Geometry of Drinfeld Modular Forms

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$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$	$\mathrm{GL}_2(A) \setminus \Omega$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

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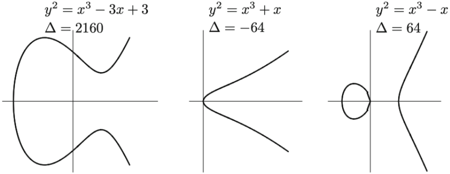

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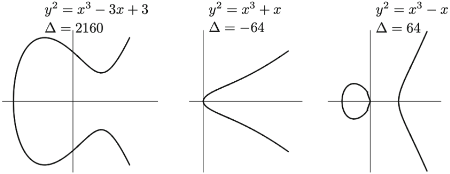

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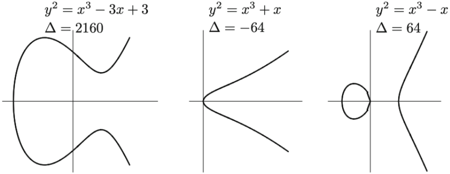

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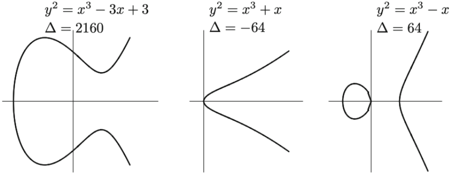

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the image of a ring homomorphism $\varphi^z : A \to C\{X^q\}$ where $C\{X^q\}$ is the non-commutative ring of $\mathbb{F}_q$-linear polynomials/C.

Moduli Problems

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For example,

$$\Gamma_0(N) \overset{def}{:=} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \pmod{N} \right\}$$

corresponds to the **moduli space** of

$$\begin{cases} \text{elliptic curves} \\ \text{Drinfeld modules of rank 2} \end{cases} \quad \text{with an } N\text{-torsion subgroup.}$$

Classical Modular Forms & Curves

Algebraic Modular Curve

$\mathscr{X}_\Gamma$

Deligne-Mumford (stacky) curve

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Definition ([DS05, 1.1.2])

A map $f : \mathcal{H} \to \mathbb{C}$ is a **modular form of weight** $k \in \mathbb{Z}$ for $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ if

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$$M(\Gamma) \overset{def}{:=} \bigoplus_{k \geq 0} M_k(\Gamma) \xrightarrow{\sim} \bigoplus_{k \geq 0} H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(\Delta)^{\otimes k/2}) \overset{def}{=:} R(\mathscr{X}_\Gamma, \Delta),$$

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4. **GAGA** - equivalences of categories:

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Theorem ([Fra23, 6.2])

We have $M(\Gamma) \cong M(\Gamma_2)$, with

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each component,

Geometry of Drinfeld Modular Forms (2/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma_2} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ_2 we find the following.

Theorem ([Fra23, 6.2])

We have $M(\Gamma) \cong M(\Gamma_2)$, with

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each component,where l_1, l_2 are the solutions to $k \equiv 2l \pmod{q-1}$.

Geometry of Drinfeld Modular Forms (3/3)

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.

Geometry of Drinfeld Modular Forms (3/3)

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma'} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

Geometry of Drinfeld Modular Forms (3/3)

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma'} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ' we find the following generalization of [Fra23, Theorem 6.2].

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{tikzcd} \mathscr{X}_{\Gamma'} \arrow[d] \\ \mathscr{X}_{\Gamma} \end{tikzcd}$$

When we compare the modular forms for Γ and Γ' we find the following generalization of [Fra23, Theorem 6.2].

Theorem ([Fra23, 6.12])

We have $M(\Gamma) \cong M(\Gamma')$,

Let q be odd;
$\Gamma \leq \mathrm{GL}_2(A)$;
$\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma'} \\ \big\downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ' we find the following generalization of [Fra23, Theorem 6.2].

Theorem ([Fra23, 6.12])

We have $M(\Gamma) \cong M(\Gamma')$, and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

Conclusion

Thank you!
Further details available at arXiv:2310.19623

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