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# A Nonproportional Premium Rating Method for Construction Risks

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**Correct pricing of nonproportional (primary or excess of loss) insurance for construction risks must consider not only how the insured property values build up over time, but also how the probable maximum loss (PML) changes. Conventional pricing methods for static property risks cannot be employed for construction risks, since the latter are characterized by PML patterns that change over time, as well as evolving loss exposures and perils arising from the various phases of the construction project. A further complication arises when delay in startup (DSU) is covered, because a DSU loss is triggered by a property damage loss and both losses contribute jointly to the erosion of an excess layer. This article describes a pricing method with analysis of specific cases of interest, including guidelines for creating practical excess of loss rating models.**

## 1. INTRODUCTION

This article deals with *construction risks*, meaning the construction of a new building, structure, plant, facility, or other project, starting from the first phase of site preparation, excavations, and foundations through to the final phase of testing, commissioning, and startup, and culminating with official handover of the project by the contractors to the owners and putting the project into commercial operation. Such risks are sometimes referred to as *construction property risks* to distinguish them from associated liability risks (e.g., employers liability, third-party liability, workmen's compensation, environmental liability, etc.) when there is possibility of confusion. A recent paper on construction risk management (Brockett, Golden, and Betak 2019) provides a modern overview of construction risk insurability and construction industry insurance products.

In contrast with the completed, operational property risk that emerges after the project has been handed over, which has an essentially static risk profile, the construction risk is extremely dynamic. First, the value exposed (at risk) starts from zero and builds up in some fashion to the full and final contract value. Second, the various phases of construction may be subject to changing and radically different loss exposures. Third, the magnitude of potential loss ("probable maximum loss") can vary significantly, not only due to the buildup of value over time, but because of the nature of the contract works and the property exposed.

Construction property risks are generally insured under a single policy for the duration of the project, with a policy limit equal to the estimated final total contract value (TCV).[1] Such policies are called *full value*. Even when conservative estimates of the maximum possible loss arising from any one event are well below the TCV (e.g., for construction of a hundred-mile-long roadway in an area with no natural catastrophe exposure), the project financiers generally require a full value policy. Consequently, construction policies often have significantly higher limits than the operational policies that are arranged to insure the completed project, and they are usually insured on a *proportional basis* by a panel of insurers, each accepting a fixed percentage of the risk (policy limit).

Regardless of whether there exists significant loss exposure all the way up to the TCV, the full value requirement creates opportunities to buy and sell *nonproportional* insurance and reinsurance. This article is concerned exclusively with excess of loss insurance, meaning insurance that mirrors the terms and conditions of a full value policy but only pays in excess of a specified deductible (the *attachment point*) up to a specified maximum amount (the *layer stretch*).[2] The insurer(s) or the policyholder's insurance broker may obtain excess of loss reinsurance to exploit varied appetites and inefficiencies in the

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[1]Construction policies typically include an "escalation allowance" that extends the policy limit above the estimated final TCV stated in the policy schedule, to cater to unexpected contract value increases that could exhaust the contingency included within the contract value.

[2]Note that a *primary* (layer) is a special case where the attachment point is $0.

marketplace and obtain the lowest overall premium. Insurers may also buy excess of loss to increase the premium income received for the maximum limit of liability they are permitted to retain.

The dynamic character of construction risks complicates the pricing of nonproportional insurance. Unfortunately, because of its specialist nature and niche position within the large suite of property insurance products, construction has received scant attention with regard to pricing analysis, and nonproportional insurance even less. Commonly cited references on excess of loss pricing for property insurance (e.g., Guggisberg 2004; Bernegger 1997; Ludwig 1991; Sanders et al. 1998) make no mention of construction risks and their peculiarities. One might expect a tract focused on construction property insurance to touch upon nonproportional rating, but the standard reference for construction underwriting (Alderton 1999) omits to do so.

Various empirical models employing curves and tables have been used to price nonproportional insurance for construction, generally relying on exposure rating curves of some kind (Guggisberg 2004). A comparatively ancient Munich Re guideline (Mack 1982) is a rare instance that specifically addresses excess of loss pricing for construction; however, it completely fails to account for the time-varying nature of loss exposures and magnitudes.

This article presents a method for pricing nonproportional insurance of construction risks and develops the ideas needed to create a practical rating model. Numerous parameter choices and other proprietary decisions must be made to construct a fully fledged rating engine. The foundation provided here can serve as the starting point for such a task.

## 2. NOTATION

We make the following definitions:

- $T$ period of insurance
- $v(t)$ total insured value exposed at time $t$, for $0 \le t \le T$ (an increasing function of $t$)
- $m(t)$ probable maximum loss (PML) at time $t$, for $0 \le t \le T$
- $V$ $\max_{0 \le t \le T} v(t) = v(T)$ ........................(i.e., the total insured value)
- $M$ $\max_{0 \le t \le T} m(t)$ ...................................(i.e., the maximum PML, usually denoted simply as PML)
- $V_0$ bottom of the layer[3] ...................(i.e., the attachment point)
- $V_1$ top of the layer ...........................(i.e., $V_1 - V_0$ is the layer stretch)
- $r$ premium per unit value per unit time (or *premium rate* per unit time)
- $P$ total premium for the risk

We denote by $E(x)$ the *exposure rating curve* applicable to the exposure. That is, for $0 \le x \le 1$, $E(x)$ is the proportion of premium for an exposure with PML $Q$ that should be allocated to the primary layer $xQ$ (see Guggisberg 2004).

## 3. GENERAL FORMULAS FOR LAYER PRICING

In this section we develop a general formula for the premium allocated to the layer between $V_0$ and $V_1$ in terms of the project premium $P$, value $V$, and period $T$, given the buildup of value $v(t)$ and PML $m(t)$.

Note that $rv(t)dt$ is the premium for the risk between $t$ and $t + dt$. Hence, $rv(t)$ is the *premium density*.[4] The total premium $P$ for the risk is

$$P = \int_0^T rv(t)dt.$$

If $r$ is constant, then

$$r = P\left\{\int_0^T v(t)dt\right\}^{-1}. \tag{1}$$

[3]We ignore underlying deductibles. If a deductible $D$ applies, then $V_0$ and $V_1$ can be replaced with $V_0 + D$ and $V_1 + D$, but then $P$ must be replaced with the premium $P'$ that would be charged with a zero deductible. This may not be easy to determine. It is not correct to write $P = \{1 - E(D/M)\}P'$ because the curves $E(x)$ are not suitable for thin layers close to zero, as they do not account for loss frequency and reinstatement.

[4]If $P(t)$ is the premium for the interval $[0, t]$, then $P'(t)$ is the premium density, since the premium for the interval $\sigma \le t \le \tau$ is given by $P(\tau) - P(\sigma) = \int_\sigma^\tau P'(t)dt$. Therefore, we may define $r(t) \equiv P'(t)/v(t)$. We are assuming that $r = r(t)$ is constant.

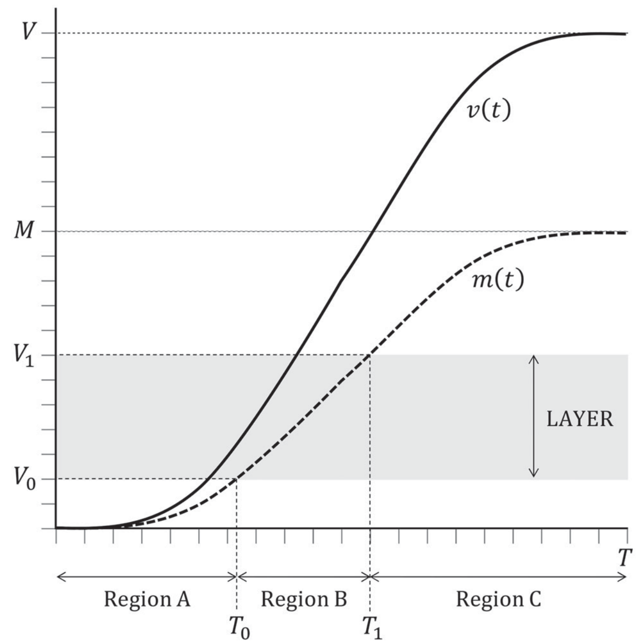


FIGURE 1. Buildup of value $v(t)$ and PML $m(t)$. *Note:* The layer from $V_0$ to $V_1$, maximum PML $M$, and final value $V$ are shown on the vertical axis. The period $T$ is shown on the horizontal axis.

In fact, $r$ may vary if the perils, coverages, or nature of the insured property change over time. This would be the case if, for example:

- Construction methods and materials change significantly during the project term.
- The project transitions from a construction phase to a testing phase.
- Windstorm exposure changes throughout the year.

We will deal with various perils and coverages separately, so we may assume that *r is constant* unless otherwise stated. So, if $\mathcal{E}_i$ is the exposure arising from a particular peril (e.g., fire, earthquake, flood) and/or during a particular phase of the project (e.g., foundations, erection, testing) between $t = t_a$ and $t = t_b$, for which a premium $P_i$ is paid, then

$$r_i = P_i \left\{ \int_{t_a}^{t_b} v(t)dt \right\}^{-1}$$

is constant.

In construction insurance, the *probable maximum loss* (PML) is a loss estimate based on a "plausible worst case" scenario (equivalent to the *maximum foreseeable loss* or *estimated maximum loss* used elsewhere in insurance).[5] A commonly accepted definition of PML (Heller et al. 2002) is "an estimate of the maximum loss which could be sustained by the insurers as a result of any one occurrence considered by the underwriter to be within the realms of probability. This ignores such coincidences and catastrophes which are remote possibilities, but which remain highly improbable."

Throughout this article we assume that any exposure in excess of the PML attracts no premium. Therefore, if there is an exposure with PML $m$ during an interval $\Delta t$, then a primary layer with limit $Q \leq m$ deserves a proportion $E(Q/m)$ of the premium for the interval $\Delta t$, where $E(x)$ is a suitably chosen exposure rating curve. We apply this principle repeatedly.

We may assume that $V_1 \leq M$ (otherwise we can redefine $V_1 = M$). We also assume for the moment that the PML function $m(t)$ is non-decreasing (which is very often the case), as shown in Figure 1. (This assumption is relaxed later.)

[5] PML estimates are subjective. Uniform rules and guidelines for PML estimation do not exist, and PML estimation depends upon available underwriting information, loss estimation models and guidelines, as well as underwriter experience and judgment. In essence, the PML should be an estimate of the maximum loss at a construction site under adverse conditions with no benefit from installed fire protection systems.

In Figure 1, $T_0$ and $T_1$ are the times when the PML $m(t)$ enters and exits the layer, respectively.[6] If $m$ is continuous, then $T_0$ and $T_1$ can be calculated by solving the equations

$$\begin{aligned} m(T_0) &= V_0 \\ m(T_1) &= V_1 \end{aligned} \tag{2}$$

Let $L_\text{A},\ L_\text{B},$ and $L_\text{C}$ be the layer premiums for the regions A, B, and C. Obviously, $L_\text{A} = 0$ since $m(t) \le V_0$. Looking at region B, we see that

$$L_\text{B} = \int_{T_0}^{T_1} \left\{ 1 - E\left(\frac{V_0}{m(t)}\right) \right\} rv(t)dt.$$

The first term $1 - E(V_0/m(t))$ is the proportion of premium that should be allocated to the layer from $V_0$ to $m(t)$. The second term $rv(t)dt$ is the premium for the interval $[t, t+dt]$.

Similarly, for region C,

$$L_\text{C} = \int_{T_1}^{T} \left\{ E\left(\frac{V_1}{m(t)}\right) - E\left(\frac{V_0}{m(t)}\right) \right\} rv(t)dt.$$

The first term $E(V_1/m(t)) - E(V_0/m(t))$ is the proportion of premium that should be allocated to the layer from $V_0$ to $V_1,$ and the second term $rv(t)dt$ is the premium for the interval $[t, t+dt]$.

Combining terms, the layer premium $L = L_\text{A} + L_\text{B} + L_\text{C}$ is:

$$L = \int_{T_0}^{T_1} \left\{ 1 - E\left(\frac{V_0}{m(t)}\right) \right\} rv(t)dt + \int_{T_1}^{T} \left\{ E\left(\frac{V_1}{m(t)}\right) - E\left(\frac{V_0}{m(t)}\right) \right\} rv(t)dt. \tag{3}$$

Rearranging terms,

$$L = \int_{T_0}^{T_1} rv(t)dt - \int_{T_0}^{T} E\left(\frac{V_0}{m(t)}\right) rv(t)dt + \int_{T_1}^{T} E\left(\frac{V_1}{m(t)}\right) rv(t)dt. \tag{4}$$

Equation (3) can also be written more compactly:

$$L = \int_0^T \left\{ E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right) \right\} rv(t)dt \tag{5}$$

since

$$E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right) = \begin{cases} 0 & m(t) < V_0 \\ 1 - E(V_0/m(t)) & V_0 \le m(t) < V_1 \\ E(V_1/m(t)) - E(V_0/m(t)) & V_1 \le m(t) \end{cases}$$

This form is suitable and preferable for programming a computer, since $L$ can be calculated without solving Equation (2) for $T_0$ and $T_1$. Note that Equation (5) *is valid for any PML function* $m(t),$ regardless of whether it is non-decreasing or continuous.

We show in Section 6 that if $v(t)$ is symmetrical under a 180° rotation (such as an S-shaped curve), then $r = 2P/VT$. In this case, Equation (5) simplifies to

$$L = \frac{2P}{VT} \int_0^T \left\{ E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right) \right\} v(t)dt. \tag{6}$$

Equation (6) expresses the premium for the layer between $V_0$ and $V_1$ in terms of $P,\ V,$ and $T,$ knowledge of $v(t)$ and $m(t),$ and the applicable exposure rating curve $E(x)$.

*Primary layers.* Note that Equation (5) or Equation (6) can be used to price a primary layer by setting $V_0 = 0,$ causing the second term in curly brackets to vanish. From Equation (5),

$$L_{\text{primary}\ \ V_1} = \int_0^T E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) rv(t)dt \tag{7}$$

[6] In other words, $T_0 = \inf\{t : m(t) > V_0\}$ and $T_1 = \sup\{t : m(t) \le V_1\}$.

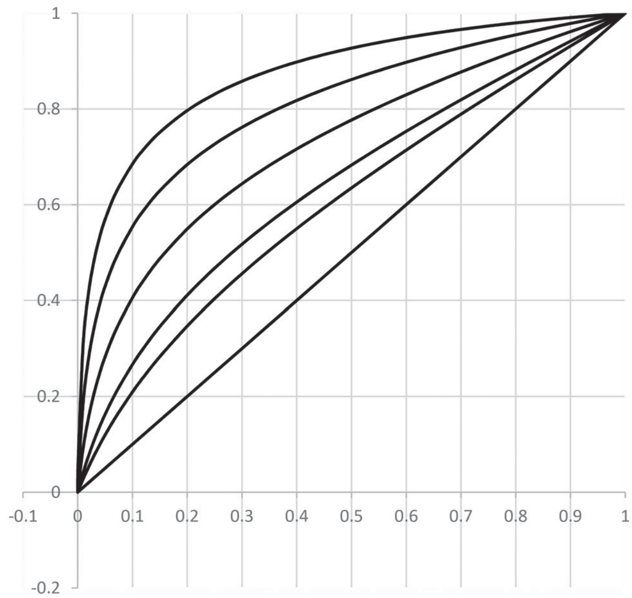


FIGURE 2. The Swiss Re $Y_c$ (Gasser) and Lloyd's curves.

and when $v(t)$ is symmetrical under a 180° rotation,

$$L_{\text{primary}\ \ V_1} = \frac{2P}{VT}\int_0^T E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) v(t)dt. \tag{8}$$

Practical use of Equation (5) or Equation (6) requires knowledge of $v(t)$, $m(t)$, and $E(x)$. In general, $v(t)$ is given by some variety of S-shaped curve, $m(t)$ is determined by the underwriter, and $E(x)$ is an increasing, concave function of the form

$$E_c(x) = \frac{\ln\left[\frac{(\alpha-1)\beta+(1-\alpha\beta)\beta^x}{1-\beta}\right]}{\ln \alpha\beta} \tag{9}$$

(see Bernegger 1997), where

$$\alpha = \alpha(c) = e^{(0.78-0.12c)c}$$
$$\beta = \beta(c) = e^{3.1-0.15(1+c)c}$$

and $c \geq 0$ is a free parameter.

Figure 2 shows $E_c(x)$ for $c = 0,\ 1,\ 2,\ 3,\ 4,$ and 5. ($E_5(x)$ is the top [most concave] curve, and the $E_0(x)$ is the bottom [diagonal] curve.) For $1 \leq c \leq 4,\ E_c(x)$ are known as the *Swiss Re $Y_c$ (or Gasser) curves*. $E_5(x)$ is known as the *Lloyd's curve.*

## 4. LINEAR BUILDUP OF VALUE CASE

When $v$ is linear, that is, $v(t) = (V/T)t$, we have from Equation (1)

$$r = \frac{2P}{VT}. \tag{10}$$

The general formula as in Equation (5) then becomes

$$L = \int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} \cdot \frac{2P}{VT} \cdot \frac{Vt}{T} dt$$

or

$$\frac{L}{P} = \frac{2}{T^2}\int_0^T t\left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} dt. \tag{11}$$

Alternatively, using the explicit formulation in Equation (4), we also have

$$\frac{L}{P}=\frac{2}{T^2}\left\{\frac{T_1^2-T_0^2}{2}-\int_{T_0}^{T} tE\left(\frac{V_0}{m(t)}\right)dt+\int_{T_1}^{T} tE\left(\frac{V_1}{m(t)}\right)dt\right\}. \tag{12}$$

The PML buildup $m(t)$ can take a variety of forms. It may increase in the same fashion as $v(t)$, or it may remain approximately constant.[7] In the case of projects with a *testing period*, $m$ may experience a jump at the start of the testing period (see Section 9). In this case $r$ also changes, so the testing period must be treated separately. Rapid changes in $m(t)$ usually indicate a change in the exposure (perils, property, or coverage) and a possible change in $r$.

## 5. SPECIAL CASES OF PML BUILDUP $m(t)$

In this section we continue to assume that $v(t)=(V/T)t$ is linear. It is usually difficult to specify the actual PML buildup $m(t)$. However, simple assumptions can often be made. We consider two cases: (A) $m$ is linear, and (B) $m$ is constant.

*Case A: $m(t)=(M/T)t$ is linear.*[8] In this case, Equation (12) becomes

$$\frac{L}{P}=\frac{2}{T^2}\left\{\frac{T_1^2-T_0^2}{2}-\int_{T_0}^{T} tE\left(\frac{V_0T}{Mt}\right)dt+\int_{T_1}^{T} tE\left(\frac{V_1T}{Mt}\right)dt\right\}. \tag{13}$$

The limits of integration $T_0$ and $T_1$ are easily determined using the linearity of $m(t)$. Since $V_i=m(T_i)=(M/T)T_i$, we have

$$\begin{aligned} T_0&=(T/M)V_0\\ T_1&=(T/M)V_1 \end{aligned} \tag{14}$$

Substituting these where they appear in Equation (13), we obtain

$$\begin{aligned}\frac{L}{P}&=\frac{2}{T^2}\left\{\frac{T^2\left(V_1^2-V_0^2\right)}{2M^2}-\int_{TV_0/M}^{T} tE\left(\frac{V_0T}{Mt}\right)dt+\int_{TV_1/M}^{T} tE\left(\frac{V_1T}{Mt}\right)dt\right\}\\ &=\frac{V_1^2-V_0^2}{M^2}-\frac{2}{T^2}\int_{TV_0/M}^{T} tE\left(\frac{V_0T}{Mt}\right)dt+\frac{2}{T^2}\int_{TV_1/M}^{T} tE\left(\frac{V_1T}{Mt}\right)dt.\end{aligned}$$

Now make the change of variable $u=Mt/V_0T$ and $u=Mt/V_1T$ in the first and second integrals, respectively, to obtain

$$\begin{aligned}\frac{L}{P}&=\frac{V_1^2-V_0^2}{M^2}-\frac{2V_0^2}{M^2}\int_1^{M/V_0} uE\left(\frac{1}{u}\right)du+\frac{2V_1^2}{M^2}\int_1^{M/V_1} uE\left(\frac{1}{u}\right)du\\ &=\frac{V_1^2}{M^2}\left\{1+2\int_1^{M/V_1} uE\left(\frac{1}{u}\right)du\right\}-\frac{V_0^2}{M^2}\left\{1+2\int_1^{M/V_0} uE\left(\frac{1}{u}\right)du\right\}.\end{aligned}$$

Notice that $T$ has disappeared. This is not surprising, since $T$ should not affect the layer price; we can always make $T=1$ by appropriate choice of units or the change of variable $t'=t/T$.

If we define for $x\geq 1$

$$\mathbb{G}(x)=\int_1^x uE\left(\frac{1}{u}\right)du$$

?

then we may write

$$\frac{L}{P}=\frac{V_1^2}{M^2}\left\{1+2\mathbb{G}\left(\frac{M}{V_1}\right)\right\}-\frac{V_0^2}{M^2}\left\{1+2\mathbb{G}\left(\frac{M}{V_0}\right)\right\}. \tag{15}$$

Unfortunately, $\mathbb{G}(x)$ cannot be expressed in terms of elementary functions when $E(x)$ is given by Equation (9).

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[7] A more interesting example is provided by the case of a hydroelectric power plant. At the start of the project, $m(t)$ increases slowly during site preparation and infrastructure works, reaches a peak midway during the period (corresponding to collapse of a cofferdam or diversion tunnel, resulting in a catastrophic flood), decreases after the main dam is completed and the diversion tunnels are closed, and increases again during the testing period. The risk exposures change significantly during the period; therefore, $r$ is not constant. An accurate layer price requires a number of separate calculations. As $m$ is not non-decreasing, Equation (4) would require modification; but Equation (5) remains valid.

[8] This may be a reasonable assumption for the construction of buildings, dams, bridges, etc.

*Example:* Suppose $v$ and $m$ build up linearly, with $M = 80$. Calculate $L/P$ for a 40 XS 10 layer, using the Lloyd's curve:

$$E(x) = E_5(x) = (2/11)\ln\left(1 + 323.4549\left(1 - e^{-1.4x}\right)\right).$$

*Solution:* From Equation (15) we have

$$\frac{L}{P} = \frac{25}{64}\left\{1 + 2\int_1^{1.6} uE\left(\frac{1}{u}\right)du\right\} - \frac{1}{64}\left\{1 + 2\int_1^{8} uE\left(\frac{1}{u}\right)du\right\} \approx 0.1873.$$

A naive calculation that assumes $m(t) = 80$ (constant) gives $L/P = E(5/8) - E(1/8) \approx 0.2320$, which overestimates the correct amount by $\sim 24\%$.

Incidentally, $\mathbb{G}(M/V_0) = \int_1^{M/V_0} uE(1/u)du$ does not converge as $V_0 \to 0$ since $uE(1/u) \geq u(1/u) = 1$. However, the product $V_0^2\mathbb{G}(M/V_0)$ does converge as $V_0 \to 0$. By L'Hôpital's rule,

$$\lim_{V_0 \to 0} V_0^2 \int_1^{M/V_0} uE\left(\frac{1}{u}\right)du = \lim_{V_0 \to 0} \frac{\frac{M}{V_0} \cdot E\left(\frac{V_0}{M}\right)\left(-\frac{M}{V_0^2}\right)}{-\frac{1}{V_0^3}} = M^2 \lim_{V_0 \to 0} E\left(\frac{V_0}{M}\right) = 0.$$

*Case B: $m(t) = M$ is constant.*[9] Strictly speaking, this is unrealistic if $v(t) = (V/T)t$ is linear. Since $v(0) = 0$, we have $m(t) > v(t)$ near $t = 0$, which is not possible. This objection can be overcome by choosing

$$m(t) = \begin{cases} (V/T)t & 0 \leq t < MT/V \\ M & MT/V \leq t \leq T \end{cases}$$

as illustrated in Figure 3.

In reality, we can often ignore this complication. The assumption that $m$ is constant usually occurs in situations where $m$ builds up quickly to a constant value $M \ll V$. We can ignore the region near $t = 0$ where $v(t) < M$, since this region has little influence on the layer price (since the premium density $rv(t)$ is small). The layer price is dominated by the region where $v(t) > M$.

If $m$ is constant, $T_0$ and $T_1$ are not well defined, since $m(t)$ does not "enter" and "exit" the layer. But we can proceed in direct fashion to write

$$\begin{aligned} L &= \int_0^T \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} rv(t)dt \\ &= \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\}\int_0^T rv(t)dt \\ &= P\left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} \end{aligned}$$

which yields the simple formula

$$\frac{L}{P} = E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right). \tag{16}$$

This is the usual formula for a static property risk. The behavior of $v(t)$ is irrelevant; what matters is that $m(t)$ is constant, so the factor $E(V_1/m(t)) - E\left(V_0/m(t)\right) = E(V_1/M) - E(V_0/M)$ is time-independent.

## 6. S-SHAPED BUILDUP OF VALUE (I)

The buildup of value at a construction project typically has a sigmoidal (S-shaped) curve over time. Costs build up slowly at the start of the project, during mobilization and site preparation. Later, costs accumulate at an almost constant rate with work crews on site and delivery of construction materials and machinery and equipment to be erected. As the project nears completion, the cost accumulation decelerates.

[9] This may be a reasonable assumption for the construction of "long linear" risks such as roads, railways, tunnels, pipelines, transmission and distribution lines, etc.

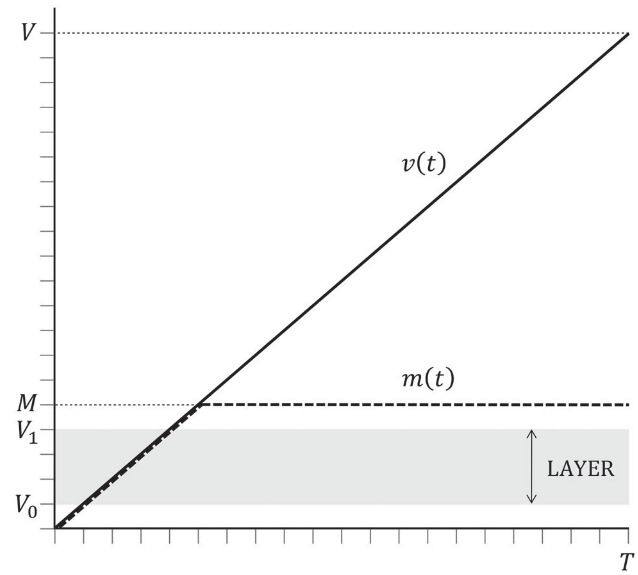


FIGURE 3. Illustration of the buildup of PML $m(t)$ in the case $m(t) = M$ constant. *Note:* $m(t)$ is adjusted at the beginning of the period to follow $v(t)$, but this short region actually has very little influence.

Recall that in Section 4 we used the linearity of $v$ to find $r = 2P/VT$. Fortunately, this remains valid when the S-curve is symmetrical under a 180° rotation, as shown in Figure 4.

To see this, note that the area under the S-shaped curve is the same as under the straight line, so $\int_0^T v(t)dt = VT/2$. Indeed, this holds for any function $v(t)$ that is symmetrical under a 180° rotation. This symmetry means $v(t) + v(T-t) = V$. Hence,

$$\begin{aligned} 2\int_0^T v(t)dt &= \int_0^T v(t)dt + \int_0^T [V - v(T-t)]dt \\ &= \int_0^T v(t)dt + VT - \int_0^T v(T-t)dt \\ &= VT \end{aligned}$$

so that $\int_0^T v(t)dt = VT/2$. Therefore, when $v(t)$ is symmetrical under a 180° rotation:

$$r = P\left\{\int_0^T v(t)dt\right\}^{-1} = \frac{2P}{VT}$$

as claimed in Section 3. This proves Equation (6).

In the case where $v(t)$ was linear, we expressed the premium density

$$rv(t) = \frac{2P}{VT} \cdot \frac{V}{T} t = \frac{2P}{T^2} t$$

but this no longer holds when $v$ is not linear. Looking at Figure 5, we see that in the region $0 \le t \le T/2$ (where the straight line is above the S-curve), the straight line gives a higher premium density $rv(t)$ than the S-shaped curve. Similarly, in the region $T/2 \le t \le T$, the straight line gives a lower premium density. (Equation (1) shows that $r$ is the same for both curves.)

These effects do not cancel each other out in the calculation of the layer price, because of the "layer allocation factor" that appears in Equations (5) and (6):

$$\theta(t) = E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right).$$

The behavior of $\theta(t)$ as $t$ advances is rather subtle (see the Appendix for details). $\theta(t)$ can favor (i.e., result in a higher layer premium) either the S-shaped or the straight-line buildup, depending on the relative widths and positions of regions A, B, and C. As the attachment point rises, region A (which contributes nothing) becomes larger, regions B and C shift further to the

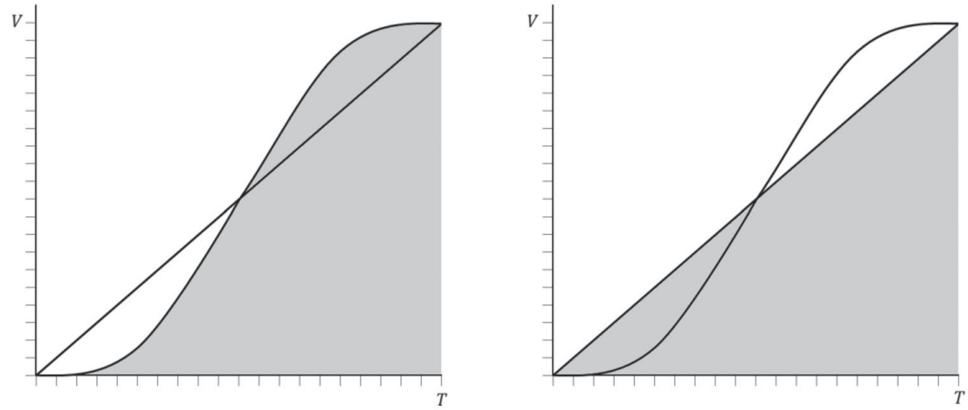


FIGURE 4. The area under an S-shaped curve (left) is the same as the area under a straight line (right).

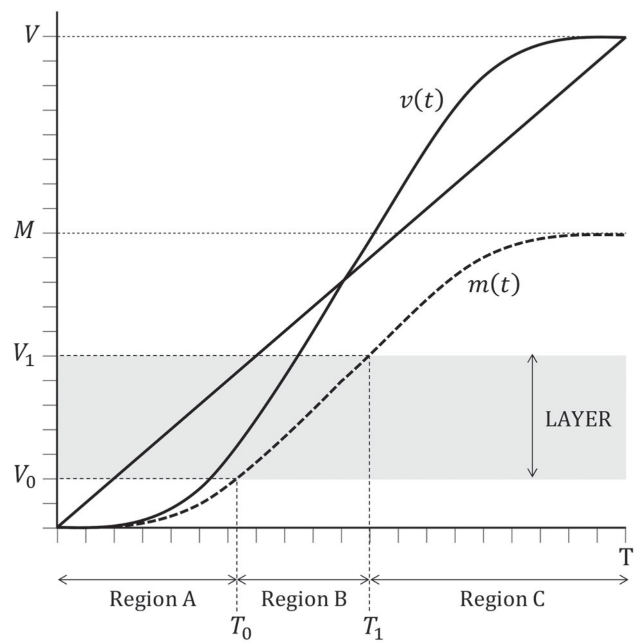


FIGURE 5. Comparison of an S-shaped buildup of value curve $v(t)$ with a straight-line approximation.

right, and region C gets smaller; all of these increase the premium for the S-shaped buildup. In most cases, the linear approximation results in a lower premium than an S-shaped function.

## 7. S-SHAPED BUILDUP OF VALUE (II)

The S-shaped curve for buildup of value at some construction projects with final value $V$ and period $T$ can be approximated by simple functions such as

$$u_1(t) = \frac{V}{T^2}\left\{3t^2 - \left(\frac{2}{T}\right)t^3\right\}$$
$$u_2(t) = \frac{V}{2}\left\{\sin\pi\left(\frac{t}{T} - \frac{1}{2}\right) + 1\right\}$$

for $0 \le t \le T$. These functions are actually quite similar on $[0, T]$, as shown in Figure 6 (with $V = T = 1$). The polynomial $u_1$ is something of a curiosity; besides possessing 180° symmetry on the interval $[0, T]$, it also approximates $u_2$ extremely well. The functions $u_1$ and $u_2$

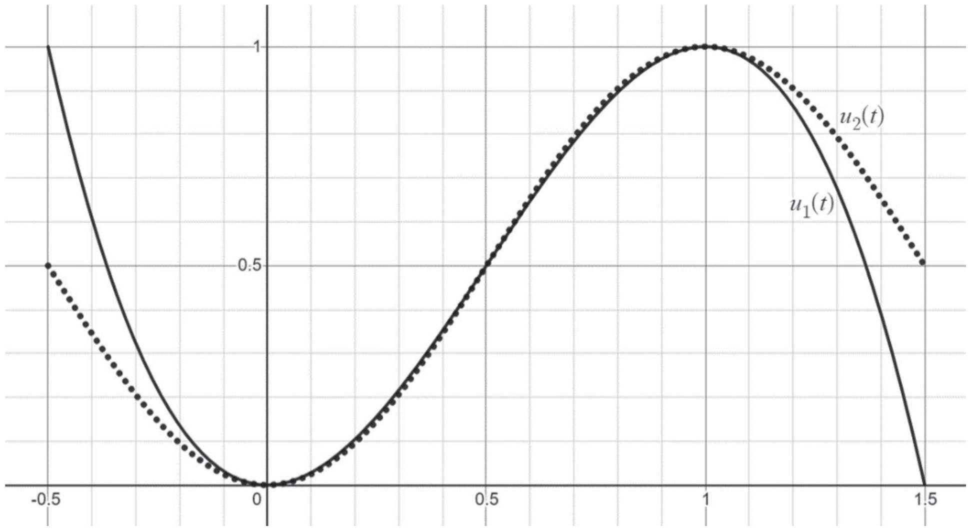


FIGURE 6. Comparison of $u_1(t) = 3t^2 - 2t^3$ and $u_2(t) = [\sin\pi(t - 1/2) + 1]/2$ on the interval $0 \le t \le 1$.

- Are symmetric under a 180° rotation.
- Satisfy $u_i'(0) = u_i'(T) = 0$.
- Have maximum slope $u_1'(T/2) = 3V/2T$ and $u_2'(T/2) = \pi V/2T$ (differing by less than ½% when $V = T = 1$).

Since $r = 2P/VT$ when $v$ is symmetrical under a 180° rotation, we have, from Equation (6),

$$\frac{L}{P} = \frac{2}{VT}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} u_i(t)dt. \tag{17}$$

Alternatively, we can use the explicit form from Equation (4) and write

$$\frac{L}{P} = \frac{2}{VT}\left\{\int_{T_0}^{T_1} u_i(t)dt - \int_{T_0}^{T} E\left(\frac{V_0}{m(t)}\right)u_i(t)dt + \int_{T_1}^{T} E\left(\frac{V_1}{m(t)}\right)u_i(t)dt\right\}. \tag{18}$$

(Note that the factor of $V^{-1}$ in front of the integrals cancels the $V$ in $u_i(t)$.) The limits of integration $T_0$ and $T_1$ can be calculated by solving the equations

$$m(T_0) = V_0$$
$$m(T_1) = V_1$$

We consider again the special cases when $m$ is proportional to $v$ and when $m$ is constant.

*Case A:* Assume $m$ is proportional to $v$, that is $m(t) = (M/V)u_i(t)$. Then Equation (17) or Equation (18) can be used to compute $L$.

*Example:* Suppose $v$ and $m$ are given by $v(t) = 100(3t^2 - 2t^3)$ and $m(t) = 80(3t^2 - 2t^3)$ for $0 \le t \le 1$. Calculate $L/P$ for a 40 XS 10 layer, using the Lloyd's curve.

*Solution:* We solve the equations

$$3T_0^2 - 2T_0^3 = 1/8$$
$$3T_1^2 - 2T_1^3 = 5/8$$

numerically to obtain $T_0 \approx 0.2211$ and $T_1 \approx 0.5841$. Using Equation (18) with $T = 1$,

$$\frac{L}{2P} = \left[t^3 - \frac{t^4}{2}\right]\Bigg|_{T_0}^{T_1} - \int_{T_0}^{1} E\left(\frac{1/8}{3t^2 - 2t^3}\right)[3t^2 - 2t^3]dt + \int_{T_1}^{1} E\left(\frac{5/8}{3t^2 - 2t^3}\right)[3t^2 - 2t^3]dt$$
$$= 0.1315 - \int_{0.2211}^{1} E\left(\frac{1/8}{3t^2 - 2t^3}\right)[3t^2 - 2t^3]dt + \int_{0.5841}^{1} E\left(\frac{5/8}{3t^2 - 2t^3}\right)[3t^2 - 2t^3]dt$$

Using the Lloyd's curve $E(x) = (2/11)\ln(1 + 323.4549(1 - e^{-1.4x}))$ we obtain

$$\frac{L}{P} \approx 2(0.1315 - 0.3804 + 0.3475) = 0.1971.$$

The figure of 0.1873 obtained in the earlier example using a straight-line buildup of value is too low by about 5%.

*Case B:* Assume $m(t) = M$ is constant.[10] From Equation (16) we know that $v(t)$ does not matter and

$$\frac{L}{P} = E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right).$$

This can be checked directly. For example, for $u_1$,

$$\begin{aligned}\frac{L}{P} &= \frac{1}{P}\int_0^T \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} ru_1(t)dt \\ &= \frac{1}{P}\int_0^T \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} \cdot \frac{2P}{VT} \cdot \frac{V}{T^2}\left\{3t^2 - \left(\frac{2}{T}\right)t^3\right\}dt \\ &= \frac{2}{T^3}\left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\}\int_0^T \left(3t^2 - \frac{2t^3}{T}\right)dt \\ &= E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right).\end{aligned}$$

## 8. OTHER S-SHAPED CURVES

We have seen that $u_1$ and $u_2$ have S-shaped curves on $[0, T]$ that are symmetric under a 180° rotation. Unfortunately, these functions do not admit easy modification of their shape to model different rates of buildup of value.

The buildup of value $v(t)$ for heavy industrial projects like power plants and oil refineries usually has a more pronounced S-shape than for buildings, dams, or bridges. At the beginning of such projects, $v(t)$ increases slowly while mobilization and site preparation take place, increases rapidly as machinery and equipment are delivered and erected, and flattens out again while numerous functional checks and tests are carried out. A collection of S-shaped curves with different degrees of "steepness" would enable us to model the buildup of value for a wide variety of construction projects.

A family of symmetric S-shaped curves $D_k(t) : [0, T] \to [0, V]$ can be generated for $k > 0$ by

$$D_k(t) = \frac{V}{2}\left\{\frac{\tanh\left[k\left(\frac{t}{T} - \frac{1}{2}\right)\right]}{\tanh\left(\frac{k}{2}\right)} + 1\right\}. \tag{19}$$

These functions have a maximum slope equal to $D'_k(T/2) = kV/2T\tanh(k/2)$. We may define

$$D_0(t) \equiv \lim_{k\to 0} D_k(t) = (V/T)t.$$

To verify this, we replace $\tanh x$ by its Taylor series $\tanh x = x - x^3/3 + \cdots,$ which converges for $|x| < \pi/2,$ and compute the limit:

$$\lim_{k\to 0}\frac{V}{2}\left\{\frac{\tanh\left[k\left(\frac{t}{T} - \frac{1}{2}\right)\right]}{\tanh\left(\frac{k}{2}\right)} + 1\right\} = \lim_{k\to 0}\frac{V}{2}\left\{\frac{k\left(\frac{t}{T} - \frac{1}{2}\right) + \text{ terms of order } k^3}{\frac{k}{2} + \text{ terms of order } k^3} + 1\right\} = \frac{V}{T}t.$$

When the buildup of value $v(t)$ is approximated by $D_k(t)$, Equation (6) becomes:

[10] As mentioned before, $v(t) < M$ near $t = 0$ is unrealistic. If desired, we could choose

$$m(t) = \begin{cases} v(t) & 0 \le t < \tau \\ M & \tau \le t \le T \end{cases}$$

where $\tau$ is chosen so that $v(\tau) = M$. However, we ignore this complication by assuming $M \ll V$.

TABLE 1
Premium Layer Allocation for Two Layers and Three Different Buildup of Value Functions

| Layer | Layer allocation $L/P$ using $v(t) =$ | | |
|---|---|---|---|
| | $\frac{Vt}{T}$ | $\frac{V}{T^2}\left[3t^2 - \frac{2t^3}{T}\right]$ | $\frac{V}{2}\left\{\frac{\tanh[7(t/T-1/2)]}{\tanh(7/2)} + 1\right\}$ |
| 50 XS 10 | 0.219 | 0.229 | 0.244 |
| 50 XS 50 | 0.317 | 0.417 | 0.562 |

$$\frac{L}{P} = \frac{2}{VT}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} D_k(t)dt. \tag{20}$$

Table 1 compares the premium allocation for two layers and three different curves $v(t)$ : the straight line $v(t) = Vt/T$ and the functions $u_1(t)$ and $D_7(t)$. We choose $M = V = 100$ and $m(t) = v(t)$.[11] Note that the linear approximation underestimates the layer price when the buildup has an S-shape, and the discrepancy becomes larger as the attachment point increases.

## 9. DIFFERENT EXPOSURES AND PERILS

*Testing periods.* Construction risks involving the erection of machinery and equipment usually have a *testing period* that commences after the construction works are complete. The buildup curve, therefore, reaches its maximum value prior to the end of the period and remains constant, as shown in Figure 7. The testing period is usually responsible for a significant proportion (possibly a majority) of the policy premium; therefore, it plays an important role in layer pricing.

The construction and testing periods must be priced separately, because $r$ is different during these two phases. Hence, separate premium calculations must be carried out for the construction period (with its buildup curve $v_{\text{con}}(t)$ and associated premium $P_{\text{con}}$) and the testing period (with $v_{\text{test}}(t) = V$ and associated premium $P_{\text{test}}$).

Note that the PML curve $m(t)$ usually jumps to a higher value at the start of the testing period, as typically $M_{\text{test}} > M_{\text{con}}$ due to more severe fire and explosion risks during testing.

*Natural catastrophe exposures.* Natural catastrophe exposures must be treated separately if they contribute significantly to the premium, because they have their own exposure rating curves, PML curves, and $r$ values.

Catastrophe perils such as earthquake have a PML curve that can be modeled as a fixed proportion of the value: $m_{\text{cat}}(t) = \alpha v(t)$. Other catastrophe perils, such as hurricanes, are seasonal and can be modeled with $m_{\text{cat}}(t) = \alpha v(t)$ during hurricane seasons and $m_{\text{cat}}(t) = 0$ between seasons, as shown in Figure 8.

Note that $r$ is not constant here: $r(t) = \tilde{r}\chi(t)$ for some constant $\tilde{r}$, where

$$\chi(t) = \begin{cases} 1 & t \in \text{hurricane season} \\ 0 & t \notin \text{hurricane season} \end{cases}$$

Hence,

$$P = \int_0^T r(t)v(t)dt = \int_0^T \tilde{r}\chi(t)v(t)dt$$

and therefore,

$$\tilde{r} = P\left\{\int_0^T \chi(t)v(t)dt\right\}^{-1}. \tag{21}$$

Unfortunately, $\chi(t)v(t)$ is not symmetrical under a 180° rotation, so we cannot simplify this expression, as we did in the symmetrical case where $r = 2P/VT$. Instead, $\tilde{r}$ must be calculated using Equation (21).

[11] The Lloyd's curve is used for $E(x)$.

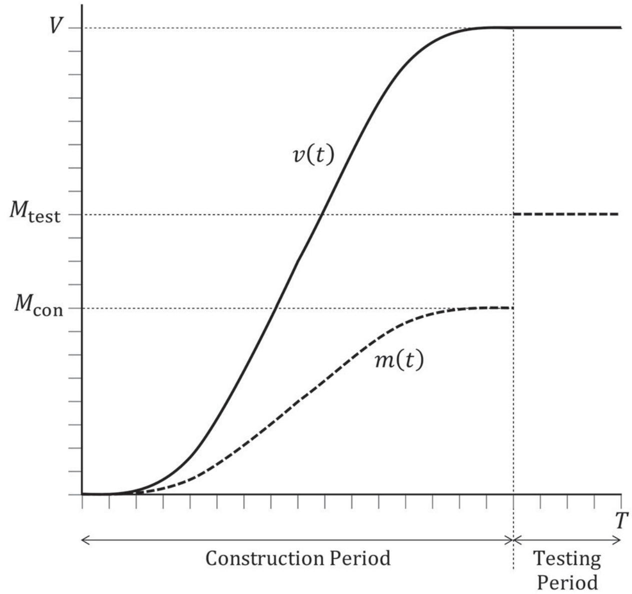


FIGURE 7. A project with a testing period. *Note:* The buildup of value $v(t)$ reaches its final value $V$ at the end of the construction period and remains constant during the testing period. The PML curve $m(t)$ has a discontinuity.

Equation (5) gives the layer premium,

$$L_{\text{cat}} = \int_0^T \left\{ E\left( \min\left\{ \frac{V_1}{m_{\text{cat}}(t)}, 1 \right\} \right) - E\left( \min\left\{ \frac{V_0}{m_{\text{cat}}(t)}, 1 \right\} \right) \right\} \tilde{r} \chi(t) v(t) dt.$$

Note that outside the hurricane season $m_{\text{cat}}(t) = 0,$ so the term in curly brackets is $1 - 1 = 0$. Hence, the integral is *over hurricane seasons only*, so the factor $\chi(t)$ is superfluous. Therefore, we can simplify this formula to

$$L_{\text{cat}} = \int_0^T \left\{ E\left( \min\left\{ \frac{V_1}{m_{\text{cat}}(t)}, 1 \right\} \right) - E\left( \min\left\{ \frac{V_0}{m_{\text{cat}}(t)}, 1 \right\} \right) \right\} \tilde{r} v(t) dt. \tag{22}$$

Note that the PML function $m_{\text{cat}}(t)$ has the form $m_{\text{cat}}(t) = \chi(t) m(t)$ for some function $m(t)$.

*Independent perils and exposures***.** Suppose the overall loss exposure is comprised of multiple, independent[12] exposures $\mathcal{E}_i$ with associated $m_i(t),\ E_i(x),\ r_i,$ and $P_i$. (The exposure rating curve $E_i(x)$ is parameterized by some $c_i$.) Then the layer price is calculated by adding the layer prices for each individual exposure:

$$L = \sum_i L_i = \sum_i \int_0^T \left\{ E_i\left( \min\left\{ \frac{V_1}{m_i(t)}, 1 \right\} \right) - E_i\left( \min\left\{ \frac{V_0}{m_i(t)}, 1 \right\} \right) \right\} r_i v(t) dt.$$

If $v(t)$ is symmetrical under a $180^\circ$ rotation, then

$$L = \frac{2}{VT} \sum_i P_i \int_0^T \left\{ E_i\left( \min\left\{ \frac{V_1}{m_i(t)}, 1 \right\} \right) - E_i\left( \min\left\{ \frac{V_0}{m_i(t)}, 1 \right\} \right) \right\} v(t) dt. \tag{23}$$

## 10. LOSS LIMITS

An insurance policy may have *loss limits* (particularly in the case of natural catastrophe perils) that cap the maximum payout for an event. Suppose there is a loss limit $Q < M$. (If $Q \geq M,$ the limit serves no purpose.) We may assume $V_1 \leq Q$. (If $V_1 > Q,$ we can simply redefine $V_1 = Q$.)

[12]*Independent* means that a loss cannot arise from more than one peril or exposure. The various phases of the project (contract works, testing, maintenance) are independent because they do not overlap. Similarly, natural catastrophe perils (earthquake, windstorm, flood) are independent of one another and independent of any other peril or exposure.

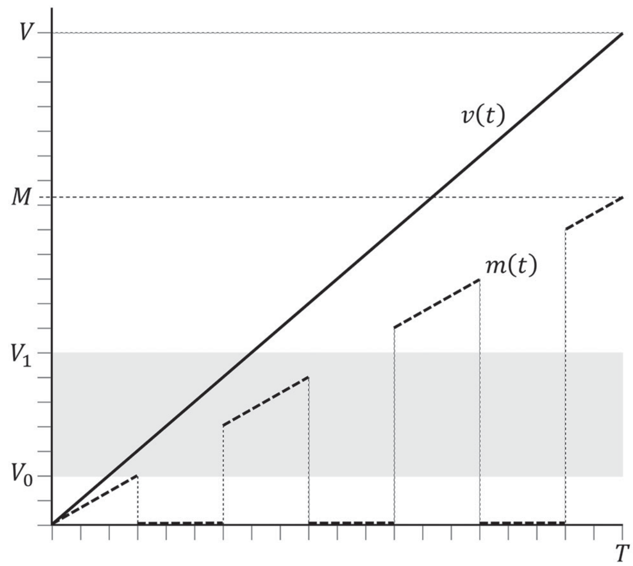


FIGURE 8. Example of a seasonal catastrophe exposure. *Note:* The PML $m(t)$ drops to zero outside the hurricane seasons.

The layer premium $L$ is *not affected by the loss limit*, since the layer lies below the limit ($V_1 \le Q$). However, the premium $\tilde{P}$ *with* the loss limit is not the same as the premium $P$ *without* the loss limit $p$ — it is naturally smaller. The underwriter can supply the actual policy premium $\tilde{P}$, but probably not $P$. Thus, the general formula of Equation (6) needs to be altered so that $\tilde{P}$ appears instead of $P$.

Consider the loss limit $Q$ as a primary layer, attracting premium $\tilde{P}$. Equation (7) for a primary $Q$ layer gives

$$\tilde{P} = \int_0^T E\left(\min\left\{\frac{Q}{m(t)}, 1\right\}\right) rv(t)dt \tag{24}$$

so

$$r = \tilde{P}\left\{\int_0^T E\left(\min\left\{\frac{Q}{m(t)}, 1\right\}\right) v(t)dt\right\}^{-1}.$$

Therefore, by Equation (5) we have

$$L = \frac{\tilde{P}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} v(t)dt}{\int_0^T E\left(\min\left\{\frac{Q}{m(t)}, 1\right\}\right) v(t)dt}. \tag{25}$$

## 11. DELAY IN STARTUP

Delay in startup (DSU) coverage indemnifies the project owner and financiers in the event that insured loss or damage causes partial or total delay in completion of the project. This insurance may cover only debt service and fixed operating costs, or may extend to include loss of profit.

As time advances during the period of construction, the *frequency and severity* of DSU claims generally increase. This is because of:

- Increasing value on site.
- Increasing scope and complexity of the project works.
- Emergence of a distinct critical path in the schedule, with various interdependences and "bottlenecks."
- Erosion or exhaustion of any buffer/margin built into the original schedule, due to inevitable, routine delays.
- Decreasing time remaining during which to make up lost time and reduce a delay.

This implies that *the premium density for DSU should increase with time.* DSU coverage has its own sum insured $\hat{V}$, but no "buildup of value" analogous to $v(t)$. (We will use a hat ˆ to denote quantities applying to DSU.) The full DSU sum insured is theoretically exposed at any time. However, we do not choose $\hat{v}(t) = \hat{V}$ (constant) for $0 \le t \le T$, since the resulting premium density $\hat{r}\hat{v}(t)$ would overweight premium in the early part in the period.

We make the simple (but not unreasonable) assumption that the premium density increases in proportion to the project value: $\hat{r}\hat{v}(t) \propto v(t)$. Since we assume $\hat{r}$ is constant, this means that $\hat{v}(t) = kv(t)$ for some constant of proportionality. Setting $t = T$ we find that

$$\hat{v}(t) = \frac{\hat{V}}{V} \cdot v(t). \tag{26}$$

With this choice, note that

$$\hat{r}\hat{v}(t) = \frac{\hat{P}\hat{v}(t)}{\int_0^T \hat{v}(t)dt} = \frac{\hat{P}v(t)}{\int_0^T v(t)dt} = \frac{\hat{P}rv(t)}{\int_0^T rv(t)dt} = \frac{\hat{P}}{P} \cdot rv(t) \tag{27}$$

which can be written symmetrically:

$$\frac{\hat{r}\hat{v}(t)}{\hat{P}} = \frac{rv(t)}{P}.$$

The layer price $\hat{L}$ for *standalone* DSU coverage follows immediately from Equation (5), using Equations (26) and (27):

$$\begin{aligned}\hat{L} &= \int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\}\hat{r}\hat{v}(t)dt \\ &= \frac{\hat{P}r}{P}\int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\}v(t)dt.\end{aligned}$$

If $v(t)$ is symmetrical under a 180° rotation, then $r = 2P/VT$, so

$$\hat{L} = \frac{2\hat{P}}{VT}\int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\}v(t)dt. \tag{28}$$

We must write $\hat{E}$, since the exposure rating curve for DSU is different than for property damage (PD). Usually, $\hat{c} < c$, since DSU has a higher proportion of large losses compared to PD; that is, $\hat{E}$ is one of the more diagonal (less concave) curves shown in Figure 2.

*Note on $\hat{m}(t)$:* An accurate PML[13] buildup curve is even more challenging to specify for DSU than for property damage, because DSU PML scenarios are difficult to determine and model. In the following discussion we leave $\hat{m}(t)$ arbitrary. However, we expect $\hat{m}(t)$ to be increasing, since the severity of DSU claims generally increases with time. One option is to choose $\hat{m}$ linear: $\hat{m}(t) = \hat{M}t/T$, where $\hat{M}$ is the maximum PML (usually, though not always, equal to $\hat{V}$). Another option is to choose $\hat{m}$ proportional to $m$ or $v$[14]:

$$\hat{m}(t) = \frac{\hat{M}}{M} \cdot m(t) \qquad \text{or} \qquad \hat{m}(t) = \frac{\hat{M}}{V} \cdot v(t).$$

Any of these assumptions may be used in Equation (28).[15]

## 12. COMBINED PROPERTY DAMAGE (PD) AND DELAY IN STARTUP (DSU) COVERAGE

As explained at the end of Section 9, when the overall loss exposure is comprised of independent exposures $\mathcal{E}_i$, the layer premium is calculated by adding the layer premiums for each individual exposure: $L = \sum_i L_i$. The DSU layer premium in Equation (28), however, *cannot be added to the PD layer premium* to arrive at the premium for combined PD and DSU

[13]In the case of delay in startup, the PML is sometimes called the *maximum probable delay* (MPD).
[14]These produce the same function $\hat{m}$ if $m \propto v$.
[15]If $m(t)$ is constant (e.g., for roads, railways, tunnels, etc.) then $\hat{m}(t) = (\hat{M}/M)m(t)$ would not be appropriate, because $\hat{m}(t)$ should still be increasing.

coverage. This is because a DSU claim necessarily occurs together with the PD claim that triggered the delay (they are *dependent* events). Both claims contribute to erosion of the attachment point and the layer.

Therefore, when pricing nonproportional insurance with PD and DSU coverage, we must consider that the layer is exposed to PD and DSU losses combined, not separately. Unfortunately, no simple combination of the layer prices calculated in Equations (5) and (28) can provide the correct price for combined PD and DSU coverage, since the time evolution of the values and PMLs must be considered.

We approach the problem by adding the PD and DSU exposures to create a *combined exposure* with probable maximum loss $m_+(t) = m(t) + \hat{m}(t)$. We also write $M_+ = M + \hat{M}$. The combined exposure has a loss distribution described by an exposure rating curve $E_+(x) = E_{c_+}(x)$, which lies between the curves $E(x) = E_c(x)$ and $\hat{E}(x) = E_{\hat{c}}(x)$. A simple way to select $c_+$ is to create a weighted average of $c$ and $\hat{c}$:

$$c_+ = c\left(\frac{M}{M+\hat{M}}\right) + \hat{c}\left(\frac{\hat{M}}{M+\hat{M}}\right) = \frac{cM + \hat{c}\hat{M}}{M_+}.$$

During the time interval $[t, t+dt]$ the layer attracts a portion $rv(t)dt$ of the PD premium and a portion $\hat{r}\hat{v}(t)dt$ of the DSU premium. Hence the layer premium $L_+$ for combined PD + DSU coverage is provided by Equation (5):

$$L_+ = \int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}(rv(t) + \hat{r}\hat{v}(t))dt. \tag{29}$$

Inserting Equation (27) into Equation (29) we obtain

$$L_+ = \frac{P_+ r}{P}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v(t)dt \tag{30}$$

where $P_+ \equiv P + \hat{P}$. Inserting Equation (1) for $r$ gives an equivalent, more symmetrical formula[16]:

$$L_+ = \frac{P_+}{\int_0^T v(t)dt}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v(t)dt. \tag{31}$$

If $v(t)$ is symmetrical under a 180° rotation, then Equation (30) or Equation (31) yields

$$L_+ = \frac{2P_+}{VT}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v(t)dt. \tag{32}$$

Note that Equation (32) reduces to Equation (5) in the case of standalone PD coverage ($P_+ = P,\ m_+ = m,\ E_+ = E$) and to Equation (28) in the case of standalone DSU coverage ($P_+ = \hat{P},\ m_+ = \hat{m},\ E_+ = \hat{E}$).

*Multiple independent exposures.* Suppose $\mathcal{E}_i$ are independent exposures with associated $m_i(t),\ E_i(x),\ r_i,\ P_i$ (for PD exposure), and $\hat{m}_i(t),\ \hat{E}_i(x),\ \hat{P}_i$ (for DSU exposure). The combined layer premium is calculated by adding the layer premiums for each PD + DSU exposure using Equation (30)[17]:

[16]Replacing $v(t)$ with any multiple $\alpha v(t)$ leaves Equation (31) unchanged, because the constants cancel in the two integrals. Therefore, we can replace $v(t)$ with $\left(1 + \hat{V}/V\right)v(t) = v(t) + \hat{v}(t) \equiv v_+(t)$ to obtain the perfectly symmetrical formula

$$L_+ = \frac{P_+}{\int_0^T v_+(t)dt}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v_+(t)dt\ .$$

[17]The underwriter may not know $P_{i+}$ (the combined PD + DSU premium for the exposure $\mathcal{E}_i$)! He should be able to supply $P_i$, but perhaps not $\hat{P}_i$ (which is needed to specify $P_{i+} = P_i + \hat{P}_i$), since DSU premiums are usually not broken down by phases and perils. It is not uncommon for DSU to be priced as a multiple of the PD price or in some other coarse fashion. If $\hat{P}_i$ is not provided, we can make the simple assumption that $\hat{P}_i = (P_i/P)\hat{P}$, and hence $P_{i+} = P_i + \hat{P}_i = P_i + (P_i/P)\hat{P} = P_i\left(1 + \hat{P}/P\right)$. The policy DSU premium $\hat{P}$ should be known.

$$L_+ = \sum_i L_{i+} = \sum_i \frac{P_{i+} r_i}{P_i} \int_0^T \left\{ E_{i+}\left( \min\left\{ \frac{V_1}{m_{i+}(t)}, 1 \right\} \right) - E_{i+}\left( \min\left\{ \frac{V_0}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt.$$

If $v(t)$ is symmetrical under a 180° rotation, then

$$L_+ = \frac{2}{VT} \sum_i P_{i+} \int_0^T \left\{ E_{i+}\left( \min\left\{ \frac{V_1}{m_{i+}(t)}, 1 \right\} \right) - E_{i+}\left( \min\left\{ \frac{V_0}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt.$$

*Loss limits with combined PD and DSU coverage.* Suppose an exposure $\mathcal{E}_i$ has a loss limit $Q$ that applies to the combined PD + DSU coverage. As noted in Section 10, we may assume without loss of generality that $V_1 \le Q < M_+$. Repeating the method of Section 10, we obtain the following obvious generalization of Equation (25):

$$L_{i+} = \frac{\tilde{P}_{i+} \int_0^T \left\{ E_{i+}\left( \min\left\{ \frac{V_1}{m_{i+}(t)}, 1 \right\} \right) - E_{i+}\left( \min\left\{ \frac{V_0}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt}{\int_0^T \left\{ E_{i+}\left( \min\left\{ \frac{Q}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt}. \tag{33}$$

## 13. CONCLUSION

This pricing model is founded on the simple utilization of industry standard exposure rating curves given by Equation (9). Straightforward algebra leads us to the general formula of Equation (5) for the allocation of premium to a specified layer, or Equation (6) in the case where the buildup of value $v(t)$ possesses the symmetry of an S-shaped curve. These formulas are well suited for computer programming to carry out the integrations. (The author participated in the creation of a working rating engine using only Microsoft Excel.)

Complications arise when the basic formula of Equation (5) is applied to realistic situations in which $r$ (the premium rate per unit time) and $E(x)$ (the exposure rating curve) are not constant. This is usually the case, since a construction project encompasses multiple phases (e.g., excavations, foundations, superstructure, testing, and commissioning) and is exposed to various perils.

Moreover, different coverages are characterized by different values of $r$ and exposure rating curves $E(x)$. A subtle complication arises because several different coverages may be triggered simultaneously by a single event and contribute jointly to the loss to a given layer. We examined the case of combined property damage and delay in startup (the most common situation); however, the same issue can arise with other coverages routinely included under a construction policy, such as third-party liability and contractors' plant and equipment.

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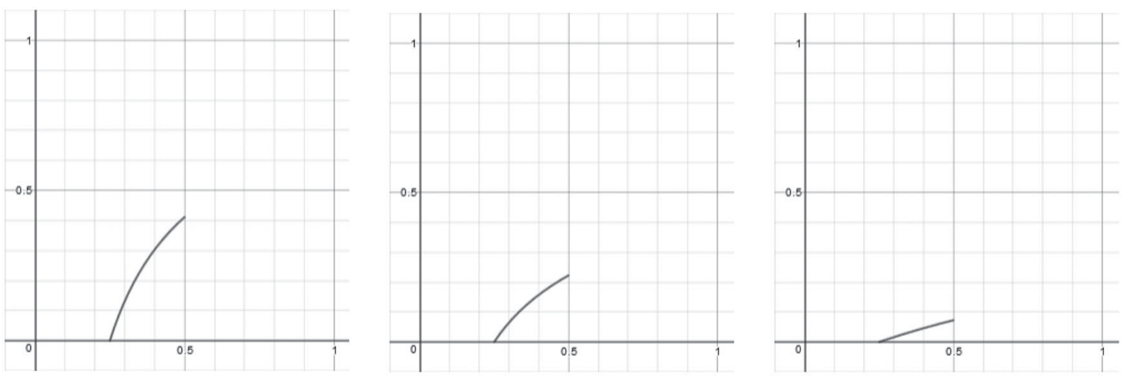
## APPENDIX. THE "ALLOCATION FACTOR" $\theta(t)$

In Section 6 we began a discussion of the "layer allocation factor"

$$\theta(t) = E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)$$
$$= \begin{cases} 0 & m(t) < V_0 & \text{(Region A)} \\ 1 - E\big(V_0/m(t)\big) & V_0 \le m(t) < V_1 & \text{(Region B)} \\ E\big(V_1/m(t)\big) - E\big(V_0/m(t)\big) & V_1 \le m(t) & \text{(Region C)} \end{cases}$$

We now examine $\theta(t)$ more carefully as $t$ advances. For the sake of simplicity, we assume that $m(t) = Mt$ is linear and take $T = 1$. Hence, we have




$c = 1$ $\qquad$ $c = 3$ $\qquad$ $c = 5$

FIGURE 9. Graphs showing the allocation factor $\theta(t)$ in region B. *Note:* Here $V_0/M = 0.25$ and $V_1/M = 0.5$, and the graph of $\theta(t)$ is shown for $c = 1, 3, 5$.

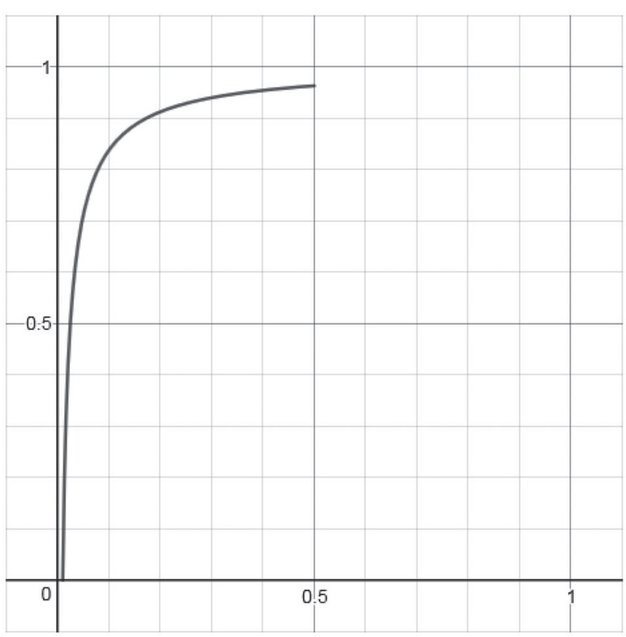



FIGURE 10. Graph of the allocation factor $\theta(t)$ in Region B when $V_0$ is close to zero. *Note:* Here $V_0/M = 0.01$, $V_1/M = 0.5$, and $c = 1$.

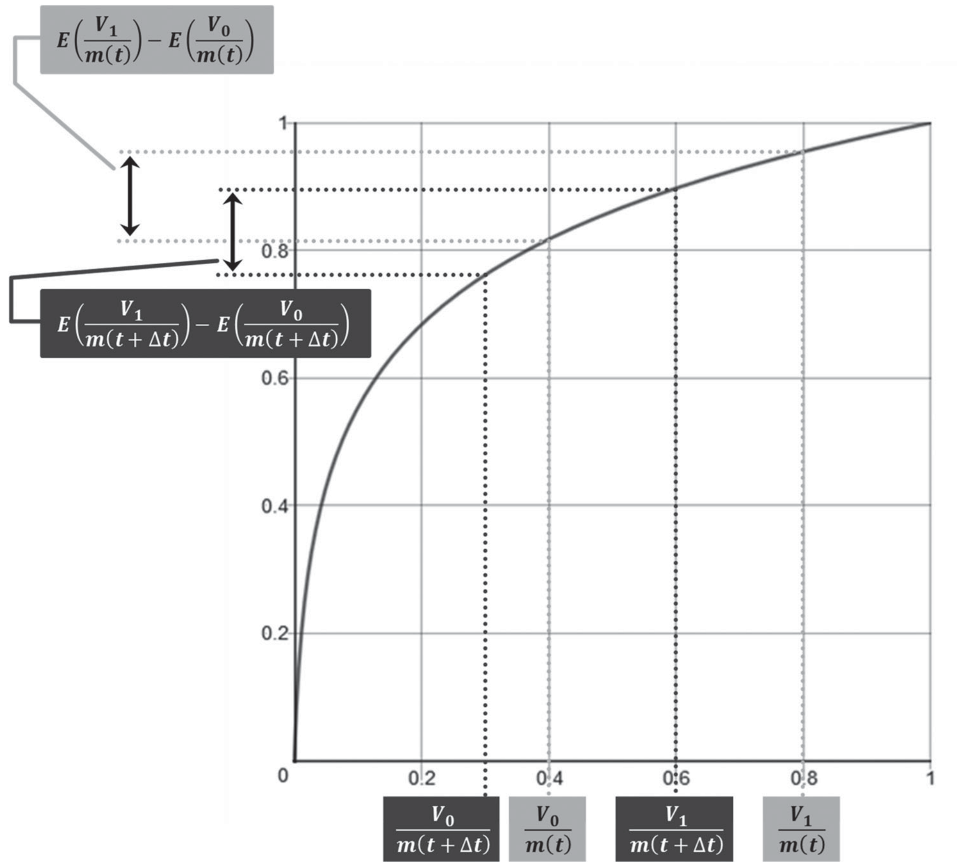



FIGURE 11. Graph illustrating the complexity of $\theta(t)$ in Region C where $\theta(t) = E(V_1/m(t)) - E(V_0/m(t))$. *Note:* Recall that $m(t)$ is the PML buildup curve, generally an increasing function.

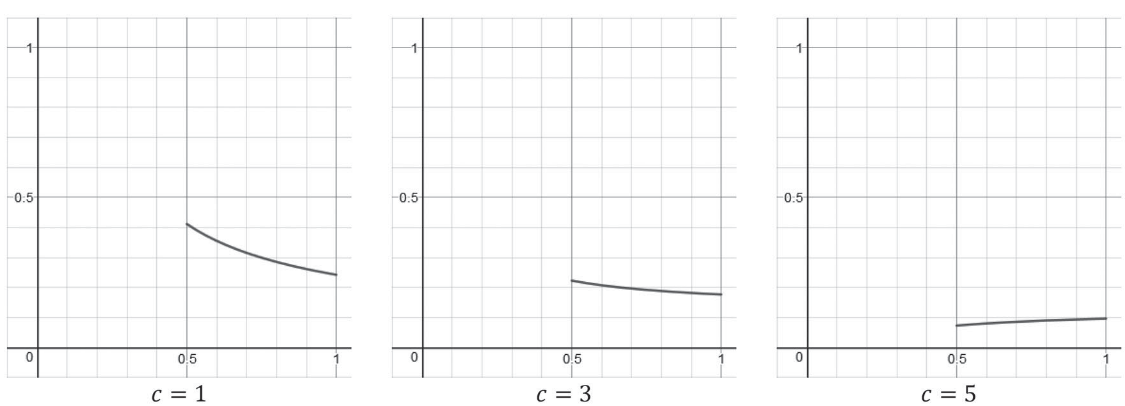



FIGURE 12. Graphs showing the allocation factor $\theta(t)$ in Region C. *Note:* Here $V_0/M = 0.25$ and $V_1/M = 0.5$, and the graph of $\theta(t)$ is shown for $c = 1$, 3, 5.

$$\theta(t) = \begin{cases} 0 & Mt < V_0 & \text{(Region A)} \\ 1 - E(V_0/Mt) & V_0 \le Mt < V_1 & \text{(Region B)} \\ E(V_1/Mt) - E(V_0/Mt) & V_1 \le Mt & \text{(Region C)} \end{cases}$$

*Region A.* In region A, $\theta(t) = 0$ so there is no contribution.

*Region B.* In region B, assuming $V_0 > 0$, the behavior of $\theta(t)$ is simple: $\theta(t)$ increases from 0 to $1 - E(V_0/V_1)$ as $Mt$ increases from $V_0$ to $V_1$. Figure 9 shows the graphs of $\theta(t)$ when $V_0/M = 0.25$ and $V_1/M = 0.5$, for various values of $c$.

The case $V_0 = 0$ (a pure primary layer) is a little anomalous. If $V_0 = 0$, then $\theta(t) \equiv 1$. Note that $\theta(t) \to 1$ as $V_0 \to 0$ for any $t > 0$, so $V_0 = 0$ is just the limiting case. This is illustrated in Figure 10, which shows $\theta(t)$ when $V_0$ is close to zero (here $V_0/M = 0.01$, $V_1/M = 0.5$, and $c = 1$).

*Region C.* The behavior of $\theta(t)$ in this region is somewhat subtle. As $t$ increases, both $E(V_1/Mt)$ and $E(V_0/Mt)$ are decreasing, so $\theta(t)$ is the difference between two decreasing numbers, as illustrated in Figure 11 (with $V_1 = 2V_0$).

The behavior of $\theta(t)$ is not obvious, even for simple $m(t)$ such as a straight line; it behaves in a surprisingly complex manner. $\theta(t)$ is rather sensitive to the values of $V_0/M$ and $V_1/M$, as well as the parameter $c$ (remember that $E(x) = E_c(x)$). It turns out that $\theta(t)$ is decreasing for $0 \le c < \left(\sqrt{753} - 3\right)/6 = 4.073\ldots$ and increasing for $c > 4.073\ldots$[18] Figure 12 shows the graphs of $\theta(t)$ for $V_0/M = 0.25$ and $V_1/M = 0.5$ for various values of $c$.

Note there is nothing anomalous about the case $V_0 = 0$ in Region C: if $V_0 = 0$, $\theta(t)$ decreases steadily from 1 to $E(V_1/M)$ as $Mt$ increases from $V_1$ to $M$.

When the behavior of $\theta(t)$ is examined under various conditions (choices of $V_0/M$, $V_1/M$, and $c$), we find that $\theta(t)$ may be increasing or decreasing in region C, but in most cases, it does not vary too greatly.

[18]This occurs because $\beta(c) = e^{3.1-0.15(1+c)c} = 1$ in Equation (9) when $c = \left(\sqrt{753} - 3\right)/6$.