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The Maximax Minimax Quotient Theorem

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Abstract
We present an optimization problem emerging from optimal control theory and situated at the intersection of fractional programming and linear max-min programming on polytopes. A naïve solution would require solving four nested, possibly nonlinear, optimization problems. Instead, relying on numerous geometric arguments we determine an analytical solution to this problem. In the course of proving our main theorem, we also establish another optimization result stating that the minimum of a specific minimax optimization is located at a vertex of the constraint set.

Keywords Optimization · Fractional programming · Max-min programming · Polytopes

Mathematics Subject Classification 49K35 · 90C32 · 90C47

1 Introduction

The field of fractional programming studies the optimization of a ratio of functions and made its debut in the 1960s with Charnes and Cooper [4]. It has since then expanded to more complex and more general problems [9]. However, outside of linear fractional programming, very few analytical results are available; the focus has now largely shifted to developing search algorithms [1, 8]. In this paper, we are interested in a specific fractional optimization problem introduced in [3] and composed of four

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nested optimization problems. For this reason, a search algorithm would have a high computational cost and would be especially wasteful since an analytical solution exists.

Our ratio of interest features a max-min optimization [5, 6] belonging to the setting of semi-infinite programming [7]. Because of the infinite number of constraints, it is not possible to immediately apply the classical results of linear max-min theory [10] stating that the maximum is attained on the boundary of the constraint set. Nonetheless, thanks to the specific geometry of our problem we are able to prove a very similar result, first mentioned as Theorem 3.1 in the authors' work [3]. However, its proof is omitted from [3].

Armed with this preliminary result on max-min programming, we formulate and establish the Maximax Minimax Quotient Theorem. This result concerns the maximization of a ratio of a maximum and a minimax over two polytopes. In the special case where these polytopes are symmetric, this result reduces to Theorem 3.2 of [3], whose proof was again omitted for length concerns.

The remainder of this paper is organized as follows. Section 2 establishes the existence of the Maximax Minimax Quotient and proves a preliminary optimization result. Section 3 states our central theorem and provides its proof. Section 4 gathers all the lemmas involved in the proof of the Maximax Minimax Quotient Theorem. Section 5 justifies the continuity of two maxima functions used during the proof of our main result. Finally, Sect. 6 illustrates the proof of our theorem on a simple example.

Notation: We use ∂X to denote the boundary of a set X and its interior is denoted $X^\circ := X \backslash \partial X$. In $\mathbb{R}^n$ we denote the unit sphere with $\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and the ball of radius ε centered on x with $B_\varepsilon(x) := \big\{y \in \mathbb{R}^n : \|y - x\| \leq \varepsilon\big\}$. The scalar product of vectors is denoted by $\langle \cdot, \cdot \rangle$. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ both nonzero, we denote as $\widehat{x, y}$ the signed angle from x to y in the 2D plane containing both of them. We take the convention that the angles are positive when going in the clockwise orientation.

2 Preliminaries

Definition 2.1 A *polytope* in $\mathbb{R}^n$ is a compact intersection of finitely many half-spaces.

Following Definition 2.1, this work only considers convex polytopes. If X and Y are two nonempty polytopes in $\mathbb{R}^n$ with $-X \subset Y^\circ$, and $d \in \mathbb{S}$, we define the *Maximax Minimax Quotient* as

$$r_{X,Y}(d) := \frac{\max\limits_{x \in X,\ y \in Y} \big\{\|x + y\| : x + y \in \mathbb{R}^+ d\big\}}{\min\limits_{x \in X} \Big\{\max\limits_{y \in Y} \big\{\|x + y\| : x + y \in \mathbb{R}^+ d\big\}\Big\}}. \tag{1}$$

The objective of the Maximax Minimax Quotient Theorem is to determine the direction d that maximizes $r_{X,Y}(d)$. Note that in the numerator of (1), x and y are chosen together to satisfy the constraint $x + y \in \mathbb{R}^+ d$, while in the denominator this constraint only applies to y. Before starting the actual proof of this theorem, we first need to justify the existence of the minimum and the maxima appearing in (1).

Proposition 2.1 *Let X, Y be two nonempty polytopes in $\mathbb{R}^n$ with $-X \subset Y^\circ$,* $\dim Y = n$ *and $d \in \mathbb{S}$. Then,*

(i) $\max_{x \in X,\, y \in Y} \left\{ \|x + y\| : x + y \in \mathbb{R}^+ d \right\}$ *exists,*
(ii) $\lambda^*(x, d) := \max_{y \in Y} \left\{ \|x + y\| : x + y \in \mathbb{R}^+ d \right\}$ *exists for all $x \in X$,*
(iii) $\min_{x \in X} \left\{ \lambda^*(x, d) \right\}$ *exists,*
(iv) and $\min_{x \in X} \left\{ \lambda^*(x, d) \right\} > 0$.

Proof (i) Let $S := \left\{ (x, y) \in X \times Y : x + y \in \mathbb{R}^+ d \right\}$. Set S is a closed subset of the compact set $X \times Y$, so S is compact. Since X is nonempty, we take $x \in X$. Using $-X \subset Y$ we have $-x \in Y$ and $x + (-x) = 0 \in \mathbb{R}^+ d$. Then, $(x, -x) \in S$, so S is nonempty. Function $f : S \to \mathbb{R}$ defined as $f(x, y) := \|x + y\|$ is continuous, so it reaches a maximum over S.

(ii) For $x \in X$ define $S(x) := \left\{ y \in Y : x + y \in \mathbb{R}^+ d \right\}$. Since $S(x)$ is a closed subset of the compact set Y, $S(x)$ is compact. Since $-X \subset Y$, we have $-x \in S(x)$ and so $S(x) \neq \emptyset$. Function $f_x : S(x) \to \mathbb{R}$ defined as $f_x(y) := \|x + y\|$ is continuous, so it reaches a maximum over $S(x)$, i.e., λ^* exists.

(iii) For $x \in X$ and $d \in \mathbb{S}$, the argument of $\lambda^*(x, d)$ is uniquely defined as $y^*(x, d) := \lambda^*(x, d) d - x$ since $\|d\| = 1$ and

$$y^*(x, d) = \arg\max_{y \in Y} \left\{ \|x + y\| : x + y \in \mathbb{R}^+ d \right\}. \tag{2}$$

Lemma 5.2 shows that λ^* is continuous in x and d, so y^* is also continuous in x and d. Then, function $f : X \to \mathbb{R}$ defined as $f(x) := \|x + y^*(x, d)\|$ is continuous, so it reaches a minimum over the compact and nonempty set X.

(iv) Note that $y^*(x, d) \in \partial Y$ for all $x \in X$. Indeed, assume for contradiction purposes that there exists $\varepsilon > 0$ such that $B_\varepsilon\left(y^*(x, d)\right) \in Y$. We required $\dim Y = n$ to make this ball of full dimension, so that $z := y^*(x, d) + \varepsilon d \in Y$. Then, $x + z = \left(\lambda^*(x, d) + \varepsilon\right) d \in \mathbb{R}^+ d$ and $\|x + z\| = \lambda^*(x, d) + \varepsilon > \lambda^*(x, d)$ contradicting the optimality of λ^*. Thus, $y^*(x, d) \in \partial Y$. Since $-X \subset Y^\circ$, we have $\|x + y^*(x, d)\| > 0$ for all $x \in X$. □

Then, with the assumptions of Proposition 2.1 the Maximax Minimax Quotient is well defined. The proof of our main theorem relies on another optimization result stating that the argument of the minimum in (1) lies at a vertex of X.

Definition 2.2 A *vertex* of a set $X \subset \mathbb{R}^n$ is a point $x \in X$ such that if there are $x_1 \in X$, $x_2 \in X$ and $\lambda \in [0, 1]$ with $x = \lambda x_1 + (1 - \lambda) x_2$, then $x = x_1 = x_2$.

With this definition, a vertex of a polytope corresponds to the usual understanding of a vertex of a polytope.

Theorem 2.1 *Let $d \in \mathbb{S}$, X and Y two polytopes of $\mathbb{R}^n$ with $-X \subset Y$ and* $\dim Y = n$. *Then, there exists a vertex v of X where* $\min_{x \in X} \left\{ \lambda^*(x, d) \right\}$ *is reached.*

Proof According to Proposition 2.1 the minimum of λ^* exists. Then, let $x^* \in X$ such that $\lambda^*(x^*, d) = \min\limits_{x \in X} \big\{\lambda^*(x, d)\big\}$, i.e., $\|y^*(x^*) + x^*\| = \min\limits_{x \in X} \|y^*(x) + x\|$. Since $-x^*$ must minimize the distance between itself and $y^*(x^*) \in \partial Y$, with $-X \subset Y$ obviously $x^* \in \partial X$. Assume now that x^* is not on a vertex of ∂X. Let S_x be the surface of lowest dimension in ∂X such that $x^* \in S_x$ and $\dim S_x \geq 1$.

Let v be a vertex of S_x and $x(\alpha) := x^* + \alpha(v - x^*)$ for $\alpha \in \mathbb{R}$. Notice that $x(0) = x^*$ and $x(1) = v$. Due to the choice of v, the convexity of S_x and x^* not being a vertex, there exists $\varepsilon > 0$ such that $x(\alpha) \in S_x$ for all $\alpha \in [-\varepsilon, 1]$. We also define the lengths $L(\alpha) := \|y^*\big(x(\alpha)\big) + x(\alpha)\|$ and $L^* := L(0)$.

Since $\|d\| = 1$ and $y^*\big(x(\alpha)\big) + x(\alpha) \in \mathbb{R}^+ d$, we have $L(\alpha) = \langle y^*\big(x(\alpha)\big) + x(\alpha), d\rangle$. By definition of x^*, we know that $L^* \leq L(\alpha)$ for all $\alpha \in [-\varepsilon, 1]$. For contradiction purposes, assume that there exists $\alpha_0 \in (0, 1]$ such that $L^* < L(\alpha_0)$. We introduce the convexity coefficient $\beta := \frac{\alpha_0}{\alpha_0 + \varepsilon} > 0$ and then

$$\begin{aligned} L^* &= \beta L^* + (1-\beta)L^* < \beta L(-\varepsilon) + (1-\beta)L(\alpha_0) \\ &< \beta\langle y^*\big(x(-\varepsilon)\big) + x(-\varepsilon), d\rangle + (1-\beta)\langle y^*\big(x(\alpha_0)\big) + x(\alpha_0), d\rangle = \langle z + x^*, d\rangle, \end{aligned}$$

with $z := \beta y^*\big(x(-\varepsilon)\big) + (1-\beta) y^*\big(x(\alpha_0)\big)$. Indeed, note that $\beta x(-\varepsilon) + (1-\beta)x(\alpha_0) = x^*$, and $z + x^* \in \mathbb{R}^+ d$. Note that $L^* = \max\limits_{y \in Y} \big\{\langle x^* + y, d\rangle : x^* + y \in \mathbb{R}^+ d\big\}$, but $L^* < \langle x^* + z, d\rangle$. Given that $z \in Y$ by convexity of Y and $x^* + z \in \mathbb{R}^+ d$, we have reached a contradiction. Thus, there is no $\alpha_0 \in (0, 1]$ such that $L^* < L(\alpha_0)$. Therefore, for all $\alpha \in [0, 1]$, $L(\alpha) = L^*$. By taking $\alpha = 1$, we have $x(\alpha) = v$, so the minimum L^* is also reached on the vertex v of X. $\square$

We have now all the preliminary results necessary to state our central theorem.

3 The Maximax Minimax Quotient Theorem

Theorem 3.1 (Maximax Minimax Quotient Theorem) *If X and Y are two polytopes in $\mathbb{R}^n$ with $-X \subset Y^\circ$, $\dim X = 1$, $\partial X = \{x_1, x_2\}$ with $x_2 \neq 0$ and $\dim Y = n$, then* $\max\limits_{d \in \mathbb{S}} r_{X,Y}(d) = \max\big\{r_{X,Y}(x_2), r_{X,Y}(-x_2)\big\}$.

Proof Since $\dim X = 1$, its extremities x_1 and x_2 are different, so at least one of them is nonzero. Then, imposing $x_2 \neq 0$ does not restrain the generality of our result. Following Proposition 2.1, $r_{X,Y}$ is well defined. Reusing y^* defined in (2), we introduce $x_M^*(d) := \arg\min\limits_{x \in X} \big\{\|x + y^*(x, d)\|\big\}$ and $x_N^*(d) := \arg\max\limits_{x \in X} \big\{\|x + y^*(x, d)\| : x + y^*(x, d) \in \mathbb{R}^+ d\big\}$. According to Theorem 2.1, $x_M^*(d) \in \partial X$ for all $d \in \mathbb{S}$ and following Lemma 5.3, x_N^* is a continuous function of d. For some $d \in \mathbb{S}$ the arg min and arg max in the definitions of x_M^* and x_N^* might not be unique; if so we take the arguments ensuring that $x_M^*(d) \in \partial X$ and that x_N^* is continuous. We also define $y_N^*(d) := y^*\big(x_N^*(d), d\big)$ and $y_M^*(d) := y^*\big(x_M^*(d), d\big)$. Then,

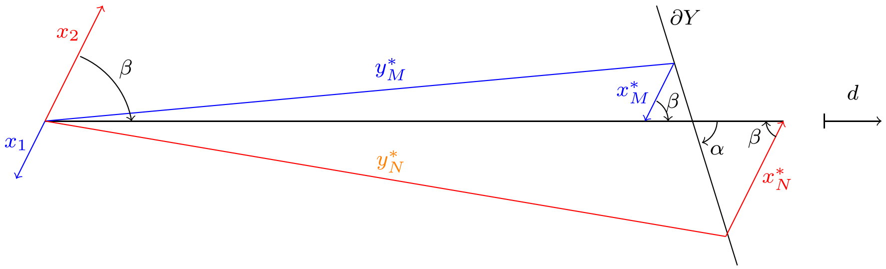


Fig. 1 Illustration of y_N^*, x_N^*, y_M^* and x_M^* for a direction d parametrized by β

$$r_{X,Y}(d) = \frac{\max\limits_{y \in Y}\left\{\|y + x_N^*(d)\| : y + x_N^*(d) \in \mathbb{R}^+ d\right\}}{\max\limits_{y \in Y}\left\{\|y + x_M^*(d)\| : y + x_M^*(d) \in \mathbb{R}^+ d\right\}} = \frac{\|x_N^*(d) + y_N^*(d)\|}{\|x_M^*(d) + y_M^*(d)\|}.$$

Since $\dim X = 1$, we can take $\mathcal{P}$ to be a two-dimensional plane containing X. Then, we will study how $r_{X,Y}(d)$ varies when d takes values in $\mathbb{S} \cap \mathcal{P}$. We introduce the signed angles $\alpha := \widehat{d, \partial Y}$ and $\beta := \widehat{x_2, d}$. These angles are represented in Fig. 1, and they take value in $[0, 2\pi)$. We parametrize all directions $d \in \mathbb{S} \cap \mathcal{P}$ by the angle β. Then, we will study how $r_{X,Y}(d)$ varies when $\beta \in [0, 2\pi)$.

We first establish in Lemma 4.1 that $x_N^*(d)$ and $x_M^*(d)$ are constant, different and both belong in ∂X when $y_M^*(d)$, d and $y_N^*(d)$ all intersect the same face of ∂Y, as illustrated in Fig. 1. In these situations, Lemma 4.2 shows that the ratio $r_{X,Y}$ is constant. Thus, $r_{X,Y}$ can only change when one of the three rays intersects a different face of ∂Y than the other two. We refer to these situations as vertex crossings. Lemma 4.3 introduces the vertices v_π and $v_{2\pi}$.

We study the crossing of vertices before v_π in Lemma 4.4. During these crossings, Lemma 4.5 shows that $r_{X,Y}$ decreases as β increases. Lemma 4.6 states that $r_{X,Y}$ reaches a local minimum during the crossing of v_π. As β increases between v_π and π, Lemmas 4.7 and 4.8 prove that $r_{X,Y}$ increases during the crossing of vertices. Finally, Lemma 4.9 completes the revolution by showing that $r_{X,Y}$ decreases after $\beta = \pi$ until a local minimum at $v_{2\pi}$ and then increases again until $\beta = 2\pi$. Thus, the directions $d \in \mathcal{P} \cap \mathbb{S}$ maximizing $r_{X,Y}(d)$ are collinear with the set X. Note that Fig. 1 implicitly assumes that $0 \in X$. Lemma 4.10 proves that even if $0 \notin X$, all above results still hold. Therefore, $\max\limits_{d \in \mathbb{S}} r_{X,Y}(d) = \max\limits_{\mathcal{P}}\left\{\max\limits_{d \in \mathcal{P} \cap \mathbb{S}} r_{X,Y}(d)\right\} = \max\left\{r_{X,Y}(x_2), r_{X,Y}(-x_2)\right\}$. □

In the special case where X and Y are symmetric polytopes, this result reduces to Theorem 3.2 of [3]. Indeed, $r_{X,Y}$ becomes an even function which leads to $r_{X,Y}(x_2) = r_{X,Y}(-x_2)$.

4 Supporting Lemmata

In this section, we establish all the lemmas involved in the proof of the Maximax Minimax Quotient Theorem.

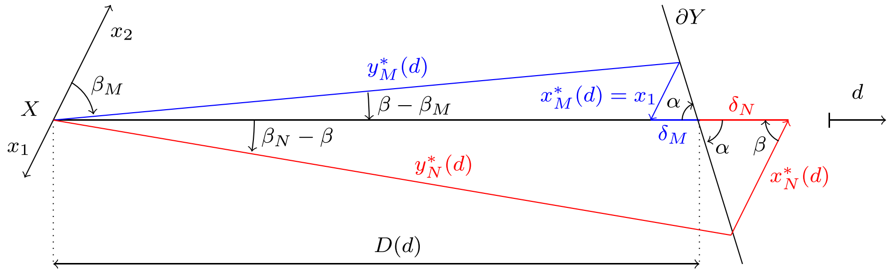


Fig. 2 Illustration of $y_N^*(d)$ leading and outside, while $y_M^*(d)$ is trailing and inside the same face of ∂Y

Lemma 4.1 *If d, $y_N^*(d)$ and $y_M^*(d)$ all intersect the same face of ∂Y, then $x_N^*(d)$ and $x_M^*(d)$ are constant, different and both belong to ∂X.*

Proof We introduce the angles $\beta_M := \widehat{x_2, y_M^*}$ and $\beta_N := \widehat{x_2, y_N^*}$. Let α_0 be the value of α when $\beta = 0$, i.e., when d is positively collinear with x_2.

We say that y_N^* is *leading* and y_M^* is *trailing* when $\beta_M < \beta_N$, and conversely when $\beta_N < \beta_M$, we say that y_M is *leading* and y_N is *trailing*.

For each $d \in \mathbb{S} \cap \mathcal{P}$ we define $D(d) := \max_{y \in Y}\left\{\|y\| : y \in \mathbb{R}^+ d\right\}$, whose existence is justified by the compactness of Y.

We say that y_N^* or y_M^* is *outside* when $\|y_N^* + x_N^*\| > D$ or $\|y_M^* + x_M^*\| > D$, respectively. Otherwise, y_N^* or y_M^* is *inside*. Directly related to the previous definition, we introduce

$$\delta_M(d) := D(d) - \|x_M^*(d) + y_M^*(d)\| \text{ and } \delta_N(d) := \|x_N^*(d) + y_N^*(d)\| - D(d). \quad (3)$$

We know from Theorem 2.1 that $x_M^*(d) \in \partial X$ for all $d \in \mathbb{S}$. In the case illustrated in Fig. 2, $x_M^*(d) = x_1$ because it maximizes δ_M.

If $\alpha + \beta \in \{\pi, 2\pi\}$, then X is parallel with a face of ∂Y making x_N^* and x_M^* not uniquely defined. Regardless, we can still take $x_N^*(d) \neq x_M^*(d)$, with $x_N^*(d) \in \partial X$ and $x_M^*(d) \in \partial X$. Otherwise, x_N^* and x_M^* are uniquely defined. Since $x_N^*(d) \in X$, $x_M^*(d) \in X$ for all $d \in \mathbb{S}$ and $\dim X = 1$, vectors $x_N^*(d)$ and $x_M^*(d)$ are always collinear. We then use Thales's theorem and obtain $\delta_N(d) = \delta_M(d)\frac{\|x_N^*(d)\|}{\|x_M^*(d)\|}$. Since $x_N^*(d)$ is chosen to maximize δ_N and is independent from δ_M, it must have the greatest norm, so $x_N^*(d) \in \partial X$. In the case where $\alpha + \beta \notin \{\pi, 2\pi\}$, $\|x + y\|$ depends on the value of x. Because $x_N^*(d)$ is chosen to maximize $\|x + y\|$ while $x_M^*(d)$ is minimizing it, we have $x_N^*(d) \neq x_M^*(d)$.

Since x_N^* is continuous according to Lemma 5.3 and $x_N^*(d) \in \left\{x_1, x_2\right\}$, then $x_N^*(d)$ is constant on the faces of ∂Y. Because $x_M^*(d) \in \partial X$ too, it must also be constant. $\square$

Lemma 4.2 *When d, $y_N^*(d)$ and $y_M^*(d)$ all intersect the same face of ∂Y, the ratio $r_{X,Y}(d)$ is constant.*

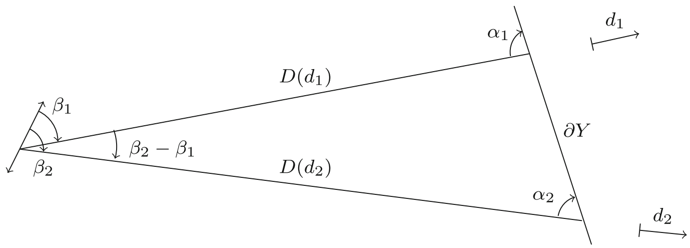


Fig. 3 Ratio $r_{X,Y}(d)$ is constant on a face of ∂Y

Proof Based on Fig. 2, we apply the sine law in the triangle bounded by ∂Y, δ_M and x_M^*

$$\frac{\|x_M^*(d)\|}{\sin\alpha} = \frac{\delta_M(d)}{\sin(\pi-\alpha-\beta)} = \frac{\delta_M(d)}{\sin(\alpha+\beta)}, \quad \text{so} \quad \frac{\delta_M(d)}{D(d)} = \frac{\|x_M^*(d)\|\sin(\alpha+\beta)}{D(d)\sin\alpha}.$$

Similarly for the triangle bounded by ∂Y, δ_N and x_N^*, the law of sines yields

$$\frac{\|x_N^*(d)\|}{\sin\alpha} = \frac{\delta_N(d)}{\sin(\pi-\alpha-\beta)} = \frac{\delta_N(d)}{\sin(\alpha+\beta)}, \quad \text{so} \quad \frac{\delta_N(d)}{D(d)} = \frac{\|x_N^*(d)\|\sin(\alpha+\beta)}{D(d)\sin\alpha}.$$

Even if the two equations above were derived for the specific situation of Fig. 2, they hold as long as y_N^*, D and y_M^* intersect the same face of ∂Y. Based on (3), we have

$$r_{X,Y}(d) = \frac{D(d)+\delta_N(d)}{D(d)-\delta_M(d)} = \frac{1+\frac{\delta_N}{D}}{1-\frac{\delta_M}{D}}. \tag{4}$$

We will now prove that the ratios δ_N/D and δ_M/D do not change on a face of ∂Y. Let $d_1 \in \mathcal{P}\cap\mathbb{S}$ and $d_2 \in \mathcal{P}\cap\mathbb{S}$ such that $D(d_1)$, $D(d_2)$, $y_M^*(d_1)$, $y_M^*(d_2)$, $y_N^*(d_1)$ and $y_N^*(d_2)$ all intersect the same face of ∂Y, as illustrated in Fig. 3.

The sum of the angles of the triangle in Fig. 3 is

$$(\beta_2-\beta_1)+\alpha_2+(\pi-\alpha_1)=\pi \qquad \text{so} \qquad \beta_2+\alpha_2=\beta_1+\alpha_1. \tag{5}$$

Therefore, $\alpha+\beta$ is constant on faces of ∂Y. We use the sine law in the triangle in Fig. 3 and obtain

$$\frac{D(d_1)}{\sin\alpha_2} = \frac{D(d_2)}{\sin(\pi-\alpha_1)} = \frac{D(d_2)}{\sin\alpha_1}, \quad so, \quad D(d_1)\sin\alpha_1 = D(d_2)\sin\alpha_2.$$

According to Lemma 4.1 we also know that $x_N^*(d_1) = x_N^*(d_2)$, thus

$$\frac{\delta_N(d_1)}{D(d_1)} = \frac{\|x_N^*(d_1)\|\sin(\alpha_1+\beta_1)}{D(d_1)\sin\alpha_1} = \frac{\|x_N^*(d_2)\|\sin(\alpha_2+\beta_2)}{D(d_2)\sin\alpha_2} = \frac{\delta_N(d_2)}{D(d_2)}.$$

The same holds for δ_M/D. Hence, (4) yields $r_{X,Y}(d_1) = r_{X,Y}(d_2)$. $\square$

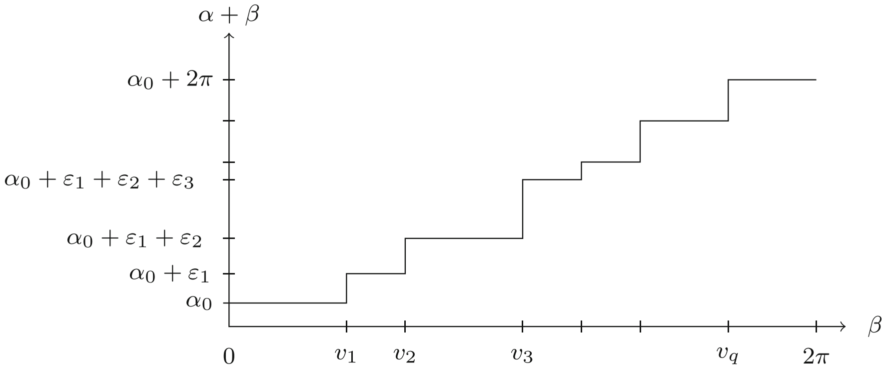


Fig. 4 Evolution of $\alpha + \beta$ with β increasing clockwise in $[0, 2\pi)$

Lemma 4.3 *There are two vertices of $Y \cap \mathcal{P}$, namely v_π and $v_{2\pi}$ whose crossing by d makes the angle $\alpha + \beta$ become greater than π and 2π, respectively.*

Proof We have taken the convention that the angles are positively oriented in the clockwise orientation. According to (5), the angle $\alpha + \beta$ is constant on a face of ∂Y. When d crosses a vertex of external angle ε as represented in Fig. 5, the value of α has a discontinuity of $+\varepsilon$. Let q be the number of vertices of ∂Y and ε_i the external angle of the i^{th} vertex v_i. Since $Y \cap \mathcal{P}$ is a polygon, $\sum_{i=1}^{q} \varepsilon_i = 2\pi$. We can then represent the evolution of $\alpha + \beta$ as a function of β with Fig. 4. Instead of labeling the horizontal axis with the values taken by β as the corresponding vector $d(\beta)$ crosses the vertex v_i, we directly use v_i with a slight abuse of notation.

Recall that α_0 is the value of α when $\beta = 0$. After a whole revolution $\alpha + \beta = \alpha_0 + 2\pi$. So there are two vertices v_π and $v_{2\pi}$ where $\alpha + \beta$ first crosses π and then 2π. In the eventuality that $\alpha + \beta = \pi$ or 2π on a face of ∂Y, we define v_π or $v_{2\pi}$ as the vertex preceding the face. □

Lemma 4.4 *During the crossing of vertices before v_π as β increases, $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. They are constant, different and both belong in ∂X.*

Proof We study the crossing of a vertex v of angle ε between the faces F_1 and F_2 of ∂Y. For each vertex v, we introduce x_v the vector collinear with X, going from v to the ray directed by d, as illustrated in Fig. 5 and we say that the crossing of v is ongoing as long as $\|x_v\| < \max\{\|x_1\|, \|x_2\|\}$. We also define $\delta_v := \|v + x_v\| - D$.

Before starting the crossing of v_π, we have $\alpha + \beta \in (\alpha_0, \pi)$. This situation is depicted in Fig. 2, where y_N^* is leading and outside, so y_N^* reaches the vertex before y_M^* and d. The length of $x_N^*(d)$ can vary to maximize δ_N, so y_N^* could still intersect F_1, even if the crossing is ongoing. We have seen in Lemma 4.1 that if y_N^* is still on F_1, then it must be the furthest possible to maximize δ_N, in that case $y_N^* = v$. Otherwise, y_N^* intersects F_2. We want to establish a criterion to distinguish these two possible scenarios.

We first consider the scenario where $y_N^* = v$ and $x_N^*(d) = x_v$. We take $y \in F_2 \backslash \{v\}$ such that $x_2 + y \in \mathbb{R}^+ d$ as represented in Fig. 6 and we define $\delta := \|x_2 + y\| - D$.

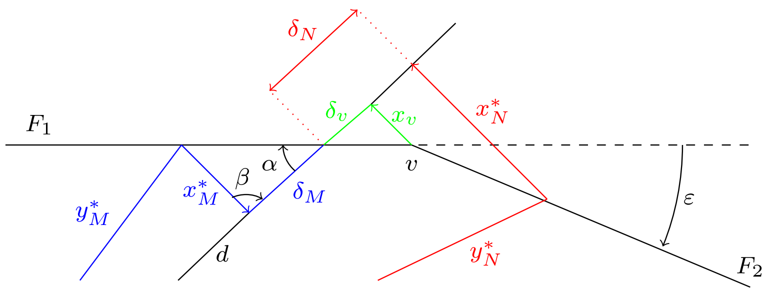


Fig. 5 Illustration of x_v during the crossing of a vertex v, with y_N^* leading

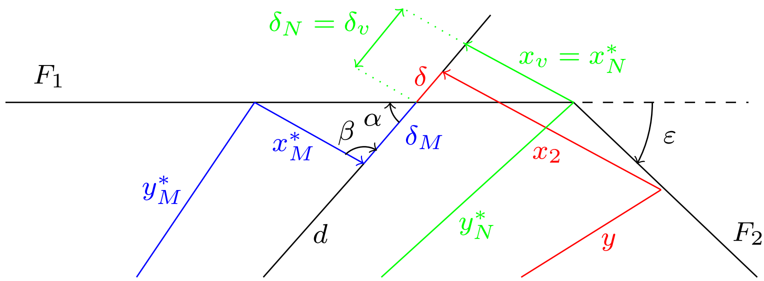


Fig. 6 Illustration of the crossing scenario where $y_N^* = v$

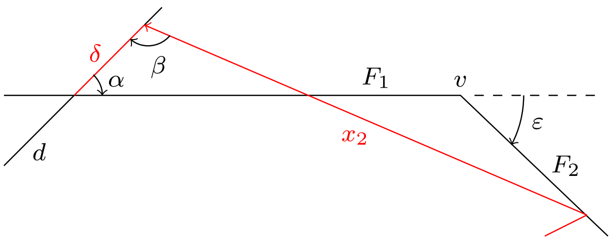


Fig. 7 Illustration of the line segment corresponding to x_2 crossing the interior of Y in Fig. 6

Since δ_N must be maximized by the choice of y_N^* and $y \neq y_N^*$, we have $\delta < \delta_N = \delta_v$. But $\|x_2\| > \|x_v\|$, so the line segment corresponding to x_2 crosses the interior of Y. Focusing on this part of Fig. 6, we obtain Fig. 7.

Two of the angles of the triangle delimited by F_1, F_2 and x_2 are $\pi - \alpha - \beta$ and $\pi - \varepsilon$. Therefore, their sum is in $(0, \pi)$ and thus $\alpha + \beta + \varepsilon > \pi$. Since we assumed that $\alpha + \beta \in (\alpha_0, \pi)$, the vertex v must in fact be v_π for this scenario to happen.

Thus, the crossing of a vertex preceding v_π follows the second scenario as depicted in Fig. 5 with $y_N^* \in F_2$. We study Fig. 8 which is a more detailed view of Fig. 5, with δ_0 depending solely on d and ε.

Since x_v and $x_N^*(d)$ are collinear, we can apply Thales's theorem in Fig. 8 and obtain that $\delta_N - \delta_0 = (\delta_v - \delta_0)\frac{\|x_N^*(d)\|}{\|x_v(d)\|}$. Then, δ_N is maximized when $\|x_N^*(d)\|$ is maximal, so $x_N^*(d) = x_2$ during the crossing. We know from Theorem 2.1 that $x_M^*(d) \in \partial X$ for

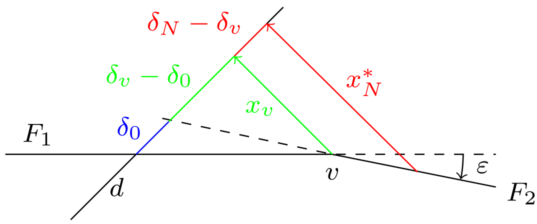


Fig. 8 Illustration of x_v and x_N^* in Fig. 5

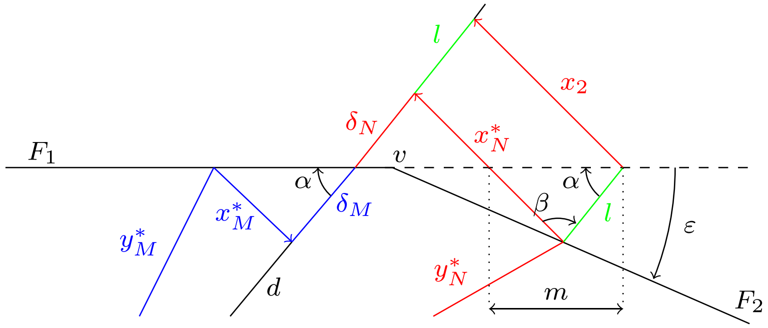


Fig. 9 Part I of the crossing of vertex v by y_N^* leading and outside as β increases

all $d \in \mathbb{S}$. Then, as in Lemma 4.1, x_N^* and x_M^* are constant and different since x_N^* is continuous in d, so $x_M^*(d) = x_1$. □

Lemma 4.5 *During the crossing of vertices before v_π as β increases, $r_{X,Y}(d)$ decreases.*

Proof The leading vector y_N^* is outside and crosses a vertex v between the faces F_1 and F_2 of ∂Y while β increases. We separate the vertex crossing into two parts: when only $y_N^* \in F_2$, and when both $d \in F_2$ and $y_N^* \in F_2$. Let $\varepsilon > 0$ be the external angle of the vertex as shown on Fig. 9.

According to Lemma 4.2, $r_{X,Y}$ is constant on faces of ∂Y and we call r_{F_1} its value on the face F_1. If F_1 was prolonged past v with a straight line (dashed line in Fig. 9), then we would have $y_N^*(d) \in F_1$ and $r_{X,Y}(d) = r_{F_1}$. But, $y_N^*(d) \in F_2$ as proven in Lemma 4.4 because the crossing occurs before v_π. We call l the resulting difference in δ_N as illustrated in Fig. 9. Notice that the two green segments of length l in Fig. 9 are parallel. We parametrize the position of y_N^* on F_2 with the length m as defined in Fig. 9. When $y_N^* = v$, $m = 0$, and m increases with β. Using the sine law, we obtain

$$\frac{m}{\sin\beta} = \frac{l}{\sin(\pi-\alpha-\beta)} = \frac{l}{\sin(\alpha+\beta)}. \tag{6}$$

Then,

$$r_{X,Y}(d) = \frac{D+\delta_N}{D-\delta_M} = \frac{D+\delta_N+l}{D-\delta_M} - \frac{l}{D-\delta_M} = r_{F_1} - \frac{m\sin(\alpha+\beta)}{(D-\delta_M)\sin(\beta)}. \tag{7}$$

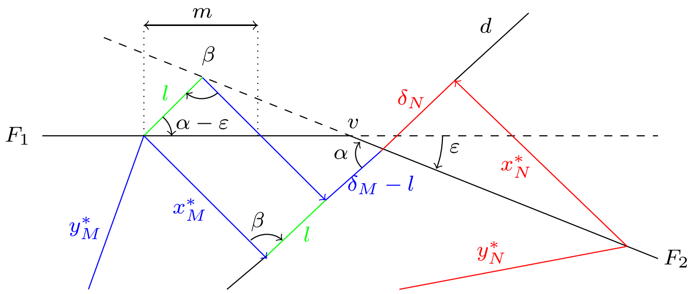


Fig. 10 Part II of the crossing of vertex v by y_N^* leading and outside as β increases

By definition, the length m is positive. Since $-x_M^* \in Y^\circ$ but $y_M^* \in \partial Y$, we have $D - \delta_M = \|y_M^* + x_M^*\| > 0$. Before v_π we have $\alpha + \beta \in (\alpha_0, \pi)$. In that case $\sin(\alpha + \beta) > 0$ and $\sin(\beta) > 0$. Therefore, the term subtracted from r_{F_1} is positive, i.e., $r_{X,Y}(d) < r_{F_1}$.

We can now tackle the second part of the crossing, when y_N^* and d both have crossed the vertex as illustrated in Fig. 10.

If F_2 was prolonged with a straight line before v and $y_M^* \in F_2$, then we would have $r_{X,Y}(d) = r_{F_2}$, value of $r_{X,Y}$ on F_2. But that is not the case, $y_M^*(d) \in F_1$ and the resulting difference in δ_M is called l. Using the sine law in Fig. 10, we can relate l to m

$$\frac{m}{\sin\beta} = \frac{l}{\sin(\pi - \beta - \alpha + \varepsilon)} = \frac{l}{\sin(\alpha + \beta - \varepsilon)}. \tag{8}$$

We have $\alpha + \beta \in (\alpha_0, \pi)$, so $\sin(\beta) > 0$. If α was still measured between d and F_1, then its value would be $\alpha_{F_1} = \alpha - \varepsilon$. Since we are before the crossing of v_π, $\alpha_{F_1} + \beta \in (\alpha_0, \pi)$, i.e., $\alpha + \beta - \varepsilon \in (\alpha_0, \pi)$. This yields $\sin(\alpha + \beta - \varepsilon) > 0$, which makes $l > 0$, because the length m is positive by definition. Then,

$$r_{F_2} = \frac{D + \delta_N}{D - (\delta_M - l)} = \frac{D + \delta_N}{D - \delta_M + l} < \frac{D + \delta_N}{D - \delta_M} = r_{X,Y}(d). \tag{9}$$

Thus, the ratio $r_{X,Y}$ decreases during the crossing of a vertex before v_π. □

Lemma 4.6 *During the crossing of v_π, the ratio $r_{X,Y}(d)$ reaches a local minimum.*

Proof Recall that before the crossing, $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. During the crossing of v_π, i.e., when $\|x_{v_\pi}\| < \max\{\|x_1\|, \|x_2\|\}$, we have $\alpha + \beta \leq \pi$ but $\alpha + \beta + \varepsilon > \pi$. The situation is illustrated in Fig. 11. We showed in Lemma 4.4 that $y_N^* = v_\pi$ and $x_N^*(d) = x_{v_\pi}$.

If F_1 was prolonged with a straight line (dashed line of Fig. 11), we would have $y_N^* \neq v_\pi$, $x_N^*(d) = x_2$ and the ratio would be $r_{F_1} = \frac{D+\delta_{v_\pi}+l}{D-\delta_M}$, which is the value of $r_{X,Y}$ on F_1. Since d has not yet crossed v_π, $\alpha + \beta < \pi$ and thus (6) and (7) still hold, leading to $r_{X,Y}(d) < r_{F_1}$.

Once d has crossed v_π, we still have $y_N^* = v_\pi$ to maximize δ_N. Then, the equality $x_N^*(d) = x_{v_\pi}$ holds during the whole crossing, i.e., as x_{v_π} goes from x_2 to x_1. The second part of the crossing is illustrated in Fig. 12.

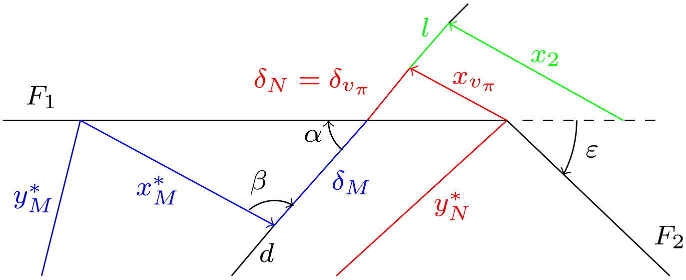


Fig. 11 Crossing of v_π, with $y^*_N = v_\pi$

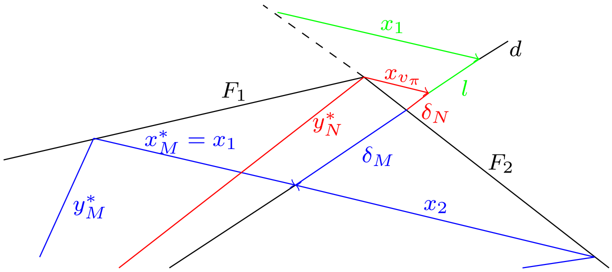


Fig. 12 Illustration of the endpoint of y^*_M switching from F_1 to F_2 during the crossing of v_π

Assume that during the entire crossing of v_π, $x^*_M(d) = x_1$. Then, at the end of the crossing we will have $y^*_M = v_\pi$ and $x^*_M(d) = x_{v_\pi} = x^*_N(d)$, which contradicts the definitions of $x^*_M(d)$ and $x^*_N(d)$, they must be different. Thus, $x^*_M(d)$ does not remain equal to x_1 during the entire crossing. Since $x^*_M \in \{x_1, x_2\}$, at some point x^*_M switches to x_2 as y^*_M switches from F_1 to F_2. This switching point is illustrated in Fig. 12, and y^*_M becomes the leading vector.

After this switch, $y^*_M \in F_2$ and $x^*_M(d) = x_2$. If F_2 was prolonged with the dashed line in Fig. 12, we would have $x^*_N = x_1$ instead of x_{v_π} with a gain of l for δ_N making the ratio equal to $r_{F_2} = \frac{D+\delta_N+l}{D-\delta_M}$, value of $r_{X,Y}$ on F_2. But $x^*_N = x_{v_\pi}$ and $l > 0$, thus $r_{F_2} > \frac{D+\delta_N}{D-\delta_M} = r_{X,Y}(d)$. Therefore, $r_{X,Y}$ reaches a local minimum during the crossing of v_π. □

Lemma 4.7 *During the crossing of vertices after v_π as β increases until π, $x^*_N(d) = x_1$ and $x^*_M(d) = x_2$. They are constant different and both in ∂X.*

Proof After the crossing of v_π, $\alpha + \beta \in (\pi, \alpha_0 + \pi)$ and y^*_M is leading and inside as established in Lemma 4.6. Thus, y^*_M is the first to reach vertex v. Since $x^*_M \in \{x_1, x_2\}$, we cannot have $x^*_M = x_v$ during the entire crossing because x_v is a continuous function of β. Thus y^*_M passes v and belongs to F_2. In Lemma 5.3, we showed that x^*_N is continuous in d. Thus, $x^*_N(d)$ cannot switch like $x^*_M(d)$ did around v_π to take the lead. Instead, $x^*_N(d)$ is trailing as illustrated in Fig. 13.

Since $y^*_N \in F_1$ during the crossing, we can apply Thales's theorem in Fig. 13 and obtain that for a fixed d, δ_N is proportional to $\|x^*_N(d)\|$. Thus, to maximize δ_N we have $x^*_N(d) \in \partial X$ and, since y^*_N is trailing, we have $x^*_N(d) = x_1$ during the entire crossing.

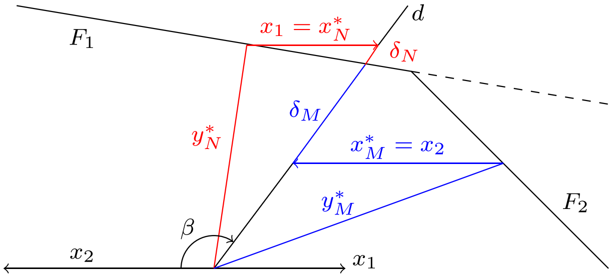


Fig. 13 Crossing of a vertex v after v_π

By the definitions of $x_N^*(d)$ and $x_M^*(d)$, we have $x_N^*(d) \neq x_M^*(d)$. Since both $x_N^*(d)$ and $x_M^*(d)$ belong to $\partial X = \{x_1, x_2\}$, then $x_M^*(d) = x_2$ during the entire crossing. □

Lemma 4.8 *During the crossing of vertices after v_π as β increases until π, $r_{X,Y}(d)$ increases.*

Proof The leading vector y_M^* is inside and crosses a vertex v between faces F_1 and F_2 as β increases. We define $\beta' := \pi - \beta$. Then, reversing the crossing illustrated in Fig. 13 is exactly the crossing illustrated in Fig. 9 with β' increasing and x_1 and x_2 exchanged. According to Lemma 4.5, in that reversed crossing $r_{X,Y}$ is decreasing. Therefore, $r_{X,Y}$ increases during the crossing of vertices after v_π as β increases until π. □

Lemma 4.9 *For $\beta > \pi$, $r_{X,Y}(d)$ decreases until $v_{2\pi}$ where it reaches a local minimum. After $v_{2\pi}$ as β increases until 2π, $r_{X,Y}(d)$ increases.*

Proof Let us change the angle convention so that angles are now positively oriented in the counterclockwise orientation. The vertex that was previously labeled as $v_{2\pi}$ becomes the new v_π. Then, we only need to apply Lemmas 4.4, 4.5, 4.6, 4.7 and 4.8 to this new configuration to conclude the proof. □

Lemma 4.10 *All above results hold even if $0 \notin X$.*

Proof In all the figures, we made the implicit assumption that $0 \in X$, so that x_1 and x_2 were negatively collinear. Let x_1 be positively collinear with x_2 and $\|x_2\| > \|x_1\|$.

In Fig. 2, we would now have $y_N^*(d)$ and $y_M^*(d)$ both outside. Then, the definition of δ_M should be adapted. Let $\delta_M(d) := \|x_M^*(d) + y_M^*(d)\| - D(d)$ and then $r_{X,Y}(d) = \frac{D+\delta_N}{D+\delta_M}$. Except for this modification, we would still have $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$. Thales theorem can be used similarly to show that $x_N^*(d) \in \partial X$. Therefore, Lemma 4.1 holds.

In the proof of Lemma 4.2, we still have δ_N/D and δ_M/D invariant with respect to d on a given face of ∂Y, so $r_{X,Y}$ is still constant on faces. Lemma 4.3 is not affected at all. The first part of the crossing of a vertex before v_π as β increases is illustrated by Fig. 14.

For δ_M to be minimized and δ_N to be maximized, the Thales theorem clearly proves that $x_M^* \in \partial X$ and $x_N^* \in \partial X$ during the crossing. We still have $x_N^*(d) = x_2$ and $x_M^*(d) = x_1$, so Lemma 4.4 holds.

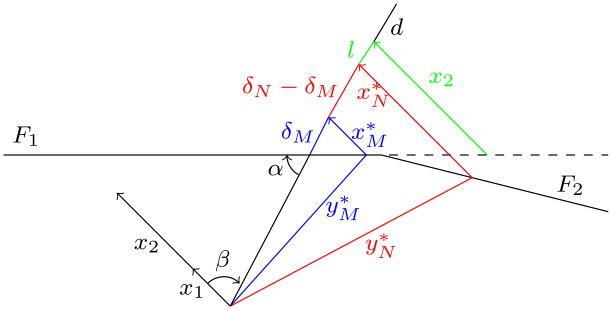
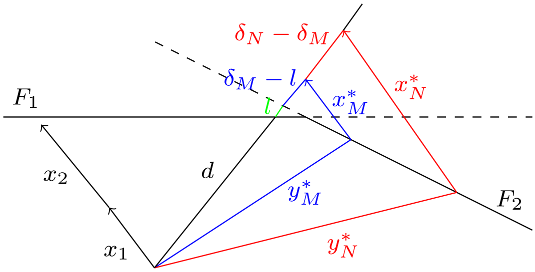


Fig. 14 Part I of the crossing of a vertex before v_π with $0 \notin X$

Fig. 15 Part II of the crossing of a vertex before v_π with $0 \notin X$



Following the reasoning in Lemma 4.5, we have $l > 0$, which leads to

$$r_{F_1} = \frac{D + \delta_N + l}{D + \delta_M} > \frac{D + \delta_N}{D + \delta_M} = r_{X,Y}(d).$$

During the second part, both $y_N^* \in F_2$ and $y_M^* \in F_2$ but $d \in F_1$. This situation is illustrated in Fig. 15.

We compare the current value of $r_{X,Y}(d)$ with r_{F_2}, its value on F_2:

$$r_{F_2} = \frac{D + (\delta_N - l)}{D + (\delta_M - l)} \qquad \text{and} \qquad r_{X,Y}(d) = \frac{D + \delta_N}{D + \delta_M}.$$

Since $l > 0$ and $\delta_N > \delta_M$, a simple calculation shows that $r_{X,Y}(d) < r_{F_2}$. Therefore, $r_{X,Y}$ is decreasing during the crossing of a vertex before v_π as β increases, Lemma 4.5 holds.

During the crossing of v_π, $y_N^* = v_\pi$ and $x_N^* = x_{v_\pi}$ with its norm decreasing continuously until $x_N^* = x_1$, while x_M^* will switch to x_2 in order to minimize δ_M. This is the same process as described in Lemma 4.6, so $r_{X,Y}$ also reaches a local minimum.

Because all the results studied so far still hold, Lemmas 4.7, 4.8 and 4.9 hold too because they rely on those earlier results. □

We have now established all the lemmas directly involved in the proof of the Maximax Minimax Quotient Theorem, but we still have a few claims of continuity to prove.

5 Continuity of Extrema

In Proposition 2.1 (iii), we needed the continuity of λ^* to prove it has a minimum, and in Lemma 4.1 we used the continuity of x_N^* and y_N^*. In this section, we will thus prove the continuity of these two maxima functions relying on the Berge Maximum Theorem [2].

Lemma 5.1 *Let X and Y be two nonempty polytopes in $\mathbb{R}^n$ with $-X \subset Y$. Then, the set-valued function $\varphi : X \times \mathbb{S} \rightrightarrows Y$ defined as $\varphi(x, d) := Y \cap \{\lambda d - x : \lambda \geq 0\}$ satisfies Definition 17.2 of [2].*

Proof We define $\Omega := X \times \mathbb{S}$, so that $\varphi : \Omega \rightrightarrows Y$. On the space Ω, we introduce the norm $\|\cdot\|_\Omega$ as $\|(x, d)\|_\Omega = \|x\| + \|d\|$. Since $\|\cdot\|$ is the Euclidean norm, $\|\cdot\|_\Omega$ is a norm on Ω. By Definition 17.2 of [2], we need to prove that φ is both upper and lower hemicontinuous at all points of Ω.

First, using Lemma 17.5 of [2] we will prove that φ is lower hemicontinuous by showing that for an open subset A of Y, $\varphi^l(A)$ is open. The lower inverse image of A is defined in [2] as

$$\begin{aligned}\varphi^l(A) &:= \{\omega \in \Omega : \varphi(\omega) \cap A \neq \emptyset\} \\ &= \{(x, d) \in X \times \mathbb{S} : Y \cap \{\lambda d - x : \lambda \geq 0\} \cap A \neq \emptyset\} \\ &= \{(x, d) \in X \times \mathbb{S} : \{\lambda d - x : \lambda \geq 0\} \cap A \neq \emptyset\},\end{aligned}$$

because $A \subset Y$. Let $\omega = (x, d) \in \varphi^l(A)$. Then, there exists $\lambda \geq 0$ such that $\lambda d - x \in A$. Since A is open, there exists $\varepsilon > 0$ such that the ball $B_\varepsilon(\lambda d - x) \subset A$. Now let $\omega_1 = (x_1, d_1) \in \Omega$ and denote $\varepsilon_x := \|x_1 - x\|$ and $\varepsilon_d := \|d_1 - d\|$. Then,

$$\|\lambda d_1 - x_1 - (\lambda d - x)\| = \|\lambda(d_1 - d) - (x_1 - x)\| \leq \lambda\varepsilon_d + \varepsilon_x.$$

Since $\lambda \geq 0$ is fixed, we can choose ε_d and ε_x positive and small enough so that $\lambda\varepsilon_d + \varepsilon_x \leq \varepsilon$. Then, we have showed that for all $\omega_1 = (x_1, d_1) \in \Omega$ such that $\|\omega - \omega_1\|_\Omega \leq \min(\varepsilon_d, \varepsilon_x)$, i.e., such that $\|x_1 - x\| \leq \varepsilon_x$ and $\|d_1 - d\| \leq \varepsilon_d$, we have $\lambda d_1 - x_1 \in B_\varepsilon(\lambda d - x) \subset A$, i.e., $\omega_1 \in \varphi^l(A)$. Therefore, $\varphi^l(A)$ is open, and so φ is lower hemicontinuous.

To prove the upper hemicontinuity of φ, we will use Lemma 17.4 of [2] and prove that for a closed subset A of Y, the lower inverse image of A is closed. Let $\{\omega_k\}$ be a sequence in $\varphi^l(A)$ converging to $\omega = (x, d) \in \Omega$. We want to prove that the limit $\omega \in \varphi^l(A)$.

For $k \geq 0$, we have $\omega_k = (x_k, d_k)$ and define $\Lambda_k := \{\lambda_k \geq 0 : \lambda_k d_k - x_k \in A\} \neq \emptyset$. Since A is a closed subset of the compact set Y, then A is compact. Thus Λ_k has a minimum and a maximum; we denote them by λ_k^{min} and λ_k^{max} respectively.

Since sequences $\{d_k\}$ and $\{x_k\}$ converge, they are bounded. The set A is also bounded, and thus sequence $\{\lambda_k^{max}\}$ is bounded. Let $\lambda^{max} := \sup_{k \geq 0} \lambda_k^{max} > 0$.

For $k \geq 0$, we define segments $S_k := \{\lambda d_k - x_k : \lambda \in [0, \lambda^{max}]\}$, and $S := \{\lambda d - x : \lambda \in [0, \lambda^{max}]\}$. These segments are all compact sets. We also introduce the sequences $a_k := \lambda_k^{min} d_k - x_k \in A \cap S_k$ and $b_k := \lambda_k^{min} d - x \in S$.

Take $\varepsilon > 0$. Since sequences $\{d_k\}$ and $\{x_k\}$ converge toward d and x respectively, there exists $N \geq 0$ such that for $k \geq N$, we have $\|d_k - d\| \leq \frac{\varepsilon}{2\lambda^{max}}$ and $\|x_k - x\| \leq \frac{\varepsilon}{2}$. Then, for any $\lambda_k \in [0, \lambda^{max}]$ as

$$\|\lambda_k d_k - x_k - (\lambda_k d - x)\| = \|\lambda_k(d_k - d) - (x_k - x)\| \leq \lambda_k \frac{\varepsilon}{2\lambda^{max}} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Since $\lambda_k^{min} \in [0, \lambda^{max}]$, we have $\|a_k - b_k\| \xrightarrow[k\to\infty]{} 0$. We define the distance between the sets A and S

$$dist(A, S) := \min\left\{\|a - s_\lambda\| : a \in A,\ s_\lambda \in S\right\}.$$

The minimum exists because A and S are both compact and the norm is continuous. Since $a_k \in A$ and $b_k \in S$, we have $dist(A, S) \leq \|a_k - b_k\|$ for all $k \geq 0$. Therefore, $dist(A, S) = 0$. So, $A \cap S \neq \emptyset$, leading to $\omega \in \varphi^l(A)$. Then, $\varphi^l(A)$ is closed and so φ is upper hemicontinuous. □

Lemma 5.2 *Let X and Y be two nonempty polytopes in $\mathbb{R}^n$ with $-X \subset Y$. Then, $\lambda^*(x, d) := \max_{y\in Y}\left\{\|x + y\| : x + y \in \mathbb{R}^+ d\right\}$ is continuous in $x \in X$ and $d \in \mathbb{S}$.*

Proof According to Proposition 2.1 (ii), whose proof does not rely on the current lemma, λ^* is well defined. We introduce the set-valued function $\varphi : X \times \mathbb{S} \rightrightarrows Y$ defined by $\varphi(x, d) := \left\{y \in Y : x + y \in \mathbb{R}^+ d\right\} = Y \cap \left(\mathbb{R}^+ d - \{x\}\right)$, where $\mathbb{R}^+ d - \{x\} = \left\{\lambda d - x : \lambda \geq 0\right\}$.

We define the graph of φ as $\mathrm{Gr}\,\varphi := \left\{(x, d, y) \in X \times \mathbb{S} \times Y : y \in \varphi(x, d)\right\}$, and the continuous function $f : \mathrm{Gr}\,\varphi \to \mathbb{R}^+$ as $f(x, d, y) = \|x + y\|$. Set $X \times \mathbb{S}$ is compact and nonempty. Since Y is compact and $\mathbb{R}^+ d - \{x\}$ is closed, their intersection $\varphi(x, d)$ is compact. Because $-X \subset Y$, for all $x \in X$ we have $-x \in \varphi(x, d)$, so $\varphi(x, d) \neq \emptyset$. According to Lemma 5.1, φ satisfies Definition 17.2 of [2]. Then, we can apply the Berge Maximum Theorem [2] and conclude that λ^* is continuous in x and d. □

Lemma 5.3 *Let X and Y be two nonempty polytopes in $\mathbb{R}^n$ with $-X \subset Y$. Then, the functions $\left(x_N^*, y_N^*\right)(d) = \arg\max_{x\in X,\, y\in Y}\left\{\|x + y\| : x + y \in \mathbb{R}^+ d\right\}$ are continuous in $d \in \mathbb{S}$.*

Proof Let $Z := X + Y = \left\{x + y : x \in X,\ y \in Y\right\}$. Then Z is the Minkowski sum of two polytopes, so it is also a polytope [11]. According to Proposition 2.1 (i), whose proof does not rely on the current lemma, $\max_{x\in X,\, y\in Y}\left\{\|x + y\| : x + y \in \mathbb{R}^+ d\right\}$ exists and thus $\max_{z\in Z}\left\{\|z\| : z \in \mathbb{R}^+ d\right\}$ is also well defined.

Since $-X \subset Y$, for all $x \in X$, $-x \in Y$ and thus $0 \in Z$. Then, $\{0\}$ and Z are two polytopes in $\mathbb{R}^n$ with $\pm 0 \in Z$. According to Lemma 5.2, the function $\lambda^*(0, d) := \max_{z\in Z}\left\{\|z + 0\| : z + 0 \in \mathbb{R}^+ d\right\}$ is continuous in $d \in \mathbb{S}$.

Then, we define the continuous function $z(d) := \lambda^*(0, d)d \in Z$ for $d \in \mathbb{S}$. Note that $z(d) = \arg\max_{z\in Z}\left\{\|z\| : z \in \mathbb{R}^+ d\right\} = \left(x_N^*, y_N^*\right)(d)$, so these functions are continuous. □

6 Illustration

We will now illustrate the Maximax Minimax Quotient Theorem on a simple example. We consider polygon X delimited by the vertices $x_1 = (0, -0.5)$ and $x_2 = (0, 1)$ in $\mathbb{R}^2$ and polygon Y with vertices $(\pm 1, \pm 2)$ and $(\pm 3, 0)$ as represented on Fig. 16.

Since $-X \subset Y^\circ$, $\dim X = 1$, $x_2 \neq 0$ and $\dim Y = 2$, the assumptions of the Maximax Minimax Quotient Theorem are satisfied. To illustrate the proof of the theorem, for all $d \in \mathbb{S}$ we define the angle $\beta := \widehat{x_2, d}$ positively oriented clockwise. We also enumerate the vertices in the clockwise direction and we note that $v_2 = v_\pi$ and $v_5 = v_{2\pi}$ as defined in Lemma 4.3. Then, we compute $r_{X,Y}$ for $\beta \in [0, 2\pi)$ as shown in Fig. 17. The red spikes denote when the ray $d(\beta)$ hits a vertex of Y.

As demonstrated by the Maximax Minimax Quotient Theorem, $r_{X,Y}$ has two local maxima achieved at $\beta = 0$ and $\beta = \pi$. These two values are different because polygon X is not symmetric. Note also that the Maximax Minimax Quotient Theorem does not state that the maximum is *only* reached when $\beta \in \{0, \pi\}$. Indeed as shown in Fig. 17 and established in Lemma 4.2, $r_{X,Y}$ is constant on the faces of ∂Y. Thus, the two local maxima are achieved on the faces $[v_1, v_6]$ and $[v_3, v_4]$. As proven in Lemma 4.6 and

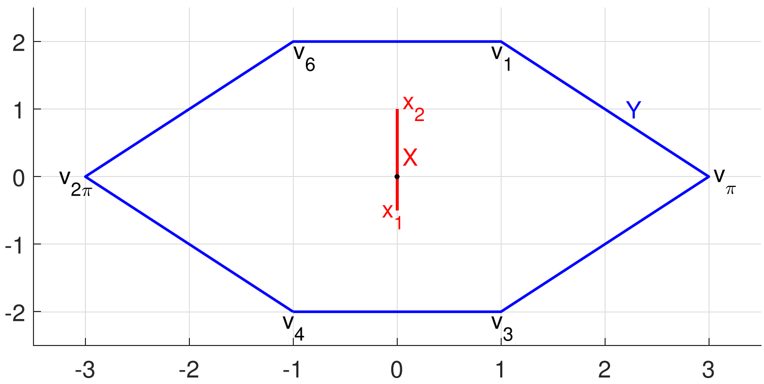


Fig. 16 Illustration of polygons X and Y

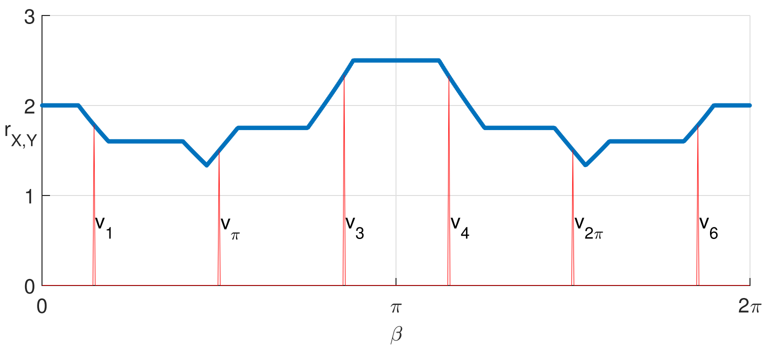


Fig. 17 Graph of $r_{X,Y}$ as a function of β

in Lemma 4.9, $r_{X,Y}$ reaches a local minimum during the crossing of the vertices v_π and $v_{2\pi}$.

7 Conclusion

In this paper, we considered an optimization problem arising from optimal control and pertaining to both fractional programming and max-min programming. We first justified the existence of the Maximax Minimax Quotient. Then, relying on numerous geometrical arguments and on the continuity of two maxima functions we were able to establish the Maximax Minimax Quotient Theorem. This result provides an analytical solution to the maximization of a ratio of a maximum and a minimax over two polytopes. We illustrated our theorem and its proof on a simple example in $\mathbb{R}^2$. This work filled the theoretical gap left in [3], and because of our less restrictive assumptions, we also open the way for a more general framework than that of [3]. A possible avenue for future work on this theorem is to study the case where $\dim X > 1$.

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