NOTES FOR SPRING 2024 TALKS

Format of the Document

Slide Title

Slide Contents

1. Workshop on Number Theory in Function Fields @ Penn State 3/13/2024

(1) **Title**

- Thank organizers!
- *The plan:* since everyone I sent this to said they didn't do stacks, we will focus on stacks
- *Intention:* invite & challenge everyone to start using the language of stacks
- *Warning:* You're welcome to interrupt with questions, but this is my first feature-length talk and first talk to experts in my field, and I am not great at doing math "live" so I may say something stupid

(2) **Notation**

- We focus on the function field of $\mathbb{P}^1/\mathbb{F}_q$ for ease of notation
- We discuss the relevance of the hypothesis that q is odd later; this is *not essential*, merely *convenient*

(3) **The classical thing we want to analogize**

- *Well-known* classical (modular forms = sections of a line bundle)
- Note: $M(\Gamma) \neq R(\mathscr{X}_\Gamma)$; *need log divisor* $M(\Gamma) = R(\mathscr{X}_\Gamma; \Delta)$
- Gekeler asks for a description of $M(\Gamma)$ for Drinfeld modular forms, in particular with generators/relations

(4) **Why Stacks? What are Stacks?**

- Stacks are *uniquely suited* to *all* modular forms
  e.g. (stacky RR) - "jumpiness" in $\dim M_k(\Gamma)$ corresponds to floors in stacky RR
- Analogy: $(\mathit{schemes}) = \begin{pmatrix} \text{locally} \\ \text{ringed spaces} \end{pmatrix} \leftrightsquigarrow (\mathit{stacks}) = \begin{pmatrix} \text{categories} \\ \text{fibered in} \\ \text{groupoids} \end{pmatrix}$
- Yoneda's Functor of Points perspective means "sheaves = stacks"

(5) **So, what are stacks?**

- *Main focus* is on stacky curves, but we also discuss closely related *gerbes* over stacky curves
- Note: Every smooth, projective curve X may be treated as a stacky curve with nothing stacky about it. The stack quotient $[X/G]$ for a finite group $G \leqslant \mathrm{Aut}(X)$ is a stacky curve, (as in Definition [LRZ16, 2.1])
- [VZB22, Remark 5.2.8] - *most* stacky curves are quotients like above
- **gerbe** - smooth, proper, geometrically connected Deligne-Mumford stack of dimension 1, i.e. stacky curve *without* dense open subscheme.
  gerbe - stack $\approx$ stacky curve, where every single point has a generic/uniform stabilizer

(6) **Stacky Curves** 101

- **signature of** $\mathscr{X}$ - (genus; orders of stabilizers of stacky points)
  signature of $(\mathscr{X}, \Delta)$ - (genus; orders of stabilizers of stacky points; degree of Δ) where Δ is a finite formal sum of distinct points of $\mathscr{X}$ called **log divisor**
- Just read the rest of the slide
- *Note:* $\mathscr{X} \cong \mathrm{sProj}(R(\mathscr{X}))$

(7) **Computing the Canonical Ring of a Stacky Curve**

- [VZB22]'s inductive result is based on considering covers of stacky curves formed by removing stacky points or changing the orders of stacky points
- [LRZ16] also has such inductive results
- In [VZB22], [LRZ16], [O'D15], [CFO24] computing canonical rings of stacky curves is a lot about the combinatorics of the floors showing up in stacky RR and giving a *ground-up* description

(8) **Old Friends**

- Example (the j-line v1.0) - recall our favorite algebras of Drinfeld modular forms (without and with type resp.); the stacky j-line is a projective line with 2 stacky points corresponding to e.g. the denominators in the valence formula: [Gek99, Equation (3.10)]:
$$\sum_{z \in GL_2(A)\backslash\Omega}^{*} v_z(f) + \frac{v_e(f)}{q+1} + \frac{v_\infty(f)}{q-1} = \frac{k}{q^2-1},$$
  where $\sum^*$ denotes a sum over non-elliptic classes of $\mathrm{GL}_2(A)\backslash\Omega$.
- *Note:* we return to the matter of stabilizers carefully later, the point of this example is below
- The problem is: the canonical ring of this stacky j-line *isn't* the algebra of modular forms for $\mathrm{GL}_2(A)$. *Need a log canonical ring instead*, but this is *not the main focus*.

(9) **What goes "Wrong" in Function Fields**

- Read the slide.
- The idea is that the proofs have too may Lemmas, so we'll discuss features in the proof instead.
- A big part of this is just phrasing familiar Drinfeld things in stacks terminology.

(10) **Drinfeld Modular Forms**

- Whip through definition of Drinfeld modular form.
- (Every talk needs one joke & one proof (& you should be able to tell the difference)) - The emphasis is that weight and type are *not* independent.

(11) **"Fourier series" for Drinfeld Modular Forms**

- Read Lemma from "right to left"
- u-series tell us about the *log* part of the log canonical ring (pole orders @ cusps)
- u-series help us decompose modular forms into differently *typed* parts

(12) **From Florian and Gebhard with Love**

- *sensitivity* of modular forms to determinants: weight-type dependence & u-series coefficients
- Breuer's Γ_2-modular forms are *easier* to recognize as sections of a log canonical divisor on a log stacky Drinfeld modular curve. In particular Breuer was the inspiration for the comparison of algebras Theorem [Fra24, 6.2]
- Bruer's forms are a *special case* of Böckle's Γ'-forms since we're taking inverse image under det of subgroups of $\mathbb{F}_q^\times$. Theorem [Fra24, 6.12] was suggested, including a proof technique, by Böckle

(13) **Cusps and Elliptic Points**

- Quickly recall cusps.
- Note: my elliptic points are *not just* $(j = 0)$*-classes on* X_Γ^{an}.
- Cusps correspond to "tails" of the graph quotient $\Gamma\backslash\mathscr{T}$ for $\mathscr{T}$ the Bruhat-Tits tree of $\mathrm{PGL}_2(K_\infty)$.
- We illustrate with Mihran's example how to form a "ramified cover" of $\mathrm{GL}_2(A)\backslash\mathscr{T}$ by $\Gamma\backslash\mathscr{T}$ and the graph of $\Gamma\backslash\mathscr{T}$
- *Advert:* in joint works with Mihran & Kevin Ho, we aim to generalize [GN95] and [PW16]

(14) **Cusps are Elliptic Points**

- For us elliptic points are no more than stacky points - the essential thing is having nontrivial stabilizers, i.e. *extra automorphisms*
- Therefore, cusps of Drinfeld modular curves are elliptic points (under this definition)
- *Example* - extra automorphisms of the Carlitz module $\rho = TX + X^q$ vs. no exta automorphisms of singular elliptic curves. *This is a purely Drinfeld-setting problem.*
- We know $M(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6]$ so why are stabilizers not orders 4 and 6? - everything in the moduli has generic μ_2-stabilizer. Likewise, every Drinfeld module has a generic μ_{q-1}-stabilizer coming from $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for $a \in \mathbb{F}_q^\times$.
- *Caution:* we're hiding something tricky here. The cusps of a Drinfeld modular curve X_Γ^{an} have isotropy groups $\{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)\}$, but if we think of compactifying $\Gamma\backslash\mathscr{F}$, for $\mathscr{F}$ the fundamental domain for Ω, we can use u^{h_s}, where h_s is the width of the cusp, as our chart at the point ∞ that we're adding in the compactification. Since we're compactifying a quotient of the fundamental domain rather than compactifying Ω and then taking a quotient, we've already removed the translations from the isotropy groups of the cusps, leaving a finite cyclic isotropy group. We *don't have a moduli interpretation* for the required $(q-1)^2$-automorphisms of the Carlitz module yet though...

(15) **Elliptic Points on Stacky Curves**

- *Claim:* cusps are elliptic points under my definition. This is *essential* for computing algebras of Drinfeld modular forms via log canonical rings
- *Question:* how stacky of stacky points are cusps? i.e. how elliptic are the elliptic points?
- We need to discuss *gerbes* in order to be sure we're talking about the right space with the right stabilizers.
- *Example* - j-lines

(16) **Rigid Stacky GAGA**

- *Recall intention:* become able to work with stacks in Drinfeld setting, i.e. our aim is to introduce the key tools
- We need to generalize rigid analytic GAGA to stacky rigid analytic GAGA via [PY16] to compare Drinfeld modular forms on $\mathscr{X}_\Gamma$ and X_Γ^{an}

(17) **Geometry of Drinfeld Modular Forms** (1/3)

- *Theorem* - the algebra of Drinfeld modular forms of Γ_2 is the log canonical ring of $(\mathscr{X}_{\Gamma_2}; 2\Delta)$
- *Formally* -

  Theorem 1.1 ([Fra24, Theorem 6.1])**.** *Let q be an odd prime and let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices of* $\mathrm{GL}_2(A)$ *and such that* $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ *for every $\gamma \in \Gamma$. Let Δ be the divisor supported at the cusps of the modular curve $\mathscr{X}_\Gamma$ with the rigid analytic coarse space $X_\Gamma^{an} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. There is an isomorphism of graded rings*

  $$M(\Gamma) \cong R(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)),$$

  where $\Omega^1_{\mathscr{X}_\Gamma}$ is the sheaf of differentials on $\mathscr{X}_\Gamma$. The isomorphism of algebras is given by the isomorphisms of components $M_{k,l}(\Gamma) \to H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(2\Delta)^{\otimes k/2})$ given by $f \mapsto f(dz)^{\otimes k/2}$.

- *Success* of the Theorem is we can answer Gekeler for Γ_2 using [VZB22], [O'D15], [CFO24], [LRZ16]
- *Failure* of the Theorem is if we can show (cusps of Γ_2) $\leftrightarrow$ (cusps of Γ) then $R(\mathscr{X}_{\Gamma_2}; 2\Delta)$ is the spin log canonical ring of $(\mathscr{X}_\Gamma; \Delta)$ in the sense of [LRZ16]
- *Key Ingredients* - dz double pole at ∞ & rigid stacky GAGA

(18) **Geometry of Drinfeld Modular Forms** (2/3)

- *Theorem* - $M(\Gamma) \cong M(\Gamma_2)$, i.e. we can recover $M(\Gamma)$ from a log canonical ring, fully answering Gekeler
- *Formally* -

  Theorem 1.2 ([Fra24, Theorem 6.2])**.** *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in* $\mathrm{GL}_2(A)$*. Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. Then $M(\Gamma) \cong M(\Gamma_2)$, with*

  $$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

  on each graded piece, where l_1, l_2 are the two solutions to $k \equiv 2l \pmod{q-1}$.

(19) **Geometry of Drinfeld Modular Forms** (3/3)

- *Theorem* - $M(\Gamma) \cong M(\Gamma')$, i.e. [Fra24, Theorem 6.2] is a special case of [Fra24, Theorem 6.12].
- *Formally* -

  Theorem 1.3 ([Fra24, Theorem 6.12])**.** *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose that Γ' is such that $\Gamma_1 \leqslant \Gamma' \leqslant \Gamma$. Then as algebras*

  $$M(\Gamma) = M(\Gamma'),$$

  and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

- This was suggested by Böckle as was the proof technique.
- Both [Fra24, Theorem 6.2] and [Fra24, Theorem 6.12] have classical analogs which come up in discussion of *nebentypes* for classical modular forms e.g.

2. Thesis Defense

(1) **Title**

- *The plan:* Penn state talk with a few more details
- *Intention:* invite & challenge everyone to start using the language of stacks

(2) **Notation**

- We focus on the function field of $\mathbb{P}^1/\mathbb{F}_q$ for ease of notation
- We discuss the relevance of the hypothesis that q is odd later; this is *not essential*, merely *convenient*

(3) **Elliptic Curves and Drinfeld Modules**

- Both elliptic curves and Drinfeld modules have a lattice-quotient (analytic) construction *and* a "Weierstrass" (algebraic) model
- Let $C\{X^q\} \overset{def}{=} \{\sum_{i=0}^n a_i X^{q^i} : a_0, \cdots, a_n \in C,\ n \geqslant 0\}$ denote the non-commutative polynomial ring of $\mathbb{F}_q$-linear polynomials/C (i.e. $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{F}_q$); multiplication given by composition
- Let $\omega \in \mathbb{C}$ be $\mathbb{R}$-linearly independent from 1. Let $\Lambda = \mathbb{Z}\omega + \mathbb{Z} \subset \mathbb{C}$ be a lattice. Then the **Weierstrass p-function** is
$$p(z, \omega, 1) = p(z, \Lambda) \overset{def}{=} \frac{1}{z^2} + \sum_{z \in \Lambda - \{0\}} \left(\frac{1}{z - \lambda} - \frac{1}{\lambda^2} \right).$$
The p-function satisfies a differential equation
$$(p')^2(z) = 4p^3(z) - g_2 p(z) - g_3,$$
where g_2 and g_3 are values of certain Eisenstein series,
i.e. the p-function gives a Weierstrass model associated to the lattice Λ.

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- Note: $M(\Gamma) \neq R(\mathscr{X}_\Gamma)$; *need log divisor* $M(\Gamma) = R(\mathscr{X}_\Gamma; \Delta)$
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(9) **Example of Section Rings**

- $S_{D'}$ generated in degrees $1, 2, 4$; $I_{D'}$ has $\mathrm{gin}_<(I_{D'}) = \langle y^2 \rangle \subset \Bbbk[u, x_1, x_2^2]$
- S_D generated in degrees $1, 2, 2$; I_D has $\mathrm{gin}_<(I_D) = \langle x_1^3 \rangle \subset \Bbbk[u, x_1, x_2]$

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- *Caution:* we're hiding something tricky here!

(17) **Isotropy** (1/2)

- The "degenerate" Drinfeld modules of rank 2 which are cusps of a Drinfeld modular curve are Drinfeld modules of rank 1.
- Up to homothety there is only one rank 1 Drinfeld module: the **Carlitz module**:
$$\rho(T) = TX + X^q \leftrightsquigarrow \overline{\pi}A \subset \Omega,$$
where $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is the **Carlitz period**, defined up to a $(q-1)$st root of unity.
- $\mathrm{Aut}(\rho) \cong \mathbb{F}_q^\times$ since $\overline{\pi}A \sim \alpha\overline{\pi}A$ for any $\alpha \in \mathbb{F}_q^\times$.
- "Extra" automorphisms come from specifying a Carlitz period.
- *Just read* the isotropy groups of cusps side.

(18) **Isotropy** (2/2)

- Classical pictures [DS05, Figures 2.3 and 2.4]
- Drinfeld fundmental domain from Tristan Phillips
- The point here is that the notation $X_\Gamma^{\mathrm{an}} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$ is misleading!
- We are *really* taking
  1. a quotient $\Gamma\backslash\mathscr{F}$ for $\mathscr{F}$ the fundamental domain (i.e. building of $\mathscr{T}(\mathbb{R})$) of Ω
  2. a quotient $\Gamma\backslash\mathbb{P}^1(K)$ separately
  3. *then* glueing the chart(s) at ∞ (resp. cusps) to the (open/affine) quotient $\Gamma\backslash\mathscr{F}$
- We can use u^{h_s}, where h_s is the width of the cusp s, as our chart at the point ∞ that we're adding in the compactification

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- *Claim:* cusps are elliptic points under my definition. This is *essential* for computing algebras of Drinfeld modular forms via log canonical rings
- *Question:* how stacky of stacky points are cusps? i.e. how elliptic are the elliptic points?
- We need to discuss *gerbes* in order to be sure we're talking about the right space with the right stabilizers.
- *Example* - j-lines
- *What is the j-line?* Every elliptic curve (resp. Drinfeld module) has a numerical invariant called its j-invariant $j(E) = \dfrac{c_4(E)^3}{\Delta(E)}$ $\left(\text{resp. } j(\varphi) = \dfrac{g(\varphi)^{q+1}}{\Delta(\varphi)}\right)$. This j comes from a **modular function** - a meromorphic function on the (compactified) upper half-plane with a tranformation rule similar to a modular form's "weak modular condition." That is, the j-function maps from a given modular curve to a projective line $\mathbb{P}^1$(base field) and the j-invariant of an elliptic curve (resp. Drinfeld module) is the image of the curve (resp. module) under this map

(20) **Rigid Stacky GAGA**

- *Recall intention:* become able to work with stacks in Drinfeld setting, i.e. our aim is to introduce the key tools
- We need to generalize rigid analytic GAGA to stacky rigid analytic GAGA via [PY16] to compare Drinfeld modular forms on $\mathscr{X}_\Gamma$ and X_Γ^{an}

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- *Theorem* - the algebra of Drinfeld modular forms of Γ_2 is the log canonical ring of $(\mathscr{X}_{\Gamma_2}; 2\Delta)$
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  Theorem 2.1 ([Fra24, Theorem 6.1]). *Let q be an odd prime and let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices of $\mathrm{GL}_2(A)$ and such that $\det(\gamma) \in (\mathbb{F}_q^\times)^2$ for every $\gamma \in \Gamma$. Let Δ be the divisor supported at the cusps of the modular curve $\mathscr{X}_\Gamma$ with the rigid analytic coarse space $X_\Gamma^{an} = \Gamma\backslash(\Omega \cup \mathbb{P}^1(K))$. There is an isomorphism of graded rings*

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- *Success* of the Theorem is we can answer Gekeler for Γ_2 using [VZB22], [O'D15], [CFO24], [LRZ16]
- *Failure* of the Theorem is if we can show (cusps of Γ_2) $\leftrightarrow$ (cusps of Γ) then $R(\mathscr{X}_{\Gamma_2}; 2\Delta)$ is the spin log canonical ring of $(\mathscr{X}_\Gamma; \Delta)$ in the sense of [LRZ16]
- *Key Ingredients* - dz double pole at ∞ & rigid stacky GAGA

(22) **Geometry of Drinfeld Modular Forms** (2/3)

- *Theorem* - $M(\Gamma) \cong M(\Gamma_2)$, i.e. we can recover $M(\Gamma)$ from a log canonical ring, fully answering Gekeler
- *Formally* -

  Theorem 2.2 ([Fra24, Theorem 6.2]). *Let q be a power of an odd prime. Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. Then $M(\Gamma) \cong M(\Gamma_2)$, with*

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References

[CFO24] Michael Cerchia, Jesse Franklin, and Evan O'Dorney, *Section rings of* $\mathbb{Q}$*-divisors on genus* 1 *curves*, 2024, https://arxiv.org/abs/2312.15128.

[DS05] Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196

[Fra24] Jesse Franklin, *The geometry of Drinfeld modular forms*, 2024, https://arxiv.org/abs/2310.19623.

[Gek99] Ernst-Ulrich Gekeler, *A survey on Drinfeld modular forms*, Turkish J. Math. **23** (1999), no. 4, 485–518. MR 1780937

[GN95] Ernst-Ulrich Gekeler and Udo Nonnengardt, *Fundamental domains of some arithmetic groups over function fields*, Internat. J. Math. **6** (1995), no. 5, 689–708. MR 1351161

[LRZ16] Aaron Landesman, Peter Ruhm, and Robin Zhang, *Spin canonical rings of log stacky curves*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 6, 2339–2383. MR 3580174

[O'D15] Evan O'Dorney, *Canonical rings of* $\mathbb{Q}$*-divisors on* $\mathbb{P}^1$, Annals of Combinatorics **19** (2015), no. 4, 765–784.

[PW16] Mihran Papikian and Fu-Tsun Wei, *On the eisenstein ideal over function fields*, Journal of Number Theory **161** (2016), 384–434.

[PY16] Mauro Porta and Tony Yue Yu, *Higher analytic stacks and GAGA theorems*, Adv. Math. **302** (2016), 351–409. MR 3545934

[VZB22] John Voight and David Zureick-Brown, *The canonical ring of a stacky curve*, Mem. Amer. Math. Soc. **277** (2022), no. 1362, v+144. MR 4403928