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Calculus of Variations and Optimal Control Theory

AE 419: Supervised Learning Project

by

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Certificate

Certified that this Supervised Learning Report titled "Calculus of Variations and Optimal Control Theory" by "Nakul Randad" is approved by me for submission. Certified further that, to the best of my knowledge, the report represents work carried out by the students.

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Prof. Debasish Chatterjee

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Abstract

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized. It has numerous applications in both science and engineering. Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.

The report is mainly based on the first four chapters "Calculus of Variations and Optimal Control Theory - A Concise Introduction" by Daniel Liberzon[1]. The Zermelo's river crossing problem solved in the last chapter is inspired from Patrick Suhm's YouTube video "Time Optimal River Crossing Tutorial Based On Pontryagins Maximum Principle".

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1. Introduction

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized. It has numerous applications in both science and engineering. Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved.

A control problem includes a cost functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost function. The optimal control can be derived using Pontryagin's maximum principle (a necessary condition also known as Pontryagin's minimum principle or simply Pontryagin's Principle), or by solving the Hamilton–Jacobi–Bellman equation (a sufficient condition).

Optimal control problems are generally nonlinear and therefore, generally do not have analytic solutions (e.g., like the linear-quadratic optimal control problem). As a result, it is necessary to employ numerical methods to solve optimal control problems. The calculus of variations is employed to obtain the first-order optimality conditions. These conditions result in a two-point (or, in the case of a complex problem, a multi-point) boundary-value problem. This boundary-value problem actually has a special structure because it arises from taking the derivative of a Hamiltonian.

This report has been divided as follows: **We need to explain what each chapter contains**

- *Chapter 2* introduces to the basic concepts required for formulating and solving an optimal control problem.
- *Chapter 3* presents the important results of Calculus of Variations, introducing the Euler-Lagrange equation and Hamilton's canonical equations among others.
- *Chapter 4* introduces us to the use of calculus of variations for solving Optimal Control problems
- *Chapter 5* formulates and proves the Maximum Principle for fixed and variable endpoint control problems
- *Chapter 6*, here we attempt to solve the classic Zermelo's river crossing problem employing the tools described in the previous chapters.

2. Background

2.1 Optimal Control Problem

In this study, the control systems will be described by ordinary differential equations (ODEs) of the form

$$x = f(t, x, u), \;\; x(t_o) = x_0 \tag{2.1}$$

where x is the state taking values in R^n, u is the control input taking values in some control set $U \subset R^m$, t is time, t_0 is the initial time, and x_0 is the initial state. Both x and u are functions of t, but we will often suppress their time arguments.

The second basic ingredient is the cost functional. It associates a cost with each possible behavior. For a given initial data $(t_0; x_0)$, the behaviors are parameterized by control functions u. Thus, the cost functional assigns a cost value to each admissible control. Cost functionals are denoted by J and are of the form:

$$J(u) := \int_{t_o}^{t_f} L(t, x(t), u(t))dt + K(t_f, x_f) \tag{2.2}$$

where L and K are given functions (running cost and terminal cost, respectively), t_f is the final (or terminal) time which is either free or fixed, and $x_f := x(t_f)$ is the final (or terminal) state which is either free or fixed or belongs to some given target set. Note again that u itself is a function of time; this is why we say that J is a functional (a real-valued function on a space of functions). The optimal control problem can then be posed as follows: Find a control u that minimizes $J(u)$ over all admissible controls (or at least over nearby controls). It can be argued that optimality is a universal principle of life, in the sense that many|if not most|processes in nature are governed by solutions to some optimization problems (although we may never know exactly what is being optimized).

2.1.1 Basic Terminology

Consider a function $f : R^n -> R$. Let D be some subset of R^n, which could be the entire Rn. We denote by $|.|$ the standard Euclidean norm on R^n. A point $x^* \in D$ is a local minimum of f over D if there exists an $\epsilon > 0$ such that for all $x \in D$ satisfying $|x - x^*| < \epsilon$ we have

$$f(x^*) \leq f(x) \tag{2.3}$$

In other words, x^* is a local minimum if in some ball around it, f does not attain a value smaller than $f(x^*)$. Note that this refers only to points in D;

2.1.2 Unconstrained optimization

The term "unconstrained optimization" usually refers to the situation where all points x sufficiently near x^* in R^n are in D, i.e., x^* belongs to D together with some $R^n - neighborhood$. The simplest case is when $D = R^n$, which is sometimes called the completely unconstrained case. However, as far as local minimization is concerned, it is enough to assume that x^* is an interior point of D. This is automatically true if D is an open subset of R^n.

2.1.2.1 First-Order Necessary Condition for Optimality

Suppose that f is a C^1 (continuously differentiable) function and x^* is its local minimum. Pick an arbitrary vector $d \in R^n$. Since we are in the unconstrained case, moving away from x^* in the direction of d or $-d$ cannot immediately take us outside *D*. In other words, we have $x^* + \alpha d \in D$ for all $\alpha \in R$ close enough to 0.

$$\bigtriangledown f(x^*) = 0 \tag{2.4}$$

This is the first-order necessary condition for optimality.
A point x^* satisfying this condition is called a stationary point. The condition is "first-order" because it is derived using the first-order expansion.

2.1.2.2 Second-Order Condtions for Optimality

We now derive another necessary condition and also a sufficient condition for optimality, under the stronger hypothesis that f is a $C^2 2$ function (twice continuously differentiable).
The matrix $\bigtriangledown^2 f(x^*)$ must be positive semidefinite:

$$\bigtriangledown^2 f(x^*) \geq 0 \tag{2.5}$$

where,

$$\nabla^2 f := \begin{pmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_n} \\ & \ddots & \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{pmatrix}$$

Figure 2.1: Hessian Matrix

is the Hessian matrix of f.
This is the second-order necessary condition for optimality. Like the previous first-order necessary condition, this second-order condition only applies to the unconstrained case. But, unlike the first-order condition, it requires f to be C^2 and not just C^1. Another difference with the first-order condition is that the second-order condition distinguishes minima from maxima: at a local maximum, the Hessian must be negative semidefinite, while the first-order condition applies to any extremum (a minimum or a maximum).

Second Order Sufficient Condition for optimality:
If a C^2 function f satisfies

$$\bigtriangledown f(x^*) = 0 \; s \; \bigtriangledown^2 f(x^*) < 0 \tag{2.6}$$

on an interior point x^* of its domain, then x^* is a strict local minimum of f.

2.1.2.3 Feasible directions, Global Minima and Convex Problems

The key fact that we used in the previous developments was that for every $d \in^n$, points of the form $x^* + \alpha d$ for sufficiently close to 0 belong to D. This is no longer the case if D has a boundary (e.g., D is a closed ball in ^{n}) and x^* is a point on this boundary. Such situations do not fit into the unconstrained optimization scenario as we defined earlier; however, for simple enough sets D and with some extra care, a similar analysis is possible. Let us call a vector $d \in^n$ a feasible direction (at x^*) if $x^* + \alpha d \in D$ for small enough $\alpha > 0$. If not all directions d are feasible, then the condition $\bigtriangledown f(x^*) = 0$ is no longer necessary for optimality.
If the set D is convex, then the line segment connecting x^* to an arbitrary other point $x \in D$ lies entirely in D. All points on this line segment take the form $x^* + \alpha d$, $\alpha \in [0; \alpha']$ for some $d \in^n$ and $\alpha' > 0$. This means that the feasible direction approach is particularly suitable for the case of a convex D. But if D is not convex, then the first-order and second-order necessary conditions in terms of feasible directions are conservative.

2.1.2.4 Weierstrass Theorem

If f is a continuous function and D is a compact set, then there exists a global minimum of f over D. For subsets of ^{n}, compactness can be defined in three equivalent ways:

- D is compact if it is closed and bounded.
- D is compact if every open cover of D has a finite subcover.
- D is compact if every sequence in D has a subsequence converging to some point in D (sequential compactness).

2.1.3 Constrained optimization

Suppose that D is a surface in ^{n} defined by the equality constraints

$$h_1(x) = h_2(x) = ... = h_m(x) = 0 \tag{2.7}$$

where h_i, i = 1,....,m are C^1 functions from ^{n} to . We assume that f is a C^1 function and study its minima over D.

2.1.3.1 First-Order Necessary Condition (Lagrange Multipliers)

Let $x^* \in D$ be a local minimum of f over D. We assume that x^* is a regular point of D in the sense that the gradients $\bigtriangledown h_i, i = 1, ..., m$ are linearly independent at x^*.

Claim: The gradient of f at x^* is a linear combination of the gradients of the constraint functions $h_1, ..., h_m$ at x^*:

$$\bigtriangledown f(x^*) \in \{\bigtriangledown h_i(x^*), i = 1, ..., m\} \tag{2.8}$$

Therefore, there exist real numbers $\lambda_1^*, ..., \lambda_m^*$ such that

$$\bigtriangledown f(x^*) + \lambda_1^* \bigtriangledown h_1(x^*) + ... + \lambda_m^* \bigtriangledown h_m(x^*) = 0 \tag{2.9}$$

This is the first-order necessary condition for constrained optimality. The coefficients $^*_i, i = 1, ..., m$ are called Lagrange multipliers.

The first-order necessary condition for constrained optimality generalizes the corresponding result we derived earlier for the unconstrained case. The condition $\bigtriangledown f(x^*) + \lambda_1^* \bigtriangledown h_1(x^*) + ... + \lambda_m^* \bigtriangledown h_m(x^*) = 0$ together with the constraints $h_1(x) = h_2(x) = ... = h_m(x) = 0$ is a system of $n + m$ equations in $n + m$ unknowns: n components of x^* plus m components of the Lagrange multiplier vector $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)^T$. For m = 0, we recover the condition (div. fx=0) which consists of n equations in n unknowns.

Consider the function $l : R^n$ x $R^m \to R$ defined by:

$$l(x, \lambda) := f(x) + \Sigma_{i=1}^m \lambda_i h_i(x) \tag{2.10}$$

which we call the augmented cost function. If x^* is a local constrained minimum of f and λ^* is the corresponding vector of Lagrange multipliers for which (1.24) holds, then the gradient of l at (x^*, λ^*) satisfies

$$\nabla^2 f := \begin{pmatrix} f_{x_1x_1} & \cdots & f_{x_1x_n} \\ & \ddots & \\ f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}$$

Figure 2.2: Hessian Matrix

where l_x, l_λ are the vectors of partial derivatives of l with respect to the components of x and λ, respectively, and $h = (h_1, ..., h_m)^T$ is the vector of constraint functions. We conclude that (x^*, λ^*) is a usual (unconstrained) stationary point of the augmented cost l. Loosely speaking, adding Lagrange multipliers converts a constrained problem into an unconstrained one, and the first-order necessary condition (1.24) for constrained optimality is recovered from the first-order necessary condition for unconstrained optimality applied to l.

2.1.3.2 Second-Order Conditions

Suppose that x^* is a regular point of D and a local minimum of f over D, where D is defined by the equality constraints as before. We let λ^* be the vector of Lagrange multipliers provided by the first-order necessary condition, and define the augmented cost l. We also assume that f is C^2. Consider the Hessian of l with respect to x evaluated at $(x^*.\lambda^*)$:

$$\nabla \ell(x^*, \lambda^*) = \begin{pmatrix} \ell_x(x^*, \lambda^*) \\ \ell_\lambda(x^*, \lambda^*) \end{pmatrix} = \begin{pmatrix} \nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla h_i(x^*) \\ h(x^*) \end{pmatrix} = 0$$

Figure 2.3: Second order conditions

The second-order necessary condition says that this Hessian matrix must be positive semidef-

inite on the tangent space to D at x^*, i.e., we must have $d^T l_{xx}(x^*; \lambda^*)d \geq 0$ for all $d \in T_{x^*}D$. The second-order sufficient condition says that a point $x^* \in D$ is a strict constrained local minimum of f if the first-order necessary condition for constrained optimality holds and, in addition, we have

$$d^T l_{xx}(x^*, \lambda^*)d > 0 \quad \forall d, \tag{2.11}$$

such that

$$\bigtriangledown h_i(x^*).d, i = 1, ..., m \tag{2.12}$$

2.1.4 Function spaces, norms, and local minima

Norm ||.|| is a real-valued function on V which is positive definite ($||y|| > 0 if y \neq 0$), homogeneous ($||\lambda y|| = |\lambda|\Delta||y||, \quad \lambda \in R, y \in V$), and satisfies the triangle inequality (ky + zk kyk + kzk). The norm gives us the notion of a distance, or metric, d(y, z) := ky zk. This allows us to define local minima and enables us to talk about topological concepts such as convergence and continuity.

On the space $C^0([a, b, R^n)$, a commonly used norm is

$$||y||_0 := max_{a \leq x \leq b}|y(x)| \tag{2.13}$$

where |.| is the standard Euclidean norm on R^n as before. Replacing the maximum by a supremum, we can extend the 0-norm to functions that are defined over an infinite interval or are not necessarily continuous. There exist many other norms, such as for example the L_p norm

$$||y||_{L_p} := (\sigma_a^b |y(x)|^p dx)^{1/p} \tag{2.14}$$

where p is a positive integer.

Let V be a vector space of functions equipped with a norm ||.||, let A be a subset of V , and let J be a real-valued functional defined on V (or just on A). A function $y^* \in A$ is a local minimum of J over A if there exists an $\epsilon > 0$ such that for all y A satisfying $||yy^*|| < \epsilon$ we

have

$$J(y^*) \leq J(y) \tag{2.15}$$

For the norm, we will typically use either the 0-norm or the 1-norm, with V being $C^0([a,b],R^n)$ or $C^1([a,b],R^n)$ respectively.

2.1.5 First variation and first-order necessary condition

The derivative of J at y, which will now be called the first variation, will also be a functional on V , and in fact this functional will be linear. To define it, we consider functions in V of the form $y+\alpha\eta$, where $\eta \in V$ and α is a real parameter (which can be restricted to some interval around 0).

A linear functional $\delta J|_y : V \to R$ is called the first variation of J at y if for all η and all we have

$$J(y+\alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + o(\alpha) \tag{2.16}$$

The first variation as defined above corresponds to the Gateaux derivative of J, which is just the usual derivative of $J(y+\alpha\eta)$ with respect to α (for fixed y and η) evaluated at $\alpha = 0$:

$$\delta J|_y(\eta) = lim_{\alpha\to 0}\frac{J(y+\alpha\eta)-J(y)}{\alpha} \tag{2.17}$$

Now, suppose that y^* is a local minimum of J over some subset A of V . We call a perturbation $\eta \in V$ admissible (with respect to the subset A) if $y^* + \alpha\eta \in A$ for all α sufficiently close to 0. It follows from our definitions of a local minimum and an admissible perturbation that $J(y^* + \alpha\eta$ as a function of has a local minimum at $alpha = 0$ for each admissible η.

The first-order necessary condition for optimality: For all admissible perturbations η, we must have

$$\delta J|_{y^*}(\eta) = 0 \tag{2.18}$$

Observe that our notion of the first variation, defined via the expansion (2.16), is independent of the choice of the norm on V . This means that the first-order necessary condition (2.18) is

valid for every norm. To obtain a necessary condition better tailored to a particular norm, we could define $\delta J|_{y^*}$ differently, by using the following expansion instead of (2.16):

$$J(y+\eta) = J(y) + \delta J|_y(\eta) + o(||\eta||) \tag{2.19}$$

The difference with our original formulation is subtle but substantial. The earlier expansion (2.16) describes how the value of J changes with for each fixed η. In (2.19), the higher-order term is a function of $||\eta||$ and so the expansion captures the effect of all η at once. The firat variation thus defined is called the Frechet derivative of J, which is stronger that the Gateaux derivative.

2.1.6 Second variation and second-order conditions

A real-valued functional B on VV is called bilinear if it is linear in each argument (when the other one is fixed). Setting $Q(y) := B(y, y)$ we then obtain a quadratic functional, or quadratic form, on V.

A quadratic form $\delta^2 J|_y : V$ is called the second variation of J at y if for all $\eta \in V$ and all α we have

$$J(\alpha\eta + y) = J(y) + \delta J|_y(\eta)\alpha + \delta^2 J|_y(\eta)\alpha^2 + o(\alpha^2) \tag{2.20}$$

Now, we easily establish the following **second order necessary condition for optimality:** If y^* is a local minimum of J over $A \in V$, then for all admissible perturbations η we have

$$\delta^2 J|_{y^*}(\eta) \geq 0 \tag{2.21}$$

In other words, the second variation must be positive semidefinite on the space of admissible perturbations.

Now when we talk about the second order necessary conditions, our old method does not go through. We know that there exists an $\epsilon > 0$ such that for all nonzero α with $|\alpha| < \epsilon$ we have $o(\alpha^2) < \delta^2 J|_y(\eta)\alpha^2$. Using this inequality and (2.18), we obtain that $J(y^* + \alpha\eta) > J(y^*)$. Note that this does not yet prove that y^* is a (strict) local minimum of J. According to the

definition of a local minimum, we must show that $J(y^*)$ is the lowest value of J in some ball around y^* with respect to the selected norm on V . The problem is that the term $o(\alpha^2$ and hence the above ϵ depend on the choice of the perturbation η. In the finite-dimensional case we took the minimum of η over all perturbations of unit length, but we cannot do that here because the unit sphere in the infinite-dimensional space V is not compact and the Weierstrass Theorem does not apply to it.

To resolve this, the first step is to strengthen our condition to:

$$\delta^2 J|_{y^*}(\eta) \geq \lambda ||\eta||^2 \tag{2.22}$$

for some $\lambda > 0$. The second step is to modify the definitions of the first and second variations by explicitly requiring that the higher-order terms decay uniformly with respect to $||\eta||$. We already mentioned such an alternative definition of the first variation via the expansion (2.19). Similarly, we could define $\delta^2 J|_y$ via the following expansion:

$$J(y+\eta) = J(y) + \delta J|_y(\eta) + \delta^2 J|_y(\eta) + o(||\eta||^2) \tag{2.23}$$

3. Calculus of Variations

3.1 Basic Calculus of Variations problem

Consider a function $L : R \times R \times R \to R$

Basic Calculus of Variations Problem: Among all C^1 curves $y : [a, b] \to R$ satisfying given boundary conditions

$$y(a) = y_0, \quad y(b) = y_1 \tag{3.1}$$

find minima of the cost functional

$$J(y) := \int_a^b L(x, y(x), y'(x))dx \tag{3.2}$$

The function L is called the Lagrangian, or the running cost. Even though y and y' are the position and velocity along the curve, L is to be viewed as a function of three independent variables.

3.1.1 Weak and Strong extrema

Extrema (minima and maxima) of J with respect to the 0-norm are called strong extrema, and those with respect to the 1-norm are called weak extrema. If a C^1 curve y^* is a strong extremum, then it is automatically a weak one, but the converse is not true. We will want to allow curves y which are continuous everywhere on [a, b] and whose derivative y' exists everywhere except possibly a finite number of points in [a, b] and is continuous and bounded between these points. Let us agree to call such curves piecewise C^1. If we use the 0-norm, then it makes no difference whether y is C^1 or piecewise C^1 or just C^0 ; this is another advantage of the 0-norm over the 1-norm.

3.2 First-order necessary conditions for weak extrema

3.2.1 Euler-Lagrange equation

To keep things simple, we assume that all derivatives appearing in our calculations exist and are continuous. Let y = y(x) be a given test curve in A. For a perturbation η to be admissible, the new curve $y + \alpha\eta$ must again satisfy the boundary conditions. Clearly, this is true if and only if

$$\eta(a) = \eta(b) = 0 \tag{3.3}$$

Now, the first-order necessary condition (1.37) says that if y is a local extremum of J, then for every η satisfying the above, we must have $\delta J|_y(\eta) = 0$.
Recall that the first variation $\delta J|_y$ was defined via

$$J(y + \alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + o(\alpha) \tag{3.4}$$

The LHS of the above equation is

$$J(y + \alpha\eta) = \int_a^b L(x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x))dx \tag{3.5}$$

We can write down its first-order Taylor expansion with respect to by expanding the expression inside the integral with the help of the chain rule:

$$J(y + \alpha\eta) = \int_a^b (L(x, y(x), y'(x)) + L_y(x, y(x), y'(x))\alpha\eta(x) + L_z(x, y(x), y'(x))\alpha\eta'(x) + o(\alpha))dx \tag{3.6}$$

matching this with the right hand side, we get the firstvariation to be:

$$\delta J|_y(\eta) = \int_a^b (L_y(x, y(x), y'(x))\eta(x) + L_z(x, y(x), y'(x))\eta'(x))dx \tag{3.7}$$

We see that the first variation depends not just on η but also on η'. However, we can eliminate the dependence on η' if we apply integration by parts to the second term on the right-hand

side of the above equation

$$\delta J|_y(\eta) = \int_a^b (L_y(x, y(x), y'(x))\eta(x) - \frac{d}{dx} L_z(x, y(x), y'(x))\eta'(x))dx + L_z(x, y(x), y'(x))\eta'(x)|_a^b \tag{3.8}$$

where the last term is 0 when η satisfies the boundary conditions. Thus we conclude that if y is an extremum, then we must have

$$\int_a^b (L_y(x, y(x), y'(x)) - \frac{d}{dx} L_z(x, y(x), y'(x)))\eta(x)dx = 0 \tag{3.9}$$

for all C^1 curves η vanishing at endpoints x=a and x=b. The onle way the above equation can hold is if the term inside the parentheses- which does not depend on η equals o for all x. Hence,

$$L_y(x, y(x), y'(x)) = \frac{d}{dx} L_z(x, y(x), y'(x)) \quad \forall x \in [a, b] \tag{3.10}$$

This is the celebrated Euler-Lagrange equation providing the first-order necessary condition for optimality. . It is often written in the shorter form

$$L_y = \frac{d}{dx} L_{y'} \tag{3.11}$$

The functional J to be minimized is given by the integral of the Lagrangian L along a path, while the Euler-Lagrange equation involves derivatives of L and must hold for every point on the optimal path; observe that the integral has disappeared. The underlying reason is that if a path is optimal, then every infinitesimally small portion of it is optimal as well (no "shortcuts" are possible).

Trajectories satisfying the Euler-Lagrange equation are called extremals. Since the Euler Lagrange equation is only a necessary condition for optimality, not every extremal is an extremum.

3.2.2 Technical remarks

Let us examine more carefully under what differentiability assumptions our derivation of the Euler-Lagrange equation is valid. The appearance of second-order partial derivatives of L

in the detailed expression of the Euler-Lagrange equation suggests that we should assume $L \in C^2$.Somewhat more alarmingly, the presence of the term $L_{zz}y''$ indicates that we should assume $y \in C^2$, and not merely $y \in C^1$ as in our original formulation of the Basic Calculus of Variations Problem. Fortunately, we can avoid making this assumption if we proceed more carefully, as follows. Let us apply integration by parts to the first term rather than the second term on the right-hand side of the original equation.

$$\delta J|_y(\eta) = \int_a^b (-\eta'(x) \int_a^x L_y(w, y(w), y'(w))dw + L_z(z, y(z), y'(x))\eta'(x))dx + \eta(x) \int_a^x L_y(w, y(w), y'(w))dw|_a^b \tag{3.12}$$

where the last term again vanishes for our class of perturbations η. Thus an extremum y must satisfy

$$\int_a^b (L_z(z, y(z), y'(x)) - \int_a^x L_y(w, y(w), y'(w))dw)\eta'(x)dx = 0 \tag{3.13}$$

3.2.3 Two special cases

$$L_y = L_{y'x} + L_{y'y}y' + L_{y'y'}y" \tag{3.14}$$

SPECIAL CASE 1 (no y) This refers to the situation where the Lagrangian does not depend on y, i.e., L = L(x, y'). The Euler-Lagrange equation becomes $\frac{d}{dx}L_{y'} = 0$, which means that $L_{y'}$ must stay constant. In other words, extremals are solutions of the first-order differential equation

$$L_{y'}(x, y'(x)) = c \tag{3.15}$$

for various values of $c \in R$. . Due to the presence of the parameter c, we expect that the family of solutions of the above equation is rich enough to contain one (and only one) extremal that passes through two given points.

The quantity $L_{y'}$, evaluated along a given curve, is called the momentum.

SPECIAL CASE 2 (no x) Suppose now that the Lagrangian does not depend on x, i.e., L = L(y, y'). In this case the partial derivative $L_{y'x}$ vanishes and the Euler-Lagrange equation

becomes

$$0 = L_{y'y}y' + L_{y'y'}y" - L_y \tag{3.16}$$

Multiplying both sides by y', we have

$$0 = L_{y'y}(y')^2 + L_{y'y'}y'y" - L_y y' = \frac{d}{dx}(L_{y'}y' - L) \tag{3.17}$$

This means that $L_{y'}y' - L$ must remain constant. Thus, similarly to Case 1, extremals are described by the family of first-order differential equations

$$L_{y'}(y(x), y'(x))y'(x) - L(y(x), y'(x)) = c \tag{3.18}$$

parametrized by $c \in R$

The quantity $L_{y'}y' - L$ is called the *Hamiltonian*

3.2.4 Variable-endpoint problems

If we change the boundary conditions for the curves of interest, then the class of admissible perturbations will also change, and in general the necessary condition for optimality will be different.

Suppose that the cost functional takes the same form as before, the initial point of the curve is still fixed by the boundary condition y(a) = y0, but the terminal point y(b) is free. The perturbations η must still satisfy $\eta(a) = 0$ but $\eta(b)$ can be arbitrary. The first variation is then given by

$$\delta J|_y(\eta) = \int_a^b (L_y(x, y(x), y'(x)) - \frac{d}{dx}L_z(x, y(x), y'(x)))\eta(x)dx + L_z(b, y(b), y'(b))\eta(b) \tag{3.19}$$

and this must be 0 if y is to be an extremum. Perturbations such that $\eta(b) = 0$ are still allowed; let us consider them first. They make the last term in the above equation disappear, leaving us with the original equation. Exactly as before, we deduce from this that the Euler-Lagrange equation must hold, i.e., it is still a necessary condition for optimality. The Euler-Lagrange equation says that the expression in the large parentheses inside the integral in the above

equation is 0. But this means that the entire integral is 0, for all admissible η (not just those vanishing at x = b). The last term must then also vanish, which gives us an additional necessary condition for optimality:

$$L_z(b, y(b), y'(b))\eta(b) = 0 \tag{3.20}$$

or, since $\eta(b)$ is arbitrary

$$L_z(b, y(b), y'(b)) = 0 \tag{3.21}$$

3.3 Hamiltonian formalism and mechanics

3.3.1 Hamilton's cannonical equations

Momentum:

$$p := L_{y'}(x, y, y') \tag{3.22}$$

Hamiltonian:

$$H(x, y, y', p) := p.y' - L(x, y, y') \tag{3.23}$$

The inner product sign $\cdot$ in the definition of H reflects the fact that in the multiple-degrees-of-freedom case, y' and p are vectors.

The variables y and p are called the canonical variables. Suppose now that y is an extremal, i.e., satisfies the Euler-Lagrange equation. It turns out that the differential equations describing the evolution of y and p along such a curve, when written in terms of the Hamiltonian H, take a particularly nice form. For y, we have

$$\frac{dy}{dx} = y'(x) = H_p(x, y(x), y'(x)) \tag{3.24}$$

For p, we have

$$\frac{dp}{dx} = \frac{d}{dx}L_{y'}(x, y(x), y'(x)) = L_y(x, y(x), y'(x)) = -H_y(x, y(x), y'(x)) \tag{3.25}$$

where the second equality is the Euler-Lagrange equation. In more concise form, the result is

$$y' = H_p, \;\; p' = -H_y \tag{3.26}$$

which is known as the system of Hamilton's canonical equations.
An important additional observation is that the partial derivative of H with respect to y' is

$$H_{y'}(x, y, y', p) = p - L_{y'}(x, y, y') = 0 \tag{3.27}$$

Mathematically, the Lagrangian L and the Hamiltonian H are related via a construction known as the Legendre transformation.

3.3.2 Legendre transformation

Consider a function $f : rR$, whose argument we denote by ζ. The Legendre transform of f will be a new function, f* , of a new variable, $p \in R$.
Let p be given. Draw a line through the origin with slope p. Take a point $\zeta = \zeta(p)$ at which the (directed) vertical distance from the graph of f to this line is maximized:

$$\zeta(p) := arg\, max_{\zeta}(p\zeta - f(\zeta)) \tag{3.28}$$

Now, define $f^*(p)$ to be this maximal value of the gap between $p\zeta$ and $f(\zeta)$:

$$f^8(p) := p\zeta(p) - f(\zeta(p)) = max_{\zeta}(p\zeta - f(\zeta)) \tag{3.29}$$

We can also write this definition more symmetrically as

$$f^*(p) + f(\zeta) = p\zeta \tag{3.30}$$

When f is differentiable, the maximization condition implies that the derivative of $p\zeta - f(\zeta)$ with respect to ζ must equal 0 at $\zeta(p)$:

$$p - f'(\zeta(p)) = 0 \tag{3.31}$$

For example, f* is a convex function even if f is not convex.. Also, for convex functions the Legendre transformation is involutive: if f is convex, then $f** = f$.

Now let us return to the Hamiltonian H defined above. We claim that it can be obtained by applying the Legendre transformation to the Lagrangian L. More precisely, for arbitrary fixed x and y let us consider $L(x, y, y')$ as a function of $= y'$. The relation between p and $\zeta(p) = y'(p)$ becomes

$$p - L_{y'}(x, y, y'(p)) = 0 \tag{3.32}$$

which corresponds to our earlier definition of the momentum p. Next,

$$L*(x, y, p) = py'(p) - L(x, y, y'(p) \tag{3.33}$$

which is essentially our earlier definition of the Hamiltonian H. The Legendre transform of L(x, y, y') as a function of y' (with x, y fixed) is H(x, y, p), which is a function of p (with x, y fixed) and no longer has y' as an argument.

3.3.3 Principle of least action and conversation laws

Newton's second law of motion in the three-dimensional space can be written as the vector equation

$$\frac{d}{dt}(m\dot{q}) = -U_q \tag{3.34}$$

where $q = (x, y, z)^T$ is the vector of coordinates, $\dot{q} = dq/dt$ is the velocity vector, and U = U(q) is the potential; consequently, $m\dot{q}$ is the momentum and $-U_q$ is the force.

Let us try to bring the Euler-Lagrange equation into this form. First, let us write t instead of x for the independent variable. Second, let us write q instead of y for the dependent variable. Then also y' becomes $\dot{q}$ and we have $L(t, q, \dot{q}$. In the new notation, the Euler-Lagrange equation becomes:

$$\frac{d}{dt}L_{\dot{q}} = L_q \tag{3.35}$$

Let us now compare the above two equations. Is there a choice of the Lagrangian L that would make these two equations the same? The following Lagrangian does the job:

$$L := \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(q) \tag{3.36}$$

which is the difference between the kinetic energy and potential energy.

Hamilton's principle of least action: Trajectories of mechanical systems are extremals of the functional

$$\int_{t_0}^{t_1} (T - U)dt \tag{3.37}$$

which is called the *action integral*

Conservation of energy. In a conservative system, the potential is fixed and does not change with time. Since the kinetic energy does not explicitly depend on time either, we have $L = L(q, \dot{q})$ In other words, the Lagrangian is invariant under time shifting.

Conservation of momentum. Suppose that no force is acting on the system (i.e., the system is closed). Since the force is given by $L_q = -U_q$, this implies that U must be constant. The kinetic energy T depends on $\dot{q}$ but not on q. Thus the Lagrangian L = T U does not explicitly depend on q, which means that it is invariant under parallel translations. This situation corresponds to the "no y" case, where we saw that the momentum, $L_{\dot{q}} = m\dot{q}$ in the present notation, is conserved. A more general statement is that for each coordinate q_i that does not appear in L, the corresponding component $L_{\dot{q}_i} = m\dot{q}_i$ of the momentum is conserved.

Conservation of angular momentum. Consider a planar motion in a central field; in polar coordinates (r,), this is defined by the property that U = U(r), i.e., the potential depends only on the radius and not on the angle. This means that no torque is acting on the system, making the Lagrangian L invariant under rotations. Now we can use the fact that the Euler-Lagrange equation looks the same in all coordinate systems. In particular, in polar coordinates we have

$$L_\theta = \frac{d}{dt}L_{\dot{\theta}} \tag{3.38}$$

and the analogous equation for r (which we do not need here). Arguing exactly as before, we can show that in the present "no " case the corresponding component of the momentum, $L_{\dot{\theta}}$,

is conserved.

3.4 Variational problems with constraints

3.4.1 Integral constraints

Suppose that we augment the Basic Calculus of Variations Problem with an additional constraint of the form

$$C(y) := \int_a^b M(x, y(x), y'(x))dx = C_0 \tag{3.39}$$

where C stands for the "constraint" functional, M is a function from the same class as L, and C_0 is a given constant.

Assume that a given curve y is an extremum. Let us consider perturbed curves of the familiar form

$$y + \alpha\eta$$

To be admissible, the perturbation η must preserve the constraint (in addition to vanishing at the endpoints as before). In other words, we must have $C(y + \alpha\eta) = C_0$ for all α sufficiently close to 0. In terms of the first variation of C

$$\delta C|y(\eta) = 0 \tag{3.40}$$

we obtain from this that

$$\int_a^b (M_y(x, y(x), y'(x)) - \frac{d}{dx} < M_{y'}(x, y(x), y'(x)))\eta(x)dx = 0 \tag{3.41}$$

So, we must also have:

$$\delta J|_y(\eta) = \int_a^b (L_y(x, y(x), y'(x)) - \frac{d}{dx} < L_{y'}(x, y(x), y'(x)))\eta(x)dx = 0 \tag{3.42}$$

This conclusion can be summarised as followss:

$$\int_a^b (L_y - \frac{d}{dx}L_{y'})\eta(x)dx = 0 \;\; \forall\eta \; such \; that \; \int_a^b (M_y - \frac{d}{dx}M_{y'})\eta(x)dx = 0 \tag{3.43}$$

There exist a constant $\lambda*$ (a Lagrange multiplier) such that

$$(L_y - \frac{d}{dx}L_{y'}) + \lambda * (M_y - \frac{d}{dx}M_{y'}) = 0 \tag{3.44}$$

for all $x \in [a, b]$. This amounts to saying that the Euler-Lagrange equation holds for the augmented Lagrangian $L + \lambda * M$. In other words, y is an extremal of the augmented cost functional

$$(J + \lambda * C)(y) = \int_a^b (L(x, y(x), y'(x)) + \lambda * M(x, y(x), y'(x)))dx \tag{3.45}$$

Hence, the **first-order necessary condition for constrained optimality:** If y$(\cdot)$ is an extremum for the constrained problem and is not an extremal of the constraint functional C (i.e., does not satisfy the Euler-Lagrange equation for M), then it is an extremal of the augmented cost functional for some $* \in R$. We can also state this condition more succinctly, combining the nondegeneracy assumption and the conclusion into one statement: y must satisfy the Euler-Lagrange equation for $\lambda_0^* + \lambda^* M$, where λ_0^* and λ^* are constants (not both 0). The number λ_0^* is called the abnormal multiplier.

3.4.2 Non-integral constraints

We now suppose that instead of the integral constraint, we have an equality constraint which must hold pointwise:

$$M(x, y(x), y'(x)) = 0 \tag{3.46}$$

for all $x \in [a, b]$.

Let y be a test curve, the Euler-Lagrange equation must hold for the augmented Lagrangian

$$L + \lambda^*(x)M$$

where $\lambda^* : [a, b] \to R$ is some function. Here we need to assume that there are at least two degrees of freedom and that everywhere along the curve we have $M_{y'}$ or, if y' does not appear in the constraint equation, $M_{y'}$.

3.5 Second-order conditions

$$J(\alpha\eta + y) = J(y) + \delta J|_y(\eta)\alpha + \delta^2 J|_y(\eta)\alpha^2 + o(\alpha^2) \tag{3.47}$$

which defines the quadratic form $\delta^2 J|_y$ called the *second variation*

3.5.1 Legendre's necessary condition for a weak minimum

Let us compute $\delta^2 J|_y$ for a given test curve y. The LHS is:

$$J(y + \alpha\eta) = \int_a^b L(x, y(x) + \alpha\eta(x), y'(x) + \alpha\eta'(x))dx \tag{3.48}$$

We need to write down its second-order Taylor expansion with respect to α. We do this by expanding the function inside the integral with respect to α (using the chain rule) and separating the terms of different orders in α. Matching the expressions term by term, we get that the second variation is given by:

$$\delta^2 J|_y(\eta) = \frac{1}{2}\int_a^b (L_{yy}\eta^2 + 2 * L_{yy'}\eta\eta' + L_{y'y'}(\eta')^2)dx \tag{3.49}$$

where the integrand is evaluated along (x,y(x),y'(x)).

d. We can simplify the expression for $\delta^2 J|_y(\eta)$ by eliminating the "mixed" term containing the product $\eta\eta'$. We do this by using—as we did in our earlier derivation of the Euler-Lagrange equation—the method of integration by parts. The first, non-integral term on the right-hand side vanishes due to the boundary conditions. Therefore, the second variation can be written as

$$\delta^2 J|_y(\eta) = \int_a^b (P(x)(\eta'(x))^2 Q(x)(\eta(x))^2)dx \tag{3.50}$$

where,

$$P(x) := \frac{1}{2}L_{y'y'}(x, y(x), y'(x)),\ \ Q(x) := \frac{1}{2}(L_{yy}(x, y(x), y'(x)) - \frac{d}{dx}L_{yy'}(x, y(x), y'(x))) \tag{3.51}$$

Note that P is continuous, and Q is also continuous at least when $y \in C^2$.

Consider a family of perturbations η_ϵ parameterized by small $\epsilon > 0$, depicted in Figure 2.12.

The function η_ϵ equals 0 everywhere outside some interval [c, d] belonging to [a, b], and inside this interval it equals 1 except near the endpoints where it rapidly goes up to 1 and back down to 0. This rapid transfer is accomplished by the derivative $\eta^{'_\epsilon}$ having a short pulse of width approximately ϵ and height approximately $1/\epsilon$ right after c, and a similar negative pulse right before d. Here ϵ is small compared to $d - c$.

We can see that

$$|\int_a^b Q(x)(\eta_\epsilon(x))^2 dx| \geq \int_c^d |Q(x)|dx \tag{3.52}$$

and this bound is uniform over ϵ. On the other hand, for nonzero P the integral $\int_a^b P(x)(\eta_\epsilon(x))^2 dx$ does not stay bounded as $\epsilon \to 0$, because it is of order $1/\epsilon$

Second-order necessary condition for optimality: For all $x \in [a, b]$ we must have

$$L_{y'y'}(x, y(x), y'(x)) \geq 0 \tag{3.53}$$

This condition is known as **Legendre's condition.**

3.5.2 Sufficient condition for a weak minimum

We are now interested in obtaining a second-order sufficient condition for proving optimality of a given test curve y. we want to have $\delta^2 J|_y(\eta) > 0$ for all admissible perturbations. In addition, we need some uniformity to be able to dominate the $o(\alpha^2)$ term.

The first step is to reduce the quadratic first-order differential equation to another differential equation, linear but of second order, by making the substitution

$$w(x) = -\frac{Pv'(x)}{v(x)} \tag{3.54}$$

where v is a new (unknown) function, twice differentiable and not equal to 0 anywhere. From this we obtain

$$P(Q - \frac{\frac{d}{dx}(Pv')v - P(v')^2}{v^2} = \frac{P(v')^2}{v^2} \tag{3.55}$$

Multiplying both sides of this equation by v (which is nonzero), dividing by P (which is positive), and canceling terms, we can bring it to the form

$$Qv = \frac{d}{dx}(Pv') \tag{3.56}$$

This is the so-called accessory, or Jacobi, equation.

Since the above equation is a second-order differential equation, the initial data at x = a needed to uniquely specify a solution consists of v(a) and v'(a). In addition, note that if v is a solution of the equation then v is also a solution for every constant . By adjusting appropriately, we can thus assume with no loss of generality that v'(a) = 1 (since we are not interested in v being identically 0). Among such solutions, let us consider the one that starts at 0, i.e., set v(a) = 0. A point c > a is said to be conjugate to a if this solution v hits 0 again at c, i.e., v(c) = v(a) = 0.

A linear differential equation that describes, within terms of higher order, the propagation of the difference between two nearby solutions of a given differential equation is called *the variational equation.* . In this sense, the Jacobi equation is the variational equation for the Euler-Lagrange equation.

We conclude that the second variation $\delta^2 J|_y$ is positive definite (on the space of admissible perturbations) if P(x) > 0 for all x [a, b] and there are no points conjugate to a on [a, b]. We remark in passing that the absence of points conjugate to a on [a, b] is also a necessary condition for $\delta^2 J|_y$ to be positive definite, and if $\delta^2 J|_y$ is positive semidefinite then no interior point of [a, b] can be conjugate to a. We are now ready to state the following second-order sufficient condition for optimality: An extremal y(·) is a strict minimum if $L_{y'y'}(x, y(x), y'(x)) > 0$ for all $x \in [a, b]$ and the interval [a, b] contains no points conjugate to a.

4. Optimal Control Formulation

4.1 Necessary conditions for strong extrema

4.1.1 Weierstrass-Erdmann corner conditions

A piecewise C1 curve y on [a, b] is C1 everywhere except possibly at a finite number of points where it is continuous but its derivative y' is discontinuous. Such points of discontinuity of y' are known as corner points. A corner point c [a, b] is characterized by the property that the left-hand derivative $y'(c^-) := lim_{x \to c} y'(x)$ and the right-hand derivative $y'(c^+) := lim_{x \to c} y'(x)$ both exist but have different values.

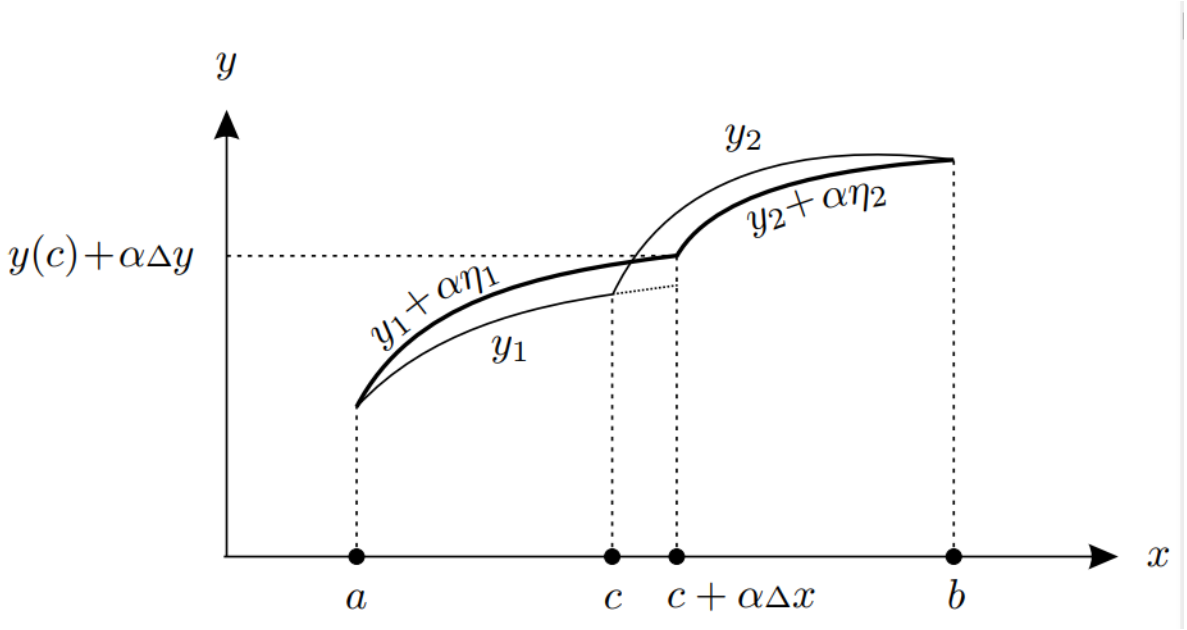

Figure 4.1: A perturbation of an extremal with a corner

The **Weierstrass-Erdmann corner conditions:** If a curve y is a strong extremum, then $L_{y'}$ and $y'L_{y'}L$ must be continuous at each corner point of y. More precisely, their discontinuities (due to the fact that y' does not exist at corner points) must be removable.

4.1.2 Weierstrass excess function

For a given Lagrangian L = L(x, y, z), the Weierstrass excess function, or E-function, is defined as

$$E(x, y, z, w) := L(x, y, w)L(x, y, z)(wz)\Delta L_z(x, y, z) \tag{4.1}$$

The **Weierstrass necessary condition for a strong minimum** states that if y(·) is a strong minimum, then

$$E(x, y(x), y'(x), w) \geq 0 \tag{4.2}$$

for all noncorner points $x \in [a, b]$ and all $w \in R$.

4.2 Optimal control problem formulation and assumptions

4.2.1 Control System

Control systems that we want to study take the form

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0 \tag{4.3}$$

First, to assure sufficient regularity of f with respect to t, we take f(·, x) to be piecewise continuous for each fixed x. Here, by a piecewise continuous function we mean a function having at most a finite number of discontinuities on every bounded interval, and possessing the limits from the right and from the left at each of these discontinuities. For convenience, we assume that the value of such a function at each discontinuity is equal to one of these one-sided limits (i.e., the function is either leftcontinuous or right-continuous at each point).

The assumption of a finite number of discontinuities on each bounded interval is actually not crucial; we can allow discontinuities to have accumulation points, as long as the function remains locally bounded (or at least locally integrable).

Second, we need to specify how regular f should be with respect to x. A standard assumption in this regard is that f is locally Lipschitz in x, uniformly over t.

A function x(·) that can be represented as an integral of another function g(·), and thus automatically satisfies $\dot{x} = g$ almost everywhere, is called absolutely continuous. This class of functions generalizes the piecewise C^1 functions that we considered earlier. Basically, the extra generality here is that the derivative can be discontinuous on a set of points that has measure zero (e.g., a countable set) rather than at a finite number of points on a bounded interval, and can approach infinity near these points. If we insist that the derivative be locally bounded, we arrive at the slightly smaller class of locally Lipschitz functions.

Coming back to the original controls problem, we finally have:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), u(s))ds \tag{4.4}$$

4.2.2 Cost functional

We will consider cost functionals of the form

$$J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t))dt + K(t_f, x_f) \tag{4.5}$$

Optimal control problems in which the cost is given by above equation are known as problems in the Bolza form, or collectively as the Bolza problem. There are two important special cases of the Bolza problem. The first one is the Lagrange problem, in which there is no terminal cost: K 0. This problem—and its name—of course come from calculus of variations. The second special case is the Mayer problem, in which there is no running cost: L 0. We can pass back and forth between these different forms by means of simple transformations.

4.2.3 Target set

The target set $S = [t_0,)(x_1)$, where x1 is a fixed point in R^n, gives a free-time, fixed-endpoint problem. A generalization of this is to consider a target set of the form $S = [t_0,)S_1$, where S1 is a surface (manifold) in R^n. Another natural target set is $S = t_1 R^n$, where t1 is a fixed time in $[t_0, \infty)$; this gives a fixed-time, free-endpoint problem. It is useful to observe that if we start with a fixed-time, free-endpoint problem and consider again the auxiliary state $x_{n+1} := t$, we recover the previous case with $S1 \in R^{n+1}$ given by $[xR^{n+1} : x_{n+1} = t_1]$. A target set $S = TS_1$, where T is some subset of $[t_0, \infty)$ and S_1 is some surface in R_n, includes as special cases all the target sets mentioned above. It also includes target sets of the form $S = t_1 x_1$, which corresponds to the most restrictive case of a fixed-time, fixed-endpoint problem. As the opposite extreme, we can have $S = [t_0, \infty)R^n$, i.e., a free-time, free-endpoint problem.

4.3 Variational approach to the fixed-time, free-endpoint problem

5. The Maximum Principle

5.1 Statement of the Maximum Principle

This section is devoted to the maximum principle, which is in some sense the focus of this study. Let us first describe two special instances of the optimal control problem, for which we want to state and then prove the maximum principle. Later we will have a look at the other cases of interest.

5.1.1 Basic fixed-endpoint control problem

Control systems that we want to study take the form $x = f(x, u)$ and the running cost is: $L = L(x, u)$ (both have no t-argument and are therefore time independent). For this special problem, the maximum principle takes the following form:

Maximum Principle for the Basic Fixed-Endpoint Control Problem:
Let $u^* : [t_0, t_f] \to U$ be an optimal control (in the global sense) and let $x^* : [t_0, t_f] \to R^n$ be the corresponding optimal state trajectory. Then there exist a function $p^* : [t_0, t_f] \to R^n$ and a constant $p_0^* \leq 0$ satisfying $(p_0^*, p^*(t)) \neq (0, 0)$ for all $t \in [t_0, t_f]$ and having the following properties:
1) x^* and p^* satisfy the canonical equations

$$\dot{x}^* = H_p(x^*, u^*, p^*, p_0^*); \qquad \dot{p}^* = -H_x(x^*, u^*, p^*, p_0^*) \tag{5.1}$$

with the boundary conditions $x^*(t_0) = x_0$ and $x^*(t_f) = x_1$, where the Hamiltonian $H : R^n \times$

$U \times R^n \times R \to R$ is defined as

$$H(x, u, p, p_0) := \langle p, f(x, u) \rangle + p_0 L(x, u) \tag{5.2}$$

2) For each fixed t, the function $u \mapsto H(x^*(t), u, p^*(t), p_0^*)$ has a global maximum at $u = u^*(t)$, i.e., the inequality

$$H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x^*(t), u, p^*(t), p_0^*)$$

holds for all $t \in [t_0, t_f]$ and all $u \in U$.
3) $H(x^*(t), u^*(t), p^*(t), p_0^*) = 0$ for all $t \in [t_0, t_f]$.

5.1.2 Basic variable-endpoint control problem

The Basic Variable-Endpoint Control Problem is the same as the Basic Fixed-Endpoint Control Problem except the target set is now of the form $S = [t_0, \infty) \times S_1$, where S_1 is a k-dimensional surface in R^n for some non-negative integer $k \leq n$. We define such a surface via equality constraints:

$$S_1 = \{x \in R^n : h_1(x) = h_2(x) = \cdots = h_{n-k}(x) = 0\}$$

where $h_i, i = 1, \ldots, n-k$ are C^1 functions from R^n to R. We also assume that every $x \in S^1$ is a regular point.

Maximum Principle for the Basic Variable-Endpoint Control Problem:
Let $u^* : [t_0, t_f] \to U$ be an optimal control and let $x^* : [t_0, t_f] \to R^n$ be the corresponding optimal state trajectory. Then there exist a function $p^* : [t_0, t_f] \to R^n$ and a constant $p_0^* \leq 0$ satisfying $(p_0^*, p^*(t)) \neq (0, 0)$ for all $t \in [t_0, t_f]$ and having the following properties:
1) x^* and p^* satisfy the canonical equations (5.1) with respect to the Hamiltonian (5.2), with the boundary conditions $x^*(t_0) = x_0$ and $x^*(t_f) \in S_1$.
2) $H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x^*(t), u, p^*(t), p_0^*)$ for all $t \in [t_0, t_f]$ and all $u \in U$.

3) $H(x^*(t), u^*(t), p^*(t), p_0^*) = 0$ *for all* $t \in [t_0, t_f]$.

4) The vector $p^*(t_f)$ is orthogonal to the tangent space to S_1 at $x^*(t_f)$:

$$\langle p^*(t_f), d\rangle = 0 \qquad \forall d \in T_{x^*(t_f)} S_1. \tag{5.3}$$

Here the tangent space can be characterized as

$$T_{x^*(t_f)} S_1 = \{d \in R^n : \langle \bigtriangledown h_i(x^*(t_f)), d\rangle = 0, \quad i = 1, \ldots, n-k\} \tag{5.4}$$

5.2 Proof of the Maximum Principle

5.2.1 From Lagrange to Mayer form

We define an additional state variable, $x^0 \in R$, to be the solution of

$$\dot{x}^0 = L(x, u), \qquad x^0(t_0) = 0$$

and arrive at the augmented system

$$\dot{x}^0 = L(x, u), \qquad \dot{x} = f(x, u), \tag{5.5}$$

with initial condition $\begin{pmatrix} 0 \\ x_0 \end{pmatrix}$. The cost can then be rewritten as

$$J(u) = \int_{t_0}^{t_f} \dot{x}^0(t) dt = x^0(t_f) \tag{5.6}$$

which means that in the new coordinates the problem is in the Mayer form (there is a terminal cost and no running cost). Also, the target set becomes $[t_0, \infty) \times R \times \{x_1\} =: [t_0, \infty) \times S'$; here S' is the line in R^{n+1} that passes through $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$ and is parallel to the x^0-axis. To simplify the notation, we define

$$y := \begin{pmatrix} x^0 \\ x \end{pmatrix} \in R^{n+1}$$

and write the system (5.5) more compactly as

$$\dot{y} = \begin{pmatrix} L(x,u) \\ f(x,u) \end{pmatrix} =: g(y,u) \tag{5.7}$$

Also, the Hamiltonian (5.2) can be equivalently represented as the following inner product in R^{n+1}:

$$H(x,u,p,p_0) = \left\langle \begin{pmatrix} p_0 \\ p \end{pmatrix}, \begin{pmatrix} L(x,u) \\ f(x,u) \end{pmatrix} \right\rangle \tag{5.8}$$

5.2.2 Temporal control perturbation

Now we introduce a small change in the terminal time t^* of the optimal trajectory, i.e., let the optimal control act on a little longer or a little shorter time interval. We formalize this as follows: for an arbitrary $\tau \in R$ and a small $\epsilon > 0$, we consider the perturbed control (which has two cases depending on the sign of τ)

$$u_\tau(t) := u^*(min\{t, t^*\}), \quad t \in [t_0, t^* + \epsilon\tau]$$

We get the value of the resulting perturbed trajectory y at the new terminal time $t^* + \epsilon\tau$. For $\tau > 0$, the first-order Taylor expansion of y around $t = t^*$ gives

$$y(t^* + \epsilon\tau) = y^*(t^*) + \dot{y}(t^*)\epsilon\tau + o(\epsilon) = y^*(t^*) + g(y^*(t^*), u^*(t^*))\epsilon\tau + o(\epsilon) =: y^*(t^*) + \epsilon\delta(\tau) + o(\epsilon) \tag{5.9}$$

As we vary τ over R, keeping ϵ fixed, the points $y^*(t^*) + \epsilon\delta(\tau)$ form a line through $y^*(t^*)$. We denote this line by $\vec{\rho}$. Every point on $\vec{\rho}$ corresponds to a control u_τ for some τ. On the other hand, the approximation of $y(t^* + \epsilon\tau)$ by $y^*(t^*) + \epsilon\delta(\tau)$ is valid only in the limit as $\epsilon \to 0$. So, $\delta(\tau)$ tells us the direction (but not the magnitude) of the terminal point deviation caused by an infinitesimal change in the terminal time. The arrow over ρ is meant to indicate that points on the line correspond to perturbation directions.

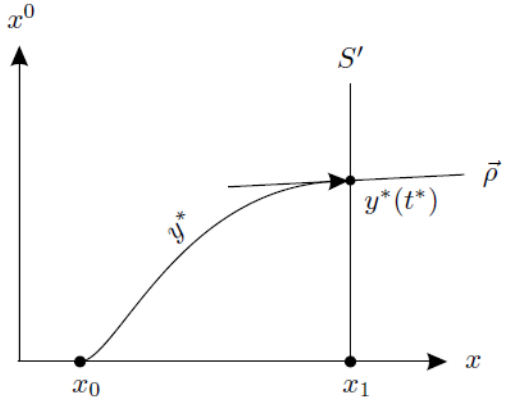

Figure 5.1: Effect of the Temporal Control Perturbation

5.2.3 Spatial control perturbation

Let us now introduce needle perturbations. Let w be an arbitrary element of the control set U. Consider the interval $I := (b - \epsilon a, b] \subset (t0, t^*)$, where $b \neq t^*$ is a point of continuity of u^*, $a > 0$ is arbitrary, and $a > 0$ is small. We define the perturbed control

$$u_{w,I}(t) := \begin{cases} u^*(t) & if\ t \notin I, \\ w & if\ t \in I. \end{cases}$$

The figure (5.2) illustrates this control perturbation and the resulting state trajectory perturbation. As the figure suggests, the perturbed trajectory y corresponding to u_w, I will deviate from y^* on the interval I and afterwards will "run parallel" to y^*.

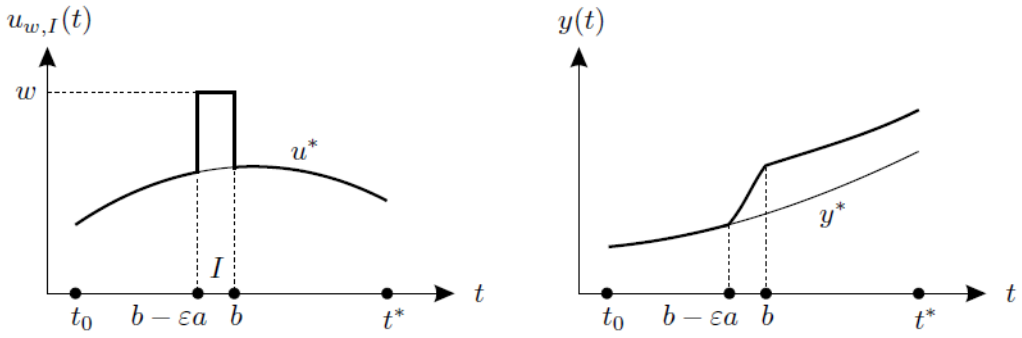

Figure 5.2: Effect of the Spacial Control Perturbation

The first-order Taylor expansion of y^* around $t = b$ and of y around $t = b - \epsilon a$ with proper rearranging gives

$$y(b) \approx y^*(b) + \nu_b(w)\epsilon a \tag{5.10}$$

where

$$\nu_b(w) := g(y^*(b), w) - g(y^*(b), u^*(b)). \tag{5.11}$$

Intuitively, this result makes sense: up to terms of order $o(\epsilon)$, the difference between the two states $y(b)$ and $y^*(b)$ is the difference (5.10) between the state velocities at $y = y^*(b)$ corresponding to $u = w$ and $u = u^*(b)$, multiplied by the length ϵa of the time interval on which the perturbation is acting.

5.2.4 Variational equation

Now we see how the difference between the trajectory y arising from a spatial (needle) perturbation and the optimal trajectory y^* propagates after the perturbation stops acting, i.e., for $t \geq b$. To study this question, let us begin by writing

$$y(t) = y^*(t) + \epsilon\psi(t) + o(\epsilon) =: y(t, \epsilon) \tag{5.12}$$

for $b \leq t \leq t^*$, where $\psi : [b, t^*] \to R^{n+1}$ is a quantity that we want to characterize. We know that $\psi(b) = \nu_b(w)a$. After manipulating the equations we get,

$$\psi(t) = \nu_b(w)a + \int_b^* t g_y(y^*(s), u^*(s))\psi(s)ds.$$

Taking the derivative with respect to t, we conclude that satisfies the differential equation

$$\dot{\psi} = g_y(y^*, u^*)\psi = g_y|_*\psi \tag{5.13}$$

The equation is used to describe how spatial perturbations propagate with time. Pictorially, the role of is illustrated in figure (5.3), with the understanding that the labels involving are accurate only up to terms of order $o(\epsilon)$.

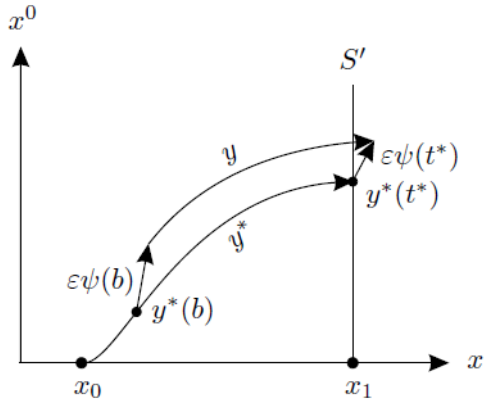

Figure 5.3: Propagation of the Spacial Control Perturbation

Let us denote by $\phi_*(.,.)$ the state transition matrix for the linear time-varying system, so that

$$\psi(t^*) = \phi_*(t^*, b)\psi(b)$$

Let us introduce the notation

$$\delta(w, I) := \phi_*(t^*, b)\nu_b(w)a \tag{5.14}$$

to arrive at the more compact formula

$$y(t^*) = y^*(t*) + \epsilon\delta(w, I) + o(\epsilon)$$

(here the interval I used for constructing the needle perturbation encodes information about the values of a and b).

5.2.5 Terminal cone

Now lets discuss the combined effects of the temporal and spacial control perturbations on the terminal state. the vector $\epsilon\delta(w, I)$ describes the infinitesimal (first-order in ϵ) perturbation with parameters w and I. We let $\vec{\rho}(w, b)$ denote the ray in this direction originating at $y^*(t^*)$. If we keep w, b, and ϵ fixed, $\vec{\rho}(w, b)$ consists of the points $y^*(t^*) + \epsilon\delta(w, I)$ for various values of a. We also let $\vec{P}$ denote the union of the rays $\vec{\rho}(w, b)$ for all possible values of w and b. Then $\vec{P}$ is a cone with vertex at $y^*(t^*)$. Now adding in the line $\vec{\rho}$ of perturbation directions arising from temporal perturbations of u^* to convex cone $co(\vec{P})$ which consists exactly of convex combinations of points in $\vec{P}$,

$$C_{t*} := y = y^*(t^*) + \epsilon\left(\beta_0\delta(\tau) + \sum_{i=1}^{m}\beta_i\delta(w_i, I_i)\right) \tag{5.15}$$

where $\epsilon > 0, \beta_0, \beta_1, \ldots, \beta_m \geq 0$. The set C_{t*} is called *terminal cone*.

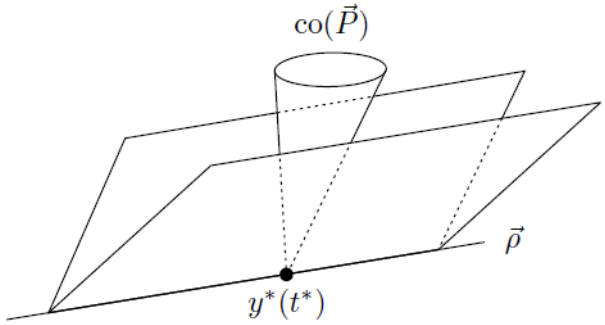

Figure 5.4: The Terminal Cone

5.2.6 Key topological lemma

Optimality means that no other trajectory y corresponding to another control u can reach the line S0 (the vertical line through $\begin{pmatrix} 0 \\ x_1 \end{pmatrix}$ in the y-space) at a point below $y^*(t^*)$. Since the terminal cone C_{t^*} is a linear approximation of the set of points that we can reach by applying perturbed controls, we expect that the terminal cone should face "upward." Now, consider the vector

$$\mu := (-1\ 0\ \cdots\ 0)^T \in R^{n+1} \tag{5.16}$$

and let $\vec{\mu}$ be the ray generated by this vector (which points downward) originating at $y^*(t^*)$. Optimality suggests that $\vec{\mu}$ should be directed outside of C_{t^*}

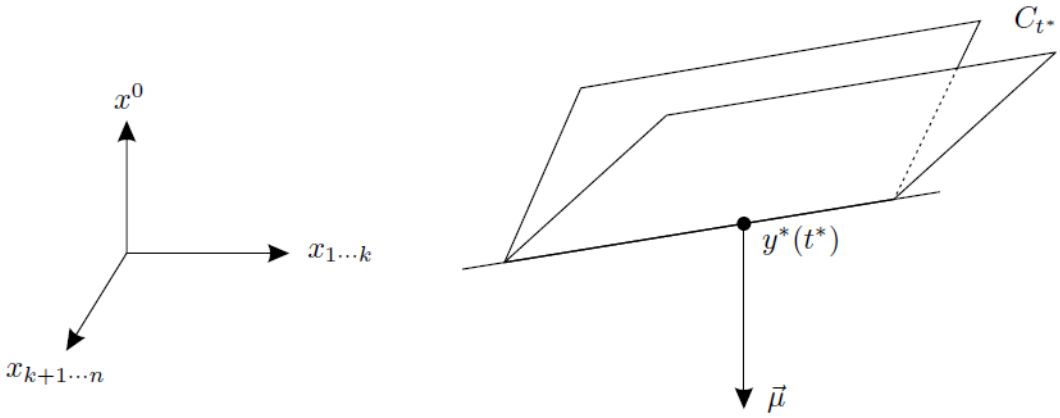

Figure 5.5: $\vec{\mu}$ does not intersect the interior of the cone C_{t^*}

5.2.7 Separating hyperplane

The ray $\vec{\mu}$ is a convex set, and from the convexity of the terminal cone C_{t^*} it is easy to see that its interior is convex as well. We already know that $\vec{\mu}$ does not intersect the interior of C_{t^*}. Therefore, we can apply the Separating Hyperplane Theorem to conclude the existence of a hyperplane separating $\vec{\mu}$ from C_{t^*}. Such a hyperplane need not be unique. The normal to

the hyperplane is a nonzero vector in R^{n+1}. Let us denote this normal vector by

$$\begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix} \tag{5.17}$$

where $p_0^* \in R$ and $p^*(t^*) \in R^n$ are, by definition, its x^0-component and x-component, respectively. Then the equation of the hyperplane is

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, y \right\rangle = \left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, y^*(t^*) \right\rangle$$

and the separation property is formally written as

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \delta \right\rangle \leq 0 \qquad \forall\ \delta\ \textit{such that}\ y^*(t^*) + \delta \in C_{t^*} \tag{5.18}$$

and

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \mu \right\rangle \geq 0 \tag{5.19}$$

In view of the definition (5.16) of μ, the inequality simply says that $p_0^* \leq 0$, as required by the statement of the maximum principle.

5.2.8 Adjoint equation

From the variational equation, we get the adjoint system of (5.13) as

$$\dot{z} = -A_*^T = \begin{pmatrix} 0 & 0 \\ -L_x|_*(t) & -(f_x)^T|_*(t) \end{pmatrix} z \tag{5.20}$$

Let us denote the first component of z by p_0 and the vector of the remaining n components of z by p. We get,

$$\dot{p} = -H_x(x^*, u^*, p, p_0)$$

Now, let us specify the terminal condition for the system (5.20) at time t^* by setting $z(t^*)$ equal to the vector (5.17)

$$\dot{p}_0(t) = \dot{p}^* = -H_x(x^*, u^*, p^*, p_0^*)$$

Since (5.20) is a homogeneous (unforced) linear time-varying system, we have

$$\begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix} \neq 0 \qquad \forall \in [t_0, t^*] \tag{5.21}$$

as required in the statement of the maximum principle.

5.2.9 Properties of the Hamiltonian

We are now ready to prove the remaining properties of the Hamiltonian of the maximum principle for the Basic Fixed Endpoint Control Problem.

Hamiltonian Maximization condition

Using inequality (5.18) and definition (5.14) for positive ϵ and a we get,

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \phi_*(t^*, b)\nu_b(w) \right\rangle \leq 0$$

Invoking the adjoint property we obtain

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}, \nu_b(w) \right\rangle \leq 0 \tag{5.22}$$

From the definition of $\nu_b(w)$ we have,

$$\nu_b(w) = g(y^*(b), w) - g(y^*(b), u^*(b)) = \begin{pmatrix} L(x^*(b), w) - L(x^*(b), u^*(b)) \\ f(x^*(b), w) - f(x^*(b), u^*(b)) \end{pmatrix}$$

Thus we can expand (5.22) as follows:

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}, \begin{pmatrix} L(x^*(b), w) \\ f(x^*(b), w) \end{pmatrix} \right\rangle \leq \left\langle \begin{pmatrix} p_0^* \\ p^*(b) \end{pmatrix}, \begin{pmatrix} L(x^*(b), u^*(b))) \\ f(x^*(b), u^*(b))) \end{pmatrix} \right\rangle$$

Recalling the expression (5.8) for the Hamiltonian, we see that

$$H(x^*(b), w, p^*(b), p_0^*) \leq H(x^*(b), u^*(b), p^*(b), p_0^*). \tag{5.23}$$

In the above derivation, w was an arbitrary element of the control set U and b was an arbitrary time in the interval (t_0, t^*) at which the optimal control u^* is continuous. Thus we have established that the Hamiltonian maximization condition holds everywhere except possibly a finite number of time instants (discontinuities of u^*).

$H|_* = \mathbf{0}$ **everywhere**

From (5.9) and (5.7) we get $\delta(\tau) = \begin{pmatrix} L(x^*(t^*), u^*(t^*))) \\ f(x^*(t^*), u^*(t^*))) \end{pmatrix} \tau$. Since τ can be either positive or negative, the inequality (5.18) can be satisfied only if

$$\left\langle \begin{pmatrix} p_0^* \\ p^*(t^*) \end{pmatrix}, \begin{pmatrix} L(x^*(t^*), u^*(t^*))) \\ f(x^*(t^*), u^*(t^*))) \end{pmatrix} \right\rangle = 0.$$

In other words, $H(x^*(t^*), u^*(t^*), p^*(t^*), p_0^*)$ or $H|_*$ equals 0 at the terminal time (t^*). Additionally the function $H(x^*(.), u^*(.), p^*(.), p_0^*)$ is continuous everywhere, and has zero time derivative almost everywhere (using canonical equations (5.1)). Thus it is identically 0, as claimed.
Our proof of the maximum principle for the **Basic Fixed-Endpoint Control Problem** is now complete.

5.2.10 Transversality condition

We now turn to the Basic Variable-Endpoint Control Problem. In this case there is an additional statement to be proved, which is the transversality condition (5.3). When the fixed endpoint x_1 is replaced by the surface S_1, we would instead have a contradiction with optimality if we were able to hit a point with a cost lower than $x^{0,*}(t^*)$ whose x-component is in S_1 (but is not necessarily $x^*(t^*)$). Let us denote the set of such points by D. We are looking to establish separation between convex sets; for this reason, just as we replace the actual set of terminal points with its linear approximation C_{t^*} , we will consider the linear approximation

of D given by the linear span of $\vec{\mu}$ and the tangent space $T_{x^*(t^*)}S_1$, i.e., the set

$$T := \left\{ y \in R^{n+1} \ : \ y = y^*(t^*) + \begin{pmatrix} 0 \\ d \end{pmatrix} + \beta\mu, \ d \in T_{x^*(t^*)}S_1, \ \beta \geq 0 \right\}. \tag{5.24}$$

As T does not intersect the interior of the cone C_{t^*} and by Separating Hyperplane Theorem, we say that there exists a hyperplane that separates T and C_{t^*}. We denote its normal vector by (5.17) as before. Thus we finally obtain that $\langle p^*(t^*), d\rangle = 0$ for all $d \in T_{x^*(t^*)}S_1$ which is precisely the desired transversality condition (5.3). Our proof of the maximum principle for the **Basic Variable-Endpoint Control Problem** is now complete.

Note that in the special case when $S_1 = R^n$ (a free-time, free-endpoint problem), the hyperplane separates C_{t^*} from the entire (n +1)-dimensional half-space that lies below $y^*(t^*)$. Clearly, this hyperplane must be horizontal, hence its normal must be vertical and we conclude that $p^*(t^*) = 0$. This is consistent with (4.3) because $T_{x^*(t^*)}S_1 = R^n$ in this case.

5.3 Discussion of the Maximum Principle

It should always be remembered that the maximum principle provides necessary conditions for optimality. Thus it only helps single out optimal control candidates, each of which needs to be further analyzed to determine whether it is indeed optimal. There is a possibility of non-existence of an optimal control.

Some control problems that we are interested in do not fit into the setting of the Basic Fixed-Endpoint or Variable-Endpoint Control Problem. We will now discuss several such scenarios and, for each one, to arrive at the correct statement of the maximum principle by adapting the proof of the maximum principle to these new situations.

5.3.1 Fixed terminal time

Suppose that the terminal time t_f is fixed, so that the terminal time t of the optimal trajectory x^* considered in the proof must equal a given value t_1. Temporal control perturbations are then no longer admissible. Accordingly, the line $\vec{\rho}$ formed by the perturbation directions $\delta(\tau)$ defined in (5.9) must not be used when generating the terminal cone C_{t^*}. These perturbation directions were used only to show that $H|_*(t^*) = 0$. Thus we conclude that the Hamiltonian will now be constant but not necessarily 0 along the optimal trajectory, while all the other conditions remain unchanged.

5.3.2 Time-dependent system and cost

In this kind of a system, f and/or the running cost L depend on t. The Hamiltonian is now time dependent:

$$H(t, x, u, p, p_0) := \langle p, f(t, x, u)\rangle + p_0 L(t, x, u). \tag{5.25}$$

The previous discussion remains valid up to and including the equation $\dot{p}^*_{n+1} = -H_t|_*$ but the right-hand side no longer equals 0. Thus, p^*_{n+1} and $H|_* = -p^*_{n+1}$ are not constant any more. Instead, we have the differential equation

$$\frac{d}{dt}H|_* = H_t|_* \tag{5.26}$$

with the boundary condition $H|_*(t_f) = -p^*_{n+1}(t_f)$. If the terminal time t_f in the original problem is free, then the final value of x_{n+1} is free and the transversality condition yields $p^*_{n+1}(t_f) = 0$. In this case we obtain $H|_*(t_f) = 0$.

5.3.3 Terminal cost

Let us now consider a situation where a terminal cost of the form $K = K(x_f)$ is present. To illustrate just one simple case, we suppose that we are dealing with a free-time, free-endpoint

problem in the Mayer form, i.e., there is no running cost ($L \equiv 0$). We assume the function K to be differentiable as many times as desired; everything else is as in the Basic Variable-Endpoint Control Problem.

Since $L \equiv 0$, there is no need to consider the x^0-coordinate. Temporal and spatial control perturbations can be used to construct the terminal cone C_{t^*} as before, except that it now lives in the original x-space: $C_{t^*} \subset R^n$. By optimality, and since the terminal state is free, no perturbed state trajectory can have a terminal cost lower than $K(x^*(t^*))$, the terminal cost of the candidate optimal trajectory. Thus we expect that K should not decrease along any direction in C_{t^*}, a property that we can write as

$$\langle -K(x^*(t^*)), \delta \rangle \leq 0 \qquad \forall \delta \text{ such that } x^*(t^*) + \delta \in C_{t^*} \tag{5.27}$$

Geometrically, this means that C_{t^*} lies on one side of the hyperplane passing through $x^*(t^*)$ with normal $-K(x^*(t^*))$ which we henceforth assume to be a nonzero vector. A comparison of (5.27) with (5.18) unmistakably suggests that we should define

$$p^*(t^*) := -K(x^*(t^*)). \tag{5.28}$$

5.3.4 Initial sets

We now want to mention how the maximum principle can be extended in another direction. We usually assume that the initial state x_0 is fixed, while the final state x_f may vary within some set S_1. Here, let us briefly consider the possibility that x_0 may vary as well, so that we have an initial set instead of a fixed initial state. We can impose separate constraints on x_0 and x_f or, more generally, we can require that $\begin{pmatrix} x_0 \\ x_f \end{pmatrix}$ belong to some surface S_2 in R^{2n}. The terminal time t_f can be either free or fixed, as before.

It turns out that this more general set-up can be easily handled by modifying the transversality condition, which will now involve the values of the costate both at the initial time and at the

final time. Namely, the transversality condition will now say that the vector $\begin{pmatrix} p^*(t_0) \\ -p^*(t_f) \end{pmatrix}$ must be orthogonal to the tangent space to S_2 at $\begin{pmatrix} x^*(t_0) \\ x^*(t_f) \end{pmatrix}$:

$$\left\langle \begin{pmatrix} p^*(t_0) \\ -p^*(t_f) \end{pmatrix}, \; d \right\rangle = 0 \qquad \forall \; d \in T_{\begin{pmatrix} x^*(t_0) \\ x^*(t_f) \end{pmatrix}} S_2. \tag{5.29}$$

The total number of boundary conditions for the system of canonical equations is still $2n$, since each additional degree of freedom for $x^*(t_0)$ leads to one additional constraint on $p^*(t_0)$.

6. Problem formulation & Results

6.1 Problem Statement

The problem we try to solve here is the famous Zelmelo's navigation problem. Consider ship that has to travel through a region of strong currents. The ship is assumed to have constant speed V but its heading angle can be varied (control input). The current is assumed to be in the y direction with maximum speed of W which varies parabolically as a function of its distance from banks of the river. The goal is to minimize the time required to reach the desired position on the other bank from the start location.

6.2 Problem Formulation

The motion of the boat is then given by the dynamics

$$\dot{x} = Vcos(u) \tag{6.1}$$

$$\dot{y} = Vsin(u) + R(x) \tag{6.2}$$

where $R(x)$ is the river velocity at the given location x.

$$R(x) = W\left(\frac{4x}{x_f} - \frac{4x^2}{x_f^2}\right)$$

here W is the peak velocity of the river which is observed at its centre.

$$\begin{aligned} x(0) &= x_0; \qquad y(0) = y_0, \\ x(t_f) &= x_f; \qquad y(t_f) = y_f. \end{aligned}$$

The breadth of the river is therefore $b = (x_f - x_0)$. The end time t_f is free and it is to be minimised. The problem can thus be characterised as the *Basic Fixed-Endpoint Control Problem.*

The cost function for the problem is

$$J = \int_0^{t_f} L \; dt \; = \int_0^{t_f} 1 \; dt \; = \; t_f \tag{6.3}$$

with boundary conditions $x_0 = y_0 = 0, \; x_f = 500, \; y_f = 150$. Let the values of boat velocity and peak river-water velocity be $V = 12$, $W = -4$.

6.3 Optimality Conditions

The Hamiltonian for the given problem is

$$H := L \; + \; p^T f = 1 + p_1 V cos(u) + p_2(V sin(u) + R(x)). \tag{6.4}$$

System dynamics are given by the equations (6.1) and (6.2).
Co-state equation is

$$\dot{p}^* = -H_x = \begin{bmatrix} p_2 W \left(\frac{8x}{x_f^2} - \frac{4}{x_f} \right) \\ 0 \end{bmatrix}. \tag{6.5}$$

Stationary Condition is

$$H_u = -p_1 V sin(u) + p_2 V cos(u) = 0 \quad \rightarrow \quad u = atan\left(\frac{p_2}{p_1}\right) \tag{6.6}$$
$$H_{uu} = -p_1 V cos(u) - p_2 V sin(u) \geq 0$$

Finally the Transversality condition is

$$H(x^*(t_f), u^*(t_f), p^*(t_f)) = 0 \tag{6.7}$$

6.4 Numeric Solution of the Boundary Value Problem

We use a shooting method (fsolve, ode45) to numerically solve[2] the BVP in MATLAB. A tranformation to a fixed endtime t_f is helpful to get numeric stability.

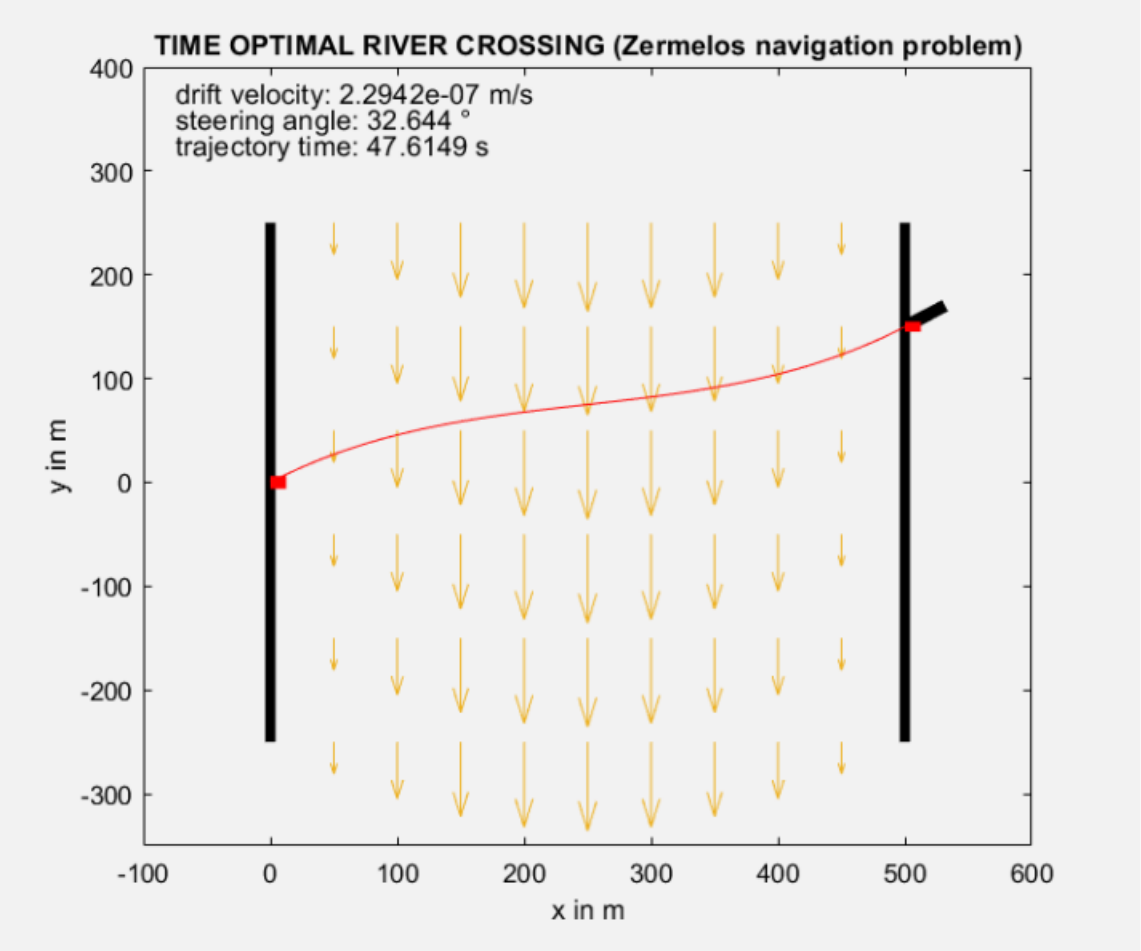

Figure 6.1: Optimal path solution

Figure (6.1) shows the optimal path traced by the boat to reach from initial position to the desired one in minimum time. The total time required to cross the river with available resources is 47.6149 sec.

Bibliography

[1] Daniel Liberzon. *Calculus of Variations and Optimal Control Theory*. Princeton University Press, 2011.

[2] Patrick Suhm. Time optimal river crossing.