Finite difference approximation for two-dimensional time fractional diffusion equation*

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ABSTRACT
Fractional diffusion equations have recently been used to model problems in physics, hydrology, biology and other areas of application. In this paper, we consider a two-dimensional time fractional diffusion equation (2D-TFDE) on a finite domain. An implicit difference approximation for the 2D-TFDE is presented. Stability and convergence of the method are discussed using mathematical induction. Finally, a numerical example is given. The numerical result is in excellent agreement with our theoretical analysis.

Key words: Two-dimensional time fractional differential equation; implicit difference approximation; stability; convergence.

1. INTRODUCTION

There has been increasing interest in the description of physical and chemical processes by means of equations involving fractional derivatives and integrals over the last decade [19]. Fractional differential equations provide a powerful instrument for the description of memory and hereditary properties of different substances. Fractional derivatives are used to model anomalous diffusion [18]. Fractional kinetic equation equations have proved particularly useful in the context of anomalous slow diffusion (subdiffusion) [18]. Subdiffusive motion is characterized by an asymptotic long-time behavior of the mean square

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displacement of the form

$$\langle x^2(t)\rangle \sim \frac{2K_\alpha}{\Gamma(1+\alpha)}t^\alpha, t \to \infty, \tag{1}$$

where $0 < \alpha < 1$ is the anomalous diffusion exponent. This process is usually referred to as subdiffusive. A fractional diffusion equation can be used to describe subdiffusion. Ordinary (or Brownian) diffusion corresponds to $\alpha = 1$ with $K_1 = D$ (the ordinary diffusion coefficient). Subdiffusive motion is particularly important in the context of complex systems such as glassy and disordered materials, in which pathways are constrained for geometric or energetic reasons. Diffusion equations that use time fractional derivatives are attractive because they describe a wealth of non-Markovian random walks [13].

The time-fractional diffusion equations (TFDE) is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order α with $0 < \alpha < 1$ (in Riemann-Liouville or Caputo sense). It represents anomalous sub-diffusion. It is a well established fact that this is equation models various phenomena. Fractional differential model is much more well-suited to physical problems than its differential parner since it makes less unnecessary or over-restricted assumptions which may change the problem being solved. Metzler and Klafter [18] have demonstrated that fractional diffusion equation describes a non-Markovian diffusion process with a memory. Ginoa et al. [5] have presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials. The fractional diffusion equation has been treated in different contexts by a number of authors. Wyss [21] considered the time fractional diffusion equation and the solution is given in closed form in terms of Fox functions. Schneider and Wyss [20] considered the time fractional diffusion and wave equations. The corresponding Green functions are obtained in closed form for arbitrary space dimensions in terms of Fox functions and their properties are exhibited. Gorenflo et al. [6] used the similarity method and the method of Laplace transform to obtain the scale-invariant solution of time-fractional diffusion-wave equation in terms of the wright function. Liu et al. [10] considered time-fractional advection-dispersion equation and derived the complete solution. Huang and Liu [8] considered the time-fractional diffusion equations in a n-dimensional whole-space and half-space. They investigate the explicit relationships between the problems in whole-space with the corresponding problems in half-space by the Fourier-Laplace transforms. Anh and Leonenko

[2] presented a spectral representation of the mean-square solution of the fractional diffusion equation with random initial condition. Enzo et al. [3] and Luisa et al. [14] considered and proved the solutions to the Cauchy problem of the fractional telegraph equation can be expressed as the distribution of a suitable composition of different processes.

However, published papers on the numerical solution of fractional partial differential equations are sparse. Liu et al. [11,12] used fractional Method of Lines to solve the space fractional diffusion equation, they transform this partial differential equation into a system of ordinary differential equations. Fix and Roop [4] developed a finite element method for a two-point boundary value problem. Meerschaert et al. have done a lot of excellent works. They proposed finite difference approximations for two-sided space-fractional partial differential equations [15] and fractional advection-dispersion flow equations [16]. Liu et al. [13] considered a discrete non-Markovian random walk approximation for the time fractional diffusion equation and discussed the stability and convergence of the approximation. Meerschaert et al. [17] also proposed a finite difference approximation for the two-dimensional space-fractional dispersion equation. The standard discretization methods (backward scheme for time domain, and central scheme for spatial domain) cannot be used for solving the time fractional partial differential equations. Yuste and Acedo [22] proposed an explicit finite difference method and a new Von Neumann-type stability analysis for the time fractional diffusion equation in one-dimension, and published their results in SIAM J. Numer Anal. (Vol.42, No.5, 2005, 1862-1874). However, they did not give the convergence analysis and pointed out that it is not such an easy task when implicit methods are considered. Langlands and Henry [9] also investigated this problem and proposed an implicit numerical scheme (L1 approximation), and discussed the accuracy and stability of this scheme. However, the global accuracy of the implicit numerical scheme has not been derived and it seems that the unconditional stability for all the fractional order has not been established. Thus effective numerical methods and error analysis for the time fractional partial differential equations are still in their infancy and quite limited. In this paper, we consider the two-dimensional time fractional diffusion equation (2D-TFDE).

This paper is organized as follows: An implicit difference approximation (IDA) is proposed in section 2. In sections 3 and 4, the stability and convergence of the IDA are analyzed respectively. Finally, some numerical results are given.

2. AN IMPLICIT DIFFERENCE APPROXIMATION FOR THE 2D-TFDE

In this section, we consider the following 2D-TFDE of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x,y,t)\frac{\partial^2 u(x,y,t)}{\partial x^2} + b(x,y,t)\frac{\partial^2 u(x,y,t)}{\partial y^2} + f(x,y,t) \tag{2}$$

with initial condition and boundary condition are given as follow:

$$u(x,y,0) = \varphi(x,y), (x,y) \in \Omega, \tag{3}$$

$$u(x,y,t)|_{\partial\Omega} = 0, \quad 0 \le t \le T \tag{4}$$

where $\Omega = \{(x, y)|0 \le x \le L, 0 \le y \le M\}$, $a(x, y, t) > 0$, $b(x, y, t) > 0$.

The fractional derivative $\dfrac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}$ in (2) is the Caputo fractional derivative of order α defined by

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^\alpha}, \quad 0 < \alpha < 1. \tag{5}$$

Define $t_k = k\tau$; $k = 0, 1, 2, \cdots, n$; $x_i = i\Delta x$, $i = 0, 1, 2, \cdots, l$; $y_j = j\Delta y$, $j = 0, 1, 2, \cdots, m$, where $\tau = \frac{T}{n}$, $\Delta x = \frac{L}{l}$ and $\Delta y = \frac{M}{m}$ are time and space steps, respectively. Let $u^k_{i,j}$ be the numerical approximation to $u(x_i, y_j, t_k)$ and $f^k_{i,j} = f(x_i, y_j, t_k)$, $\varphi_{i,j} = \varphi(x_i, y_j)$, $a^k_{i,j} = a(x_i, y_j, t_k)$, $b^k_{i,j} = b(x_i, y_j, t_k)$.

In the differential equation (2), using

$$\frac{\partial u(x,y,\xi)}{\partial \xi} = \frac{\partial u(x,y,t_s)}{\partial \xi} + O(\tau), t_s \le \xi \le t_{s+1}$$

and

$$\frac{\partial u(x,y,t_s)}{\partial \xi} = \frac{u(x,y,t_{s+1}) - u(x,y,t_s)}{\tau} + O(\tau),$$

the time fractional derivative term can be approximated by the following scheme:

$$
\begin{aligned}
\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\int_{s\tau}^{(s+1)\tau}\frac{\partial u(x_i,y_j,\xi)}{\partial \xi}\frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\int_{s\tau}^{(s+1)\tau}\frac{\partial u(x_i,y_j,t_s)}{\partial \xi}\frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&\approx \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{u(x_i,y_j,t_{s+1})-u(x_i,y_j,t_s)}{\tau}\int_{s\tau}^{(s+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{u(x_i,y_j,t_{s+1})-u(x_i,y_j,t_s)}{\tau}\int_{(k-s)\tau}^{(k-s+1)\tau}\frac{d\eta}{\eta^\alpha} \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{s=0}^{k}\frac{u(x_i,y_j,t_{k+1-s})-u(x_i,y_j,t_{k-s})}{\tau}\int_{s\tau}^{(s+1)\tau}\frac{d\eta}{\eta^\alpha} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{k}\frac{u(x_i,y_j,t_{k+1-s})-u(x_i,y_j,t_{k-s})}{\tau}[(s+1)^{1-\alpha}-s^{1-\alpha}] \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[u(x_i,y_j,t_{k+1})-u(x_i,y_j,t_k)] \\
&+ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=1}^{k}[u(x_i,y_j,t_{k+1-s})-u(x_i,y_j,t_{k-s})][(s+1)^{1-\alpha}-s^{1-\alpha}]
\end{aligned}
$$

where $b_s = (s+1)^{1-\alpha} - s^{1-\alpha}$; $s = 0, 1, 2, \cdots, n$. Now, define

$$L^\alpha_{h,\tau}u(x_i,y_j,t_k) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{s=0}^{k} b_s[u(x_i,y_j,t_{k+1-s})-u(x_i,y_j,t_{k-s})]. \tag{6}$$

Then we have

$$\left|\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} - L^\alpha_{h,\tau}u(x_i,y_j,t_k)\right| \le C_1\tau\int_0^{t_{k+1}}\frac{d\xi}{(t_{k+1}-\xi)^\alpha} \le C\tau \tag{7}$$

where C_1, C are constants.

We have adopted a symmetric second difference quotient in space at level $t = t_{k+1}$ for approximating the second-order space derivative. Thus, the discretized (2) takes the following form

$$
\begin{aligned}
u^{k+1}_{i,j} - u^k_{i,j} &+ \sum_{s=1}^{k} b_s(u^{k+1-s}_i - u^{k-s}_i) \\
&= \mu_1\Gamma(2-\alpha)a^{k+1}_{i,j}(u^{k+1}_{i+1,j} - 2u^{k+1}_{i,j} + u^{k+1}_{i-1,j}) \\
&+ \mu_2\Gamma(2-\alpha)b^{k+1}_{i,j}(u^{k+1}_{i,j+1} - 2u^{k+1}_{i,j} + u^{k+1}_{i,j-1}) + \tau^\alpha\Gamma(2-\alpha)f^{k+1}_{i,j}
\end{aligned} \tag{8}
$$

for $i = 1, 2, \cdots, m-1$; $k = 0, 1, 2, \cdots, n-1$, where $\mu_1 = \dfrac{t^\alpha}{(\Delta x)^2}$ and $\mu_2 = \dfrac{t^\alpha}{(\Delta y)^2}$.

Let $r_1 = r_1(i, j, k) = \mu_1\Gamma(2-\alpha)a_{i,j}^{k+1}$ and $r_2 = r_2(i, j, k) = \mu_1\Gamma(2-\alpha)b_{i,j}^{k+1}$, we can obtain the following implicit difference approximation (IDA):

$$\begin{aligned}-r_1(u_{i+1,j}^{k+1}+u_{i-1,j}^{k+1}) &+ (1+2r_1+2r_2)u_{i,j}^{k+1} - r_2(u_{i,j+1}^{k+1}+u_{i,j-1}^{k+1})\\ &= u_{i,j}^{k} - \sum_{s=1}^{k} b_s u_{i,j}^{k+1-s} + \sum_{s=1}^{k} b_s u_{i,j}^{k-s} + \tau^\alpha\Gamma(2-\alpha)f_{i,j}^{k+1}.\end{aligned}$$

Hence, for $k = 0$:

$$\begin{aligned}-r_1(u_{i+1,j}^{1}+u_{i-1,j}^{1}) &+ (1+2r_1+2r_2)u_{i,j}^{1} - r_2(u_{i,j+1}^{1}+u_{i,j-1}^{1})\\ &= u_{i,j}^{0} + \tau^\alpha\Gamma(2-\alpha)f_{i,j}^{1};\end{aligned} \tag{9}$$

for $k > 0$:

$$\begin{aligned}-r_1(u_{i+1,j}^{k+1}+u_{i-1,j}^{k+1}) &+ (1+2r_1+2r_2)u_{i,j}^{k+1} - r_2(u_{i,j+1}^{k+1}+u_{i,j-1}^{k+1})\\ &= (2-2^{1-\alpha})u_{i,j}^{k} + \sum_{s=1}^{k-1} u_{i,j}^{k-s}(b_{s-1}-b_s)\\ &+ b_k u_{i,j}^{0} + \tau^\alpha\Gamma(2-\alpha)f_{i,j}^{k+1}\end{aligned} \tag{10}$$

where $i = 1, 2, \cdots, l$; $j = 1, 2, \cdots, m$.

Let

$$\mathbf{u}^k = \begin{bmatrix}\mathbf{u}_1^k\\ \mathbf{u}_2^k\\ \vdots\\ \mathbf{u}_{l-1}^k\end{bmatrix}, \mathbf{f}^k = \begin{bmatrix}\mathbf{f}_1^k\\ \mathbf{f}_2^k\\ \vdots\\ \mathbf{f}_{l-1}^k\end{bmatrix}, \Phi = \begin{bmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_{l-1}\end{bmatrix} \tag{11}$$

where

$$\mathbf{u}_i^k = \begin{bmatrix}u_{i,1}^k\\ u_{i,2}^k\\ \vdots\\ u_{i,m-1}^k\end{bmatrix}, \mathbf{f}_i^k = \begin{bmatrix}f_{i,1}^k\\ f_{i,2}^k\\ \vdots\\ f_{i,m-1}^k\end{bmatrix}, \Phi_i = \begin{bmatrix}\varphi_{i,1}\\ \varphi_{i,2}\\ \vdots\\ \varphi_{i,m-1}\end{bmatrix}, i = 1, 2, \cdots, l-1; k = 0, 1, \cdots, n.$$

The above equation can be written in matrix form

$$\begin{cases} A\mathbf{u}^1 = \mathbf{u}^0 + \tau^\alpha\Gamma(2-\alpha)\mathbf{f}^1,\\ A\mathbf{u}^{k+1} = \sum\limits_{j=0}^{k-1}(b_j - b_{j+1})\mathbf{u}^{k-j} + b_k\mathbf{u}^0 + \tau^\alpha\Gamma(2-\alpha)\mathbf{f}^{k+1}\ ,\\ \mathbf{u}^0 = \Phi, \end{cases} \tag{12}$$

where $A = [A_{i,j}]$ is the matrix of coefficients. We can obtain the following result.

Lemma 1 *In (10), the coefficients* $b_s (s = 0, 1, 2, \cdots)$ *satisfy:*

$$(1) b_s > b_{s+1},\ s = 0, 1, 2, \cdots;$$

$$(2) b_0 = 1,\ b_s > 0,\ s = 0, 1, 2, \cdots.$$

3. STABILITY ANALYSIS OF THE IMPLICIT DIFFERENCE APPROXIMATION

We suppose that $\tilde{u}_{i,j}^k, (i = 0, 1, 2, \cdots, l; j = 0, 1, 2, \cdots, m; k = 0, 1, 2, \cdots, n)$ is the approximate solution of (9) and (10), the error

$$\varepsilon_{i,j}^k = \tilde{u}_{i,j}^k - u_{i,j}^k, (i = 0, 1, 2, \cdots, l; j = 0, 1, 2, \cdots, m; k = 0, 1, 2, \cdots, n)$$

satisfies

$$-r_1(\varepsilon_{i+1,j}^1 + \varepsilon_{i-1,j}^1) + (1 + 2r_1 + 2r_2)\varepsilon_{i,j}^1 - r_2(\varepsilon_{i,j+1}^1 + \varepsilon_{i,j-1}^1) = \varepsilon_{i,j}^0, \tag{13}$$

$$\begin{aligned} -r_1(\varepsilon_{i+1,j}^{k+1} + \varepsilon_{i-1,j}^{k+1}) &+ (1 + 2r_1 + 2r_2)\varepsilon_{i,j}^{k+1} - r_2(\varepsilon_{i,j+1}^{k+1} + \varepsilon_{i,j-1}^{k+1}) \\ &= (2 - 2^{1-\alpha})\varepsilon_{i,j}^k + \sum_{s=1}^{k-1} \varepsilon_{i,j}^{k-s}(b_{s-1} - b_s) + b_k \varepsilon_{i,j}^0, \end{aligned} \tag{14}$$

which can be written as

$$\begin{cases} A\mathbf{E}^1 = \mathbf{E}^0, \\ A\mathbf{E}^{k+1} = (b_0 - b_1)\mathbf{E}^k + (b_1 - b_2)\mathbf{E}^{k-1} + \cdots + (b_{k-1} - b_k)\mathbf{E}^1 + b_k \mathbf{E}^0, \\ \mathbf{E}^0, \end{cases} \tag{15}$$

where

$$\mathbf{E}^k = \begin{bmatrix} \mathbf{E}_1^k \\ \mathbf{E}_2^k \\ \vdots \\ \mathbf{E}_{l-1}^k \end{bmatrix} \tag{16}$$

and

$$\mathbf{E}_i^k = \begin{bmatrix} \varepsilon_{i,1}^k \\ \varepsilon_{i,2}^k \\ \vdots \\ \varepsilon_{i,m-1}^k \end{bmatrix}, i = 1, 2, \cdots, l.$$

Hence, the following result can be proved using mathematical induction.
Theorem 2 $\|\mathbf{E}^k\|_\infty \le \|\mathbf{E}^0\|_\infty$, $k = 1, 2, 3, \cdots$.

PROOF. For $k = 1$,

$$-r_1(\varepsilon^1_{i+1,j} + \varepsilon^1_{i-1,j}) + (1 + 2r_1 + 2r_2)\varepsilon^1_{i,j} - r_2(\varepsilon^1_{i,j+1} + \varepsilon^1_{i,j-1}) = \varepsilon^0_{i,j}.$$

Let $|\varepsilon^1_{p,q}| = \max\limits_{1\le i\le l-1;1\le j\le m-1} |\varepsilon^1_{i,j}|$, we have

$$\begin{aligned}
|\varepsilon^1_{p,q}| &= -r_1(|\varepsilon^1_{p,q}| + |\varepsilon^1_{p,q}|) + (1 + 2r_1 + 2r_2)|\varepsilon^1_{p,q}| - r_2(|\varepsilon^1_{p,q}| + |\varepsilon^1_{p,q}|)\\
&\le -r_1(|\varepsilon^1_{p+1,q}| + |\varepsilon^1_{p-1,q}|) + (1 + 2r_1 + 2r_2)|\varepsilon^1_{p,q}| - r_2(|\varepsilon^1_{p,q+1}| + |\varepsilon^1_{p,q-1}|)\\
&\le |-r_1(\varepsilon^1_{p+1,q} + \varepsilon^1_{p-1,q}) + (1 + 2r_1 + 2r_2)\varepsilon^1_{p,q} - r_2(\varepsilon^1_{p,q+1} + \varepsilon^1_{p,q-1})|\\
&= |\varepsilon^0_{p,q}| \le \|\mathbf{E}^0\|_\infty,
\end{aligned} \tag{17}$$

also, $\|\mathbf{E}^1\|_\infty \le \|\mathbf{E}^0\|_\infty$.

Suppose that $\|\mathbf{E}^s\|_\infty \le \|\mathbf{E}^0\|_\infty$, $s = 1, 2, \cdots; k$. Let $|\varepsilon^{k+1}_{p,q}| = \max\limits_{1\le i\le l-1;1\le j\le m-1} |\varepsilon^{k+1}_{i,j}|,$, we also have

$$\begin{aligned}
|\varepsilon^{k+1}_{p,q}| &= -r_1(|\varepsilon^{k+1}_{p,q}| + |\varepsilon^{k+1}_{p,q}|) + (1 + 2r_1 + 2r_2)|\varepsilon^{k+1}_{p,q}| - r_2(|\varepsilon^{k+1}_{p,q}| + |\varepsilon^{k+1}_{p,q}|)\\
&\le -r_1(|\varepsilon^{k+1}_{p+1,q}| + |\varepsilon^{k+1}_{p-1,q}|) + (1 + 2r_1 + 2r_2)|\varepsilon^{k+1}_{p,q}| - r_2(|\varepsilon^{k+1}_{p,q+1}| + |\varepsilon^{k+1}_{p,q-1}|)\\
&\le |-r_1(\varepsilon^{k+1}_{p+1,q} + \varepsilon^{k+1}_{p-1,q}) + (1 + 2r_1 + 2r_2)\varepsilon^{k+1}_{p,q} - r_2(\varepsilon^{k+1}_{p,q+1} + \varepsilon^{k+1}_{p,q-1})|\\
&= |(b_0 - b_1)\varepsilon^k_{p,q} + \sum_{s=1}^{k-1}(b_s - b_{s+1})\varepsilon^{k-s}_{p,q} + b_k\varepsilon^0_{p,q}|\\
&\le (b_0 - b_1)|\varepsilon^k_{p,q}| + \sum_{s=1}^{k-1}(b_s - b_{s+1})|\varepsilon^{k-s}_{p,q}| + b_k|\varepsilon^0_{p,q}|\\
&\le (b_0 - b_1)\|\mathbf{E}^k\|_\infty + \sum_{s=1}^{k-1}(b_s - b_{s+1})\|\mathbf{E}^{k-s}\|_\infty + b_k\|\mathbf{E}^0\|_\infty\\
&\le \{b_0 - b_1 + \sum_{s=1}^{k-1}(b_s - b_{s+1}) + b_k\}\|\mathbf{E}^0\|_\infty\\
&= \|\mathbf{E}^0\|_\infty,
\end{aligned} \tag{18}$$

also, $\|\mathbf{E}^{k+1}\|_\infty \le \|\mathbf{E}^0\|_\infty$.

Hence, the following theorem is obtained.

Theorem 3 *The implicit difference approximation defined by (9) and (10) is unconditionally stable.*

4. CONVERGENCE ANALYSIS OF THE IMPLICIT DIFFERENCE APPROXIMATION

Let $u(x_i, y_j, t_k)$, $i = 0, 1, \cdots, l$; $j = 0, 1, \cdots, m$; $k = 0, 1, \cdots, n$ be the exact solution of the fractional partial differential equation (2) at mesh point (x_i, t_k). Define

$\eta_{i,j}^k = u(x_i, y_j, t_k) - u_{i,j}^k$, $i = 0, 1, \cdots, l$; $j = 0, 1, \cdots, m$; $k = 0, 1, \cdots, n$ and $\mathbf{e}^k = (\mathbf{e}_1^k, \mathbf{e}_2^k, \cdots, \mathbf{e}_{m-1}^k)^T$. Using $\mathbf{e}^0 = 0$, where

$$\mathbf{e}_i^k = \begin{bmatrix} \eta_{i,1}^k \\ \eta_{i,2}^k \\ \vdots \\ \eta_{i,m-1}^k \end{bmatrix}, i = 1, 2, \cdots, l-1.$$

Substitution into (9) and (10) leads to

$$-r_1(\eta_{i+1,j}^1 + \eta_{i-1,j}^1) + (1 + 2r_1 + 2r_2)\eta_{i,j}^1 - r_2(\eta_{i,j+1}^1 + \eta_{i,j-1}^1) = R_{i,j}^0, \tag{19}$$

$$\begin{aligned} &-r_1(\eta_{i+1,j}^{k+1} + \eta_{i-1,j}^{k+1}) + (1 + 2r_1 + 2r_2)\eta_{i,j}^{k+1} - r_2(\eta_{i,j+1}^{k+1} + \eta_{i,j-1}^{k+1}) \\ &\quad = (2 - 2^{1-\alpha})\eta_{i,j}^k + \sum_{s=1}^{k-1} \eta_{i,j}^{k-s}(b_{s-1} - b_s) + b_k\eta_{i,j}^0 + R_{i,j}^{k+1} \end{aligned} \tag{20}$$

where

$$\begin{aligned} R_{i,j}^{k+1} &= u(x_i, y_j, t_{k+1}) - u(x_i, y_j, t_k) + \sum_{s=1}^{k-1} b_s \left[u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-s})\right] \\ &\quad -r_1 \left[u(x_{i+1}, y_j, t_{k+1}) - 2u(x_i, y_j, t_{k+1}) + u(x_{i-1}, y_j, t_{k+1})\right] \\ &\quad -r_2 \left[u(x_i, y_{j+1}, t_{k+1}) - 2u(x_i, y_j, t_{k+1}) + u(x_i, y_{j-1}, t_{k+1})\right] \\ &= \sum_{s=0}^{k} b_s \left[u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-j})\right] \\ &\quad -r_1 \left[u(x_{i+1}, y_j, t_{k+1}) - 2u(x_i, y_j, t_{k+1}) + u(x_{i-1}, y_j, t_{k+1})\right] \\ &\quad -r_2 \left[u(x_i, y_{j+1}, t_{k+1}) - 2u(x_i, y_j, t_{k+1}) + u(x_i, y_{j-1}, t_{k+1})\right] \\ &\quad -\tau^\alpha\Gamma(2-\alpha) f_{i,j}^{k+1}. \end{aligned} \tag{21}$$

From (7), we have

$$\begin{aligned} &\tfrac{1}{\Gamma(2-\alpha)\tau^\alpha} \sum_{s=0}^{k} b_s \left[u(x_i, y_j, t_{k+1-s}) - u(x_i, y_j, t_{k-s})\right] \\ &\quad = \tfrac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} + O(\tau), \end{aligned} \tag{22}$$

$$\begin{aligned} &\tfrac{u(x_{i+1},y_j,t_{k+1}) - 2u(x_i,y_j,t_{k+1}) + u(x_{i-1},y_j,t_{k+1})}{(\Delta x)^2} \\ &\quad = \tfrac{\partial^2 u(x_i,y_j,t_{k+1})}{\partial x^2} + O((\Delta x)^2), \end{aligned} \tag{23}$$

$$\begin{aligned} &\tfrac{u(x_i,y_{j+1},t_{k+1}) - 2u(x_i,y_j,t_{k+1}) + u(x_i,y_{j-1},t_{k+1})}{(\Delta y)^2} \\ &\quad = \tfrac{\partial^2 u(x_i,y_j,t_{k+1})}{\partial y^2} + O((\Delta y)^2). \end{aligned} \tag{24}$$

Hence,

$$R_{i,j}^{k+1} = O(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2) \tag{25}$$

also

$$\begin{gathered} |R_{i,j}^{k+1}| \leq C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2), \\ i = 1, 2, \cdots, l-1; j = 1, 2, \cdots, m-1; k = 1, 2, \cdots, n \end{gathered} \tag{26}$$

where C is a constant.

Consequently, we obtain

Theorem 4 $\|\mathbf{e}^k\|_\infty \leq Cb_{k-1}^{-1}(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2), k = 1, 2, \cdots, n,$ *where* $\|\mathbf{e}^k\|_\infty = \|\mathrm{e}^k\|_\infty = \max\limits_{1 \leq i \leq l-1; 1 \leq j \leq m-1} |e_{i,j}^k|$ *and C is a constant.*

PROOF. Using mathematical induction method. For $k = 1$, let $\|\mathbf{e}^1\|_\infty = |e_{p,q}^1| = \max\limits_{1 \leq i \leq l-1; 1 \leq j \leq m-1} |e_{i,j}^1|$, we have

$$\begin{aligned} |e_{p,q}^1| &\leq -r_1(|e_{p+1,q}^1| + |e_{p-1,q}^1|) + (1 + 2r_1 + 2r_2)|e_{p,q}^1| - r_2(|e_{p,q+1}^1| + |e_{p,q-1}^1|) \\ &\leq |-r_1(e_{p+1,q}^1 + e_{p+1,q}^1) + (1 + 2r_1 + 2r_2)e_{p,q}^1 - r_2(e_{p,q+1}^1 + e_{p,q-1}^1)| \\ &= |R_{p,q}^1| \\ &\leq Cb_0^{-1}(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2). \end{aligned} \tag{27}$$

Suppose that $\|\mathbf{e}^s\|_\infty \leq Cb_{s-1}^{-1}(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2)$; $s = 0, 1, 2, \cdots, k-1$ and $|e_{p,q}^{k+1}| = \max\limits_{1 \leq i \leq l-1; 1 \leq j \leq m-1} |e_{i,j}^{k+1}|.$. Note that $b_s^{-1} \leq b_k^{-1}$, $s = 0, 1, \cdots, k$. We have

$$\begin{aligned} e_{p,q}^{k+1}| &\leq -r_1(|e_{p+1,q}^{k+1}| + |e_{p-1,q}^{k+1}|) + (1 + 2r_1 + 2r_2)|e_{p,q}^{k+1}| - r_2(|e_{p,q+1}^{k+1} + |e_{p,q-1}^{k+1}|) \\ &\leq |-r_1(e_{p+1,q}^{k+1} + e_{p-1,q}^{k+1}) + (1 + 2r_1 + 2r_2)e_{p,q}^{k+1} - r_2(e_{p,q+1}^{k+1} + e_{p,q-1}^{k+1})| \\ &= |\sum_{s=0}^{k-1}(b_s - b_{s+1})e_{p,q}^{k-s} + R_{p,q}^{k+1}| \\ &\leq \sum_{s=0}^{k-1}(b_s - b_{s+1})|e_{p,q}^{k-s}| + |R_{p,q}^{k+1}| \\ &\leq \sum_{s=0}^{k-1}(b_s - b_{s+1})|e_{p,q}^{k-s}| + C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2) \\ &\leq \sum_{s=0}^{k-1}(b_s - b_{s+1})\|\mathrm{e}^{k-s}\|_\infty + C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2) \\ &\leq \left[\sum_{s=0}^{k-1}(b_s - b_{s+1}) + b_k\right] b_k^{-1}C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2) \\ &= b_k^{-1}C(\tau^{1+\alpha} + \tau^{\alpha}(\Delta x)^2 + \tau^{\alpha}(\Delta y)^2). \end{aligned} \tag{28}$$

Because

$$\begin{aligned}\lim_{k\to\infty}\frac{b_k^{-1}}{k^\alpha} &= \lim_{k\to\infty}\frac{k^{-\alpha}}{(k+1)^{1-\alpha}-k^{1-\alpha}}\\ &= \lim_{k\to\infty}\frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha}-1}\\ &= \lim_{k\to\infty}\frac{k^{-1}}{(1-\alpha)k^{-1}}\\ &= \frac{1}{1-\alpha}.\end{aligned} \tag{29}$$

Hence, there is a constant C,

$$\|\mathrm{e}^k\|_\infty \le Ck^\alpha(\tau^{1+\alpha}+(\tau^{1+\alpha}+\tau^\alpha(\Delta x)^2+\tau^\alpha(\Delta y)^2)).$$

If $k\tau \le T$ is finite, then we obtain the following theorem:

Theorem 5 *Let $u_{i,j}^k$ be the approximate value of $u(x_i, y_j, t_k)$ computed by use of the difference scheme (9) and (10). Then there is a positive constant C, such that*

$$\begin{aligned}&|u_{i,j}^k - u(x_i,y_j,t_k)| \le C(\tau+(\Delta x)^2+(\Delta y)^2),\\ &\quad i=1,2,\cdots,l-1; j=1,2,\cdots,m-1; k=1,2,\cdots,n.\end{aligned} \tag{30}$$

5. NUMERICAL RESULTS

Example 1. Consider the following two-dimensional time fractional diffusion equation

$$\begin{aligned}&\frac{\partial^{0.4}u(x,t)}{\partial t^{0.4}} = \frac{2t^{1.6}}{\pi^2\Gamma(0.6)}\frac{\partial^2 u}{\partial x^2} + \frac{t^{1.6}}{12\pi^2\Gamma(0.6)}\frac{\partial^2 u}{\partial y^2} + f(x,y,t), (x,y)\in\Omega\times(0,T],\\ &u|_{\partial\Omega}=0, u(x,y,0)=\sin\pi x\sin\pi y, (x,y)\in\Omega\end{aligned} \tag{31}$$

where $f(x, y, t) = \dfrac{25t^{1.6}}{12\Gamma(0.6)}(t^2+2)\sin\pi x\sin\pi y$, $\Omega = \{(x, y)|0 < x < 1, 0 < y < 1\}$ and $\partial\Omega$ is the the boundary of Ω. The exact solution of the above equation is $u(x, y, t) = (t^2 + 1)\sin\pi x\sin\pi y$.

The maximum error of the exact solution and numerical solution is defined as follows:

$$E_\infty = \max_{0\le j\le M}\max_{0\le k\le N}\left\{|u(x_j,t_k)-u_j^k|\right\}.$$

Table 1. The maximum error $|u_{i,j}^k - u(x_i, y_j, t_k)|$ at $t = 1.0$

Δt	$\Delta x = \Delta y$	E_∞
$\frac{1}{16}$	$\frac{1}{4}$	5.39188E-2
$\frac{1}{64}$	$\frac{1}{8}$	1.30699E-2
$\frac{1}{100}$	$\frac{1}{10}$	8.26645E-3
$\frac{1}{400}$	$\frac{1}{20}$	1.67537E-3

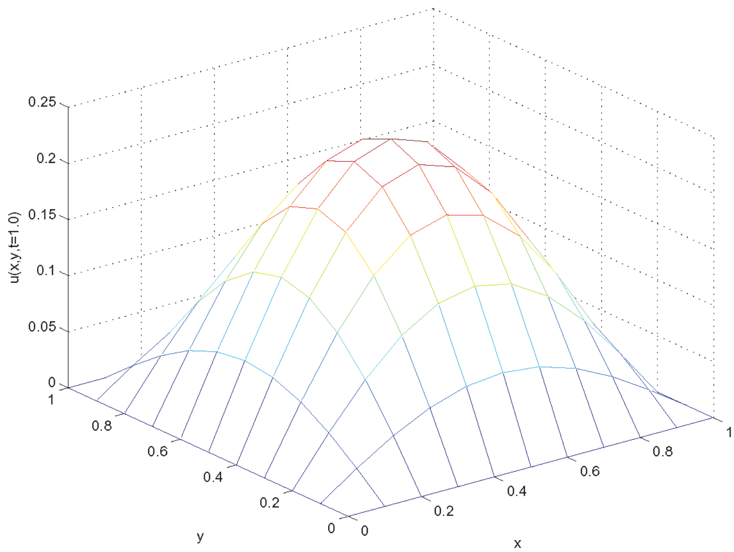

Figure 1. Numerical solution of the equation (32) when $\alpha = 0.6$ and $t = 1.0$.

Table 1 shows the maximum absolute numerical error, at time t=0.1, between the exact solution and the numerical solution of the IDA. From Table 1, it can be seen that our numerical method (IDA) is in excellent agreement with numerical solution, and our numerical method yields convergence with $O(\tau + h^2)$. These results confirm our theoretical analysis.

Example 2. Consider the following two-dimensional time fractional diffusion equation

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x,y) \in \Omega \times (0,T], \\ &u|_{\partial\Omega} = 0, u(x,y,0) = \sin \pi x \sin \pi y, (x,y) \in \Omega \end{aligned} \tag{32}$$

Figures 1-4 give the numerical simulation of the solution of the equation (32) and compare the response of the diffusion system at $t = 1.0$ when $\alpha = 0.6, 0.7, 0.8, 0.9$, respectively. From Figures 1-4, it can be seen that the solution continuously depends on the time fractional derivative.

6. CONCLUSIONS

In this paper, the implicit finite difference approximation for the 2D-TFDE in a bounded domain have been described and demonstrated. The implicit difference approximation is unconditionally stable and convergence. This method and technique can be also applied to solve fractional differential equations.

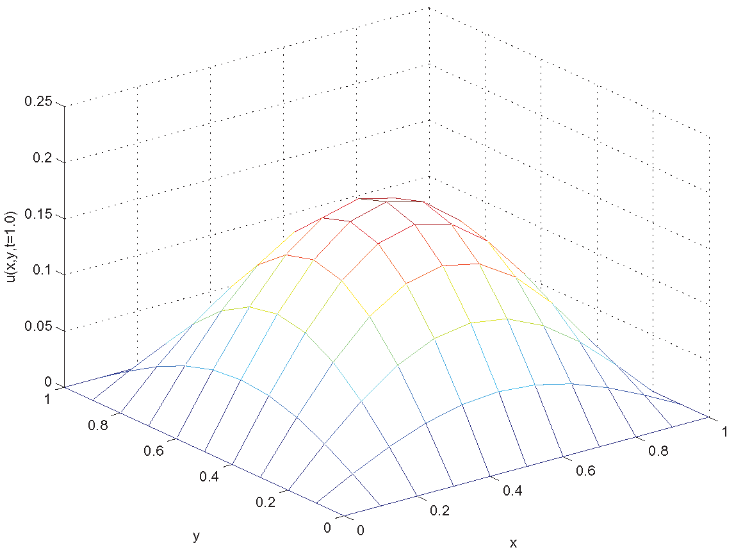

Figure 2. Numerical solution of the equation (32) when $\alpha = 0.7$ and $t = 1.0$.

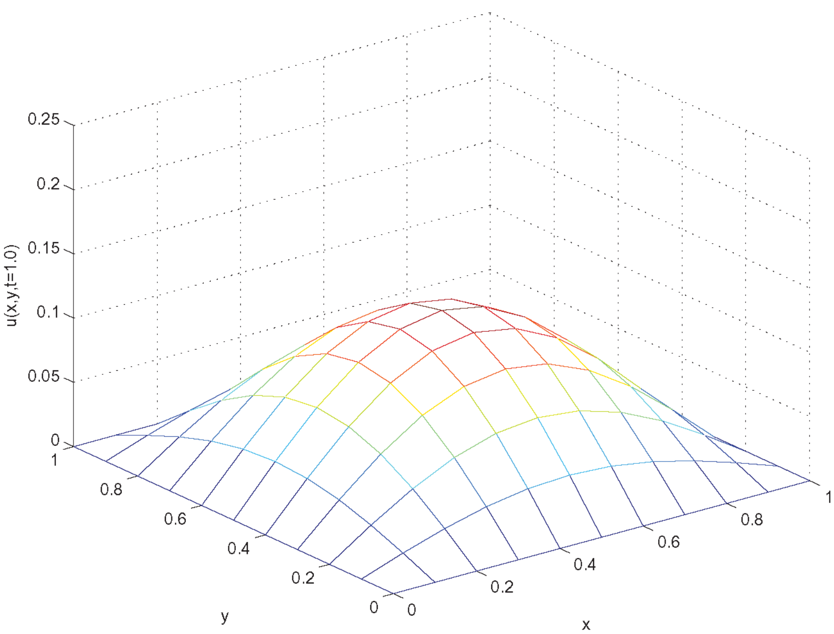

Figure 3. Numerical solution of the equation (32) when $\alpha = 0.8$ and $t = 1.0$.

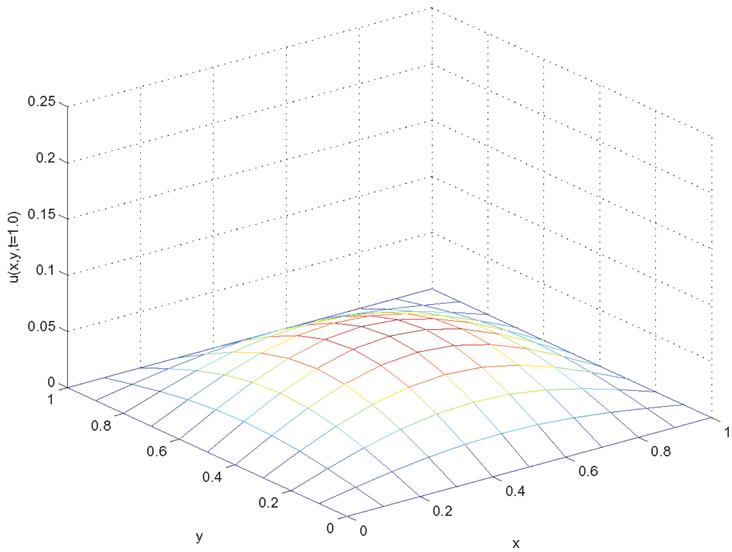

Figure 4. Numerical solution of the equation (32) when $\alpha = 0.9$ and $t = 1.0$.

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