CHAPTER 3

Some Generalized Functions for the Size Distribution of Income[†]

James B. McDonald [‡]

Abstract

Many distributions have been used as descriptive models for the size distribution of income. This paper considers two generalized beta distributions which include many of these models as special or limiting cases. These generalized distributions have not been used as models for the distribution of income and provide a unified method of comparing many models previously considered.

Expressions are reported which facilitate parameter estimation and the analysis of associated means, variances, and various measures of inequality.

The distributions considered are fit to US family income and their relative performance is compared.

1 Introduction

Many distributions have been considered as descriptive models for the distribution of income. These include, among others, the lognormal, gamma, beta, Singh-Maddala, Pareto, and Weibull distributions. In many applications, the Singh-Maddala distribution provides a better fit than the gamma which performs much better than the lognormal (McDonald and Ransom (1979); Salem and Mount (1974); Singh and Maddala (1976)). Thurow (1970) adopted the beta distribution as a model for the distribution of income, and this model includes the gamma as a limiting case;

[†] Reprint of McDonald, J.B. (1984), Some Generalized Functions for the Size Distribution of Income, *Econometrica*, 52, 647-663. © The Econometric Society, 1984.

[‡] The author appreciates comments and helpful suggestions from Alan Harrison. Dwight Israelsen, and the referees. The author appreciates the assistance of Jeff Green with the computer programming.

hence, the beta will provide at least as good a fit as the gamma. The Singh-Maddala distribution includes the Weibull and Fisk distributions as special cases.

Recently the generalized gamma has been used by Atoda *et al.* (1980); Esteban (1981); Kloek and van Dijk (1978); Taille (1981). Esteban (1981) demonstrates that the generalized gamma has similar tail behavior or includes the lognormal, Weibull, gamma, exponential, normal, and Pareto distributions as special or limiting cases. However, the beta, Singh-Maddala, and Fisk distributions are not included as members of this class of distributions.

In this paper two generalized beta distributions are considered. One of these includes the Singh-Maddala *and* the generalized gamma as special or limiting cases. This distribution provides a useful extension which facilitates a comparison of alternative models within the framework of a generalized model. The second includes the beta used by Thurow and the generalized gamma as special cases.

Section 2 includes a discussion of the generalized beta distributions and the relationships between these distributions and other widely used models for income distribution. Formulas describing associated population characteristics which are useful in the estimation and analysis of empirical data are also reported. Section 3 illustrates some applications of these results.

2 The Models

The generalized gamma (GG) and generalized beta of the first and second kind (GB 1, GB2) are defined by

$$f(y;a,\beta,p) = \frac{ay^{ap-1}e^{-(y/\beta)^a}}{\beta^{ap}\Gamma(p)}, \quad 0 \leq y, \tag{3.1}$$

$$g(y;a,b,p,q) = \frac{ay^{ap-1}(1-(y/b)^a)^{q-1}}{b^{ap}B(p,q)}, \quad 0 \leq y \leq b, \tag{3.2}$$

$$\begin{aligned} h(y;a,b,p,q) &= \frac{ay^{ap-1}}{b^{ap}B(p,q)(1+(y/b)^a)^{p+q}}, \quad 0 \leq y \\ &= 0 \quad otherwise. \end{aligned} \tag{3.3}$$

These distributions can be shown to include the beta of the first kind (B1) considered by Thurow, the beta of the second kind (B2), the Singh-Maddala (SM), the lognormal (LN), gamma (GA), Weibull (W), Fisk or Sech^2, and exponential (Exp) distributions as special or limiting cases. These relationships are depicted in Figure 3.1. The special cases of the generalized gamma distribution are carefully developed in the paper by Esteban (1981). Esteban characterizes density functions $f(y)$ in terms of an elasticity $-yf'(y)/f(y)$ and demonstrates that

$$\eta_f(y) = -yd(\ln f(y))/dy = 1 - ap + a(y/\beta)^a \tag{3.4}$$

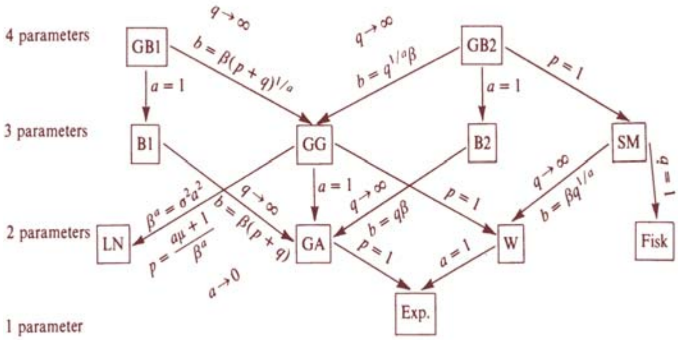

Fig. 3.1: Distribution trees.

uniquely characterizes the generalized gamma in equation (3.1). The corresponding elasticities for (3.2) and (3.3) are given by

$$\eta_g(y) = 1 - ap + \frac{a(q-1)(y/b)^a}{(1-(y/b)^a)} \tag{3.5}$$

and

$$\eta_h(y) = 1 - ap + a(p+q)\frac{(y/b)^a}{1+(y/b)^a}.^4 \tag{3.6}$$

From Figure 3.1, we observe that both of the generalized beta distributions include the generalized gamma as a limiting case:

$$\begin{aligned} f(y;a,\beta,p) &= \lim_{q\to\infty} g(y;\ a,\ \beta(p+q)^{1/a},\ p,\ q) \\ &= \lim_{q\to\infty} h(y;\ a,\ \beta(q)^{1/a},\ p,\ q) \end{aligned}\quad. \tag{3.7}$$

The details associated with these results are included in the Appendix. The *generalized beta of the second kind* is a particularly useful family of distributions and includes the generalized gamma, beta of the second kind, the Singh-Maddala, and all of the previously mentioned associated special cases as members. The distributions of the F statistic (variance ratio) are also a special case of the generalized beta of the second kind.[5]

---

[4] A referee pointed out that the monotonicity of (3.4), (3.5), and (3.6)) implies that associated graphs showing the relationship of the natural logarithm of the density function and $\ln y$ must be concave.

[5] The author has found that some of these distributions are known by different names in other disciplines. For example, Arnold (1980) refers to a GB2 with a non-zero threshold as a Feller-Pareto distribution. The Singh-Maddala function is a member of the Burr family (type 12) and has

Table 3.1: Distributions and Population Characteristics[a]

Model	Distribution Function	Moments	Gini Coefficient
GB1	$\frac{(y/b)^{ap}}{pB(p,q)}\,{}_2F_1\left[\begin{matrix} p,\ 1-q;\ (y/b)^a \\ p+1; \end{matrix}\right]$	$\frac{b^h B(p+q,h/a)}{B(p,h/a)}$	$\frac{B(2p+1/a,q)}{B(p,q)B(p+1/a,q)p(ap+1)}\,{}_4F_3\left[\begin{matrix} 2p+1/a,p,p+1/a,1-q;1 \\ 2p+q+1/a,p+1,p+1/a+1; \end{matrix}\right]$
GG	$\frac{e^{-(y/\beta)^a}(y/\beta)^{ap}}{\Gamma(p+1)}\,{}_1F_1\left[\begin{matrix} 1;\ (y/\beta)^a \\ p+1; \end{matrix}\right]$	$\frac{\beta^h \Gamma(p+h/a)}{\Gamma(p)}$	$\frac{1}{2^{2p+1/a}B(p,p+1/a)}\left\{\left(\frac{1}{p}\right)\,{}_2F_1\left[\begin{matrix} 1,2p+1/a;\ \frac{1}{2} \\ p+1; \end{matrix}\right] - \left(\frac{1}{p+1/a}\right)\,{}_2F_1\left[\begin{matrix} 1,2p+1/a;\frac{1}{2} \\ p+1/a+1; \end{matrix}\right]\right\}$
GB2	$\frac{\left(\frac{(y/b)^a}{1+(y/b)^a}\right)^p}{pB(p,q)}\,{}_2F_1\left[\begin{matrix} p,1-q;\ \frac{(y/b)^a}{1+(y/b)^a} \\ p+1; \end{matrix}\right]$	$\frac{b^h B(p+h/a,q-h/a)}{B(p,q)}$	$\frac{B(2q-1/a,2p+1/a)}{B(p,q)B(p+1/a,q-1/a)}\left\{\left(\frac{1}{p}\right)\,{}_3F_2\left[\begin{matrix} 1,p+q,2p+1/a;\ 1 \\ p+1,2(p+q); \end{matrix}\right] - \left(\frac{1}{p+1/a}\right)\,{}_3F_2\left[\begin{matrix} 1,p+q,2p+1/a;\ 1 \\ p+1/a+1,2(p+q); \end{matrix}\right]\right\}$
B1	$\frac{(y/b)^p}{pB(p,q)}\,{}_2F_1\left[\begin{matrix} p,1-q;\ (y/b) \\ p+1; \end{matrix}\right]$	$\frac{b^h B(p+q,h)}{B(p,h)}$	$\frac{B(p+q,\frac{1}{2})B(p+\frac{1}{2},\frac{1}{2})}{B(q,\frac{1}{2})\pi}$

[a] ${}_pF_q$ and B(,), respectively denote generalised hypergeometric series and beta functions which are defined in the Appendix. $\Lambda(y|\mu,\sigma^2)$ is standard notation for the distribution of a lognormal random variable; cf. Aitchison and Brown (1969). The results for the exponential can be obtained by letting $a=1$ in the row corresponding to the Weibull distribution

Table 3.1: Cont.

Model	Distribution Function	Moments	Gini Coefficient
B2	$\frac{\left(\frac{(y/b)}{1+y/b}\right)^p}{pB(p,q)}\, {}_2F_1\left[\begin{matrix} p, 1-q; & \frac{y/b}{1+y/b} \\ p+1; & \end{matrix}\right]$	$\frac{b^h B(q-h,p+h)}{B(p,q)}$	$\frac{2B(2p,2q-1)}{pB^2(p,q)}$
SM	$1-\frac{1}{(1+(x/b)^a)^q}$	$\frac{b^h B(1+h/a,q-h/a)}{B(1,q)}$	$1-\frac{\Gamma(q)\Gamma(2q-1/a)}{\Gamma(q-1/a)\Gamma(2q)}$
LN	$\Lambda(y\mid\mu,\sigma^2)$	$e^{h\mu+h^2\sigma^2/2}$	$2N\left[\sigma/\sqrt{2};0,1\right]-1$
GA	$\frac{e^{-(y/\beta)}(y/\beta)^p}{\Gamma(p+1)}\, {}_1F_1\left[\begin{matrix} 1; y/\beta \\ p+1; \end{matrix}\right]$	$\frac{\beta^h\Gamma(p+h)}{\Gamma(p)}$	$\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p+1)\sqrt{\pi}}$
W	$1-e^{-x^a/\beta^a}$	$\beta^h\Gamma(1+h/a)$	$1-(1/2^{1/a})$
F	$1-\frac{1}{(1+(y/b)^a)}$	$b^h\Gamma(1+h/a)\Gamma(1-h/a)$	$1/a$

Given an arbitrary estimation criterion, the higher a distribution is on a branch in Figure 3.1, the better it will perform as measured by the same criterion, e.g., least squares estimators of the generalized beta distribution of the first kind will have a sum of squared errors at least as small as the corresponding sum of squared errors for the lognormal distribution. However, no such conclusions can be drawn about the relative performance of distributions on different branches, e.g., the generalized gamma and Singh-Maddala distributions. If estimation is based upon one criterion such as maximum likelihood, distributions higher on a branch in Figure 3.1 will not necessarily perform better when compared according to another criterion such as sum of squared errors.

Expressions for the moments, distribution functions, and several measures of inequality corresponding to (3.1), (3.2), and (3.3) can be expressed in terms of

$$I(x,h) = \int_0^x y^h f(y)dy, \tag{3.8}$$

$$I^*(i,j) = \int_0^\infty x^i f(x) \int_0^x y^j f(y)dydx. \tag{3.9}$$

The associated moments, conditional on their existence, and the distribution function are defined by

$$E(y^h) = \lim_{x\to\infty} I(x,\ h), \tag{3.10}$$

$$F(y) = I(y,0), \tag{3.11}$$

respectively. The relative mean deviation of Pietra (P) and Gini (G) measures [6] of inequality can be expressed in terms of (3.8) and (3.9) by

$$P = E(|y-\mu|)/2\mu = I(\mu,0) - I(\mu,1)/\mu \tag{3.12}$$

and

$$G = E(|y-x|)/2\mu = (1/\mu)(I^*(1,0) - I^*(0,1)) \tag{3.13}$$

where $\mu = E(y)$. Expressions for $I(x,h)$ and $I^*(i,j)$ corresponding to (3.1), (3.2), and (3.3) are derived in the Appendix. In each case, these expressions are functions of the parameters defining the distribution function under consideration (a,b,β,p,q). Table 3.1 includes expressions for the distribution function, moments, and Gini coefficients for each of the distributions discussed. These expressions are functions of the parameters in the respective models and are useful in estimation and analysis of

also been referred to as a Beta-P distribution (Cronin (1979); Johnson and Kotz (1970)). Another special case of the generalized beta of the second kind encountered in other areas of application corresponds to q = 1 and is known as a three-parameter kappa distribution, Beta-*k* or Burr distribution of the third type (Tadikamalla, 1980). If p and a are both equal to one, then the corresponding distribution has been referred to as the Lomax distribution. The power and uniform distributions, among others, are special cases of the generalized beta of the first kind and B1.

[6] Gastwirth (1972) discusses some issues associated with nonparametric estimation of the Pietra index or relative mean deviation and Gini coefficient and their interpretation and historical background. McDonald and Ransom (1981) discuss some related inferential issues. Also see Kendall and Stuart (1961).

population characteristics. The expressions for the distribution function facilitate parameter estimation based upon data in a grouped format. Given parameter estimates, these results can then be used to estimate corresponding population characteristics of interest as well as providing indirect checks on the validity of the parameter estimates. This point will be illustrated by means of an example in the next section. The estimated population characteristics depend upon the estimation technique and the assumed distribution function. This point is covered in more detail in McDonald and Ransom (1979).

3 Applications

Estimation of the models discussed in Section 2 involves nonlinear techniques. Numerous estimation problems can arise in nonlinear estimation which will yield questionable results. An indirect check of the validity of the parameter estimates obtained from a nonlinear optimization routine is provided by comparing estimated population characteristics such as the mean with independently obtained results where available. The expressions in Table 3.1 facilitate such a comparison, and Section 3.1 provides an example of this. The relative performance of alternative models for the distribution of income is compared in Section 3.2.

3.1 Analysis of Parameter Estimates

Thurow's (1970) widely cited paper provides an example of estimation problems and an application of expressions in Table 3.1 in the detection of questionable results. The underlying distribution of income is assumed to be modeled by a beta density function.

$$g(y;\, a=1,\, b,\, p,\, q) = \frac{y^{p-1}(b-y)^{q-1}}{B(p,q)b^{p+q}}, \quad 0<y<b, \quad p,\, q>0, \tag{3.14}$$

which corresponds to (3.2) with $a = 1$. Thurow assumed that the maximum income (b) was equal to \$15,000 and obtained separate estimates of p and q for the distribution of income (1959 dollars) of families and unrelated individuals for whites and nonwhites for the period 1949-1966. Income characteristics associated with the estimated parameter values for (p,q,b) are inferred and their relationship with hypothesized explanatory variables considered. Thurow's results raise questions as to whether economic growth is associated with a more egalitarian distribution as well as suggesting that inflation may lead to a more equal distribution of income for whites. The accuracy of the estimated (p,q)'s is a critical element in the validity of the analysis of the estimated relationship between the hypothesized explanatory variables and the distribution of income. Thurow's estimates of $(p,\, q)$ were not reported in his paper, but were provided on request and are given in Table 3.2[7] The mean and Gini coefficient associated with the beta function (B1) in Table 3.1 are given by

$$E(y) = \frac{bp}{p+q}, \tag{3.15}$$

$$G = \frac{\Gamma(p+q)\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})}{\Gamma(p+q+\frac{1}{2})\Gamma(p+1)\Gamma(q)\Gamma(\frac{1}{2})}. \tag{3.16}$$

The mean income level and Gini coefficients implied by Thurow's estimates can be readily obtained from equations (3.15) and (3.16) and are reported in Table 3.2.

The corresponding estimates reported in census publications are also given in this table to provide a useful comparison.

An analysis of the entries in Table 3.2 suggests that the distribution of income for whites is more egalitarian with a higher mean than for nonwhites. This qualitative result is consistent with Thurow's estimates as well as those reported in the current population reports; however, other implications of these two sets of estimates are not. For example, all of the associated estimated density functions are either "$\cup$" shaped $(p < 1, q < 1)$ or "$\imath$" shaped $(p < 1, q > 1)$ rather than "$\cap$" shaped. The agreement between the implied and census estimates of the mean is much closer than for the Gini coefficients. The magnitude and inter-temporal behavior (reductions in excess of 30 per cent) of the associated Gini coefficients implied by the estimated parameters (p, q) for the period under consideration are inconsistent with the census estimates and provide additional evidence of an estimation problem.

Thus there is relatively close agreement between the two estimates of mean income, but very poor agreement between the measures of inequality. The estimation procedure appears to have roughly preserved the mean characteristic, but implicitly modeled intra and/or inter-group variation incorrectly. The results could also have been partially due to the conjunction of the nature of the income groups and treatment of the maximum income.

The next section includes examples in which estimated distributions of the form considered by Thurow provide relatively accurate estimates of population characteristics. The parameters p, q, and b are estimated. The corresponding densities are "$\cap$" shaped and the associated estimates of the mean and Gini coefficient are very close to those reported by the census publications. The large values of b and q merely correspond to the estimated beta density (B1) being closely approximated by its limiting form, the gamma.

---

[7] The author appreciates Professor Thurow's assistance in providing these estimates and suggestions, but has been unable to replicate the results.

Table 3.2: Thurow's estimates of p, q*

	Whites						Nonwhites					
Year	p	q	mean**		Gini**		p	q	mean**		Gini**	
1949	.258	.666	4.188	(4.052)	.615	(.404)	.160	.930	2.202	(2.044)	.752	(.443)
1950	.279	.687	4.332	(4.294)	.600	(.407)	.172	.930	2.341	(2.245)	.738	(.438)
1951	.269	.625	4.513	(4.441)	.596	(.387)	.182	.908	2.505	(2.334)	.725	(.433)
1952	.298	.649	4.720	(4.579)	.576	(.398)	.205	.921	2.731	(2.518)	.702	(.407)
1953	.327	.667	4.935	(4.798)	.556	(.395)	.228	.920	2.979	(2.638)	.679	(.428)
1954	.334	.697	4.859	(4.741)	.557	(.401)	.217	.929	2.84	(2.579)	.691	(.456)
1955	.368	.718	5.083	(5.034)	.536	(.397)	.225	.913	2.966	(2.679)	.681	(.431)
1956	.411	.731	5.398	(5.311)	.511	(.391)	.249	.978	3.044	(2.875)	.666	(.427)
1957	.406	.728	5.370	(5.229)	.513	(.385)	.269	1.025	3.118	(2.860)	.652	(.435)
1958	.411	.750	5.310	(5.291)	.514	(.388)	.276	1.075	3.064	(2.893)	.651	(.448)
1959	.460	.765	5.633	(5.571)	.488	(.396)	.286	1.051	3.209	(2.977)	.641	(.452)
1960	.504	.815	5.732	(5.646)	.473	(.398)	.330	1.061	3.559	(3.276)	.608	(.459)
1961	.622	.979	5.828	(5.817)	.443	(.408)	.346	1.173	3.417	(3.268)	.607	(.462)
1962	.663	.971	6.086	(5.987)	.426	(.395)	.338	1.107	3.509	(3.278)	.607	(.443)
1963	.712	.985	6.294	(6.167)	.411	(.396)	.356	1.073	3.737	(3.513)	.591	(.440)
1964	.785	1.017	6.534	(6.333)	.391	(.400)	.406	1.095	4.057	(3.788)	.562	(.444)
1965	.842	1.029	6.750	(6.552)	.376	(.393)	.452	1.124	4.302	(3.859)	.538	(.427)
1966	.955	1.044	6.166	(6.912)	.348	(.390)	.514	1.104	4.756	(4.192)	.504	(.426)

* Thurow did not estimate the parameter b, but rather assumed it to be 15 ($ 15,000) and included any higher Incomes in the group with an upper bound of $15,000. The mean and Gini coefficients were evaluated using equations (15) and (16).

** The numbers in parentheses are the corresponding census estimates reported in current population reports (P60). The nominal figures for mean Income were adjusted by the CPI to obtain the figures in I959 dollars.

3.2 Estimation and Comparison of Alternative Distributions

The generalized gamma and generalized beta of the first and second kinds and special cases previously discussed were fit to US family nominal income for 1970-1980. The data were in a grouped format and the corresponding multinomial likelihood function is given by

$$L(\Theta) = N! \prod_{i=1}^{g} \frac{(P_i(\Theta))^{n_i}}{n_i!}$$

where $P_i(\Theta) = \int_{I_i} f(y;\ \Theta)dy$ denotes the predicted fraction of the population in the ith of g income groups defined by $I_i = [y_{i-1}, y_i)$. The $(n_i/N)$are the corresponding observed relative frequencies $(N = \sum n_i)$. The estimators obtained by maximizing the multinomial likelihood function will be asymptotically efficient relative to other estimators based on grouped data; however, they may be less efficient than maximum likelihood estimators based on individual observations (cf. Cox and Hinckley (1974)).[8] The results of this estimation for 1970, 1975, and 1980 are reported in Tables 3.3, 3.4, and 3.5. The reported values for the mean and Gini coefficients were obtained by substituting the estimated parameters into the relevant expressions given in Table 3.1.

The generalized beta of the second kind provides a better fit than the generalized gamma of the first kind based upon a sum of squared or absolute errors criterion (SSE, SAE), chi-square (χ^2), or log-likelihood criterion[9]. The values of the log-likelihood functions are consistent with the logical relationships between the

[8] The program GQOPT obtained from Richard Quandt was used to maximize the multinomial log-likelihood function. A convergence criterion of 10^{-8} was specified. The data used are given in the following table, and were taken from the Census Population Reports.

Endpoint (in thousands)	1970	1975	1980
2.5	6.6	3.5	2.1
5.0	12.5	8.5	4.1
7.5	15.2	10.6	6.2
10	16.6	10.6	6.5
12.5	15.8	11.4	7.3
15	11.0	10.9	6.9
20	13.1	18.8	14.0
25	4.6	11.6	13.7
35	3.0	9.5	19.8
50	1.1	3.2	12.8
∞	0.5	1.4	6.7

For cases in which the percentages do not add to 100, the percentages used in estimation were obtained by transforming the reported figures by multiplying them by (100/sum of percentages).

[9] The SSE, SAE, and χ^2 values are obtained by evaluating $\sum_{i=1}^{11}\left(\frac{n_i}{N} - p_i\left(\widehat{\Theta}\right)\right)^2$, $\sum_{i=1}^{11}\left|\frac{n_i}{N} - P_i\left(\widehat{\Theta}\right)\right|$ and $N\sum_{i=1}^{11}\left(\frac{n_i}{N} - P_i\left(\widehat{\Theta}\right)\right)^2 / P_i\left(\widehat{\Theta}\right)$, respectively.

Table 3.3: Estimated Distribution Functions 1970 Family Income

	GB1	GB2	B1	GG	B2	SM	LN	G	W	F	E
a	.8954	5.0573	1.0000	0.8944	1.0000	1.9652	$\mu = 2.1924$	1.0000	1.5603	2.5123	1.0000
$b(\beta)$	649292.	13.5815	389112	(3.4106)	95.0194	18.7288	$\sigma = 0.6977$	(4.8274)	(12.3587)	9.3067	(11.1339)
p	2.8164	0.2961	2.3026	2.8228	2.5556	1.0000	–	2.3026	1.0000	1.0000	1.0000
q	53205.0	0.6708	80621.0	–	22.8234	2.9388	–	–	–	1.0000	–
Mean*	11.119	11.168	11.113	11.122	11.127	11.121	11.425	11.115	11.108	12.261	11.134
Gini**	.354	.337	.352	.354	.355	.350	.379	.352	.359	.398	.500
SSE	.0025	0.0001	0.0022	0.0025	0.0026	0.0014	0.0080	0.0022	0.0027	0.0051	0.0350
SAE	.1399	0.0307	0.1308	0.1399	0.1434	0.0998	0.2429	0.1307	0.1410	0.2043	0.4894
χ^2	1686.9	48.4	1997.6	1683.2	1326.0	644.1	3455.5	1994.9	7621.1	2619.9	11430.4
-ln L	710.6	79.9	722.6	710.6	654.7	363.5	1692.3	722.6	1180.6	1321.9	6051.5

* Census estimate : 11.106

** Census estimate: .354.

Table 3.4: Estimated Distribution Functions 1975 Family Income

	GB1	GB2	B1	GG	B2	SM	LN	G	W	F	E
a	1.0643	3.4977	1.0000	1.0643	1.0000	1.8648	μ=2.5157	1.0000	1.5923	2.4290	1.0000
$b(\beta)$	631397.	20.9572	393308	(8.0305)	984.238	31.5176	σ=0.7248	(6.8115)	(17.2351)	12.9153	(15.6136)
p	2.0376	0.4433	2.2726	2.0378	2.3038	1.0000	–	2.2729	1.0000	1.0000	1.0000
q	162350.	1.1372	57742.4	–	147.444	3.7657	–	–	–	–	1.0000
Mean*	15.472	15.593	15.479	15.475	15.484	15.506	16.094	15.482	15.460	17.368	15.614
Gini**	.353	.352	.354	.353	.355	.353	.392	.354	.353	.412	.500
SSE	.0012	0.0003	0.0013	0.0012	0.0014	0.0008	0.0063	0.0013	0.0014	0.0046	0.0281
SAE	.0912	0.0502	0.0963	0.0913	0.0982	0.0733	0.2226	0.0962	0.0964	0.1994	0.4322
χ^2	553.6	122.6	548.1	553.3	540.6	280.3	3108.7	547.8	1213.2	2200.9	9287.3
-ln L	319.9	117.6	323.3	319.9	322.3	196.2	1453.6	323.3	529.2	1169.4	5191.1

* Census estimate: 15.546

** Census estimate: .358.

Table 3.5: Estimated Distribution Functions 1980 Family Income

	GB1	GB2	B1	GG	B2	SM	LN	G	W	F	E
a	1.4008	2.5373	1.0000	1.4008	1.0000	1.6971	μ=2.9372	1.0000	1.6057	2.2768	1.0000
$b(\beta)$	273102.	40.7667	163.757	(21.8145)	3535660.	87.6981	σ=0.7797	(11.0473)	(26.3368)	19.7450	(24.5954)
p	1.2454	0.6117	1.9173	1.2454	2.1555	1.0000	–	2.1557	1.0000	1.0000	1.0000
q	549517.	2.1329	11.3828	–	320081.	8.3679	–	–	–	1.0000	–
Mean*	23.644	23.931	23.604	23.646	23.810	23.730	25.564	23.815	23.065	27.749	24.596
Gini**	.353	.359	.353	.353	.363	.355	.419	.363	.351	.439	.500
SSE	.0004	0.0002	0.0006	0.0004	0.0008	0.0003	0.0070	0.0008	0.0005	0.0053	0.0234
SAE	.0545	0.0385	0.0609	0.0545	0.0775	0.0495	0.2326	0.0775	0.0561	0.2038	0.4186
χ^2	143.0	84.3	188.4	143.0	353.5	114.2	4620.7	353.6	183.3	2438.9	9107.8
-ln$L$	129.1	99.1	151.6	129.1	228.3	114.5	1859.4	228.3	150.0	1219.5	5108.2

* Census estimate: 23.974

** Census estimate: .365.

distribution functions, i.e., the higher a distribution is on a branch in Figure 3.1, the larger is the associated likelihood function. The other measures of goodness of fit reported need not provide the same rankings.

It is interesting to note that the Singh-Maddala distribution function provides a better fit to the data than any of the distribution functions except for the generalized beta of the second kind regardless of the criterion used for comparison. It even performs better than the four-parameter generalized beta of the first kind for the data set considered. This result is of particular interest due to the relative simplicity of the distribution function. Atoda *et al.* (1980) found that the estimated Singh-Maddala distribution generally outperformed the corresponding B1 or GG for the distribution of Japanese income considered. They used a nonlinear least squares estimation technique and adopted chi-square and SSE criteria for purposes of comparing the distributions. For the US data considered here, the generalized gamma was generally the second best of the three-parameter models regardless of the criterion used for 1975 and 1980.

The lognormal is, with few exceptions, worse than any of the other two-parameter models. The lognormal overstates income in the upper tail (last group) and has a larger estimated mean income and suggests greater dispersion than for the other models with two or more parameters.

The likelihood ratio test provides the basis for comparing nested models. The asymptotic distribution of $2\left[\ln L(\Theta_{ML}) - \ln L(\Theta_R)\right]$ is chi-square with degrees of freedom equal to the number of independent restrictions imposed on the more general model in order to yield the nested model. Θ_{ML}, and Θ_R, respectively, denote maximum likelihood estimators of the general and restricted model. The literature dealing with nonnested hypotheses provides an approach for comparing distributions on different branches.

The differences between GB1 and GG are not statistically significant for any of the three years and have almost identical characteristics. Similarly, GB1, GG, B1, and GA have almost identical characteristics for 1975. Other differences between the nested models appeared to be statistically significant using either a likelihood ratio or Wald test.

The chi-square statistic provides a test of "goodness of fit" and has an asymptotic distribution which is chi-square with degrees of freedom equal to one less than the difference between the number of income groups and number of parameters. There is considerable variation in the value of chi-square across distributions, but all must be rejected at conventional levels of significance. This result is common in applications involving large sample sizes (Kloek and van Dijk, 1978; McDonald and Ransom, 1979), and suggests that it might be productive to consider the impact of sample size upon the power of such tests.

In summary, the generalized beta of the second kind provided the best relative fit and included many other distributions as special or limiting cases. The differences were statistically significant. The Singh-Maddala (or Burr) distribution provided a better fit than the generalized beta of the first kind (four parameters) and all of the two- and three-parameter models considered. The Singh-Maddala distribution function has a closed form which greatly facilitates estimation and analysis of results.

Appendix

Derivation of $I(x,h)$ *and* $I^*(i,j)$

The incomplete moments $I(x,h)$ for the generalized gamma can be obtained by substituting (3.1) into (3.8) to obtain

$$I(x,y) = \int_0^x \frac{ay^{ap+h-1}}{b^{ap}\Gamma(p)} e^{-(y/b)^a} dy. \tag{3.17}$$

Equation (3.17) can be evaluated by making the change of variable $s = (y/b)^a$; hence,

$$\begin{aligned} I(x,h) &= b^h \int_0^{(x/b)^a} \frac{s^{p+h/a-1}}{\Gamma(p)} e^{-s} ds \\ &= \frac{b^h e^{-(x/b)^a} (x/b)^{ap+h}}{\Gamma(p)(p+\frac{h}{a})} {}_1F_1\left[\begin{matrix} 1;\ (x/b)^a \\ p+\frac{h}{a}+1; \end{matrix}\right] \end{aligned} \tag{3.18}$$

(cf. McDonald and Jensen (1979); Rainville (1960)).

The

$${}_1F_1\left[\begin{matrix} a; \\ b;y \end{matrix}\right]$$

is the confluent hypergeometric series and is a special case of the generalized hypergeometric series defined by

$${}_pF_q\left[\begin{matrix} a_1,\ldots,a_p;\ x \\ b_1,\ldots,b_q; \end{matrix}\right] = \sum_{i=0}^{\infty} \frac{(a_1)_i \ldots (a_p)_i}{(b_1)_i \ldots (b_q)_i} \frac{x^i}{i!}$$

where $(a)_i = (a)(a+1)\ldots(a+i-1)$. (A.2) can be used to evaluate $I^*(i,j)$.

$$\begin{aligned} I^*(i,j) &= \int_0^\infty x^i f(x) \int_0^x y^j f(y) dy dx = \int_0^\infty x^i f(x) I(x,j) dx \\ &= \frac{b^j}{\Gamma(p)(p+j/a)} \int_0^\infty x^i \frac{ax^{ap-1} e^{-(x/b)^a}}{b^{ap}\Gamma(p)} \frac{e^{-(x/b)^a} x^{ap+j}}{b^{ap+j}} {}_1F_1\left[\begin{matrix} 1;\ (x/b)^a \\ p+\frac{j}{a}+1; \end{matrix}\right] dx \qquad (3.19) \\ &= \frac{a}{\Gamma^2(p)(p+j/a)b^{2ap}} \int_0^\infty x^{2ap+i+j-1} e^{-2(x/b)^a} {}_1F_1\left[\begin{matrix} 1;\ (x/b)^a \\ p+j/a+1; \end{matrix}\right] dx. \end{aligned}$$

Making the change of variable $s = (x/b)^a$,

$$I^*(i,j) = \frac{b^{i+j}}{\Gamma^2(p)(p+j/a)} \int_0^\infty s^{2p+(i+j)/a-1} e^{-2s} {}_1F_1 \left[\begin{matrix} 1;\ s \\ p+\frac{j}{a}+1; \end{matrix} \right] ds$$

$$= \frac{b^{i+j}\Gamma(2p+\frac{i+j}{a})}{\Gamma^2(p)(p+j/a)2^{2p+(i+j)/a}} {}_2F_1 \left[\begin{matrix} 1, 2p+\frac{i+j}{a};\ \frac{1}{2} \\ p+\frac{j}{a}+1; \end{matrix} \right]. \tag{3.20}$$

See Gradshteyn and Rhyzik (1965, p. 851, 7.5229).

The derivation for the generalized beta for the first kind is similar to that for the generalized gamma except that the evaluation of $I(x,h)$ makes use of the incomplete beta function (Rainville, 1960) and the evaluation of $I^*(i,j)$ makes use of an integral reported in Gradshteyn and Rhyzik (1965, p. 850). The corresponding results can be shown to be

$$I(x,h) = \frac{b^h(x/b)^{ap+h}}{B(p,q)\left(p+\frac{h}{a}\right)} {}_2F_1 \left[\begin{matrix} p+\frac{h}{a}, 1-q;\ \left(\frac{x}{b}\right)^a \\ p+\frac{h}{a}+1; \end{matrix} \right],$$

$$I^*(i,j) = \frac{b^{i+j}B\left(2p+\frac{i+j}{a}, q\right)}{B^2(p,q)\left(p+\frac{j}{a}\right)} {}_3F_2 \left[\begin{matrix} 2p+\frac{i+j}{a}, p+\frac{j}{a}, 1-q;\ 1 \\ 2p+q+\frac{i+j}{a}, p+\frac{j}{a}+1; \end{matrix} \right], \tag{3.21}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$.

The derivation of $I\ (x,h)$ for the generalized beta of the second kind also makes use of the incomplete beta function. The evaluation of the corresponding $I^*(i,j)$ involves the integral reported in Gradshteyn and Rhyzik (1965, p. 849, #5). $I(x,h)$ and $I^*(i,j)$ are given by

$$I(x,h) = \frac{b^h z^{p+h/a}}{B(p,q)\left(p+\frac{h}{a}\right)} {}_2F_1 \left[\begin{matrix} p+\frac{h}{a}, 1+\frac{h}{a}-q;\ z \\ p+\frac{h}{a}+1; \end{matrix} \right] \tag{3.22}$$

where

$$z = \frac{(x/b)^a}{1+(x/b)^a}$$

and

$$I^*(i,j) = \frac{b^{i+j}B\left(2p+\frac{i+j}{a}, q-\frac{i}{a}\right)}{B^2(p,q)\left(p+\frac{j}{a}\right)} {}_3F_2 \left[\begin{matrix} p+\frac{j}{a}, 1+\frac{j}{a}-q, 2p+\frac{i+j}{a};\ 1 \\ p+\frac{j}{a}+1, 2p+q+\frac{j}{a}; \end{matrix} \right].$$

Expressions for the moments and distributions can be easily obtained from equations (3.10) and (3.11) using the expressions for $I(x,h)$ in (3.18), (3.21), and (3.22).

The expressions for $I^*(i,j)$ can be substituted into (3.13) to yield formulas for the Gini measure of inequality. In some instances these equations have been transformed into simpler representations reported in Table 3.1.

Limiting behavior of the generalized beta of the first kind

The generalized beta density of the first kind is given by

$$g(y) = \frac{ay^{ap-1}(1-(y/b)^a)^{q-1}}{b^{ap}B(p,q)}. \tag{3.23}$$

This density function approaches the generalized gamma density as $q \to \infty$ where the scale factor changes with q as

$$b = \beta(p+q)^{1/a}. \tag{3.24}$$

Making this substitution into (3.23) for b yields

$$g(y) = \left(\frac{ay^{ap-1}}{\Gamma(p)\beta^{ap}}\right)\left(\frac{\Gamma(p+q)}{\Gamma(q)(p+q)^p}\right)\left(1-\frac{y^a}{\beta^a(p+q)}\right)^{q-1}. \tag{3.25}$$

For large values of q, the gamma function can be approximated by Stirling's formula,

$$\Gamma(x) \doteq e^{-x}x^{x-1/2}\sqrt{2\pi}. \tag{3.26}$$

See Kendall and Stuart (1961, v. 1, p. 811).

The second bracketed expression in (3.25) can be shown to approach 1 by making the substitution (3.26) for the gamma functions and taking the limit as $q \to \infty$. Similarly, the last bracketed expression in (3.25) approaches $e^{-(y/\beta)^a}$ as $q \to \infty$. Therefore the generalized beta in (3.23) approaches the generalized gamma (3.1) as $q \to \infty$.

Limiting behavior of the generalized beta of the second kind

Substituting $b = q^{1/a}\beta$ into (3.3) and grouping terms yields

$$h(y) = \left(\frac{ay^{ap-1}}{\beta^{ap}\Gamma(p)}\right)\left(\frac{\Gamma(p+q)}{\Gamma(q)q^p}\right)\left[\frac{1}{\left(1+\frac{y^a}{\beta^a q}\right)^{p+q}}\right]. \tag{3.27}$$

Using (3.26) in the second bracketed expression in (3.27) and taking the limit of (3.27) as $q \to \infty$ yields the expression for the generalized gamma density given in equation (3.1).

References

Aitchison, J. and J. A. C. Brown (1969) *The Lognormal Distribution with Special References to its Uses in Economics*, Cambridge University Press, Cambridge.

Arnold, B. (1980) Pareto Distributions: Pareto and Related Heavy-tailed Distributions, mimeographed manuscript, University of California at Riverside.

Atoda, N., T. Suruga and T. Tachibanaki (1980) Statistical Inference of Functional Forms for Income Distribution, manuscript. Kyoto University.

Cox, D. R. and D. V. Hinckley (1974) *Theoretical Statistics*, Chapman and Hall, London.

Cronin, D. C. (1979) A Function for the Size Distribution of Income: A Further Comment, *Econometrica*, **47**, 773–774.

Esteban, J. (1981) Income-Share Elasticity, Density Functions and the Size Distribution of Income, mimeographed manuscript, University of Barcelona.

Gastwirth, J. L. (1972) The Estimation of the Lorenz Curve and Gini Index, *Review of Economics and Statistics*, **54**, 306–316.

Gradshteyn, I. S. and I. M. Rhyzik (1965) *Tables of Integrals, Series, and Products*, Academic Press, New York.

Johnson, N. L. and S. Kotz (1970) *Continuous Univariate Distributions*, vol. 1, John Wiley and Sons, New York.

Kendall, M. G. and A. Stuart (1961) *The Advanced Theory of Statistics*, vol. 1, 2nd ed., Hafner, New York.

Kloek, T. and H. K. van Dijk (1978) Efficient Estimation of Income Distribution Parameters, *Journal or Econometrics*, **8**, 61–74.

McDonald, J. B. and B. C. Jensen (1979) An Analysis of Estimators of Alternative Measures of Income Inequality Associated with the Gamma Distribution Function, *Journal of the American Statistical Association*, **74**, 856–860.

McDonald, J. B. and M. J. Ransom (1979) Functional Forms, Estimation Techniques and the Distribution of Income, *Econometrica*, **47**, 1513–1526.

McDonald, J. B. and M. J. Ransom (1981) An Analysis of the Bounds for the Gini Coefficient, *Journal of Econometrics*, **17**, 177–188.

Rainville, E. D. (1960) *Special Functions*, MacMillan, New York.

Salem, A. B. Z. and T. D. Mount (1974) A Convenient Descriptive Model of Income Distribution: the Gamma Density, *Econometrica*, **42**(6), 1115–1127.

Singh, S. K. and G. S. Maddala (1976) A Function for Size Distribution of Incomes, *Econometrica*, **44**(5), 963–970.

Tadikamalla, P. R. (1980) A Look at the Burr and Related Distributions, *International Statistical Review*, **48**, 337–349.

Taille, C. (1981) Lorenz Ordering Within the Generalized Gamma Family of Income Distributions, in G. P. Taille, C. and B. Balderssari (eds.) *Statistical Distributions in Scientific Work*, vol. 6, pp. 181–192, Reidel, Boston.

Thurow, L. C. (1970) Analyzing the American Income Distribution, *American Economic Review*, **48**, 261–269.

US Bureau of the Census (1961-1971) Income of Families and Persons in the United States, Current Population Reports, Series P-60, US Government Printing Office, Washington.