1. Translator's Notes

This document is a translation of chapter 6 in [1], written by A. Beauville. My translation focuses mostly on preserving the mathematics in the original document as accurately as possible. It is not a very consistent or idiomatic translation, but should be very literal when adapting the French.

Title: **The Minimal Number of Singular Fibers of a Stable Curve over $\mathbb{P}^1$.**

2. Introduction

This section attempts to answer the following question posed by Szpiro. A non-constant family of stable curves of genus g over $\mathbb{P}^1_{\mathbb{C}}$ admits a certain number of singular fibers, what is the smallest possible number (as a function of g)?

Unless otherwise mentioned, the varieties considered are complex; but we will say some words about the situation in characteristic p, which is more complicated.

3. Results Without the Hypothesis of Stability

If we remove the condition of stability, I think the response is well known.

Proposition 3.1.

(1) *All families of curves over $\mathbb{P}^1$ with variable modules admit at least 3 singular fibres.*
(2) *For all $g \geqslant 1$, there exists a family of curves of genus g over $\mathbb{P}^1$ with variable modules that admit exactly 3 singular fibres.*

By "family of curves" we mean a proper and flat morphism $f : X \to \mathbb{P}^1$ whose fibres are connected curves. "Variable modules" means that the smooth fibres of f are not all isomorphisms, or as some would say, [the family is] nonisotrivial.

Proof. Let U be the biggest open set in $\mathbb{P}^1$ above which f is smooth, and let $\widetilde{U}$ be its universal cover. The local system $R^1 f_*(\mathbb{Z})$ becomes constant on $\widetilde{U}$. By choosing a symplectic base we define a morphism from $\widetilde{U}$ to the Seigel half-space H_g, by Torelli's theorem and the hypothesis that the family has variable modules, the morphism is not constant. Since H_g is isomorphic to a terminal domain, this means U is not isomorphic to $\mathbb{C}^1$ nor $\mathbb{P}^1$ by Liouville's theorem. So $\mathbb{P}^1 - U$ contains three points.

For $t \in \mathbb{P}^1$ and $n \geqslant 3$, consider the curve C_t with affine equation $y^2 = x^n - ntx + (n-1)t$. For $t \neq 0, 1, \infty$ this is a smooth hyperelliptic curve of genus $\frac{n-1}{2}$ and we easily check that the family $(C_t)_{t \in \mathbb{P}^1}$ is nonisotrivial. This completes the proof. □

What can we say about characteristic p? Part (2) of the Proposition 3.1 is still valid if $p \neq 2$ and we can do the former construction; the case $p = 2$ is left as an exercise to the reader. Part (1) of Proposition 3.1 is not true in characteristic p, but we have the following.

Proposition 3.2. *Let k be an algebraically closed field of characteristic p. If $p > 2g + 1$ then all nonisotrivial families of genus g curves over $\mathbb{P}^1$ admit at least 3 singular fibres.*

Proof. We first show that the first hypothesis $p > 2g + 1$ implies that there exists a prime number $l \neq p$ and $l \neq 2$ such that the order of the group $\mathrm{GL}_{2g}(\mathbb{Z}/l\mathbb{Z})$ is not divisible by p. In effect, the order of $\mathrm{GL}_{2g}(\mathbb{Z}/l\mathbb{Z})$ equals $(l^{2g} - 1)(l^{2g-1} - 1) \cdots (l - 1) l^{g(2g-1)}$. Just choose l in a way that the class modulo p is a generator for the cyclic group $\mathbb{F}_P^\times$, and therefore is of order $p - 1 > 2g$. We can choose l an odd prime by the Theorem of arithmetic progression.

Now suppose $g \geqslant 2$. Denote by $\mathbb{G}_m$ the affine line (over k) with the origin removed. By our proof of the Proposition 3.1 it suffices to prove that a family of curves $f : X \to \mathbb{G}_m$ of smooth curves of genus g is necessarily isotrivial. We choose l as above, and consider the locally constant sheaf $R^1f_*(\mathbb{Z}/l\mathbb{Z})$. It corresponds to a homomorphism

$$\rho : \pi_1(\mathbb{G}_m, *) \to \mathrm{GL}_{2g}(\mathbb{Z}/l\mathbb{Z}).$$

Since $\mathrm{GL}_{2g}(\mathbb{Z}/l\mathbb{Z})$ is of order prime to p, ρ factors through π mod $\mathbb{G}_m$, which means that there is an integer k such that under the fibration $f' : X' \to \mathbb{G}_m$, the reciprocal image of f by the morphism $x \mapsto x^k$, the sheaf $R^1f'_*(\mathbb{Z}/l\mathbb{Z})$ is constant. Now this sheaf is identified with the sheaf of points of order l of the relative Jacobian $\mathrm{Pic}^0(X'/\mathbb{G}_m)$; the fact that this is constant implies that the abelian scheme with semi-stable reduction at 0 and at infinity by 8 (D) Corollary 5.18. It is therefore the same for the family of curves $f' : X' \to \mathbb{G}_m$ by 8 Proposition 5.7, in other words, there is a semi-stable fibration $\tilde{f} : \widetilde{U} \to \mathbb{G}_m$ which prolongs f'. Having at most two singular fibres, the fibration $\tilde{f}$ is isotrivial by 8 (S) Theorem 3.3, and it is the same for the fibration f.

It remains to treat the case $g = 1$. Let $f : X \to \mathbb{P}^1$ be a fibration of curves of genus 1 smooth apart from $\{0, \infty\}$. Replacing f by the associated Jacobian fibration, we can assume f admits a section. Consider the classifying morphism $j : \mathbb{P}^1 \to \mathbb{P}^1$. It factors into $\mathbb{P}^1 \xrightarrow{r} \mathbb{P}^1 \xrightarrow{j'} \mathbb{P}^1$, where r is a root and where j' is separable, ramified at $0, 1728$ and at ∞. Moreover, since $p > 3$, the ramification index of a point of $f'^{-1}(0)$ (resp. $j'^{-1}(1728)$) is divisible by 2 (resp. 3). If we denote the degree of j' by n, the Riemann-Hurwitz formula gives the inequality

$$-2 \geqslant -2n + \frac{n}{2} + \frac{2n}{3} + n - 2.$$

Let $n \leqslant 0$ and we have a contradiction.

It is easy to see the condition $p > 2g + 1$ is necessary: in characteristic p the equation

$$y^2 = x^p + tx^{p-1} + 1,$$

for $t \in \mathbb{G}_m$, describes a smooth, nonisotrivial family of hyperelliptic genus g curves over $\mathbb{G}_m$ with $p = 2g + 1$. It is even more delicate to construct a smooth, nonisotrivial family over the affine line. Raynaud told me in characteristic 2 and genus $g \geqslant 2$ the curve with equation $y^2 + y = x^{2g+1} + tx$, for $t \in \mathbb{A}^1$. We can also consider the curve $y^p + y = x^{kp-1} + tx$ for $t \in \mathbb{A}^1$.

On the other hand the argument of the proposition (modified to account for characteristic 2 and 3) shows that there does not exist a nonisotrivial family of smooth elliptic curves the affine line. ☐

4. Statement of the Main Result

We now only consider stable families $Y \to \mathbb{P}^1$. The surface Y in this case can have double points (of type A_n). By resolving them in successive bursts, we arrive at a semi-stable fibration $X \to \mathbb{P}^1$, which for us means

(1) the surface X is smooth,
(2) the fibres are connected curves of genus $\geqslant 1$ having at most ordinary double points,
(3) there does not exist an exceptional curve on X contained in a fibre.

Such a fibration is said to be trivial if there exists a curve C and an isomorphism $u : X \to C \times \mathbb{P}^1$ such that $\mathrm{pr}_2 \circ u = f$.

Theorem 4.1. *Let $f : X \to \mathbb{P}^1$ be a nontrivial semi-stable fibration.*

(1) *f admits at least 4 singular fibres,*
(2) *Suppose f has exactly 4 singular fibres. Then the surface X is algebraically simply connected and (check) $p_g = 0$. The irreducible components of singular fibres are rational curves (eventually singular); the classes of these curves generate a hyperplane in the affine $\mathbb{Q}$-vector space* $\mathrm{Pic}(X) \otimes \mathbb{Q}$.

The conditions imposed for (2) are very restrictive, both for the surfaces and for the pencil of curves that it must contain. The surface X can be:

(1) a rational surface
(2) an elliptic surface over $\mathbb{P}^1$ such that the elliptic fibration contains exactly two mutliple fibres of coprime multiplicities, and such that the associated Jacobian bundles are rational. (See 8 (Do))
(3) a surface of general type with $p_g = 0$ and $\pi_1^{\mathrm{alg}} = \{1\}$.

An example of a surface of type (3) was just given by R. Barlow. We will see an example below of a semi-stable fibration with 4 singular fibres, with fibres genus 1 and the surface X rational. I don't know of an example with fibres of genus $g \geqslant 2$ and I tend to think it does not exist.

5. Some Lemmas

In this section we will bring together some well-known facts that we will need.

Lemma 5.1. *Under the hypotheses of Theorem 4.1, let C be a singular fibre, let N be its normalization and let g be the genus of the generic fibre. Note $g(N) = \dim_{\mathbb{C}} H^1(N, \mathcal{O}_N)$ and $q = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X)$. We have*
(1) $rg_{\mathbb{Z}} H^1(C, \mathbb{Z}) = g + g(N)$
(2) $g(N) \geqslant q$.

Proof. Denote by $\pi : N \to C$ the canonical morphism, by Σ the collection of double points of C by $\mathbb{Z}_C$ (resp. $\mathbb{Z}_N$) the constant sheaf of the fibre $\mathbb{Z}$ over C (resp. N), by $\mathbb{Z}(s)$ the sheaf on C which is 0 outside of s and the fibre $\mathbb{Z}$ (resp. $\mathbb{C}$) on s.

Consider the exact sequences

$$0 \to \mathbb{Z}_C \to \pi_* \mathbb{Z}_N \to \oplus_{s \in \Sigma} \mathbb{Z}(s) \to 0 \tag{5.1}$$

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_N \to \oplus_{s \in \Sigma} \mathbb{C}(s) \to 0 \tag{5.2}$$

as well as the associated exact sequences in cohomology

$$0 \to H^0(C, \mathbb{Z}) \to H^0(N, \mathbb{Z}) \to \mathbb{Z}^{\Sigma} \to H^1(C, \mathbb{Z}) \to H^1(N, \mathbb{Z}) \to 0 \tag{5.3}$$

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(N, \mathcal{O}_N) \to \mathbb{C}^{\Sigma} \to H^1(C, \mathcal{O}_C) \to H^1(N, \mathcal{O}_N) \to 0 \tag{5.4}$$

Denote by n the number of double points of C and let c denote the number of irreducible components and we deduce

$$rg_{\mathbb{Z}} H^1(C, \mathbb{Z}) = 2g(N) + n - c + 1 \tag{5.5}$$

$$g = g(N) + n - c + 1 \tag{5.6}$$

from which by subtraction we prove the assertion (1).

To prove part (2) it suffices to prove that the homomorphism $\mathrm{J}(N) \to \mathrm{Alb}(X)$ is surjective. Let Q be the cokernel and let $\alpha : X \to Q$ be the map induced by the morphism to the Albanese. By construction, the image of the fibre C under α is reduced to a point and a rigidity lemma from 8 (M) Proposition 6.1 leads to the same conclusion for all fibres of f, so in particular there is a morphism $\beta : \mathbb{P}^1 \to Q$ such that $\alpha = \beta \circ f$. Such a morphism is necessarily trivial, the image of α is reduced to a point; since $\alpha(X)$ generates Q we have $Q = \{0\}$, hence the assertion (2). $\square$

Recall that we denote by $\mathrm{NS}(X)_{\mathbb{Q}}$ the vector subspace of $H^2(X, \mathbb{Q})$ generated by the algebraic classes, and $\rho(X)$ its dimension (Picard number).

Lemma 5.2. *Let $C_1, \cdots, C_p$ be the reducible fibres of f and let c_i be the number of irreducible components of C_i. The components of the C_i for $1 \leqslant i \leqslant p$ generates in $\mathrm{NS}(X)_{\mathbb{Q}}$ a subspace of dimension $1 + \sum_i (c_i - 1)$. In particular we have*

$$\rho(X) \geqslant 2 + \sum_i (c_i - 1).$$

Proof. Let P be orthogonal to C in $\mathrm{NS}(X)_{\mathbb{Q}}$ (under the intersection form). Equip the quotient space $\overline{P} = P/\mathbb{Q}C$ of the form φ induced by the intersection form. For $1 \leqslant i \leqslant p$, let P_i be the subspace of $\mathrm{NS}(X)_{\mathbb{Q}}$ generated by the components of C_i and let $\overline{P_i} = P_i/\mathbb{Q}C$ be their images in $\overline{P}$. We have $\dim \overline{P_i} = c_i - 1$. The subspaces $\overline{P_i}$ are pairwise orthogonal in $\overline{P}$ and the restriction of φ to $\overline{P_i}$ is non-degenerate by 8 (B) Corollary 8.4. This entails that the subspaces $\overline{P_i}$ are in a direct sum such that

$$\dim \sum_i \overline{P_i} = \sum_i (c_i - 1)$$

and subsequently $\dim \sum_i P_i = 1 + \sum_i (c_i - 1)$. The last assertion follows from the fact that the inclusion $P \subset \mathrm{NS}(X)_{\mathbb{Q}}$ is strict. □

6. Proof of Theorem

Denote by $C_1, \cdots, C_r$ the singular fibres of f. We suppose $r \leqslant 4$. If $r < 4$ then denote by $C_{r+1}, \cdots, C_4$ any smooth fibres of f. We write N_i for the normalization of C_i and write c_i for the number of irreducible components of C_i.

6.1. **The Basic Calculation.** Let's calculate the topological Euler-Poincaré characteristic $\chi(X)$ of X. Denoting by C a smooth fibre of f we have

$$\chi(X) = \chi(\mathbb{P}^1)\chi(C) + \sum_{i=1}^{4} (\chi(C_i) - \chi(C))$$

by 8 (B) Lemma 6.4.

Let's calculate the difference $\chi(C_i) - \chi(C)$. We have

$$b_0(C_i) = b_0(C) = 1, \text{ so } b_0(C_i) - b_0(C) = 0$$

$$b_2(C_I) = c_i \text{ and } b_2(C) = 1 \text{ so } b_2(C_i) - b_2(C) = c_i - 1,$$

and by Lemma 5.1 $b_1(C_i) - b_1(C) = g - g(N_i)$. As a consequence we get

$$\chi(X) = 4 - 4g + \sum_{i=1}^{4} (g - g(N_i)) + \sum_{i=1}^{4} (c_i - 1).$$

On the other hand we have

$$\chi(X) = 2 - 2b_1(X) + b_2(X) = 2 - 4q + b_2(X).$$

By comparing we get the equation

$$b_2(X) = 2 + \sum (c_i - 1) + \sum (q - g(N_i)) \tag{6.1}$$

We deduce from Lemmas 5.1 and 5.2 the inequalities

$$2 + \sum_i (c_i - 1) \leqslant \rho(X) \leqslant b_2(X) \leqslant 2 + \sum_i (q - g(N_i)),$$

and the terms at either end are equal so we actually have an equality. This has several consequences:

1. We have $p_g(X) = 0$ from $\rho(X) = b_2(X)$.
2. Given Lemma 5.2 the equality $\rho(X) = 2 + \sum_i (c_i - 1)$ means that the components of C_i for $1 \leqslant i \leqslant p$ generate a hyperplane in $\mathrm{NS}(X)_{\mathbb{Q}}$.
3. We have $g(N_i) = q$ for $1 \leqslant i \leqslant p$ since $b_2(X) = 2 + \sum_i (c_i - 1)$ (compare with 6.1).

6.2. **Proof of the First Assertion of Theorem 4.1.** Suppose f has at most 3 singular fibres. The curve C_4 is therefore smooth, we have $g(C_4) = q$ as above, hence $g = q$. So we have $\dim \mathrm{J}(N_i) = g$ for $1 \leqslant i \leqslant 3$ which means that the generalized Jacobian $\mathrm{J}(C_i)$ is an abelian variety of dimension g. The Jacobians $\mathrm{J}(f^{-1}(t))$ for $t \in \mathbb{P}^1$ therefore form a principally polarized family of abelian varieties over $\mathbb{P}^1$ (in other words $\mathrm{Pic}^0(X/\mathbb{P}^1)$ is an abelian scheme over $\mathbb{P}^1$). Such a family is constant (which can be seen from the reasoning of the proof of Proposition 3.1). By Torelli's theorem, the smooth fibres of f are all isomorphic. Since f is a semi-stable fibration over $\mathbb{P}^1$, we conclude f is isotrivial, contrary to the assumption.

Now we assume f has exactly 4 singular fibres.

6.3. **The case $q \geqslant 2$.** The surface X satisfying $p_g = 0$ and $q \geqslant 2$ is settled by 8 (B) Lemma 6.1 and Proposition 6.2; there exists some smooth curve B of genus q and a morphism $p : X \to B$ whose fibres F_b for $b \in B$ are rational. Since $g(N_i) = q$, it follows from the Riemann-Hurwitz formula that C_i is a meeting of a section of p and a certain number of components of curves F_b ($b \in B$). But then we have $C \cdot F_b = 1$, which implies that the smooth fibres of f are the sections of p, in particular those are all isomorphisms to B, which shows that as above a contradiction.

We will need in the sequel the following observation.

Lemma 6.1. *Let $f : X \to \mathbb{P}^1$ be a semi-stable fibration and let $r : \widetilde{X} \to X$ be the associated étale covering. Then the fibration $f \circ r : \widetilde{X} \to \mathbb{P}^1$ is semi-stable and admits the same number of singular fibres as f.*

Proof. Effectively the fibres of $\tilde{f} = f \circ r$ are étale coverings of fibres of f, the only nontrivial point to prove is the fact that the fibres are connected. If they weren't, $\tilde{f}$ would factorize as $\widetilde{X} \to R \xrightarrow{u} \mathbb{P}^1$, where R is a smooth curve and u is a ramified cover: the fibre of $\tilde{f}$ over a point of ramification of u would be unreduced, which is absurd. ☐

6.4. **The case $q = 1$.** The Albanese fibration still provides a morphism $p : X \to B$ having connected fibres with $g(B) = 1$ by 8 (B) Proposition 5.15. Let's fix an index i ($1 \leqslant i \leqslant 4$) and put $C = C_i,\ N = N_i$. Since $g(N) = 1$, the curve C is an union of rational curves, contained in the fibres of p, and an irreducible curve C' which has normalization N' an elliptic curve. The morphism p induces an étale cover $\pi : N' \to B$.

If π is of degree 1, we can apply the reasoning of part (3), so suppose $\deg(\pi) > 1$. Suppose $\widetilde{X} = X \times_B N'$ and denote the first projection $r : \widetilde{X} \to X$. Suppose $\widetilde{C'} = r^{-1}(C')$, $\widetilde{C} = r^{-1}(C)$, let $\widetilde{N'}$ (resp. $\widetilde{N}$) be the normalization of $\widetilde{C'}$ (resp. $\widetilde{C}$), so we have a commutative/Cartesian diagram

$$\begin{array}{ccc} \widetilde{N'} & \longrightarrow & N' \\ \big\downarrow & & \big\downarrow \\ \widetilde{C'} & \xrightarrow{r} & C'. \end{array}$$

By construction the étale cover $\widetilde{N'} \to N'$ admits a section; since $\deg r > 1$ we have that $g(\widetilde{N'}) \geqslant 2$. Thus the normalization $\widetilde{N}$ of the fibre $\widetilde{C}$ of $\tilde{f}$ is of genus $\geqslant 2$.

On the other hand, by Lemma 6.1, the fibration $\tilde{f}$ has the properties (1) through (3) of part (1) of the Theorem 4.1. So we have $p_g(\widetilde{X}) = 0$ from which $q(\widetilde{X}) = 1$ since $\chi(\mathcal{O}_{\widetilde{X}}) = \deg r \cdot \chi(\mathcal{O}_X) = 0$. But we also have (from property (3)) $g(\widetilde{N}) = q(\widetilde{X}) = 1$, so a contradiction with the above.

6.5. **The end of the proof.** So we have $q = 0$. Note that the property (3) of part (1) of Theorem 4.1 implies that $g(N_i) = 0$, in other words, the irreducible components of fibres are rational curves. It remains to show that X is simply algebraically connected, i.e. does not admit a nontrivial étale cover. So let $r : \widetilde{X} \to X$ be such a cover, and by Lemma 6.1 we have $p_g(\widetilde{X}) = 0$ from which $\mathcal{O}_{\widetilde{X}} \leqslant 1$. But on the other hand

$$\chi\mathcal{O}_{\widetilde{X}} = \deg(r) \cdot \chi(\mathcal{O}_X) = \deg(r) \geqslant 2,$$

which gives a contradiction. This completes the proof of Theorem 4.1.

Remark. Lemmas 5.1 and 5.2 remain valid in characteristic p as well as the first part 6.1 of this proof and the beginning of the second part 6.2 (it is necessary to take $q = \frac{1}{2}b_1(X) = \dim \mathrm{Alb}(X)$; note that we can no longer conclude $p_g(X) = 0$ from $\rho(X) = b_2(X)$). We therefore obtain, in any characteristic, the following result:

Let $f : X \to \mathbb{P}^1$ be a semi-stable fibration admitting at most 3 singular fibres. Then $\mathrm{Pic}^0(X/\mathbb{P}^1)$ is an abelian scheme over $\mathbb{P}^1$. In other words, the components of a singular fibre are smooth and the graph of the unions of these components is a tree.

Semi-stable fibrations having this last property were constructed by 8 (MB); his examples have more than 3 singular fibres. On the other hand, Szpiro showed that a nontrivial fibration over $\mathbb{P}^1$, in any characteristic, has at least 3 singular fibres 8 (S).

7. Examples

Example. This example is a semi-stable fibration of genus 1 with 4 singular fibres.
It's about the "Hasse cubic," given by

$$C_t : (x^3 + y^3 + z^3) - 3txyz = 0, \ (t \in \mathbb{P}^1).$$

This equations defines a pencil of cubics in $\mathbb{P}^2$ admitting 9 distinct base points. Let X be the surface obtained by blowing up these 9 points in $\mathbb{P}^2$ and let $f : X \to \mathbb{P}^1$ be the morphism defined by the pencil: it is a semi-stable fibration with genus 1 fibres. An easy calculation shows that f admits exactly 4 singular fibres (for $t = 0, rho, \rho^2, \infty$, where $\rho = e^{2\pi i/3}$) which are isomorphic to the union of 3 nonconcurrent lines in $\mathbb{P}^2$.

This family of cubics has a very special property: the 9 inflection points of a curve smooth C_t are the base points of the pencil (the Hessian of C_t is itself a cubic of the pencil). In more scholarly words, we have constructed the moduli family of elliptic curves over $\mathcal{H}/\Gamma(3) \cong \mathbb{P}^1 - \{1, \rho, \rho^2, \infty\}$.

Example. This example is a semi-stable fibration with 6 singular fibres and any genus.
We start with a smooth curve C, a morphism $\varphi : C \to \mathbb{P}^1$ of degree n, and an automorphism u of $\mathbb{P}^1$. We now make the following assumptions:

(1) the points of ramification of φ are all of index 2
(2) the branching locus R of φ in $\mathbb{P}^1$ is stable under u, but contains no fixed point of u.

In $C \times \mathbb{P}^1$ consider the divisors Γ_φ and $\Gamma_{u\circ\varphi}$, the graphs of φ and $u \circ \varphi$ respectively. They are linearly equivalent, for example because the group $\mathrm{PGL}(2)$ is a rational variety. There is therefore a double covering $\pi : X' \to C \times \mathbb{P}^1$ ramified along $\Gamma_\varphi \cup \Gamma_{u\circ\varphi}$. Denote by $g : X' \to \mathbb{P}^1$ the morphism composed of π and the second projection. For $t \in \mathbb{P}^1$ the fibre $g^{-1}(t)$ is a double cover of C ramified along the divisor $\varphi^{-1}(t) + \varphi^{-1}(u^{-1}(t))$. This divisor is reduced except for $t \in R$ or when t is one of the two fixed points of u, in which case the non-reduced points are of multiplicity 2.

As a result, the fibres of g are stable. By blowing up the double points of X' (namely the points above $\Gamma_\varphi \cap \Gamma_{u\circ\varphi}$), we obtain a semi-stable fibration $X \to \mathbb{P}^1$, which admits $\mathrm{Card}(R) + 2$ singular fibres. The genus of the fibres is $n - 1 + 2g(C)$.

We will now construct morphisms $\varphi : C \to \mathbb{P}^1$ satisfying hypothesis (1) with $\mathrm{Card}(R) = 4$; it is clear that we can then find an automorphism u of $\mathbb{P}^1$ satifying (2). If n is even, let $r : E \to \mathbb{P}^1$ be a double cover ramified at 4 points with (so $g(E) = 1$), and let $s : C \to E$ be an étale cover of degree

$\frac{n}{2}$, and the morphism $\varphi = r \circ s$ answers the question. The semi-stable fibration is associated with 6 singular fibres and its fibres are of genus $n+1$.

If n is odd, we show that there exists a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree n, satisfying (1) and with $\mathrm{Card}(R) = 4$ (we construct φ as a covering associated with a suitable homomorphism $\pi_1(\mathbb{P}^1 - R) \to \mathcal{G}_n$.) We thus obtain a semi-stable family of curves of genus $n-1$ with 6 singular fibres.

Remark.

(1) Applying the above to the morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 defined by $\varphi(t) = t^2 + \frac{1}{t^2}$ (quotient of $\mathbb{P}^1$ by $(\mathbb{Z}/2) \times (\mathbb{Z}/2)$), which satisfies (1) with $\mathrm{Card}(R) = 3$ we obtain a semi-stable fibration in $\mathbb{P}^1$ with fibres of genus 3 and 5 singular fibres.

(2) Considering the same way the degree 2 morphism $t \mapsto t^2$ of $\mathbb{P}^1$ with itself, we obtain another semi-stable family of elliptic curves with 4 singular fibres defined by the equation:

$$y^2 = (x^2 - t)(x^2 - \frac{1}{t}),\ (t \in \mathbb{P}^1).$$

This is the modular curve for an index 2 subgroup of $\Gamma(2)$ containing $\Gamma(4)$.

8. (Beauville's) References

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