between ξ and α is $\xi = 1/\alpha$. When $0 < \alpha < 2$ the distribution is also tail equivalent to an α-stable distribution. A critical issue is the selection of an appropriate threshold u as defined before. We fit recursively the GPD (Generalized Pareto distribution) to those data point, which exceed high thresholds from 600 million lire to 3,500 million lire with a step of 10 million lire. We obtain maximum likelihood estimates for the shape parameter, ξ, and plot it respect the number of exceedances and the threshold (Figure 4 and 5). One conclusion is that the distribution of fire claims is long-tailed as the exploratory analysis suggests. As can be observed from the figures, the choice of the tail index is crucial also for the simulation process explained in the next section. If the threshold is set too high, there will not be enough observations over u to calculate good estimates of ψ and ξ. However we do not want u to be too low, because we want to analyse "large claims". The ideal situation would be that shape estimates' pattern is stable.

Some of the most frequent questions concerning risk management in finance involve the extreme quantile estimation. A typical example of such tail related risk measures is Value at Risk (VaR) calculation and the expected shortfall. From VaR it is possible to calculate the risk capital sufficient to cover losses from a portfolio over a holding period for a fixed number of days and is defined as the p^{th} quantile of the distribution of the possible losses. Another informative measure of risk is the expected shortfall or tail conditional expectation which estimates the potential size of loss exceeding VaR or p^{th} quantile. The higher the value of the shape parameter the heavier the tail and the higher the quantile estimates we derive. We plot the 99^{th} quantile and expected shortfall respect to the threshold (Figure 6 and 7) according to the formulas shown in Appendix A. We observe that the quantile plot is relatively stable in the range form 1,500 to 2,000 million lire with a decreasing trend. On the other hand the expected shortfall is relatively unstable and reveals an increasing trend across an increasing threshold.

The expected shortfall plays a central role in rating of an excess of loss reinsurance in an excess of a priority level r to ∞ but we can provide it in a more efficient way. In considering the issue of the best choice of threshold we can also investigate how the price of a layer varies with threshold. To give an indication of the prices we get from our model we calculate $E\left[y|x > \delta\right]$ for a layer running from 7,000 to 100,000 million lire (Figure 8)

$$E\left[y|x > \delta\right] = \int_r^R (x - r) f_{x^\delta}(x) dx + (R - r)(1 - F_{x^\delta}(R)) \tag{4}$$

where $F_{x^\delta}(x)$ and $f_{x^\delta}(x) = dF_{x^\delta}(x)/dx$ denotes the cumulative density function and the probability density function respectively for the losses truncated at δ, that is 200 million lire for our analysis. Then we calculate $E\left[y|x > \delta\right]$ recursively fixing $r = 7,000$ and increasing R from 7,000 to 100,000 million lire with a step of 50 million lire with $u = 1,500$ and we observe the concavity of our expression (Figure 9).

We select the threshold equal to 1,500 and 3,000 million lire. We have 141 and 48 exceedance observations respectively and we estimate the Generalized Pareto distributions: $\xi = 0.492$ for $u = 1,500$ and $\xi = 0.715$ for $u = 3,000$. We provide the QQ-plot for the two thresholds (Figure 10 and 11). We represent graphically the empirical distribution and the two estimated GPD to make a comparative analysis of fitting (Figure 12). The GPD with $u = 1,500$ is a reasonable fit, although its tail is just a little too thin to capture the behavior of the highest observed losses and seems to slightly underestimate the probabilities of medium-to-large losses respect to the GPD with $u = 3,000$. Anyway we choose a threshold equal to 1,500 million lire for the simulation.

6 Simulation Method

We concentrate on the estimation of the expected insurer's loss cost and we select the number of simulations necessary to estimate with an acceptable degree of precision. One simulation is equivalent to the aggregate loss experience for a one-year period. First, we generate a random number n claims in excess of 200 million lire taken from the Negative Binomial distribution. Secondly we generate n claims values; all these values are taken from the Generalized Pareto distribution as specified in section 5. Next each claim value is apportioned to reinsurance layers according to a quota share and an excess of loss treaty we want to test. An excess of loss treaty with $r = 7,000$ and $R = \infty$ million is imposed according to the payout expressed in Section 2, while we impose a ratio q for the quota share treaty equal to the ratio of mean of total cost netting the excess of loss treaty on total cost. We calculate $q = 0.855$ and the insurance company transfers the $(1 - q)\,\%$ of losses to the reinsurer. This allows us to compare the impact of the two reinsurance structures in terms of economic risk capital. Finally the aggregate loss each reinsurance layers is calculated by adding the appropriate portions of n individual claim values. We repeat 100,000 independent simulations resulting in samples for the annual aggregate loss in reinsurance layers; then we use the sample mean as an estimate for the expected insurer's total cost.

For each treaty simulated we calculate the economic risk capital as the difference between the expected loss, defined as the expected annual claims amount, and the 99.93^{th} quantile of the distribution corresponding to a Standard&Poor's A rating. We analyse the reinsurance effect on claims distribution and capture asymmetry and heavy taildness of total cost from the insurer point of view (Table 1).

Table 1 - Simulation results and comparison of reinsurance structures

	Tot.Cost	Net Quota Share$^{(*)}$	Net Excess of Loss$^{(*)}$
Mean	54,836	46,910	46,910
St.dev	47,603	40,723	35,817
Risk Capital	352,650	301,680	190,980
Skewness	3.48	3.48	1.4

$^{(*)}$Net to insurance company after reinsurance

We estimate a non-parametric empirical probability density function and we represent graphically a normal kernel density distribution of the three possible loss amounts. The distributions present a significant left asymmetry expressed by the skewness coefficient and are characterized by right fat tails expressing relevant risk capital coefficients (Figure 13). We observe, graphically, that excess of loss treaty "cut" the tail of the empirical distribution as confirmed by the risk capital and skewness coefficient and reduces the volatility of total loss amount. The quota share treaty, as we expect, produces a simple shift of the total loss amount.

As explained in the previous sections, we choose a threshold equal to 1,500 million lire on fitting loss severity and frequency distribution. In our experience the choice of threshold seems not to be so crucial to the results of the simulation but this will be the subject for further investigation in a future extension of this work. The question of which threshold is ultimately best depends on the use to which the results are to be put as outlined by McNeil [11]. If we are concerned with answering the overall risk problem in terms of economic risk capital on the company, we may want to be conservative and bring answers which are too high rather then too low. On the other hand there may be business reasons for keeping the attachment point or premium low from the reinsurance point of view.

7 Conclusion

The viability of an insurance company depends critically on the size and frequency of large claims. An accurate modelling of the distribution of large claims helps an insurance company with pricing and reserving decisions, for example maintaining at an acceptable level the unexpected fluctuations in the results through reinsurance. In this paper we are particularly interested in finding an effective model for large losses in the past so that we can extrapolate through simulation a scenario based on separate estimation of loss frequency and loss severity according to extreme value theory. We present an application to the fire claims of an insurance company. We discuss the probability distribution of large claims with particular reference to generalized Pareto approximations. One conclusion is that the distribution of fire claims is long-tailed and an accurate exploratory data analysis is done to detect heavy tailed behavior and stability of parameters and statistics across different thresholds.

The analysis highlights the importance of a reinsurance program in term of "capital absorption&release" because " what happens in the tail of the loss distribution" -where things can go very wrong and where the inevitable could sadly happen- has relevant impact on the capital base. We simulate the impact of a quota share and an excess of loss reinsurance structure on the distribution of total loss and on economic risk capital. We provide also a tool to price and investigate how different reinsurance programs can affect economic risk capital and explain the rationale of the choice of the optimal reinsurance programs to smooth economic results.

The rationale behind this simulation is one that leads to price different reinsurance structures and combination of these. The comparison could allow us to choose among different reinsurance programmes the one that is the best (optimal scheme) in the "volatility of results-capital absorption&release" space. We implicitly demonstrate that reinsurance decisions based on costs comparison could lead to wrong conclusions.

Appendix A

We define the conditional excess distribution function F_u as

$$F_u(x) = \Pr\{x \le u + y | x > u\} \qquad 0 \le x \le x_M \tag{A.1}$$

where x is the random variable, u is the threshold and $y = x - u$ are the excedancees and $x_M < \infty$ is the higher value of x. The probability associated to x is $\Pr\{x \le u\} = F(u)$ and $\Pr\{x \le u + y\} = F(u+y)$. $F_u(x)$ expresses the conditional probability so

$$F_u(x) = \Pr\{x \le u + y | x > u\} = \frac{\Pr\{x \le u + y \cap x > u\}}{\Pr\{x > u\}} = \frac{F(u+y) - F(u)}{1 - F(u)}. \tag{A.2}$$

We assume that adjusted historical loss data are realizations of independent, identically distributed, truncated random variables of the Generalized Pareto distribution (GPD) according to the Pickands [16] and Balkema and de Haan [1] theorem

$$\widetilde{F_u}(x) \approx G_{\xi\psi}(x) = \begin{cases} 1 - \left(1 + \frac{\widetilde{\xi}(x-u)}{\widetilde{\psi}}\right)_+^{-\frac{1}{\xi}} & \text{if } \xi \ne 0 \\ 1 - e^{-(x-u)/\varphi} & \text{if } \xi = 0 \end{cases} \quad for \quad x \ge 0. \tag{A.3}$$

Replacing in (A.3) and assuming that $1 - F(u)$, the not conditional probability that an observation exceeds n, can be approximated as $1 - F(u) \cong \frac{n}{N}$, where n is the number of the extreme observations and N is the total number of observations, we obtain

$$F(x) = F(u) + \widetilde{F_u}(x)\left[1 - F(u)\right] = 1 - \frac{n}{N}\left(1 + \frac{\widetilde{\xi}(x-u)}{\widetilde{\psi}}\right)^{-\frac{1}{\xi}}. \tag{A.4}$$

Inverting the expression (A.4), we obtain for a given probability p the VaR (Value at Risk)

$$VaR_p = u + \frac{\widetilde{\psi}}{\widetilde{\xi}}\left[\left(\frac{pN}{n}\right)^{\widetilde{\xi}} - 1\right] \tag{A.5}$$

and so we can derive the expected shortfall as

$$ES_p = VaR_p + E\left[\underbrace{x - VaR_p}_{y} | x > VaR_p\right]. \tag{A.6}$$

The second addend when $\xi < 1$ is

$$e(u) = E\left(x - u | x > u\right) = \frac{\psi + \xi u}{1 - \xi} \tag{A.7}$$

and then we derive

$$ES_p = VaR_p + \frac{\psi + \xi\left(VaR_p - u\right)}{1 - \xi}. \tag{A.8}$$

The associated density function is

$$\frac{\partial F_u\left(x\right)}{\partial x} = g\left(x, \xi, \psi\right) = \begin{cases} \frac{1}{\psi}\left(1 + \frac{\xi}{\psi}\left(x - u\right)\right)^{-\frac{1}{\xi} - 1} & \text{if } \xi \neq 0 \\ \frac{1}{\psi} e^{-(x-u)/\varphi} & \text{if } \xi = 0 \end{cases} \tag{A.9}$$

and the maximum likelihood function associated and maximizing respect to the two parameters ξ and ψ is

$$\max_{\xi,\psi} L\left(\xi, \psi\right) = \begin{cases} -n \ln \psi - \left(\frac{1}{\xi} + 1\right) \sum\limits_{i=1}^{n} \ln\left(1 + \frac{\xi}{\psi}\left(x - u\right)\right) & \text{if } \xi \neq 0 \\ -n \ln \psi - \frac{1}{\psi}\left(\sum\limits_{i=1}^{n} x + nu\right) & \text{if } \xi = 0 \end{cases}. \tag{A.10}$$

References

[1] Balkema, A. and de Haan, L. (1974), "Residual Life Time at Great Age", *Annals of Probability*, 2, 792-804.

[2] Beirlant, J. and Teugels, J. (1992), "Modelling Large Claims in Non-Life Insurance", *Insurance Mathematics and Economics*, 11, 17-29.

[3] Beirlant, J., Matthys, G. and Dierckx, G. (2000), "Heavy Tailed Distributions and Rating", Working Paper, Katholieke University Leuven, University Center of Statistics, available www.kuleuven.ac.be/ucs/research/publi.htm.

[4] Beirlant, J. and Matthys, G. (2001), "Extreme Quantile Estimation for Heavy-Tailed Distributions", Working Paper, Katholieke University Leuven, University Center of Statistics, available www.kuleuven.ac.be/ucs/research/publi.htm.

[5] Coutts, S. and T.R.H. Thomas (1997), "Capital and Risk and their Relationship to Reinsurance Programs", Casualty Actuarial Society, Forum Summer 1997, Reinsurance Call Papers, available http://www.casact.org/pubs/forum/97spforum/97spftoc.htm.

[6] Danielsson, J. and de Vries,G.C. (2000), "Value at Risk and Extreme Returns", *Annals d'Economie et de Statistique*, Vol. 60, pp. 239-269.

[7] Doherty, N.A. (1997), "Financial Innovation in the Management of Catastrophe Risk", *Joint Day Proceedings Volume of XXVIIIth International Astin Colloquium/7th International AFIR Colloquium*, 1-26.

[8] Embrechts,P., Kluppelberg, C. and Mikosch,T. (1997), *Modelling Extremal Events for Insurance and Finance*, Springer Verlag, Berlin.

[9] Embrechts, P., Resnik S.I. and G.Samorodnitsky (1999), "Extreme Value Theory as a Risk Management Tool", *North American Actuarial Journal*, 3, 30-41.

[10] Embrechts, P. (2000), "Extreme Value Theory: Potential and Limitations as an Integrated Risk Management Tool", *Derivatives Use, Trading & Regulation*, 6, 449-456.

[11] McNeil, A. (1997), "Estimating the Tails of Loss Severity Distributions Using Extreme Value Theory", *Astin Bulletin*, 27, 117-137, available www.math.ethz.ch/∼mcneil.

[12] McNeil, A. and Saladin,T. (1997), "The Peaks over Thresholds Method for Estimating High Quantiles of Loss Distribution", in *Joint Day Proceedings Volume of XXVIIIth International Astin Colloquium/7th International AFIR Colloquium*,. 23-43, available www.math.ethz.ch/∼mcneil.

[13] McNeil, A. and Saladin,T. (2000), "Developing Scenarios for Future Extreme Losses Using the POT Model", in *Extremes and Integrated Risk Management*, edited by P.Embrechts PME, published by RISK books, London, available www.math.ethz.ch/∼mcneil.

[14] Neftci, S.N. (2000), “Value at Risk Calculation, Extreme Events, and Tail Estimation”, Working Paper, City University of New York (CUNY).

[15] Papush, D.E. (1997), “A Simulation Approach in Excess Reinsurance Pricing”, Casualty Actuarial Society, Forum Summer 1997, Reinsurance Call Papers, available http://www.casact.org/pubs/forum/97spforum/97spftoc.htm.

[16] Pickands, J. (1975), ”Statistical Inference Using Extreme Order Statisitcs”, *The Annals of Statistics*, 3, 119-131.

[17] Punter, A. (1997), ”A Critical Evaluation of Progress to Date”, in *Practical Applications of Financial Market Tools to Corporate Risk Management*, International Risk Management, London: Emap Finance Ltd., 7-12.

[18] Resnik, S.I. (1997), “Discussion of the Danish Data on Large Fire Insurance Losses”, *Astin Bulletin*, 27, 139-151, available www.casact.org/library/astin/vol27no1/index.htm.

[19] Këllezi, E. and Gilli, M. (2000), “Extreme Value Theory for Tail-Related Risk Measures”, Working Paper, Department of Economics and FAME, University of Geneva.

[20] Smith, R.L. (1989), “Extreme Value Analysis of Environmental Time Series: An Application to Trend Detection in Ground Level Zone”, *Statistical Science*, 4, 367-393.

[21] Smith, R.L. (1996), “Extreme Value Analysis for Insurance Risk”, *Statistical Science*, 4, 367-393.

[22] Smith, R.L. (2000), “Extreme Value Statistics in Meteorology and the Environment”, in *Environmental statistics,* Chapter 8, Spring 2000*,* draft.

[23] Tilley, J.A. (1997), ”The Securization of Catastrophic Property Risks”, *Joint Day Proceedings Volume of XXVIIIth International Astin Colloquium/7th International AFIR Colloquium*, 27-53.

[24] Venter, G.G. (2001), “Measuring Value in Reinsurance”, Casualty Actuarial Society, Forum Summer 2001, Reinsurance Call Papers, available www.casact.org/pubs/forum/01sforum/01sftoc.htm.

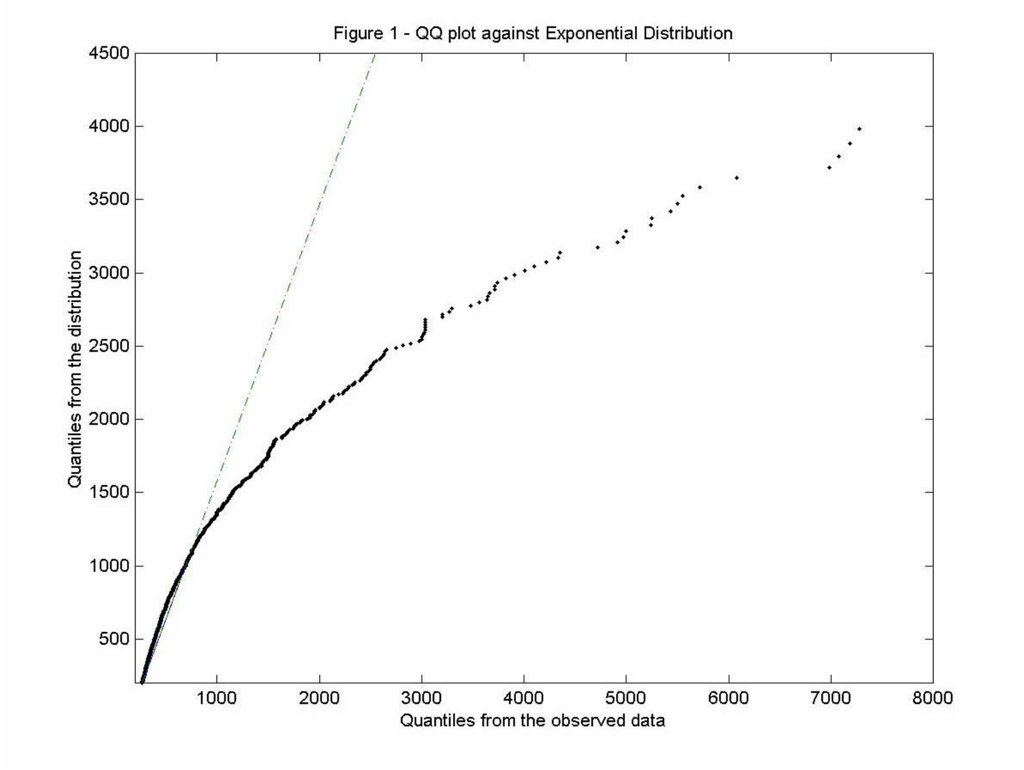


Figure 1 - QQ plot against Exponential Distribution
Quantiles from the distribution
Quantiles from the observed data
4500
4000
3500
3000
2500
2000
1500
1000
500
1000
2000
3000
4000
5000
6000
7000
8000

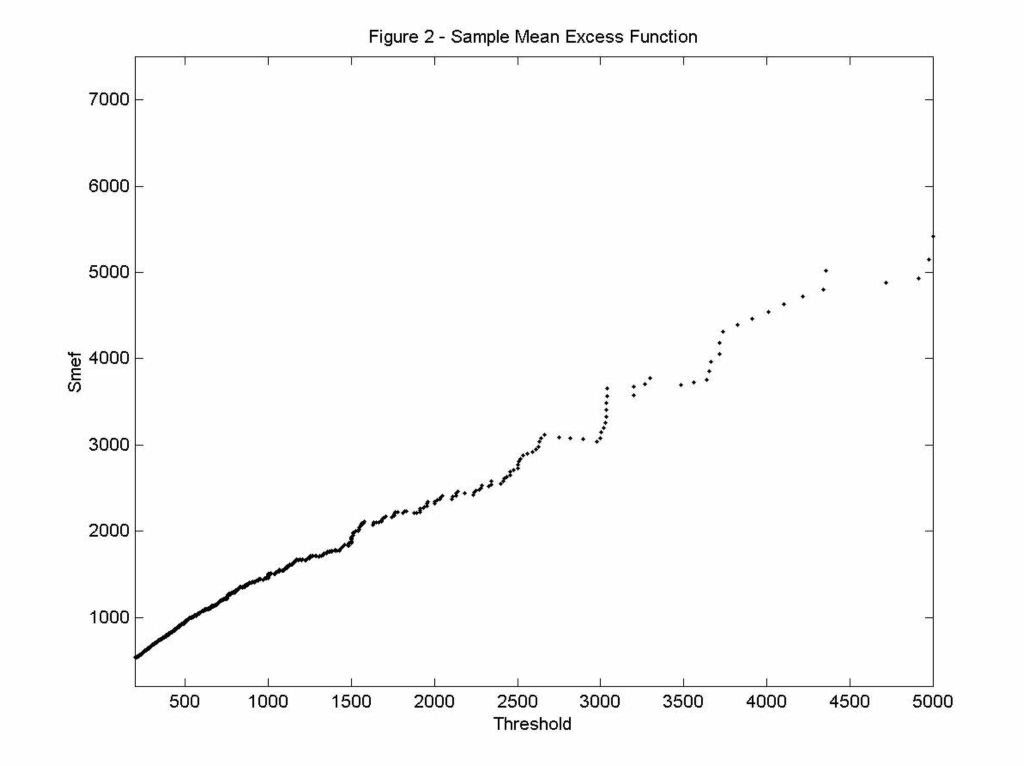


Figure 2 - Sample Mean Excess Function
Smef
Threshold
7000
6000
5000
4000
3000
2000
1000
500
1000
1500
2000
2500
3000
3500
4000
4500
5000

Figure 3 - Hill Plot

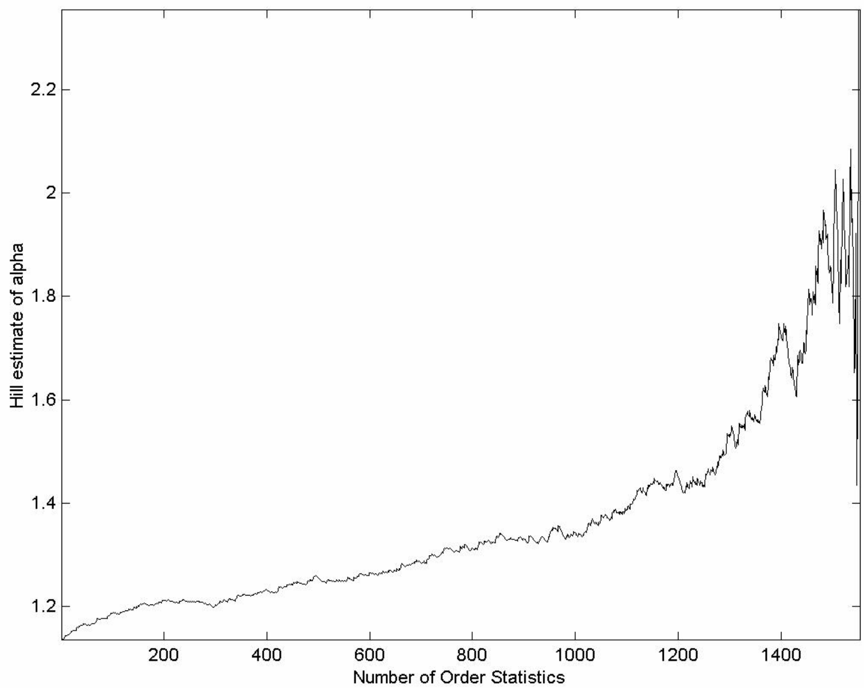
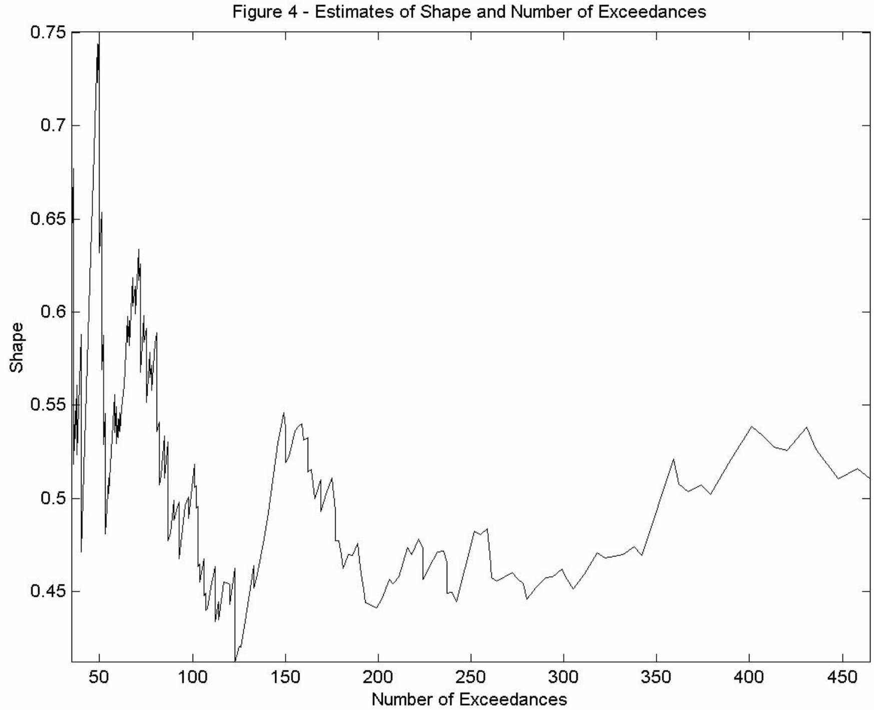


Figure 4 - Estimates of Shape and Number of Exceedances



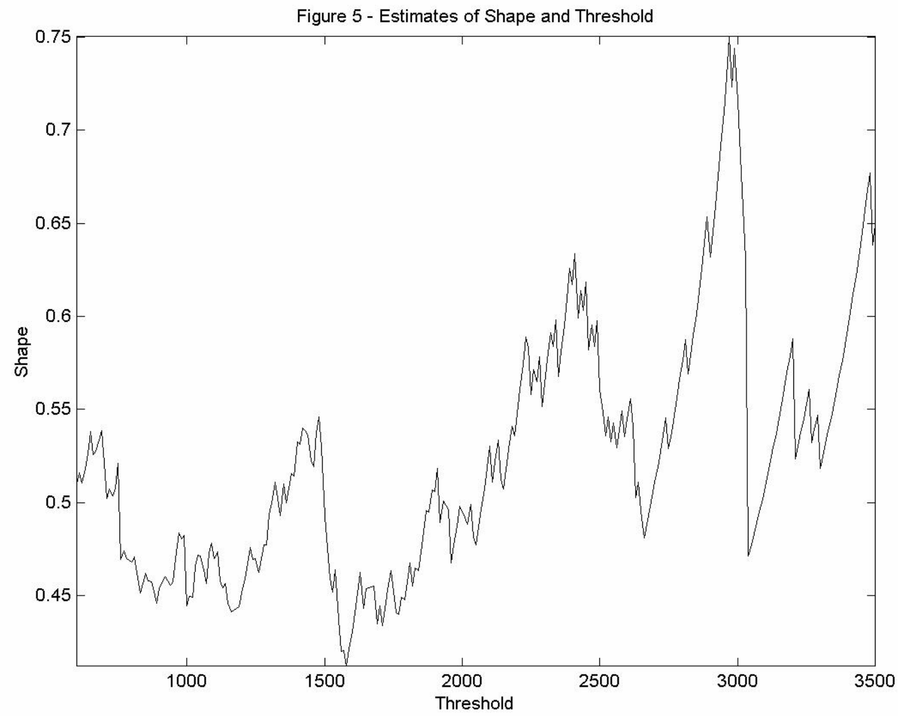


Figure 5 - Estimates of Shape and Threshold
Shape
Threshold
0.75
0.7
0.65
0.6
0.55
0.5
0.45
1000
1500
2000
2500
3000
3500

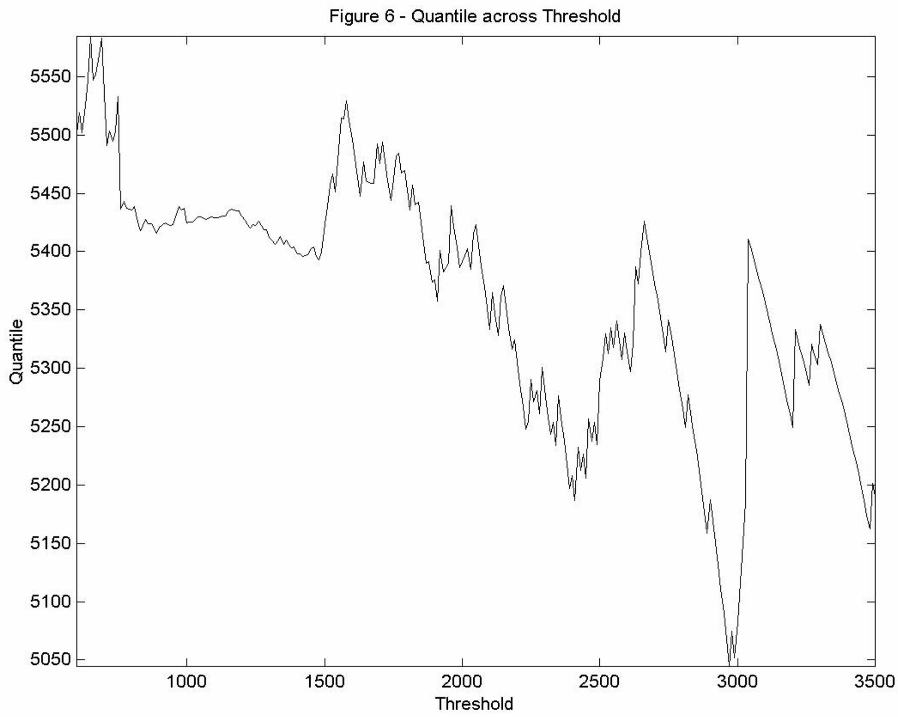


Figure 6 - Quantile across Threshold
Quantile
Threshold
5550
5500
5450
5400
5350
5300
5250
5200
5150
5100
5050
1000
1500
2000
2500
3000
3500

Figure 7 - Expected Shortfall across Threshold

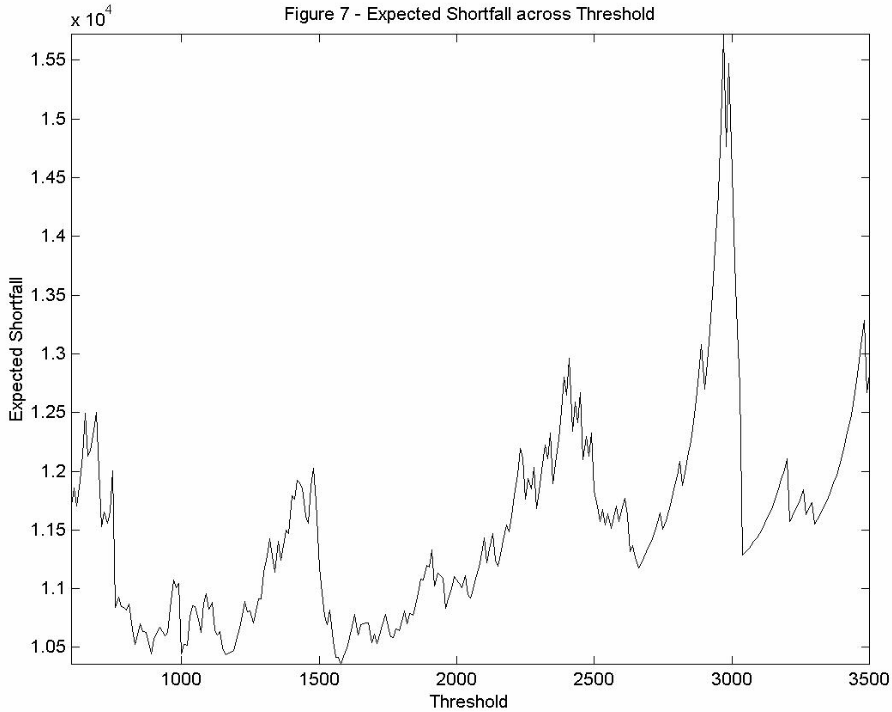


Figure 8 - E[y] across Threshold

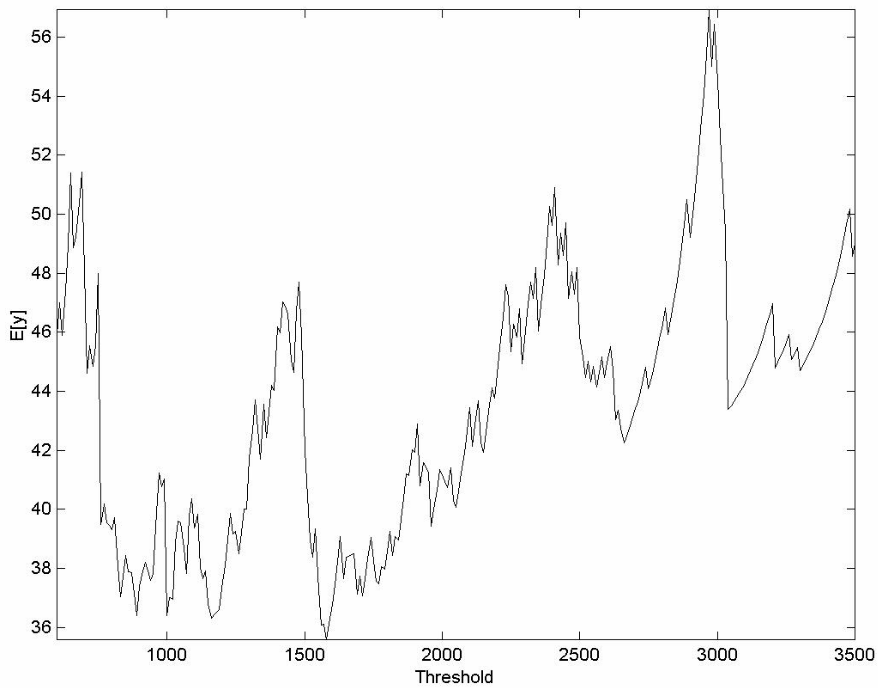


Figure 9 - E[y] across Attachment Point

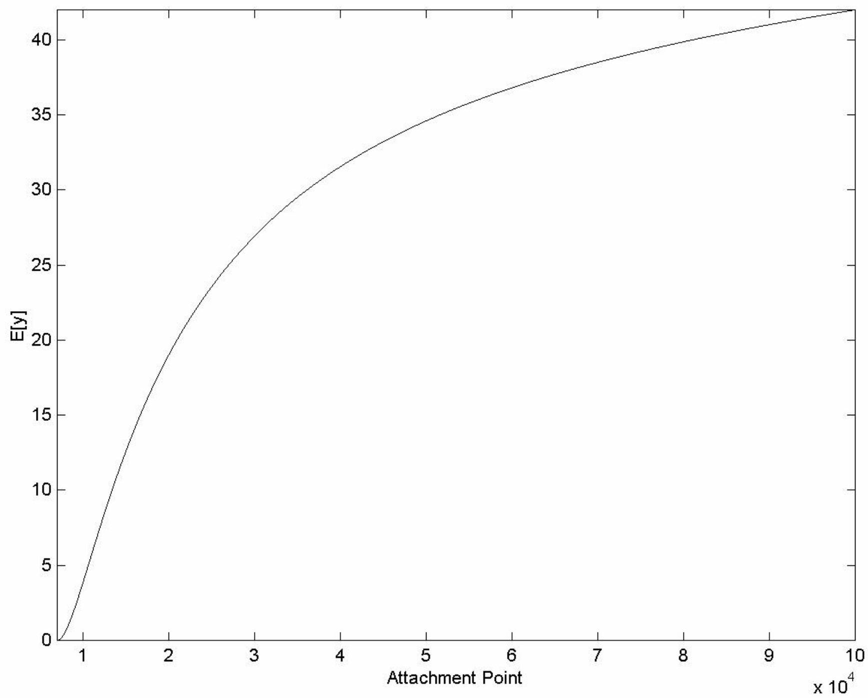
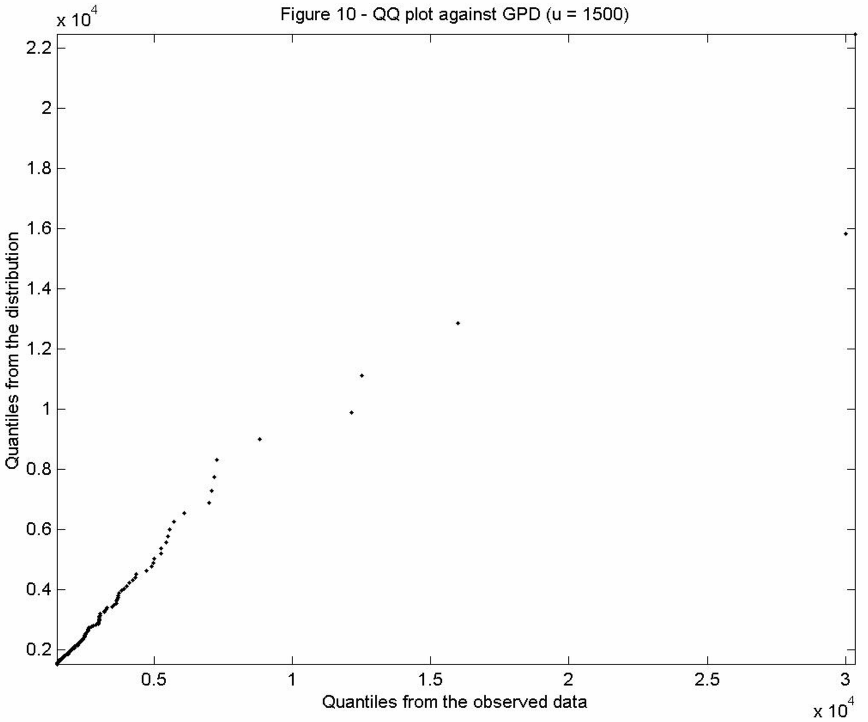


Figure 10 - QQ plot against GPD (u = 1500)



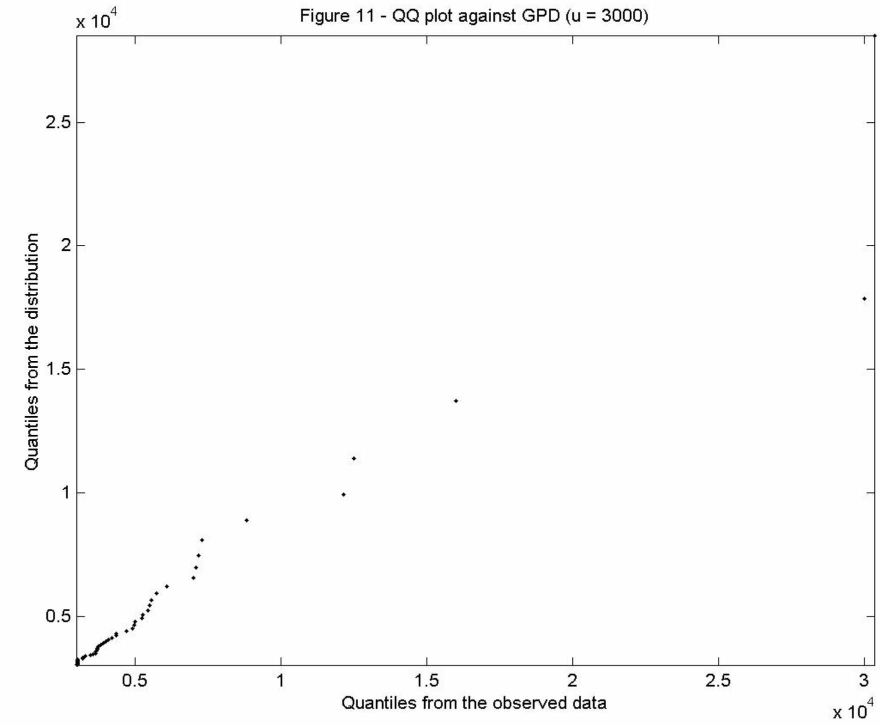


Figure 11 - QQ plot against GPD (u = 3000)
x 10^4
2.5
2
1.5
1
0.5
Quantiles from the distribution
0.5
1
1.5
2
2.5
3
Quantiles from the observed data
x 10^4

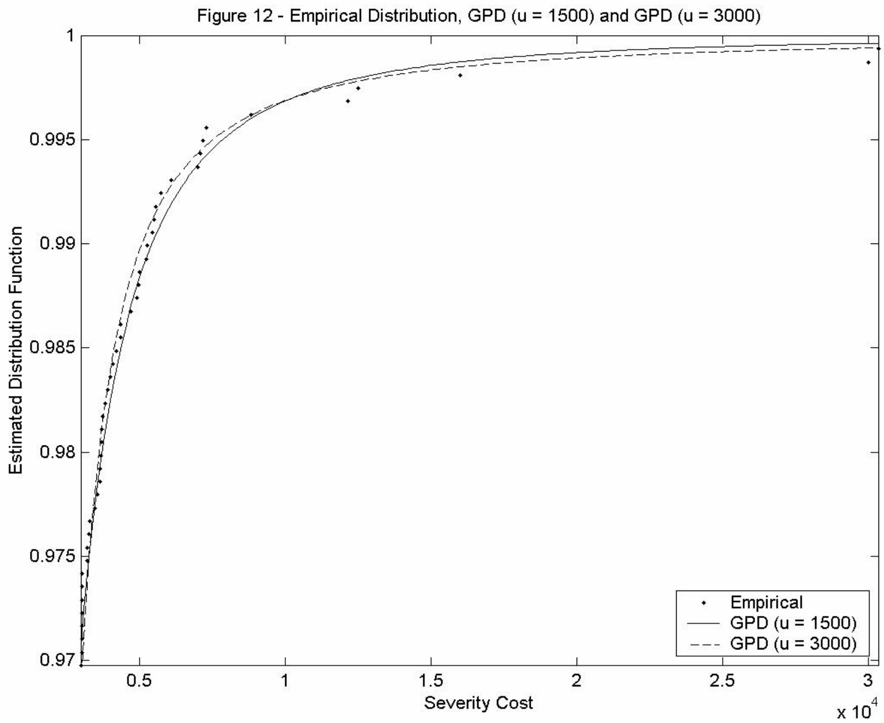


Figure 12 - Empirical Distribution, GPD (u = 1500) and GPD (u = 3000)
1
0.995
0.99
0.985
0.98
0.975
0.97
Estimated Distribution Function
Empirical
GPD (u = 1500)
GPD (u = 3000)
0.5
1
1.5
2
2.5
3
Severity Cost
x 10^4

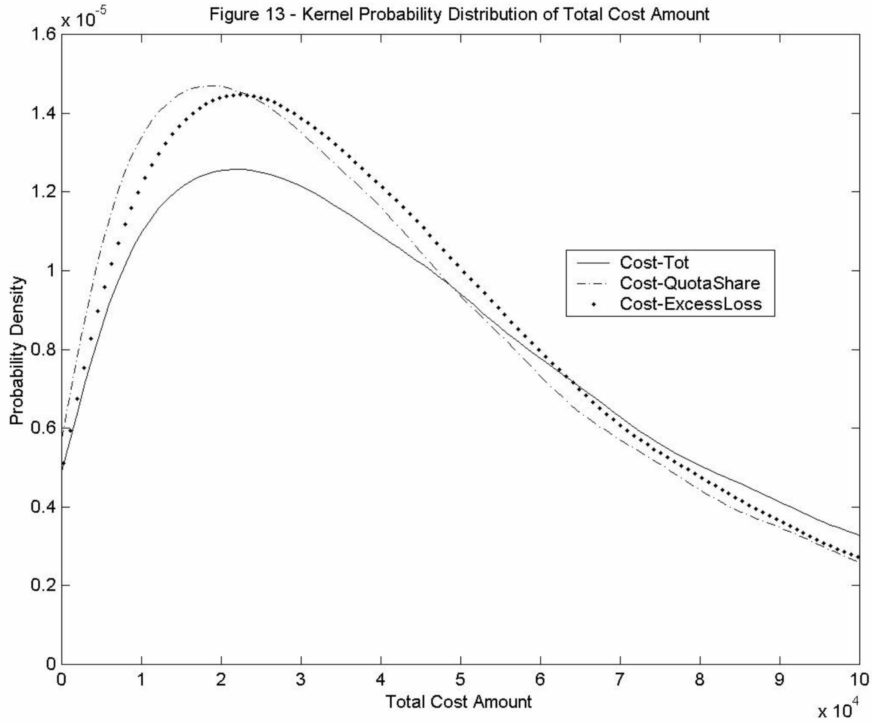
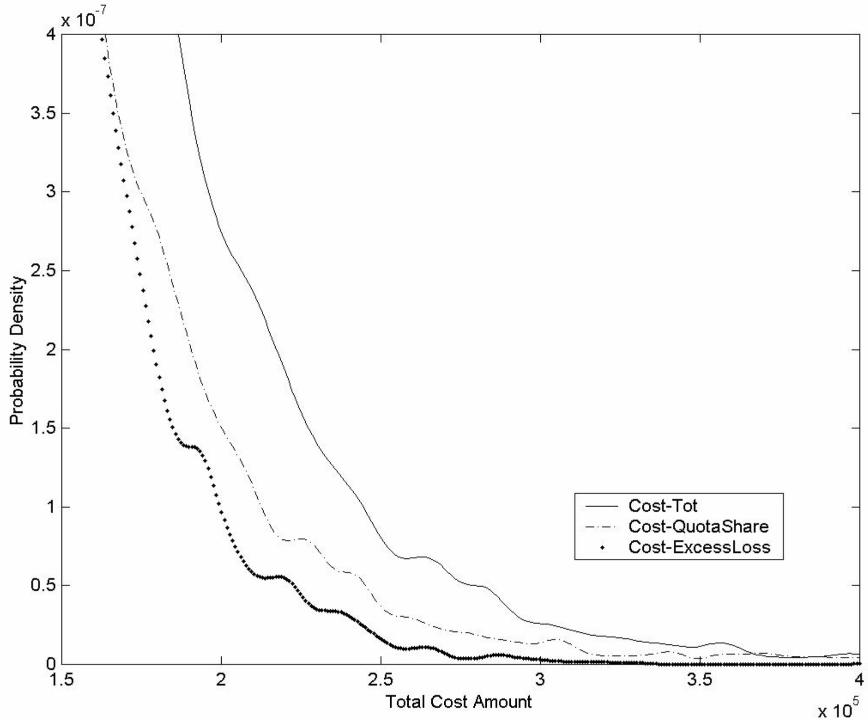


Figure 13 - Kernel Probability Distribution of Total Cost Amount
x 10^-5
Probability Density
Total Cost Amount
x 10^4
Cost-Tot
Cost-QuotaShare
Cost-ExcessLoss


x 10^-7
Probability Density
Total Cost Amount
x 10^5
Cost-Tot
Cost-QuotaShare
Cost-ExcessLoss