RESEARCH STATEMENT

JESSE FRANKLIN

1. Research Areas

My primary research areas are Number Theory, and Algebraic Geometry, AMS Subject Classification numbers **[11]** (primary interest) and **[14]** (secondary interest). A new interest of mine is Differential Algebra, AMS classification **[16]**.

As each of these subjects is particularly background intensive, I have written this research statement with as little notation as possible and the intention that it is possible for mathematicians of any background to understand something about the work that interests me.

2. Recent Achievements

A summary of my thesis problem and its solution is an easy way to give context for my other papers. My thesis was concerned with a question in E. U. Gekeler's 1986 monograph [Gek86, Page XIII], where the author asks for a description of algebras of Drinfeld modular forms in terms of generators and relations. The "Drinfeld setting" where my thesis work takes place refers broadly to doing arithmetic geometry over the function field of a smooth, projective curve, such as the field $\mathbb{F}_q(T)$, where $\mathbb{F}_q$ is the finite field with q elements and T is an indeterminant in the case when the curve in question is a projective line, for example. Drinfeld introduced the study of what he called "elliptic modules," which we now call "Drinfeld modules," in [Dri74] in order to address problems in the Langlands program over function fields. Many objects from classical number theory (over number fields) such as modular curves and modular forms have analogs over function fields, and we refer to the function-field side of this analogy as the "Drinfeld setting." This study of modularity broadly has been of immense interest to number theorists since this theory was used to prove Fermat's last theorem.

My paper [Fra24b] and my thesis [Fra24a] resolve Gekeler's question by showing the desired generators and relations for an algebra of Drinfeld modular forms may be determined by computing a presentation for the canonical ring of a certain stacky curve in the style of [VZB22]. That is, the content of my main results [Fra24b, Theorems 6.1 and 6.11] is that we can answer Gekeler's question by computing a canonical ring, this by using only geometric invariants. Just as in the case of classical modular forms over number fields, which are known to be sections of a line bundle on some stacky curve, [Fra24b, Theorem 6.1] shows that Drinfeld modular forms are sections of a particular line bundle on a specified stacky curve.

The theory of computing these canonical rings of stacky curves is best covered in the very comprehensive results of [VZB22]. However, in joint work with Evan O'Dorney and Michael Cerchia, suggested to us by Voight and Zurieck-Brown, we have generalized some of their results, and some of O'Dorney's results from his [O'D15]. Our paper [CFO24] computes the section ring of any effective $\mathbb{Q}$-divisor (a divisor with non-negative rational coefficients) on an elliptic curve. There are other papers which deal with similar ring presentations for stacky curves such as [LRZ16]. But, as Voight put it, [VZB22] sets a very high bar for such a theory and the results of [CFO24] are so complicated that it is not necessarily worthwhile to pursue the matter of section rings of $\mathbb{Q}$-divisors on stacky curves further. It seems unlikely that aesthetic results will follow as the geometry of the curve in question gets more complicated, but it is worth noting that computing the section ring of any particular $\mathbb{Q}$-divisor on a specified curve is not hard compared with describing these results in general. This to say, there is a wealth of information which makes solving Gekeler's problem in the manner suggested by [Fra24b] quite practical.

The other aspect of my solution to Gekeler's problem is the input for this program of computing a canonical ring: some geometric invariants of a particular stacky curve - a Drinfeld modular curve. In joint work with Mihran Papikian and Sheng-Yang Kevin Ho, [FHP24], we compute several such invariants for a special class of Drinfeld modular curves, generalizing Gekeler and Nonnengardt's [GN95] to subgroups of SL_2 from the original GL_2-subgroups. The results of this paper are genus formulas for certain Drinfeld modular curves (see [FHP24, Section 3]) and Weierstrass equations for such Drinfeld modular curves in the only cases when those curves are elliptic curves ([FHP24, Section 4]). Thanks to the combinatorial nature of our theory we also describe the cusps of the corresponding modular curves, another essential input required to solve Gekeler's problem explicitly following [Fra24b].

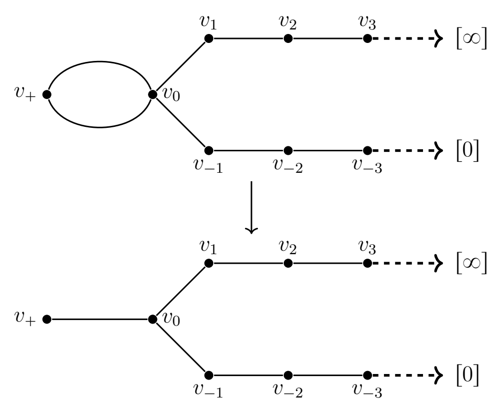

FIGURE 1. ([FHP24, Figure 3]) - Sketch of a graph cover where arrows correspond to cusps of Drinfeld modular curves.

3. Current and Future Research Projects

In [Mum78] the author describes a remarkable dictionary, first discovered by Krichever in [Kri83] (we have cited the English translation of the 1976 original), which relates the data consisting of a curve with at least one point on it and a vector bundle over it, and a commutative subring of a ring of non-commutative operators. Mumford explains that there are three distinct cases of operators for which this correspondence exists: difference, field, and differential operators. This field operator case gives a correspondence between shtukahs and what we have since come to call Drinfeld modules. The case I am most interested in is the differential operator correspondence, where we replace Drinfeld modules by their differential-ring analogs: Krichever modules.

It is clear from [Lau] that there is a theory of a moduli space of Krichever modules much like the corresponding theory for Drinfeld modules, and so by extension like the theory of the moduli of elliptic curves. In particular, the moduli of Krichever modules is a Deligne-Mumford stack like the other moduli spaces mentioned above. So, a natural direction is to consider how much modularity theory we can develop for Krichever moduli. For example, adopting the algebraist's point of view that modular forms are sections of a line bundle on the moduli space, we may be able to make sense of a genuine theory of Krichever modular forms with appropriate treatment of the moduli stack. This theory is interesting in its own right and introduces a new geometric language for describing differential equations.

Mumford also describes concrete applications of Krichever's dictionary and Krichever modules to differential equations. In the difference operator case of the dictionary there is a correspondence between a singular curve with p ordinary double points and whose smooth model is a curve of genus 0, and the p-soliton solutions to the non-linear differential equations which are the Toda lattice equations describing a force-repelled spring arrangement of n particles in a circle. For differential operators, there is a correspondence between Jacobian flows and solutions to Korteweg-de Vries equations, which also appear as solitons in the case of certain singular curves. This all to say, there is genuine application for the theory of Krichever modules both within the realms of differential algebra and arithmetic geometry for their own sakes and in the solving of differential equations.

I would also like to briefly mention several other projects which have been suggested to me and which I hope to pursue in the future.

Of primary interest to myself, the organizers, and the other participants of the 2023 Explicit Computations with Stacks MRC is the question of how one might do explicit computations of canonical rings of stacky surfaces. It is well-known that like ('nice') genus $g \geqslant 3$ curves have canonical embeddings, ('nice') surfaces have 'tri-canonical' embeddings using the line bundle $3K$, where K is a canonical divisor on the surface. These embeddings allow for techniques like the classical computation of canonical rings of curves in Green and Lazarsfeld's proof of Petri's Theorem [GL85] to extend to certain kinds of surfaces. Authors such as Catanese, Reid, and Shafarevich, to name only a few major contributors, have some partial results in e.g. [Rei78b], [Rei78a], [Sha13] and [CFHR99] which describe canonical and similar rings for some classes of surfaces. The general theory of canonical rings of surfaces is however, as Reid puts it, "pathological;" the very fact that this question has remain unresolved for decades despite being an object of persitent study by experts such as Reid throughout their careers is testament to its difficulty. The modern theory of stacks with its higher level of generality can offer some means of attacking this problem. With stacks experts like Michael Cerchia and Eran Assaf working on this problem there is bound to be new and exciting theory to develop in this area.

It was suggested to me by Federico Pellarin that my theory of the geometry of Drinfeld modular forms might be applied to the study of vectorial (vector-valued) Drinfeld modular forms which he describes in [Pel23]. Similarly, Matt Papanikolas suggested that it would be interesting to consider quasi-modular forms in the Drinfeld setting as in [CG23] using the theory of stacks to describe the corresponding geometry.

4. About TIFR and the Importance of Pure Math Research

I want to mention briefly my unique circumstances as I apply for research positions this year. In April 2024, I was offered a research postdoc position at the Tata Institute for Fundamental Research in Mumbai. India. However, even after obtaining a security clearance from the Indian government and applying no less than five separate times over the course of three months and spending thousands of dollars, the Indian consulate in the U. S. would not grant me a visa so that I could go do my research. If it is of interest to a potential employer, I am happy to provide extensive documentation of my battle with the consulate to get a visa.

Eventually, after the reasons for my visa rejections reached some sufficiently egregious level, I took a job teaching math at Salt Lake Community College. The sharing of math via instruction is one of the most important aspects of my pursuit of this beautiful subject, and to conclude this research statement I wish to mention two last points related to this choice. First, my decision to leave a highly prestigious research job for a teaching job in no way should reflect that I am not commited to research. Since taking this teaching job I have completed a list of nearly 100 revisions to [Fra24b] suggested to me by editors at the *Journal of Number Theory* and have been invited to speak at the University of Utah's Representation and Number Theory Seminar. This to say I am still actively pursuing my own math, even if I am not being paid to do so. Second, incorporating outreach into my work, in other words bringing math to the people who need it, is something I wish to prioritize moving forward as a mathematician, as in my choice to teach the highly diverse student body at Salt Lake Community College. The purpose of pure math as I understand it is nothing less than the promotion of world peace: a world filled with people who have their curiosity, understanding, and communication skills enhanced by the rigorous foundations of mathematics is a world where we may resolve our problems without violence. It is my hope that I will not only extend the cutting edge of human knowledge with my research, but that I will also help others share the joys of exploring these horizons.

References

[CFHR99] Fabrizio Catanese, Marco Franciosi, Klaus Hulek, and Miles Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. **154** (1999), 185–220. MR 1689180

[CFO24] Michael Cerchia, Jesse Franklin, and Evan O'Dorney, *Section rings of $\mathbb{Q}$-divisors on genus 1 curves*, 2024, `https://arxiv.org/abs/2312.15128`.

[CG23] Yen-Tsung Chen and Ouz Gezmi, *Nearly holomorphic drinfeld modular forms and their special values at cm points*, 2023.

[Dri74] V G Drinfel'd, *Elliptic modules*, Mathematics of the USSR-Sbornik **23** (1974), no. 4, 561.

[FHP24] Jesse Franklin, Sheng-Yang Kevin Ho, and Mihran Papikian, *On drinfeld modular curves for sl(2)*, 2024.

[Fra24a] Jesse Franklin, *Computing the canonical ring of certain stacks*, Ph.D. thesis, 2024, Copyright - Database copyright ProQuest LLC; ProQuest does not claim copyright in the individual underlying works; Last updated - 2024-04-05, p. 218.

[Fra24b] Jesse Franklin, *The geometry of Drinfeld modular forms*, 2024, `https://arxiv.org/abs/2310.19623`.

[Gek86] Ernst-Ulrich Gekeler, *Drinfeld modular curves*, Lecture Notes in Mathematics, vol. 1231, Springer-Verlag, Berlin, 1986. MR 874338

[GL85] Mark Green and Robert Lazarsfeld, *A simple proof of Petri's theorem on canonical curves*, Geometry today (Rome, 1984), Progr. Math., vol. 60, Birkhäuser Boston, Boston, MA, 1985, pp. 129–142. MR 895152

[GN95] Ernst-Ulrich Gekeler and Udo Nonnengardt, *Fundamental domains of some arithmetic groups over function fields*, Internat. J. Math. **6** (1995), no. 5, 689–708. MR 1351161

[Kri83] I. M. Krichever, *Rational solutions of the zakharov-shabat equations and completely integrable systems of n particles on a line*, Journal of Soviet Mathematics **21** (1983), no. 3, 335–345.

[Lau] Gérard Laumon, *Sur les modules de Krichever*.

[LRZ16] Aaron Landesman, Peter Ruhm, and Robin Zhang, *Spin canonical rings of log stacky curves*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 6, 2339–2383. MR 3580174

[Mum78] David Mumford, *An algebro-geometric construction of commuting operators and of solutions to the toda lattice equation, korteweg devries equation and related nonlinear equation*, 01 1978, pp. 115–153.

[O'D15] Evan O'Dorney, *Canonical rings of $\mathbb{Q}$-divisors on $\mathbb{P}^1$*, Annals of Combinatorics **19** (2015), no. 4, 765–784.

[Pel23] Federico Pellarin, *The analytic theory of vectorial Drinfeld modular forms*, To appear in Memoirs AMS, July 2023.

[Rei78a] Miles Reid, *Surfaces with $p_g = 0, k^2 = 1$.*, `www.maths.warwick.ac.uk/~miles/Reprint/pg0K21.pdf`, 1978.

[Rei78b] ———, *Surfaces with $p_g = 0, k^2 = 2$.*, `http://homepages.warwick.ac.uk/staff/Miles.Reid/Unpub/five_early/K2=2.pdf`, 1978.

[Sha13] Igor R. Shafarevich, *Basic algebraic geometry. 2*, third ed., Springer, Heidelberg, 2013, Schemes and complex manifolds, Translated from the 2007 third Russian edition by Miles Reid. MR 3100288

[VZB22] John Voight and David Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), no. 1362, v+144. MR 4403928