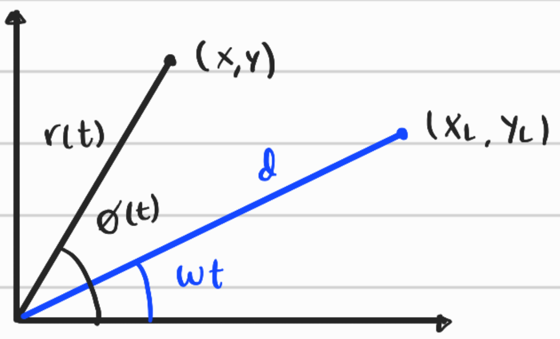
Solución Tarea Semana 4 - Teórico

c) Usando la figura dada, se ve que se necesita calcular una distancia en coordenadas polares.



El punto (x,y) está dado en polares por:

$$x = r\cos(\phi) \quad , \quad y = r\sin(\theta)$$

El punto (x_L, y_L) está dado en polares por:

$$x_L = d\cos(\omega t) \quad , \quad y_L = d\sin(\omega t)$$

Con esto:

$$\begin{aligned} r_L^2 &= [r\cos(\phi) - d\cos(\omega t)]^2 + [r\sin(\phi) - d\sin(\omega t)]^2 \\ &= r^2\cos^2(\theta) + r^2\sin^2(\theta) - 2rd\cos(\theta)\cos(\omega t) - 2rd\sin(\theta)\sin(\omega t) \\ &\quad + d^2\cos^2(\omega t) + d^2\sin^2(\omega t) \end{aligned}$$

$$\Rightarrow r_L^2 = r^2[\cos^2(\theta) + \sin^2(\theta)] - 2rd[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)]$$
$$+ d^2[\cos^2(\omega t) + \sin^2(\omega t)]$$

$$\Rightarrow r_L^2 = r^2 + d^2 - 2rd\cos(\phi - \omega t)$$

$$\boxed{r_L = \sqrt{r^2 + d^2 - 2rd\cos(\phi - \omega t)}} \quad //$$

d) Hallando el Lagrangiano $\mathcal{L} = T - V$:

Para T: La velocidad del cohete es polar. Así:

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\phi}\,\hat{\theta}$$

$$T = \frac{1}{2}mv^2 \longrightarrow \left\{ T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \right\}$$

Para V: Debido a la interacción con la luna y la tierra:

$$V = -\frac{G m_T m}{\sqrt{X_R^2 + Y_R^2}} - \frac{G m_L m}{r_L} \Rightarrow \left\{ V = -Gm\left(\frac{m_T}{r} + \frac{m_L}{r_L}\right) \right\}$$

Así: $$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r\dot{\phi}^2) + Gm\left(\frac{m_T}{r} + \frac{m_L}{r_L}\right)$$

También: $P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$, $P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr\dot{\phi}$

Se ve que: $\frac{P_r^2}{2m} = \frac{1}{2}m\dot{r}^2$ y $\frac{P_\phi^2}{2mr^2} = \frac{1}{2}mr\dot{\phi}^2$. Como $H = T + V$:

$$H = \frac{P_r^2}{2m} + \frac{P_\phi^2}{2mr^2} - Gm\left(\frac{m_T}{r} + \frac{m_L}{r_L}\right)$$ //

e) Como $\dot{r} = \frac{\partial H}{\partial P_r}$: $\dot{r} = \frac{P_r}{m}$ Como $\dot{\phi} = \frac{\partial H}{\partial P_\phi}$: $\dot{\phi} = \frac{P_\phi}{mr}$

Paréntesis:

(1): Hallando $\frac{\partial}{\partial r}(r_L^{-1})$: $= -r_L^{-2} \cdot \frac{\partial r_L}{\partial r} \longrightarrow -\frac{1}{r_L^2} \cdot \frac{2r - 2d\cos(\phi - \omega t)}{2r_L}$

$$\Rightarrow -\frac{(r - d\cos(\phi - \omega t))}{r_L^3}$$

(2) Hallando $\frac{\partial}{\partial \phi}(r_L^{-1})$: $= -\frac{1}{r_L^2} \cdot \frac{[2rd\sin(\phi - \omega t)]}{2r_L}$

$$\Rightarrow -\frac{rd\sin(\phi - \omega t)}{r_L^3}$$

Como $\dot{P_r} = -\frac{\partial H}{\partial r}$: $$\dot{P_r} = \frac{P_\phi^2}{mr^3} - \frac{Gmm_T}{r^2} - \frac{Gmm_L}{r_L^3}[r - d\cos(\phi - \omega t)]$$

Como $\dot{P_\phi} = -\frac{\partial H}{\partial \phi}$: $$\dot{P_\phi} = -\frac{Gmm_L}{r_L^3}[rd\sin(\phi - \omega t)]$$ //

F) Sea $\tilde{r} = \frac{r}{d}$ y $\tilde{P_r} = \frac{P_r}{md}$, vemos que $\dot{r} = d\dot{\tilde{r}}$ y $P_r = md\dot{\tilde{r}}$

Así: $md\tilde{P_r} = md\dot{\tilde{r}} \longrightarrow \tilde{P_r} = \dot{\tilde{r}}$

Sea $\tilde{P_\phi} = \frac{P_\phi}{md^2}$, como $P_\phi = mr^2\dot{\phi}$: $\tilde{P_\phi} = \frac{r^2\dot{\phi}}{d^2}$, con $\frac{1}{\tilde{r}^2} = \frac{d^2}{r^2}$

$\Rightarrow \dot{\phi} = \frac{\tilde{P_\phi}}{\tilde{r}^2}$

Reemplazando $P_\phi = md^2\tilde{P_\phi}$, $r = \tilde{r}d$ y $\dot{P_r} = md\dot{\tilde{P_r}}$ en la ecuación de momentum en r:

$$\cancel{m}\cancel{d}\dot{\tilde{P_r}} = \frac{\cancel{m^2}\cancel{d^4}\tilde{P_\phi}^2}{\cancel{m}\tilde{r}^3\cancel{d^3}} - \frac{\Delta\cancel{m}}{\tilde{r}^2 d^2} - \frac{Gm_L\cancel{m}\,[\tilde{r}\cancel{d} - \cancel{d}\cos(\phi-\omega t)]}{(\tilde{r}d^2 + d^2 - 2\tilde{r}d^2\cos(\phi-\omega t))^{3/2}}$$

$$\Rightarrow \dot{\tilde{P_r}} = \frac{\tilde{P_\phi}^2}{\tilde{r}^3} - \frac{\Delta}{\tilde{r}^2} - \frac{Gm_L}{d^3}\frac{[\tilde{r} - \cos(\phi-\omega t)]}{[\tilde{r} + 1 - 2\tilde{r}\cos(\phi-\omega t)]^{3/2}}$$

Como $\tilde{r}'^2 = 1 + \tilde{r}^2 - 2\tilde{r}\cos(\phi-\omega t)$ y $\Delta\nu = Gm_L/d^3$:

$$\Rightarrow \dot{\tilde{P_r}} = \frac{\tilde{P_\phi}^2}{\tilde{r}^3} - \Delta\left\{\frac{1}{\tilde{r}^2} + \frac{\nu}{\tilde{r}'^3}[\tilde{r} - \cos(\phi-\omega t)]\right\}$$

Reemplazando en $\dot{\tilde{P_\phi}}$:

$$\cancel{m}\cancel{d^2}\dot{\tilde{P_\phi}} = -\frac{G\cancel{m}m_L\cdot\tilde{r}\cancel{d^2}\sin(\phi-\omega t)}{[\tilde{r}^2d^2 + d^2 - 2\tilde{r}d^2\cos(\phi-\omega t)]^{3/2}}$$

$\Rightarrow \dot{\tilde{P_\phi}} = -\frac{Gm_L}{d^3}\frac{\tilde{r}\sin(\phi-\omega t)}{r'^3}$, usando $\Delta\nu = \frac{Gm_L}{d^3}$:

$$\dot{\tilde{P_\phi}} = -\frac{\Delta\nu\tilde{r}}{r'^3}\sin(\phi-\omega t)$$

9) Sea $\tilde{P_r}^{o} = \frac{P_r}{md}$, y sabiendo que $P_r = m\dot{r}$:

$$\tilde{P_r}^{o} = \frac{\dot r}{d} \longrightarrow \tilde{P_r}^{o} = \frac{1}{d}\frac{dr}{dt}$$, como $r = \sqrt{x^2+y^2}$:

$$\tilde{P_r}^{o} = \frac{1}{d}\frac{d\sqrt{x^2+y^2}}{dt} \Rightarrow \frac{1}{d}\frac{2x\dot x + 2y\dot y}{2\sqrt{x^2+y^2}} \Rightarrow \frac{x\dot x + y\dot y}{rd} = \tilde{P_r}^{o}$$

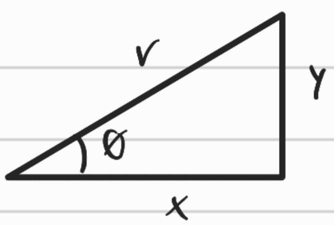
Sabiendo $\cos(\phi) = \frac{x}{r}$ y $\sin(\phi) = \frac{y}{r}$: $\frac{\dot x}{d}\cos(\phi) + \frac{\dot y}{d}\sin(\phi) = \tilde{P_r}^{o}$

Como $\tilde{V_x} = \frac{\dot x}{d} = \tilde{V_0}\cos(\theta)$ y $\tilde{V_y} = \frac{\dot y}{d} = \tilde{V_0}\sin(\theta)$:

$$\tilde{V_0}\cos(\theta)\cos(\phi) + \tilde{V_0}\sin(\theta)\sin(\phi) = \tilde{P_r}^{o} \Rightarrow \boxed{\tilde{V_0}\cos(\theta-\phi) = \tilde{P_r}^{o}}$$

Para $\tilde{P_\phi}^{o}$: $\tilde{P_\phi}^{o} = \frac{P_\phi}{md^2} \longrightarrow \tilde{P_\phi}^{o} = \frac{mr^2\dot\phi}{md^2} \Leftrightarrow \tilde{r}^2\dot\phi$

Por geometría:



$$\Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right) \xrightarrow{d/dt} \dot\phi = \frac{d}{dt}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Tomando la derivada:

$$\tilde{r}^2\dot\phi = \frac{1}{1+(y/x)^2}\cdot\frac{d}{dt}\left(\frac{y}{x}\right) \Leftrightarrow \frac{1}{1+(y/x)^2}\cdot\frac{\dot y x - \dot x y}{x^2} = \left(\frac{\dot y x - \dot x y}{x^2+y^2}\right)\tilde{r}^2$$

Como $r^2 = x^2+y^2$

$$\left(\frac{\tilde r}{r}\right)^2(\dot y x - \dot x y) = \tilde{r}^2\dot\phi$$, usando $\frac{\dot y}{r} = \tilde{V_0}\sin(\theta)$ y $\frac{\dot x}{r} = \tilde{V_0}\cos(\theta)$:

$$\frac{\tilde{r}^2}{r}\left(\tilde{V_0}\sin(\theta)\cos(\phi) - \tilde{V_0}\cos(\theta)\sin(\phi)\right) = \frac{\tilde{r}^2}{r}\tilde{V_0}\sin(\theta-\phi)$$

Finalmente: $\frac{\tilde{r}^2}{r} = \frac{r}{d} = \tilde{r}$. Así: $\boxed{\tilde{P_\phi}^{o} = \tilde{r}\cdot\tilde{V_0}\sin(\theta-\phi)}$