A MATHEMATICAL APPROACH TO FIRE PROTECTION CLASSIFICATION RATES

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I. INTRODUCTION

A. *The Problem.*

The actuarial core of the fire protection classification rate relativity problem is the actuarial core of any fire rating problem: The fire rate structure must be (or, at least, for generations, by custom and usage, has been) refined far beyond the refinement of the fire statistical plan. Entirely apart from the detail of recently-publicized shortcomings of the current most widespread fire statistical plan, the National Board of Fire Underwriters *Standard Classification of Occupancy Hazards,*[1] further refinement of the statistical plan is no answer of itself because, very simply, of credibility considerations. A fact well known to any experienced fire ratemaker has been formalized by Dr. Almer in the statement: "Statistical experience proves that most claims in any branch [of nonlife insurance] will be concentrated in some few statistical risk groups (or tariff partitions), leaving most tariff groups without effective statistics, even if a five-year experience is utilized."[2]

Specifically in present instance, the actuarial problem is to support classification rates and rate relativities for as many as ten or more public fire protection classifications upon a statistical plan which, credibility considerations aside, spans the entire range of protection classifications with only two statistical classifications, "Protected" and "Unprotected." It is submitted that extension of theories already proposed[3,4] not only will permit a mathematical approach to this problem, but also leads to certain working formulas which are completely and immediately practical of application in cook-book fashion to save laborious trial-and-error calculation in rate revision operations.

B. *Fire Protection Classifications.*

In general, the relative efficacy of public fire defenses is evaluated for

[1] Among others, *The National Underwriter,* June 19, 1964, p. 2.

[2] Almer, (11). P. 341. (Bibliography is appended.)

[3] McIntosh, (14).

[4] McIntosh, (15). Specifically the section: *Variable Hazard,* p. 15.

rate making purposes by application of the National Board of Fire Underwriters *Standard Schedule for Grading Cities and Towns of the United States with reference to Their Fire Defenses and Physical Conditions.*[5] This document, seldom designated by its full official title, has been described elsewhere in some detail,[6] but one particular feature is pertinent to what follows here. Application of the *Standard Schedule* to the public fire defense facilities maintained by a given community does *not* produce a protection *classification* directly; it produces a protection *"grading,"* which subsequently is converted to a classification for rate making and underwriting purposes.

In the complete absence of public fire defenses recognizable as such, a maximum grading of 5000 "points of deficiency" is assessed. For recognizable fire defense facilities, the 5000-point maximum is reduced to some lesser figure, depending upon the detail of conditions found by inspection to exist. Theoretical perfection, never yet approached, would result in a point grading of zero. The protection grading actually assigned to any given community will be some number of points of deficiency from zero (theoretically) to 5000; the better the public fire defenses, the lower will be the deficiency-point total, or "grading."

The present significance of this fact is that the protection grading, although necessarily expressed in discrete units, the "points of deficiency," must be considered a *continuous variable.* Any grading from zero to 5000 is theoretically possible, although for practical reasons a grading of less than 1000 points is extremely difficult to achieve, and no city in the United States currently enjoys a grading of 500 points or less. In theory the fire rate must be a *continuous function* of this *continuous variable,* despite the fact that, for obvious reasons, it cannot be treated as such in practice.

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[5] A notable exception is found in the rating system of the State of Texas, whereunder public protection is evaluated by a very different approach. There are other minor exceptions.

[6] Riegel & Miller, (19). p. 564.

The conversion of *grading* to *classification* is illustrated in the table below.

Grading (Point Total)	N.B.F.U. Protection Class*	N.B.F.U. Statistical Class**
0 – 500	1	"Protected"
501 – 1000	2	" " "
1001 – 1500	3	" " "
1501 – 2000	4	" " "
2001 – 2500	5	" " "
2501 – 3000	6	" " "
3001 – 3500	7	" " "
3501 – 4000	8	" " "
4001 – 4500	9	"Unprotected"
4501 – 5000	10	" " "

* According to the N.B.F.U. *Standard Schedule.*
** According to the N.B.F.U. *Standard Classification of Occupancy Hazards.*

Two points should be noted for reference. First, mathematically speaking, the classification is a step function of the continuous grading, hence the rate as a function of classification becomes a step function of the grading. This represents the imposition of an artificially discrete mathematical model upon what in actuality is a continuum. The practical necessity of this departure from actuality is not questioned. Any rating system whereunder the rate must vary with variation of a single grading point anywhere in the 0-5000 range would be impossible of practical application, if for no reason other than that it would drive the ratemaker insane in very short order. But whatever the practical necessity, the artificiality of the model must be recognized, to focus attention upon the problem of just how great a departure from actuality can be tolerated before the inevitable and extremely practical consequences of the fact of the departure itself may become unacceptably severe. In other words, for ratemaking purposes, how refined should the protection classification system be to attain the maximum simplicity of practical operations consistent with avoidance of practical problems of unacceptable severity? The question is not academic.

Secondly, both the number of the classifications and the exact locations of interclass boundaries are arbitrary. Other classification systems can be, and in fact have been, formulated by subdividing the grading range into brackets differing markedly from those shown above. Provided, of course, that stability of the rate structure is not destroyed by too-frequent revision, there is absolutely nothing to prevent the fire ratemaker from establishing protection classifications in whatever number, with inter-class boundaries at whatever locations, may prove most expedient and appropriate to the problem at hand, so long as it is specified just which protection classes are "Protected" and which are "Unprotected" for statistical reporting purposes. The N.B.F.U. protection classifications tabulated above are a generally (but not universally) recognized standard of reference, but it is not unknown for a simplified, variant system to underlie Dwelling rates in the same jurisdiction wherein the N.B.F.U. classes may underlie commercial risk rates. At this writing a six-class system to underlie Dwelling fire rates has been recommended to all fire rating bureaus nationwide.[7]

The importance of these points, first, that any protection classification system is an artificial model and, secondly, that the detail of any such system is arbitrary, will be developed in Section IV. A.2, following.

C. *Designation of Classes.*

Three categories of classifications are involved in what follows. For present purposes, the term *"underwriting class"* will be used to designate either an occupancy class, *e.g.,* "Dwellings," "Metalworkers," etc., or a construction-occupancy class, *e.g.,* "Frame Dwellings," "Brick Metalworkers," etc. The present development is not concerned with relationships between underwriting classes, but only with certain relationships between sub-classes within any given underwriting class.

A *"statistical class,"* or a *"statistical sub-class"* of an underwriting class, will be that sub-class of the underwriting class upon which loss experience is reported separately as "Protected" or, alternatively, as "Unprotected."

A *"protection class"* is a sub-class either of the *"Protected"* or of the *"Unprotected"* statistical class. The precise definition of a given protection class must be in terms of grading-point brackets, as illustrated above, but there will be no occasion in what follows here to specify such brackets, nor even again to refer to the grading except in general terms in discussion of continuity. In particular instance, it must and will be specified which protection classes belong to the "Protected" statistical class,

[7] Memorandum: *Recommended Schedule of Fire Insurance Rates for Dwellings and Private Outbuildings Appertaining Thereto,* dated December 9, 1959. The Inter-Regional Insurance Conference (now known as The Fire Insurance Research and Actuarial Association). p. 3.

and which to the "Unprotected" statistical class. The term, "protected class" or "unprotected class," *without* capitalization or quotation marks refers to a protection class which is a member of the "Protected" or of the "Unprotected" *statistical* class. Protection classes will be specifically designated by number, and in particular instance the ranges of numbers assigned to protection classes belonging to the "Protected" and "Unprotected" statistical classes, respectively, will be specified. Invariably, the *lower* the numerical designation of a protection class, the *better* the quality of public fire protection associated therewith. *Higher* class numbers denote *inferior* protection.

D. *The Presentation.*

The development proper may be said to begin with the consideration of rate structures in Section IV. Sections II and III are concerned primarily with essential background material, definitions and notation. To support developments presented here, it has been necessary to reformulate in precise mathematical expression certain theoretical material previously presented by McIntosh in somewhat loose statement.[5]

If the working formulas of Section VI, dealing with practical applications, are accepted on faith, then Section VI (page ??) may be read independently of all else save only reference, as necessary, to definitions and notation to be found in Sections II – IV.

II. FIRE PROTECTION CLASSIFICATION RATES AND RATE RELATIVITIES

A. *The Fire Protection Classification Rate*

Fire rating terminology contains no exact equivalents of the casualty terms: "classification rate;" and, "classification rate relativity." Partly this is because terminology must be fitted in particular instance to the detail of a particular rating schedule, and the variations of detail among the several rating schedules in use are too great to permit any sort of standardized expression completely unambiguous out of context. More to the point, however, is the fact that a true "classification rate" is virtually unknown in fire. The fire "classification rate" will be an average rate in nearly all cases. Furthermore, the fire class average rate may reflect fortuitous variation of conditions of hazard completely extraneous to the particular hazards definitive of the class. For example, in a given state, the N.B.F.U. Class x "Mercantile Building" average rate may reflect a significantly disproportionate concentration in Class x of a particular

[5] McIntosh, (14).

type of construction not uniformly distributed among the several fire protection classes. The possible severity of such distortion is exemplified by the fact that, in the State of Louisiana, the average Mercantile Building rate of N.B.F.U. Class 9 is appreciably *higher* than the average rate of N.B.F.U. Class 10, although by provisions of the applicable schedule, the Class 9 Mercantile Building rate must be exactly 5% *lower* than the Class 10 rate wherever and whenever all hazard conditions other than the public protection are equal, whatever those extraneous conditions may be.

That such considerations apply not only to the fire "schedule" rate, but also to the fire "class" rate, may not be obvious. However, many states surcharge the Private Dwelling "class" rate for additional families in occupancy, and whether this be done by "schedule charge" or by separate basis rate tables seems a distinction of convenience without substance. In an actual case known to the author, a disproportionate concentration of multiple-family occupancy in N.B.F.U. Class 3 resulted in a distortion of Dwelling protection classification rate relativities of better than 10%, in any comparisons of Class 3 with other protection classes. If the so-called "loss constant rating method" is used, whereunder the "effective" rate becomes a function of policy size, the distribution of policy size among the several protection classes may not be, and in general will not be, uniform. Again using Louisiana as example, the mean "effective" Brick Dwelling Building rate of N.B.F.U. Class 3 is *lower* than that for Class 2, precisely because the average Dwelling policy size in the City of New Orleans (which dominates Class 3) is appreciably higher than the average policy size elsewhere in the State.

In any given instance, the variation of extraneous hazard conditions from class to protection class may be insignificant; or the variation may be of a nature such that it is not reflected in the rates produced by applicable schedule. Where this is the case, protection classification rate relativities may be determined by direct comparisons among the classification rates themselves. But in many instances direct interprotection-class rate comparison is useless for the purpose of determining the effect upon rate of public fire protection of itself and by itself. Another concept is needed.

For present purposes, the *"protection classification rate"* of Class x is defined to be the appropriately weighted average of individual rates respectively applicable to each of the several risks in Class x. (How this shall be determined in the case of existing rates is of no present concern.) It is these *rates* which *collectively* must be reconciled in the course of rate revision to statistical classification *premiums* developed by application of the rate level adjustment formulas independently of the actual rate revision calculations.

B. *The Fire Protection Classification Normal.*

To isolate for study the protection component of the protection classification rate, the *"protection classification normal"* of Class x in general is defined to be that value which the protection classification rate would have assumed had all extraneous conditions of hazard throughout Class x been identical to those actually existing throughout the highest-numbered class of the protection classification system, *except* when dealing with the "loss constant method" of rating private dwellings. In that particular case, it may be desirable to normalize the effective rate to the statewide mean policy size. The classification *normal* may be and should be conceived as the classification rate "normalized" to a standard set of extraneous conditions.

Choice of the highest-numbered protection class to be in general the standard-of-reference is not entirely arbitrary. Since the rate of this class reflects no recognition whatever of public protection, it already is self-decomposed into an extraneous component equal to the rate itself, and a protection component which (depending upon form of the calculation) will be zero in summations or unity as a factor in products. Entirely apart from any theoretical significance, the self-decomposition of the rate of the highest numbered class may prove extremely convenient in practical calculation involving certain rating schedules.

The difficulty of calculating the classification normals, once the classification rate of the highest-numbered class has been determined, will vary widely according to the detail of the rating schedule. In some cases, precise calculation may be tedious to a point of practical impossibility. In general, where accurate calculation is not practicable, at least acceptably accurate estimates can be made. It is here assumed that either accurate calculation or acceptably accurate estimate of normals can be made in all cases, given adequate data of field conditions which must be obtained in any case.

The *"rate-normal ratio"* is defined to be the quotient of the classification rate divided by the normal, or, alternatively, the reciprocal quotient, of normal divided by rate. Although separate notation for each of these reciprocals will prove convenient to avoid negative exponents, in general discussion there will be no need to distinguish between them, and the term "rate-normal ratio" will be applied indiscriminately to either.

Where distinction may be necessary in particular instance, it will be made by notation if not clear from context.

For what is to follow, it is not sufficient that the rate-normal ratio be obtainable by direct division of the normal into the rate, or *v.v.* Its purpose is to permit calculation of the rate from the normal, or *v.v.*, when *one only* of these quantities is known independently. As with the normal itself, the difficulty of calculating or estimating the rate-normal ratio solely upon the basis of schedule provisions and known field conditions, will vary widely from schedule to schedule. It must be assumed for what follows that calculation or acceptable estimate of the rate-normal ratios can be made by some method *other* than direct division between rate and normal developed independently of each other.

It is further assumed here that the rate-normal ratio will be a constant, characteristic of class and not necessarily the same for all protection classes. No generalizations can be made concerning special methods required when the rate-normal ratio becomes a function of the normal itself, except to say that in the author's experience graphical methods prove expedient and usually will yield satisfactory solutions.

C. *The Rate Revision Problem.*

The *"underwriting target rate"* is defined to be that rate which, if applied indiscriminately to each and every risk of the underwriting class ("Frame Dwelling," "Mercantile Building", etc., etc.), will produce the underwriting classification premium required by the rate level adjustment formulas. (Here assumed to have been pre-determined.)

The *"protected target rate"* is defined by analogy, with specific reference to the "Protected" statistical sub-class of the underwriting class. (Here assumed to have been pre-determined.)

The *"unprotected target rate"* is defined by analogy, with specific reference to the "Unprotected" statistical sub-class of the underwriting class. (Here assumed to have been pre-determined.)

The *"underwriting trial average"* is defined to be the average of the several protection classification rates for the underwriting class over the entire range of protection classifications. This average is to be weighted in accordance with that proportion attributable respectively to each protection class of the total amount of insurance written throughout the underwriting class.[9]

[9] In practice the exact distribution of liability among the several protection classes may not be ascertainable. In such instance, the distribution is approximated by the best available set of indices, *e.g.* risk count by class, etc.

The *"protected trial average"* is defined by analogy, with specific reference to the "Protected" statistical sub-class of the underwriting class.

The *"unprotected trial average"* is defined by analogy, with specific reference to the "Unprotected" statistical sub-class of the underwriting class.

From the foregoing definitions, it will follow by straightforward algebra (if not obvious) that a given set of protection classification rates will produce upon field application the required classification premiums if and only if the trial averages produced by those rates are respectively equal to the corresponding target rates. The problem of developing adjusted protection classification rates which will produce required underwriting and statistical classification premiums thus resolves itself into the problem of developing adjusted protection classification rates which will produce trial averages equal to pre-determined target rates.

Where only the underwriting target rate is specified, and where pre-existing protection classification rate relativities are to be left undisturbed, the immediate solution is, of course, simply to multiply all existing protection classification rates by the percentage quotient of the required underwriting classification premium divided by the most recently available reported classification premium. However, if separately-specified protected and unprotected targets require respective adjustments in differing percentages, simple multiplication of the protected and unprotected rates by the respectively indicated percentage factors will distort relativities, and may produce inversions such that the rates in a given community will *decrease* if the fire department is disbanded and the fire engines are sold for scrap.[10] In any case, simple multiplication of all existing rates by a constant percentage factor is inappropriate where for any reason the protection classification relativities are to be revised regardless of any premium adjustment. Where uniform percentage adjustment of all protection classification rates is inappropriate, solutions may be obtained by trial and error.

There are less-tedious methods.

D. *General Notation.*

The following general notation will be used throughout what follows, excepting only where superseded by special notation to be defined when introduced.

[10] McIntosh, (16). Specifically the section: *Rate Adjustment*, p. 131. The principle here involved is not restricted to protection classification rates, but is completely general in application whenever related classes are to be adjusted.

Let:

x: – The class number designating a particular protection class: "Class x."[11]

a: – The highest class number assigned to a protected class. To be specified in particular instance.

β: – The lowest class number assigned to an unprotected class. To be specified in particular instance.

z: – The highest class number assigned under the protection classification system. To be specified in particular instance.[12]

R_x: – The protection classification rate of Class x.

Q_x: – The protection classification normal of Class x. (By definition of Q_x; then: $Q_z \equiv R_z$, *except* when dealing with the "loss constant rating method." Choice of "Q_z" *vs.* "R_z" as appropriate to immediate context.)

$\left.\begin{array}{l} q_x = Q_x/R_x \\ r_x = R_x/Q_x \end{array}\right\}$: – The rate-normal ratio of Class x. (Choice of "r_x" *vs.* "q_x" as convenient. By definition of Q_x; then: $r_z = q_z \equiv 1$, *except* when dealing with the "loss constant rating method.")

$\left.\begin{array}{l} \sum_T = \sum_{x=1}^{z} \\ \sum_P = \sum_{x=1}^{a} \\ \sum_U = \sum_{x=\beta}^{z} \end{array}\right\}$: – For reasons of convenience to become apparent.

T: – The underwriting target rate.

P: – The protected target rate.

U: – The unprotected target rate.

v_x: – The pro-rata portion attributable to Class x of the total amount of insurance written throughout the underwriting class. $\sum_T v_x = 1$

$v_P := \sum_P v_x$; $v_U = \sum_U v_x$

[11] Under some classification systems, the several classes are lettered rather than numbered, but for what follows it is necessary that numbers replace any non-numerical class designations.

[12] For consistency, the Greek omega, "ω", probably should be used here, but this is avoided because of the typographical similarity of "ω" to Roman "w", frequently used in what follows.

$$w_x := \begin{cases} v_x/v_P; \text{ if } 1 \leq x \leq \alpha. & \text{Then: } \sum_P w_x = 1 \\ v_x/v_U; \text{ if } \beta \leq x \leq z. & \text{Then: } \sum_U w_x = 1 \end{cases}$$

$\hat{T} = \sum_T v_x R_x$: – The underwriting trial average rate.

$\hat{P} = \sum_P w_x R_x$: – The protected trial average rate.

$\hat{U} = \sum_U w_x R_x$: – The unprotected trial average rate.

$\hat{T}_Q = \sum_T v_x Q_x$: – The unprotected trial average normal.

$\hat{P}_Q = \sum_P w_x Q_x$: – The protected trial average normal.

$\hat{U}_Q = \sum_U w_x Q_x$: – The unprotected trial average normal.

Rate notation as given above invariably refers to the *"adjusted"* rates, *i.e.* those to be placed into effect upon completion of rate revision calculations. Corresponding notation with reference to the *"existing"* rates in effect immediately prior to rate revision is obtained by superscript, thus:

"R_x^e"; "Q_x^e"; "T^e"; "P^e"; "U^e". (See Section IV.B., following.)

$\overline{Y}(\underline{Y})$: – The maximum (minimum) value of whichever of the foregoing quantities (except "x") may replace "Y", *e.g.* $\overline{R}_x$ ($\underline{R}_x$).

III. RATE VECTORS; PROTECTION CURVES

A. *Sets and Vectors.*

That highly useful concept which permeates the structure of modern mathematics, and which a friend of the author has christened, "The Great God, Set", in impious reference to the fraticidal villain of the Pharaonic pantheon,[13] appears to be the mathematical key to the fire rating problem, just as already it has proved the key to other problems long considered invulnerable to systematic, mathematical attack. In simpler applications, *e.g.* the solution of simultaneous linear equations, the villain need not be formally identified.[14] As the problem becomes more complex, a point is reached where either he must stand forth in his own true shape, or else the development at best becomes interminably tedious and at worst becomes sheer impossibility. It is suggested that the critical point already may have been passed in fire rating theory.[15] In any event, it will be reached here.

A completed jigsaw puzzle presents a picture not inherent to any single one of its pieces, nor even collectively inherent to all of its pieces

[13] Among others, Müller, (17). p. 114.

[14] But see, for example, Kemeny *et al.*, (6). Ch. 4, Sect. 3, p. 223, for a set-theoretic approach to simultaneous linear equations.

[15] *Cf.* McIntosh, (14).

except when these are arranged in particular relationships to each other. It is *not* the exact value individually assigned to any one protection classification rate, R_x, nor even the *combination* of values assigned respectively to each protection classification rate, which will produce required underwriting classification premiums. It is only by the assignment to each protection classification rate, respectively and *in order*, of the values represented in some *permutation* of some combination of possible values that the rate structure can be reconciled to a premium structure independently pre-determined. Except in special instance, the appropriate permutation will be unique to any given combination, and except in trivial instance[16] the choice of appropriate combinations will *not* be unbounded.

Two puzzles are readily identified and distinguished from each other by reference to the one as, *e.g.*, "the ship picture," and to the other as, *e.g.*, "the horse picture." Equally unambiguous identification and distinction by meticulously cataloging the shape, size, coloration and place in the pattern of each individual piece of each respective puzzle, will prove an endless and fruitless task with any but the simplest of those puzzles designed for amusement of the pre-school-age toddler. A vector exhibits a particular permutation of a particular combination of values. A pair of ordinary Cartesian coordinates, (a,b), which is a very simple vector, does not represent the same point as the pair (b,a) unless it happens that $a = b$ under all possible circumstance. When the vector itself is identified, there is no need to catalogue the individual components, and the latter task may prove quite a chore when these components must be treated as variables to be subsequently evaluated.

Finally, when a jigsaw puzzle must be moved, it is easier and quicker to move it assembled upon a biscuit board than to carry it piece by piece across the room. There will be no need for laborious re-assembly to re-form the picture; and there is no chance of a piece being accidentally dropped in transit, to be unintentionally kicked out of sight under the sofa.

A fire rate structure expressed in terms of rate vectors is easily transformed mathematically from what it is to what it should be. Systematic

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[16] With highly specialized underwriting classes, it may happen that in a given territory no risks in class will exist, yet a rate structure for the underwriting class may be desired either for the sake of formal completion of a comprehensive rating schedule, or in anticipation of future establishment of risks in class within the territory. In such cases, normally the ratemaker will incorporate into the schedules the rate levels of other states where the class is represented, but obviously there are no bounds to his judgment in this instance.

mathematical approach to the fire rating problem on any basis other than in terms of rate vectors seems impossible.

B. *Rate Vectors. Notation and Definitions.*

(Superscript Convention: – The convention of tensor notation, omission of the parentheses distinguishing a superscript index, "$R^{(i)}$", will be followed as a matter of convenience. If it is remembered that, throughout what follows, a letter superscript is an index, *not* an exponent, there will be no occasion for confusion.)

$\boldsymbol{R}^i = (R_1{}^i, R_2{}^i, \ldots, R_a{}^i; R_\beta{}^i, \ldots, R_z{}^i)$: – *A rate vector.* The superscript identifies the classification rate, $R_x{}^i$, as a component of the vector, $\boldsymbol{R}^i$. The superscript does *not* designate a predetermined value of R_x. Also, it is quite possible that for some x, then: $R_x{}^i = R_x{}^j$, where $\boldsymbol{R}^i \neq \boldsymbol{R}^j$. It will be true that $\boldsymbol{R}^i = \boldsymbol{R}^j$ only if $R_x{}^i = R_x{}^j$ for *all* x. The semicolon indicates the break between the protected and the unprotected classifications.

$\boldsymbol{R}^{P:j} = (R_1{}^j, \ldots, R_a{}^j; O, \ldots, O)$: – *A protected rate vector.* The number of terminal, zero components equals $z - a$ (except as specified later).

$\boldsymbol{R}^{U:k} = (O, \ldots, O; R_\beta{}^k, \ldots, R_z{}^k)$: – *An unprotected rate vector.* The number of initial, zero components equals a (except as specified later).

$\boldsymbol{R}_P{}^i = (R_1{}^i, \ldots, R_a{}^i; R_\beta{}^i, \ldots, R_z{}^i)$: – A "*P-reconciled*" rate vector, such that: $\sum_P w_x R_x{}^i = \hat{P}^i = P$; *but:* $\sum_U w_x R_x{}^i = \hat{U}^i \neq U$.

$\boldsymbol{R}_P{}^{P:i} = (R_1{}^i, \ldots, R_a{}^i; O, \ldots, O)$: – A P-reconciled protected rate vector.

$\boldsymbol{R}_U{}^j = (R_1{}^j, \ldots, R_a{}^j; \ R_\beta{}^j, \ldots, R_z{}^j)$: – A "$U$-reconciled rate vector, such that: $\sum_U w_x R_x{}^j = \hat{U}^j = U$; *but:* $\sum_P w_x R_x{}^j = \hat{P}^j \neq P$.

$\boldsymbol{R}_U{}^{U:j} = (O, \ldots, O; R_\beta{}^j, \ldots, R_z{}^j)$: – A U-reconciled unprotected rate vector.

$\boldsymbol{R}_T{}^i = (R_j{}^i, \ldots, R_a{}^i; R_\beta{}^i, \ldots, R_z{}^i)$: – A "*T-reconciled*" rate vector, such that: $\sum_T v_x R_x{}^i = \hat{T}^i = T$; *but:* $\sum_P w_x R_x{}^i = \hat{P}^i \neq P$; *and:* $\sum_U w_x R_x{}^i = \hat{U}^i \neq U$.

$\tilde{\boldsymbol{R}}^i = (R_1{}^i, \ldots, R_a{}^i; R_\beta{}^i, \ldots, R_z{}^i)$: – A "*feasible*" rate vector, such that: $\sum_P w_x R_x = \hat{P}^i = P$; *and;* $\sum_U w_x R_x = \hat{U}^i = U$.

It follows from definitions that $T = v_P P + v_U U$; whence, if $\tilde{\boldsymbol{R}}^i$ is feasible, as above, then also: $\sum_T v_x R_x = \hat{T}^i = T$.

(a) $\tilde{\boldsymbol{R}}^i = \boldsymbol{R}_P^{P:j} + \boldsymbol{R}_U^{U:k} = (R_1^j, \ldots, R_a^j;\ R_\beta^k, \ldots,\ R_z^k)$; where possibly but not necessarily: $i = j = k$. If, in the middle member, either the protected vector is *not* P-reconciled; *or,* the unprotected vector is *not* U-reconciled, then $\boldsymbol{R}^i$ in the left member is *not* feasible.

Note that the individual component rates do *not* carry the reconciliation subscript, *"P"*, *"U"* or *"T"*, or the "feasible" tilde, "$\sim$". It is the vector, *as a vector,* which is reconciled, *not* the individual component rates. If $\boldsymbol{R}^i$ is feasible, $\tilde{\boldsymbol{R}}^i$, and $\boldsymbol{R}^j$ is feasible, $\tilde{\boldsymbol{R}}^j$; then $\tilde{\boldsymbol{R}}^k = \boldsymbol{R}_P^{P:i} + \boldsymbol{R}_U^{U:j}$ will also be feasible, $\tilde{\boldsymbol{R}}^k$. But the vectors:

$$\boldsymbol{R}^l = \boldsymbol{R}_P^{P:i} + \boldsymbol{R}^{U:l} = (R_1^i, \ldots, R_a^i;\ R_\beta^j, \ldots, R_\mu^i, \ldots, R_z^j) \neq \tilde{\boldsymbol{R}}^l$$

and:

$$\boldsymbol{R}^m = \boldsymbol{R}^{P:m} + \boldsymbol{R}_U^{U:j} = (R_1^i, \ldots, R_\xi^j, \ldots R_a^i;\ R_\beta^j, \ldots, R_z^j) \neq \tilde{\boldsymbol{R}}^m$$

will *not* be feasible except possibly in special cases, although every individual component, R_x^i or R_x^j, of $\boldsymbol{R}^l$ and $\boldsymbol{R}^m$ appears also as a component of one or the other of the feasible vectors $\tilde{\boldsymbol{R}}^i$ and $\tilde{\boldsymbol{R}}^j$.

$\boldsymbol{Q}^i = (Q_1^i, \ldots, Q_a^i; Q_\beta^i; \ldots, Q_z^i)$: – A *normal vector.*

$\boldsymbol{Q}^{P:j}; \boldsymbol{Q}^{U:k}$: — A *protected normal vector;* an *unprotected normal vector.* Definitions by analogy to definitions of $\boldsymbol{R}^{P:j}$ and $\boldsymbol{R}^{U:k}$.

$\boldsymbol{Q}^i \overset{r}{\longleftrightarrow} \boldsymbol{R}^i$: – For all x; then $R_x^i = r_x Q_x^i$. Then $\boldsymbol{Q}^i$ *"underlies"* its *"resting"* vector, $\boldsymbol{R}^i$. If $\boldsymbol{R}^i$ is T- P- or U-reconciled, $\boldsymbol{R}_T^i$, etc.; or is feasible, $\tilde{\boldsymbol{R}}^i$, then $\boldsymbol{Q}^i$ is reconciled, $\boldsymbol{Q}_T^i$, etc.; or feasible, $\tilde{\boldsymbol{Q}}^i$, accordingly.

$\boldsymbol{R}^i \overset{q}{\longleftrightarrow} \boldsymbol{Q}^i$: – For all x; then $\boldsymbol{Q}_x^i = q_x R_x^i$. Then $\boldsymbol{R}^i$ *"rests upon"* its *"underlying"* vector, $\boldsymbol{Q}^i$.

C. *Protection Curves.*

A *"protection curve"* is either a rate curve or a normal curve.

A *"rate curve"* is any smoothly continuous curve passing through the plot of the component rates of a rate vector plotted against class number.

A *"normal curve"* is a smoothly continuous curve passing through the plot of the component normals of a normal vector; provided that the slope of a normal curve must be non-negative throughout the interval $1 \leq x \leq z$.

The reflection in protection classification rates of variation of extraneous hazard conditions may produce negative slope to the rate curve over part of its length. Negative slope to the normal curve indicates *increase* of rate with *improvement* of protection, or *v.v.*, throughout the interval of the protection grading where the negative slope occurs. Remembering that in theory, the rate is a continuous function of grading, which is a continuous variable (see Section I, preceding), this represents a logically indefensible violation of consistency which may result in $Q_\mu < Q_\zeta$, where $\mu > \zeta$, under the classification system currently in use, and is certain to result in $Q_\mu < Q_\zeta$ where $\mu > \zeta$ under some classification system possible of adoption.

A protection curve uniquely "determines" its "defining" vector. A rate vector or normal vector does not define a unique protection curve. In the absence of further specification, the vector defines an entire family of curves, but this is of no practical consequence. The French curves and ships' curves of Mr. Carlson's nostalgic reference[17] are still very much in evidence upon the fire ratemaker's desk. He is sufficiently calloused to the implications to lose no sleep over the fact that a particular squiggle which gives him an appropriate rate pattern will have an infinite number of siblings, any of whom would be equally obliging.

IV. RATE STRUCTURES

A. *Adjusted Rate Structures.*[18]

1. *The Feasible Adjusted Rate Structure.*

The *"feasible adjusted rate structure"*, $\{\tilde{\boldsymbol{R}}\}$, is the set of all feasible adjusted rate vectors. It is completely bounded.

Let:

$$\dot{\boldsymbol{R}}_1{}^P = (P/w_1, \ldots, O;O, \ldots, O) = (\dot{R}_1, \ldots, O;O, \ldots, O)$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$
$$\dot{\boldsymbol{R}}_\xi{}^P = (O, \ldots, P/w_\xi, \ldots, O;O, \ldots, O) = (O, \ldots, \dot{R}_\xi, \ldots, O;O, \ldots, O)$$
$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

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[17] Carlson, (13). p. 76.

[18] See APPENDIX A for development of equations presented below without proof, and for further discussion of the concepts summarized below. For the practical significance of these concepts, in addition to APPENDIX A, see also Section VI.D, to follow.

$$\left.\begin{array}{l}\dot{\boldsymbol{R}}_{\alpha}{}^{P} = (O, \ldots, P/w_{\alpha}; O, \ldots, O) = (O, \ldots, \dot{R}_{\alpha}; O, \ldots, O) \\ \dot{\boldsymbol{R}}_{\beta}{}^{U} = (O, \ldots, O; U/w_{\beta}, \ldots, O) = (O, \ldots, O; \dot{R}_{\beta}, \ldots, O)\end{array}\right\}; (\beta \equiv \alpha + 1)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\dot{\boldsymbol{R}}_{\mu}{}^{U} = (O, \ldots, O; O, \ldots, U/w_{\mu}, \ldots, O) = (O, \ldots, O; O, \ldots, \dot{R}_{\mu}, \ldots, O)$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\dot{\boldsymbol{R}}_{z}{}^{U} = (O, \ldots, O; O, \ldots, U/w_{z}) = (O, \ldots, O; O, \ldots, \dot{R}_{z})$$

Then the feasible adjusted rate structure will be formally defined: If $\boldsymbol{R}^{i}$ is feasible, $\tilde{\boldsymbol{R}}^{i}$ (*i.e.*, R^{i} is an element of $\{\tilde{\boldsymbol{R}}\}$), then necessarily:[19]

$$\tilde{\boldsymbol{R}}^{i} = \boldsymbol{R}_{P}{}^{P:j} + \boldsymbol{R}_{U}{}^{U:k} = \textstyle\sum_{P} a_{x}{}^{P:j} \dot{\boldsymbol{R}}_{x}{}^{P} + \sum_{U} a_{x}{}^{U:k} \dot{\boldsymbol{R}}_{x}{}^{U} \tag{1}$$

$$(a_{x}{}^{P:j} \geq O; \textstyle\sum_{P} a_{x}{}^{P:j} = 1. \quad a_{x}{}^{U:k} \geq O; \sum_{U} a_{x}{}^{U:k} = 1)$$

(Possibly, not necessarily: $i = j = k$)

and the components of $\tilde{\boldsymbol{R}}^{i}$ will be given by:

$$\begin{array}{l} R_{x}{}^{i} = a_{x}{}^{P:j} \dot{R}_{x} = a_{x}{}^{P:j} P/w_{x}; \quad (x = 1, \ldots, \alpha.) \\ R_{x}{}^{i} = a_{x}{}^{U:k} \dot{R}_{x} = a_{x}{}^{U:k} U/w_{x}; \quad (x = \beta, \ldots, z.) \end{array} \tag{1.a}$$

By implications of definitions given, $P > O$, $U > O$; and for all x, then $w_{x} \geqslant O$ and $R_{x}{}^{i} \geqslant O$. By hypothesis, henceforth for all x, then $w_{x} > O$ in all equations presented. If, for any x, then $w_{x} = O$, *i.e.* if no insurance is written in Class x,[20] then the class must be dropped from all calculation, and the rate must be established by judgment alone, with reference to the rates of other classes. In consequence, $\dot{R}_{x} > O$ for all x. Therefore, for all x, the coefficients, $a_{x}^{:\cdot}$ of Eq.(1) must be non-negative to avoid $R_{x}{}^{i} < O$ for some x.

The restriction that the two sets of coefficients, $\{a_{x}{}^{P:j}\}$ and $\{a_{x}{}^{U:k}\}$, each sum to unity is justified in Appendix A. For the moment it may be noted that since by Eqs. (1.a):

$$w_{x} R_{x}{}^{i} = a_{x}{}^{P:j} P; \; (x = 1, 2, \ldots, \alpha)$$

and:

$$w_{x} R_{x}{}^{i} = a_{x}{}^{U:j} U; \; (x = \beta, \ldots, z)$$

then the summation-to-unity restriction on $a_{x}^{:\cdot}$ is sufficient to insure that $\tilde{\boldsymbol{R}}^{i}$ will be feasible.

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[19] *Cf. McIntosh*, (14). p. 151, Eq. (8)

[20] If, in a given state, no community is classified as Class x, or if no risks of a given underwriting class are found in any Class x community, then $w_{x} = O$. At this writing, no N.B.F.U. Class 1 city exists in the United States. (See Section VI, to follow.)

Obviously, however, not all vectors possible of calculation by Eq.(1) will be acceptable solutions to the rate revision problem. To begin with, by Eq.(1.a), for one or more x, it is possible that for some i, then $R_x{}^i = O$, which is an absurdity in practice. Secondly, rate inversions may be produced, *i.e.* it would possible for a given community to suffer *increase* in rates solely by virtue of *improvement* in its fire defenses, or *v.v.*, which again is an absurdity. As noted above in Section II.B, reflection in the rate, R_x, of a disproportionate concentration in Class x of extraneous hazards may properly result in $R_x > R_{x+1}$, but in considering the normals, the condition that $Q_x > Q_{x+1}$ constitutes a serious violation of consistency by the implication therein that *improvement* of protection will *increase* loss expectation, or *v.v.* For many vectors calculable by Eq.(1), it will happen that $R_x/r_x = Q_x > Q_{x+1} = R_{x+1}/r_{x+1}$.

The virtue of defining $\{\tilde{\boldsymbol{R}}\}$ and of formulating Eq.(1) is to establish a basis for further development.

2. *The Operational Adjusted Rate Structure.*

Consistency, as above, requires only that for all x, then $Q_x \leq Q_{x+1}$, but if $Q_x = Q_{x+1}$ a triviality results. In such a case, Class x and Class $(x + 1)$ should be consolidated into a single class. Therefore, the consistency requirement, that $Q_x \leq Q_{x+1}$, properly may be and should be modified by hypothesis to the strict inequality, $Q_x < Q_{x+1}$, but even this is not sufficient. It has been noted in Section I.B. that the adoption of any protection classification system constitutes imposition of an arbitrarily discrete model upon an actual continuum, which leads to the question of inter-class differentials. In theory the model is inappropriate, whence it follows that the results of application of the model will be inaccurate. Mr. Pruitt's statement that: "*** in this area, as in so many others, simplicity and accuracy are mutually antagonistic. To the degree that we require a mathematical and clearly defined accuracy, we must perforce sacrifice simplicity and ease of operation,"[21] seems entirely appropriate here, although the original context is presently irrelevant. The question, very simply, is: How great a departure from "mathematical and clearly defined accuracy" can be tolerated for the sake of "simplicity and ease of operation"?

If the classification normal relativity Q_x/Q_{x+1}, between adjacent normals, is trivial, the rate structure becomes unnecessarily complex.[22] At

[21] Pruitt, (18). p. 154

[22] It should be noted, however, that in considering elements of the fire rate other than reflection of public protection, it may become necessary to retain trivial rate differ-

the other extreme, if Q_x/Q_{x+1} is excessive, the result well may be Mr. Pruitt's "horse and rabbit stew – 'one rabbit, one horse' ";[23] in any case, the spectrum of hazard within the individual class will be so broad as to constitute open invitation to rate deviation and cream-skimming.[24] There also may be other extremely practical complications of a kind such that some rating jurisdictions on occasion have refined the protection classification system by insertion of additional classes when excessive differences could be reduced in no other fashion. Exact figures at which Q_x/Q_{x+1} passes from "reasonable" to "trivial," or, alternatively, to "excessive" cannot be specified; nevertheless it seems necessary, and on occasion has proved necessary, to establish bounds to Q_x/Q_{x+1}.

Strictly speaking, excess or triviality in this regard cannot be judged on the basis of the ratio Q_x/Q_{x+1} alone. The ratio Q_x/Q_{x+1} and the value of the difference, $Q_{x+1} - Q_x$, must be examined together, for all practical purposes.[25] To incorporate simultaneous consideration of Q_x/Q_{x+1} and $Q_{x+1} - Q_x$ into what follows here, however, would require that the bounds to be hypothecated as applicable to Q_x/Q_{x+1} be made functions of Q_{x+1}, which in turn would materially complicate the development to no good purpose. In practical operations, fore-knowledge of the general level of rates to be obtained (though not, of course, of the exact values) normally allows the ratemaker to estimate ratios which will produce reasonable differences, or *v.v.* if he prefers. What follows in terms of Q_x/Q_{x+1} could have been developed in terms of $Q_{x+1} - Q_x$, though obviously the form of the development would have differed.

To exclude from the rate structure values of Q_x/Q_{x+1} either excessive or trivial, let $c_x = Q_x/Q_{x+1}$, and let the constraint, $O < \underline{c}_x \le c_x \le \overline{c}_x < 1$, be introduced into the calculation. By definition of the rate-normal ratio,

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ences to avoid violation of consistency. The ultimate cause of such circumstances is the fact that the contribution of a given hazard to the total expectation of loss will vary according to the presence or absence of other given hazards. *Cf.* McIntosh, (14), p. 152; also (16), p. 118 *ff.* (The solution given in the latter reference is an alternative to retention of a trivial differential, but is not always practicable.)

[23] Pruitt, (18). p. 153

[24] To untangle the metaphor, cream the rabbit.

[25] For example, the author once was involved in a rather heated controversy with the officials of a certain municipality over the question of whether or not a rate reduction of \$0.20 per \$1,000 of insurance was an insultingly "trivial" return for money spent by the city to improve its protection classification, although the flat sum amounted to 11% of the pre-existing rate of \$1.80, a percentage normally considered quite reasonable.

r_x, it then follows that since f_x is defined by $f_x = R_x/R_{x+1}$, then $O < \underline{f}_x \leq f_x \leq \overline{f}_x < r_x/r_{x+1}$, by the equation:

(2) $$\underline{f}_x = \underline{c}_x r_x/r_{x+1}; \text{ and: } \overline{f}_x = \overline{c}_x r_x/r_{x+1}$$

The hypothesis of bounds then may be stated completely as the constraint:

(I) $$O < \underline{c}_x \leq c_x \leq \overline{c}_x < 1; \text{ and: } O < \underline{f}_x \leq f_x \leq \overline{f}_x < r_x/r_{x+1}$$

It should be noted that Constraint (I) also implies that $Q_x < Q_{x+1}$ and that $R_x > O$, as required.

The *"operational adjusted rate structure"*, $op\{\tilde{\boldsymbol{R}}\}$, now may be defined informally as the set of all feasible vectors whose component rates may be appropriate for application, and may be defined formally as a proper subset of the feasible rate structure $\{\tilde{\boldsymbol{R}}\}$ such that if $\boldsymbol{R}^i$ is a member of $op\{\tilde{\boldsymbol{R}}\}$, then:

(3) $$op\tilde{\boldsymbol{R}}^i = op\boldsymbol{R}_P{}^{P:j} + op\boldsymbol{R}_U{}^{U:k} = \sum_P a_x{}^{P:j}\dot{\boldsymbol{R}}_x{}^P + \sum_U a_x{}^{U:k}\dot{\boldsymbol{R}}_x{}^U$$

$$(a_x{}^{P:j} > O; \sum_P a_x{}^{P:j} = 1. \quad a_x{}^{U:k} > O; \sum_U a_x{}^{U:k} = 1)$$

(For $x \neq a$ or z[20]:

$$\underline{f}_x a_{x+1}^{\cdot\cdot} w_x/w_{x+1} \leq a_x^{\cdot\cdot} \leq \overline{f}_x a_{x+1}^{\cdot\cdot} w_x/w_{x+1})$$

$$(\underline{f}_a a_\beta{}^{U:k} w_a U/w_\beta P \leq a_a{}^{P:j} \leq \overline{f}_a a_\beta{}^{U:k} w_a U/w_\beta P)$$

(Possibly, not necessarily: $i = j = k$)

The component rates, $R_x{}^i$, of $op\boldsymbol{R}^i$, are given by Eqs. (1.a) subject to the restrictions imposed in Eq. (3) upon the coefficients $a_x^{\cdot\cdot}$.

Henceforth, an operational rate vector, $op\tilde{\boldsymbol{R}}^i$ will be denoted simply as "$\tilde{\boldsymbol{R}}^i$", except when it may be necessary to emphasize in particular context that a given vector not only is feasible but also is operational.

Very obviously, the bounds $\underline{f}_x$ and $\overline{f}_x$ are not mathematically rigorous, but the degree of rigidity exhibited will vary with practical circumstance in a particular case.

3. *The Final Adjusted Rate Structure.*

The *"final adjusted rate structure"*, consists of a single vector, $\tilde{\boldsymbol{R}}^*$, which is that particular vector whose components, $R_x{}^*$, are the rates to be placed into effect. Obviously, $\tilde{\boldsymbol{R}}^*$ must be an operational vector.

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[20] By previously-given definition of z, f_z does not exist, hence the restriction cannot apply to a_z.

B. *The Existing Rate Structure.*

The *"existing rate structure"* consists of a single vector, $\boldsymbol{R}^e$, whose components, R_x^e, are the rates actually in effect at the time the operation of rate revision is initiated. Superscript "e" identifies quantities associated with the existing rate structure, thus: U^e, P^e, Q_x^e, etc.

The only present concern with the existing rate structure is the utilization of U^e, P^e, R_x^e, etc., as the parameters and arguments of rating formulas appearing in Section VI, to follow.

V. RATE STRUCTURE ALGEBRA

A. *The Problem.*

It is, of course, obvious that, given pre-determined target rates, U and P, a feasible vector always will result if the components, R_x^i, of any rate vector, $\boldsymbol{R}^i$ are multiplied by the ratio $P/\hat{P}^i$ for $x = 1,2, \ldots, \alpha$, and by the ratio $U/\hat{U}^i$ for $x = \beta, \ldots, z$. There are, however, circumstances under which this simple solution either is inadequate or produces undesirable side effects, perhaps intolerable side effects.

It may be that for some x, say $x = \zeta$, that the value to be assumed by the final adjusted rate, R_ζ^*, is pre-determined within narrow bounds by underwriting or other considerations, and an interminable number of trials with successive rate vectors, $\boldsymbol{R}^i$, $\boldsymbol{R}^j, \ldots$ may be required before a vector $\boldsymbol{R}^k$ is found such that $R_\zeta^* = R_\zeta^k P/P^k$ or $R_\zeta^* = R^k U/\hat{U}^k$, accordingly as $\zeta \leq \alpha$ or $\zeta \geqslant \beta$, and also such that for all $x \neq \zeta$, the rates $R_x^* = R_x^k P/\hat{P}^k$ or $R_x^* = R_x^k U/\hat{U}^k$, as $x \leq \alpha$ or $x \geqslant \beta$, are considered appropriate. The problem becomes particularly difficult if bounding values of two or more of the final adjusted rates are pre-determined by side conditions.

Also, when this method of solution is used, the ratemaker has no control over the boundary ratio, $c_\alpha = Q_\alpha/Q_\beta$. Not only may c_α become either obviously and completely trivial or obviously and intolerably excessive, but uncritical and exclusive reliance upon this method has been known to produce in actual practice the weird situation where a community could secure wholesale fire rate reductions by disbanding the fire department and selling off the apparatus. In theory, remembering that R_x is in actuality a continuous function of protection grading, separate adjustment of the premiums for the "Protected" and "Unprotected" statistical classes (where $P/P^e \neq U/U^e$) should be accomplished by rotating the rate curve, not by breaking it into two pieces and translating each piece up or down the vertical axis independently of the position of the other.[27] Although in

[27] See Note 10, *sup.*

many cases this theory is academic, in other cases it definitely will not be so. Whether or not it is academic will depend entirely on the actual values of P^e, U^e, P/P^e and U/U^e in particular instance.

Two systematic methods of solution which avoid both the theoretical and the practical difficulties involved here are given in Section VI, to follow, but first it may be well to explore the implications of Constraint (I) imposed upon the vectors of the operational rate structure.

B. *The Simplest Non-Trivial Case.*

Assume a system of four protection classes. Classes 1 and 2 belong to the "Protected" statistical class, and Classes 3 and 4 belong to the "Unprotected" statistical class.

It follows from the definition, $f_x = R_x/R_{x+1}$, that since $z = 4$, then:

$$(4) \qquad \begin{aligned} R_3 &= f_3 R_4 \\ R_2 &= f_2 R_3 = f_2 f_3 R_4 \\ R_1 &= f_1 R_2 = f_1 f_2 R_3 = f_1 f_2 f_3 R_4 \end{aligned}$$

whence:

$$(5) \qquad \begin{aligned} \textstyle\sum_P w_x R_x &= (f_1 w_1 + w_2)\, R_2 = f_2 f_3\, (f_1 w_1 + w_2)\, R_4 = \hat{P} \\ \textstyle\sum_U w_x R_x &= (f_3 w_3 + w_4)\, R_4 = (f_3 w_3 + w_4)\, R_2/f_2 f_3 = \hat{U} \end{aligned}$$

whence:

$$(6) \qquad \frac{\hat{P}}{\hat{U}} = f_2 f_3 \, \frac{f_1 w_1 + w_2}{f_3 w_3 + w_4} = \hat{\mathfrak{p}}$$

where by definition: $\hat{\mathfrak{p}} = \hat{P}^i/\hat{U}^i$; and for reference to follow, let $\mathfrak{p}^* = P/U$; whence it follows that if $\boldsymbol{R}^i$ is feasible, $\tilde{\boldsymbol{R}}^i$, then: $\hat{\mathfrak{p}}^i = \mathfrak{p}^* = P/U$.

It further follows from Eqs. (4) and (5), by rearrangement following direct substitution of corresponding terms, that if $\boldsymbol{R}^i$ is feasible, $\tilde{\boldsymbol{R}}^i$, then:

$$(7) \qquad \begin{aligned} R_1{}^i &= \frac{f_1{}^i P}{f_1{}^i w_1 + w_2} = \frac{f_1{}^i f_2{}^i f_3{}^i U}{f_3{}^i w_3 + w_4} \\ R_2{}^i &= \frac{P}{f_1{}^i w_1 + w_2} = \frac{f_2{}^i f_3{}^i U}{f_3{}^i w_3 + w_4} \\ R_3{}^i &= \frac{P}{f_2{}^i\, (f_1{}^i w_1 + w_2)} = \frac{f_3{}^i U}{f_3{}^i w_3 + w_4} \\ R_4{}^i &= \frac{P}{f_2{}^i f_3{}^i\, (f_1{}^i w_1 + w_2)} = \frac{U}{f_3{}^i w_3 + w_4} \end{aligned}$$

Imposing the bounds of Constraint (1) upon f_1^* and f_3^*, it follows from Eqs. (7) that extremal values of R_2 and R_3 are given by:

$$\text{(8)} \qquad \overline{R}_2 = \frac{P}{\underline{f}_1^{\,i} w_1 + w_2}\text{; and: } \underline{R}_2 = \frac{P}{\overline{f}_1^{\,i} w_1 + w_2}$$

$$\text{(9)} \qquad \underline{R}_3 = \frac{\underline{f}_3 U}{\underline{f}_3 w_3 + w_4}\text{; and: } \overline{R}_3 = \frac{\overline{f}_3 U}{\overline{f}_3 w_3 + w_4}$$

whence:

$$\text{(10)} \qquad f_2^i \le \frac{\overline{R}_2}{\underline{R}_3} = \frac{(\underline{f}_3 w_3 + w_4) P}{\underline{f}_3 (\underline{f}_1 w_1 + w_2) U}$$

$$\text{(11)} \qquad f_2^i \ge \frac{\underline{R}_2}{\overline{R}_3} = \frac{(\overline{f}_3 w_3 + w_4) P}{\overline{f}_3 (\overline{f}_1 w_1 + w_2) U}$$

The implications of Ineq. (10) are that for *any* choices of $\underline{f}_1 > O$, and of $\underline{f}_3 > O$, as required by Constraint (1), there may be encountered values of U and P, which are beyond the ratemaker's control, such that necessarily $f_2^i > r_2/r_3$, whence, by Eq. (IV.A.2),[28] then $Q_2^i > Q_3^i$ and possibly, even $Q_2^i > Q_4^i$. Conversely, the implications of Ineq. (11) are that for *any* choice of $\overline{f}_1 < r_1/r_2$ and of $\overline{f}_3 < r_3/r_4$, there will exist values of U and P such that necessarily $\epsilon \ge c_2^i > O$, where ϵ is arbitrarily small, which implies that $\eta \ge | Q_3 - Q_2 | > O$, where η is arbitrarily small which is the very essence of triviality. By appropriate rearrangement of Ineqs. (10) and (11), comparable implications concerning the value of f_1^i can be demonstrated to result from any choice of bounds to f_2^i and f_3^i, and concerning the value of f_3^i for any choice of bounds to f_1^i and f_2^i.

It is to be noted that realization of the possibilities implied, as above, by Ineqs. (10) and (11) depends upon the ratio $p^* = {}^P/_U$, and not upon the actual value of either P or U. This fact may be turned to practical advantage.

There are eight possible combinations: $\underline{f}_1, \underline{f}_2, \underline{f}_3$; $\underline{f}_1, \underline{f}_2, \overline{f}_3$; etc., of the extremal ratios, $\underline{f}_x$ and $\overline{f}_x$. Entering each of these combinations in turn into Eq. (6), let:

$$\text{(12)} \qquad \begin{aligned} p^{I} &= \underline{f}_2 \underline{f}_3 \frac{\underline{f}_1 w_1 + w_2}{\underline{f}_3 w_3 + w_4} = p(\underline{f}_1, \underline{f}_2, \underline{f}_3) \\ p^{II} &= p(\underline{f}_1, \overline{f}_2, \underline{f}_3) \\ &\cdots\cdots\cdots\cdots \\ p^{VIII} &= p(\overline{f}_1, \overline{f}_2, \overline{f}_3) \end{aligned}$$

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[28] "Eq. (IV.A.2)": – Eq. (2) introduced in Section IV.A.

From the first and last of Eqs. (12), it will be seen that for any given choice of values for $\underline{f}_x$ and $\bar{f}_x$, the value of $\mathcal{P}^I$ is the minimum value and the value of $\mathcal{P}^{VIII}$ is the maximum value which can be assumed by $\hat{\mathcal{P}}^i = \mathcal{P}(f_1{}^i, f_2{}^i, f_3{}^i)$ subject to Constraint (I). It then follows from Eqs. (7) (whereby it is seen that $R_x{}^i$ is for all x a function of either U or P, together with one or more of the ratios $f_x{}^i$) that if $\mathcal{P}^* < \mathcal{P}^I$ or $\mathcal{P}^* > \mathcal{P}^{VIII}$, no operational rate vector will exist. To be feasible in such instance, there must be associated with the vector $\boldsymbol{R}^i$, values of $f_x{}^i$ such that for at least one x, then $f_x{}^i > \bar{f}_x$ if $\mathcal{P}^* > \mathcal{P}^{VIII}$, or $f_x{}^i < \underline{f}_x$ if $\mathcal{P}^* < \mathcal{P}^I$.

If $\mathcal{P}^* = \mathcal{P}^I$ or $\mathcal{P}^* = \mathcal{P}^{VIII}$, then there will exist exactly one operational rate vector, $\tilde{\boldsymbol{R}}^*$, which may be calculated directly by Eqs. (7), entering as arguments of the equations the values of $\underline{f}_1, \underline{f}_2, \underline{f}_3$ if $\mathcal{P}^* = \mathcal{P}^I$, and the values of $\bar{f}_1, \bar{f}_2, \bar{f}_3$ if $\mathcal{P}^* = \mathcal{P}^{VIII}$.

It is to be demonstrated[29] that if $\mathcal{P}^I < \mathcal{P}^* < \mathcal{P}^{VIII}$, then the final rate vector, $\tilde{\boldsymbol{R}}^*$, may be calculated directly as a linear convex combination[30] of certain vectors to be associated with $\mathcal{P}^I, \ldots, \mathcal{P}^{VIII}$, *provided* that side conditions imposed upon $\tilde{\boldsymbol{R}}^*$ (*e.g.*, predetermined bounds to the value to be assumed by $R_x{}^*$ for some one or more x) do not render solution impossible in particular instance. The smaller the value of $\mathcal{P}^* - \mathcal{P}^I$, or, alternatively, of $\mathcal{P}^{VIII} - \mathcal{P}^*$, the more restricted will be the ratemaker's freedom of choice.

C. *The General Case.*

For notational convenience, let:

$$f_x = \begin{cases} R_x/R_{x+1}; & \text{if: } x < z \\ 1; & \text{if: } x = z \end{cases}$$

$$f_{\zeta:\mu} = \prod_{x=\zeta}^{\mu} f_x = \begin{cases} R_\zeta/R_{\mu+1}; & \text{if: } \zeta \leq \mu < z \\ R_\zeta/R_z; & \text{if: } \zeta \leq \mu = z \end{cases}$$

With the above definition of $f_{\zeta:\mu}$, and extension of the previously-given definition of f_x,[31] under a generalized classification system of Class

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[29] See APPENDIX A and also Section VI.D., to follow.

[30] "Linear convex combination": —*E.g.*, each of the two summations in the right member of Eq.(IV.A.1) is a "linear convex combination" of vectors by virtue of the restrictions upon the coefficients a_x^i. (If these restrictions are removed, the entire right member becomes simply a "linear combination" of vectors.)

[31] It should be noted that this extension of the definition of f_x requires appropriate qualifying extension of the statement of Constraint (I).

1, . . . , Class a; Class β, . . . , Class z, Eqs. (4) become:

$$(13) \qquad R_x = f_x R_{x+1} = f_{x:a-1} R_a = f_{x:z} R_z$$

whence Eqs. (5) become:

$$(14) \qquad \begin{aligned} \sum_P w_x R_x &= \left(\sum_{x=1}^{a-1} f_{x:a-1} w_x + w_a\right) R_a \\ &= f_{a:z} \left(\sum_{x=1}^{a-1} f_{x:a-1} w_x + w_a\right) R_z = \hat{P} \\ \sum_U w_x R_x &= (f_{a:z})^{-1} \left(\sum_U f_{x:z} w_x\right) R_a \\ &= \left(\sum_U f_{x:z} w_x\right) R_z = \hat{U} \end{aligned}$$

whence:

$$(15) \qquad \hat{\mathfrak{p}} = \frac{\hat{P}}{\hat{U}} = f_{a:z} \frac{\sum_{x=1}^{a-1} f_{x:a-1} w_x + w_a}{\sum_U f_{x:z} w_x}$$

and by analogy to Eqs. (7), for any feasible vector, $\tilde{\boldsymbol{R}}^i$:

$$(16) \qquad \begin{aligned} R_\zeta{}^i &= \frac{f_{\zeta:a}{}^i P}{\sum_P f_{x:a}{}^i w_x} = \frac{f_{\zeta:z}{}^i U}{\sum_U f_{x:z}{}^i w_x}; \quad (\zeta \leq a) \\ R_\mu{}^i &= \frac{P}{f_{\beta:\mu}{}^i \left(\sum_P f_{x:a}{}^i w_x\right)} = \frac{f_{\mu:z}{}^i U}{\sum_U f_{x:z}{}^i w_x}; \quad (\mu \geqslant \beta) \end{aligned}$$

Imposing Constraint (I) upon $f_x{}^i$, by analogy to Eqs. (8) and (9):

$$(17) \qquad \overline{R}_a = \frac{P}{\sum_{x=1}^{a-1} \underline{f}_{x:a-1} w_x + w_a}; \text{ and: } \underline{R}_a = \frac{P}{\sum_{x=1}^{a-1} \overline{f}_{x:a-1} w_x + w_a}$$

$$(18) \qquad \underline{R}_\beta = \frac{\underline{f}_{\beta:z} U}{\sum_U \underline{f}_{x:z} w_x}; \text{ and: } \overline{R}_\beta = \frac{\overline{f}_{\beta:z} U}{\sum_U \overline{f}_{x:z} w_x}$$

From Eqs. (17) and (18), inequalities analogous to Ineqs. (10) and (11) may be formulated, and these inequalities will carry exactly the same implications under a generalized classification system as do Ineqs. (10) and (11) under the 4-class system assumed in Section B, above.

There will be 2^{z-1} possible combinations of the extremal ratios: $\underline{f}_1, \underline{f}_2, \ldots, \underline{f}_{z-1}; \ldots; \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_{z-1}$. Hence the analogue of Eqs. (12) will be a system of 2^{z-1} equations:

$$
\begin{aligned}
\mathcal{P}^{I} &= \underline{f}_{\alpha}\underline{f}_{\beta:z}\,\frac{\sum_{x=1}^{\alpha-1} \underline{f}_{x:\alpha-1} w_x + w_\alpha}{\sum_U \underline{f}_{x:z} w_x} \\
&= \mathcal{P}(\underline{f}_1, \ldots, \underline{f}_{\alpha-1}; \underline{f}_\alpha; \underline{f}_\beta, \ldots, \underline{f}_{z-1})
\end{aligned}
\tag{19}
$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\mathcal{P}^{\phi} = \mathcal{P}(f'_1, \ldots, f'_{\alpha-1}; f'_\alpha; f'_\beta, \ldots, f'_{z-1}); \quad (f'_x = \underline{f}_x \text{ or } \bar{f}_x)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\mathcal{P}^{\Omega} = \mathcal{P}(\bar{f}_1, \ldots, \bar{f}_{\alpha-1}; \bar{f}_\alpha; \bar{f}_\beta, \ldots, \bar{f}_{z-1}).$$

It will not follow that if $\phi \neq \psi$, then necessarily $\mathcal{P}^{\phi} \neq \mathcal{P}^{\psi}$; and for $I < \phi < \Omega$, the order of relative magnitude among the several $\mathcal{P}^{\phi}$ may vary with the actual values of $\underline{f}_x$, $\bar{f}_x$ and w_x in particular instance. In all cases, however, regardless of the parameter values, the value of $\mathcal{P}^{I}$ will be less than, and the value of $\mathcal{P}^{\Omega}$, greater than, the value of any $\mathcal{P}^{\phi}$ for $\phi \neq I$ or Ω. Also, as under the 4-class system previously displayed, in the completely general case: if $\mathcal{P}^{*} < \mathcal{P}^{I}$ or $\mathcal{P}^{*} > \mathcal{P}^{\Omega}$, there will be no solution to the rate revision problem except in violation of Constraint (1); if $\mathcal{P}^{\phi} = \mathcal{P}^{I}$ or $\mathcal{P}^{\phi} = \mathcal{P}^{\Omega}$ there will be a unique solution to the problem; if $\mathcal{P}^{I} < \mathcal{P}^{\phi} < \mathcal{P}^{\Omega}$, then operational rate vectors may be calculated directly as linear convex combinations of not more than z vectors of certain ones to be associated with the several $\mathcal{P}^{\phi}$;[32] finally, the smaller the value of $\mathcal{P}^{*} - \mathcal{P}^{I}$ or, alternatively, of $\mathcal{P}^{\Omega} - \mathcal{P}^{*}$, the narrower will be the bounds of the operational rate structure, $op\{\tilde{\mathbf{R}}\}$, *i.e.* the more restricted will be the ratemaker's freedom of choice.

See APPENDIX A for further discussion.

VI. RATE CALCULATION

A. *The Classification System.*

Throughout what follows, it is assumed that the protection classification system is the N.B.F.U. system described in Section I.B., preceding. However, as no city in the United States presently is classified as N.B.F.U. Class 1, then $w_1 = 0$, whence Class 2 is the lowest-numbered class to be considered in numerical examples. This is of absolutely no consequence in connection with *Method I,* to follow in Section C, below, except to explain the absence of Class 1 rates throughout the calcula-

[32] Normally, the number of vectors required for this purpose will not exceed three or four, but in no case will more than z vectors be required in the combination. See APPENDIX A and also Section VI.D, to follow.

tion. For the significance of the missing Class 1 with respect to *Method II*, Section D, below, see APPENDIX A, Section 3, following.

B. *Application to "Loss Constant Rates." Data Tables.*

The data used in all numerical examples to follow is given in Tables 1, 2 and 3. The data of Table 1 is to be used in all cases. It will be specified in particular cases whether Table 2 or, alternatively, Table 3 is to be used.

The rates and parameters given in these tables are based upon the Frame Dwelling Building rates in effect as of this writing in the State of Louisiana. The only modification of the actually-existing rate structure has been to eliminate a so-called "Country Dwelling" rate (higher than R_{10}), and to combine the actual weighting factors for "Country" and for N.B.F.U. Class 9 into the value shown for w_9 in Table 1. The true value of w_9 would be less than 2%, since N.B.F.U. Class 9 is virtually non-existent in the state.[33]

The Dwelling rate structure in Louisiana embodies the so-called "loss constant rating method," under which the *"effective rate," E_x*, is given by the formula:

(20) $$E_x = (C_x + D_x V)/V$$

where:

C_x = The *"loss constant."* (Possibly the same for two or more classes.)

D_x = The *"residual rate."*[34]

V = *"Policy size," i.e.* the amount of insurance under a given policy.

Let:

V_x^{av} = The mean policy size in Class x.

V_T^{av} = The mean policy size statewide.

It then follows by Eq. (20) that:

(21) $E_x^{av} = (C_x + D_x V_x^{av})/V_x^{av}$ = The mean effective rate of Class x.

(22) $E_x^{Q} = (C_x + D_x V_T^{av})/V_T^{av}$ = The mean effective rate of Class x "normalized" to the statewide mean policy size, V_T^{av}.

---

[33] Whenever an 8th class community in Louisiana slips, it seems to slip all the way through N.B.F.U. Class 9 into Class 10. When a Class 10 town decides to improve its status, momentum usually carries it up through Class 9 into Class 8 or better.

[34] The "residual rate" commonly is denoted by "*R*," rather than by "*D*," in expressing Eq. (20), but in present context this notation obviously would cause confusion.

TABLE 1: — Weighting Factors.

Stat. Class.	"Protected"							"Unprotected"	
Prot. Class	2	3	4	5	6	7	8	9	10
w_x	0.066	0.461	0.052	0.148	0.137	0.079	0.057	0.215	0.785

$w_1 = 0$. min. Class Number in calculation $= 2$. $\alpha = 8$; $\beta = 9$; $z = 10$.

TABLE 2: — Existing Rates.

Prot. Class	2	3	4	5	6	7	8	9	10
R_x^e	1.81	2.01	2.21	2.89	3.09	3.57	3.77	4.24	5.02
$\underline{f}_x$	0.75	0.75	0.75	0.80	0.80	0.80	0.80	0.80	* *
$\bar{f}_x$	0.90	0.90	0.90	0.95	0.95	0.95	0.95	0.95	* *

$P^e = 2.509$; $U^e = 4.852$. $r_x = q_x = 1$, for all x.

TABLE 3: — Existing Rates, Existing Normals.

Prot. Class	2	3	4	5	6	7	8	9	10
R_x^e	1.72	1.73	2.24	2.75	3.22	3.85	3.98	4.77	5.47
$\underline{f}_x$	0.83	0.64	0.80	0.73	0.77	0.82	0.74	0.83	* *
$\bar{f}_x$	1.00	0.77	0.96	0.87	0.92	0.98	0.88	0.99	* *
Q_x^e	1.81	2.01	2.21	2.89	3.09	3.57	3.77	4.24	5.02
$\underline{c}_x$	0.75	0.75	0.75	0.80	0.80	0.80	0.80	0.80	* *
$\bar{c}_x$	0.90	0.90	0.90	0.95	0.95	0.95	0.95	0.95	* *
r_x	0.951	0.861	1.014	0.952	1.042	1.078	1.056	1.125	1.090
q_x	1.052	1.161	0.986	1.050	0.960	0.928	0.947	0.889	0.917

$P^e = 2.407$; $U^e = 5.320$. $P_Q^e = 2.509$; $U_Q^e = 4.852$.
$q_P = 1.069$; $q_U = 0.911$.

It will follow by straightforward algebra that E_x^{av} as calculated by Eq.(21) conforms exactly to the definition of the protection classification rate, R_x, as given in Section II.D., preceding, and hence may be substituted for R_x in any equation so far developed or to be developed below, without affecting the validity of the equation.

In general, as noted in Section II, R_x is "normalized" to the conditions of Class z, *i.e.* of Class 10 in present instance, to obtain Q_x. If V_{10}^{av} is substituted for V_T^{av} in Eq.(22), then E_x^Q will conform exactly to the general definition of Q_x. The fact that it does not so conform to the general definition (unless by coincidence $V_T^{av} = V_{10}^{av}$, which is unlikely) is presently irrelevant. The basic concept of Q_x is simply the normalization of R_x to some common set of extraneous hazard conditions, and the choice of Class 10 in the general case is not mandatory, though convenient. (See Section II.B, preceding) Normalization of E_x^{av} to the statewide average, V_T^{av}, rather than to the Class 10 average, V_{10}^{av}, is not mandatory. However, choice of V_T^{av} normally will give more conveniently-handled values for low-numbered classes, and also a truer picture of rate distribution, than will V_{10}^{av}. In any equation so far developed or to be developed below, E_x^Q may be substituted for Q_x without affecting the validity of the equation in the least.

The values of R_x in Table 2 are the actual values of E_x^Q for Frame Dwelling Buildings, calculated from current Louisiana rates-in-effect on the basis of the actual policy-size sampling underlying the rate structure. These same E_x^Q are entered as "Q_x" in Table 3, wherein E_x^{av} becomes "R_x". Equations (21) and (22) form the bridge which links the present development with the loss constant rating method.

It should be noted that the values of $\underline{f}_x$, $\bar{f}_x$, $\underline{c}_x$ and $\bar{c}_x$ shown in the tables are assumed for illustrative purposes only. There is no intent to suggest that these values are necessarily appropriate in any given instance.

C. *Method I.*[35]

Conditions of Application.

(a) Neither the final value to be assumed by any individual adjusted rate, R_x^*, nor the percentage value of the adjustment to any individual protection class, is pre-determined; *and:*

(b) The bounds to the inter-class ratios, $\underline{f}_x$ and $\bar{f}_x$, or, $\underline{c}_x$ and $\bar{c}_x$, are considered to be extremely elastic.

[35] See APPENDIX B for derivation of all equations employed in this section.

Case 1.

Supplemental Conditions: – The rate-normal ratio, $r_x = R_x/Q_x$, may be taken equal to unity for all x; *and*: the shape of the existing rate curve is to remain unchanged.

Algebraic Solution:

For each protection class in turn, calculate the final adjusted rate, R_x^*, by the equation:

$$R_x^* = R_x^e \frac{U - P}{U^e - P^e} + \frac{PU^e - UP^e}{U^e - P^e} \tag{23.a}$$

Graphical Solution:

(1) Plot the points $(P^e;P)$ and $(U^e;U)$, labeling the horizontal axis, "R_x^e," and the vertical axis, "R_x^*". Draw the straight line through these points.

(2) Read the final adjusted rates, R_x^*, as the ordinates of those points on the line, whose abcissas are the respective existing rates, R_x^e.

EXAMPLE 1.

Premium Adjustments Required: To the "Protected" statistical class: – 10% increase. To the "Unprotected" statistical class: – 25% increase.

Data Reference: Tables 1 and 2.

Algebraic Solution:

The complete rate calculation is shown in Table 4, together with the values of f_x^* and the verification.

The small differences, $\hat{P}^* - P = 0.001$ and $\hat{U}^* - U = 0.003$ are due solely to rounding error, as may be seen by carrying at least five decimals at each stage of the overall calculation. The form of the calculation is exact.

Whether or not the ratio $f_4^* < \underline{f}_4$ is to be accepted is a matter of judgment. In dollars and cents: $\underline{f}_4 R_5^* - R_4^* = \$2.422 - \$2.339 = \$0.083$ per $1,000 of insurance.

TABLE 4.

Solution of Example 1.

x	R_x^e		R_x^*	f_x^*	Verification	
*	* *		* *	* *	w_x	$w_xR_x^*$
2	1.81		1.775	0.863	0.066	0.117
3	2.01		2.057	0.897	0.461	0.948
4	2.21		2.339	#0.709	0.052	0.112
5	2.89		3.299	0.921	0.148	0.488
6	3.09	$\times 1.411 - 0.779 =$	3.581	0.841	0.137	0.491
7	3.57		4.258	0.938	0.079	0.336
8	3.77		4.520	0.872	0.057	0.259
*	* *		* *	* *	$\hat{P}^* = 2.761$	
*	* *		* *	* *	$\hat{P}^* - P = 0.001$	
9	4.24		5.204	0.856	0.215	1.119
10	5.02		6.304	* *	0.785	4.949
$P = 1.10P^c = 2.760$; $U = 1.25U^e = 6.065$					$\hat{U}^* = 6.068$	
$(U\text{-}P)/(U^e\text{-}P^e) = 1.411$; $(PU^e\text{-}UP^e)/(U^e\text{-}P^e) = -0.779$					$\hat{U}^* - U = 0.003$	

$f_4^* = 0.709 < 0.75 = \underline{f}_4$

Graphical Solution:

Figure 1 represents the graphical solution of the problem. The final adjusted rates obtained from the original of the graph are:

$$R_2^* = 1.78\ ;\ R_5^* = 3.30\ ;\ R_8^* = 4.50$$
$$R_3^* = 2.06\ ;\ R_6^* = 3.58\ ;\ R_9^* = 5.20$$
$$R_4^* = 2.34\ ;\ R_7^* = 4.26\ ;\ R_{10}^* = 6.29$$

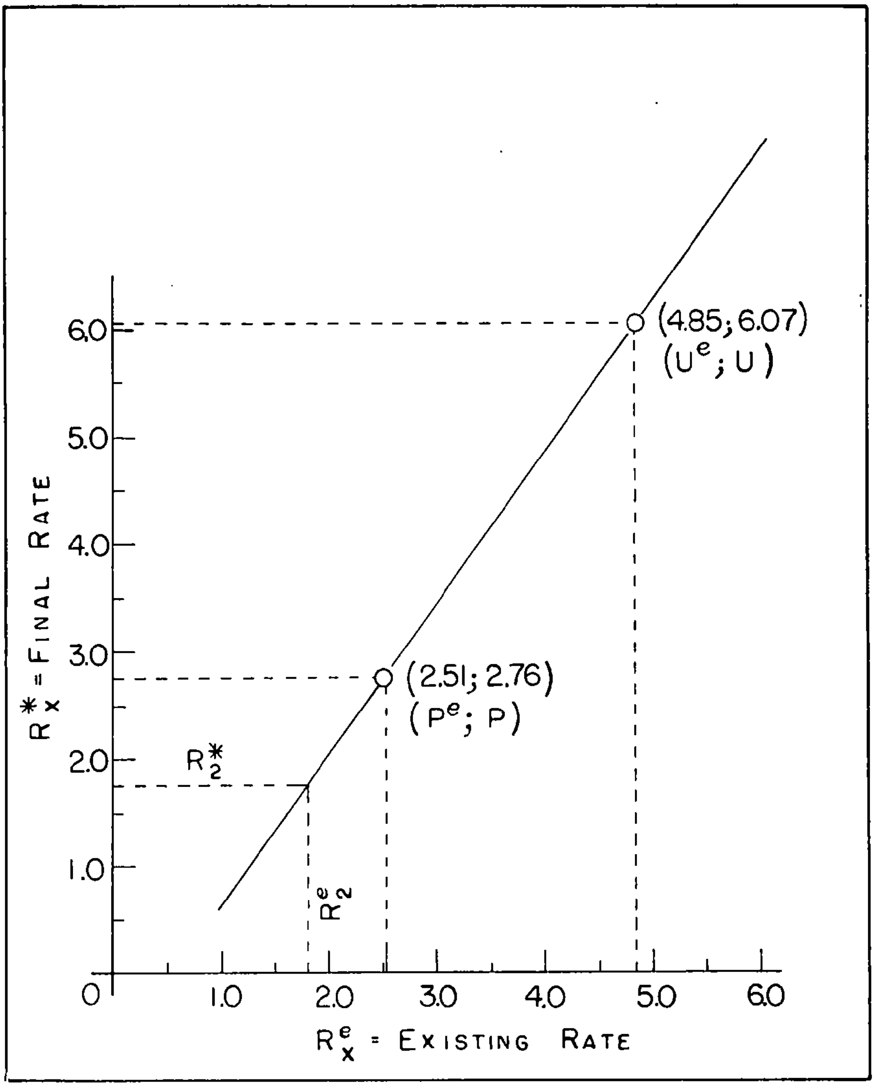
and in verification:

$$\textstyle\sum_P w_x R_x^* = \hat{P}^* = 2.731;\ \text{whence: } \hat{P}^* - P = -0.029$$
$$\textstyle\sum_U w_x R_x^* = \hat{U}^* = 6.016;\ \text{whence: } \hat{U}^* - U = -0.049$$

The graphical solution follows immediately from the fact that Eq. (23.a) is simply the slope-intercept form of a linear equation in R_x^e and R_x^*.

FIGURE 1.

Graphical Solution of Example 1.



Case 2.

Supplemental Conditions: — The rate-normal ratio, r_x, may be taken equal to unity for all x; *but:* the shape of the existing rate curve is to be revised.

Algebraic Solution:

(1) Determine by any convenient method a set of trial rates, $R_x{}^i$, which define a rate curve of the desired shape. Calculate $\hat{U}^i = \sum_U w_x R_x{}^i$ and $\hat{P}^i = \sum_P w_x R^i$.

(2) Substitute $R_x{}^i$, $\hat{U}^i$ and $\hat{P}^i$ respectively for $R_x{}^e$, U^e and P^e in Eq.(23.a), and compute the final rates, $R_x{}^*$, as in Case 1.

Note: Although the exact values of $R_x{}^i$, $\hat{U}^i$ and $\hat{P}^i$ are immaterial, the difference $\hat{U}^i - \hat{P}^i$ should contain at least as many significant figures as does the difference $U - P$. This normally will result if the several $R_x{}^i$ are so chosen that $R_{10}{}^i > U$, $R_8{}^i > P$, and $R_\zeta{}^i < P$ for at least one value of ζ such that $\zeta < 8$.

Graphical Solution:

Proceed as in the graphical solution of Case 1, except that the points to be plotted are $(\hat{P}^i; P)$ and $(\hat{U}^i; U)$, and the horizontal axis is to be labeled "$R_x{}^i$."

EXAMPLE 2.

Premium Adjustment Required: To the "Protected" statistical class: — 25% increase. To the "Unprotected" statistical class: — no adjustment of presently reported premium.[36]

Data Reference: Tables 1 and 2.

Assumption: The following trial rates define a rate curve of the desired shape:

$$R_2{}^i = 1.37\ ;\ R_5{}^i = 2.22\ ;\ R_8{}^i = 3.61$$
$$R_3{}^i = 1.61\ ;\ R_6{}^i = 2.61\ ;\ R_9{}^i = 4.25$$
$$R_4{}^i = 1.89\ ;\ R_7{}^i = 3.07\ ;\ R_{10}{}^i = 5.00$$

Algebraic Solution:

$$U^c = 4.852\ ;\ \text{whence: } U = 1.000U^c = 4.852$$
$$P^c = 2.509\ ;\ \text{whence: } P = 1.250P^c = 3.136$$
$$\hat{U}^i = \textstyle\sum_U w_x R_x{}^i = 4.839$$
$$\hat{P}^i = \textstyle\sum_P w_x R_x{}^i = 2.066$$

whence by Eq.(23.a) with appropriate substitutions:

(23.a.1) $R_x{}^* = 0.619R_x{}^i + 1.858$.

The final adjusted rates calculated by Eq.(23.a) are:

$$R_2{}^* = 2.706\ ;\ R_5{}^* = 3.232\ ;\ R_8{}^* = 4.093$$
$$R_3{}^* = 2.855\ ;\ R_6{}^* = 3.474\ ;\ R_9{}^* = 4.489$$
$$R_4{}^* = 3.028\ ;\ R_7{}^* = 3.758\ ;\ R_{10}{}^* = 4.953$$

and in verification:

$$\textstyle\sum_P w_x R_x{}^* = \hat{P}^* = 3.136;\ \text{whence: } \hat{P}^* - P = 0$$
$$\textstyle\sum_U w_x R_x{}^* = \hat{U}^* = 4.853;\ \text{whence: } \hat{U}^* - U = 0.001$$

The rate curves defined respectively by the trial rates $R_x{}^i$ and the final rates, $R_x{}^*$, are shown in Figs. 2.a and 2.b. The relationship (see the figures) between $R_x{}^i - L_x{}^i$ and $R_x{}^* - \tilde{L}_x$ should be noted.

Case 3.

Supplemental Conditions: — The rate-normal ratio, r_x, cannot be taken equal to unity for all *x; but:* the shape of the normal curve is to remain unchanged.

Algebraic Solution:

(1) Calculate the *"unprotected target normal,"* U_Q, and the *"protected target normal,"* P_Q, by the equations:

(24)
$$U_Q = U_Q{}^c \frac{U - P}{U^c - P^c} + q_U \frac{PU^c - UP^c}{U^c - P^c}\ ;\quad (q_U = \textstyle\sum_U w_x q_x)$$
$$P_Q = P_Q{}^c \frac{U - P}{U^c - P^c} + q_P \frac{PU^c - UP^c}{U^c - P^c}\ ;\quad (q_P = \textstyle\sum_P w_x q_x)$$

---

[36] It may be noted that if the values of R_x shown in Table 2 simply are increased each by 25% for Classes 2-8 while leaving R_x at present value for Classes 9 and 10, the result will be not only: $R_8{}^* = 4.71 > 4.24 = R_9{}^*$; but also: $R_7{}^* = 4.46 > 4.24 = R_9{}^*$. Remembering that by hypothesis, $r_x = 1$, this solution is unacceptable even though the required premium volume would be obtained.

FIGURE 2.a.
Rate Curves. Example 2.

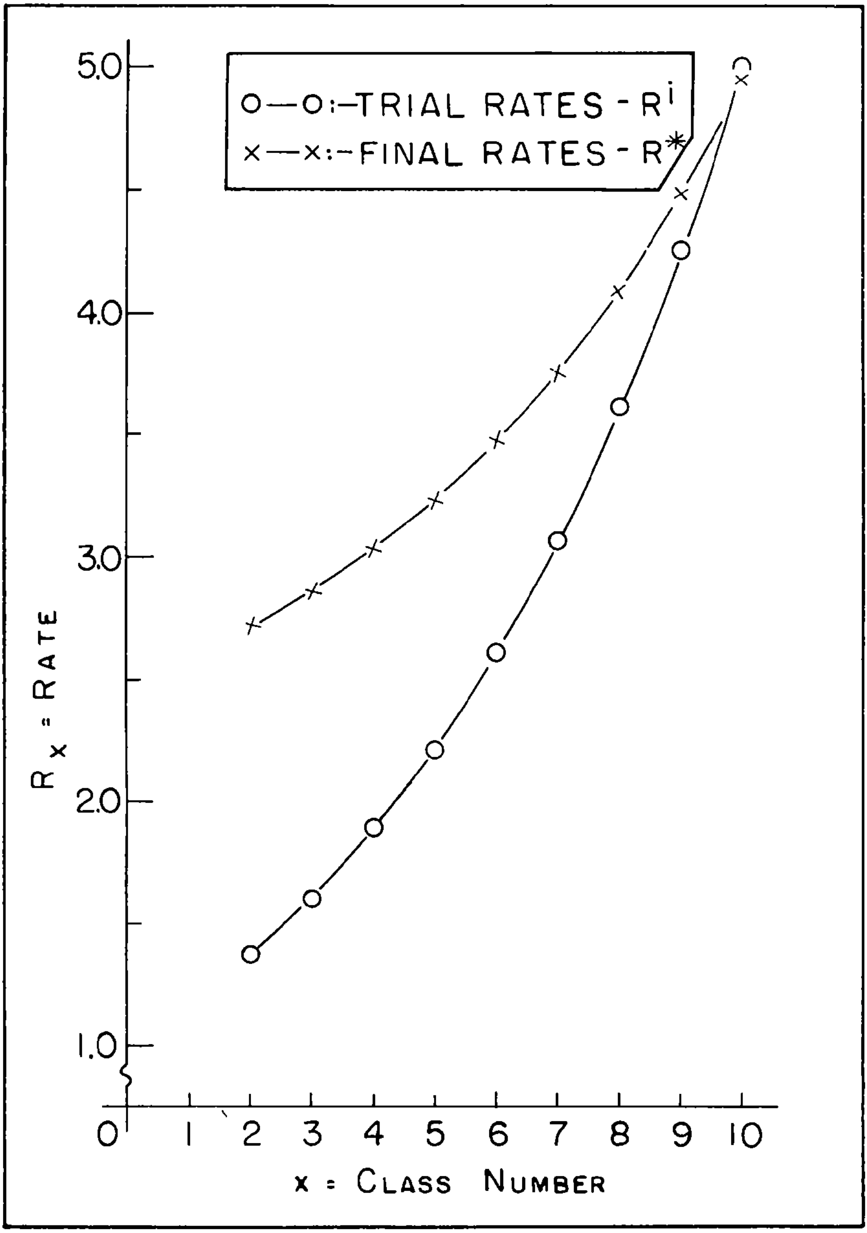
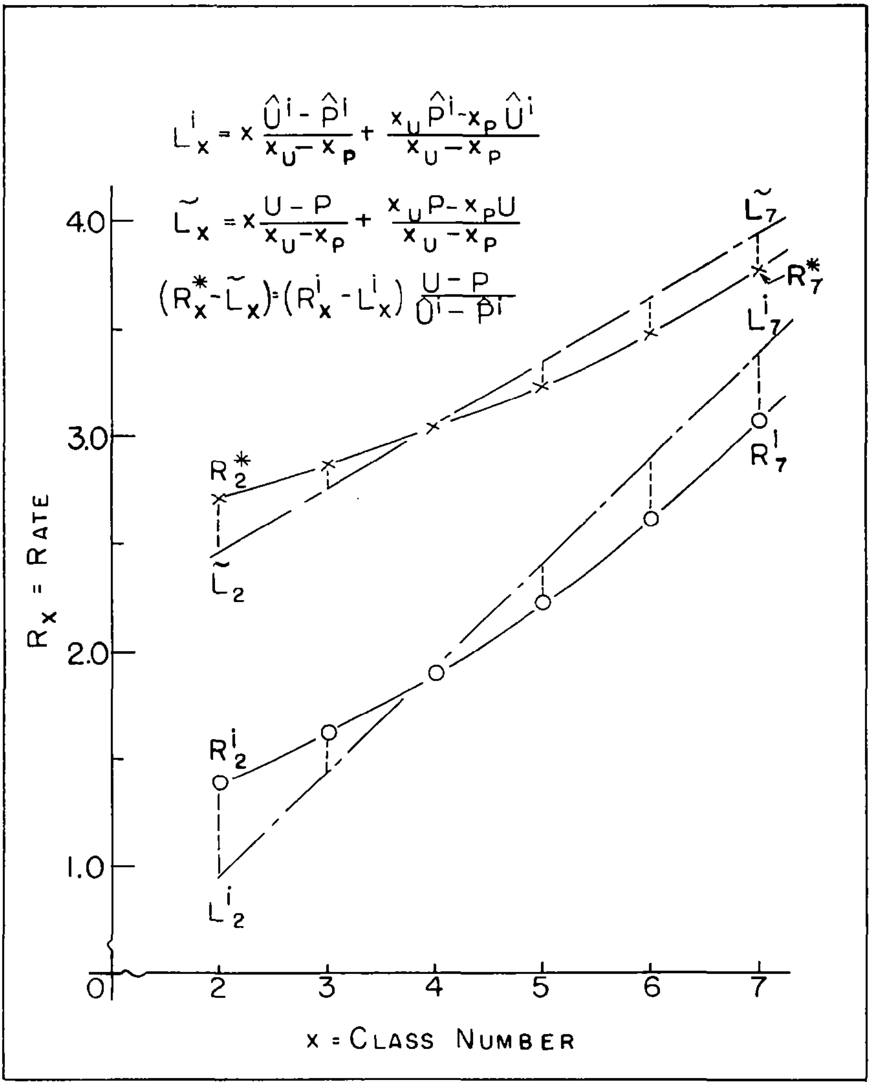


FIGURE 2.b.

Rate Curves. Example 2.
Expanded Scale with Reference Lines.



(2) For each protection class in turn, calculate the final adjusted normals $Q_x{}^*$ by the equation (*cf.* Eq.(23.a)):

$$(23.b) \qquad Q_x{}^* = Q_x{}^c \frac{U_Q - P_Q}{U_Q{}^c - P_Q{}^c} + \frac{P_Q U_Q{}^c - U_Q P_Q{}^c}{U_Q{}^c - P_Q{}^c}$$

(3) Calculate the final adjusted rates by the equation:

$$(25) \qquad R_x{}^* = r_x Q_x{}^*$$

It may be noted that Eqs.(23.b) and (25) can be combined if both sides of Ep.(23.b) are multiplied by r_x, yielding *(Cf.* Eq.(23.a)):

$$(25.a) \qquad R_x{}^* = R_x{}^c \frac{U_Q - P_Q}{U_Q{}^e - P_Q{}^e} + r_x \frac{P_Q U_Q{}^e - U_Q P_Q{}^e}{U_Q{}^e - P_Q{}^c}$$

whence $R_x{}^*$ is obtained directly without intermediate calculation of $Q_x{}^*$. Offsetting the immediate operational economies of Eq.(25.a) is the fact that unless significant changes are to be expected in the distribution of sums insured, as reflected in the several weighting factors, w_x, the final adjusted normals, $Q_x{}^*$, of the current rate revision may be stored to become the existing normals to be used in the next subsequent rate revision. Thus the immediate use of Eqs.(23.b) and (25) in preference to Eq. (25.a) may save calculation at a later date.

It also may be noted that it is possible to obtain a solution by the method of Case 1 which will produce the required premium. However, direct adjustment of $R_x{}^e$ to $R_x{}^*$ by Eq.(23.a) when $r_x \neq 1$ is very likely to result in unacceptable inversion of the normals, *i.e.* for some x, then $Q_x > Q_{x+1}$. If $r_x \neq 1$, then $R_x > R_{x+1}$ is permissible, but never the inconsistency of $Q_x > Q_{x+1}$.

Graphical Solution:

By analogy to the graphical solution of Case 1, the final adjusted normals $Q_x{}^*$ are obtained from the plot of a straight line through the points $(P_Q{}^c;\ P_Q)$ and $(U_Q{}^c;\ U_Q)$, where the horizontal axis represents $Q_x{}^c$ and the vertical axis represents $Q_x{}^*$. The final adjusted rates then follow by Eq.(25).

Case 4.

Supplemental Conditions: – The rate normal ratio, r_x, cannot be taken equal to unity for all *x; and:* the shape of the normal curve is to be revised.

Algebraic Solution:

(1) Calculate U_Q and P_Q by Eqs. (24) as in Case 3.

(2) Determine by any convenient method a set of trial normals, Q_x^i, which define a normal curve of the desired shape. Calculate: $\hat{U}_Q^i = \sum_U w_x Q_x^i$; and: $\hat{P}_Q^i = \sum_P w_x Q_x^i$.

(3) Calculate the final adjusted normals, Q_x^*, by Eq.(23.b).

(4) Calculate the final adjusted rates, R_x^*, by Eq.(25), as in Case 3.

Graphical Solution:

Case 4 may be solved graphically for Q_x^*, by analogy to the graphical solution of Case 3, whence R_x^* then follows by Eq.(25).

EXAMPLE 3.

Premium Adjustments Required: To the "Protected" statistical class: – 30% increase. To the "Unprotected" statistical class: – 5% increase.

Data Reference: Tables 1 and 3.

Assumption: The following trial normals define a normal curve of the desired shape:

$$Q_2^i = 1.37\ ;\ Q_5^i = 2.22\ ;\ Q_8^i = 3.61$$
$$Q_3^i = 1.61\ ;\ Q_6^i = 2.61\ ;\ Q_9^i = 4.25$$
$$Q_4^i = 1.89\ ;\ Q_7^i = 3.07\ ;\ Q_{10}^i = 5.00$$

Algebraic Solution:

$$U^e = 5.320;\ \text{whence: } U = 1.05 \times 5.320 = 5.586$$
$$P^e = 2.407;\ \text{whence: } P = 1.30 \times 2.407 = 3.129$$

whence by Eqs.(24):

$$U_Q = 5.092;\ \text{and: } P_Q = 3.289$$

and by hypothesis:

$$\hat{U}_Q^i = \textstyle\sum_U w_x Q_x^i = 4.839$$
$$\hat{P}_Q^i = \textstyle\sum_P w_x Q_x^i = 2.066$$

whence by Eq.(23.b) the final adjusted normals are:

$$Q_2^* = 2.836\ ;\ Q_5^* = 3.389\ ;\ Q_8^* = 4.293$$
$$Q_3^* = 2.993\ ;\ Q_6^* = 3.643\ ;\ Q_9^* = 4.709$$
$$Q_4^* = 3.175\ ;\ Q_7^* = 3.942\ ;\ Q_{10}^* = 5.196$$

and in verification of the normals, Q_x*:

$$\Sigma_P w_x Q_x^* = \hat{P}_Q^* = 3.287;\ \text{whence: } \hat{P}_Q^* - P_Q = -0.002$$
$$\Sigma_U w_x Q_x^* = \hat{U}_Q^* = 5.091;\ \text{whence: } \hat{U}_Q^* - U_Q = -0.001$$

From the final adjusted normals, as above, the final adjusted rates are, by Eq.(25):

$$R_2^* = 2.694\ ;\ R_5^* = 3.226\ ;\ R_8^* = 4.533$$
$$R_3^* = 2.577\ ;\ R_6^* = 3.795\ ;\ R_9^* = 5.298$$
$$R_4^* = 3.219\ ;\ R_7^* = 4.249\ ;\ R_{10}^* = 5.664$$

and in verification of the final adjusted rates, R_x*:

$$\Sigma_P w_x R_x^* = \hat{P}^* = 3.124;\ \text{whence: } \hat{P}^* - P = -0.005$$
$$\Sigma_U w_x R_x^* = \hat{U}^* = 5.585;\ \text{whence: } \hat{U}^* - U = -0.001$$

D. *Method II.*[37]

Conditions of Application.

(a) The bounds to the inter-class ratios, $\underline{f}_x$ and $\overline{f}_x$, are considered relatively inelastic as between one or more pairs of adjacent classes; *or:*

(b) The final values to be assumed by some one or more of the adjusted rates R_x^* are pre-determined by underwriting or other considerations; *and:*

(c) The shape of the final rate curve (or final normal curve) is immaterial.

Pre-calculation of Parameters

Pre-calculate and store for use in successive rate revisions over a period of years the parameter vectors N^ϕ whose component rates are shown in Table 5. For each vector N^ϕ, calculate $P^\phi = \Sigma_P w_x N_x$. (It will be found that for all ϕ, then $\Sigma_U w_x N_x^\phi = 1$.) Once calculated, these parameters need not be re-calculated unless and until either significant change occurs in the distribution of sums insured (*i.e.* in the values of w_x) or the extremal ratios $\underline{f}_x$ and $\overline{f}_x$ are revised.

Tables 6A and 6B show a sample calculation of these parameters from data given in Tables 1 and 3. Table 6B serves also as the table of parameters for use in illustrative examples to follow.

[37] See APPENDIX A for full discussion of Method II and derivation of equations to follow.

TABLE 5: – Parameter Formulas.

* *	N_x^ϕ $(x \leq 7)$	N_8^ϕ	N_9^ϕ	N_{10}^ϕ	* *	N_x^ϕ $(x \leq 7)$	N_8^ϕ	N_9^ϕ	N_{10}^ϕ
N_x^{I}	$\prod_{x=1}^{7} \underline{f}_x N_8^\phi$	$\underline{f}_8 \underline{N}_9$	$\underline{N}_9$	$\overline{N}_{10}$	N_x^{V}	$\prod_{x=1}^{7} \bar{f}_x N_8^\phi$	$\underline{f}_8 \underline{N}_9$	$\underline{N}_9$	$\overline{N}_{10}$
N_x^{II}		$\underline{f}_8 \overline{N}_9$	$\overline{N}_9$	$\underline{N}_{10}$	N_x^{VI}		$\underline{f}_8 \overline{N}_9$	$\overline{N}_9$	$\underline{N}_{10}$
N_x^{III}		$\bar{f}_8 \underline{N}_9$	$\underline{N}_9$	$\overline{N}_{10}$	N_x^{VII}		$\bar{f}_8 \underline{N}_9$	$\underline{N}_9$	$\overline{N}_{10}$
N_x^{IV}		$\bar{f}_8 \overline{N}_9$	$\overline{N}_9$	$\underline{N}_{10}$	N_x^{VIII}		$\bar{f}_8 \overline{N}_9$	$\overline{N}_9$	$\underline{N}_{10}$

$$\overline{N}_{10} = \frac{1}{\underline{f}_9 w_9 + w_{10}}$$

$$\underline{N}_9 = \underline{f}_9 \overline{N}_{10}$$

$$\underline{N}_{10} = \frac{1}{\bar{f}_9 w_9 + w_{10}}$$

$$\overline{N}_9 = \bar{f}_9 \underline{N}_{10}$$

TABLE 6A: – Parameter Calculations.

$$\overline{N}_{10} = \frac{1}{0.83 \times 0.215 + 0.785} = 1.0380$$

$$\underline{N}_{10} = \frac{1}{0.99 \times 0.215 + 0.785} = 1.0022$$

$$\underline{N}_9 = 0.83\ \overline{N}_{10} = 0.8615$$

$$\overline{N}_9 = 0.99\ \underline{N}_{10} = 0.9921$$

$$\underline{N}_9 \times \begin{cases} 0.88 = 0.7581 = \bar{f}_8 \underline{N}_9 \\ 0.74 = 0.6375 = \underline{f}_8 \underline{N}_9 = \underline{N}_8 \end{cases}$$

$$\overline{N}_9 \times \begin{cases} 0.88 = 0.8730 = \bar{f}_8 \overline{N}_9 = \overline{N}_8 \\ 0.74 = 0.7341 = \underline{f}_8 \overline{N}_9 \end{cases}$$

TABLE 6B: – Parameter Calculations.

* *	N^{I}	N^{II}	N^{III}	N^{IV}	N^{V}	N^{VI}	N^{VII}	N^{VIII}
N_{10}^{ϕ}	$\bar{N}_{10}$	$\underline{N}_{10}$	$\bar{N}_{10}$	$\underline{N}_{10}$	$\bar{N}_{10}$	$\underline{N}_{10}$	$\bar{N}_{10}$	$\underline{N}_{10}$
N_{9}^{ϕ}	$\underline{N}_{9}$	$\bar{N}_{9}$	$\underline{N}_{9}$	$\bar{N}_{9}$	$\underline{N}_{9}$	$\bar{N}_{9}$	$\underline{N}_{9}$	$\bar{N}_{9}$
N_{8}^{ϕ}	$\underline{f}_8\underline{N}_9$	$\underline{f}_8\bar{N}_9$	$\bar{f}_8\underline{N}_9$	$\bar{f}_8\bar{N}_9$	$\underline{f}_8\underline{N}_9$	$\underline{f}_8\bar{N}_9$	$\bar{f}_8\underline{N}_9$	$\bar{f}_8\bar{N}_9$
N_{10}^{ϕ}	1.0380	1.0022	1.0380	1.0022	1.0380	1.0022	1.0380	1.0022
N_{9}^{ϕ}	0.8615	0.9921	0.8615	0.9921	0.8615	0.9921	0.8615	0.9921
N_{8}^{ϕ}	0.6375	0.7341	0.7581	0.8730	0.6375	0.7341	0.7581	0.8730
	$\times$ (0.82 = $\underline{f}_7$)				$\times$ (0.98 = $\bar{f}_7$)			
N_{7}^{ϕ}	0.5227	0.6019	0.6216	0.7158	0.6247	0.7194	0.7429	0.8555
	$\times$ (0.77 = $\underline{f}_6$)				$\times$ (0.92 = $\bar{f}_6$)			
N_{6}^{ϕ}	0.4024	0.4634	0.4786	0.5511	0.5747	0.6618	0.6834	0.7870
	$\times$ (0.73 = $\underline{f}_5$)				$\times$ (0.87 = $\bar{f}_5$)			
N_{5}^{ϕ}	0.2937	0.3382	0.3493	0.4023	0.4999	0.5757	0.5945	0.6846
	$\times$ (0.80 = $\underline{f}_4$)				$\times$ (0.96 = $\bar{f}_4$)			
N_{4}^{ϕ}	0.2349	0.2705	0.2794	0.3218	0.4799	0.5526	0.5707	0.6572
	$\times$ (0.64 = $\underline{f}_3$)				$\times$ (0.77 = $\bar{f}_3$)			
N_{3}^{ϕ}	0.1503	0.1731	0.1788	0.2059	0.3695	0.4255	0.4394	0.5060
	$\times$ (0.83 = $\underline{f}_2$)				$\times$ (1.00 = $\bar{f}_2$)			
N_{2}^{ϕ}	0.1247	0.1436	0.1484	0.1708	0.3695	0.4255	0.4394	0.5060
* *	$p^{\phi} = \sum_{P} w_x N_x^{\phi}$							
p^{ϕ}	0.2656	0.3058	0.3160	0.3640	0.4577	0.5272	0.5444	0.6269

Properties of the Parameter Vectors.

Regardless of the values of w_x and of $\underline{f}_x < \overline{f}_x$ for any or all x, it will be found that always: $\mathfrak{p}^I < \mathfrak{p}^\phi$ for any $\phi \neq I$; $\mathfrak{p}^{VIII} > \mathfrak{p}^\psi$ for any $\psi \neq VIII$; $\mathfrak{p}^I < \mathfrak{p}^{III} < \mathfrak{p}^{VI} < \mathfrak{p}^{VIII}$. Also, for $x < 8$, always: $N_x^I < N_x^{III} < N_x^{VI} < N_x^{VIII}$. The ordering of the remaining $\mathfrak{p}^\phi$ and (for $x < 8$) of N_x^ϕ will depend upon the actual values of w_x, $\underline{f}_x$ and $\overline{f}_x$ in a given case.

On the assumption that the extremal ratios $\underline{f}_x$ and $\overline{f}_x$ are rigid bounds to f_x^i; then, letting $\mathfrak{p}^* = P/U$; for any value of U and for any values of w_x, $\underline{f}_x$ and $\overline{f}_x$:

(a) If: $\mathfrak{p}^{III} \leq \mathfrak{p}^* \leq \mathfrak{p}^{VI}$; then:

$$N_x^{III} = \overline{N}_x \geq R_x^*/U = N_x^* \geq N_x^{VI} = \underline{N}_x;\ (x = \alpha = 8;\ \text{or: } x = z = 10)$$

$$N_x^{III} = \underline{N}_x \leq R_x^*/U = N_x^* \leq N_x^{VI} = \overline{N}_x;\ (x \neq \alpha = 8;\ \text{or: } x \neq z = 10)$$

(b) If: $\mathfrak{p}^{VI} < \mathfrak{p}^* \leqslant \mathfrak{p}^{VIII}$; then:

$$N_x^{VI} < R_x^*/U = N_x^* \leq N_x^{VIII},\ (x \leq \alpha = 8)$$

(c) If: $\mathfrak{p}^I \leq \mathfrak{p}^* < \mathfrak{p}^{III}$; then:

$$N_x^I \leq R_x^*/U = N_x^* < N_x^{III};\ (x \leq \alpha = 8)$$

(d) If:

$\mathfrak{p}^* = \mathfrak{p}^I$, then necessarily: $\tilde{\boldsymbol{R}}^* = U\boldsymbol{N}^I$

$\mathfrak{p}^* = \mathfrak{p}^{VIII}$, then necessarily: $\tilde{\boldsymbol{R}}^* = U\boldsymbol{N}^{VIII}$

(e) If $\mathfrak{p}^* < \mathfrak{p}^I$ or $\mathfrak{p}^* > \mathfrak{p}^{VIII}$, there will be *no* solution to the rate revision problem unless and until the bound $\underline{f}_x$ or $\overline{f}_x$ is relaxed for at least one x.

Additional and comparable properties will depend on the values of w_x, $\underline{f}_x$ and $\overline{f}_x$. For example, in the assumed instance of $\mathfrak{p}^{VII} > \mathfrak{p}^{VI}$ (see Table 6B), if $\mathfrak{p}^* > \mathfrak{p}^{VII}$, then $N_x^* > N_x^{VII}$ for $x \leq 8$, but this would not necessarily be the case were $\mathfrak{p}^{VII} < \mathfrak{p}^{VI}$, as it might be in particular instance.

Since the exact values of $\underline{f}_x$ and $\overline{f}_x$ depend upon judgment, these bounds may, of course, be relaxed to obtain a solution when and if the ratemaker runs afoul of one of the inequalities above. This will be a matter of judgment in a given case. In extreme cases, revision of the classification system may be necessary. The listed properties can be useful, however, in that before actual rate calculation is started, direct comparison of $\mathfrak{p}^*$ with $\mathfrak{p}^\phi$ gives immediate indication of what may be expected in the course of the rate revision.

Procedure.

The procedure will be illustrated by examples.

EXAMPLE 4.

Target Rates: $P = 3.129$; $U = 5.586$

Solution:

$$p^* = 3.129/5.586 = 0.5601$$

Choose any $p^\phi < p^*$, say p^{VI}. The only $p^\psi > p^*$ in this case is p^{VIII}, so there is no choice, but in general, any $p^\psi > p^*$ could be chosen. Let:

(26) $$bp^\phi + (1 - b)\, p^\psi = p^*;\ p^\phi < p^* < p^\psi)$$
$$(0.5272\, b + 0.6269\, (1 - b) = 0.5601)$$

whence:

$$b = 0.6700;\ \text{and:}\ (1 - b) = 0.3300$$

Then:

(27) $$\tilde{N}^* = bN^\phi + (1 - b)\, N^\psi$$

whence:

(28) $$\tilde{R}^* = U\tilde{N}^*$$

The calculation of this example is completed and verified in Table 7. There are no worksheets other than Table 7 (unless the tape from a standard model desk calculator be counted as such).

It may be noted that this problem is exactly the problem solved in Method I, Case 4, as Example 3, preceding. It will be found that although both solutions are feasible vectors, the solution of this example is such that for all x, then $\underline{f}_x < f_x^* < \overline{f}_x$, which is not the case with the solution of Example 3.

EXAMPLE 5.

Target Rates: $P = 3.129$; $U = 5.586$

Side Condition: R_3^* is to assume the value of $2.000 = R_3^o$.

Solution:

$$p^* = 3.129/5.586 = 0.5601$$

(29) $$N_3^* = N_3^o = R_3^o/U$$
$$(N_3^o = 2.000/5.586 = 0.3580)$$

There is no solution to this problem. See Property (b) of the parameter vectors, and compare N_3^o with N_3^{VI} from Table 6B.

TABLE 7: – Solution of Example 4.

$$\overset{p^{VI}}{0.5257}\, b + \overset{p^{VIII}}{0.6269}\,(1 - b) = \overset{p^{*}}{0.5601} \longrightarrow \begin{cases} b = 0.6700 \\ 1 - b = 0.3300 \end{cases}$$

$$\begin{array}{c} b \\ 0.6700 \end{array} \begin{array}{c} N^{VI} \\ \begin{bmatrix} 0.4255 \\ 0.4255 \\ 0.5526 \\ 0.5757 \\ 0.6618 \\ 0.7194 \\ 0.7341 \\ 0.9921 \\ 1.0022 \end{bmatrix} \end{array} \begin{array}{c} (1 - b) \\ + 0.3300 \end{array} \begin{array}{c} N^{VIII} \\ \begin{bmatrix} 0.5060 \\ 0.5060 \\ 0.6572 \\ 0.6846 \\ 0.7870 \\ 0.8555 \\ 0.8730 \\ 0.9921 \\ 1.0022 \end{bmatrix} \end{array} = \begin{array}{c} \tilde{N}^{*} \\ \begin{bmatrix} 0.4519 \\ 0.4519 \\ 0.5870 \\ 0.6166 \\ 0.7031 \\ 0.7642 \\ 0.7798 \\ 0.9921 \\ 1.0022 \end{bmatrix} \end{array}$$

$$\begin{array}{c} U \\ 5.586 \end{array}\tilde{N}^{*} = \begin{array}{c} \tilde{R}^{*} \\ \begin{bmatrix} R_2^{*} = 2.5243 \\ R_3^{*} = 2.5243 \\ R_4^{*} = 3.2789 \\ R_5^{*} = 3.4163 \\ R_6^{*} = 3.9275 \\ R_7^{*} = 4.2688 \\ R_8^{*} = 4.3559 \\ R_9^{*} = 5.5418 \\ R_{10}^{*} = 5.5982 \end{bmatrix} \end{array}$$

$$\hat{P}^{*} = 3.130$$
$$\hat{P}^{*} - P = 0.001$$

$$\hat{U}^{*} = 5.586$$
$$\hat{U}^{*} - U = 0$$

EXAMPLE 6.

Target Rates: $P = 2.680;\ U = 5.586$

Side Condition: $R_3{}^*$ is to assume the value of $2.000 = R_3{}^o$

Solution:

$$p^* = 2.680/5.586 = 0.4800$$
$$N_x{}^o = 2.000/5.586 = 0.3580$$

In general, a solution to this problem will be given by the equation:

$$\tilde{N}^* = \sum b_\phi{}^* N^\phi; \quad (b_\phi{}^* \geq 0; \sum b_\phi{}^* = 1) \tag{30}$$

whence the final rate vector is obtained by Eq. (28)[38]. The coefficients of Eq.(30) are given by:[39]

$$\begin{aligned} &\sum b_\phi{}^* p^\phi + (1 - \sum b_\phi{}^*)\, p^\psi = p^* \\ &\sum b_\phi{}^* N_x{}^\phi + (1 - \sum b_\phi{}^*)\, N_x{}^\psi = N_x{}^o \end{aligned} \qquad (b_\phi{}^* \geq 0; \sum b_\phi{}^* \leq 1) \tag{31}$$

Although *a* solution to Eqs. (31) always may be found by choosing not less than three values each of P^ϕ and $N_x{}^\phi$ from among the eight listed in Table 6B, not all combinations of three or more values will give a *non-negative* solution as required. The simplest approach to the problem is as follows:

Choose a value of $N_x{}^{\phi:i} < N_x{}^o$ and a value of $N_x{}^{\psi:i} > N_x{}^o$.[40] Determine b^i by the equation:

$$b^i N_x{}^{\phi:i} + (1 - b^i)\, N_x{}^{\psi:i} = N_x{}^o \tag{32}$$

and calculate:

$$\hat{p}^{*:i} = b^i p^{\phi:i} + (1 - b^i)\, p^{\psi:i} \tag{33}$$

If $\hat{p}^{*:i} = p^*$, the problem is solved by entering b^i and $(1 - b^i)$ as coefficients in Eq.(30). If $\hat{p}^{*:i} \neq p^*$, chose a value of $N_x{}^{\phi:j} < N_x{}^o$ and of $N_x{}^{\psi:j} > N_x{}^o$, where possibly (not necessarily) $\phi{:}j = \phi{:}i$ or $\psi{:}j = \psi{:}i$, but not both. Calculate b^j by Eq.(32) and, thence, $\hat{p}^{*:j}$ by Eq.(33). If $\hat{p}^{*:j} = p^*$, the problem is solved. If $\hat{p}^{*:j} \neq p^*$ and also *either* $\hat{p}^{*:i} < p^*$ and $\hat{p}^{*:j} < p^*$, *or* $\hat{p}^{*:i} > p^*$ and $\hat{p}^{*:j} > p^*$, repeat the

[38] Despite the formal similarity, Eq. (30) does NOT follow by simple change of notation in Eq. (1) of Section IV.A. See APPENDIX A.

[39] *Cf.* McIntosh, (14). p. 152, Eq. (9). Equation (30), above, *DOES* follow from Eq. (9) of the reference by simple change of notation accompanied by re-definition of terms.

[40] If $N_x{}^o < N_x{}^I$, or $N_x{}^o > N_x{}^{VIII}$, the problem is insoluble.

operation until values of $\hat{p}^{*:k}$ and $\hat{p}^{*:l}$ are obtained such that $\hat{p}^{*:k} < p^* < \hat{p}^{*:l}$. (Normally, if: $p^{\phi:i} = p^{I}$ and $p^{\psi:j} = p^{VIII}$, then $\hat{p}^{*:i} < p^* < \hat{p}^{*:j}$ or $\hat{p}^{*:i} > p^* > \hat{p}^{*:j}$; but at most not more than three of four trials should be required to bracket the value of p^* with values of $\hat{p}^*$).

Assume $\hat{p}^{*:i} < p^* < \hat{p}^{*:j}$. Calculate t by the equation:

$$t\hat{p}^{*:i} + (1 - t)\,\hat{p}^{*:j} = p^* \tag{34}$$

Thence calculate:

$$\begin{aligned} b_{\phi:i} &= tb^{i}; \text{ and: } b_{\psi:i} = t\,(1 - b^{i}) \\ b_{\phi:j} &= (1 - t)\,b^{j}; \text{ and } b_{\psi:j} = (1 - t)\,(1 - b^{j}) \end{aligned} \tag{35}$$

Thence a solution to the problem will follow upon entering the coefficients calculated by Eq.(35), together with the associated parameter vectors, $N^{\phi:i}$, etc., into Eq.(30).

The complete solution of Example 6 is given in Table 8.

Extension of Application

The procedures indicated under the Examples 4-6 may be extended to more complicated cases, *e.g.* where values of R_x^{o} are predetermined for two or more classes, or where for some class the value of f_x^{c} already is so extreme that any further movement of the value either upward or, alternatively, downward, cannot be tolerated. Such extensions involve techniques of finding directly a non-negative solution of Eqs.(31), and very possibly involve pre-calculation of additional parameter vectors beyond those given in Table 6B. Although a solution to Eqs.(30) and (31) must always exist, utilizing not more than eight parameter vectors under the classification system assumed here for illustrative purposes, the total number of possible parameter vectors, no two of which are equal, will be $2^8 = 256$, and the practical difficulty lies in determining *which* eight out of that total will serve in particular instance.

It seems probable that the full potential of Method II can be exploited in application only if computer facilities are utilized. However, although they cannot be presented simply in empirical fashion, nor be applied properly without at least a basic understanding of theory sum-

TABLE 8: – Solution of Example 6.

$P = 2.680;\ U = 5.586;\ \mathfrak{p}^* = 2.680/5.586 = 0.4800$

$R_3^* = R_3^0 = 2.000;\ N_3^0 = 2.000/5.586 = 0.3580$

$$\overset{N_3^I}{0.1503}b^i + \overset{N_3^{VI}}{0.4255}(1-b^i) = 0.3580 \longrightarrow \begin{cases} b^i = 0.2452 \\ (1-b^i) = 0.7548 \end{cases}$$

$$\overset{N_3^{IV}}{0.2059}b^j + \overset{N_3^{VIII}}{0.5060}(1-b^j) = 0.3580 \longrightarrow \begin{cases} b^j = 0.4931 \\ (1-b^j) = 0.5069 \end{cases}$$

$$b^i\mathfrak{p}^I + (1-b^i)\,\mathfrak{p}^{VI} = 0.4630 = \hat{\mathfrak{p}}^{*:i}$$

$$b^j\mathfrak{p}^{IV} + (1-b^j)\,\mathfrak{p}^{VIII} = 0.4971 = \hat{\mathfrak{p}}^{*:j}$$

$$0.4630t + 0.4971(1-t) = 0.4800 \longrightarrow \begin{cases} t = 0.5014 \\ (1-t) = 0.4986 \end{cases}$$

$$0.5014 \times \begin{cases} b^i = 0.1229 = b_I^* \\ (1-b^i) = 0.3784 = b_{VI}^* \end{cases} \qquad 0.4986 \times \begin{cases} b^j = 0.2458 = b_{IV}^* \\ (1-b^j) = 0.2527 = b_{VIII}^* \end{cases}$$

$$\overset{U}{5.586}(b_I^*N^I + b_{IV}^*N^{IV} + b_{VI}^*N^{VI} + b_{VIII}^*N^{VIII} = \tilde{N}^*) = \overset{\tilde{R}^*}{\begin{bmatrix} R_2^* = 1.9327 \\ R_3^0 = 1.9986 \\ R_4^* = 2.6794 \\ R_5^* = 2.9354 \\ R_6^* = 3.5415 \\ R_7^* = 4.0688 \\ R_8^* = 4.4324 \\ R_9^* = 5.4502 \\ R_{10}^* = 5.6206 \end{bmatrix}}$$

$$\hat{P}^* = 2.660$$
$$\hat{P}^* - P = -0.020$$

$$\hat{U}^* = 5.584$$
$$\hat{U}^* - U = -0.002$$

marized in APPENDIX A, extensions of Method II beyond the elementary applications illustrated above most certainly seem entirely practical.

VII. CONCLUSION

No basic concept new to fire rating theory has been offered in Sections I-VI, preceding, nor is to be offered in appendices to follow. The substance of the entire development is reformulation and extension of theory previously suggested in forms not only incomplete, but also unfortunately imprecise. In Section VI.C., public fire protection facilities simply are treated as a specific example of the variable "hazard r" earlier discussed in general terms by McIntosh, both from a theoretical and from a practical standpoint.[41] Section VI.D., preceding, and APPENDIX A, to follow, are foreshadowed by an earlier application of the theory of polyhedral sets to the fire schedule rating problem on the mathematically acceptable but actuarially unrealistic assumption that the problem will be essentially linear,[42] which it is not. The probable severity of fire loss contingent upon occurrence is not stochastically independent of the probability of occurrence, whence it will follow that the charges and credits of a fire rating schedule cannot be strictly additive, except as an approximation over a very limited range of variation.

The utility of Eqs.(IV. A.1) and (VI.D.31) is that, taken together, these transformations permit reduction of the problem to linear forms,[43] for which ready-made solutions usually will be available by the theorems of linear algebra.[44]

The tool marks can be polished off of the final product. All equations of Section VI.C. are conventional and somewhat elementary algebraic expressions, and the vector equations of Section VI.D. could be replaced by ordinary simultaneous equations at no sacrifice other than of typographical economy. On the other hand, a proof necessary to the support of Method I, Case 4, is complete in five short matrix equations,[45] whereas it is extremely tedious to prove the same result by ordi-

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[41] McIntosh, (15), p. 15, and (16), p. 131. Equation (9), p. 17 of the reference is the forerunner of Eqs. (VI. C. 23 –) presented here. The graphical method of curve adjustment cited without description on p. 20 of the reference is a graphical solution of Eqs.(VI.C. 23 –), though not same method presented in Section VI.C., preceding here.

[42] McIntosh, (14). pp. 140-146, & pp. 150-152. See also Note 39, *sup.*

[43] By permitting immediate introduction of curvilinear coordinates. See APPENDIX A.

[44] See BIBLIOGRAPHY, to follow.

[45] Eqs. (B.9) – (B.13) of APPENDIX B.

nary algebra. Further, although it is easily demonstrated by conventional algebra *why* Method II works *when* it works, it is only in terms of the properties of polyhedral convex sets[46] that the conditions under which solution is possible are expeditiously found. If the properties of Eqs. (VI.D.30) and (VI.D.31) (of which Eq.(A.38) of APPENDIX A is the generalized form) can be completely investigated by ordinary algebra, the operation certainly will be interminable, as will be the problem of distinguishing between Eqs.(IV. A.1) and (VI.D.30), which, though similar in form, are by no means equivalent to each other.[47]

Even so, it must be admitted that Method II can be demonstrated in a fashion much simpler than by the full, formal developments to be found in APPENDIX A. The support of Method II is not, however, the sole reason, nor even the primary reason, for APPENDIX A. When the development there presented was begun (in search of a method to define rate bounds under conditions such that all schedule charges are *not* to be assumed as additive) there was no faint suspicion that Method II would begin to take shape on the work-bench almost immediately; the original concept of Section VI was restricted to Section VI.C., before early drafts of APPENDIX A relegated Method I to by-product status.

The main purpose of APPENDIX A is to submit for evaluation, in all detail, an actuarial research tool which seems of not inconsiderable potential utility. Section 1 of APPENDIX A is intended to stand on its own feet. When the transformations defined in that section are further compounded with the particular transformation defined in Section 2 of APPENDIX A, then Method II is the result. However, other transformations can be grafted onto the development of Section 1 as circumstance may dictate; the transformation of the coefficients of Eq.(IV. A.1) into the parameter vectors of Method II is by no means the only direction the extension of Section 1 could have taken.

Once a transformation is defined, it can then be compounded in almost any desired direction to achieve almost any desired result with a minimum of effort. There is no reason why the transformations F^* and F^k of equations (A.28) and (A.32) must be defined in terms of Eqs.(A.26) except for present purposes only. It would be interesting to see what might result from Eq.(A.32) were F^k defined in terms of those equations "not of a simple rational form" which have caused Messrs. Bailey and Simon

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46 *Cf.* among others, Kemeny *et al.*, (6). Ch. 5.

47 See APPENDIX A.

to express a plaintive wish for a "small computer."[48] Were F^k to be so re-defined, very obviously the remaining elements of Eq.(A.32) must be appropriately re-defined also, which might prove difficult or perhaps impossible. But the idea seems worth a try.

A second possible line of research would seem to lie in formal recognition of P and of U as the stochastic variables which actually they are, instead of as the constants which here they are unrealistically assumed to be. It does not seem certain that this line is entirely divorced from the problem attacked by Bailey and Simon (cited above), namely that of determining the best set of classification and sub-classification relativities under a multiple-classification system.

APPENDIX A

1. The Adjusted Rate Structures (Sections IV. A. & V.)

a. *The Feasible Rate Structure,*

Implicitly by definitions given, for all x, then:

$w_x \geqslant 0; R_x \geq 0;$ if $\boldsymbol{R}^{P:j}$ is P-reconciled, $\boldsymbol{R}_P^{P:j}$, then: $\sum_P w_x R_x^j = P.$

If $w_\zeta = 0$, drop the ζ^{th} term from the summation.[49] Then $w_x > 0$ for all x remaining. It is assumed below that $w_x > 0$ for all $x \leq a$.

Let $R_x^j = 0$ for all $x \neq \mu$. Then $R_\mu^j = P/w_\mu;$ whence it follows that, since never: $R_x^j < 0$; then never: $R_\mu^j > P/w_\mu;$ whence always:

(A.1) $$0 \leq R_x^j \leq P/w_x; \; (x \leq a)$$

Thence it follows that:

(a) *The set* $\{\mathbf{R}_P{}^P\}$ *is bounded.* It is contained in a hypersphere by virtue of Ineq.(A.1).[50]

(b) *The set* $\{\boldsymbol{R}_P{}^P\}$ *is polyhedral and convex.* It is the intersection of the closed half spaces defined by Ineq.(A.1).[51]

Let $\dot{\boldsymbol{R}}_\zeta{}^P = (0, \ldots, P/w_\zeta, \ldots, 0; 0, \ldots, 0)$, where $\zeta \leq a$. Then the vector $\dot{\boldsymbol{R}}_\zeta{}^P$ is an extreme point (or "extremal vector") of $\{\boldsymbol{R}_P{}^P\}$, whence: If and only if $\boldsymbol{R}^{P:j}$ is a member of $\{\boldsymbol{R}_P{}^P\}$, then:

(A.2) $$\boldsymbol{R}_P{}^{P:j} = \sum_P \mathrm{a}_x{}^{P:j} \dot{\boldsymbol{R}}_x{}^P; \; (\mathrm{a}_x{}^{P:j} \geq 0; \sum_P \mathrm{a}_x{}^{P:j} = 1)^{52}$$

[48] Bailey & Simon, (12). Specifically: Section B, pp. 11 & 13.
[49] See Section 3, following.
[50] Taylor, (20). p. 70.
[51] Kemeny *et al.* (6). pp. 340-341.
[52] *Ibid.* p. 347.

The vectors $\dot{R}_x^P$ form a basis for the space S_P of all protected rate vectors R^P, and the vectors $\dot{R}_x^P$ are α in number. Hence S_P is α-dimensional and R^P is α-dimensional.[53]

Eq.(A.2) may be rewritten:

$$\text{(A.3)} \quad R_P^{P:j} = \sum_{x=1}^{\alpha-1} a_x^{P:j} (\dot{R}_x^P - \dot{R}_\alpha^P) + \dot{R}_\alpha^P; \quad (a_x^{P:j} \geqslant 0; \sum_{x=1}^{\alpha-1} a_x^{P:j} \leq 1)$$

whence $\{R_P^P\}$ is spanned by the $(\alpha - 1)$ linearly independent vectors $(\dot{R}_x^P - \dot{R}_\alpha^P)$, hence $\{R_P^P\}$ is an $(\alpha - 1)$-dimensional[54] affine[55] subset of S_P.

By exact analogy to the foregoing:

$$\text{(A.4)} \qquad 0 \leq R_x^k \leq U/w_x; \quad (x \geq \beta)$$

The set of all U-reconciled rate vectors, $\{R_U^U\}$, is a $(z - \beta)$-dimensional affine subset of the space S_U of all unprotected rate vectors.

The set $\{R_U^U\}$ is a bounded, polyhedral convex set having as extreme points the $(z - \beta)$ vectors $\dot{R}_\mu^U = (0, \ldots 0; 0, \ldots, U/w_\mu, \ldots, 0)$, where $\mu \geq \beta$, whence: If and only if $R^{U:k}$ is a member of $\{R_U^U\}$, then:

$$\text{(A.5)} \qquad R^{U:k} = \textstyle\sum_U a_x^{U:k} \dot{R}_x^U; \quad (a_x^{U:k} \geq 0; \sum_U a_x^{U:k} = 1)$$

Any feasible vector, $\tilde{R}^i$, may be written uniquely as: $\tilde{R}^i = R_P^{P:j} + R_U^{U:k}$, whence it follows that the feasible rate structure, $\{\tilde{R}\}$ is the direct sum of $\{R_P^P\}$ and $\{R_U^U\}$:

$$\text{(A.6)} \qquad \{\tilde{R}\} = \{R_P^P\} \oplus \{R_U^U\}$$

whence the dimension of $\{\tilde{R}\} = (\alpha - 1) + (z - \beta) = z - 2.$[56]

Equation (IV. A.1) follows from Eqs.(A.2), (A.5) and (A.6).

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[53] Birkhoff & MacLane, (1). pp. 168-169 & 188. It may be noted that this basis is orthogonal.

[54] *Ibid.* pp. 164 & 168-169. Designation of the initial point with subscript "α" is arbitrary here. The usual designation of the initial point is with subscript "zero," but in present instance this would require re-numbering of the vectors, any one of which could have been chosen as initial point.

[55] *Ibid.* p. 291.

[56] *Ibid.* p. 185. (The direct sum is denoted by "$\dot{+}$" in the reference (see p. 472), but "$\oplus$" seems a more common symbol.)

Let:

$$B = \begin{bmatrix} P/w_1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \cdot & \dots & \cdot & \dots \\ 0 & \dots & P/w_a & 0 & \dots & 0 \\ 0 & \dots & 0 & U/w_\beta & \dots & 0 \\ \dots & \cdot & \dots & \cdot & \dots \\ 0 & \dots & 0 & 0 & \dots & U/w_z \end{bmatrix} = \text{The \textit{“basis matrix.”}}$$

$$i = j = k; \text{ and: } Y = \begin{cases} P; (x \leq a) \\ U; (x \geq \beta) \end{cases}$$

$a_x^{Y:i} = a_x^i$; and $\mathbf{a}^i = a_1^i, \dots, a_z^i) =$ A *“primary coefficient vector.”*

Then Eq.(IV. A.1) my be written:

(A.7)[57] $\tilde{R}^i = \tilde{\mathbf{a}}^i B; (a_x^i \geq 0; \sum_P a_x^i = 1; \sum_U a_x^i = 1)$

Let:

$$W = \begin{bmatrix} w_1 & 0 \\ \dots & \dots \\ w_a & 0 \\ 0 & w_\beta \\ \dots & \dots \\ 0 & w_z \end{bmatrix} = \text{The \textit{“weighting matrix.”}}$$

$Y^* = (P\ ;\ U) =$ The *“target vector”*
$\hat{Y}^i = (\hat{P}^i; \hat{U}^i) =$ A *“trial average vector.”*

Then in general:

(A.8) $R^i W = \mathbf{a}^i B W = \hat{Y}^i;\ (a_x^i \geq 0;\ \sum_P a_x^i = 1;\ \sum_U a_x^i = 1)$ and in particular:

[57] The validity of Eq.(A.7) depends upon the symmetry of the matrix B. The equation is not general for arbitrary choice of basis vectors.

(A.9) $\tilde{R}^i W = \tilde{a}^i B W = \hat{Y}^i = Y^*$; $(a_x^i \geq 0;\ \sum_P a_x^i = 1;\ \ \sum_U a_x^i = 1)$

Where $\tilde{a}^i$ is defined by the condition that $a^i = \tilde{a}^i$ if and only if $R^i = \tilde{R}^i$.

Let B denote the linear transformation whose matrix is B, and W denote the linear transformation whose matrix is W.

Then:

(A.10) $$a^i \mathsf{B} = R^i$$

and the transformation, B, is one-one.[58]

Also:

(A.11) $$R^i \mathsf{W} = a^i \mathsf{B} \mathsf{W} = \hat{Y}^i$$

b. *The Operational Rate Structure, op* $\{\tilde{\mathbf{R}}\}$.

By Eqs. (IV. A.1.a) and definition of f_x:

(A.12)
$$\begin{aligned} f_x^i &= (a_x^{P:i} w_{x+1} P)/(a_{x+1}^{P:i} w_x P);\ (x < \alpha) \\ f_\alpha^i &= (a_\alpha^{P:i} w_\beta P)/(a_\beta^{U:i} w_\alpha U) \\ f_x^i &= (a_x^{U:i} w_{x+1} U)/(a_{x+1}^{U:i} w_x U);\ (x \geq \beta) \end{aligned}$$

whence:

(A.13) $$\underline{f}_x a_{x+1}^i w_x / w_{x+1} \leq a_x^i \leq \bar{f}_x a_{x+1}^i w_x / w_{x+1};\ (x \neq \alpha)$$

(A.14) $$\underline{f}_\alpha a_\beta^i w_\alpha U / w_\beta P \leq a_\alpha^i \leq \bar{f}_\alpha a_\beta^i w_\alpha U / w_\beta P$$

whence Eq. (IV. A.3) follows from Eq. (IV. A.1) upon imposition of Constraint (1): $0 < \underline{f}_x \leq f_x^i \leq \bar{f}_x < r_x/r_{x+1}$.

By Eqs. (A.2) and (A.5), and by Ineq. (A.12)

(A.15)
$$\begin{aligned} &op R_P^{P:i} = a^{P:i} B \\ &(\underline{f}_x a_{x+1}^{P:i} w_x / w_{x+1} \leq a_x^{P:i} \\ &\qquad \leq \bar{f}_x a_{x+1}^{P:i} w_x / w_{x+1};\ \sum_P a_x^{P:i} = 1) \end{aligned}$$

(A.16)
$$\begin{aligned} &op R_U^{U:i} = a^{U:i} B \\ &(\underline{f}_x a_{x+1}^{U:i} w_x / w_{x+1} \leq a_x^{U:i} \\ &\qquad \leq \bar{f}_x a_{x+1}^{U:i} w_x / w_{x+1};\ \sum_U a_x^{U:i} = 1) \end{aligned}$$

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[58] Birkhoff & MacLane, (1). p. 121. To avoid notational confusion, see paragraph near the top of the page, beginning: "In the choice of notation for transformations***." The present author will follow Birkhoff & MacLane in writing the point under transformation to the *left* of the transformation symbol.

where, by analogy to definitions of $\boldsymbol{R}^P$ and $\boldsymbol{R}^U$, let:

$$\mathbf{a}^P = (a_1{}^P, \ldots, a_\alpha{}^P; 0, \ldots, 0)$$
$$\mathbf{a}^U = (0, \ldots, 0; a_\beta{}^U, \ldots, a_z{}^U)$$

By Eqs. (IV. A.3), (A.7), (A.14) and (A.15):

(A.16.a) $op\tilde{\boldsymbol{R}}^i = op\boldsymbol{R}_P^{P:i} + op\boldsymbol{R}_U^{U:i}; \quad (\underline{f}_\alpha a_\beta{}^i w_\alpha U / w_\beta P \leq a_\alpha{}^i \leq \bar{f}_\alpha{}^i a_\beta{}^i w_\alpha U / w_\beta P)$

Note that Ineq. (A.14), appearing as a constraint in Eqs. (IV. A.3) and (A.16.a) does *not* appear either in Eq. (A.15) or in Eq. (A.16). The ratio $f_\alpha{}^i$ is a "link," so to speak, *between* $op\{\boldsymbol{R}_P^P\}$ and $op\{\boldsymbol{R}_U^U\}$; it is not associated exclusively with either subset of the direct sum, $op\{\tilde{\boldsymbol{R}}\}$. It is to be demonstrated that $op\{\tilde{\boldsymbol{R}}\}$ may be the empty set. Let: $\tilde{\mathbf{a}}^i = {}_{op}\tilde{\boldsymbol{a}}^i$ if $a_x{}^i$ conforms to Ineq. (A.13) for all $x \neq \alpha$; *and*: $a_\alpha{}^i$ conforms to Ineq. (A.14). Then by Eq. (A.7):

(A.17) $$op\tilde{\boldsymbol{R}}^i = {}_{op}\tilde{\mathbf{a}}^i\boldsymbol{B}$$

Let a *"ratio vector"*, $\boldsymbol{f}^i$, be defined by:

$$\boldsymbol{f}^i = (f_1{}^i, \ldots, f_{\alpha-1}{}^i; f_\alpha{}^i; f_\beta{}^i, \ldots, f_{z-1}{}^i)$$ [59]

Then the inner inequalities of Constraint (1): $\underline{f}_x \leq f_x \leq \bar{f}_x$, define a bounded, polyhedral convex set, $op\{\boldsymbol{f}\}$, the extreme points of which are:

$$\boldsymbol{f}^\phi = (f'_1, \ldots f'_{\alpha-1}; f'_\alpha; f'_\beta, \ldots, f'_{z-1}); \quad (f'_x = \underline{f}_x \text{ or } \bar{f}_x)$$

The number of extreme points of $op\{\boldsymbol{f}\}$ is 2^{z-1}, and the set is $(z-2)$-dimensional.[60]

Now, the ratio notation, $f_{\zeta:\mu} = R_\zeta / R_{\mu+1}$, adopted for convenience in Section V.C., may obscure the development henceforward. Returning to conventional product notation and simplifying, Eqs. (V.C.15) and (V.C.16) may be rewritten:

(A.18) $$f_\alpha{}^i \prod_{x=\beta}^{z-1} f_x \frac{\sum_P (w_x \prod_{(1)})}{\sum_U (w_x \prod_{(2)})} = \frac{\hat{P}^i}{\hat{U}^i} = \hat{p}^i$$

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[59] The artificial ratio $f_z = 1$, defined in Section V.C. for notational convenience only, may be introduced as the z^{th} component of $\boldsymbol{f}^i$ if desired, but this is not necessary.

[60] The demonstration is analogous to the demonstration by Eqs. (A.2) and (A.3) that $\{\boldsymbol{R}_P{}^P\}$ is $(\alpha - 1)$-dimensional.

$$\text{(A.19-a)}\qquad R_\zeta{}^i = \frac{\prod_{(1)}\hat{P}^i}{\sum_P(w_x\prod_{(1)})} = R_\zeta(f^i;\hat{P}^i)$$

$$(\zeta \leq a)$$

$$\text{-b)}\qquad = \frac{f_a{}^i\prod_{(1)}\prod_{(3)}\hat{U}^i}{\sum_U(w_x\prod_{(2)})} = R_\zeta(f^i;\hat{U}^i)$$

$$\text{(A.20-a)}\qquad R_\mu{}^i = \frac{\hat{P}^i}{f_a\prod_{(3)}\sum_P(w_x\prod_{(1)})} = R_\mu(f^i;\hat{P}^i)$$

$$(\mu \geq \beta)$$

$$\text{-b)}\qquad = \frac{\prod_{(2)}\hat{U}^i}{\sum_U(w_x\prod_{(2)})} = R_\mu(f^i;\hat{U}^i)$$

where:

$$\prod_{(1)} = \begin{cases} \prod_{x=\zeta}^{a-1} f_x \text{ ; if: } \zeta \leq a-1 \\ 1 \quad \text{ ; if: } \zeta = a \end{cases}$$

$$\prod_{(2)} = \begin{cases} \prod_{x=\mu}^{z-1} f_x \text{ ; if: } \beta \leq \mu \leq z-1 \\ 1 \quad \text{ ; if: } \mu = z \end{cases}$$

$$\prod_{(3)} = \prod_{x=\beta}^{\mu} f_x \text{ ; if: } \beta \leq \mu \leq z-1$$

and Eqs.(A.19-) and (A.20-) may be consolidated into:

$$\text{(A.21)}\qquad \boldsymbol{R}^{i:j} = \boldsymbol{R}(f^i\,;\hat{\boldsymbol{Y}}^j)$$

$$= (\ldots, R_\zeta(f^i;\hat{\boldsymbol{Y}}^j), \ldots; \ldots, R_\mu(f^i;\hat{\boldsymbol{Y}}^j), \ldots)$$

where possibly, but not necessarily, $j = i$. It does not follow that if $\hat{\boldsymbol{Y}}^j = \hat{\boldsymbol{Y}}^i$, then necessarily $f^j = f^i$.

Define a set $\{\bar{f}\}$ of *"feasible ratio vectors,"* $\bar{f}^j$ by the condition that if and only if $\bar{f}^j$ is a member of $\{\bar{f}\}$, then by Eq.(A.21) necessarily: $\hat{\boldsymbol{Y}}^i = \boldsymbol{Y}^*$. By the previously-given definition of $op\{f\}$, it follows that if f^i is a member of $op\{f\}$, then $\boldsymbol{R}^{i:j}$ will necessarily conform to Constraint (I) but will not necessarily be feasible. By this definition it follows that if f^i is a member of $\{\bar{f}\}$, then $\boldsymbol{R}^{i:j}$ necessarily will be feasible but will not necessarily conform to Constraint (I). Therefore, if a set $op\{\tilde{f}\}$ be defined as the intersection of $op\{f\}$ and $\{\tilde{f}\}$, then necessarily $\boldsymbol{R}^{i:j}$ will conform to Constraint (I) *and* will be feasible. Hence, if f^i is a member of $op\{\tilde{f}\}$, then: $\boldsymbol{R}^{i:j} = op\boldsymbol{R}^{i:*} = op\tilde{\boldsymbol{R}}^i$, where the feasible tilde, "$\sim$", now replaces the second superscript.

If $\tilde{f}$ is a member of $op\{\tilde{f}\}$, let $\tilde{f} = op\tilde{f}$; and if $\tilde{\boldsymbol{R}}^i = op\tilde{\boldsymbol{R}}^i$, let $\tilde{\mathbf{a}}^i = {}_{op}\tilde{\mathbf{a}}^i$. Thence by Eqs.(A.7) – (A.9) and (A.21):

In general:

$$R(f^k;\, \hat{Y}^j) = \boldsymbol{R}^{k:j} = \mathbf{a}^{k:j}\, \boldsymbol{B} \tag{A.22}$$

and in particular:

$$R(opf^i;\, Y^*) = op\tilde{\boldsymbol{R}}^i = {}_{op}\tilde{a}^i\, \boldsymbol{B} \tag{A.23}$$

It follows from Eqs.(A.8) and (A.9) that a set of *"secondary coefficient vectors,"* $\{\tilde{b}\}$, will exist such that:

$$\textstyle\sum_j b_j{}^k\,(\hat{\boldsymbol{Y}}^j;\, \mathbf{a}^{k:j}) = (\boldsymbol{Y}^*;\, \mathbf{a}^{k:*}) = (\boldsymbol{Y}^*;\, \tilde{\mathbf{a}}^k) \tag{A.24}$$

where $b_j{}^k$ is the j^{th} component of the vector $\tilde{b}^k$; and it follows further that $\{\tilde{b}\}$ will be the solution set of the system of simultaneous equations:

$$\begin{aligned} \textstyle\sum_j b_j{}^k \hat{P}^j &= P \\ \textstyle\sum_j b_j{}^k \hat{U}^j &= U \end{aligned} \tag{A.25}$$

Hence a feasible rate vector, $\tilde{\boldsymbol{R}}^k$, always may be obtained from two or more estimated trial coefficient vectors, $\mathbf{a}^{k:\gamma}$ $\mathbf{a}^{k:\delta}$, ... This is no guarantee, however, that $\tilde{\boldsymbol{R}}^k$ will conform to Constraint (I), and if Constraint (I) may be violated, then $\tilde{\boldsymbol{R}}^k$ can be obtained directly and with less effort by Method I (Section VI.C. and APPENDIX B), from any one of the trial vectors, $\boldsymbol{R}^{k:\gamma}$, $\boldsymbol{R}^{k:\delta}$, etc., particularly if $\boldsymbol{R}^e$ may be taken as $\boldsymbol{R}^{k:\gamma}$. But if Constraint (I) may not be discarded, or if for some one or more classes, the final adjusted rate, $R_x{}^*$, is to assume a predetermined value, $R_x{}^o$, then Method I will not give $\tilde{\boldsymbol{R}}^*$ directly, except by coincidence or after lengthy trial and error to determine an appropriate trial vector.

By Eqs. (V. A.1.a.), (A.12) and (A.20-b),[61] it will follow in straightforward fashion that the components, $a_x{}^i$, of any feasible primary coefficient vector, $\bar{a}^i$, will be given by:

(A.26-a) $$a_z{}^i = \frac{w_z}{\sum_U (w_x \prod_{(z)})}$$

-b) $$a_\mu{}^i = \prod_{(z)} a_z{}^i \left(\frac{w_\mu}{w_z}\right);\ (\beta \leq \mu \leq z - 1)$$

-c) $$a_a{}^i = f_a{}^i \prod_{x=\beta}^{z-1} f_x{}^i a_z{}^i \left(\frac{w_a}{w_z}\right)\left(\frac{U}{P}\right)$$

-d) $$a_\zeta{}^i = \prod_{x=\zeta}^{z-1} f_x{}^i a_z{}^i \left(\frac{w_a}{w_z}\right)\left(\frac{U}{P}\right);\ (\zeta \leq a - 1)$$

and Eqs. (A.26-) may be consolidated into:

(A.27) $$\bar{\mathbf{a}}^i = \mathbf{a}(\bar{f}^i; \mathcal{P}^*)$$

Equation (A.27) defines a one-one transformation:[62]

$F^*: \bar{f}^i \rightarrow \bar{\mathbf{a}}^i$. The transformation F^* serves to introduce curvilinear coordinates,[63] whereby the linear transformation of Eq. (A.10), $B: \bar{\mathbf{a}} \rightarrow \bar{\boldsymbol{R}}$ may be substituted in calculation for the non-linear transformation of $\{\bar{f}\}$ onto $\{\bar{\boldsymbol{R}}\}$ defined by Eq. (A.21) when $\hat{\boldsymbol{Y}}^i = \boldsymbol{Y}^*$. By Eq. (A.27):

(A.28) $$\bar{f}^i F^* = \bar{\mathbf{a}}^i$$

whence by Eqs. (A.10) and (A.11):

(A.29) $$(\bar{f}^i F^*) B = \bar{\mathbf{a}}^i B = \bar{\boldsymbol{R}}^i$$
$$(\bar{f}^i F^*) B W = \bar{\mathbf{a}}^i B W = \boldsymbol{Y}^*$$

To establish the validity of Eqs. (A.28) and (A.29), it is sufficient to

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[61] The practical reasons for selecting Eq. (A.20-b) specifically from among the four equations, Eqs. (A.19-) and (A.20-), will become apparent in Section 3, to follow.

[62] Despite the formidable appearance of the function $\mathbf{a}(f^i; \mathcal{P}^*)$, the demonstration that F^* is one-one, is very easy. If Eqs. (A.26-b) – (A.26-d) are expanded by substitution of the value of $a_z{}^i$ from Eq. (A.26-a), and a system of simultaneous equations is set up from the recursion formulas obtained by solving Eqs. (A.12) for $a_x{}^i$ in terms of $a_{x+1}{}^i$, then all denominators cancel out immediately, and the rest will follow in simple and straightforward fashion.

[63] Kaplan, (5) pp. 96 & 151; but see also pp. 132 *ff*. For any fixed value of $\hat{\mathcal{P}}^i = \hat{P}^i / \hat{U}^i$, the several ratios $f_x{}^i$ are functionally dependent.

prove that F^* is one-one; and see *Note* 62, preceding. It is *not* necessary (fortunately) to define the matrix of F^*, thence to proceed by Eq. (A.7) and (A.9).

Upon substitution of $\hat{U}^j/\hat{P}^j = 1/\hat{\mathcal{P}}^j$ for $U/P = 1/\mathcal{P}^*$ in Eqs. (A.26-c) and (A.26-d),[64] the complete generalization of Eqs. (A.27) – (A.29) follows by exact analogy to the development of those equations, whence:

(A.30) $$\mathbf{a}^{k:j} = \mathbf{a}(f^k;\ \hat{\mathcal{P}}^j)$$

(A.31) $$f^k F^j = \mathbf{a}^{k:j}$$

(A.32) $$(f^k F^j) BW = \mathbf{a}^{k:j} BW = \hat{\mathbf{Y}}^j$$

In particular, if $f^k = f^\phi =$ an extreme point of $op\{f\}$, then

(A.33) $$\mathbf{a}^{\phi:\phi} = \mathbf{a}^\phi = \mathbf{a}(f^\phi; \mathcal{P}^\phi)$$

where $\mathcal{P}^\phi$ is calculated by Eq. (A.18), letting $f^i = f^\phi$; and where arbitrarily by convention it is required that $\mathcal{P}$ carry the superscript of f, regardless of the fact that possibly $\mathcal{P}^\phi = \mathcal{P}^\psi$ where $f^\phi \neq f^\psi$.

By Eq. (A.33), the extreme points of $op\{f\}$ are mapped in one-one correspondence onto the vectors $\mathbf{a}^\phi$. Extending to any value of k the convention that $\mathcal{P}$ always must carry the superscript of f, then all remaining points, f^k of the set $op\{f\}$ are mapped by Eq. (A.30) onto the vectors $\mathbf{a}^k$.

Thus there exists a family of transformations, $\{F^k\}$, whereby every f^k belonging to $op\{f\}$ is mapped in one-one correspondence into a coefficient vector, $\mathbf{a}^k$; where possibly but not necessarily, $k = \phi$. By the definition of $op\{f\}$ and the derivation of Eq. (A.30) it follows that any $\mathbf{a}^k$ will conform to Constraint (I) if $\mathbf{a}^k = f^k F^k$ and f^k is a member of $\{f\}$. Let the set of all such coefficient vectors, $\mathbf{a}^k$, be designated $c\{\mathbf{a}\}$.

It can be shown that $c\{\mathbf{a}\}$ is a bounded, polyhedral convex set whose extreme points are the vectors $\mathbf{a}^\phi = f^\phi F^\phi$.[65] Thence it follows that any

[64] Justified by Eqs. (A.19-) and (A.20-), and by the formal similarity to Eqs. (IV.A.1.a) of the equations: $R_x{}^i = a^{P:i}\hat{P}^i/w_x$, where $x \leq \alpha$; and: $R_x{}^i = a_x{}^{U:i}\,\hat{U}^i/w_x$, where $x \geq \beta$. The substitution is equivalent to re-definition of the basis vectors in terms of $\hat{P}^i$ and $\hat{U}^i$, though the operation is not identical in concept to a change of basis.

[65] The fact that the set is bounded and convex follows immediately from the facts that $op\{f\}$ is bounded; and that, by its form, $a(f; \mathcal{P})$ is a continuous function. The rest will follow from Eqs. (A.18) and (A.30) by the use of Lagrange multipliers. (Kaplan, (5). p. 128 *ff*.)

$\mathbf{a}^k$ which is a member of $c\{\mathbf{a}\}$ will be given by:

(A.34) $$\mathbf{a}^k = \Sigma_\phi b_\phi{}^k \mathbf{a}^\phi; \quad (b_\phi{}^k \geq 0; \ \Sigma_\phi b_\phi{}^k = 1)$$

Thence it follows that, if by Eq. (A.24):

(A.35) $$\Sigma_\phi b_\phi{}^k (\hat{\mathbf{Y}}^\phi; \mathbf{a}^\phi) = (\mathbf{Y}^*; \tilde{\mathbf{a}}^k); \quad (b_\phi{}^k \geq 0; \ \Sigma_\phi b_\phi{}^k = 1)$$

then necessarily: $\tilde{a}^k = {}_{op}\tilde{a}^k$; whence the rate vector $\widetilde{opR}^k = {}_{op}\tilde{a}^k \boldsymbol{B}$ must be feasible and also must conform to Constraint (I); whence the rate revision problem is solved.

Since by Eq.(A.7), then: $a_x{}^k \geq 0; \ \Sigma_P a_x{}^k = 1; \ \Sigma_U a_x{}^k = 1;$ then the system of ordinary simultaneous equations equivalent to Eq.(A.35) need never contain more than z rows, the first two of which give the target rates, U and P. The remaining rows may be formulated to contain as the constant terms not more than $(z - \beta)$ pre-selected values, $a_x{}^{U:o} = w_x R_x{}^o / U$, of the coefficients $a_x{}^{U:k}$, plus $(\alpha - 1)$ values, $a_x{}^{P:o} = w_x R_x{}^o / P$, of the coefficients $a_x{}^{P:k}$,[66] where $R_x{}^o$ is a pre-selected value of $R_x{}^k$.

A solution to Eq.(A.35) always must exist, since, as noted, there will be 2^{z-1} extreme points, f^ϕ, of $op\{f\}$, and hence there will be 2^{z-1} choices among the vectors a^ϕ from which to select at least $(z + 1)$ vectors to give a system of not more than z equations in not less than $z + 1$ unknown secondary coefficients, $b_\phi{}^k$. However, a *non-negative* solution may not exist. By the form of Eq.(A.18) (remembering that for all x, then $f_x > 0$ by Constraint (I), and $w_x > 0$ by hypothesis) it follows that $\mathcal{P}^\psi = \max\{\mathcal{P}^\phi\}$ when $f_x{}^\psi = \bar{f}_x$ for all x, and $\mathcal{P}^\psi = \min\{\mathcal{P}^\phi\}$ when $f_x{}^\psi = \underline{f}_x$ for all x. Thence it will follow that if $\mathcal{P}^* = P/U > \max\{\mathcal{P}^\phi\}$ or $\mathcal{P}^* < \min\{\mathcal{P}^\phi\}$, then necessarily $b_\phi{}^k < 0$ for at least one ϕ in the solution of Eq.(A.35); whence $\mathbf{a}^k = \Sigma_\phi b_\phi{}^k \mathbf{a}^\phi$ will not belong to $c\{\mathbf{a}\}$ and hence will not conform to Constraint (I).

If $\min\{\mathcal{P}^\phi\} \leq \mathcal{P}^* \leq \max\{\mathcal{P}^\phi\}$, a non-negative solution to Eq.(A.35) will exist, which will be given by some combination of not more than z of the 2^{z-1} vectors $(\hat{\mathbf{Y}}^\phi; \mathbf{a}^\phi)$. It will be a unique solution if $\mathcal{P}^* = \max\{\mathcal{P}^\phi\}$ or $\mathcal{P}^* = \min\{\mathcal{P}^\phi\}$. This follows from the properties of $c\{\mathbf{a}^k\}$ as a bounded, polyhedral, convex set.

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[66] See Subsection a., preceding. The nature of $\{\tilde{\boldsymbol{R}}\}$ as the direct sum of $\{\boldsymbol{R}_P{}^P\}$ and $\{\boldsymbol{R}_U{}^U\}$ precludes the selection of more than $(z - \beta)$ of the $a_x{}^{U:k}$; or of more than $(\alpha - 1)$ of the $a_x{}^{P:k}$.

2. *Rate Calculation Method II.*

For purposes of practical application, Eq. (A.35) is greatly simplified by one final transformation, $G^k\!:\mathbf{a}^k \rightarrow N^k$; where N^k is an *"abstract rate vector."* Let:

$$\hat{\boldsymbol{\gamma}}^i = \hat{\mathbf{Y}}^i/\hat{U}^i = (\hat{\mathfrak{p}}^i; 1); \quad \boldsymbol{\gamma}^* = \mathbf{Y}^*/U = (\mathfrak{p}^*; 1)$$

and let the matrix of G^k be:

$$G^k = \begin{bmatrix} \hat{\mathfrak{p}}^k/w_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \ddots & \cdots & & \ddots & \cdots \\ 0 & \cdots & \hat{\mathfrak{p}}^k/w_a & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1/w_\beta & \cdots & 0 \\ \cdots & \ddots & \cdots & & \ddots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1/w_z \end{bmatrix}$$

where if $\hat{\mathfrak{p}}^k = \mathfrak{p}^*$, then $G^k = G^*$

Then:

(A.36) $\quad (\mathbf{a}^k G)W = N^k W = \hat{\boldsymbol{\gamma}}^k$; (if $\mathbf{a}^k = \tilde{\mathbf{a}}^k$; then $\hat{\boldsymbol{\gamma}}^k = \boldsymbol{\gamma}^*$)

To prove Eq. (36), by definition of G:

(A.37)
$$\sum_{P} w_x N_x^k = \sum_{P} w_x a_x^k \hat{\mathfrak{p}}^k / w_x = \sum_{P} a_x^k \hat{\mathfrak{p}}^k = \hat{\mathfrak{p}}^k$$
$$\sum_{U} w_x N_x^k = \sum_{U} w_x a_x^k / w_x = \sum_{U} a_x^k = 1$$

whence Eq. (A.36) follows immediately. Equation (A.36) simply is the matrix form of Eqs. (A.37) with the first and second members transposed.

Thence by analogy to Eq. (A.35):

(A.38) $\quad \sum_\phi b_\phi{}^k (\hat{\boldsymbol{\gamma}}^\phi; N^\phi) = (\boldsymbol{\gamma}^*; \tilde{N}^k); \quad (b_\phi{}^k \geq 0)$, where $N^\phi = a^\phi G^\phi$.

Since by definitions of $\hat{\boldsymbol{\gamma}}^\phi$ and of $\boldsymbol{\gamma}^*$, the second equation of the System (A.38) always will be $\sum_\phi b_\phi{}^k = 1$, it is not necessary to include this equation in the constraints.

The discussion following Eq. (A.35), concerning the existence of solutions thereto, applies in its entirety to Eq. (A.38).

It follows from the definition of $\boldsymbol{N}^{\phi}$ that if a non-negative solution to Eq. (A.38) exists, then $\tilde{\boldsymbol{N}}^{k}$ conforms to Constraint (I), and hence may be written: $op\tilde{\boldsymbol{N}}^{k}$. And by Eq. (A.36) and definition of $\boldsymbol{\Upsilon}^{*}$:

$$(A.39) \qquad U(op\tilde{\boldsymbol{N}}^{k})\boldsymbol{W} = U\boldsymbol{\Upsilon}^{*} = \boldsymbol{Y}^{*}$$

Thence it follows that if $\boldsymbol{R}^{k} = U(op\tilde{\boldsymbol{N}}^{k})$, then necessarily: $\boldsymbol{R}^{k} = op\tilde{\boldsymbol{R}}^{k}$; whence Method II follows immediately.

Letting $N_x^{o} = R_x^{o}/U$, where R_x^{o} is a pre-selected value of R_x^{k}, then up to $(\alpha - 1)$ values of N_x^{o} for $x \leq \alpha$, plus up to $(z - \beta)$ values for $x \geq \beta$, may be entered as the constant terms of the system of Eq. (A.38). Also, pre-selected ratios may be substituted for preselected rates, on a one-for-one exchange of choice, *i.e.*, if f_{ζ}^{o} is chosen, where $\zeta \leq \alpha$, then only $(\alpha - 2)$ of the protected rates may be pre-selected. If a ratio f_x^{o} is pre-selected, the product $f_x^{o}N_{x+1}^{k}$ is substituted for N_x^{k} in the equation for N_x^{k}.

The nature of the entire foregoing development from Eq. (A.7) forward now may be indicated by the compound-transformation equation:

$$(A.40) \qquad U(f^{k}F^{k}GBW) = Uf^{k}T^{k} = \hat{\boldsymbol{Y}}^{k}$$

where $T^{k} = F^{k}G^{k}BW$; and if $f^{k} = opf^{k}$ and $F^{k} = F^{*}$, then $\hat{\boldsymbol{Y}}^{k} = \boldsymbol{Y}^{*}$.

Once the transformations have been appropriately defined, the rest follows in straightforward fashion.

3. *Practical Considerations.*

It follows from the "abstract"[67] nature of $\boldsymbol{P}^{\phi}$ and $\boldsymbol{N}^{\phi}$ that once calculated, the values may be stored and used in the course of successive rate adjustments over a period of years. Re-calculation of these parameters is required only following significant change in the distribution of sums insured among the classes, revision of the original estimates of $\underline{f}_x$ and $\bar{f}_x$ or revision of the classification system itself.

Throughout the formal development, the fixed class numbers, α, β, z, defined in Section II.D as indices, have been used also as parameters, *e.g.* in stating that $\{\tilde{\boldsymbol{R}}\}$ is "$(z - 2)$-dimensional". So long as $w_x > 0$

[67] A more appropriate term here would be "dimensionless," in the sense that a trigonometric sine is a "dimensionless" ratio; but since the abstract vector $\boldsymbol{N}$ is "z-dimensional" in a mathematical sense; the term "abstract" is used to avoid semantic difficulties.

for all $x = 1, 2, \ldots, z$, this presents no problem. In such instance, "α" not only designates a class, but also equals the number of protection classes within the "Protected" statistical class, etc., but if $w_x = 0$ for any x, as in Section VI where all calculations contemplate $w_1 = 0$, then z will not indicate the number of classes as well as the designation of the highest numbered class; and in reference to the incomplete system of Section VI, the feasible rate structure, $\{\tilde{R}\}$, is *not* $(z-2)$-dimensional, but is $(z-3)$-dimensional. Although $z = 10$ still designates, as an index, the highest-numbered class, the number of classes is not $z = 10$ but is $z - 1 = 9$, etc. In practical application, general expressions in which the fixed class numbers, α, β and z, represent the number of classes, rather than indices designating particular classes, must be modified according to circumstances where $w_x = 0$ for one or more of the classes included in the system currently involved.

If $w_\eta = 0, \ldots, w_\theta = 0$, for one or for two or more consecutive class numbers, $\eta, \ldots \theta$, but $w_\zeta > 0$ *and* $w_\mu > 0$, where $\zeta = \eta - 1$ and $\mu = \theta + 1$, a further modification must be made in all expressions involving c_x and f_x to avoid distortion of the results of practical calculations. In such instance, for:

c_ζ;	substitute:	$d_\zeta = Q_\zeta / Q_\mu$
f_ζ;	"	: $g_\zeta = R_\zeta / R_\mu$
$\underline{c}, \overline{c}$;	"	: $\underline{d}, \overline{d}$
$\underline{f}, \overline{f}$;	"	: $\underline{g}, \overline{g}$

and "$\overline{g} < r_\zeta / r_\mu$" replaces "$\overline{f} < r_x / r_{x+1}$" in Constraint (I).

The choice of Eq.(A.20-b) from which to develop Eqs.(A.26-) rests upon the fact that normally there will be not more than two, or at most three, unprotected classes, *vs.* at least three, and probably six to eight protected classes. Hence in practice, the denominators will be simpler if either Eq.(A.20-b) or Eq.(A.19-b) is used in preference to the other choices. Of these two, choice of Eq.(A.20-b) results in the simplest form of the recursion equations, Eqs.(A.26-), which in turn simplifies the formulas for pre-calculation of the several N^ϕ. In particular cases, it may prove expedient to choose Eq.(A.20-a) or one of Eqs. (A.19-). Theoretically, it makes no difference whatever in the final result, whichever of the four possibilities may be chosen; it is not even necessary that the same one of the four equations, Eqs.(A.19-) and (A.20-) be chosen to calculate each of the several coefficients a_x^k in turn.

APPENDIX B

RATE CALCULATION METHOD I

(SECTION VI. C.)

By rearrangement of the standard "two-point" formula, the equation of a line through the points $(x_P; \hat{P}^i)$ and $(x_U; \hat{U}^i)$ will be:

$$L_x{}^i = x\frac{\hat{U}^i - \hat{P}^i}{x_P - x_U} + \frac{x_U\hat{P}^i - x_P\hat{U}^i}{x_P - x_U} \tag{B.1}$$

where $x_P = \sum_P x\, w_x$ and $x_U = \sum_U x\, w_x$. It follows by straightforward algebra that if x is restricted to integral values, then:

$$\textstyle\sum_P w_x L_x{}^i = \hat{P}^i; \text{ and: } \sum_U w_x L_x{}^i = \hat{U}^i \tag{B.2}$$

When $\hat{P}^i = P$ and $\hat{U}^i = U$, let $L_x{}^i = \tilde{L}_x$ in Eqs.(B.1) and (B.2).

It follows immediately from the definition of $\hat{P}^i$ and $\hat{U}^i$, and from Eqs.(B.2), that:

$$\textstyle\sum_P w_x(R_x{}^i - L_x{}^i) = 0; \text{ and: } \sum_U w_x(R_x{}^i - L_x{}^i) = 0 \tag{B.3}$$

Let:

$$R_x{}^* = \frac{U - P}{\hat{U}^i - \hat{P}^i}(R_x{}^i - L_x{}^i) + \tilde{L}_x \tag{B.4}$$

then by Eqs.(B.2) and (B.3):

$$\begin{aligned} \sum_P w_x R_x{}^* &= \frac{U - P}{\hat{U}^i - \hat{P}^i}\sum_P w_x(R_x{}^i - L_x{}^i) + \sum_P w_x\tilde{L}_x \\ &= \sum_P w_x\tilde{L}_x = P \\ \sum_U w_x R_x{}^* &= \sum_U w_x\tilde{L}_x = U \end{aligned} \tag{B.5}$$

whence $\tilde{\boldsymbol{R}}^*$ as defined by Eq.(B.4) is feasible.

Substituting into Eq.(B.4) the values of $L_x{}^i$ and $\tilde{L}_x$, and simplifying, it follows immediately that:

$$R_x{}^* = R_x{}^i\,\frac{U - P}{\hat{U}^i - \hat{P}^i} + \frac{P\hat{U}^i - U\hat{P}^i}{\hat{U}^i - \hat{P}^i} \tag{B.6}$$

It should be noted that (unless $\hat{U}^i - \hat{P}^i = 0$, in which case the problem is degenerate) the actual values of the trial averages, $\hat{P}^i$ and $\hat{U}^i$

are completely immaterial. Any trial rate vector whatever will be transformed into a feasible vector by Eq.(B.6). Upon substitution of R_x^e for R_x^i, etc., Eq.(VI. C.23.a) is obtained.

Eq.(VI. C.23.b) follows by exact formal analogy to the above derivation of Eq.(VI. C.23.a).

Now let $\boldsymbol{R}^*$ be feasible by hypothesis. Then by definition of q_x and by Eq.(VI. C.23.a):

$$(B.7) \qquad q_x R_x^* = q_x R_x^e \frac{U - P}{U^e - P^e} + q_x \frac{PU^e - UP^e}{U^e - P^e}$$

whence by definition of q_x, Eq.(VI. C.24) follows immediately by summation of both sides of Eq.(B.7) following multiplication by w_x:

$$(B.8) \qquad \begin{aligned} \textstyle\sum_U w_x Q_x^* &= U_Q = U_Q^e \frac{U - P}{U^e - P^e} + q_P \frac{PU^e - UP^e}{U^e - P^e} \\ \textstyle\sum_P w_x Q_x^* &= P_Q = P_Q^e \frac{U - P}{U^e - P^e} + q_U \frac{PU^e - UP^e}{U^e - P^e} \end{aligned}$$

where $q_P = \sum_P w_x q_x$ and $q_U = \sum_U w_x q_x$. But for insertion of the summations on the left, Eq.(B.8) is Eq.(VI. C.24).

To prove that for an arbitrary vector, $\boldsymbol{Q}^j$, if $\sum_P w_x Q_x^j = P_Q$ and $\sum_U w_x Q_x^j = U_Q$, then necessarily the rate vector $\boldsymbol{R}^j$ which rests upon $\boldsymbol{Q}^j$ will be feasible, let Eqs.(B.8) be expressed in the form:

$$(B.9) \qquad \tilde{\boldsymbol{Q}}^* \boldsymbol{W} = (P_Q; U_Q) = \boldsymbol{Y}_Q^*$$

where $\boldsymbol{W}$ is the weighting matrix defined in Section 1.a., of APPENDIX A. (page .) Let the matrix $\boldsymbol{M}$ be defined as the $z \times z$ matrix whose entries along the main diagonal are r_x, and elsewhere than along the main diagonal, are zero. Then:

$$(B.10) \qquad \boldsymbol{R}^* = \boldsymbol{Q}^* \boldsymbol{M}$$

By derivation of Eq.(B.8), $\boldsymbol{R}^*$ is feasible, $\tilde{\boldsymbol{R}}^*$, whence by Eq.(B.10):

$$(B.11) \qquad \tilde{\boldsymbol{R}}^* \boldsymbol{W} = (P; U) = \boldsymbol{Y}^* = \tilde{\boldsymbol{Q}}^* \boldsymbol{M} \boldsymbol{W}$$

By hypothesis: $\boldsymbol{Q}^j \boldsymbol{W} = \boldsymbol{Y}_Q^*$; whence by Eq.(B.9):

$$(B.12) \qquad \boldsymbol{Q}^j \boldsymbol{W} = \boldsymbol{Y}_Q^* = \tilde{\boldsymbol{Q}}^* \boldsymbol{W}$$

whence immediately:

$$(B.13) \qquad \boldsymbol{R}^j \boldsymbol{W} = \boldsymbol{Q}^j \boldsymbol{M} \boldsymbol{W} = \tilde{\boldsymbol{Q}}^* \boldsymbol{M} \boldsymbol{W} = \tilde{\boldsymbol{R}}^* \boldsymbol{W} = \boldsymbol{Y}^*$$

whence $\boldsymbol{R}^j$ must be feasible, $\tilde{\boldsymbol{R}}^j$.

It may be noted that a feasible solution always will result from application of Eq.(VI. C.23.a) to the rates, $R_x{}^i = r_x Q_x{}^i$, without consideration of the normals as prescribed in Cases 3 and 4. However, by definitions given:

(B.14) $\qquad c_x = f_x q_x / q_{x+1}$

It follows that if $q_x > q_{x+1}$, it may happen that values of $c_x{}^*$ will be obtained which not only exceed $\overline{c}_x$ but also exceed unity, which implies $Q_x > Q_{x+1}$, which in turn implies *increase* of rate with *improvement* of protection. To illustrate, in Example 2, of Section VI. C., it was assumed that $r_x = 1$, but on the alternative assumption that r_x and q_x are as shown in Table 3, then by Eq.(B.14) it will be found that the c_x ratios associated with the solution of Example 2 are such that $c_5 = 1.02$ and $c_3 = 1.11$. Unless $r_x \doteq r_{x+1} \doteq 1$ for all x, then the normals, rather than the rates themselves, must be used in the calculation to preclude possibility of inversions such as the foregoing; except in the special case where $P\hat{U}^i - U\hat{P}^i = 0$, in which case $f_x{}^*$ will equal $f_x{}^i$ for all x, since the additive term then disappears from Eq.(VI. C.23.a).

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DISCUSSION BY LESTER B. DROPKIN

For several years now, writers and reviewers of papers presented to this Society have stressed the desirability, and indeed, the inevitability of utilizing theory, methods, techniques and procedures derived from what may be broadly referred to as the field of Finite Mathematics. During these same years the Society has also seen an increasing number of papers dealing with the ratemaking problems of the fire actuary. Recalling Mr. McIntosh's earlier work, it is not unexpected that he would again bring these two lines together in the paper now under review.

In his current paper, "A Mathematical Approach to Fire Protection Classification Rates," Mr. McIntosh deals with the problem of determining a set of rates such that they will, in the language of the paper, simultaneously fulfill the conditions of "feasibility" and "operational constraint." These two terms, although coming from the language of linear programming, represent simple and familiar concepts. The feasibility property will be readily recognized as that old friend: a rate structure in balance by part and in total. The question of operational constraints may similarly be recognized as coming within considerations of rate relativity, albeit the rate relativities here are not specifically given. Rather, each of the rate relativities is fixed only to the extent of having given lower and upper limits, such limits being predetermined by judgment or other outside factors. It is, of course, the simultaneous existence of the feasibility and constraint conditions that make the problem a real and interesting one.

The definition of the problem and the treatment of its solution (including therein those cases where no solution is possible) proceeds via