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Quantitative Resilience of Linear Systems*

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Abstract— **Actuator malfunctions may have disastrous consequences for systems not designed to mitigate them. We focus on the loss of control authority over actuators, where some actuators are uncontrolled but remain fully capable. To counteract the undesirable outputs of these malfunctioning actuators, we use real-time measurements and redundant actuators. In this setting, a system that can still reach its target is deemed resilient. To quantify the resilience of a system, we compare the shortest time for the undamaged system to reach the target with the worst-case shortest time for the malfunctioning system to reach the same target, i.e., when the malfunction makes that time the longest. Contrary to prior work on driftless linear systems, the absence of analytical expression for time-optimal controls of general linear systems prevents an exact calculation of quantitative resilience. Instead, relying on Lyapunov theory we derive analytical bounds on the nominal and malfunctioning reach times in order to bound quantitative resilience. We illustrate our work on a temperature control system.**

I. INTRODUCTION

Redundancy is the most effective remedy to actuator malfunctions [1], but it is also the most onerous, which leads to our questions of interest: can the system still reach its target even after an actuator malfunction? If so, how much is it delayed? We focus on the malfunction consisting in the *loss of control authority over actuators* [2]. These malfunctioning actuators do not respond to the controller, but they produce uncontrolled and possibly undesirable inputs within their full range of actuation. A system is *resilient* if it can reach its target despite such a malfunction [3].

Robust control generally cannot synthesize an appropriate controller because we consider undesirable inputs that can have the same magnitude as the controls [3], [4]. To identify and counteract faulty actuators, sensors measure in real-time the outputs produced by each actuator [1].

The resilience of systems has first been studied in [5] for unbounded inputs. The work [6] shows that bounding the inputs levies a death sentence to the simple *max-min controllability* condition of [5]. Similarly, the approaches of [6]–[8] all lead to overly complex conditions for resilience of control systems. A simpler approach comes from differential games with Hájek's duality theorems [9]. Based on these results and with the controllability conditions of [10], we establish straightforward resilience conditions for linear systems.

However, resilience only guarantees reachability despite the malfunction. It does not say how much delay is caused by the malfunction. To measure this delay, *quantitative resilience* was introduced for driftless linear systems in [11], [12] as the maximal ratio over all targets of the minimal reach times for the nominal and malfunctioning systems. However, the geometric approach developed for those works [13] does not translate to general linear systems.

To calculate quantitative resilience, we first need to determine the fastest time for a linear system to reach its target. Because this time has no general analytical expression [14], research focused on algorithms [14]–[16] relying on Pontryagin's Maximum Principle [17]. We also need the fastest time for the malfunctioning system to reach its target under the undesirable input making that time the longest. Since initial works focused on generic perturbations [18], [19], we prefer the differential games approach [20], as it considers specifically the worst undesirable input.

Using these algorithms, we can evaluate the ratio of minimal times for the nominal and malfunctioning systems to reach a target x_{goal} from an initial state x_0. However, calculating quantitative resilience by testing every possible pair (x_0, x_{goal}) is not meaningful. Instead, we use Lyapunov theory [21] to establish analytical bounds on the minimal reach times and approximate quantitative resilience.

The main contributions of this work are twofold. Firstly, we establish simple necessary and sufficient conditions to verify the resilience of linear systems. Secondly, relying on Lyapunov theory, we establish analytical bounds on the quantitative resilience of linear systems.

The remainder of this paper is organized as follows. Section II states the two problems of interest. In Section III, building on previous work, we derive necessary and sufficient resilience conditions. In Section IV, we establish analytical bounds on the quantitative resilience of linear systems. We apply our work on a temperature control system in Section V.

Notation: For a set $X \subseteq \mathbb{R}^n$, $co(X)$ denotes its convex hull, ∂X its boundary and X° its interior. For a set $Z \subseteq \mathbb{C}$, we say that $Re(Z) \leq 0$ (resp. $Re(Z) = 0$) if the real part of each $z \in Z$ verifies $Re(z) \leq 0$ (resp. $Re(z) = 0$). The set of eigenvalues of a matrix A is $\lambda(A)$. If $\lambda(A) \subseteq \mathbb{R}$, then λ_{min}^A and λ_{max}^A are the minimal and maximal eigenvalues. The nullspace and image of A are $\ker(A)$ and $\mathrm{Im}(A)$. Positive definite matrix P is denoted by $P \succ 0$ and defines a *P-norm* as $\|x\|_P^2 := x^\top P x$. The closed ball of the Euclidean norm $\|\cdot\|_2$ of radius $\varepsilon > 0$ and center $c \in \mathbb{R}^n$ is denoted by $\mathbb{B}_\varepsilon(c)$, and the unit sphere is $\mathbb{S} := \partial \mathbb{B}_1(0)$. For $x \in \mathbb{R}^n$, $\|x\|_\infty = \max |x_i|$. The function $u : [0, +\infty) \to \mathcal{U}$ is alternatively denoted by $u(\cdot) \in \mathcal{U}$. If $B \in \mathbb{R}^{n \times m}$ and $\mathcal{U}$ is a set, then $B\mathcal{U} := \{Bu : u \in \mathcal{U}\}$. We also use $\mathbf{1} = (1, \dots, 1)$ to denote the vector of 1.

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II. PROBLEM STATEMENT

We consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + \bar{B}\bar{u}(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad \bar{u}(t) \in \bar{\mathcal{U}}, \qquad (1)$$

with $A \in \mathbb{R}^{n\times n}$ and $\bar{B} \in \mathbb{R}^{n\times(m+p)}$ constant matrices and $\bar{\mathcal{U}}$ an hyperreactangle of $\mathbb{R}^{m+p}$. After a loss of control authority over p of the $m+p$ actuators, the input signal $\bar{u}(\cdot)$ is split between the undesirable inputs $w(\cdot) \in \mathcal{W}$ and the controlled inputs $u(\cdot) \in \mathcal{U}$, with $\mathcal{U} \subseteq \mathbb{R}^m$ and $\mathcal{W} \subseteq \mathbb{R}^p$ such that $\mathcal{U}\times\mathcal{W} = \bar{\mathcal{U}}$. Matrix $\bar{B}$ is accordingly split into two constant matrices $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times p}$ so that the dynamics become

$$\dot{x}(t) = Ax(t) + Bu(t) + Cw(t), \quad x(0) = x_0 \in \mathbb{R}^n. \qquad (2)$$

Definition 1: A target $x_{goal} \in \mathbb{R}^n$ is *resiliently reachable* from $x_0 \in \mathbb{R}^n$ by system (2) if for all piecewise continuous $w(\cdot) \in \mathcal{W}$, there is $T > 0$ and piecewise continuous $u(\cdot) \in \mathcal{U}$ such that the solution to (2) exists, is unique and $x(T) = x_{goal}$.

Definition 2: System (2) is *resilient* to the loss of the actuators corresponding to C if every $x_{goal} \in \mathbb{R}^n$ is resiliently reachable from any $x_0 \in \mathbb{R}^n$ by system (2).

We are now led to our first problem.

Problem 1: Determine whether system (2) is resilient.

If system (2) is indeed resilient, it can reach any target despite malfunctions, but it might need an excessively long time, making its resilience useless in practice. To avoid this scenario, previous work [11] established a metric quantifying the resilience of linear driftless systems. We now generalize this metric to any control system. For a target $x_{goal} \in \mathbb{R}^n$, the *nominal reach time* introduced in [11] becomes

$$T_N^*(x_0, x_{goal}) := \inf_{\bar{u}(\cdot)\in\bar{\mathcal{U}}} \left\{ \begin{matrix} T > 0 : x(T) = x_{goal} \\ \text{following (1)} \end{matrix} \right\}, \qquad (3)$$

the *malfunctioning reach time* becomes

$$T_M^*(x_0, x_{goal}) := \sup_{w(\cdot)\in\mathcal{W}} \left\{ \inf_{u(\cdot)\in\mathcal{U}} \left\{ \begin{matrix} T > 0 : x(T) = x_{goal} \\ \text{following (2)} \end{matrix} \right\} \right\}, \qquad (4)$$

and the *quantitative resilience*

$$r_q(x_{goal}) := \inf_{x_0\in\mathbb{R}^n} \frac{T_N^*(x_0, x_{goal})}{T_M^*(x_0, x_{goal})}. \qquad (5)$$

If $x_0 = x_{goal}$, then $T_N^* = T_M^* = 0$ and we take the convention that their ratio is 1. By definition, if (2) is resilient, then T_M^* is finite and the sup, inf in (4) become max, min achieved with optimal signals w^* and u^*. In that case, T_N^* is also finite and inf becomes min in (3). Focusing on the case $x_{goal} = 0$, we write T_N^*, T_M^* and r_q without their argument x_{goal}.

Our resilience framework assumes that the controller has only access to the past and current values of w, but not to their future. Then, the optimal control u^* of (4) cannot anticipate a random undesirable input w^*, and T_M^* is not likely to be time-optimal as required by Definition 2.4 of [11].

The only way to calculate u^* without any future knowledge of w^* is to solve the intractable Isaac's main equation [22]. This equation is the differential games counterpart of the Hamilton-Jacobi-Bellman (HJB) equation. According to the review [23], Isaac's main equation is even more difficult to solve than the HJB equation, which usually results in intractable partial differential equations [17]. Hence, [22] determines only suboptimal solutions in this setting, the paper itself concludes that its practical contribution is minimal.

Instead, we adopt the setting of [20], where u^* and w^* are unique, *bang-bang* [24], and make the transfer from x_0 to x_{goal} time-optimal. The controller knows that w^* will be chosen to make T_M^* the longest. Thus, u^* is chosen to react optimally to this worst undesirable input. Then, w^* is chosen, and to make T_M^* the longest, it is the same as the controller had predicted. Hence, from an outside perspective it appears as if the controller built u^* knowing w^* in advance, as reflected by (4). Then, T_M^* is time-optimal and can be compared with T_N^*, leading to the following problem.

Problem 2: Quantify resilience of system (2).

III. PRELIMINARIES

Following [9], we introduce the associated system to (2)

$$\dot{y}(t) = Ay(t) + z(t), \qquad y(0) = x_0, \qquad z(t) \in \mathcal{Z}, \qquad (6)$$

$\mathcal{Z} := \mathcal{BU} - (-\mathcal{CW}) = \{ z \in \mathcal{BU} : z - Cw \in \mathcal{BU} \text{ for all } w \in \mathcal{W} \}$. If $0 \in \mathcal{W}$, then $z \in \mathcal{Z}$ if and only if for all $w \in \mathcal{W}$, there exists $u \in \mathcal{U}$ such that $z - Cw = Bu$. Informally, z represents the control available after counteracting any undesirable input.

System (6) is related to Problem 1 through the following duality theorem of [9], whose proof is extended to non-zero states x_{goal} in [24], and reformulated in [7].

Theorem 1 (Hájek Duality theorem): System (2) can be driven to x_{goal} at time T for all $w(\cdot) \in \mathcal{W}$ by control signal $u(\cdot) \in \mathcal{U}$, if and only if system (6) can be driven to x_{goal} at time T by a control signal $z(\cdot) \in \mathcal{Z}$, and $Bu(t) = z(t) - Cw(t)$.

Theorem 1 transforms the problem of resilient reachability to one of *bounded controllability*. Using Corollary 3.7 of [10] we obtain the following resilience condition.

Theorem 2: System (2) is resilient if and only if $\ker(B) \cap \mathcal{Z} \neq \emptyset$, $co(\mathcal{Z})^\circ \neq \emptyset$, $\operatorname{rank}\big([B \;\; AB \;\; \dots \;\; A^{n-1}B]\big) = n$, $Re\big(\lambda(A)\big) = 0$ and there is no real eigenvector v of $A^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$.

The first three conditions ensure that the controls can move the system in any direction, while $Re\big(\lambda(A)\big) = 0$ guarantees that the controls can always overcome the drift.

Corollary 1: If $0 \in \mathcal{Z}^\circ$ and $Re\big(\lambda(A)\big) = 0$, then system (2) is resilient.

Proof: Obviously $co(\mathcal{Z})^\circ \neq \emptyset$ because $\mathcal{Z} \subseteq co(\mathcal{Z})$. Since $0 \in \ker(B)$, we have $\ker(B) \cap \mathcal{Z} \neq \emptyset$. Let v be any eigenvector of $A^\top$. Since $0 \in \mathcal{Z}^\circ$, there exists $\varepsilon > 0$ such that $\mathbb{B}_\varepsilon(0) \subseteq \mathcal{Z}$. Take $z = \varepsilon \frac{v}{\|v\|}$, then $v^\top z = \varepsilon\|v\| > 0$. Note that $\mathbb{B}_\varepsilon(0) \subseteq \mathcal{Z} \subseteq \mathcal{BU} \subseteq \operatorname{Im}(B)$. Since $\mathbb{B}_\varepsilon(0)$ is of full dimension, so is the image of B, i.e., $\operatorname{rank}(B) = n$. Then, all the conditions of Theorem 2 are satisfied, so the system is resilient. ■

For driftless systems, as in [11], [12], Corollary 1 becomes

Corollary 2: Let $A = 0$ and assume that $0 \in \mathcal{Z}^\circ$. Then, driftless system (2) is resilient.

Similarly, if a driftless system is resilient, then $-\mathcal{CW} \subseteq \mathcal{BU}^\circ$ and $\operatorname{rank}(B) = n$ [12]. We show that this condition is also sufficient for resilience due to the compactness of $\bar{\mathcal{U}}$.

Proposition 1: If $-\mathcal{CW} \subseteq \mathcal{BU}^\circ$ and $\operatorname{rank}(B) = n$, then $0 \in \mathcal{Z}^\circ$.

Proof: The compactness of $\bar{\mathcal{U}}$ implies that of $\mathcal{U}$ and $\mathcal{W}$, as well as of $\mathcal{BU}$ and $\mathcal{CW}$. By assumption, $-\mathcal{CW}$ and $\partial\mathcal{BU}$ are disjoint. Since they are both compact, there is a positive distance between them [25], in the sense that there exists $\delta > 0$ such that $(-\mathcal{CW}) + \mathbb{B}_\delta(0) \subseteq \mathcal{BU}$, because $\text{rank}(B) = n$. Thus, for all $w \in \mathcal{W}$ and all $z \in \mathbb{B}_\delta(0)$, $-Cw + z \in \mathcal{BU}$, i.e., $z \in \mathcal{Z}$. Then, $\mathbb{B}_\delta(0) \subseteq \mathcal{Z}$, i.e., $0 \in \mathcal{Z}^\circ$. ∎

Proposition 1 simplifies Corollary 1 as the set $\mathcal{Z}$ is more difficult to characterize than $\mathcal{BU}$ and $\mathcal{CW}$. The price of this simplification is that the system needs full actuation. We will now focus on the classical case $x_{goal} = 0$, as in [20].

Definition 3: System (2) is *resiliently stabilizable* if for all $x_0 \in \mathbb{R}^n$ and all piecewise continuous $w(\cdot) \in \mathcal{W}$, there exists $T > 0$ and piecewise continuous $u(\cdot) \in \mathcal{U}$ such that the solution to (2) exists, is unique and $x(T) = 0$.

Using Corollary 3.6 of [10] we trivially transform Theorem 1 into the following resilient stability condition.

Theorem 3: System (2) is resiliently stabilizable if and only if $\text{rank}\big(\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}\big) = n$, $\ker(B) \cap \mathcal{Z} \neq \emptyset$, $co(\mathcal{Z})^\circ \neq \emptyset$, $Re\big(\lambda(A)\big) \leq 0$ and there is no real eigenvector v of $A^\top$ satisfying $v^\top z \leq 0$ for all $z \in \mathcal{Z}$.

As before, a simpler sufficient condition can be derived with an analoguous proof.

Corollary 3: If $0 \in \mathcal{Z}^\circ$ and $Re\big(\lambda(A)\big) \leq 0$, then system (2) is resiliently stabilizable.

We have established several conditions to verify the resilience of linear systems, let us now quantify their resilience.

IV. QUANTITATIVE RESILIENCE

We consider unit bounded controls, $\bar{\mathcal{U}} = [-1,1]^{m+p}$, in line with previous works [15], [16], [18]. Unlike with driftless systems [11], the reach times T_N^* and T_M^* are not homogeneous in x_0, i.e., $T_N^*(\alpha x_0) \neq |\alpha| T_N^*(x_0)$ for $\alpha \in \mathbb{R}$, as $x(T) - x_0$ is not linear in T. Thus, the optimization domain of the infimum in (5) cannot be scaled down to $\mathbb{S}$ as in [11].

A. Nominal reach time

In order to calculate r_q, we need $T_N^*(x_0)$ for all $x_0 \in \mathbb{R}^n$. However, $T_N^*(x_0)$ has no general closed-form solution [14], and it cannot be computed for all $x_0 \in \mathbb{R}^n$. Thus, we establish analytical bounds on T_N^* so that we can approximate r_q.

Proposition 2: If (1) is controllable and A is Hurwitz, then

$$T_N^*(x_0) \;\geq\; 2\frac{\lambda_{min}^P}{\lambda_{max}^Q} \ln\left(1 + \frac{\lambda_{max}^Q \|x_0\|_P}{2\lambda_{min}^P b_{max}^P}\right), \tag{7}$$

for any $P \succ 0$ and $Q \succ 0$ such that $PA + A^\top P = -Q$ and with $b_{max}^P := \max\big\{\|\bar{B}\bar{u}\|_P : \bar{u} \in \bar{\mathcal{U}}\big\}$.

Proof: Since A is Hurwitz, by the Lyapunov theorem [21], for any $Q \succ 0$ there is $P \succ 0$ such that $PA + A^\top P = -Q$. Let us consider any such pair (P,Q). We define the Lyapunov function $V(x) := x^\top P x = \|x\|_P^2$. Then,

$$\begin{aligned}\dot{V}(x) &= \dot{x}^\top P x + x^\top P \dot{x} = x^\top (A^\top P + PA)x + 2x^\top P\bar{B}\bar{u} \\ &= -x^\top Q x + 2x^\top P \bar{B}\bar{u}.\end{aligned}$$

Since $P \succ 0$, there exists $M \in \mathbb{R}^{n\times n}$ such that $P = M^\top M$ [26]. Then, $x^\top P\bar{B}\bar{u} = (Mx)^\top M\bar{B}\bar{u} \geq -\|Mx\|_2 \|M\bar{B}\bar{u}\|_2$, by the Cauchy-Schwarz inequality [26]. Note that $\|Mx\|_2^2 = x^\top M^\top M x = x^\top P x = \|x\|_P^2$. Similarly, $\|M\bar{B}\bar{u}\|_2 = \|\bar{B}\bar{u}\|_P$.

The maximum in b_{max}^P exists since $\bar{\mathcal{U}}$ is compact and the map : $\bar{u} \mapsto \|\bar{B}\bar{u}\|_P$ is continuous. Since $Q \succ 0$, we have $x^\top Q x \leq \lambda_{max}^Q \|x\|_2^2$ and $\|x\|_2^2 \leq \|x\|_P^2 / \lambda_{min}^P$ because $P \succ 0$. Finally, for $x \neq 0$ we obtain

$$\dot{V}(x) = \frac{d}{dt}\|x\|_P^2 \geq -\frac{\lambda_{max}^Q}{\lambda_{min}^P}\|x\|_P^2 - 2b_{max}^P \|x\|_P. \tag{8}$$

Let $y := \|x\|_P$, $\alpha := \frac{\lambda_{max}^Q}{2\lambda_{min}^P} > 0$ and $\beta := b_{max}^P > 0$. We divide (8) by $2y$ so that $\dot{y} \geq w(y) := -\alpha y - \beta$. Inspired by the proof of the Bihari inequality in [27], for $r \geq r_0 := \frac{-\beta}{1+\alpha}$, define

$$\begin{aligned} G(r) &:= \int_{r_0}^{r} \frac{ds}{w(s)} = \int_{r_0}^{r} \frac{-ds}{\alpha s + \beta} = \frac{-1}{\alpha}\Big[\ln(\alpha s + \beta)\Big]_{r_0}^{r} \\ &= \frac{-1}{\alpha}\ln\left(\frac{\alpha r + \beta}{\alpha r_0 + \beta}\right). \end{aligned}$$

The integral is well-defined since $\alpha s + \beta > 0$ for $s \in [r_0, r]$. Note that $\frac{d}{dt}G\big(y(t)\big) = \frac{\dot{y}(t)}{w(y(t))} \leq 1$ because $\dot{y}(t) \geq w\big(y(t)\big)$ and $w\big(y(t)\big) < 0$ since $\alpha > 0$ and $\beta > 0$. Thus, for any $T > 0$,

$$G\big(y(T)\big) - G\big(y(0)\big) = \int_0^T \frac{d}{dt}G\big(y(t)\big)\,dt \leq \int_0^T 1\,dt = T. \tag{9}$$

Because $\bar{\mathcal{U}}$ is compact and convex, and (1) is controllable, there exists a time-optimal control signal $\bar{u}^*(\cdot) \in \bar{\mathcal{U}}$ driving the state from x_0 to the origin in time $T_N^*(x_0)$ [17]. Then, applying (9) to this trajectory yields

$$\begin{aligned} T_N^*(x_0) &\geq \frac{-1}{\alpha}\ln\left(\frac{\alpha\|0\|_P + \beta}{\alpha r_0 + \beta}\right) + \frac{1}{\alpha}\ln\left(\frac{\alpha\|x_0\|_P + \beta}{\alpha r_0 + \beta}\right) \\ &\geq \frac{1}{\alpha}\ln\left(1 + \frac{\alpha}{\beta}\|x_0\|_P\right) = 2\frac{\lambda_{min}^P}{\lambda_{max}^Q}\ln\left(1 + \frac{\lambda_{max}^Q \|x_0\|_P}{2\lambda_{min}^P b_{max}^P}\right). \end{aligned}$$

∎

We now want to find an upper bound to $T_N^*(x_0)$.

Proposition 3: If (1) is controllable, $\text{rank}(\bar{B}) = n$ and A is Hurwitz, then

$$T_N^*(x_0) \;\leq\; 2\frac{\lambda_{max}^P}{\lambda_{min}^Q}\ln\left(1 + \frac{\lambda_{min}^Q \|x_0\|_P}{2\lambda_{max}^P b_{min}^P}\right), \tag{10}$$

for any $P \succ 0$ and $Q \succ 0$ such that $PA + A^\top P = -Q$ and with $b_{min}^P := \min\big\{\|\bar{B}\bar{u}\|_P : \bar{u} \in \partial\bar{\mathcal{U}}\big\}$.

Proof: Since A is Hurwitz, by the Lyapunov theorem, for any $Q \succ 0$ there exists $P \succ 0$ such that $PA + A^\top P = -Q$. Let $V(x) := x^\top P x$. As in the proof of Proposition 2, $\dot{V}(x) = -x^\top Q x + 2x^\top P \bar{B}\bar{u}$. The minimum in b_{min}^P exists since the map : $\bar{u} \mapsto \|\bar{B}\bar{u}\|_P$ is continuous and $\partial\bar{\mathcal{U}}$ is compact. Because $\text{rank}(\bar{B}) = n$, we can choose $\bar{B}\bar{u}(t) = -\frac{x(t)}{\|x(t)\|_P} b_{min}^P$ for $x(t) \neq 0$. Indeed, assume for contradiction purposes that $\bar{u} \notin \bar{\mathcal{U}}$. Then, $\|\bar{u}\|_\infty > 1$. Let $\hat{u} := \frac{\bar{u}}{\|\bar{u}\|_\infty}$. Then, $\|\hat{u}\|_\infty = 1$, so $\hat{u} \in \partial\bar{\mathcal{U}}$, but $\|\bar{B}\hat{u}\|_P = \frac{\|\bar{B}\bar{u}\|_P}{\|\bar{u}\|_\infty} = \frac{b_{min}^P}{\|\bar{u}\|_\infty} < b_{min}^P$, which is a contradiction. Hence, the proposed control signal is admissible. However, it is not always time-optimal and thus only leads to an upper bound of $T_N^*(x_0)$. We obtain $2x^\top P\bar{B}\bar{u} = -2b_{min}^P \|x\|_P$, so that

$$\frac{d}{dt}\|x\|_P^2 = \dot{V}(x) \leq -\frac{\lambda_{min}^Q}{\lambda_{max}^P}\|x\|_P^2 - 2b_{min}^P \|x\|_P. \tag{11}$$

Let $y := \|x\|_P$, $\alpha := \frac{\lambda^Q_{min}}{2\lambda^P_{max}} > 0$ and $\beta := b^P_{min} > 0$. For $x \neq 0$, dividing (11) by $2y > 0$, yields $\dot{y} \leq w(y) := -\alpha y - \beta < 0$.

Since $w(y) < 0$, $\frac{\dot{y}(t)}{w(y(t))} \geq 1$. With the same definition of G as in the proof of Proposition 2 and similarly to (9), we obtain $G\big(y(T)\big) - G\big(y(0)\big) \geq \int_0^T 1\, dt = T$ for any $T > 0$.

We need to prove that our control law ensures a finite time convergence to 0. Note that $\dot{V} \leq -2\alpha V - 2\beta\sqrt{V} := r(V)$, and that $v(t) = e^{-2\alpha t}\left(\frac{\beta}{\alpha}(1 - e^{\alpha t}) + \|x_0\|_P\right)^2$ is a solution to $\dot{v} = r(v)$ with $v(0) = \|x_0\|^2_P$. At $\tau = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha}{\beta}\|x_0\|_P\right)$, we have $v(\tau) = 0$. Since $\alpha > 0$ and $\beta > 0$, v converges in finite time to 0. Thus, according to Proposition 1 of [28], x converges in a finite time T to the origin with the control law $\bar{B}\bar{u} = \frac{-x}{\|x\|_P} b^P_{min}$.

Since this control law is not time-optimal, $T \geq T^*_N(x_0)$. With the expression of G calculated previously, we obtain $\frac{1}{\alpha}\ln\left(1 + \frac{\alpha}{\beta}\|x_0\|_P\right) \geq T$. Substituting α and β yields (10). ■

Remark 1: It is more complicated to calculate b^P_{min} than its counterpart b^P_{max} since the maximum of $\|\bar{B}\bar{u}\|_P$ is reached on a vertex of $\bar{\mathcal{U}}$, while its minimum is not.

We can now investigate the malfunctioning reach time T^*_M.

B. Malfunctioning reach time

In this section we will bound T^*_M following similar methods we applied to T^*_N.

Proposition 4: If (2) is resiliently stabilizable, then

$$T^*_M(x_0) \;\geq\; 2\frac{\lambda^P_{min}}{\lambda^Q_{max}} \ln\left(1 + \frac{\lambda^Q_{max}\|x_0\|_P}{2\lambda^P_{min} z^P_{max}}\right), \tag{12}$$

for any $P \succ 0$ and $Q \succ 0$ such that $PA + A^\top P = -Q$ and with $z^P_{max} := \max\big\{\|z\|_P : z \in \mathcal{Z}\big\}$.

Proof: Since $\mathcal{BU}$ and $\mathcal{CW}$ are compact, $\mathcal{Z}$ is compact [29], so z^P_{max} exists. According to Theorem 3, A is Hurwitz, so for $Q \succ 0$ there exists $P \succ 0$ such that $A^\top P + PA = -Q$. Let $V(x) := x^\top Px$. Then, $\dot{V}(x) = -x^\top Qx + 2x^\top P(Bu + Cw)$. Since (2) is resiliently stabilizable, we can take $w^*(\cdot)$ and $u^*(\cdot)$ to be the optimizers in (4). For $w^*(\cdot) \in \mathcal{W}$, the control signal $Bu^*(\cdot) \in \mathcal{BU}$ drive the state of (2) to 0. Then, according to Theorem 1, $z(\cdot) = Cw^*(\cdot) + Bu^*(\cdot) \in \mathcal{Z}$. Then, $\|Cw^*(t) + Bu^*(t)\|_P \leq z^P_{max}$, which yields

$$\dot{V}(x) \geq -\frac{\lambda^Q_{max}}{\lambda^P_{min}}\|x\|^2_P - 2z^P_{max}\|x\|_P.$$

We now proceed as in the second half of the proof of Proposition 2 to obtain (12). ■

Similarly, we upper bound the malfunctioning reach time.

Proposition 5: If (2) is resiliently stabilizable, then

$$T^*_M(x_0) \;\leq\; 2\frac{\lambda^P_{max}}{\lambda^Q_{min}} \ln\left(1 + \frac{\lambda^Q_{min}\|x_0\|_P}{2\lambda^P_{max} z^P_{min}}\right), \tag{13}$$

for any $P \succ 0$ and $Q \succ 0$ such that $PA + A^\top P = -Q$ and with $z^P_{min} := \min\big\{\|z\|_P : z \in \partial\mathcal{Z}\big\}$.

Proof: Since $\mathcal{Z}$ is compact, so is $\partial\mathcal{Z}$ and thus z^P_{min} exists. According to Theorem 3, A is Hurwitz, so for $Q \succ 0$ there exists $P \succ 0$ such that $A^\top P + PA = -Q$. Let $V(x) := x^\top Px$. Then, $\dot{V}(x) = -x^\top Qx + 2x^\top P(Bu + Cw)$. Let $w^*(\cdot)$ be the argument of the supremum in (4), which is a maximum since the system is resiliently stabilizable.

Since $\mathcal{BU}$ and $\mathcal{CW}$ are convex, so is $\mathcal{Z}$ [29]. Because (2) is resiliently stabilizable, according to Theorem 3, $co(\mathcal{Z})^\circ \neq \emptyset$, i.e., $\mathcal{Z}^\circ \neq \emptyset$. Then, a ball of dimension n fits in $\mathcal{Z}$, so we can choose the control signal $z(t) := \frac{-x(t)}{\|x(t)\|_P} z^P_{min} \in \mathcal{Z}$. By definition of $\mathcal{Z}$, for any $w(\cdot) \in \mathcal{W}$ and thus for $w^*(\cdot) \in \mathcal{W}$ there exists $u(\cdot) \in \mathcal{U}$ such that $z(t) = Cw^*(t) + Bu(t)$. Then, applying w^* and u leads to an upper bound of T^*_M since u is not necessarily optimal, while w^* is optimal. Hence,

$$\dot{V}(x) \leq -\frac{\lambda^Q_{min}}{\lambda^P_{max}}\|x\|^2_P - 2z^P_{min}\|x\|_P.$$

We now proceed as in the second half of the proof of Proposition 3 to obtain (13). ■

We are now able to bound $T^*_N(x_0)/T^*_M(x_0)$ for all x_0 and thus obtain an approximate of quantitative resilience.

C. Bounding Quantitative Resilience

With a lower bound to quantitative resilience $r_q \geq b$, in the worst case, the malfunctioning system will take less than $1/b$ times longer than the nominal system to reach the origin from the same initial state.

Theorem 4: If (2) is resiliently stabilizable, then

$$r_q \geq \min\left(\frac{\lambda^P_{min}\lambda^Q_{min}}{\lambda^P_{max}\lambda^Q_{max}},\ \frac{z^P_{min}}{b^P_{max}}\right), \tag{14}$$

for any $P \succ 0$ and $Q \succ 0$ such that $A^\top P + PA + Q = 0$.

Proof: For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, (7) and (13) yield

$$\frac{T^*_N(x_0)}{T^*_M(x_0)} \geq \frac{\lambda^P_{min}\lambda^Q_{min}}{\lambda^P_{max}\lambda^Q_{max}} \frac{\ln\left(1 + \frac{\lambda^Q_{max}\|x_0\|_P}{2\lambda^P_{min} b^P_{max}}\right)}{\ln\left(1 + \frac{\lambda^Q_{min}\|x_0\|_P}{2\lambda^P_{max} z^P_{min}}\right)} := f(\|x_0\|_P).$$

Then, according to (5), $r_q \geq \inf_{x_0 \in \mathbb{R}^n} f(\|x_0\|_P)$. To alleviate the notation, we define the positive constants $a := \frac{\lambda^P_{min}\lambda^Q_{min}}{\lambda^P_{max}\lambda^Q_{max}}$, $b := \frac{\lambda^Q_{max}}{2\lambda^P_{min} b^P_{max}}$ and $c := \frac{\lambda^Q_{min}}{2\lambda^P_{max} z^P_{min}}$, so that $f(s) = a\frac{\ln(1+bs)}{\ln(1+cs)}$.

If $b = c$, then $f(s) = a$ for all $s \geq 0$, so $r_q \geq a$. If $b > c$, then f is increasing in s, so $\inf\big\{f(s) : s > 0\big\} = \lim\limits_{s\to 0} f(s)$. Using the first order expansion of ln for small s, we have

$$f(s) = a\frac{\ln(1+bs)}{\ln(1+cs)} \approx a\frac{bs + o(s)}{cs + o(s)} = a\frac{b}{c} + o(1).$$

Then, $f(0) = a\frac{b}{c} = \frac{z^P_{min}}{b^P_{max}}$. If $c > b$, then f is decreasing, so $\inf\big\{f(s) : s \geq 0\big\} = \lim\limits_{s\to+\infty} f(s)$. Note that

$$\frac{\ln(1+bs)}{\ln(1+cs)} = \frac{1 + \frac{\ln(\frac{1}{s}+b)}{\ln(s)}}{1 + \frac{\ln(\frac{1}{s}+c)}{\ln(s)}} \xrightarrow[s\to+\infty]{} \frac{1 + \frac{\ln(b)}{+\infty}}{1 + \frac{\ln(c)}{+\infty}} = 1.$$

Then, $\lim\limits_{s\to+\infty} f(s) = a$. To sum up, $f(s) \geq \min\left(a, a\frac{b}{c}\right)$. ■

We can upper bound r_q using a similar approach.

Theorem 5: If (2) is resiliently stabilizable, then

$$r_q \leq \min\left(\frac{\lambda^P_{max}\lambda^Q_{max}}{\lambda^P_{min}\lambda^Q_{min}}, \frac{z^P_{max}}{b^P_{min}}\right), \tag{15}$$

for any $P \succ 0$ and $Q \succ 0$ such that $A^\top P + PA = -Q$.

Proof: For $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, (10) and (12) yield

$$\frac{T^*_N(x_0)}{T^*_M(x_0)} \leq \frac{\lambda^P_{max}\lambda^Q_{max}}{\lambda^P_{min}\lambda^Q_{min}} \frac{\ln\left(1 + \frac{\lambda^Q_{min}\|x_0\|_P}{2\lambda^P_{max}b^P_{min}}\right)}{\ln\left(1 + \frac{\lambda^Q_{max}\|x_0\|_P}{2\lambda^P_{min}z^P_{max}}\right)} := g(\|x_0\|_P).$$

Then, according to (5), $r_q \leq \inf\limits_{x_0 \in \mathbb{R}^n} g(\|x_0\|_P)$. We define the positive constants $a := \frac{\lambda^P_{max}\lambda^Q_{max}}{\lambda^P_{min}\lambda^Q_{min}}$, $b := \frac{\lambda^Q_{min}}{2\lambda^P_{max}b^P_{min}}$ and $c := \frac{\lambda^Q_{max}}{2\lambda^P_{min}z^P_{max}}$, so that $g(s) = a\frac{\ln(1+bs)}{\ln(1+cs)}$.

This function g is similar to f in the proof of Theorem 4, and thus $r_q \leq \inf\limits_{x_0 \in \mathbb{R}^n} g(\|x_0\|_P) = \min\left(a, a\frac{b}{c}\right)$ yields (15). ∎

Note that we used the same pair (P,Q) to bound both T^*_N and T^*_M. Employing different pairs (P_N,Q_N) and (P_M,Q_M) would make f depend on both $\|x_0\|_{P_N}$ and $\|x_0\|_{P_M}$. Then, we would need to take $x_0 \in \mathbb{R}^n$ instead of $\|x_0\|_P \in \mathbb{R}^+$ as the argument of f, which would significantly complicate the minimum search. We leave this more convoluted approach for possible future work.

V. NUMERICAL RESULTS

We illustrate our work on a room temperature control system motivated by [30] and schematized in Figure 1.

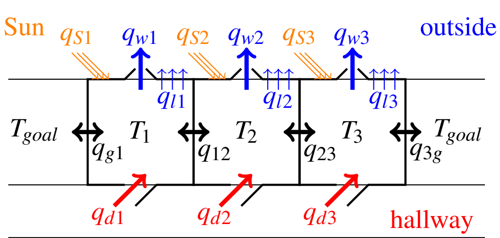

Fig. 1. Scheme of the rooms and of the heat transfers. The heater q_h and AC transfers q_{AC} are not shown for clarity.

The objective is to make the rooms 1, 2 and 3 reach temperature T_{goal}, which is the temperature of the neighboring rooms as shown on Figure 1. The central conditioning unit controls the heater q_h and the AC q_{AC} in all the rooms. Outside of work hours, the controller can also incrementally open doors q_d and windows q_w for room specific adjustments. Because each room is fitted with large windows and blinds, one can also use solar heating q_S. The heat loss through the outside wall is q_l and the heat transfer between rooms i and j is q_{ij}. The temperature dynamics are

$$m C_p \dot{T}_1 = (q_h - q_{AC}) + (q_{d1} - q_{w1}) + (q_{S1} - q_{l1}) + q_{g1} + q_{12},$$
$$m C_p \dot{T}_2 = (q_h - q_{AC}) + (q_{d2} - q_{w2}) + (q_{S2} - q_{l2}) - q_{12} + q_{23},$$
$$m C_p \dot{T}_3 = (q_h - q_{AC}) + (q_{d3} - q_{w3}) + (q_{S3} - q_{l3}) - q_{23} + q_{g3},$$

with m the mass of air in each room, C_p is the specific heat capacity of air, $q_{gi} = aU_{gi}(T_{goal} - T_i)$, $q_{ij} = aU_{ij}(T_j - T_i)$, with a the area of the wall between rooms and U_{ij} the overall heat transfer coefficient between rooms i and j, which depends on the wall materials.

We combine the heat transfers in pairs: $q_h - q_{AC} =: Q_{hAC}u_{hAC}$, $q_{di} - q_{wi} =: Q_{dw}u^i_{dw}$ and $q_{Si} - q_{li} =: Q_{Sl}u^i_{Sl}$ with $u_{hAC} \in [-1,1]$, $u^i_{dw} \in [-1,1]$, $u^i_{Sl} \in [-1,1]$ and $i \in \{1,2,3\}$.

TABLE I

NUMERICAL VALUES FOR THE SIMULATION, BASED ON [30].

Parameter	Value	Unit	Parameter	Value	Unit
a	12	m^2	mCp	42186	$J \cdot K^{-1}$
U_{g1}	6.27	$W \cdot K^{-1}$	U_{12}	5.08	$W \cdot K^{-1}$
U_{23}	5.41	$W \cdot K^{-1}$	U_{3g}	6.27	$W \cdot K^{-1}$
Q_{hAC}	350	W	Q_{dw}	300	W
Q_{Sl}	200	W	T_{goal}	293	K

We write the dynamics as $\dot{T} = AT + \bar{B}\bar{u} + DT_{goal}$, with

$$A = \frac{1}{mCp}\begin{pmatrix} -U_{g1} - U_{12} & U_{12} & 0 \\ U_{12} & -U_{12} - U_{23} & U_{23} \\ 0 & U_{23} & -U_{23} - U_{3g} \end{pmatrix},$$

$$\bar{B} = \frac{1}{mCp}\begin{pmatrix} Q_{Sl} & 0 & 0 & Q_{dw} & 0 & 0 & Q_{hAC} \\ 0 & Q_{Sl} & 0 & 0 & Q_{dw} & 0 & Q_{hAC} \\ 0 & 0 & Q_{Sl} & 0 & 0 & Q_{dw} & Q_{hAC} \end{pmatrix},$$

$$\bar{u}^\top = \begin{pmatrix} u^1_{Sl} & u^2_{Sl} & u^3_{Sl} & u^1_{dw} & u^2_{dw} & u^3_{dw} & u_{hAC} \end{pmatrix},$$

$$T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} \quad \text{and} \quad D = \frac{1}{mCp}\begin{pmatrix} U_{g1} \\ 0 \\ U_{3g} \end{pmatrix}.$$

Let $x^\top := \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = T - \mathbf{1}T_{goal}$. Then, $\dot{x} = \dot{T} = Ax + \bar{B}\bar{u} + DT_{goal} + A\mathbf{1}T_{goal} = Ax + \bar{B}\bar{u}$ and $x_{goal} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.

Since $\lambda(A) = \{-0.052, -0.033, -0.010\} \subseteq \mathbb{R}^-$, A is Hurwitz. Then, according to Theorem 2, the system cannot be resilient, but it might be resiliently stabilizable. For the loss of any one column C, $\text{rank}(B) = 3$ and we numerically verify that $-\mathcal{CW} \subseteq \mathcal{BU}^\circ$. Then, following Proposition 1 and Corollary 3, the system is resiliently stabilizable.

We consider a situation where a worker remains in their room after hours and manually opens or closes their door and window, thus overriding the controller. Let us quantify the resilience of the system to the loss of control over the door and window in room 1, i.e., over u^1_{dw}.

To choose P and Q, we generate a stochastic $Q \succ 0$ and we solve the Lyapunov equation $A^\top P + PA + Q = 0$ for $P \succ 0$. Another approach relies on the fact that for y small, $\ln(1+y) \approx y$. Thus, the lower bound of T^*_M in (12) can be approximated by $\frac{\|x_0\|_P}{z^P_{max}}$. To maximize this lower bound, we minimize $z^P_{max} = \max\{\|z\|_P : z \in \mathcal{Z}\}$, i.e., we choose $P \succ 0$ generating the tightest ellipsoid outer approximation of $\mathcal{Z}$. Similarly, to minimize the upper bound (13), we need P to generate the largest ellipsoid inside $\mathcal{Z}$. Then, we take $Q = -A^\top P - PA$, but there is no guarantee that $Q \succ 0$.

We take the initial state $x_0^\top = \begin{pmatrix} 0.8°C & 0.7°C & 0.9°C \end{pmatrix}$. We compute $T^*_N(x_0)$ with Eaton's algorithm [15] and its lower and upper bounds with (7) and (10): $35.5s \leq T^*_N(x_0) = 42.5s \leq 54.1s$. We compute $T^*_M(x_0)$ with Sakawa's algorithm [20] and its lower and upper bounds with (12) and (13): $53s \leq T^*_M(x_0) = 110.5s \leq 135s$.

Then, the rooms can take up to $T_M^*(x_0)/T_N^*(x_0) = 2.6$ times longer to all reach T_{goal} after the loss of control authority over the door and window in room 1. Our bounds lead to a worst-case time increase by a factor of up to 3.8. As can be seen on Figure 2, the bounds generated with the tight ellipsoidal approximations of $\mathcal{Z}$ are better than the bounds obtained with stochastic pairs (P,Q).

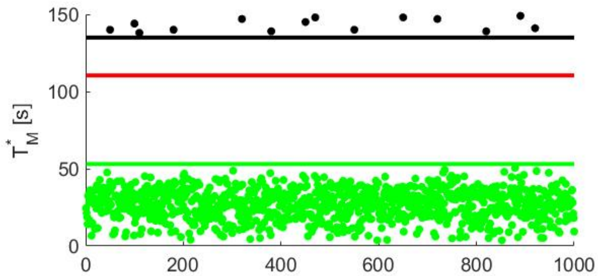

Fig. 2. Bounds on the malfunctioning reach time $T_M^*(x_0)$ represented by the red line. The green and black dots are the lower bounds (12) and the upper bounds (13) for 1000 randomly generated solution pairs (P,Q). The green and black line are the lower and upper bounds (12) and (13) generated with the ellipsoid approximations of $\mathcal{Z}$.

Using Theorem 4 and 5 we can also bound the resilience of the system starting from any initial condition x_0 and we obtain $0.097 \leq r_q \leq 2.79$. The lower bound means that the loss of control over u_{dw}^1 can make the damaged system up to $1/0.097 = 10.3$ times slower than the nominal system to reach T_{goal} in the worst case. Since the upper bound is larger than 1, it does not convey any information.

If the system loses control authority over the central heating/AC unit, i.e., u_{hAC}, the rooms can take as much as $T_M^*(x_0)/T_N^*(x_0) = 4.7$ times longer to reach T_{goal} from the same initial temperature, while our bound predicts a ratio of up to 9.3. These ratios are larger than for the loss of control over the window and door of room 1 because $Q_{hAC} > Q_{dw}$ and the central heating/AC affects directly all 3 rooms.

VI. CONCLUSION

This paper explores and quantifies the resilience of control systems to the loss of control authority over some of their actuators. We established novel necessary and sufficient conditions for the resilience of general linear systems. Based on Lyapunov theory we derived analytical bounds on the nominal and malfunctioning reach times to approximate quantitative resilience of linear systems.

There are several avenues of future work. We want to investigate thoroughly how to determine the pair of matrices (P,Q) generating the best bounds on the reach times T_N^* and T_M^* and on the quantitative resilience r_q. We also desire to extend our resilience theory to nonlinear systems and to more complex and more natural scenarios where the system must visit a succession of targets. Ensuring the safety of critical systems by preventing them from visiting dangerous locations while completing their mission even after enduring a loss of control is also among our future objectives.

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