Исправление доказательства достаточного условия дифф-ти.

Пусть $f: E \to \mathbb{R}^m$; $E \subseteq \mathbb{R}^n$, $a \in \operatorname{Int} E$,

1) $\exists$ окр. $U(a)$: $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ определены в $U(a)$

2) $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ непр. в т. a.

Тогда f дифф. в т. a.

<u>Док-во.</u> НУО $m=1$. $\frac{\partial f}{\partial x_1} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} \\ \vdots \\ \frac{\partial f_m}{\partial x_m} \end{pmatrix}, \dots$

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

если $f_1, \dots, f_m$ дифф, то f дифф.

$$df(h) = \begin{pmatrix} df_1(h) \\ \vdots \\ df_m(h) \end{pmatrix}.$$

Разберем случай $n=2$, $a=0$

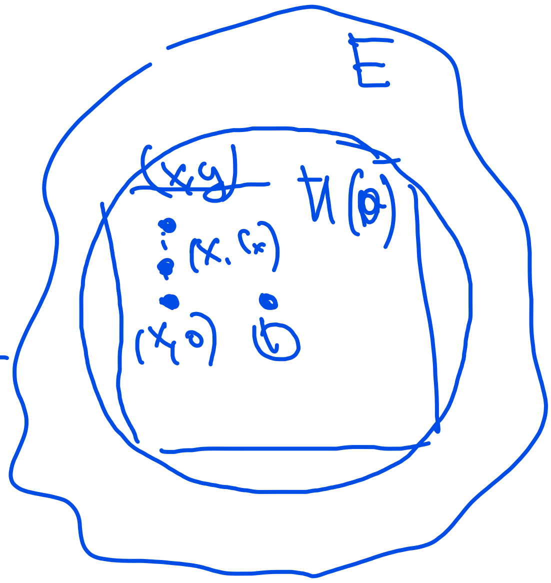
$$\underbrace{f(x,y) - f(0,0) - \langle \nabla f_{(0,0)}, (x,y) \rangle}_{= \Delta} = o(\|h\|) \iff f \text{ дифф. в т. } (0,0).$$

$h=(x,y)$ при $h \to 0$

$\Updownarrow$

$$\frac{\Delta}{\|h\|} \to 0 \text{ при } h \to 0$$

$\exists\, \delta > 0$: $(-\delta, \delta) \times (-\delta, \delta) \subset U(0)$



$$\Delta = \left(f(x,y) - f(x,0)\right) + \left(f(x,0) - f(0,0)\right) - \langle \nabla f, (x,y) \rangle =$$

$=$

фиксир. x $\varphi(y) = f(x,y)$

$f(x,y) - f(x,0) = \varphi(y) - \varphi(0)$; $\varphi'(y) = \frac{\partial f}{\partial y}(x,y)$ по определению

φ дифф. на $(-\delta, \delta)$

по классич. теор. Лагранжа $\exists\ c_x$ — между $0, y$

$$\varphi(y)-\varphi(0)=\varphi'(c_x)(y-0)=\frac{\partial f}{\partial y}(x,c_x)\cdot y$$

$$\psi(x)=f(x,0),\qquad \psi'(x)=\frac{\partial}{\partial x}f(x,0)\ \text{— опред. на } (-\delta,\delta)$$

$\Rightarrow\ \exists\ c_y$, леж. между $0, x$:

Теор. Лагр. $\underbrace{\psi(x)-\psi(0)}_{f(x,0)-f(0,0)}=\psi'(c_y)\cdot x$

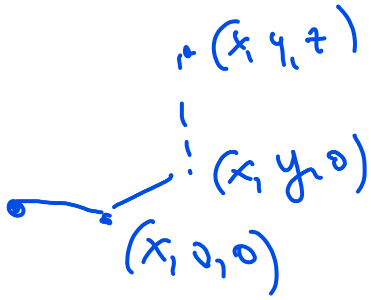
$$\frac{\Delta(x,y)}{\|(x,y)\|}=\frac{1}{\sqrt{x^2+y^2}}\left(\frac{\partial f}{\partial y}(x,c_x)y+\frac{\partial f}{\partial x}(c_y,0)x-\frac{\partial f}{\partial x}(0,0)\cdot x-\frac{\partial f}{\partial y}(0,0)\cdot y\right)$$

$$=\cancel{\frac{1}{\sqrt{x^2+y^2}}}\underbrace{\left(\frac{\partial f}{\partial x}(c_y,0)-\frac{\partial f}{\partial x}(0,0)\right)}_{\to 0}\underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\text{огр.}}+\underbrace{\left(\frac{\partial f}{\partial y}(x,c_x)-\frac{\partial f}{\partial y}(0,0)\right)}_{\to 0}\underbrace{\frac{y}{\sqrt{x^2+y^2}}}_{\text{огр.}}$$

$\frac{\partial f}{\partial x}$ непр. в т. 0

т.е. — беск. малая

Случай $n>2$ и $a\in E$ — аналогично.



Если $f(x,y): O \subseteq \mathbb{R}^2 \to \mathbb{R}$ в окр. т. a $\exists\ \frac{\partial^2 f}{\partial x\partial y}$ и $\frac{\partial^2 f}{\partial y\partial x}$ и непр. в т. a, то

$$\frac{\partial^2 f}{\partial x\partial y}(a)=\frac{\partial^2 f}{\partial y\partial x}(a).$$

$f(x_1,\dots,x_n)$

$$\frac{\partial^2 f}{\partial x_i\partial x_j}(a)=\frac{\partial^2 f}{\partial x_j\partial x_i}(a)\ \Longleftarrow\ \frac{\partial^2 f}{\partial x_i\partial x_j},\ \frac{\partial^2 f}{\partial x_j\partial x_i}\ \text{непр. в т. } a.$$

$r \in \mathbb{Z}_+$, O – откр в $\mathbb{R}^n$

$$C^r(O) = \left\{ f : O \to \mathbb{R} : \ \forall\, i_1,\dots,i_r \in \{1,\dots,n\} \ \ \frac{\partial^r f}{\partial x_{i_r}\dots\partial x_{i_1}} \in C(O) \right\}$$

$$C^\infty(O) = \bigcap_{r\in\mathbb{Z}_+} C^r(O)$$

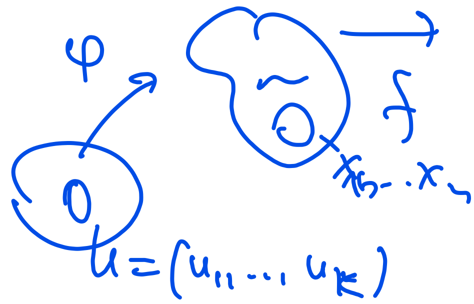
$C^r(O)$ – лин. простр-во; $f,g \in C^r(O)$, то $\lambda f + \mu g \in C^r(O)$

2). $f\cdot g \in C^r$ $\qquad \frac{\partial}{\partial x_i}(fg) = g\frac{\partial f}{\partial x_i} + f\frac{\partial g}{\partial x_i}$

3). φ $\quad C^r(O\to\mathbb{R}^n) = \{ f : f_1,\dots,f_n \in C^r(O) \}$

$\varphi \in C^r(O)$, $\quad \varphi(O) \subset \tilde{O}$, $\quad \forall\, f \in C^r(\tilde{O})$

$f\circ\varphi \in C^r(\tilde{O})$



$$\frac{\partial(f\circ\varphi)}{\partial u_j} = \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\varphi(u))}_{\in C^{r-1}} \underbrace{\frac{\partial \varphi_i}{\partial u_j}}_{C^{r-1}}$$

В частности, $\quad f \in C^r$, (x) $\qquad g(t) = f(a+th)$

a, h – фиксир.

$g = f\circ\varphi$, $\quad \varphi : t \to a+th \in C^\infty(\mathbb{R})$

$g \in C^r$(Окрест. 0).

Теорема о равенстве смешанных производных в кл. C^r.

Если $f \in C^r(O)$, $O \subset \mathbb{R}^n$ откр.; $r \in \mathbb{Z}_+$

$i_1, i_2,\dots, i_l \in \{1,\dots,n\}$, $\quad l \le r$, $\quad (j_1,\dots,j_l)$ получены из $(i_1,\dots,i_l)$

Тогда $\quad \frac{\partial^l f}{\partial x_{i_l}\dots\partial x_1}(a) = \frac{\partial^2 f}{\partial x_{j_l}\dots\partial x_{j_1}}(a)$

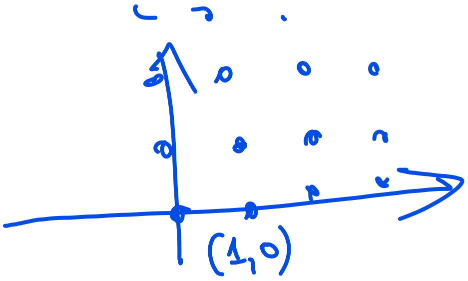
$\forall\, a \in O$.

Мультииндексы.

Мультииндексом называется элемент $\mathbb{Z}_+^n$

$j \in \mathbb{Z}_+^n \iff j=(j_1,\dots,j_n),\ j_1,\dots,j_n\in\mathbb{Z}_+$.

$n=2$



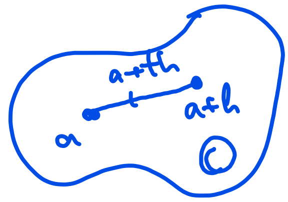
$|j| = j_1+j_2+\dots+j_n$

$j! = j_1!\cdot\ldots\cdot j_n!$

$h\in\mathbb{R}^n$, $h = (h_1,\dots,h_n)$, $h^j = h_1^{j_1}\cdot\ldots\cdot h_n^{j_n}$

$$f^{(j)}(a)=\frac{\partial^{|j|} f}{\partial x_n^{j_n}\ldots\partial x_1^{j_1}}(a).$$

Лемма. $f: O\to\mathbb{R}$, $O\subseteq\mathbb{R}^n$, $[a,a+h]\subset O$



$f\in C^r(O)$, $g(t)=f(a+th)$. Тогда $\forall\, l=0,\dots,r$

$$g^{(l)}(t)=\sum_{\substack{j\in\mathbb{Z}_+^n\\ |j|=l}}\frac{l!}{j!}f^{(j)}(a+th)\cdot h^j$$

Док-во. По индукции по l.

$l=0$. л.ч. $g^{(0)}(t)=g(t)=f(a+th)$

$|j|=0 \Rightarrow j\in\mathbb{Z}_+,\ j=0$ пр.ч. $f^{(0)}(a+th)=f(a+th)$

(база есть).

$l\to l+1$ переход. $g^{(l+1)}(t)=\left(g^{(l)}(t)\right)'_t \overset{\text{индукц. предп.}}{=}$

$$\sum_{\substack{j\in\mathbb{Z}_+^n\\ |j|=l}}\frac{l!}{j!}\left(f^{(j)}(a+th)\,h^j\right)'_t=\sum_{|j|=l}\frac{l!}{j!}\left(\sum_{i=1}^n\frac{\partial f^{(j)}}{\partial x_i}(a+th)\cdot h_i\right)h^j=$$

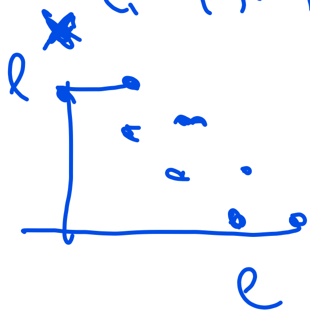
$\mu(x)=f^{(j)}(x)$

$$\left(\mu(a+th)\right)'_t=\sum_{i=1}^n\frac{\partial\mu}{\partial x_i}\cdot h_i$$

$$= \sum_{i=1}^{n} \ell! \sum_{|j|=\ell} f^{(j+e_i)}(a+th)\, h^{(j+e_i)}\, p_i$$

$$e_i = (0, \dots, 0, 1, 0, \dots, 0);$$

$$j + e_i = p \in \mathbb{Z}_+^n$$



$$\sum_{i=1}^{n} \frac{\ell!}{p!} \Bigg(\sum_{\substack{p \in \mathbb{Z}_+^n \\ |p| = \ell+1 \\ \cancel{p_i \neq 0}}} f^{(p)}(a+th)\, h^{p} \cdot p_i \Bigg) =$$

$$p! = j! \cdot p_i$$

$$(p_i = j_i + 1)$$

$$= \sum_{\substack{p \in \mathbb{Z}_+^n \\ |p| = \ell+1}} \underbrace{\frac{\ell!}{p!} \underbrace{\left(\sum_{i=1}^{n} p_i \right)}_{|p| = \ell+1}}_{\frac{(\ell+1)!}{p!}} \cdot f^{(p)}(a+th) \cdot h^{p}$$

Теор (глобальная формула Тейлора (Лагранжа)) для функции неск. переменных

$O \subseteq \mathbb{R}^n$; $f \in C^{r+1}(O)$

опер: $r \in \mathbb{Z}_+$

$\exists\ a, h: \ \forall t \in [0, 1] \quad a + th \in O$

Тогда $\exists\ \theta \in (0, 1)$:

$$f(a+h) = \sum_{\substack{j \in \mathbb{Z}_+^n \\ |j| \le r}} \frac{f^{(j)}(a)}{j!} h^j + \sum_{\substack{j \in \mathbb{Z}_+^n \\ |j| = r+1}} \frac{f^{(j)}(a + \theta h)}{j!} h^j$$