Fluid dynamics Assignment 01

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1. Sketching contours

(i) $f(x, y) = x^2 + y^2$

We draw the curves $f(x, y) = constant \in \mathbb{R}_0^+$.
If $constant = 0$ then we get the origin $O(0, 0)$.
If $constant > 0$ then we have a circle.

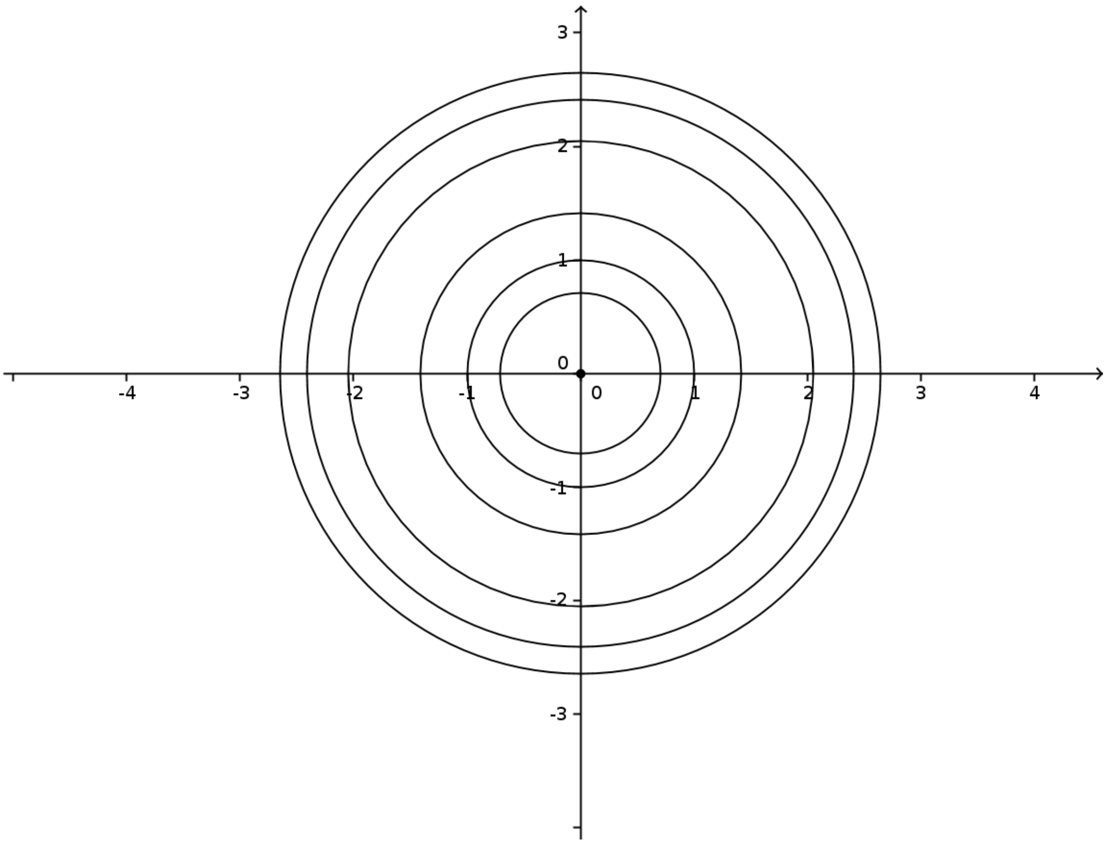

Figure 1: Contours of $f(x, y) = x^2 + y^2$

(ii) $f(x,y) = x^2 - y^2$

Here again we draw the curves $f(x,y) = \mathit{constant}$

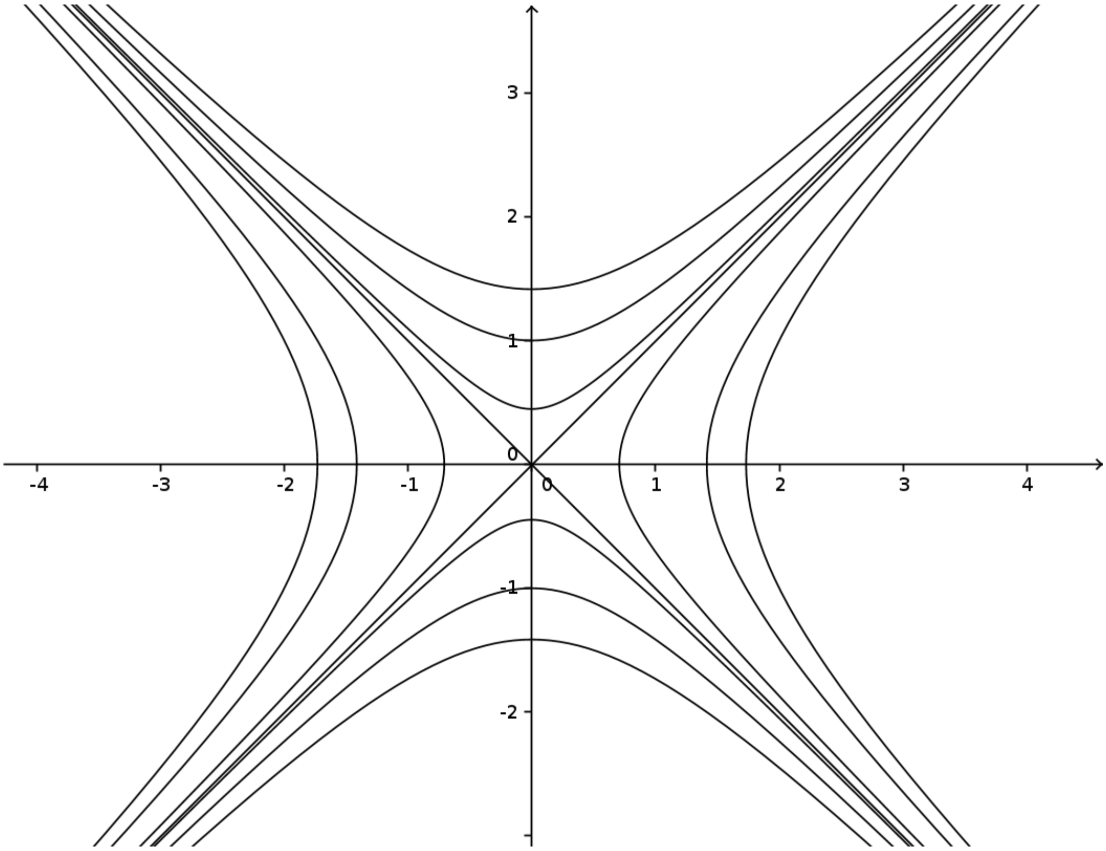

Figure 2: Contours of $f(x,y) = x^2 - y^2$

(iii) $f(x,y) = xy$

Here again we draw the curves $f(x,y) = \mathit{constant}$

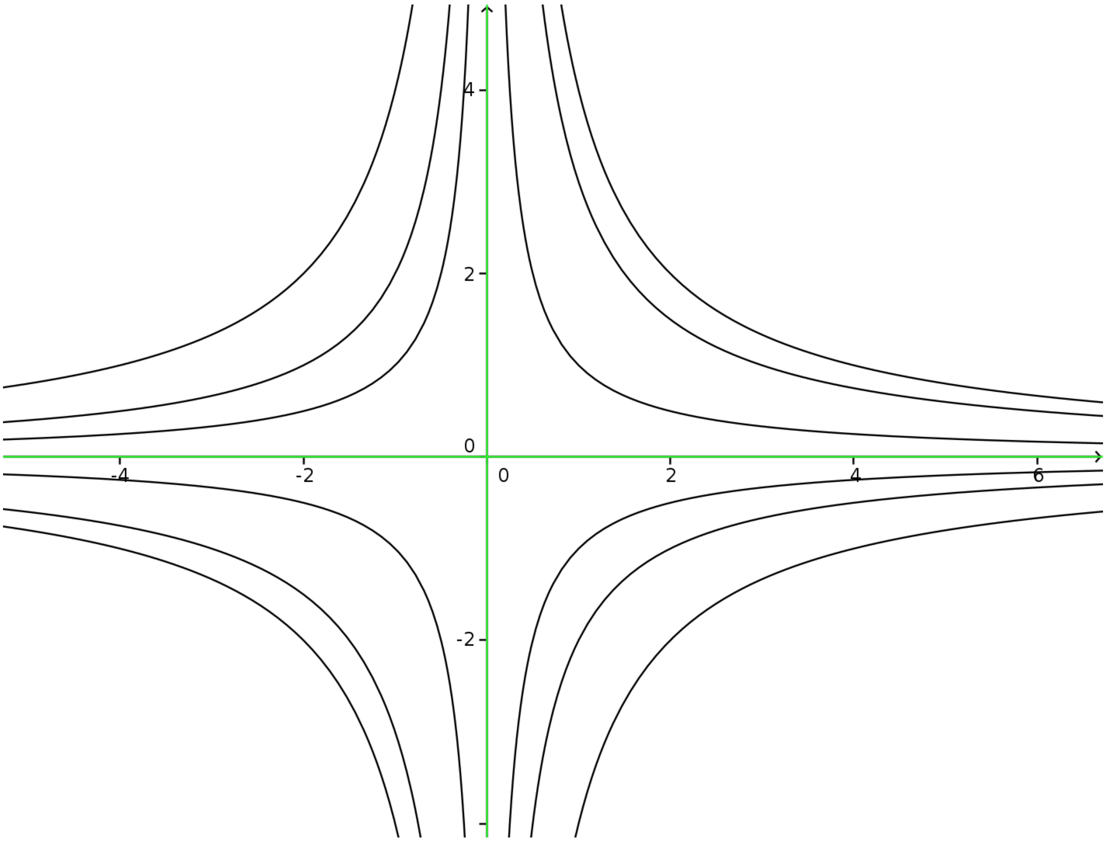

Figure 3: Contours of $f(x,y) = xy$

2. Gradient and directional derivative of $f(x,y) = 3x^2 - 2xy + y^3$

• $\nabla f(x,y)$
We have:

$$\begin{aligned}\nabla f(x,y) &= \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} \\ &= \begin{pmatrix} 6x - 2y \\ -2x + 3y^2 \end{pmatrix}\end{aligned}$$

$$\boxed{\nabla f(x,y) = \begin{pmatrix} 6x - 2y \\ -2x + 3y^2 \end{pmatrix}}$$

• Directional derivative of f in the direction of the vector $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ at the point $x = (1,1)$. We denote by $D_u f(1,1)$ that derivative.

We have

$$\begin{aligned}
D_u f(1,1) &= \lim_{h\to 0} \frac{f((1,1)+hu) - f(x,y)}{h|u|} \\
&= \nabla f(1,1).\frac{u}{|u|} \text{ because } f \text{ is differentiable at } (1,1) \\
&= \frac{1}{5}\left(4,1\right).(3,4) \\
&= \frac{1}{5}(4\times 3 + 1\times 4) \\
&= \frac{1}{5}\times 16 \\
&= \frac{16}{5}
\end{aligned}$$

$$\boxed{D_u f(1,1) = \frac{16}{5}}$$

3. Vector velocity field sketch

- $u = (x,y)$

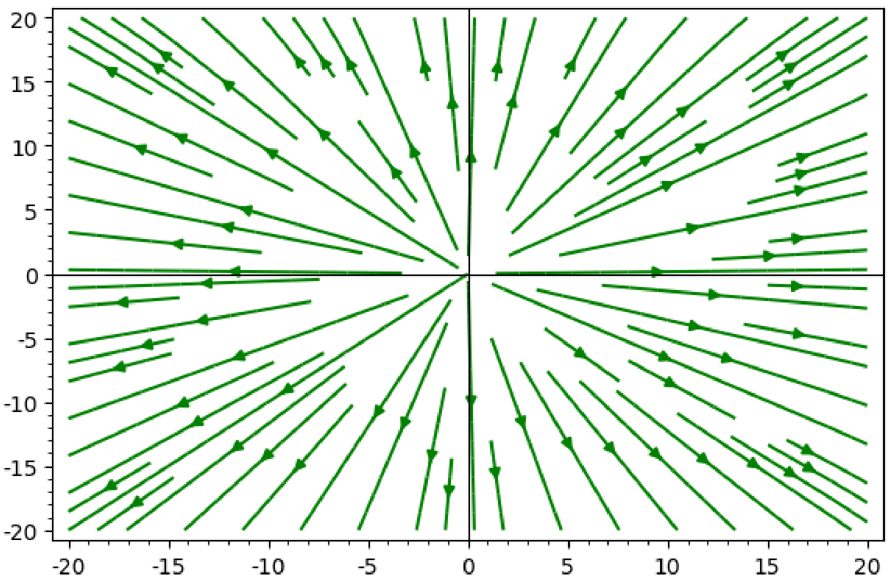

Figure 4: velocity field for $u = (x,y)$

- $u = (x, -y)$

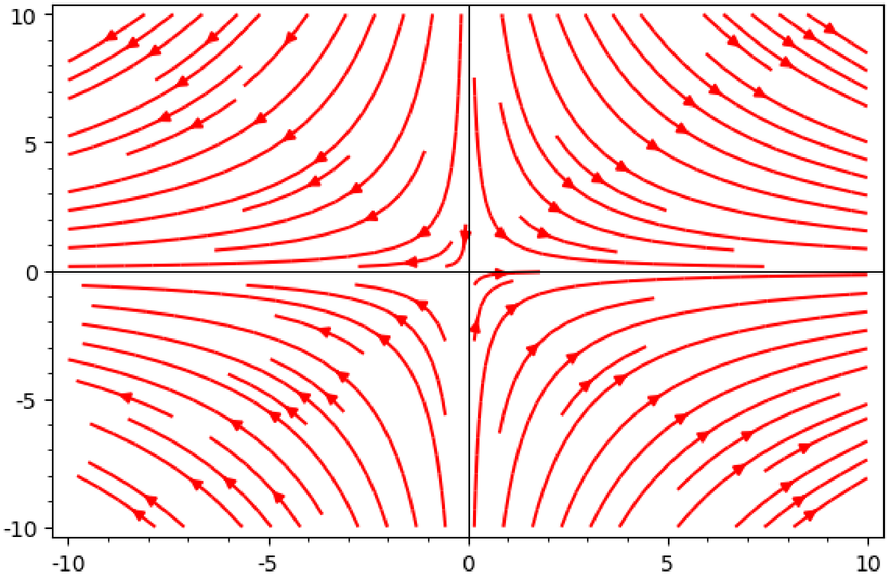

Figure 5: velocity field for $u = (x, -y)$

- $u = (-y, x)$

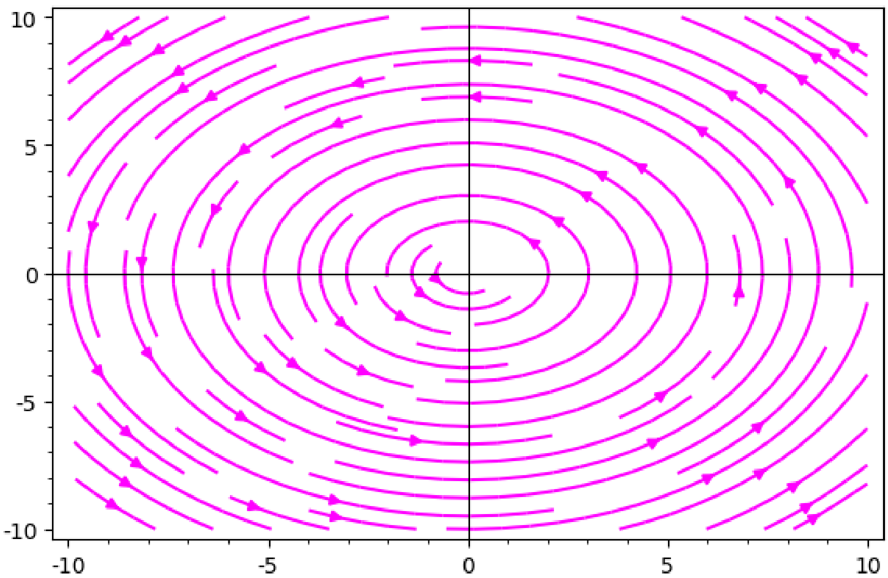

Figure 6: velocity field for $u = (-y, x)$

4.

4.a) Sketch of the streamline passing through the point (1,1) at $t = 0$.

The streamline is a curve instantaneously tangent to the velocity vector of the flow.
Let $S(t) = (x(t), y(t))$ be a streamline.
We have $det\Big(S'(t), U(t)\Big) = 0$ where $U(t) = \Big(\frac{1}{t+1}, 1\Big)$ is the velocity.
So

$$\begin{aligned}
& det\Big(S'(t), U(t)\Big) = 0 \\
& \Leftrightarrow det\begin{pmatrix} x' & \frac{1}{t+1} \\ y' & 1 \end{pmatrix} = 0 \\
& \Leftrightarrow x' - \frac{1}{t+1}y' = 0 \\
& \Leftrightarrow \frac{dx}{dt} - \frac{1}{t+1}\frac{dy}{dt} = 0 \\
& \Leftrightarrow (t+1)dx - dy = 0 \text{ multiplying by (1+t)dt} \\
& \Leftrightarrow \frac{dy}{dx} = 1 + t \\
& \Leftrightarrow y = (1+t)x + c, \;\; c \in \mathbb{R}
\end{aligned}$$

Since at $t = 0, \;\; x = y = 1$, then we have $1 = 1 + c \Rightarrow c = 0$.
Now we choose $t = 0$ to find the streamline passing through the point $(1, 1)$ at $t = 0$ which is the straight line of equation $y = x$.
The sketch of that streamline is below:

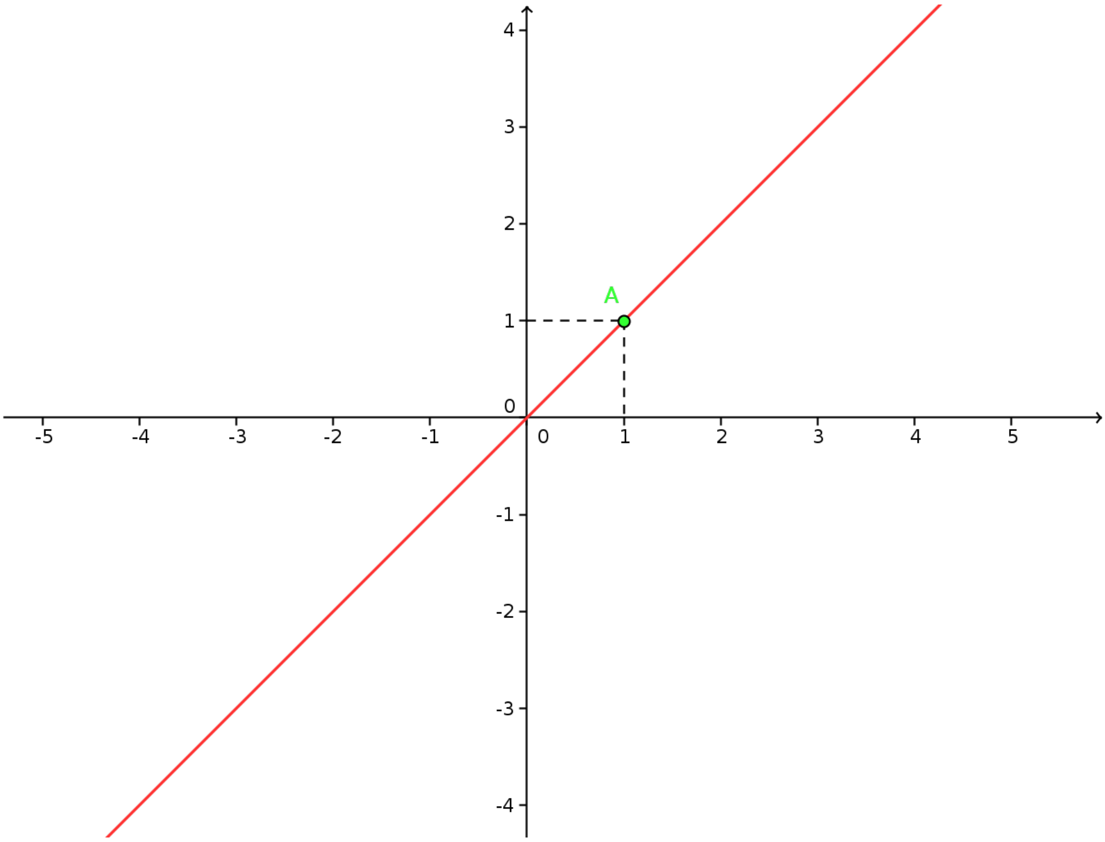

Figure 7: Streamline at t=1

0.1 4.b) The path of a fluid particle that is released from (1,1) at $t = 0$.

Let $X(t) = \Big(x(t), y(t)\Big)$ be the instantaneous position of the particle.

We have:

$$\begin{cases} \frac{dx}{dt}(t) = \frac{1}{1+t} \\ \frac{dy}{dt}(t) = 1 \end{cases} \Rightarrow \begin{cases} x(t) = ln(1+t) + c_1 \\ y(t) = t + c_2 \end{cases}$$

At $t = 0, x = y = 1$.

So $c_1 = 1, c_2 = 1$ and we have

$$\begin{cases} x(t) = 1 + ln(1+t) \\ y(t) = 1 + t \end{cases} \quad \text{Then } x(t) = 1 + ln(y(t)), \text{ that is } y(t) = \exp(x(t) - 1)$$

It follows that the path equation is $y = \exp(x - 1)$ and we draw it below:

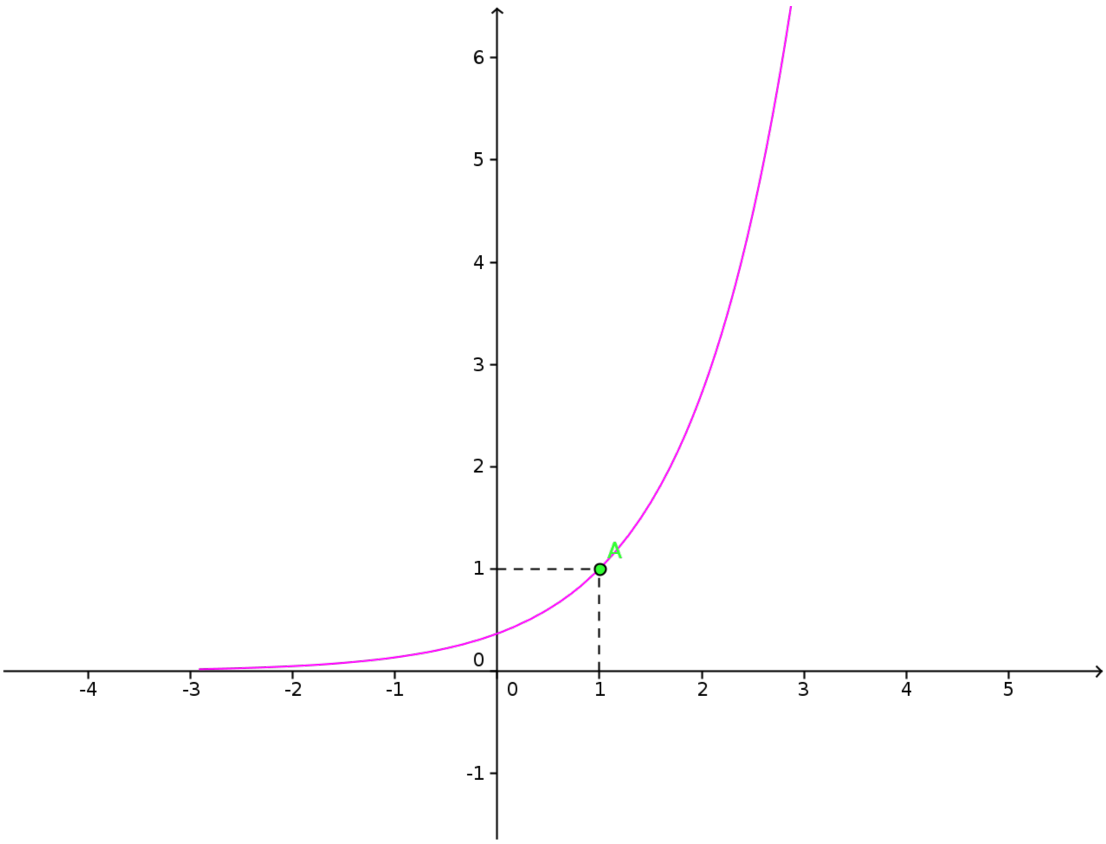

Figure 8: Path of a fluid particle

The position at time t = 0 of a streak of dye:

For the pathline trajectory, we got

$$\begin{cases} x = \ln(1+t) + 1 \\ y = t + 1 \end{cases}$$

Now consider: $\begin{cases} x_\tau = \ln(1+t_\tau)+1 \\ y\tau = t_\tau + 1 \end{cases}$

We have:

$$\begin{cases} x_\tau - \ln(1+t_\tau) = 1 \\ y_\tau - t_\tau = 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x = x_\tau - \ln(1+t_\tau) - 1 + \ln(1+t) + 1 \\ y = y_\tau - t_\tau - 1 + t + 1 \end{cases}$$

$$\Longrightarrow \begin{cases} x = x_\tau - \ln(1+t_\tau) + \ln(1+t) \\ y = y_\tau - t_\tau + t \end{cases}$$

$$\Longrightarrow \begin{cases} x = x_\tau - \ln(1+t_\tau) + \ln(1+t) \text{ (*)} \\ t_\tau = y_\tau + t - y \quad \text{(**)} \end{cases}$$

Then

$x = x_\tau - \ln(1+t_\tau) + \ln(1+t) = x_\tau - \ln(1+y_\tau+t-y) + \ln(1+t)$ using **(**)**

Also $-1 < t \leq 0$ and at $t = 0$, $(x_\tau, y\tau) = (1, 1)$. As a result,

$$x = 1 - \ln(2-y) \Longrightarrow 1 - x = \ln(2-y) \Longrightarrow e^{1-x} = 2 - y \Longrightarrow y = 2 - e^{1-x}$$

We draw that curve below:

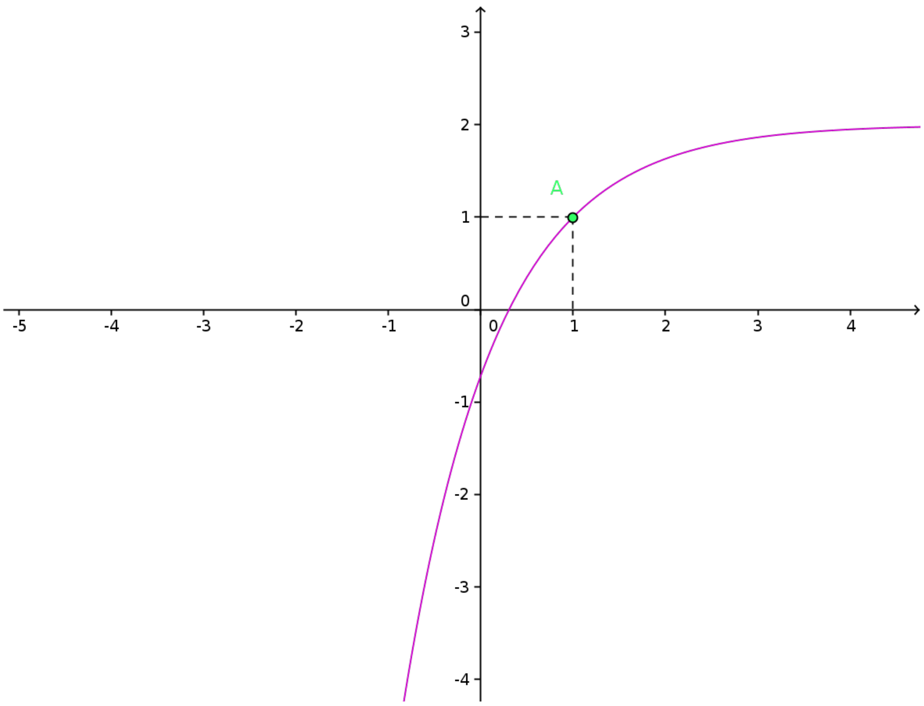

Figure 9: Position at $t = 0$ of a streak of dye released at the point $(1, 1)$

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5. Two dimensional flow represented by $\psi(x, y)$ with $u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$

Let's show that:

5.a) The streamlines are given by $\psi = const$

Let $S(t) = \Big(x(t), y(t)\Big)$ be a streamline.
Like we did in 4.a), we have:

$$\begin{aligned}
det \begin{pmatrix} \frac{dx}{dt} & u \\ \frac{y}{dt} & v \end{pmatrix} = 0 \Leftrightarrow det \begin{pmatrix} \frac{dx}{dt} & \frac{\partial \psi}{dy} \\ \frac{dy}{dt} & -\frac{\partial \psi}{\partial x} \end{pmatrix} = 0 \\
&\Leftrightarrow -\frac{dx}{dt}\frac{\partial \psi}{\partial x} - \frac{dy}{dt}\frac{\partial \psi}{\partial y} = 0 \\
&\Leftrightarrow \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \text{ multiplying by -dt} \\
&\Leftrightarrow d\psi = 0 \\
&\Leftrightarrow \psi = const.
\end{aligned}$$

So the streamlines are given by $\psi = const.$

5.b) $|\mathbf{u}| = |\nabla \psi|$

We have

$$\begin{aligned}
|\mathbf{u}| &= \sqrt{u^2 + v^2} \\
&= \sqrt{\Big(\frac{\partial \psi}{\partial y}\Big)^2 + \Big(-\frac{\partial \psi}{\partial x}\Big)^2} \\
&= \sqrt{\Big(\frac{\partial \psi}{\partial y}\Big)^2 + \Big(\frac{\partial \psi}{\partial x}\Big)^2}
\end{aligned}$$

Also,

$$\nabla \psi = \begin{pmatrix} \frac{\partial \psi}{\partial x} \\ \\ \frac{\partial \psi}{\partial y} \end{pmatrix}$$

$$\text{So } |\nabla \psi| = \sqrt{\Big(\frac{\partial \psi}{\partial y}\Big)^2 + \Big(\frac{\partial \psi}{\partial x}\Big)^2}$$

Therefore $|\nabla \psi| = |\mathbf{u}|$

5.c)

Let q be the volume integral crossing a curve from x_0 to x_1.
We have q $= \int_{x_0}^{x_1} \mathbf{u}.\mathbf{n}ds$ with $\mathbf{n}\ ds = \Bigg(dy, -dx\Bigg)$ and $\mathbf{u} = \Bigg(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\Bigg)$
Thus:

$$\begin{aligned} \mathrm{q} &= \int_{x_0}^{x_1} \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy \\ &= \int_{x_0}^{x_1} d\psi \\ &= \psi(x_1) - \psi(x_0) \end{aligned}$$

We conclude that the volume integral crossing a curve from x_0 to x_1 is: $\psi(x_1) - \psi(x_0)$.

5.d) ψ is constant along any fixed boundary

The flow can not cross a boundary and the velocity is O on the boundary. So we have:

$$u = 0i \text{ and } v = 0j \Rightarrow \begin{cases} \frac{\partial\psi}{\partial y} = 0 \\ \frac{\partial\psi}{\partial x} = 0 \end{cases}$$

$\Rightarrow \psi$ does not depend on x and does not depend on y.

$\Rightarrow \psi$ **is constant on any fixed boundary.**