ON KRICHEVER MODULES

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Translator's note

This translation of Gérard Laumon's [Lau] was made using Google translate and my own poor French. I have included a screenshot of his original bibliography at the end to avoid tracking down the .bib information required to copy his citations perfectly. I did my best to employ proper English grammar while respecting the original French, and have only changed a sentence if it began with "Then" or "Alors" because this is not proper usage, as Calum Buchanan would be quick to remind us. Where Laumon, or perhaps more accurately, Mme. Bonnardel has underlined words in definitions, I have used bold font as this is easier to read for my eyes. I have added a table of contents as this translation is intended to be a working document of notes, and this improves its readability.

Introduction

In this note, we try to better understand the similarities observed by Drinfeld and Mumford between Krichever modules and Drinfeld modules in [Dri77a] and [Mum78].

We show that it is possible to pose a moduli problem for Krichever modules, with level structure, over a differential scheme and that it is possible to represent this moduli problem with a universal differential scheme, in a manner completely parallel to what is done in [Dri74] and [Dri77b]. We also define a differential analog of Drinfeld's shutkas (from Drinfeld's "Three letters on moduli schemes" in 1976).

These constructions suggest very strongly that there must exist a Langlands-style functoriality between rank n representations of differential Galois groups and a certain type of "automorphic forms" over $\mathrm{GL}_n(A)$, where A is the ring of adeles of a field of functions of equal characteristic 0. Unfortunately, it was not possible for us to come up with a conjectural statement, even in the abelian case ($n = 1$).

The content of this note is the following.

In paragraph 1 we recall certain results on the additive differential group.

Paragraphs 2 to 5 are dedicated to the study of Krichever modules over a differential field (level structures, endomorphisms, symbols, ...); this study is completely parallel to that of Drinfeld for his modules ([Dri74] and [Dri77b]).

In paragraphs 6 and 7 we study a part of the preceding results in the case of an arbitrary differential base. We formulate and resolve a moduli problem for Krichever modules.

In paragraph 8 we recall Kirchever's dictionary between Krichever modules and Krichever bundles.

In paragraph 9 we treat the duality between Krichever modules and bundles.

The definition of differential shtukas is given in paragraph 10.

Finally, we conclude this note with two appendices on the construction of "jets" and the logarithmic derivative.

I would like to thank L. Breen for many discussions on the topics covered in this note. I also thank Ms. Bonnardel for the remarkable typing of this manuscript.

1. Endomorphisms of $\mathbb{G}_a$ over a differential $\mathbb{Q}$-algebra

Let (R, ∂) be a differential $\mathbb{Q}$-algebra and

$$\mathbb{G}_{a,(R,\partial)} = \mathrm{Spec}(R[x, x', \ldots, x^{(j)}, \ldots], \partial)$$

be the **additive (differential) group scheme over** (R, ∂), where $\partial(x^{(j)}) = x^{(j+1)}$ for all $j \in \mathbb{N}$, and where the group law is given by the comultiplication

$$\mu^* : (R[x, x', \ldots], \partial) \to (R[x, x', \ldots], \partial) \otimes_{(R,\partial)} (R[x, x', \ldots], \partial)$$

with

$$\mu^*(x^{(j)}) = x^{(j)} \otimes 1 + 1 \otimes x^{(j)}.$$

An (R, ∂)-morphism of differential schemes $\varphi : \mathbb{G}_{a,(R,\partial)} \to \mathbb{G}_{a,(R,\partial)}$ is given by a polynomial $\varphi^*(x) \in R[x, x', \ldots]$ and φ is a morhism of differential group schemes if and only if $\varphi^*(x)$ is linear in each of the variables $x^{(j)}$ (we use here the fact that $\mathbb{Q} \subset R$), i.e.

$$\varphi^*(x) = \alpha_0 x + \alpha_1 x' + \cdots + \alpha_n x^{(n)}$$

with $\alpha_0, \dots, \alpha_n \in R$, and $n \geqslant 0$.

Let

$$R[\partial] = \left\{ \sum_{i \geqslant 0} \alpha_i \partial^i : \alpha_i \in R \ \forall i \geqslant 0 \text{ and } \alpha_i = 0 \text{ for } i >> 0 \right\}$$

be equipped with the usual addition and multiplication defined by the commutation rule

$$\partial\alpha - \alpha\partial = \partial(\alpha), \quad \forall \alpha \in R.$$

We verify immediately that the map

$$\varphi \mapsto \alpha_0 + \alpha_1 \partial + \cdots + \alpha_n \partial^n,$$

where

$$\varphi^*(x) = \alpha_0 x + \alpha_1 x' + \cdots + \alpha_n x^{(n)}$$

defines an isomorphism of rings

$$\mathrm{End}_{(R,\partial)}(\mathbb{G}_{a,(R,\partial)}) \xrightarrow{\sim} R[\partial].$$

We have a degree map

$$\deg : R[\partial] \to \mathbb{N} \cup \{-\infty\}$$

defined by

$$\begin{cases} \deg(\alpha_0 + \alpha_1 \partial + \cdots + \alpha_n \partial^n) = n, & \text{if } \alpha_n \neq 0 \\ \deg(0) = -\infty, \end{cases}$$

such that for all $u, v \in R[\partial]$,

(1) $\deg(u + v) \leqslant \max(\deg u, \deg v)$ with equality if $\deg(u) \neq \deg(v)$
(2) $\deg(u + v) \leqslant \deg(u) + \deg(v)$ with equality if the coefficient of $\partial^{\deg u}$ in u (or that of $\partial^{\deg(v)}$ in v) is not a zero divisor in R.

Consequently, if we put, for $i \in \mathbb{Z}$,

$$R[\partial]_i = \{u \in R[\partial] : \deg(u) \leqslant i\},$$

$(R[\partial], R[\partial]_i)$ is a ring filtered by an increasing, exhaustive and separated filtration with $R[\partial]_{[-1]} = 0$ and $R[\partial]_0 = R$; moreover, we immediately check that $\mathrm{gr}\, R[\partial]$ is a commutative R-algebra isomorphic to $R[\zeta]$ (the usual commutative polynomials) via $\partial \mapsto \zeta$.

The subring of constants of (R, ∂) is

$$R^\partial = \{\alpha \in R : \partial(\alpha) = 0\},$$

which is also the center of $R[\partial]$. If $R = k$ is a field, then $R^\partial = k^\partial$ too.

For $\alpha_1, \dots, \alpha_n \in R$ with $n \geqslant 1$, the **wronskian** $\mathrm{W}(\alpha_1, \dots, \alpha_\mathrm{n}) \in R$ is the determinant

$$\mathrm{W}(\alpha_1, \dots, \alpha_\mathrm{n}) = \det \left(\left[\partial^i(\alpha_j) \right]_{\substack{i=0,\dots,n-1 \\ j=0,\dots,n}} \right).$$

Proposition 1.1 (Kaplansky)**.** *Let $\alpha_1, \dots, \alpha_n \in R$, where $n \geqslant 1$. If $\mathrm{W}(\alpha_1, \dots, \alpha_\mathrm{n})$ is invertible in R, then $\alpha_1, \dots, \alpha_n$ are linearly independent over R^∂. Conversely, if $R = k$ is a field and if $\alpha_1, \dots, \alpha_n$ are linearly independent over k^∂, then $\mathrm{W}(\alpha_1, \dots, \alpha_\mathrm{n})$ is invertible (i.e. is non-zero).*

Corollary 1.2. *Let (k, ∂) be a differential field of characteristic* 0.

(i) For all $u \in k[\partial] - \{0\}$, we have the inequality

$$\deg(u) \geqslant \dim_{k^\partial}(\ker u)$$

or

$$\ker u = \{\alpha \in k : u(\alpha) = 0\};$$

(ii) *For all k^∂-sub (vector) spaces V of k of finite dimension n, there exists a unique*

$$u = \alpha_0 + \alpha_1\partial + \cdots + \alpha_n\partial^n = k[\partial]$$

with $\alpha_n = 1$ and $\ker u = V$.

We will say that $u \in R[\partial]$ is **elliptic** if the coefficient of $\partial^{\deg(u)}$ in u is invertible in R.

Proposition 1.3. (i) *For $u, v \in R[\partial]$ with u elliptic, there exists unique $w, \tilde{v}$ in $R[\partial]$ such that*

$$v = wu + \tilde{v}, \quad \deg(\tilde{v}) < \deg u$$

(resp. $v = uw + \tilde{v}$, $\deg(\tilde{v}) < \deg(u)$).

(ii) *If $R = k$ is a field, then every left (resp. right) ideal of $k[\partial]$ is principal.*

The ring of differential operators $R[\partial]$ embeds in the ring of **psuedo-dfferential operators** $R(\!(\partial^{-1})\!)$ constructed as follows:

$$R(\!(\partial^{-1})\!) = \left\{ \sum_{i=-\infty}^{n} \alpha_i\partial^i : n \in \mathbb{Z} \text{ and } \alpha_i \in R \right\}$$

with the obvious addition and multiplication given by the commutation rules

$$\partial\alpha = \alpha\partial + \partial(\alpha)$$
$$\partial^{-1}\alpha = \sum_{i \geqslant 0} (-1)^i \partial^i(\alpha)\partial^{-i-1}$$

for all $\alpha \in R$.

The degree function evidently continues to

$$\deg : R(\!(\partial^{-1})\!) \to \mathbb{Z} \cup \{-\infty\}$$

and if we put for all $i \in \mathbb{Z}$

$$R(\!(\partial^{-1})\!)_i = \{u \in R(\!(\partial^{-1})\!) : \deg(u) \leqslant i\},$$

then $(R(\!(\partial^{-1})\!), R(\!(\partial^{-1})\!)_i)$ is still a filtered ring with an increasing, exhaustive, separated filtration with grading $\operatorname{gr} R(\!(\partial^{-1})\!)$ which is a commutative R-algebra isomorphic to $R[\zeta, \zeta^{-1}]$ via $\partial \mapsto \zeta$. We will say that $u \in R(\!(\partial^{-1})\!)$ is **elliptic** if the coefficient of $\partial^{\deg(u)}$ in u is invertible. Now, $R[\![\partial^{-1}]\!] = R(\!(\partial^{-1})\!)_0$ is a subring of $R(\!(\partial^{-1})\!)$, and for each $i \in \mathbb{Z}$,

$$R(\!(\partial^{-1})\!)_i = R[\![\partial^{-1}]\!]\partial^i = \partial^i R[\![\partial^{-1}]\!]$$

is a free $R[\![\partial^{-1}]\!]$-module of rank 1 both left and right; for $i \leqslant 0$, $R(\!(\partial^{-1})\!)_i$ is therefore a bilateral principal ideal of $R[\![\partial^{-1}]\!]$.

Lemma 1.4. *Is $u \in R(\!(\partial^{-1})\!)$ is elliptic, u is invertible in $R(\!(\partial^{-1})\!)$ with an inverse of degree $-\deg u$ which is also elliptic; conversely, if for $u \in R(\!(\partial^{-1})\!)$ there exists $v \in R(\!(\partial^{-1})\!)$ with $\deg(v) = -\deg(u)$ and $vu = 1$ or $uv = 1$, then u is ellitptic and v is the inverse of u.*

Proof. (Manin)
The converse is trivial if we admit the direct part; so we demonstrate this one. It suffices to treat the case with $\deg u = 0$ and with

$$u = 1 + \alpha_{-1}\partial^{-1} + \alpha_{-2}\partial^{-2} + \cdots ;$$

so, if

$$n = \alpha_{-1}\partial^{-1} + \alpha_{-2}\partial^{-2} + \cdots ,$$

the series

$$u^{-1} = 1 - n + n^2 - n^3 + \cdot$$

converges in $R(\!(\partial^{-1})\!)$, which completes the proof. ∎

Remark 1.5. *We will pay attention to the fact that in general there exist invertible elements of $R(\!(\partial^{-1})\!)$ which are non-elliptic: for example, if $\alpha \in R - \{0\}$ has $\alpha^2 = 0$, $\alpha\partial + 1$ is invertible with inverse $-\alpha\partial + 1$.*

Corollary 1.6. *If (k, ∂) is a differential field of characteristic 0, $k(\!(\partial^{-1})\!)$ is a left field and $-\deg : k(\!(\partial^{-1})\!) \to \mathbb{Z} \cup \{+\infty\}$ is a discrete valuation for which $k(\!(\partial^{-1})\!)$ is complete; $k[\![\partial^{-1}]\!]$ is the ring of the valuation $-\deg$, $k(\!(\partial^{-1})\!)_{-1}$ is the maximal ideal and the residue field is canonically isomorphic to k.*

Lemma 1.7. *Let $\beta_1 \in R^\times$. Every*

$$u = \alpha_n\partial^n + \alpha_{n-1}\partial^{n-1} + \cdots \in R(\!(\partial^{-1})\!),$$

with $n > 0$ and $\alpha_n = \beta_1^n$, admits in $R(\!(\partial^{-1})\!)$ a unique nth root of the form

$$u^{1/n} = \beta_1\partial + \beta_0 + \beta_{-1}\partial^{-1} + \cdots$$

and the centralizer of u in $R(\!(\partial^{-1})\!)$ is no other than $R^\partial(\!(u^{-1/n})\!)$.

Proof. (Manin)
For $v = \beta_1\partial + \beta_0 + \beta_{-1}\partial^{-1} + \cdots$ in $R(\!(\partial^{-1})\!)$, we verify by reccurence (TL Note: induction?) for $n > 0$ that

$$v^n = \sum_{j=-\infty}^{n} \gamma_{n,j}\partial^j$$

where

$$\gamma_{n,n} = \beta_1^n$$

and

$$\gamma_{n,j} = n\beta_1^{n-1}\beta_{n-j+1} + P_{n,j}(\beta_1, \beta_0, \ldots, \beta_{j-n+2})$$

with $P_{n,j}$ a differential polynomial, for $j \leqslant n-1$. Therefore, we can solve by descending reccurence the system of equations in the $\beta_0, \beta_{-1}, \beta_{-2}, \ldots,$

$$\alpha_j = \gamma_{n,j} \quad (j \leqslant n-1),$$

which proves existence (and uniqueness) of $u^{1/n}$. Moreover, it is clear that for all $v = \beta_1\partial + \beta_0 + \beta_{-1}\partial^{-1} + \cdots$ in $R(\!(\mu^{-1})\!)$ with β_1 invertible, we have that

$$R(\!(\partial^{-1})\!) = R(\!(v^{-1})\!);$$

therefore we have

$$R(\!(\partial^{-1})\!) = R(\!(u^{-1/n})\!).$$

If $w \in R(\!(\partial^{-1})\!)$, we can write w uniquely as

$$w = \sum_{k \leqslant N} \gamma_k u^{k/n}$$

with $\gamma_N \neq 0$ and it remains to show that w is in the centralizer of u only if $\gamma_N \in R^\partial$ ($u^{k/n}$ is trivially in the centralizer for all $k \in \mathbb{Z}$).

$$[u, v] = \sum_{k \leqslant N} [u, \gamma_k]u^{k/n}$$

and $\deg([u, w]) \leqslant n-1$, so $[u, w] = 0$ only if the coefficient of ∂^{n-1} in $[u, \gamma_N]$ is zero, i.e. $n\alpha_n\partial(\gamma_N) = 0$, i.e. $\partial(\gamma_N) = 0$, i.e. $\gamma_N \in R^\partial$, hence the conclusion. □

Corollary 1.8. *If $u \in R[\partial]$ is elliptic of degree > 1, the centralizer $Z(u)$ of u in $R[\partial]$ is a commutative sub-R^∂-algebra of $R[\partial]$; moreover, if $(R, \partial) = (k, \partial)$ is a differential field, for all $i \geqslant 0$ $k[\partial]_i \cap Z(u)$ is a finite dimensional k^∂-vector space of dimension $\leqslant i$.*

Proof. We have $Z(u) = R[\partial] \cap R^\partial(\!(u^{-1/n})\!)$ and $R(\!(\partial^{-1})\!)_i \cap R^\partial(\!(u^{-1/n})\!) = R^\partial[\![u^{-1/n}]\!]u^{i/n}$ for all $i \in \mathbb{Z}$. □

2. Kricher Modules over a differential field

Let k_0 be a field of characteristic 0, let X be a smooth, projective, geometerically connected curve over k_0 and let ∞ be a closed point of X. We put

$$A = H^0(X - \{\infty\}, \mathcal{O}_X),$$

we denote by F the field of fractions of A, and let

$$\mathrm{ord} : A \to \mathbb{N} \cup \{-\infty\}$$

be the map defined by

$$\begin{cases} \mathrm{ord}(a) = \dim_{k_0}(A/aA) & \text{if } a \in A - \{0\} \\ \mathrm{ord}(0) = -\infty. \end{cases}$$

If F_∞ is the completion of F at the place ∞, $\mathcal{O}_\infty$ the ring of integers of F_∞, m_∞ is the maximal ideal of $\mathcal{O}_\infty$, $k_\infty = \mathcal{O}_\infty/m_\infty$ is the residue field and $v_\infty : F_\infty \to \mathbb{Z} \cup \{+\infty\}$ the discrete valuation of F_∞ (normalized so that $v_\infty(\pi_\infty) = 1$ for a uniformizer π_∞ of $\mathcal{O}_\infty$), we have

$$\mathrm{ord}(a) = -[k_\infty : k_0]v_\infty(a)$$

for all $a \in A$.

On the other hand, let (k, ∂) be a diferential field with $k_0 \subset k^\partial$.

Definition 2.1. *Let r be an integer > 0. A **Krichever module of rank** r over (k, ∂) for the pointed curve (X, ∞) is a k_0-algebra homomorphism*

$$\varphi : A \to \mathrm{End}_{(k,\partial)}(\mathbb{G}_{a,(k,\partial)}) \cong k[\partial]$$

such that

$$\deg \varphi(a) = r\,\mathrm{ord}(a), \quad \forall a \in A.$$

If $\mathrm{Algdiff}_{(k,\partial)}$ is the category of (k, ∂)-differential algebras and if $\mathrm{Mod}_{A\otimes_{k_0}k^\partial}$ is the category of k^∂-modules, to any Krichever module φ we associate a functor

$$E_\varphi : \mathrm{Algdiff}_{(k,\partial)} \to \mathrm{Mod}_{A\otimes_{k_0}k^\partial}$$

which associates $\mathbb{G}_{a,(k,\partial)}(R, \partial) = R$ with (R, ∂), provided with the structure of an A-module given by φ (and the evident k^∂-vector space structure); the functor E_φ uniquely determines φ.

Definition 2.2. *Let φ and φ' be two Krichever modules of the same rank r over (k, ∂) for (X, ∞). A **generalized morphism** $u : \varphi \to \varphi'$ is an element $u \in k[\partial]$ such that $u\varphi(a) = \varphi'(a)$ for all $a \in A$. A **morphism** $u : \varphi \to \varphi'$ is a generalized morphism $u \in k \subset k[\partial]$, i.e. of degree 0. An **isogeny** $u : \varphi \to \varphi'$ is a generalized morphism $u \in k[\partial]$ which is elliptic; the degree of the isogeny is the degree of u.*

We will write $\mathrm{End}(\varphi)$ (resp. $\mathrm{End}^*(\varphi)$) for the ring of endomorphisms (resp. generalized endomorphisms) of φ; we have

$$k^\partial = \mathrm{End}(\varphi) \subset \mathrm{End}^*(\varphi).$$

We will use, in the study of $\mathrm{End}^*(\varphi)$ the following results from "Drinfeld's Three Letters":

Proposition 2.3. *Let φ be a Krichever module of rank r over (k, ∂) for (X, ∞) and let $H \subset E_\varphi(k, \partial) = k$ a sub-$A \otimes_{k_0} k^\partial$-module of finite dimension n over k^∂. There exists a unique Krichever module (up to isomorphism) φ' of rank r over (k, ∂) for (X, ∞) and an isogeny $u : \varphi \to \varphi'$ of degree n such that $H = \ker u$.*

Proof. By Corollary 1.2 (ii) there exists $u \in k[\partial]$ elliptic of degree n such that $H = \ker u$ and by Proposition 1.3 (i) there also exists, for each $a \in A - \{0\}$ unique $\varphi'(a)$ and $v \in k[\partial]$ such that $u\varphi(a) = \varphi'(a)u + v$ and $\deg(v) < \deg(u) = n$; $\varphi(a)H \subset H$ by hypothesis, therefore $H \subset \ker v$ and by Corollary 1.2 (i) this implies that $v = 0$; it is clear now that $\varphi' : A \to k[\partial]$ ($\varphi'(0) = 0$) and u answer the question. ∎

Proposition 2.4. *If $u : \varphi \to \varphi'$ and $w : \varphi \to \varphi''$ are two isogenies between Krichever modules of rank r over (k, ∂) for (X, ∞), if*

$$\dim_{k^\partial}(\ker u) = \deg u$$

and if

$$\ker u \subset \ker w$$

then there exists an isogeny (and only one) $v : \varphi' \to \varphi''$ such that $w = vu$.

Proof. We divide Euclideanly on the right w by u (see Proposition 1.3 i) then we conclude with the help of Corollary 1.2 (i). □

Corollary 2.5. *Suppose that k^∂ is a finite extension of k_0. If $u : \varphi \to \varphi'$ is an isogeny between Krichever modules of rank r over (K, ∂) for (X, ∞), there exists $a \in A - \{0\}$ and an isogeny $v : \varphi' \to \varphi$ such tha $vu = \varphi(a)$.*

Proof. Suppose first of all that $\dim_{k^\partial} = \deg u$; then since $\ker u$ is an A-module of finite dimension over k_0, $\ker u$ is torsion and so there exists $a \in A - \{0\}$ such that $\varphi(a)(\ker(u)) = \{0\}$ and this does not depend on the choice of map from Proposition 2.4. For the general case, there exists, by Kolchin (TL Note: [Ko.1] in Laumon's bibligraphy) IV 5 Cor. 2, a differential extension (k_1, ∂) of (k, ∂) such that

$$\dim_{k_1^\partial}(\ker(u : k_1 \to k_1)) = \deg u$$

and that

$$[k_1^\partial : k^\partial] < +\infty;$$

according to the case covered above there exists therefore $a \in A - \{0\}$ and $v_1 \in k_1[\partial] - \{0\}$ such that $v_1 u = \varphi(a)$ in $k_1[\partial]$, which clearly implies that $v_1 = v \in k[\partial]$ and the conclusion. □

3. Division points and level structures

TL Note: From here onwards, I will change "division points" to their more modern name "torsion points."

Let $\varphi : A \to k[\partial]$ be a Krichever modules of rank $r > 0$ over (k, ∂) for the pointed curve (X, ∞) and let $E = E_\varphi$ be the corresponding functor from $\mathrm{Algdiff}_{(k,\partial)}$ to $\mathrm{Mod}_{A \otimes_{k_0} k^\partial}$.

Definition 3.1. *For $a \in A$, the **subfunctor** $E_a \subset E$ **of a-torsion points of the Krichever module** φ is the functor defined by*

$$E_a(B, \partial) = \ker(\varphi(a) : B \to B).$$

*For $I \subset A$ an ideal, the **subfunctor** $E_I \subset E$ **of I-torsion points of the Kirchever module** φ is the functor defined by*

$$E_I(B, \partial) = \bigcap_{a \in I} E_a(B, \varphi).$$

If $I = Aa_1 + \cdots + Aa_n$, we trivially have $E_I = E_{a_1} \cap \cdots \cap E_{a_n}$; we have $E_I = E$ if and only if $I = (0)$ and $E_I = 0$ (i.e. $E_I(B, \partial) = 0$ $\forall (B, \partial) \in \mathrm{Algdiff}_{(k,\partial)}$) if and only if $I = A$.

We can consider E_I like a functor

$$E_I : \mathrm{Algdiff}_{(k,\partial)} \to \mathrm{Mod}_{(A/I) \otimes_{k_0} k^\partial}.$$

Remark 3.2. *If $I = Aa_1 + \cdots + Aa_n$ and if $\varphi(a_i) = \sum_j \alpha_i \partial^j \in k[\partial]$, the $\sum_j \alpha_{ij} x^{(j)}$ for $i = 1, \ldots, n$ generate a differential ideal of $(k[x, x', \ldots, x^{(j)}], \partial)$ which does not depend on I and not on the choice of a_i and if we write $\langle I \rangle$ for this differential ideal, we still have*

$$E_I = \mathrm{Spec}(k[x, x', \ldots]/\langle I \rangle, \partial)$$

in an obvious sense.

Recall that a differential field (k, ∂) of characteristic 0 is **differentially closed** if for all $u \in k[\partial] - \{0\}$ we have

$$\dim_{k^\partial}(\ker u) = \deg u;$$

if this is so, we easily verify that $u : k \to k$ is surjective for all $u \in k[\partial] - \{0\}$.

Theorem 3.3. *Let $\varphi : A \to k[\partial]$ be an injective homomorphism of k_0-algebras such that $\varphi(A) \not\subset k$. There exists an integer $r > 0$ such that φ is a Kirchever module of rank r over (k, ∂) for (X, ∞). Furthermore, if we also suppose that (k, ∂) is differentially closed, for every nontrivial ideal I of A, $E_I(k, \partial)$ is a free $(A/I) \otimes_{k_0} k^\partial$-module of rank r.*

Proof. Let us first notice that for φ as in the statement, there exists $r \in \mathbb{Q}_+^\times$ such that

$$\deg \varphi(a) = r \operatorname{ord}(a), \quad \forall a \in A.$$

Indeed, if we put

$$\mu(a) = -\deg \varphi(a), \quad \forall a \in A$$

then we check the following properites of μ

(1) $\mu(a + b) \geqslant \inf(\mu(a), \mu(b))$ with equality if $\mu(a) \neq \mu(b)$, $\forall a, b \in A$;
(2) $\mu(ab) = \mu(a) + \mu(b)$, $\forall a, b \in A$;
(3) $\mu(0) = +\infty$, $\mu(a) \in \mathbb{Z}$, and $\mu(a) \leqslant 0$ $\forall a \in A$ and $\mu(a) < 0$ for at least one $a \in A - \{0\}$;

as a result, μ extends to a nontrivial valuation of F, also denoted μ, negative on $A - \{0\}$; such a valuation is necessarily equivalent to v_∞ in the sense that there exists $s \in \mathbb{R}_+^\times$ with $\mu(a) = sv_\infty(a)$, $\forall a \in F$; for $r = s/[k_\infty : k_0]$ we have the equality sought, from which it follows that $r \in \mathbb{Q}$.

To prove the first assertion it remains to show that $r \in \mathbb{N}$ and for this we can replace at will (TL Note: without loss of generality?) (k, ∂) with a differential extension; in particular we can suppose that (k, ∂) is differentially closed. We will show under the latter hypothesis that $r \in \mathbb{N}$ and the second assertion of the theorem.

So, let $a \in A - \{0\}$ have $\operatorname{ord}(a) > 0$ and also let $\deg \varphi(a) > 0$; we have

$$\deg \varphi(a) = \dim_{k^\partial}(\ker \varphi(a))$$

so that

$$\dim_{k^\partial}(\ker \varphi(a^2)) = \deg \varphi(a^2) = 2 \deg \varphi(a) = 2 \dim_{k^\partial}(\ker \varphi(a));$$

now we have the well-known lemma:

Lemma 3.4. *Let B be a Dedekind ring, $b \in B$ and L a (B/b^2B)-module of finite length. We have an inequality*

$$2 \operatorname{length}(\ker(L \xrightarrow{b} L)) \geqslant \operatorname{length}(L)$$

and for $L \neq 0$, equality occurs if and only if L is a free (B/b^2B)-module of finite rank.

We apply this lemma to $B = A \otimes_{k_0} k^\partial$ and to $L = \ker \varphi(a^2)$ which has finite length because of its finite dimension over k^∂; given the equality

$$\dim_{k^\partial}(\ker \varphi(a^2)) = 2 \dim_{k^\partial}(\ker \varphi(a))$$

proved above, we deduce that $\ker \varphi(a)$ is a free $(A/aA) \otimes_{k_0} k^\partial$-module of finite rank; let $d > 0$, therefore

$$\begin{aligned} r \operatorname{ord}(a) = \deg \varphi(a) &= \dim_{k^\partial}(\ker \varphi(a)) \\ &= d \dim_{k^\partial}((A/aA) \otimes_{k_0} k^\partial) = d \operatorname{ord}(a); \end{aligned}$$

from where $r \in \mathbb{N}$ since $\operatorname{ord}(a) \neq 0$ by hypothesis and the second assertion follows for $I = aA$.

For arbitrary I there exists a proper ideal J of A such that $I + J = A$ and $I \cap J = aA$ for some $a \in A$, so then $A/I \oplus A/J = A/aA$ and $E_I \oplus E_J = E_a$ and we are brought back to the case where I is a principal ideal, which is already treated. $\square$

Definition 3.5. *A* ***level structure*** I, *for* I *a proper, nonzero ideal of* A *on a Krichever module* φ *of rank* r *over* (k,∂) *for* (X,∞) *is the data of an isomorphism*

$$\gamma : (I^{-1}/A)^r \otimes_{k_0} k^\partial \xrightarrow{\sim} E_I(k,\partial)$$

of $(A/I)\otimes_{k_0} k^\partial$*-modules, where* $E = E_\varphi$.

As $\mathrm{End}(\varphi) = k$, the data of such a level structure on φ completely rigidifies it; $\mathrm{Aut}(\varphi,\gamma)$ is reduced to the identity. On the other hand, $\mathrm{GL}_r(A/I)$ acts on the isomorphisms classes of pairs (φ,γ) via its natural action on γ.

We can also define the **Tate module** of a Krichever module of rank r over (k,∂) in every place $x \neq \infty$ of F. To simplify, we suppose k_0 is algebraically closed, $k^\partial = k_0$ and that (k,∂) admits a strongly normal extension (in the sense of Kolchin's "Galois Theory of Differential Fields"; Laumon's [Ko. 2]) which is differentially closed and with $\overline{k}^\partial = k^\partial$, so that we can talk about the differential Galois group G of $(\bar{k},\partial)$ over (k,∂); this is an algebraic pro-group over k_0.

Let $\varphi : A \to k[\partial]$ be a Kirchever module of rank r over (k,∂) for (X,∞) and let $s \in X - \{\infty\}$ be a closed point defined by a maximal ideal $\mathfrak{p}$ of A; we denote by $\mathcal{O}_x = \varprojlim_n A/\mathfrak{p}^n$ the complete local ring of X in x and F_x its field of fractions. For $E = E_\varphi$, we have a inductive system

$$E_{\mathfrak{p}}(\bar{k},\partial) \subset E_{\mathfrak{p}^2}(\bar{k},\partial) \subset E_{\mathfrak{p}^3}(\bar{k},\partial) \subset \cdots,$$

where each $E_{\mathfrak{p}^n}(\bar{k},\partial)$ is a free $A/\mathfrak{p}^n$-module of rank r by Theorem 3.3, so that

$$D_x(\varphi) = \varinjlim_n E_{\mathfrak{p}^n}(\bar{k},\partial)$$

is a divisible $\mathcal{O}_x$-module isomorphic to $(F_x/\mathcal{O}_x)^r$ (where $F_x/\mathcal{O}_x = \varinjlim_n \mathfrak{p}^{-n}/A$).

Definition 3.6. *The* ***Tate module of*** φ *is the free* $\mathcal{O}_x$*-module of rank* r

$$T_x(\varphi) = \mathrm{Hom}_{\mathcal{O}_x}(F_x/\mathcal{O}_x, D_x(E)),$$

equipped with the natural action of the group G.

If I is a proper, nonzero ideal of A, we put

$$T_I(\varphi) = \prod_{x\in V(I)} T_x(\varphi);$$

it is a free $\prod_{x\in V(I)} \mathcal{O}_x$-module of rank r. In particular, if $I = aA$ is principal, we have

$$T_I(\varphi) = T_a(\varphi) = \varprojlim_n F_{a^n}(\bar{k},\partial)$$

where the transition maps are

$$E_{a^{n+1}}(\bar{k},\partial) \xrightarrow{\varphi(a)} E_{a^n}(\bar{k},\partial).$$

4. Ring structure of the generalized endomorphisms of a Krichever module over a closed differential field

Preliminaries. Fix a real $\rho > 1$ and consider over the field of fractions F of A the non-archimedean absolute value

$$|a| = \rho^{-v_\infty(a)}, \quad \forall a \in F.$$

All of the norms we consider in the following over F-vector spaces will be non-archimedean and compatible with $||$ over F.

For every F-vector space V of finite dimension n, a norm $\|\| : V \to \mathbb{R}_+^\times$ will be called **admissible** if there exists a basis $\{v_1, \dots, v_n\}$ of V with

$$\left\| \sum_{i=1}^n a_i v_i \right\| = \sup(|a_i|, i = 1, \dots, n)$$

for all $a_1, \dots, a_n \in F$; two admissible norms over V of course define the same topology over V; moreover every norm over V is increased by (TL Note: induced by?) an admissible norm.

If W is an F-vector space (eventually of infinite dimension) equipped with a norm $\| \ \| : W \to \mathbb{R}_+^\times$, each open ball

$$B_W(0, r) = \{w \in W : \|w\| < r\}, \quad r \in \mathbb{R}_+^\times,$$

is a sub-k_0-vector space of W. A sub-k_0-vector space $H \subset W$ is called **discrete** if for each $r \in \mathbb{R}_+^\times$ we have

$$\dim_{k_0}(B_W(0, r) \cap H) < +\infty;$$

for each sub-F-vector space V of finite dimension of W, $H \cap V \subset V$ is then discrete in V not only for the norm induced by that of W over V but also for each admissible norm over V.

Lemma 4.1. *Let V be an F-vector space of finite dimension and let $H \subset V$ be a sub-A-module of V (and so a sub-k_0-vector space). If H is discrete in V for any norm over V, H is a projective A-module of finite type and* $\mathrm{rank}_A(H) \leqslant \dim_F(V)$.

Proof. We first choose a basis $\{v_1, \dots, v_n, v_{n+1}, \dots, v_{n+m}\}$ of V such that, via the indentification $V \cong F^n \oplus F^m$ given by this basis, we are

$$A^n \subset H^n \subset F^n = F^n \oplus (0) \subset F^n \oplus F^m$$

($\{v_1, \dots, v_n\}$ is a basis for $FH \subset V$). To demonstrate the proposition, it suffices to show that $\dim_{k_0}(H/A^n) < +\infty$ (effectively, this implies a part of H is an A-module of finite type and so projective since H is torsion-free over the Dedekind ring A, and on the other hand that the A-module H/A^n is torsion so that $\mathrm{rank}_A(H) = n \leqslant n + m = \dim_F(V)$).

Or, $H \subset F^n$ is discrete for the admissible norm $\|(a_1, \dots, a_n)\| = \sup(|a_i|, i = 1, \dots, n)$, i.e.

$$\dim_{k_0}(B_F(0, r)^n \cap H) < +\infty, \quad \forall r \in \mathbb{R}_+^\times,$$

and so

$$\dim_{k_0}\left(\frac{B_F(0, r)^n + A^n}{A^n} \cap \frac{H}{A^n}\right) < +\infty, \quad \forall r \in \mathbb{R}_+^\times$$

and we conclude by noting that

$$B_F(0, r) + A = F$$

for $r >> 0$. □

Now, let (k, ∂) be a differential field with $k^\partial \supset k_0$, which we suppose is differentially closed and let $\varphi : A \to k[\partial]$ be a Krichever module of rank r over (k, ∂) for (X, ∞). We propose to determine the ring $\mathrm{End}^*(\varphi)$ of generalized endomorphisms of φ.

As $\mathrm{End}^*(\varphi) \subset k[\partial]$ and $k[\partial]$ is integral, $\mathrm{End}^*(\varphi)$ is a torsion-free $A \otimes_{k_0} k^\partial$-module (we have $\varphi(A) \subset \mathrm{End}^*(\varphi) \subset k[\partial]$, hence the A-module structure of $\mathrm{End}^*(\varphi)$) which embeds into the $F \otimes_{k_0} k^\partial$-vector space $F \otimes_A \mathrm{End}^*(\varphi)$.

Over $F \otimes_A \mathrm{End}^*(\varphi)$ we have a norm

$$\| \ \| : F \otimes_A \mathrm{End}^*(\varphi) \to \mathbb{R}_+^\times,$$

characterized by

$$\|u\| = \rho^{\deg u/(r[k_\infty : k_0])}$$

for all $u \in \mathrm{End}^*(\varphi)$ and compatible with the absolute value $||$ of $F \otimes_{k_0} k^\partial$, the trivial extension of $||$ over F to $F \otimes_{k_0} k^\partial$ (we have $\deg(\varphi(a)u) = -r[k_\infty : k_0]v_\infty(a) + \deg(u)$ for all $a \in A$ and all $u \in \mathrm{End}^*(\varphi)$).

Theorem 4.2. *(i)* $\mathrm{End}^*(\varphi)$ *is a commutative ring; as an* $A \otimes_{k_0} k^\partial$*-module,* $\mathrm{End}^*(\varphi)$ *is projective of rank* $< r$*; as a* k^∂*-vector space,* $\mathrm{End}^*(\varphi)$ *is discrete in* $F \otimes_A \mathrm{End}^*(\varphi)$ *equipped with the norm* $\|\|$.

(ii) $F \otimes_A \mathrm{End}^*(\varphi)$ *is a commutative field, an extension of* $F \otimes_{k_0} k^\partial$ *(via* φ*) of degree* $\leqslant r$.

Proof. Even if it means extending the scalars from k_0 to k^∂ we may suppose (and we do suppose) that $k^\partial = k_0$.

First of all, let us show that $\mathrm{End}^*(\varphi)$ is commutative. Let $a \in A$ with $\mathrm{ord}(a) > 0$. $\mathrm{End}^*(\varphi)$ is a subring of the centralizer of $\varphi(a)$ in $k[\partial]$, consequently the assertion follows by Corollary 1.8.

It is clear on the other hand that $\mathrm{End}^*(\varphi)$ is discrete in $(F \otimes_A \mathrm{End}^*(\varphi), \| \|)$, also according to Corollary 1.8. Subsequently, for all sub-F-vector spaces V of finite dimension of $F \otimes_A \mathrm{End}^*(\varphi)$, $\mathrm{End}^*(\varphi) \cap V$ is a projective A-module of finite type by Lemma 4.1.

That being so, for $a \in A$ with $\mathrm{ord}\, a > 0$, we have a canonical homomorphism of (A/aA)-algebras

$$(A/aA) \otimes_A \mathrm{End}^*(\varphi) \to \mathrm{End}(E_a(k, \partial))$$

and in fact this homomorphism is injective; if $w \in \mathrm{End}^*(\varphi)$ is such that

$$\ker \varphi(a) \subset \ker w,$$

there exists, by Proposition 2.4, $v \in \mathrm{End}^*(\varphi)$ such that $v\varphi(a) = w$ (we recall that (k, ∂) is differentially closed). Moreover, $\ker \varphi(a) = E_a(k, \partial)$ is a free (A/aA)-module of rank r by Theorem 3.3, therefore $\mathrm{End}(E_a(k, \partial))$ is a free (A/aA)-module of rank r^2. Consequently, for each sub-F-vector space V of finite dimension of $F \otimes_A \mathrm{End}^*(\varphi)$, $(A/aA) \otimes_A (\mathrm{End}^*(\varphi) \cap V)$ is an (A/aA)-module which is at the same time a free, projective A-module of the same rank $\mathrm{End}^*(\varphi) \cap V$ and contained in a free (A/aA)-module of rank r^2; we see that

$$\mathrm{rank}_A(\mathrm{End}^*(\varphi) \cap V) \leqslant r^2$$

for all V, since

$$\dim_F(F \otimes_A \mathrm{End}^*(\varphi)) \leqslant r^2$$

and finally $\mathrm{End}^*(\varphi)$ is a projective A-module of rank $\leqslant r^2$.

To complete the proof of part (i) of the Theorem, we consider again the inclusion

$$(A/aA) \otimes_A \mathrm{End}^*(\varphi) \subset \mathrm{End}(E_a(k, \partial)) \cong M_r(A/aA)$$

and recall that $\mathrm{End}^*(\varphi)$ is a commutative ring; as $(A/aA) \otimes_A \mathrm{End}^*(\varphi)$ is free of finite rank over A/aA, we see that

$$\mathrm{rank}_{A/aA}((A/aA) \otimes_A \mathrm{End}^*(\varphi)) \leqslant r$$

and so that

$$\mathrm{rank}(\mathrm{End}^*(\varphi(a))) \leqslant r.$$

Finally, part (ii) of the Theorem follows from part (i) and from Corollary 2.5. □

Remark 4.3. *We saw during the proof of Theorem 4.2 that for each* $a \in A$ *with* $\mathrm{ord}(a) > 0$ *the canonical map*

$$(A/aA) \otimes_A \mathrm{End}^*(\varphi) \to \mathrm{End}(E_a(k, \partial))$$

is injective. We can deduce, as for abelian varieties, the following result:

Proposition 4.4. *For each closed point* $x \in X - \{\infty\}$ *the* $\mathcal{O}_X$*-linear, canonical map*

$$\mathcal{O}_X \otimes_A \mathrm{End}^*(\varphi(a)) \to \mathrm{End}_{\mathcal{O}_X}(T_x(\varphi))^G$$

(see Definition 3.6 for hypothesis and notation) is injective.

Proof. We start by generalizing the statement for a proper, nonzero ideal I of A (at the place x), then we come back to the case I is principal like at the end of the proof of Theorem 3.3 and in the latter case we use Remark 4.3 and the fact that $\mathrm{End}^*(\varphi)$ is projective over A. □

5. Pseudo Krichever modules; Krichever module symbols

We keep the notation of Section 2.

Definition 5.1. *A **psuedo-Krichever module of rank** r **(an integer** > 0**) over** (k, ∂) **for** F_∞ is a k_0-algebra morphism*

$$\psi : F_\infty \to k(\!(\partial^{-1})\!)$$

such that for each $a \in F_\infty$ we have

$$\deg \psi(a) = -r[k_\infty : k_0]v_\infty(a).$$

*A generalized morphism between two such psuedo-Kirchever modules $u : \psi \to \psi'$ is an element $u \in k(\!(\partial^{-1})\!)$ such that $u\psi(a) = \psi'(a)u$ $\forall a \in F_\infty$. If in fact $u \in k[\![\partial^{-1}]\!]$ then we call the generalized morphism simply a **morphism**.*

Proposition 5.2. *Each Krichever module $\varphi : A \to k[\partial]$ of rank r for (X, ∞) extends to a unique psuedo-Krichever module $\tilde{\psi} : F_\infty \to k(\!(\partial^{-1})\!)$ of rank r for F_∞.*

Proof. For each $a \in A - \{0\}$, $\varphi(a) \in k[\partial]$ is invertible in $k(\!(\partial^{-1})\!)$, therefore φ extends to a morphism of k_0-algebras, also denoted φ, from F into $k(\!(\partial^{-1})\!)$, verifying $\deg \varphi(a) = -r[k_\infty : k_0]v_\infty(a)$ $\forall a \in F_\infty$, and this extension is unique. This latter relation ensures the continuity of φ in the topologies over F and $k(\!(\partial^{-1})\!)$ respectively for the valuations v_∞ and $-\deg$; consequently the proposition results from the fact that $k(\!(\partial^{-1})\!)$ is complete for $-\deg$. □

A pseudo-Krichever module ψ of rank r over (k, ∂) for F_∞ induces, by transitions between gradings, an injective morphism of graded k_0-algebras of degree r

$$\operatorname{gr} \psi : \oplus_{i \in \mathbb{Z}} m_\infty^i / m_\infty^{i+1} \to \operatorname{gr} k(\!(\partial^{-1})\!) \cong k[\zeta, \zeta^{-1}]$$

and therefore, by passing to spectra, a morphism of k_0-schemes

$$\sigma : \mathbb{G}_{m,k} \to \dot{T}_\infty X,$$

where $T_\infty X = \operatorname{Spec}(\operatorname{Sym}_{k_0}(m_\infty / m_\infty^2))$ is the Zariski tangent space to X at ∞ and where $\dot{T}_\infty X = X - \{0\}$ (we have $\operatorname{Sym}_{k_0}(m_\infty / m_\infty^2) = \oplus_{i \in \mathbb{N}} m_\infty^i / m_\infty^{i+1}$ since X is smooth over k_0 at ∞); this morphism is in fact well-defined for any r and and its restriction to the unit section of $\mathbb{G}_{m,k}$ we also denote σ

$$\sigma : \operatorname{Spec}(k) \to \dot{T}_\infty X.$$

Definition 5.3. *The morphism σ above is called the **symbol of the pseudo-Krichever module** ψ; if $\psi = \tilde{\varphi}$ for some Krichever module φ we also say σ is the **symbol of** φ.*

Theorem 5.4. *Suppose that (k, ∂) is differentially closed and that k is algebraically closed. Let ψ and ψ' be two pseudo-Krichever modules over (k, ∂) for F_∞ of the same rank r. Such pseudo-modules ψ and ψ are isomorphic if and only if they have the same symbol $\sigma = \sigma' : \operatorname{Spec}(k) \to \dot{T}_\infty X$.*

Proof. Even if it means identifying k_∞ with the algebraic closure of k_0 in F_∞ and fixing a uniformizer π_∞ of F_∞, we can identify F_∞ and $k_\infty(\!(\pi_\infty)\!)$. Saying $\sigma = \sigma'$ then amounts to saying that

$$\psi|_{k_\infty} \equiv \psi'|_{k_\infty} \pmod{k(\!(\partial^{-1})\!)_{-1}}$$

and that

$$\psi^{(\prime)}(\pi_\infty) \equiv \alpha^{(\prime)} \partial^{-s} \pmod{k(\!(\partial^{-1})\!)_{-s-1}}$$

with

$$\alpha = \alpha' \in k - \{0\}$$

(we put $s = r[k_\infty : k_0]$).

The "only if" part of the theorem is immediate.

For the "if" part, which to replace ∂ with $\beta^{-1}\partial$ where β is a sth root of $\alpha = \alpha'$ in $k - \{0\}$, we can suppose that $\alpha = \alpha' = 1$. Now, we have the lemma:

Lemma 5.5. *Let*
$$w = \partial^n + \alpha_{n-1}\partial^{n-1} + \cdots \in k(\!(\partial^{-1})\!)$$
with $n \neq 0$. There exists
$$v = \beta_0 + \beta_{-1}\partial^{-1} + \cdots \in k[\![\partial^{-1}]\!]^\times$$
such that
$$v^{-1}wv = \partial^n.$$
Furhtermore, if $Z(w)$ is the centralizer of w in $k(\!(\partial^{-1})\!)$, then we have
$$v^{-1}Z(w)v = k^\partial(\!(\partial^{-1})\!).$$

Provisionally admitting this lemma, we can deduce that there exist $v, v' \in k[\![\partial^{-1}]\!]^\times$ such that
$$v^{-1}\psi(\pi_\infty)v = v'^{-1}\psi'(\pi_\infty)v' = \partial^{-s}$$
and that for each $a \in F_\infty$,
$$v^{-1}\psi(a)v \text{ and } v'^{-1}\psi'(a)v'$$
belong to $k^\partial(\!(\partial^{-1})\!)$ (a and π_∞ commute). In particular, we have k_0-morphisms $v^{-1}\psi v$ and $v'^{-1}\psi' v'$ from k_∞ into $k^\partial[\![\partial^{-1}]\!] \subset k^\partial(\!(\partial^{-1})\!)$, congruent modulo $k^\partial[\![\partial^{-1}]\!]_{-1}$; as k^∂ is algebraically closed in $k^\partial(\!(\partial^{-1})\!)$ and k_∞ is algebraic over k_0, we can deduce that $v^{-1}\psi v|_{k_\infty} = v'^{-1}\psi' v'|_{k_\infty}$. It is not clear that $u = v'v^{-1} \in k^\partial[\![\partial^{-1}]\!]^\times$ is an isomorphism of ψ and ψ'. □

(Proof of Lemma 5.5). We come back to $\alpha_{n-1} = 0$ by noticing that
$$\beta_0^{-1}\left(\partial^n + \alpha_{n-1}\partial^{n-1} + \cdots\right)\beta_0 = \partial^n + \left(n\frac{\partial(\beta_0)}{\beta_0} + \alpha_{n-1}\right)\partial^{n-1} + \cdots$$
and that there exists $\beta_0 \in k - \{0\}$ such that $n\dfrac{\partial(\beta_0)}{\beta_0} + \alpha_{n-1} = 0$. We then look, for
$$w = \partial^n + \alpha_{n-2}\partial^{n-2} + \cdots$$
an element
$$v = 1 + \beta_{-1}\partial^{-1} + \cdots \in k[\![\partial^{-1}]\!]^\times$$
such that
$$v^{-1}wv = \partial^n,$$
which amounts to solving a system of differential equations
$$\partial(\beta_{-i}) = P_i(\beta_{-1}, \beta_{-2}, \ldots, \beta_{-i+1}) \quad (i \geqslant 1),$$
where the P_i are differential polynomials, and which we do by induction on i. The last assertion of the lemma results from the equality $Z(\partial^n) = k^\partial(\!(\partial^{-1})\!)$ for $n \neq 0$. □

For each integer $r > 0$ and each morphism of k_0-schemes
$$\sigma : \mathrm{Spec}(k) \to \dot{T}_\infty X$$
it is easy to construct pseudo-Krichever modules $\psi : F_\infty \to k(\!(\partial^{-1})\!)$ of rank r and with symbol σ: indeed, if we fix a uniformizer π_∞ of F_∞, the data of σ is equivalent to the data of a k_0-morphism $k_\infty \to k^\partial$ and of $\alpha \in k - \{0\}$ and then the data ψ comes down to giving itself
$$\psi(\pi_\infty^{-1}) = L_{\pi_\infty} = \alpha_s\partial^s + \alpha_{s-1}\partial^{s-1} + \cdots \in k(\!(\partial^{-1})\!)$$
with $s = r[k_\infty : k_0]$ and
$$\alpha_s = \alpha^{-1}.$$
In particular, we write $\psi_{r,\sigma,\pi_\infty}$ for the pseudo-Krichever of rank r and symbol σ with data
$$\psi_{r,\sigma,\pi_\infty}(\pi_\infty^{-1}) = L_{r,\sigma,\pi_\infty} = (\alpha^{-1/s}\partial)^s$$
(still with $s = r[k_\infty : k_0]$ and in fact $L_{r,\sigma,\pi_\infty} \in \mathbb{Q}[\alpha^{-1}, \partial(\alpha^{-1}), \ldots, \partial^{s-1}(\alpha^{-1})][\partial]$ which does not depend on our choice of sth root of α).

On the other hand, for σ as above, we write $F_{\sigma,\infty}$ for the completion of $F \otimes_{k_0} k^{\partial}$ in the place given in the data of the inclusion $k_\infty \hookrightarrow k^{\partial}$ associated to σ.

Proposition 5.6. *Let $r > 0$ be an integer, let $\sigma : \mathrm{Spec}(k) \to \dot{T}_\infty X$ be a k_0-morphism and let π_∞ be a uniformizer of F_∞. Suppose that k^{∂} is algebraically closed. The generalized ring of endomorphisms*

$$\mathrm{End}^*(\psi_{r,\sigma,\pi_\infty}) = \{v \in k(\!(\partial^{-1})\!) : v\psi_{r,\sigma,\pi_\infty} = \psi_{r,\sigma,\pi_\infty} v\}$$

is a commutative field, a cyclic extension of $F_{\sigma,\infty}$ of degree dividing $r[k_\infty : k_0]$.

Proof. The symbol σ is given by $k_\infty \hookrightarrow k^{\partial}$ and $\alpha \in k - \{0\}$; then let $k_1 = k(\alpha^{1/s})$ and also write ∂ for the natural extension of ∂ from k to k_1. We have k_1^{∂} algebraic over k^{∂} and therefore $k_1^{\partial} = k^{\partial}$ and we have

$$F_{\sigma,\infty} \cong k(((\alpha^{-1/s}\partial)^{-s}))$$

and

$$\mathrm{End}^*(\psi_{r,\sigma,\pi_\infty}) = k(\!(\partial^{-1})\!) \cap k_1^{\partial}(((\alpha^{-1/s}\partial)^{-s}))$$

where the conclusion, with

$$[\mathrm{End}^*(\psi_{r,\sigma,\pi_\infty}) : F_{\sigma,\infty}] = s/[k_1 : k]$$

(we put $s = r[k_\infty : k_0]$). □

6. Differential schemes and groups locally isomorphic to $\mathbb{G}_a$ (in characteristic 0)

(TL Note: my numbering of results from this section until the end of the paper differs from Laumon's)

Let (S, ∂) be a differential $\mathbb{Q}$-scheme (S is a $\mathbb{Q}$-scheme and $\partial : \mathcal{O}_S \to \mathcal{O}_S$ is a derivation (TL Note: derivative)); then we have on S a filtered ring $(\mathcal{O}_S[\partial], \mathcal{O}_S[\partial]_i)$ (defined like in Section 1); $\mathcal{O}_S[\partial]$ (resp. $\mathcal{O}_S[\partial]_i$, where $i \geqslant -1$) is a left $\mathcal{O}_S$-module and is quasi-coherent (resp. locally free of rank $i+1$) on the right, and $\mathrm{gr}\,\mathcal{O}_S[\partial]$ is canonically isomorphic to $\mathcal{O}_S[\zeta]$ (commutative polynomials) via $\partial \mapsto \zeta$.

To each quasi-coherent $\mathcal{O}_S$-module E_0 we associate the left quasi-coherent $\mathcal{O}_S[\partial]$-module

$$E = \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0$$

equipped with the growing and exhaustive filtration of quasi-coherent sub-$\mathcal{O}_S$-modules

$$E_i = \mathcal{O}_S[\partial]_i \otimes_{\mathcal{O}_S} E_0$$

and the differential $\mathbb{Q}$-scheme above (S, ∂),

$$(E, \partial) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(E), \partial),$$

where ∂ is the derivation on $\mathrm{Sym}_{\mathcal{O}_S}(E)$ induced by that of $\mathcal{O}_S$ and by the connection ∂ over the $\mathcal{O}_S$-module E.

It is clear that (E, ∂) is a differential, abelian (S, ∂) group scheme, with "like" (TL Note: the air quotes are mine. What is "avec comme comultipliation celle induite par"???) comultiplication induced by

$$\begin{aligned} E &\to \mathrm{Sym}_{\mathcal{O}_S}(E) \otimes_{\mathcal{O}_S} \mathrm{Sym}_{\mathcal{O}_S}(E) \\ e &\mapsto 1 \otimes e + e \otimes 1. \end{aligned}$$

For example, for $E_0 = \mathcal{O}_S$, we have

$$(E, \partial) = \mathbb{G}_{a,(S,\partial)} = \mathrm{Spec}(\mathcal{O}_S[x, x', x'', \ldots], \partial)$$

with $\partial(x^{(i)}) = x^{(i+1)}\ \forall i \geqslant 0$; as a result, if E_0 is locally free of rank 1, (E, ∂) is locally isomorphic to $\mathbb{G}_{a,(S,\partial)}$ for the Zariski topology on S.

Remark 6.1. *If S is reduced (i.e. has no nilpotents), it is easy to show that we get all the (S,∂)-schemes in groups locally isomorphic to $\mathbb{G}_{a,(S,\partial)}$ by the procedure above; the point is that $\mathcal{O}_S \subset \mathcal{O}_S[\partial]$ have the same invertible elements by virtue of the following lemma:*

Lemma 6.2. *Let (R,∂) be a differential $\mathbb{Q}$-algebra and let*

$$u = \alpha_m \partial^m + \cdots + \alpha_0$$

$$v = \beta_n \partial^n + \cdots + \beta_0$$

be two elements of $R[\partial]$ such that

$$vu = 1.$$

These $\alpha_m \ldots, \alpha_1$ are nilpotents. Moreover, if u is in fact invertible (i.e. if we also have $uv = 1$), α_0 is invertible.

Proof. The second assertion follows easily from the first since from the relation $uv = 1$ we have

$$\sum_{i=0}^{m} \alpha_i \partial^i(\beta_0) = 1$$

and therefore $\alpha_0\beta_0 - 1$ is nilpotent.

As $\sqrt{u}$ is a differential ideal of R (by Kaplanksy [Ka; i]), it suffices, for the first assertion to show that $m \geqslant 1$ implies $\alpha_m^N = 0$ for $N. > 0$. Or, we have

$$vu = \gamma_{m+n}\partial^{m+n} + \cdots + \gamma_m \partial^m + \cdots + \gamma_0$$

with

$$\gamma_{m+j} = \beta_j \alpha_m + \sum_{k=j+1}^{n} \beta_k \delta_k$$

$(j = 0, 1, \ldots, n)$, where $\delta_k \in R\ \forall k$, and

$$\gamma_0 = \sum_{j=0}^{n} \beta_j \partial^i(\alpha_0).$$

As $\gamma_{m+j} = 0$ for $j = 0, 1, \ldots, n$ if $m \geqslant 1$, we see by descending induction over j that $\alpha_m^{n-j+1}\beta_j = 0$ and therefore that $\alpha_m^{n+1}\beta_j = 0$ for $j = 0, 1, \ldots, n$. As a result, as $\gamma_0 = 1$, we have, if $m \geqslant 1$,

$$\alpha_m^{n+1} = \alpha_m^{n+1}\gamma_0 = 0,$$

from where the conclusion. □

For (S,∂) arbitrary again and for E_0, E'_0 two locally free $\mathcal{O}_S$-modules of rank 1, the hypothesis that the characteristic of S is 0 implies:

Proposition 6.3. *We have canonically*

$$\mathrm{Hom}_{(S,\partial)}((E,\partial),(E',\partial)) = \underline{\mathrm{Hom}}_{\mathcal{O}_S[\partial]}(E',E) = E_0'^{-1} \otimes_{\mathcal{O}_S} \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0,$$

where $\underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial))$ is the Zariski sheaf over S of homomorphisms of differential (S,∂) group schemes from (E,∂) to (E',∂) and where $E_0'^{-1} = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(E_0,\mathcal{O}_S)$.

In particular, $\underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial))$ is canonically equipped with an increasing exhaustive filtration by the

$$\underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial)) = E_0'^{-1} \otimes_{\mathcal{O}_S} \mathcal{O}_S[\partial]_i \otimes_{\mathcal{O}_S} E_0,$$

with grading canonically isomorphic to $E_0'^{-1} \otimes_{\mathcal{O}_S} \mathcal{O}_S[\zeta] \otimes_{\mathcal{O}_S} E_0$ (commutative polynomials), via $\partial \mapsto \zeta$ (if $E_0 = E'_0$, this last ring is isomorphic to $\mathcal{O}_S[\zeta]$).

Definition 6.4. *A section u of $\underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial))$ is called* ***elliptic of degree*** *n if it is in fact a section of $\underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial))_n$ and if the $\mathcal{O}_S$-linear compound morphism*

$$E'_o \xrightarrow{u} E_n \twoheadrightarrow E_n/E_{n-1}$$

is an isomorphism (for $E_0 = E'_0$, this amounts to saying that the image of u in $\mathrm{gr}\,\underline{\mathrm{End}}_{(S,\partial)}(E,\partial)$ is produced by ζ^n of an invertible section of $\mathcal{O}_S$).

Let us now move on to the study of differential (S,∂) group schemes obtained as the core (TL Note: kernel?) of an elliptic morphism from (E,∂) to (E',∂).

Definition 6.5. *Let $\mathcal{V}$ be a quasi-coherent, left $\mathcal{O}_S[\partial]$-module and let*

$$(\mathcal{V},\partial) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{V}),\partial)$$

be the corresponding differential (S,∂) group scheme. We say that $\mathcal{V}$ or $(\mathcal{V},\partial)$ is ***cyclic of rank*** *n if as an $\mathcal{O}_S$-module, $\mathcal{V}$ is free of rank n ($n \in \mathbb{N}$) and if there exists a section v of $\mathcal{V}$ over S such that*

$$(v,\partial(v),\dots,\partial^{n-1}(v))$$

is a basis for the $\mathcal{O}_S$-module $\mathcal{V}$; one such section is then called a ***cyclic vector for*** *$\mathcal{V}$. We say that $\mathcal{V}$ or $(\mathcal{V},\partial)$ is* ***locally cyclic of rank*** *n if it is cyclic of rank n over each open in a Zariski open cover of S.*

With the notation as above, if $u \in \underline{\mathrm{Hom}}_{(S,\partial)}((E,\partial),(E',\partial))$ is elliptic of degree n, the cokernel of $E' \xrightarrow{u} E$ (or what amounts to the same thing, the kernel of $(E,\partial) \xrightarrow{u} (E',\partial)$) is locally cyclic of rank n.

For each differential $\mathbb{Q}$-scheme (S,∂) we write $S_\partial \subset S$ for all of the closed points $s \in S$ such that the derivation $\partial_s : \mathcal{O}_{S,s} \to \mathcal{O}_{S,s}$ (fiber from ∂ to s) has

$$\partial_s(m_{S,s}) \subset m_{S,s}$$

where $m_{S,s}$ is the maximal ideal of $\mathcal{O}_{S,s}$; for $s \in S_\partial$, $\kappa(s) = \mathcal{O}_{S,s}/m_{S,s}$ is then a differential field of characteristic 0 (∂_s induces a derivation on $\kappa(s)$); for each differential field (κ,∂) of characteristic 0 and each differential morphism

$$\sigma : \mathrm{Spec}(\kappa,\partial) \to (S,\partial)$$

the image of σ is a point $s \in S_\partial$.

Proposition 6.6. *Let E_0 be a locally free $\mathcal{O}_S$-module of rank 1, let $\mathcal{V}$ be a locally cyclic $\mathcal{O}_S[\partial]$-module of rank n and let $E = \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0 \overset{\theta}{\twoheadrightarrow} \mathcal{V}$ be an epimorphism of $\mathcal{O}_S[\partial]$-modules. Put, for each $i \geqslant 0$,*

$$E'_i = \ker(E_{n+i} = \mathcal{O}_S[\partial]_{n+1} \otimes_{\mathcal{O}_S} E_0 \xrightarrow{\theta} \mathcal{V})$$

and

$$E' = \varinjlim = \ker(E \overset{\theta}{\twoheadrightarrow} \mathcal{V}).$$

Even if it means replacing (S,∂) by (S',∂) where $S_\partial \subset S' \subset S$ is a Zariski open of S containing S_∂, we have the following conclusions:

(i) $E_{n-1} \overset{\theta}{\to} \mathcal{V}$ is an isomorphism of locally free $\mathcal{O}_S$-modules of rank n,

(ii) for each $i \geqslant 0$, $E_{n+i} \overset{\theta}{\to} \mathcal{V}$ is an epimorphism and E'_i is a locally free $\mathcal{O}_S$-module of rank $i+1$,

(iii) the sheaf produced by

$$\mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E'_0 \to E$$

is injective with image E' and induces, for each $i \geqslant 0$, an isomorphism of locally free $\mathcal{O}_S$-modules of rank $i+1$

$$\mathcal{O}_S[\partial]_i \otimes_{\mathcal{O}_S} E'_0 \xrightarrow{\sim} E'_i.$$

Remark 6.7. *In general, one has to localize at $S_\partial \subset S$ to get the conclusions* (i), (ii), (iii)*: if $R = \mathbb{Q}[x]$, if $\partial = \partial/\partial x$ and if*

$$\theta : R[\partial] \to R = R[\partial]/R[\partial]\partial$$

is the $R[\partial]$-linear map such that $\theta(1) = x$, θ is surjective but $\theta|_{R[\partial]_0}$ is not an isomorphism because x is not invertible in R.

(Proof of Proposition 6.6). We start by remarking that is checked for $(S, \partial) = \mathrm{Spec}(k, \partial)$ where (k, ∂) is a differential field (of characteristic 0): indeed, each left ideal of $k[\partial]$ is principal by Proposition 1.3 (iii) and therefore, if $\Gamma(S, E) = k[\partial]$, we have $\Gamma(S, E') = k[\partial]u$ with u necessarily elliptic of degree n, from where the conclusion.

Now let us consider the general case. The $\mathcal{O}_S$-linear map of locally free $\mathcal{O}_S$-modules of rank n,

$$\theta : E_{n-1} \to \mathcal{V},$$

is an isomorphism at each point $s \in S_\partial$ according to the argument above, therefore an isomorphism of the neighborhood of S_∂ in S (EGA I (5.5)) which shows (i) and also (ii). For assertion (iii), we consider the composite $\mathcal{O}_S$-linear map

$$E'_0 \xrightarrow{\theta} E_n \twoheadrightarrow E_n/E_{n-1}$$

which is again bjiective at each point $s \in S_\partial$ and therefore an isomorphism of the neighborhood of S_∂ in S; then, even if it means restricting to a neighborhood of S_∂ in S, we see that

$$\mathcal{W} = E/\mathcal{O}_S[\partial]E'_0$$

and

$$E'_i = \mathcal{O}_S[\partial]_i E'_0, \quad \forall i \geqslant 0$$

from where we get assertion (iii). □

The following criteria is used to check if an $\mathcal{O}_S$-module $\mathcal{V}$, which is locally free of finite rank over $\mathcal{O}_S$, is locally cyclic; it is essentially due to Deligne and Katz.

Proposition 6.8. *Let $\mathcal{V}$ be an $\mathcal{O}_S$-module, locally free of finite rank as an $\mathcal{O}_S$-module.*

(i) If ∂ does not cancel on S, i.e. if $\partial_s(\mathcal{O}_{S,s}) \not\subset m_{S,s}$, $\mathcal{V}$ is locally cyclic;

(ii) If $\mathcal{V}$ embeds as an $\mathcal{O}_S[\partial]$-module in $\mathcal{V}'$ which is locally cyclic and if $\mathcal{V}$ is, as an $\mathcal{O}_S$-module, locally a direct factor of $\mathcal{V}'$, $\mathcal{V}$ is also locally cyclic.

Proof. Straight away we will come back to the case where $(S, \partial) = \mathrm{Spec}(R, \partial)$ with R a local $\mathbb{Q}$-algebra with maximal ideal $\mathfrak{p}$, and $\partial : R \to R$ is a derivation. We distinguish three cases:

(a) $\partial(\mathfrak{p}) \not\subset \mathfrak{p}$, so that there exists $t \in \mathfrak{p}$ with $\partial(t) \in R - \mathfrak{p}$; then let $\delta = \partial(t)^{-1}\partial$, it remains to show that the $R[\delta]$-module $V = \Gamma(S, \mathcal{V})$ is cyclic; well, by Katz ([Kz]), for each R-basis $(v_0, \ldots, v_{n-1})$ of $\mathcal{V}$,

$$v = \sum_{j=0}^{n-1} \frac{t^j}{j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \delta^k(v_{j-k})$$

is a cyclic vector of V;

(b) $\partial(\mathfrak{p}) \subset \mathfrak{p}$ but $\partial(R) \not\subset \mathfrak{p}$; then by Deligne ([De]) the $(R/\mathfrak{p}R)[\partial]$-module $V/\mathfrak{p}V$ admits a cyclic vector and we conclude by Nakayama's lemma;

(c) $\partial(R) \subset \mathfrak{p}$ and the condition (ii) is realized; then ∂ passes to the quotient in $R/\mathfrak{p}$-linear maps $\bar{\partial}$ over

$$V/\mathfrak{p}V \hookrightarrow V'/\mathfrak{p}V'$$

$(V' = \Gamma(S, \mathcal{V}'))$ and, if v' is a cyclic vector for the $R[\partial]$-module V', $\bar{v}'$ is a cyclic vector for $(V'/\mathfrak{p}V', \bar{\partial})$; moreover, the minimal polynomial of $\bar{\partial}$ over $V'/\mathfrak{p}V'$ coincides with the characteristic polynomial of $\bar{\partial}$ over $V'/\mathfrak{p}V'$ and therefore it is the same over $V/\mathfrak{p}V$; this implies that there exists a cyclic vector for $(V/\mathfrak{p}V, \bar{\partial})$ and we again conclude by Nakayama's lemma.

□

Definition 6.9. *A **trivialization of an $\mathcal{O}_S[\partial]$-module** $\mathcal{V}$, which is locally free of finite rank n as an $\mathcal{O}_S$-module, is the data of an isomorphism of $\mathcal{O}_S[\partial]$-modules*

$$\gamma : \mathcal{V} \xrightarrow{\sim} \mathcal{O}_S^n,$$

where $\mathcal{O}_S$ is equipped with the natural $\mathcal{O}_S[\partial]$-module structure (on the left).

If $\mathcal{V}$ is cyclic of rank n and if v is a cyclic vector for $\mathcal{V}$, giving ourselves a trivialization γ amounts to giving ourselves n solutions $\beta_1, \dots, \beta_n$ in $\Gamma(S, \mathcal{O}_S)$ of the differential equation

$$x^{(n)} + \alpha_{n-1}x^{(n-1)} + \cdots + \alpha_0 x = 0$$

defined by

$$\alpha^{(n)}(v) = -\sum_{i=0}^{n-1} \alpha_i \partial^i(v), \quad \alpha_i \in \Gamma(S, \mathcal{O}_S),$$

in such a way that the wronskian $\mathrm{W}(\beta_1, \dots, \beta_{\mathrm{n}})$ is invertible over S ($\gamma(v) = (\beta_1, \dots, \beta_n)$); furthermore, if we put

$$u = \partial^n + \sum_{i=0}^{n-1} \alpha_i \partial^i \in \Gamma(S, \mathcal{O}_S[\partial]),$$

$\underline{\mathrm{Hom}}_{\mathcal{O}_S[\partial]}(\mathcal{V}, \mathcal{O}_S)$ is identified with the kernel of $u : \mathcal{O}_S \to \mathcal{O}_S$.

Remark 6.10. *If $(S, \partial) = \mathrm{Spec}(k, \partial)$, for (k, ∂) a differential field of characteristic 0, it is the same to give oneself a trivialization γ of $\mathcal{V}$ and to give oneself a k-isomorphism*

$$(k^\partial)^n \xrightarrow{\sim} \mathrm{Hom}_{k[\partial]}(V, k)$$

(where $V = \Gamma(S, \mathcal{V})$ by Proposition 1.1).

7. Kirchever modules over a differential scheme

The notation is the same as in Section 2. Let (S, ∂) be a differential k_0-scheme.

Definition 7.1. *A **Krichever module of rank** $r > 0$ **over** (S, ∂) **for the pointed curve** (X, ∞) is a pair (E_0, φ) consisting of*

(1) a locally free $\mathcal{O}_S$-module of rank 1, defining a differential (S, ∂) group scheme

$$(E, \partial) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0), \partial)$$

(2) a k_0-algebra homomorphism

$$\varphi : A \to \mathrm{End}_{(S,\partial)}(E, \partial)$$

such that, for each $a \in A - \{0\}$, $\varphi(a)$ is elliptic of degree $r\,\mathrm{ord}(a)$ as in Definition 6.4.

*A **morphism** $u : (E_0', \varphi') \to (E_0, \varphi)$ **between two such Krichever modules of the same rank** r **over** (S, ∂) is the data of an $\mathcal{O}_S$-linear map $u_0 : E_0' \to E_0$ inducing a morphism of differential (S, ∂) group schemes $u : (E, \partial) \to (E', \partial)$ such that $u\varphi(a) = \varphi'(a)u$ $\forall a \in A$.*

We denote by $\underline{\mathrm{Krich}}^r_{(S,\partial)}$ the category of Krichever modules of rank r over (S, ∂) (for (X, ∞)); for each morphism $f : (S, \partial) \to (T, \partial)$ of differential k_0-schemes, we then have a reciprocal image functor

$$f^* : \underline{\mathrm{Krich}}^r_{(T,\partial)} \to \underline{\mathrm{Krich}}^r_{(S,\partial)}.$$

Definition 7.2. *A **generalized morphism of degree** $\leqslant n$, $u : (E_0', \varphi') \to (E_0, \varphi)$ between Krichever modules of rank r over (S, ∂) for (X, ∞) is an $\mathcal{O}_S$-linear map $u_0 : E_0' \to E_n \subset E$ inducing a morphism of differential (S, ∂) group schemes $u : (E, \partial) \to (E', \partial)$ such that $u\varphi(a) = \varphi'(a)u$ $\forall a \in A$.*

*One such generalized morphism is called an **isogeny of degree** n if the composite morphism $E_0' \xrightarrow{u_0} E_n \twoheadrightarrow E_n/E_{n-1}$ is an isomorphism.*

Remark 7.3. *Suppose that S is connected and the Zariski sheaf $\mathcal{O}_S^\partial = \ker(\mathcal{O}_S \xrightarrow{\partial} \mathcal{O}_S)$ is constant with value a field $k_1 \supset k_0$. Each non-zero generalized endomorphism of a Krichever module over (S, ∂) is an isogeny; this results from the following lemma:*

Lemma 7.4. *Let (R, ∂) be a differential $\mathbb{Q}$-algebra with R^∂ a field and let $v \in R[\partial]$ be elliptic of degree > 0. Each non-zero $u \in R[\partial]$ such that $uv = vu$ is elliptic.*

Proof. By Lemma 1.7, even if it means adjoining to R a root of an invertible elements of R and canonically extending ∂ to this extension of R, which does not change the constants, there exists $w \in R(\!(\partial^{-1})\!)$ such that $v = w^{\deg(v)}$ and $Z(v) = R^\partial(\!(w^{-1})\!) \cap R[\partial]$; as w is still elliptic the lemma follows immediately. □

For (S, ∂) again an arbitrary differential k_0-scheme we have:

Proposition 7.5. *Let (E, φ) be Krichever module of rank r over (S, ∂) for (X, ∞) and let $\theta : E = \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0 \twoheadrightarrow \mathcal{V}$ be a locally cyclic quotient $\mathcal{O}_S[\partial]$-module of rank n. Suppose that the A-module structure φ of E passes to the quotient an A-module structure $\tilde{\varphi} : A \to \mathrm{End}_{\mathcal{O}_S[\partial]}(\mathcal{V})$ over $\mathcal{V}$. Even if it means replacing S by a Zariski open neighborhood of S_∂ in S, there exists an isogeny of degree n (and only one in a obvious sense)*

$$u : (E, \partial) \to (E', \partial)$$

between Krichever modules of rank r over (S, ∂) for (X, ∞) with kernel

$$(V, \partial) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{V}), \partial) \hookrightarrow \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(E), \partial) = (E, \partial)$$

(the closed immersion being induced by θ).

Proof. This is an immediate consequence of Proposition 6.6. □

For each ideal I of A, $0 \subsetneq I \subsetneq A$, and each Krichever module (E_0, φ) (or (E, ∂)) over (S, ∂) for (X, ∞), we put

$$(E_I, \partial) = \bigcap_{a \in I} \ker(\varphi(a) : (E, \partial) \to (E, \partial));$$

(E_I, ∂) is therefore a differential (S, ∂)-scheme of (A/I)-modules.

Lemma 7.6. *(E_I, ∂) is a differential (S, ∂) group scheme which is locally cyclic; more precisely*

$$(E_I, \partial) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S}(E \otimes_A (A/I)), \partial \otimes 1_{A/I})$$

and $E \otimes_A (A/I)$ is a locally cyclic $\mathcal{O}_S[\partial]$-module.

Proof. The lemma is immediate in the case where I is principal. In the general case, there exists an ideal J of A such that $I + J = A$ and where $I \cap J = IJ$ is principal; then, we have $A/I \oplus A/J = A/(I \cap J)$ and the lemma follows from Proposition 6.8. □

For each (A/I)-module M of finite length, we write $M^\vee = \mathrm{Hom}_{k_0}(M, k_0)$ for the k_0-dual of M, equipped with the natural structure of an (A/I)-module (also of finite length), and we write

$$\underline{M} = \mathrm{Spec}(\mathrm{Sym}_{k_0}(M^\vee))$$

for the affine k_0-scheme of (A/I)-modules characterized by

$$\underline{M}(B) = M \otimes_{k_0} B$$

for each k_0-algebra B; then

$$(S, \partial) \times_{k_0} \underline{M}$$

is a differential (S, ∂)-scheme of (A/I)-modules.

Definition 7.7. *A **level structure** I for a Krichever module of rank $r > 0$ over (S, ∂) for (X, ∞), $\varphi : A \to \mathrm{End}_{(S,\partial)}(E, \partial)$, is the data of an isomorphism of differential (S, ∂)-schemes of (A/I)-modules*

$$\gamma : (S, \partial) \times_{k_0} \underline{(I^{-1}/A)^r} \xrightarrow{\sim} (E_I, \partial).$$

Remark 7.8. *It is equivalent to give γ as above or as an (A/I)-linear trivialization (as in Definition 6.9)*

$$\gamma : E \otimes_A (A/I) \xrightarrow{\sim} \mathcal{O}_S \otimes_{k_0} ((I^{-1}/A)^\vee)^r$$

of locally cyclic $\mathcal{O}_S[\partial]$-modules $E \otimes_A (A/I)$ (as in Lemma 7.6); in particular, it follows from Remark 6.10 that, for $(S, \partial) = \mathrm{Spec}(k, \partial)$, with (k, ∂) a differential field, that Definitions 3.5 and 7.7 coincide.

Lemma 7.9. *Let (E_0, φ, γ) be a Krichever module of rank r over (S, ∂) for (X, ∞), equipped with a level structure I, $0 \subsetneq I \subsetneq A$. There exists a Zariski open neighborhood of S_∂ in S over which E_0 is trivial and (E_0, φ, γ) has no automorphisms other than the identity.*

Proof. Fix a non-zero element of $(I^{-1}/A)^r$, then γ induces (via this element) an epimorphism of $\mathcal{O}_S[\partial]$-modules

$$E = \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0 \twoheadrightarrow \mathcal{O}_S$$

and this epimorphism is uniquely determined by its restriction to E_0, $E_0 \to \mathcal{O}_S$, which is $\mathcal{O}_S$-linear. For each $s \in S_\partial$, it is clear that $(E_0)_{(s)} \to \kappa(s)$ (where $(E_0)_{(s)} = E_{0,s} \otimes_{\mathcal{O}_{S,s}} \kappa(s)$ and $\kappa(s) = \mathcal{O}_{S,s}/m_{S,s}$) is bijective because $E_{(s)} \to \kappa(s)$ is surjective and therefore non-zero; as a result $E_0 \to \mathcal{O}_S$ is an isomorphism over a Zariski open neighborhood of S_∂ in S, from where the lemma. $\square$

For (X, ∞), $r > 0$ and I with $0 \subsetneq I \subsetneq A$ fixed, we can then pose and resolve a moduli problem for Krichever modules of rank r for (X, ∞), equipped with a level structure I. More precisely, we search for a differential k_0-scheme (Σ, ∂) and a Krichever module (ε, ∂) of rank r over (Σ, ∂) for (X, ∞), equipped with a level structure I, having the following universal property:

- (U) For each differential k_0-scheme (S, ∂) and each Krichever module (E, ∂) of rank r over (S, ∂) for (X, ∞), equipped with level structure I, there exists a Zariski open neighborhood S' of S_∂ in S and a k_0-morphism

$$f' : (S', \partial) \to (\Sigma, \partial)$$

  such that $(E, \partial)|_{(S',\partial)}$ is isomorphic to $f'^*(\varepsilon, \partial)$ as a Krichever module with level structure; moreover, we demand that the pair (S', f') is unique in the sense that if (S'', f'') is another such pair, there exists a Zariski open neighborhood of S_∂ in $S' \cap S''$ on which f' and f'' coincide.

Remark 7.10. *For each differential field (k, ∂) with $k \supset k_0$, $(\Sigma, \partial)(k, \partial)$ is the collection of all of the isomorphism classes of Krichever modules of rank r over (k, ∂) for (X, ∞) with level structure I; in fact, if $\sigma \in (\Sigma, \partial)(k, \partial)$, $(\varepsilon, \partial) \otimes_{(\Sigma,\partial)} (\sigma, \partial)$ is a representative for the corresponding isomorphism class.*

Remark 7.11. *The pair $((\Sigma, \partial), (\varepsilon, \partial))$ is not unique, only its germ along Σ_∂ is (in a sense the reader will easily specify).*

Theorem 7.12. *The moduli problem posed above admits a solution $((\Sigma, \partial), (\varepsilon, \partial))$ with (Σ, ∂) an affine differential k_0-scheme of finite type (in the differential sense).*

Proof. Fix a presentation of A,

$$A = k_0[y_1, \dots, y_m]/(f_1(\underline{y}), \dots, f_n(\underline{y})),$$

in such a way that if a_μ is the image of y_μ in A, I is generated by $a_1, \dots, a_l$ for an $l \in \{1, \dots, m-1\}$ and fix a basis $(e_1, \dots, e_{\mathfrak{p}})$ of $(I^{-1}/A)^r$, which we denote by $(e_1^*, \dots, e_{\mathfrak{p}}^*)$ the dual basis of $((I^{-1}/A)^\vee)^r$.

Even if we have to replace S by a Zariski open neighborhood of S_∂ in S each Krichever module of rank r over (S, ∂) equipped with a level structure I is isomorphic to a Krichever module of form

$$\varphi : A \to \operatorname{End}(\mathbb{G}_{a,(S,\partial)}) = \Gamma(S, \mathcal{O}_S)[\partial]$$

equipped with a level structure I

$$\gamma : \mathcal{O}_S[\partial]/\mathcal{O}_S[\partial]\varphi(I) \xrightarrow{\sim} \bigoplus_{\pi=1}^{\mathfrak{p}} \mathcal{O}_S e_\pi^*$$

such that

$$\gamma(\bar{1}) = \sum_{i=1}^{\mathfrak{p}} \gamma_\pi e_\pi^*, \quad \gamma_\pi \in \Gamma(S, \mathcal{O}_S)$$

with $\gamma_1, \dots, \gamma_{\mathfrak{p}}$ and $\mathrm{W}(\gamma_1, \dots, \gamma_{\mathfrak{p}})$ invertible in S (first think about the case when (S, ∂) is the spectrum of a differential field as in Lemma 7.9); moreover, if we impose also that $\gamma_1 = 1$ (for example), this last Krichever module is unique up to isomorphism.

Moreover, a pair (φ, γ) as above is obtained from the data of

$$u_\mu = \varphi(a_\mu) = \sum_{i=0}^{r \operatorname{ord}(a_\mu)} \beta_{\mu,i} \partial^i, \quad \mu = 1, \dots, m,$$

where $\beta_{\mu,i} \in \Gamma(S, \mathcal{O}_S)$, $\beta_{\mu, r \operatorname{ord}(a_\mu)} \in \Gamma(S, \mathcal{O}_S^\times)$ for each pair (μ, i), and

$$\gamma_\pi \in \Gamma(S, \mathcal{O}_S^\times), \quad \pi = 1, \dots, \mathfrak{p},$$

with

$$\mathrm{W}(\gamma_1, \dots, \gamma_{\mathfrak{p}}) \in \Gamma(S, \mathcal{O}_S^\times),$$

these data must satify the following additional conditions:

(a) $u_\mu u_{\mu'} = u_{\mu'} u_\mu \ \forall \mu, \mu' = 1, \dots, m,$
(b) $f_\nu(u_1, \dots, u_m) = 0 \ \forall \nu = 1, \dots, n,$
(c) $u_\mu(\gamma_\pi) = 0 \ \forall \mu = 1, \dots, l \ \forall \pi = 1, \dots, \mathfrak{p},$
(d) if

$$e_\pi a_\mu = \sum_{\pi'=1}^{\mathfrak{p}} \alpha_{\mu,\pi,\pi'} e_{\pi'},$$

$\forall \mu = l+1, \dots, m \ \forall \pi = 1, \dots, \mathfrak{p}$, or the $\alpha_{\mu,\pi,\pi'} \in k_0$ are the structural constants of the A-module $(I^{-1}/A)^r$, we have

$$u_\mu(\gamma_\pi) = \sum_{\pi'=1}^{\mathfrak{p}} \alpha_{\mu,\pi,\pi'} \gamma_{\pi'}$$

$\forall \mu = l+1, \dots, m \ \forall \pi = 1, \dots, \mathfrak{p}.$

Now it is clear that (Σ, ∂) and (ε, ∂) exist, in fact (Σ, ∂) is the closed differential sub-scheme of

$$\operatorname{Spec}\left(k_0\left[\beta_{\mu,i}^{(j)}, \beta_{\mu, r\operatorname{ord}(a_\mu)}^{-1}, \gamma_\pi^{(j)}, \gamma_\pi^{-1}, \frac{1}{\mathrm{W}(\gamma_2, \dots, \gamma_{\mathfrak{p}})}\right], \partial\right),$$

where $\mu \in \{1, \dots, m\}$, $i \in \{1, \dots, r\,\mathrm{ord}(a_\mu)\}$, $\pi \in \{2, \dots, \mathfrak{p}\}$ and $j \in \mathbb{N}$ and where $\partial(\beta_{\mu,i}^{(j)}) = \beta_{\mu,i}^{(j+1)}$ and $\partial(\gamma_\pi^{(j)}) = \gamma_\pi^{(j+1)^\mu}$, defined by the differential equations which translate the conditions (a)-(d) above, for $\gamma_1 = 1$.

We will write in the following (M_I^r, ∂) for the differential k_0-scheme of the module (Σ, ∂) above.

Let $\underline{\mathrm{GL}}_r(A/I)$ be the algebraic group over k_0 whose points with values in a k_0-algebra R are $\mathrm{GL}_r((A/I) \otimes_{k_0} R)$; it is clear that $\underline{\mathrm{GL}}_r(A/I)$ (equipped with the derivation 0) acts on (M_I^r, ∂) (action on the level structure γ only). Moreover, if $0 \subsetneq I \subset J \subsetneq A$ are two proper ideals of A, we have $((I^{-1}/A)^\vee)^r \twoheadrightarrow ((J^{-1}/A)^\vee)^r$, from where we get a differential k_0-morphism

$$\pi_{J,I} : (M_I^r, \partial) \to (M_J^r, \partial)$$

which is equivariant for the actions of $\underline{\mathrm{GL}}_r(A/I)$ and $\underline{\mathrm{GL}}_r(A/J)$, linked together by the canonical k_0-morphism

$$\rho_{I,J} : \underline{\mathrm{GL}}_r(A/I) \to \underline{\mathrm{GL}}_r(A/J).$$

Passing to the projective limit following the $\pi_{J,I}$, we therefore have an affine differential k_0-scheme

$$(M^r, \partial) = \varprojlim_I ((M_I^r, \partial), \pi_{I,J})$$

equipped with an action of algebraic k_0-groups

$$\underline{\mathrm{GL}}_r(\bar{A}) = \varprojlim_I (\underline{\mathrm{GL}}_r(A/I), \rho_{I,J}).$$

□

Remark 7.13. *If $G_{I,J} = \ker(\rho_{I,J})$, G is a reductive algebraic k_0-group and a quotient $(M_I^r, \partial)/G_{I,J}$ exists in the category of affine differential k_0-schemes and $\pi_{I,J}$ induces a differential k_0-morphism*

$$\bar{\pi}_{I,J} : (M_I^r, \partial)/G_{I,J} \to (M_J^r, \partial)$$

(by Mumford and Fogarty; Laumon's [Mu. 2]). It is clear that this morphism induces a bijection between the points of its source and of its target in any differential field (k, ∂), with $k^\partial \supset k_0$, which is differentially closed. It should be true that $\bar{\pi}_{I,J}$, or at least its germ (as in Remark 7.11), is an isomorphism.

Now, let $\mathbb{A}_f$ be the ring of "closed" (TL Note: the air quote here are from Laumon) adeles of F,

$$\mathbb{A}_f = \prod_{\substack{x \in |X| \\ x \neq \infty}} (F_x, \mathcal{O}_x);$$

we can consider the group $\mathrm{GL}_r(\mathbb{A}_f)$ and its quotient $F^\times \backslash \mathrm{GL}_r(\mathbb{A}_f)$ ($F^\times$ identified with the center of $\mathrm{GL}_r(F)$, embedded diagonally into $\mathrm{GL}_r(\mathbb{A}_f)$). We check without difficulty that

$$F^\times \backslash \mathrm{GL}_r(\mathbb{A}_f) = (A - \{0\}) \backslash (M_{r\times r}(\hat{A}) \cap \mathrm{GL}_r(\mathbb{A}_f))$$

which allows us to consider $F^\times \backslash \mathrm{GL}_r(\mathbb{A}_f)$ like the group of points with values in k_0 of an algebraic k_0-group which we denote, by abuse of notation, $F^\times \backslash \underline{\mathrm{GL}}_r(\mathbb{A}_f)$.

Proposition 7.14. *The action of $\underline{\mathrm{GL}}_r(\hat{A})$ on (M^r, ∂) extends to an action of $F^\times \backslash \underline{\mathrm{GL}}_r(\mathbb{A}_f)$ on the germ of (M^r, ∂) along M_∂^r.*

Proof. If $g \in M_{r\times r}(\hat{A}) \cap \mathrm{GL}_r(\mathbb{A}_f)$, g acts on $(F/A)^r = \varinjlim_I (I^{-1}/A)^r$ and the kernel of

$$g : (F/A)^r \to (F/A)^r$$

is an A-module of finite length contained in $(I^{-1}/A)^r$ for I quite small. On the other hand, if (E, ∂) is a Krichever module of rank r over $(S, , \partial)$, a compatible level structure system (of finite distance, i.e. for the proper, non-zero ideals I of A) induces a morphism of differential (S, ∂)-schemes

$$\gamma : (S, \partial) \times_{k_0} \underline{(F/A)^r} \hookrightarrow (E, \partial)$$

by which we can transport the kernel of a g as above to a differential sub-(S,∂)-scheme of locally cyclic A-modules of (E,∂). Taking into account Proposition 7.5, we deduce, even if it means localizing (S,∂) in the neighborhood of S_∂, an isogeny $(E,\partial) \xrightarrow{u} (E',\partial)$ and a compatible system of level structures $\gamma' : (S,\partial)\times_{k_0} (\underline{F/A})^r \hookrightarrow (E',\partial)$ such that the following commutes:

$$\begin{tikzcd}
(S,\partial)\times_{k_0} (\underline{F/A})^r \arrow[r, hook, "\gamma"] \arrow[d, "g"] & (E,\partial) \arrow[d, "u"] \\
(S,\partial)\times_{k_0} (\underline{F/A})^r \arrow[r, hook, "\gamma'"] & (E',\partial)
\end{tikzcd}$$

and $(g,(\varphi,\gamma)) \mapsto (E',\gamma')$ is the action we were searching for. That $A-\{0\}$ acts trivially results from the fact that for $g = a \in A-\{0\}$, we can take $E' = E$, $u = \varphi(a)$, $\gamma' = \gamma$ in the construction above. $\square$

8. Kirchever bundles over a differential base

The notation is always the same as in Section 2; as in Section 7, let (S,∂) be a differential k_0-scheme; the derivation ∂ of $\mathcal{O}_S$ naturally extends to a derivation, also denoted ∂, of $\mathcal{O}_{X\times_{k_0}S}$ (trivial over $\mathcal{O}_X$).

Definition 8.1. *A **Krichever sheaf of rank** $r>0$ for the pointed curve (X,∞) is a triple $(F,(F_i)_{i\in\mathbb{Z}},\partial)$ consisting of a quasi-coherent $\mathcal{O}_{X\times_{k_0}S}$-module F, an increasing and exhaustive (but not separated) filtration*

$$\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F$$

of F by locally free sub-$\mathcal{O}_{X\times_{k_0}S}$-modules of rank r, and of a ∂-connection

$$\partial : F \to F$$

such that

- *(a) $F_{i+r[k_\infty:k_0]} = F_i(\infty\times_{k_0} S)$, $\forall i\in\mathbb{Z}$,*
- *(b) $\mathrm{pr}_{S*}(F_i/F_{i-1})$ is a locally free $\mathcal{O}_S$-module of rank 1, $\forall i\in\mathbb{Z}$ (pr_S is the canonical projection onto the second coordinate of $X\times_{k_0} S$),*
- *(c) $R^0\,\mathrm{pr}_{S*} F_0 = R^1\,\mathrm{pr}_{S*} F_0 = 0$,*
- *(d) $\partial(F^i)\subset F_{i+1}$ and ∂ induces an $\mathcal{O}_{X\times_{k_0}S}$-linear isomorphism $\bar\partial : F_i/F_{i-1} \xrightarrow{\sim} F_{i+1}/F_i$, $\forall i\in\mathbb{Z}$.*

*A **morphism of Krichever sheaves** of rank $r>0$ over (S,∂) for (X,∞) is a morphism between the underlying $\mathcal{O}_{X\times_{k_0}S}$-modules F which respects the filtration and commutes with the connections.*

Remark 8.2. *Let $(F,(F_i)_{i\in\mathbb{Z}},\partial)$ be a triple as in Definition 8.1 satisfying the conditions (a)-(d) and the condition (c') below:*

(c') for each point $s\in S_\partial$, $\chi(X\times_{k_0} s, F_0) = 0$.

There exists a Zariski open of S^∂ in S above which $(F,(F_i)_{i\in\mathbb{Z}},\partial)$ is a Krichever sheaf of rank r.

Indeed, it suffices to show that $H^0(X\times_{k_0} s, F_0) = 0$ for each $s\in S_\partial$ (by the semi-continuity theorem and Grauert's Corollary [Ha.1] III 12.8 and 12.9 and [SGA 6] III to get rid of the noetherian hypothesis). Let us reason through the absurd by supposing that there exists an integer $i\leqslant 0$ and $f\in H^0(X\times_{k_0} s, F_i)$ with non-zero image in $H^0(X\times_{k_0} s, F_i/F_{i-1})$; then, for each $m\geqslant 0$, it follows from condition (d) that $(f,\partial(f),\ldots,\partial^m(f))$ is a free family in $H^0(X\times_{k_0} s, F_{i+m})$ ($\partial^m(f)$ is well-defined because $s\in S_\partial$); or, according to condition (a), $H^1(X\times_{k_0} s, F_{i+m}) = 0$ for $m >> 0$ and, according to conditions (b) and (c), $\chi(X\times_{k_0} s, F_{i+m}) = i+m$ for each m; as a result, $i+m\geqslant m+1$ for $m >> 0$, which contradicts $i\leqslant 0$.

Definition 8.3. *A* ***generalized morphism of degree*** $\leqslant n$, $u : (F', (F'_i)_{i\in\mathbb{Z}}, \partial') \to (F, (F_i)_{i\in\mathbb{Z}}, \partial)$ *between two Krichever sheaves of rank* r *over* (S, ∂) *for* (X, ∞) *is a morphism of* $\mathcal{O}_{X\times_{k_0}S}$*-modules* $u' : F' \to F$ *which shifts the filtration by* n *(*F'_i *goes into* F_{i+n}*) and which commutes with connections. One such generalized morphism is called an* ***isogeny of degree*** n *if* u *induces an isomorphism* $\tilde{u} : F'_0/F'_{-1} \xrightarrow{\sim} F_n/F_{n-1}$.

The following theorem is due to Drinfeld in [Dri77a].

Theorem 8.4 (Drinfeld)**.** *The categories of Krichever modules and Krichever sheaves of rank* r *over* (S, ∂) *for* (X, ∞) *are equivalent. Furthermore, via this equivalence of categories, generalized morphisms of degree* $\leqslant n$ *(resp. isogenies of degree* n*) between Krichever modules and between Krichever sheaves are in bijection.*

Let us recall the key point of Drinfeld's proof:

Proposition 8.5. *Let* $r > 0$ *be an integer and let* (X, ∞) *be as in Section 2. We consider the following two categories for a* k_0*-scheme* S*:*

(1) the category of quasi-coherent $A \otimes_{k_0} \mathcal{O}_S$*-modules* E *equipped with an increasing, exhaustive (and separated) filtration*

$$0 = E_{-1} \subset E_0 \subset E_1 \subset \cdots \subset E$$

by coherent sub-$\mathcal{O}_S$*-modules, such that for each* $i \geqslant 0$, E_i/E_{i-1} *is a locally free* $\mathcal{O}_S$*-module of rank* 1 *and such that regardless of* $a \in A - \{0\}$, *the multiplication by* a *in* E *sends* E_i *into* $E_{i+r\,\mathrm{ord}(a)}$ *and induces a isomorphism*

$$E_i/E_{i-1} \xrightarrow{\sim} E_{i+r\,\mathrm{ord}(a)}/E_{i+r\,\mathrm{ord}(a)-1};$$

(2) the category of quasi-coherent $\mathcal{O}_{X\times_{k_0}S}$*-modules* F *equipped with an increasing, exhaustive (not-separated) filtration*

$$\cdots F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F$$

by locally free sub-$\mathcal{O}_{X\times_{k_0}S}$*-modules of rank* r *such that, for each* $i \in \mathbb{Z}$,

$$F_{i+r[k_\infty:k_0]} = F_i(\infty \times_{k_0} S)$$

and $\mathrm{pr}_{S*}(F_i/F_{i-1})$ *is a locally free* $\mathcal{O}_S$*-module of rank* 1 *and*

$$R^0\,\mathrm{pr}_{S*}\,F_0 = R^1\,\mathrm{pr}_{S*}\,F_0 = 0.$$

The functor from category (2) into the category (1) which sends $(F, (F_i)_{i\in\mathbb{Z}})$ *to* $(\mathrm{pr}_{S*}\,F, (\mathrm{pr}_{S*}\,F_{i+1})_{i\in\mathbb{N}})$ *is an equivalence of categories.*

Via the dictionary above the notion of a level structure I, for I a proper, non-zero ideal of A, for Krichever modules becomes the notion of level structure D, for $D = \mathrm{Spec}(A/I)$ a closed sub-scheme of finite type over k_0, of $X - \{\infty\}$, for Krichever sheaves. More precisely:

Definition 8.6. *A* ***level structure*** D ***for a Krichever sheaf*** $(F, (F_i)_{i\in\mathbb{Z}}, \partial)$ *of rank* r *over* (S, ∂) *for* (X, ∞) *is the data of a isomorphism of connection* $(\mathcal{O}_{D\times_{k_0}S}, \partial)$*-modules*

$$\gamma : (F_{D\times_{k_0}S}, \partial) \xrightarrow{\sim} (\mathrm{pr}_X^*(\omega_{X,D}), \partial)^r$$

where $F_{D\times_{k_0}S} = F/F(-D \times_{k_0} S)$, $\omega_X = \Omega^1_{X/k_0}$, $\omega_{X,D} = \omega_X/\omega_X(-D)$ *and* pr_X *is the canonical projection onto the first coordinate map of* $X \times_{k_0} S$.

Remark 8.7. *If* $D = \mathrm{Spec}(A/I)$, *we have, by Serre duality ([Ha.1]* III 7.6*),*

$$(I^{-1}/A)^\vee = H^0(X, \mathcal{O}_X(D)/\mathcal{O}_X)^\vee = \mathrm{Ext}^1(\mathcal{O}_X(D)/\mathcal{O}_X, \omega_X) = H^0(X, \omega_{X,D}).$$

9. Duality of Krichever modules and Kirchever bundles

The notation is the same as in Sections 2, 7 and 8. We consider a Krichever module (E_0, φ) of rank $r > 0$ over (S, ∂) for (X, ∞) and we write $(F, (F_i)_{i\in\mathbb{Z}}, \partial)$ for the corresponding Krichever sheaf.

The ring (non-commutative in general) $\mathcal{O}_S[\partial]$ is canonically isomorphic to its opposite. More precisely, we have an involution

$$(-)^* : \mathcal{O}_S[\partial] \xrightarrow{\sim} \mathcal{O}_S[\partial]^{\mathrm{opp}}$$

defined by

$$\begin{aligned} a^* &= a, \quad \forall a \in \mathcal{O}_S \\ \partial^* &= -\partial \\ (u+v)^* &= u^* + v^*, \quad \forall u, v \in \mathcal{O}_S[\partial] \\ (uv)^* &= v^* u^*, \quad \forall u, v \in \mathcal{O}_S[\partial] \end{aligned}$$

and commonly called the **formal transposition of differential operators**.

For each invertible $\mathcal{O}_S$-module E_0 we also have an "involution" (TL Note: the quotes are Laumon's)

$$(-)^* : E_0^{-1} \otimes_{\mathcal{O}_S} \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0 \xrightarrow{\sim} (E_0 \otimes_{\mathcal{O}_S} \mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0^{-1})^{\mathrm{opp}}$$

defined by

$$(e_0^\vee \otimes u \otimes e_0)^* = e_0 \otimes u^* \otimes e_0^\vee$$

for each $e_0 \in E_0$, $e_0^\vee \in E_0^{-1}$ and $u \in \mathcal{O}_S[\partial]$.

Definition 9.1. *The* ***dual of the Krichever module*** (E_0, φ) *of rank* r *over* (S, ∂) *for* (X, ∞) *is the Krichever module* (E_0^{-1}, φ^*) *of the same rank over* (S, ∂) *for* (X, ∞)*, where*

$$\varphi^* : A \to \mathrm{End}_{\mathcal{O}_S[\partial]}(\mathcal{O}_S[\partial] \otimes_{\mathcal{O}_S} E_0^{-1})$$

is the composite of φ *and the involution* $(-)^*$ *defined above (*A *is commutative, therefore* $A^{opp} = A$*).*

Remark 9.2. *This duality can be expressed, like for abelian varieties, in terms of representability of the functor* Ext^1 *. Indeed, suppose for simplicity that* (S, ∂) *is the spectrum of a differential field* (k, ∂) *of characteristic* 0*, then we have*

$$\mathrm{Ext}^1(\mathbb{G}_{a,(k,\partial)}, (\mathbb{G}_{a,k}, 0)) \cong k :$$

at each $y \in k$*, we associated the extension of differential* (k, ∂)*-schemes in groups*

$$\begin{aligned} 0 \longrightarrow (\mathbb{G}_{a,k}, 0) \longrightarrow &\mathbb{G}_{a,(k,\partial)} \longrightarrow \mathbb{G}_{a,(k,\partial)} \longrightarrow 0 \\ x &\mapsto y^{-1}x \end{aligned}$$

if $y \neq 0$*, and the trivial extension if* $y = 0$ *(the differential* (k, ∂) *group scheme* $(\mathbb{G}_{a,k}, 0)$ *respresents the functor* $(\beta, \partial) \mapsto (\beta^\partial, +)$ *from the category of differential* (k, ∂)*-algebras into that of abelian groups). In fact, the* Ext^1 *above is representable by the differential* (k, ∂) *group scheme* $\mathbb{G}_{a,(k,\partial)}$ *with coordinate* y*. It is then easy to verify that the action of* $\mathrm{End}(\mathbb{G}_{a,(k,\partial)}) = k[\partial]$ *on the left on* $\mathbb{G}_{a,(k,\partial)}$ *(with coordinate* x*) induces a right action of* $\mathrm{End}(\mathbb{G}_{a,(k,\partial)})$ *on* $\mathrm{Ext}^1(\mathbb{G}_{a,(k,\partial)}, (\mathbb{G}_{a,k}, 0))$ *which is none other than the composite of the action of* $k[\partial]$ *on the left of* $\mathbb{G}_{a,(k,\partial)}$ *(with coordinate* y*) and the formal transposition* $k[\partial] \xrightarrow[\sim]{*} k[\partial]^{opp}$*. In particular, if* $\mathbb{G}_{a,(k,\partial)}$ *(with coordinate* x*) is equipped with an* A*-module structure* φ*,* $\mathrm{Ext}^1(\mathbb{G}_{a,(k,\partial)}, (\mathbb{G}_{a,k}, 0)) \cong \mathbb{G}_{a,(k,\partial)}$ *(with coordinate* y*) is equipped with the structure of an* A*-module* φ^* *(for all of this, see [Ca]). (TL Note: [Ca] does not appear in Laumon's bibliography, but maybe he means Carlitz?)*

Serre duality then allows us to translate the duality for Krichever modules to a duality for Krichever sheaves. More precisely:

Proposition 9.3. *If $(F, (F_i)_{i\in\mathbb{Z}}, \partial)$ is a Krichever sheaf corresponding to a Krichever module (E_0, φ) of rank r over (S, ∂) for (X, ∞), then the Krichever sheaf $(F^*, (F_i^*)_{i\in\mathbb{Z}}, \partial^*)$ corresponding to (E_0^{-1}, φ^*), the dual of (E_0, φ), is defined by*

$$
\begin{aligned}
F^* &= \omega_X \otimes_{\mathcal{O}_X} F^\vee, \\
F_i^* &= \omega_X \otimes_{\mathcal{O}_X} F_{-i}^\vee, \quad \forall i \in \mathbb{Z},
\end{aligned}
$$

and

$$\partial^* = 1_{\omega_X} \otimes \partial^\vee,$$

where

$$
\begin{aligned}
F_j^\vee &= \underline{\mathrm{Hom}}_{\mathcal{O}_{X\times_{k_0}S}}(F_j, \mathcal{O}_{X\times_{k_0}S}), \quad \forall j \in \mathbb{Z}, \\
F^\vee &= \varinjlim_i F_{-i}^\vee
\end{aligned}
$$

and

$$\langle f, \partial^\vee(f^\vee)\rangle + \langle \partial(f), f^\vee\rangle = \partial\langle f, f^\vee\rangle, \quad \forall f \in F, f^\vee \in F^\vee.$$

Proof. For each $i \in \mathbb{Z}$ and $j = 0, 1$, we have

$$R^j \operatorname{pr}_{S*} F_i^* = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(R^{1-j} \operatorname{pr}_{S*} F_{i-1}, \mathcal{O}_S)$$

(by [Ha.1] III 7, [Ha.2] III 11 and [SGA 6] III to get rid of the noetherian hypothesis). □

Remark 9.4. *If a Krichever module (or sheaf) is equipped with a level structure I (or D), γ, its dual is canonically equipped with a level structure I (or D), γ^*, called the **dual of** γ and which the reader will find easy to explain.*

10. Differential Shtukahs

The notation is the same as in Section 2; let (S, ∂) be a differential k_0-scheme and we will also write ∂ for the natural $\mathcal{O}_X$-linear extension of ∂ to $\mathcal{O}_{X\times_{k_0}S}$.

Definition 10.1. *A **lower (resp. upper)differential shtuka** of rank $r > 0$ over (S, ∂) for the curve X, with pole $\infty : S \hookrightarrow X \times_{k_0} S$ (a section of pr_S) is the data of a diagram*

$$
\begin{tikzcd}
 & F_0 \\
F_{-1} \arrow[ur, hook, "i"] \arrow[dr, "\partial"'] & \\
 & F_0
\end{tikzcd}
\qquad \textit{(resp.)} \qquad
\begin{tikzcd}
F_0 \arrow[dr, "\partial"] & \\
 & F_1 \\
F_0 \arrow[ur, hook, "i"'] &
\end{tikzcd}
$$

where F_0 is a locally free $\mathcal{O}_{X\times_{k_0}S}$-module of rank r, $F_0(-\infty) \xrightarrow{j} F_{-1} \xrightarrow{i} F_0$ (resp. $F_0 \xrightarrow{i} F_1 \xrightarrow{j} F_0(\infty)$) is a lower (resp. upper) elementary modification of F_0 along ∞ and where $\partial : F_{-1} \to F_0$ (resp. $\partial : F_0 \to F_1$) is a ∂-connection.

We recall that, by definition of elementary modifications, F_0/F_{-1} (resp. F_1/F_0) is the direct image under $\infty : S \hookrightarrow X \times_{k_0} S$ of a locally free $\mathcal{O}_S$-module of rank 1.

It is clear that a Krichever sheaf of rank r over (S, ∂) for (X, ∞) induces both a lower and an upper differential shtuka of rank r over (S, ∂) for X with pole the "constant" $\infty : S \hookrightarrow X \times_{k_0} S$ with value ∞ (TL Note: the quotes are Laumon's). Furthermore, if $(F, (F_i)_{i\in\mathbb{Z}}, \partial)$ is such a Krichever sheaf, we verify that for each $i \in \mathbb{Z}$, the following commutative diagram

$$\begin{tikzcd} F_i \arrow[r, "\partial"] & F_{i+1} \\ F_{i-1} \arrow[u, hook] \arrow[r, "\partial"] & F_i \arrow[u, hook] \end{tikzcd}$$

is at the same time cartesian and cocartesian (this is condition (d) of Definition 8.1), so that the lower or upper differential shtuka associated with a Krichever sheaf is completely determined this way. In fact, we have:

Proposition 10.2. *A lower (resp. upper) differential shtuka (F_{-1}, F_0, ∂) (resp. (F_0, F_1, ∂)) of rank $r > 0$ over (S, ∂) for X comes from a Krichever sheaf of rank r over (S, ∂) for (X, ∞) if and only if the following conditions are realized:*

(1) ∂ induces an isomorphism

$$\bar{\partial} : F_{-1}/F_{-1}(-\infty) \to F_0/F_0(-\infty)$$
$$\textit{(resp.}\quad \bar{\partial} : F_0(\infty)/F_0 \to F_1(\infty)/F_1),$$

(2) the composite map

$$\bar{i} \circ \bar{\partial}^{-1} : F_0/F_0(-\infty) \to F_0/F_0(-\infty)$$
$$\textit{(resp.}\quad \bar{\partial}^{-1} \circ \bar{i} : \bar{i_0}(F)/F_0 \to F_0(\infty)/F_0),$$

where $\bar{i}$ is induced by i, is nilpotent,

(3) $R^0 \operatorname{pr}_{S*} F_0 = R^1 \operatorname{pr}_{S*} F_0 = 0$.

Now let us consider the composite morphism

$$F_0(-\infty) \xrightarrow{j} F_{-1} \xrightarrow{\partial} F_0 \twoheadrightarrow F_0/F_{-1}$$
$$(\text{resp.}\quad F_1 \xrightarrow{j} F_0(\infty) \xrightarrow{\partial(\infty)} F_1(\infty) \twoheadrightarrow (F_1/F_0)(\infty));$$

it is clear that this morphism is $\mathcal{O}_{X\times_{k_0}S}$-linear and passes to the quotient in a morphism

$$(F_0/F_{-1})(-\infty) \longrightarrow F_0/F_{-1}$$
$$(\text{resp.}\quad F_1/F_0 \longrightarrow F_1/F_0(\infty));$$

or the data of this last morphism is equivalent to the data of a section of $H^0(X\times_{k_0}S, \mathcal{O}_{X\times_{k_0}S}(\infty)/\mathcal{O}_{X\times_{k_0}S})$, i.e. to the data of a lifting

$$\begin{tikzcd} & T(X\times_{k_0} S/S) \arrow[d] \\ S \arrow[ur, "{(\infty,0)}"] \arrow[r, hook, "\infty"] & X\times_{k_0} S \end{tikzcd}$$

from the pole ∞ to the tangent fiber of $X\times_{k_0} S$ relative to S.

Definition 10.3. *The recovery $(\infty, 0)$ of the pole ∞ will be called the* ***zero of a differential shtuka****. We say that the* ***pole and the zero of a differential shtuka are disjoint*** *if $(\infty,0)(S)$ does not meet the zero section of $T(X\times_{k_0} S/S)$.*

Now let $D \subset X$ be a closed subscheme of finite type over k_0; ∂ induces a derivation, also written ∂ of $\mathcal{O}_{D\times_{k_0}S}$. For a lower (resp. upper) differential shtuka of pole $\infty : S \hookrightarrow X\times_{k_0} S$ disjoint from $D\times_{k_0} S$, we have $F_{-1,D\times_{k_0}S} \xrightarrow[\sim]{i} F_{0,D\times_{k_0}S}$ (resp. $F_{0,D\times_{k_0}S} \xrightarrow[\sim]{i} F_{1,D\times_{k_0}S}$), furthermore, the locally free $\mathcal{O}_{D\times_{k_0}S}$-module of finite rank $F_{0,D\times_{k_0}}S$ is equipped with a ∂-connection.

Definition 10.4. *A* ***level structure*** D ***of a lower (resp. upper) differentia shtuka*** *of rank* r *over* (S,∂) *for the curve* X *is the data of an isomorphism of* $(\mathcal{O}_{D\times_{k_0}S},\partial)$*-modules with connection*

$$\gamma:(F_{0,D\times_{k_0}S},\partial)\xrightarrow{\sim}(\mathrm{pr}_X^*(\omega_{X,D}),\partial)^r$$

(as in Defintion 8.6).

Definition 10.5. *Two differential shtukas of rank* r *over* (S,∂) *for the curve* X*, of pole* $\infty:S\hookrightarrow X\times_{k_0}S$*, one lower* (F'_{-1},F'_0,∂')*, the other upper* (F''_0,F''_1,∂'') *will be said to be* ***perfect duals*** *if*

(i) the locally free $\mathcal{O}_{X\times_{k_0}S}$*-modules of finite rank* F'_0 *and* F''_0 *first, and* F'_{-1} *and* F''_1 *second, are perfect duals with value in* $\omega_{X\times_{k_0}S/S}=\mathrm{pr}_X^*\,\omega_X$*, duality denoted* $\langle,\rangle$*,*
(ii) $\langle i'(f'),f''\rangle=\langle f',i''(f'')\rangle$ *for all* $f'\in F'_{-1}$*,* $f''\in F''_0$*,*
(iii) $\langle\partial'(f'),f''\rangle+\langle f',\partial''(f'')\rangle=\partial\langle i'(f'),f''\rangle$ *for all* $f'\in F'_{-1}$*,* $f''\in F''_0$*.*

It is immediate that every differential shtuka admits a dual in the sense above, that duality establishes a "bijection" (TL Note: Laumon's quotes) between lower and upper differential shtukas, that duality exchanges the zero of a differential shtuka and its opposite (in the sense of the additive group of $T(X\times_{k_0}S/S)\to X\times_{k_0}S$) and that duality extends to level structures.

Let us fix a closed subscheme $D\subset X$ of finite type over k_0. By completely neglecting problems of representability, we can formally describe the moduli space of (lower) differential shtukas of rank r for X, with level structure D as follows. We first consider the usual k_0-schemes of modules Fib^r_D and Hecke^r_D defined as follows: for any typical k_0-scheme S, $\mathrm{Hom}_{\mathrm{Sch}_{k_0}}(S,\mathrm{Fib}^r_D)$ and $\mathrm{Hom}_{\mathrm{Sch}_{k_0}}(S,\mathrm{Hecke}^r_D)$ are all of the isomorphism classes of pairs

$$(F,\gamma)$$

and sextuplets

$$(F_{-1},F_0,\gamma_{-1},\gamma_0,\infty,i)$$

where F, F_{-1}, F_0 are locally free $\mathcal{O}_{X\times_{k_0}S}$-modules of rank r, γ, γ_{-1}, γ_0 are level structures D over F, F_{-1}, F_0 respectively (γ is an isomorphism of $\mathcal{O}_{D\times_{k_0}S}$-modules $F_{D\times_{k_0}S}\xrightarrow[\sim]{\gamma}(\mathrm{pr}^*(\omega_{X,D}))^r,\ldots$), where $\infty:S\hookrightarrow X\times_{k_0}S$ is a section of pr_S disjoint from $D\times_{k_0}S$, where $F_{-1}\xrightarrow{i}F_0$ is an elementary modification along ∞ and where $\gamma_{-1}=\gamma_0\circ i_{D\times_{k_0}S}$. We have k_0-morphisms

$$\begin{tikzcd}\mathrm{Fib}^r_D & \mathrm{Hecke}^r_D \arrow[l, shift right, "\pi_0"'] \arrow[l, shift left, "\pi_{-1}"] \arrow[r, "\alpha"] & X-D,\end{tikzcd}$$

where α (resp. π_{-1}, π_0) send $(F_{-1},F_0,\gamma_{-1},\gamma_0,\infty,i)$ over ∞ (resp. (F_{-1},γ_{-1}), (F_0,γ_0)), and furthermore, the k_0-morphisms

$$\begin{tikzcd}\mathrm{Hecke}^r_D\times_{\mathrm{Fib}^r_D}\mathrm{Hecke}^r_D \arrow[r, "(\pi'_0,\pi''_0)"] \arrow[d, shift right, "\pi'_{-1}"'] \arrow[d, shift left, "\pi''_{-1}"] & \mathrm{Fib}^r_D\times_{k_0}\mathrm{Fib}^r_D \\ \mathrm{Fib}^r_D \arrow[d, "(\alpha',\alpha'')"] & \\ (X-D)\times_{k_0}(X-D) & \end{tikzcd}$$

which preserve the diagonal. Passing to morphisms induced between normal fibers along these diagonals, we obtain the k_0-morphisms

$$\begin{tikzcd} T_{\mathrm{Hecke}}(\mathrm{Hecke}^r_D \times_{\mathrm{Fib}^r_D} \mathrm{Hecke}^r_D) \arrow[r, "T(\pi_0',\pi_0'')"] \arrow[d, "T(\alpha',\alpha'')"] & T_{\mathrm{Fib}^r_D}(\mathrm{Fib}^r_D \times_{k_0} \mathrm{Fib}^r_D) \\ T_{(X-D)}((X-D)\times_{k_0}(X-D)) & \end{tikzcd}$$

which can be identified by projection on to the first coordinate with morphisms

$$\begin{tikzcd} T(\mathrm{Hecke}^r_D \xrightarrow{\pi_{-1}} \mathrm{Fib}^r_D) \arrow[r, "T\pi_0"] \arrow[d, "T\alpha"] & T(\mathrm{Fib}^r_D) \\ T(X-D). & \end{tikzcd}$$

We then apply the construction of "jets" from Appendix 11.1 to these k_0-morphisms, we obtain the morphisms of differential k_0-schemes

$$\begin{tikzcd} (\mathrm{Shtuka}^r_D, \partial) \arrow[r] \arrow[d] & (\mathrm{J}\,\mathrm{Fib}^r_D, \partial) \arrow[d, "can"] \\ (\mathrm{J}(T(\mathrm{Hecke}^r_D \xrightarrow{\pi_{-1}} \mathrm{Fib}^r_D)), \partial) \arrow[r, "\mathrm{J}(T\pi_0)"] \arrow[d, "\mathrm{J}(T\alpha)"] & (\mathrm{J}(T\,\mathrm{Fib}^r_D), \partial) \\ (\mathrm{J}(T(X-D)), \partial) & \end{tikzcd}$$

where the arrow "can" (TL Note: my quotes) is defined in Remark 11.6 and where the diagram is cartesian. For any differential k_0-scheme (S, ∂)

$$\mathrm{Hom}_{\mathrm{Schdiff}_{k_0}}((S,\partial), (\mathrm{Shtuka}^r_D, \partial))$$

is all of the isomorphism classes of (lower) differential shtukas of rank r over (S, ∂) for the curve X, equipped with a level structure D.

Remark 10.6. *For $r = 1$, we simply have (in degree 0)*

$$\begin{tikzcd} (\mathrm{Shtuka}^{1,0}_D, \partial) \arrow[r] \arrow[d] & (\mathrm{J}\,\mathrm{Pic}^0_D, \partial) \arrow[d, "\mathrm{dlog}"] \\ (\mathrm{J}(T(X-D)), \partial) \arrow[r] & (\mathrm{J}\,\mathrm{Pic}^0_D, \partial) \end{tikzcd}$$

where $\mathrm{Pic}^0_D = \mathrm{Lie}\,\mathrm{Pic}^0_D = H^1(X, \mathcal{O}_X(-D))$, where dlog is the logarithmic derivative and where the horizontal arrow from the bottom is obtained from the "jets" construction from the arrow of k_0-schemes

$$T(X-D) \to \mathrm{Pic}^r_D$$

which, over

$$T(X-D) = H^1\left(X, \frac{\mathcal{O}_X(-D+x)}{\mathcal{O}_X(-D)}\right)$$

for $x \in (X-D)(k_0)$ *is the map obtained from the corresponding co-edge of the following short exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(-D+x) \longrightarrow \frac{\mathcal{O}_X(-D+x)}{\mathcal{O}_X(-D)} \longrightarrow 0.$$

11. Appendices

11.1. **Appendix A - A Variation of the Construction of "Jets".** Fix a field k_0 of characteristic 0 and consider the "forgetful functor of the derivation" (TL Note: Laumon's quotation marks)

$$\begin{aligned} \mathrm{Algdiff}_{k_0} &\longrightarrow \mathrm{Alg}_{k_0} \\ (R,\partial) &\mapsto R \end{aligned}$$

from the category of differential k_0-algebras into that of k_0-algebras (commutative, unitary algebras).

Proposition 11.1. *The functor above admits a left adjoint*

$$\begin{aligned} \mathrm{Alg}_{k_0} &\longrightarrow \mathrm{Algdiff}_{k_0} \\ A &\mapsto (\mathrm{J}\,A,\partial). \end{aligned}$$

Proof. For any $a \in A$, let x_a be an indeterminate; then A is a quotient k_0-algebra of the k_0-algebra of polynomials $k_0[x_a : a \in A]$,

$$0 \longrightarrow I(A) \longrightarrow k_0[x_a : a \in A] \longrightarrow A \longrightarrow 0$$

$$x_a \mapsto a.$$

Put

$$(\mathrm{J}\,A,\partial) = (k_0[x_a^{(j)} : a \in A, j \in \mathbb{N}]/\langle I(A)\rangle, \partial)$$

where the $x_a^{(j)}$ are indeterminates, where $\partial(x_a^{(j)}) = x^{(j+1)}$, $\forall a \in A$, $j \in \mathbb{N}$, and where $\langle I(A)\rangle$ is the differential ideal of $(k_0[x_a^{(j)} : a \in A, j \in \mathbb{N}], \partial)$ generated by the ideal $I(A)$ of $k_0[x_a : a \in A]$ (we identify $x_a^{(0)}$ and x_a $\forall a \in A$). We clearly have a homomorphism of k_0-algebras

$$A \to \mathrm{J}\,A$$

induced by the indentification $x_a^{(0)} = x_a$ and it is immediate to check that, for any differential k_0-algebra (R,∂), the map

$$\mathrm{Hom}_{\mathrm{Algdiff}_{k_0}}((\mathrm{J}\,A,\partial),(R,\partial)) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(A,R)$$

induced by this homomorphism is bijective. The completes the proof of the proposition. □

Remark 11.2. *The functor* $B \mapsto (B,0)$ *from* Alg_{k_0} *to* $\mathrm{Algdiff}_{k_0}$ *also admits a left adjoint, namely the functor* $(S,\partial) \mapsto S/S\partial(S)$*; in particular, we have*

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{Algdiff}_{k_0}}((\mathrm{J}\,A,\partial),(B,0)) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(A,B) \\ \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(\mathrm{J}\,A/\mathrm{J}\,A\partial(\mathrm{J}\,A),B) \arrow[u, "\wr"] & \end{tikzcd}$$

where the vertical arrow is induced by the quotient map $\mathrm{J}\,A \to \mathrm{J}\,A/\mathrm{J}\,A\partial(\mathrm{J}\,A)$*. Furthermore, the composite arrow*

$$A \longrightarrow \mathrm{J}\,A \twoheadrightarrow \mathrm{J}\,A\partial(\mathrm{J}\,A)$$

is an isomorphism; in the following we will identify A *and* $\mathrm{J}\,A\partial(\mathrm{J}\,A)$ *with this isomorphism so that we have an epimorphism of differential* k_0*-algebras*

$$(\mathrm{J}\,A,\partial) \twoheadrightarrow (A,0)$$

which admits a section $A \longrightarrow \mathrm{J}\,A$ *as a morphism of* k_0*-algebras.*

Remark 11.3. *For each differential* k_0*-algebra* (R,∂) *and each* k_0*-algebra* B*, we have*

$$\mathrm{Hom}_{\mathrm{Algdiff}_{k_0}}((R,\partial),(B[\![t]\!],\partial_t)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(R,B)$$

via the canonical augmentation $B[\![t]\!] \twoheadrightarrow B$ *(*$k_0 \supset \mathbb{Q}$ *and Taylor's formula). Furthermore, we have*

$$\mathrm{Hom}_{\mathrm{Alg}_{k_0}}(\mathrm{J}\,A,B) \cong \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(A,B[\![t]\!]),$$

for any k_0*-algebras* A *and* B*, and in fact*

$$\mathrm{J}\,A = \varinjlim_n \mathrm{J}^n\,A$$

where $\mathrm{J}^n\,A$ *is the* k_0*-algebra which represent the functor* $B \mapsto \mathrm{Hom}_{\mathrm{Alg}_{k_0}}(A,B[\![t]\!]/(t^{n+1}))$ *and where the transition map* $\mathrm{J}^n\,A \longrightarrow \mathrm{J}^{n+1}\,A$ *is induced by* $B[\![t]\!]/(t^{n+2}) \twoheadrightarrow B[\![t]\!]/(t^{n+1})$*; of course,* $\mathrm{J}^0\,A = A$ *and* $\mathrm{J}^1\,A = \mathrm{Sym}_A(\Omega^1_{A/k_0})$*.*

Example 11.4. *For example, if* $A = k_0[x_1,\dots,x_r]$*, we clearly have*

$$(\mathrm{J}\,A,\partial) = (k_0[x_1^{(j)},\dots,x_r^{(j)} : j \in \mathbb{N}],\partial),$$

with $\partial(x_p^{(j)}) = x_p^{(j+1)}$ $\forall p \in \{1,\dots,r\}$*,* $j \in \mathbb{N}$*, and*

$$\mathrm{J}^n\,A = k_0[x_1^{(j)},\dots,x_r^{(j)} : j = 0,1,\dots,n]$$

with the obvious inclusion of $\mathrm{J}^n\,A$ *in* $\mathrm{J}^{n+1}\,A$ *by a transition arrow.*

The preceding can be globalized without difficulty within the framework of schemes.

Proposition 11.5. *The forgetful functor of the derviation from the category of differential* k_0*-schemes into the category of* k_0*-schemes admits a right adjoint*

$$\begin{aligned}\mathrm{Sch}_{k_0} &\longrightarrow \mathrm{Schdiff}_{k_0}\\ X &\mapsto (\mathrm{J}\,X,\partial).\end{aligned}$$

Now we have $(X,0) \hookrightarrow (\mathrm{J}\,X,\partial)$ which is the locus of zeros of the vectorfield ∂ on $\mathrm{J}\,X$ and

$$\mathrm{J}\,X = \varprojlim_n \mathrm{J}^n\,X$$

where $\mathrm{J}^0\,X = X$, $\mathrm{J}^1\,X = TX$ is the tangent fiber to X (over k_0) and $\mathrm{J}^n\,X$ is the "fiber of jets of curves traced on X" ($\mathrm{J}^n\,X$ is obtained by Greenburg's construction of "jets" applied to $X \otimes_{k_0} k[\![t]\!] \longrightarrow \mathrm{Spec}(k_0[\![t]\!])$; [Gr]); (TL Note: Laumon's quotes).

Remark 11.6. *For each* k_0*-scheme* X *and each integer* $n \geqslant 0$*, the canonical map* $\mathrm{J}\,X \longrightarrow \mathrm{J}^n\,X$ *additionally provides a morphism of differential* k_0*-schemes*

$$\mathrm{can} : (\mathrm{J}\,X,\partial) \longrightarrow (\mathrm{J}(\mathrm{J}^n\,X),\partial);$$

this is the identity for $n = 0$ *and, for* $n = 1$*, the map corresponds to*

$$\mathrm{Hom}_{\mathrm{Sch}_{k_0}}(S,X) \longrightarrow \mathrm{Hom}_{\mathrm{Sch}_{k_0}}(S,TX)$$

between points with value in a differential k_0*-scheme* (S,∂) *sending* $f : S \longrightarrow X$ *to the composite* $S \xrightarrow{\partial} TS \xrightarrow{Tf} TX$*.*

11.2. **Appendix B - The Logarithmic Derivative.** Let k_0 be a field of characteristic 0 and let G be a k_0 group scheme of finite type. We write G for the Lie k_0-algebra of G; we have an exact sequence of k_0 group schemes

$$0 \longrightarrow G \longrightarrow TG \xrightarrow{\pi} G \longrightarrow 0,$$

where $TG \xrightarrow{\pi} G$ is the tangent fiber to G, and the zero section $G \xrightarrow{\sigma} TG$ defines a k_0-morphism $TG \xrightarrow{p} G$, $\zeta \mapsto \sigma(\pi(\zeta)^{-1})\zeta$.

The construction of "jets" from Appendix 11.1 applied to G provides a differential k_0 group scheme $(\mathrm{J}\, G, \partial)$, equipped with a morphism of differential k_0-schemes

$$(\mathrm{J}\, G, \partial) \longrightarrow (\mathrm{J}(TG), \partial)$$

(see Remark 11.6); we will write

$$\mathrm{dlog} : (\mathrm{J}\, G, \partial) \longrightarrow (\mathrm{J}\, G, \partial)$$

for the composite of this morphism with $\mathrm{J}(p) : (\mathrm{J}(TG), \partial) \longrightarrow (\mathrm{J}\, G, \partial)$.

Definition 11.7. *The differential k_0-morphism* $\mathrm{dlog} : (\mathrm{J}\, G, \partial) \longrightarrow (\mathrm{J}\, G, \partial)$ *is called the* ***logarithmic derivative of*** G *([Ko.1]).*

The locus of zeros of the vectorfield ∂ on $\mathrm{J}\, G$ is a differential sub k_0 group scheme of $(\mathrm{J}\, G, \partial)$ with a zero derivation and is identified in fact with $(G, 0)$ (by Appendix 11.1). In particular, $(G, 0)$ acts on the left by translation on $(\mathrm{J}\, G, \partial)$, this action leaves dlog invariant. So, we have a morphism of differential schemes

$$(G, 0) \times_{k_0} (\mathrm{J}\, G, \partial) \longrightarrow (\mathrm{J}\, G, \partial) \otimes_{(\mathrm{J}\, G, \partial)} (\mathrm{J}\, G, \partial)$$

above on $(\mathrm{J}\, G, \partial)$ (via the second canonical projections: $(g, j) \mapsto (gj, j)$).

Proposition 11.8. $\mathrm{dlog} : (\mathrm{J}\, G, \partial) \longrightarrow (\mathrm{J}\, G, \partial)$ *is a "principal homogeneous fiber of the group* $(G, 0)$*" in the sense where the morphism above is an isomorphism.*

(TL Note: quotes are Laumon's).

Proof. It suffices to show that the morphism in question induces a bijection between points with value in an arbitrary differential k_0-scheme (S, ∂), from $(G, 0) \times_{k_0} (\mathrm{J}\, G, \partial)$ and from $(\mathrm{J}\, G, \partial) \times_{(\mathrm{J}\, G, \partial)} (\mathrm{J}\, G, \partial)$; or, if we symbolically write $\mathrm{dlog}(g) = g^{-1}g' \in G(S)$ for $g \in G(S)$, we have

$$\mathrm{dlog}(hg) = g^{-1}(h^{-1}h')g + g^{-1}g'$$

and therefore $\mathrm{dlog}(hg) = \mathrm{dlog}(g)$ implies $\mathrm{dlog}(h) = 0$, i.e. $h' = 0$, i.e. $h \in (G, 0)(S, \partial)$, hence the desired surjectivity; injectivity is trivial, we are finished. $\square$

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