Fluid dynamics Assignment 03

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1. Water in cylindrical container

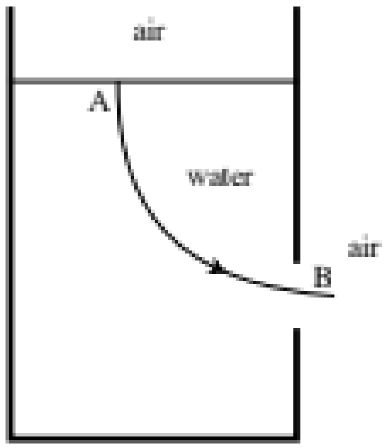

Figure 1: Water in the container exiting by the hole B

• **Velocity at B**.
Let H be Bernoulli energy density. The flow is steady. So H is constant along streamlines.
Let $H(A) = H_A$ and $H(B) = H_B$. Then we have $H_A = H_B$.
But $H_A = \frac{1}{2}\rho u_A^2 + P_A + \phi_A$ and $H_B = \frac{1}{2}\rho u_B^2 + P_B + \phi_B$.
Thus

$$\frac{1}{2}\rho u_A^2 + P_A + \phi_A = \frac{1}{2}\rho u_B^2 + P_B + \phi_B, \text{ where } \phi \text{ is the potential energy}$$

We have: $P_A = P_{atm}, \phi_A = \rho g h,\ P_B = P_{atm}, \phi_B = 0$ (we choose the reference to be B).

$$\text{So } \frac{1}{2}\rho u_A^2 + P_A + \phi_A = \frac{1}{2}\rho u_B^2 + P_B + \phi_B \Leftrightarrow \frac{1}{2}\rho u_A^2 + \rho g h = \frac{1}{2}\rho u_B^2$$
$$\Leftrightarrow \frac{1}{2}\rho(u_A^2 - u_B^2) = -\rho g h \ (*)$$

From the mass conservation we have $\rho A u_A = \rho a u_B \Leftrightarrow u_A = \frac{a}{A} u_B$ $(**)$
Using $(**)$ in $(*)$ we have

$$\frac{1}{2}\rho(\frac{a^2}{A^2}u_B^2 - u_B^2) = -\rho g h$$
$$\Leftrightarrow u_B^2\left(\frac{(a^2 - A^2)\rho}{2A^2}\right) = -\rho g h$$
$$\Leftrightarrow u_B = \sqrt{\frac{2A^2 g h}{A^2 - a^2}}.$$

But we have $a << A,\ A^2 - a^2 \sim A^2$

So: $\boxed{u_B = \sqrt{2gh}}$

• **Time taken for the water to stop flowing.**
We have

$$u_{Bfinal} - u_{Binitial} = \frac{h_{final} - h_{initial}}{t_{final} - t_{initial}}$$
$$\Leftrightarrow 0 - \sqrt{2gh_0} = \frac{0 - h_0}{t_{final} - 0}$$
$$\Leftrightarrow -\sqrt{2gh_0} = \frac{-h_0}{t_{final}}$$
$$\Leftrightarrow t_{final} = \frac{h_0}{\sqrt{2gh_0}}$$
$$\Leftrightarrow \boxed{t_{final} = \sqrt{\frac{h_0}{2g}}}$$

2. The wind blows on a solid board

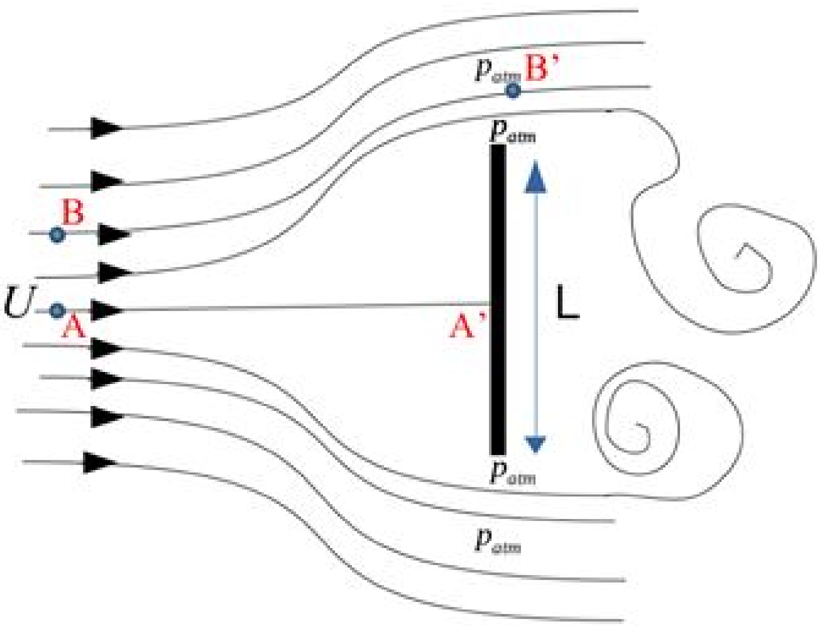

Figure 2: Wind blowing on a board

• Let's use Bernoulli's theorem to estimate the pressure at the centre of the board.
LetA' be the centre of the board.
On the streamline AA', we have:

$$\frac{1}{2}\rho U_A^2 + P_A + \phi_A = \frac{1}{2}\rho u_{A'}^2 + P_{A'} + \phi_{A'},\ U_A = U.$$

We have $u_{A'} = 0$ because there is no flow at the centre. Also $\phi_A = \phi_{A'} = 0$ (the potential energy is not relevant).

It follows that $\frac{1}{2}\rho U_A^2 + P_A = P_{A'}$ (*)

On the streamline BB', applying Bernoulli theorem we have:
$\frac{1}{2}\rho U_B^2 + P_B = \frac{1}{2}\rho u_{B'}^2 + P_{B'},\ U_B = U,\ (**)$ with $P_{B'} = P_{atm}$ (by assumption).
The point B 'is located in a region where the flow becomes uniform again (the streamlines become straight and parallel again). So $u_{B'} \simeq U_B = U$. Moreover, since A and B are very close to each other, we have: $P_A \simeq P_B$.
From $(**)$ we get:

$$\frac{1}{2}\rho U^2 + P_A = \frac{1}{2}\rho U^2 + P_{atm}$$
$$\Leftrightarrow P_A = P_{atm.}$$

$(*)$ becomes $\boxed{P_{A'} = \frac{1}{2}\rho U^2 + P_{atm}.}$

We have $\rho = \rho_{air} = 1.225Kg/m^3,\ P_{atm} = 101325Pa,\ U = 10m/s.$
So
$$P_{A'} = 101,386.25Pa$$

• Force at the centre of the board:
Let F be that force and S the area of the board. Then $F = P_{A'} \times S$.
$S = 2 \times 10 = 20m^2$.
Therefore, $\boxed{F = 2,027,725N}$.

3. Volume flux flowing through the pipe

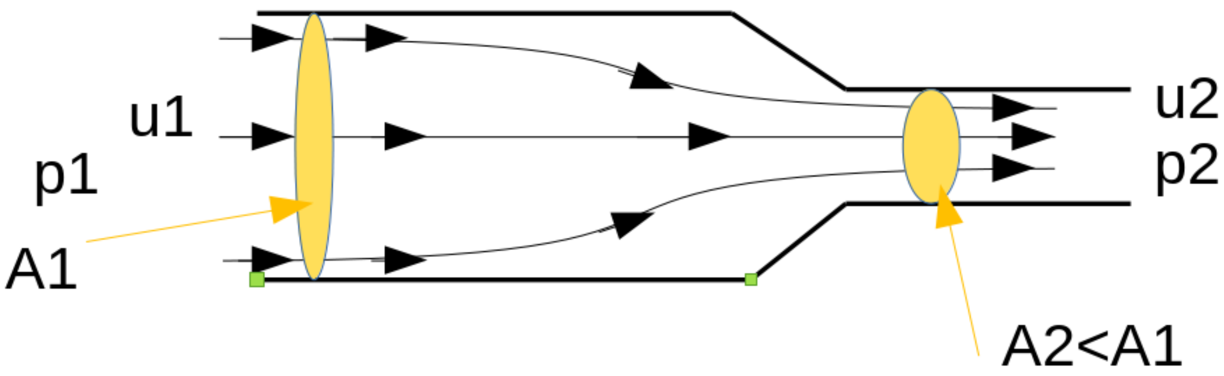

Figure 3: inviscid fluid flowing through a pipe of cross section A_1

Using Bernoulli theorem, we have:
$$\frac{1}{2}\rho u_1^2 + p_1 = \frac{1}{2}\rho u_2^2 + p_2$$
$$\Rightarrow \frac{1}{2}\rho(u_1^2 - u_2^2) = (p_2 - p_1)$$

By mass conservation, we have: $A_1u_1 = A_2u_2 \Rightarrow u_2 = \frac{A_1}{A_2}u_1$.
Also $u_1 = \frac{Q}{A_1} \Leftrightarrow u_1A_1 = Q$ (the volume flux).

Then:

$$\frac{1}{2}\rho u_1^2\left(1-\frac{A_1^2}{A_2^2}\right)=(p_1-p_2)$$
$$\Rightarrow \frac{1}{2}\rho\left(\frac{Q^2}{A_1^2}\right)\left(\frac{A_2^2-A_1^2}{A_2^2}\right)=(p_2-p_1)$$
$$\Rightarrow \rho Q^2\left(A_2^2-A_1^2\right)=2A_1^2A_2^2(p_2-p_1)$$
$$\Rightarrow Q^2=\frac{2A_2^2A_1^2(p_2-p_1)}{\rho(A_2-A_1)^2}$$

Therefore:

$$\boxed{Q=\sqrt{\frac{2A_2^2A_1^2(p_2-p_1)}{\rho(A_2-A_1)^2}}}$$

4. Two dimensional jet of water

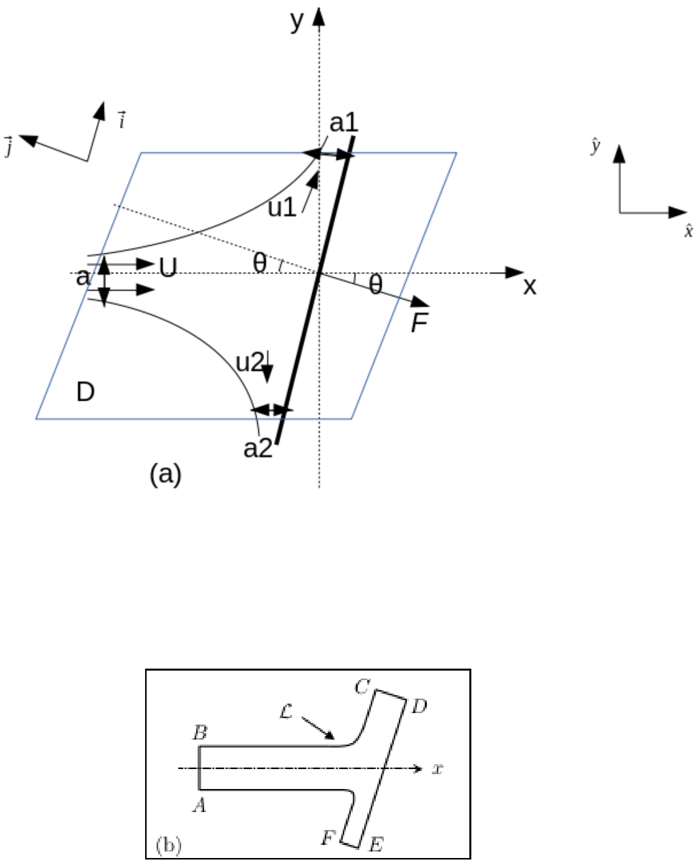

Figure 4: Two dimensional jet of water

Momentum integral: $\int_D \rho\mathbf{U}(\mathbf{n}.\mathbf{U})dS=\int_{\mathcal{L}}\rho\mathbf{U}(\mathbf{n}.\mathbf{U})dl=\int_{\mathcal{L}}-p\mathbf{n}dl$
But we have:

$$\int_{\mathcal{L}}\rho\mathbf{U}(\mathbf{n}.\mathbf{U})dl=\int_{AB}\rho(U\hat{x})(-Udl)+\int_{CD}\rho(u_1\vec{i})(u_1dl)+\int_{EF}\rho(-u_2\vec{i})(u_2dl)$$
$$\int_{\mathcal{L}}\rho\mathbf{U}(\mathbf{n}.\mathbf{U})dl=-\rho U^2a\hat{x}+\rho u_1^2a_1\vec{i}-\rho u_2^2a_2\vec{i}\ (*)$$

and $\int_{\mathcal{L}}-p\mathbf{n}dl=\vec{F}$ the force normal to the wall.

Let's apply Bernoulli theorem on the streamlines AF and BC. We then have:

$$\frac{1}{2}\rho U^2 + p_{atm} = \frac{1}{2}\rho u_1^2 + p_{atm} \quad \text{(streamline } BC\text{)},$$
$$\frac{1}{2}\rho U^2 + p_{atm} = \frac{1}{2}\rho u_2^2 + p_{atm} \quad \text{(streamline } AF\text{)}.$$

From these two equations we have: $u_1^2 = U^2 = u_2^2 \Rightarrow u_1 = u_2 = U$.
From the mass conservation, we have $\rho U a = \rho u_1 a_1 + \rho u_2 a_2$.
So we have $a = a_1 + a_2$ and $(*)$ becomes:

$$\vec{F} = \int_{\mathcal{L}} \rho \mathbf{U}(\mathbf{n}.\mathbf{U})dl = -\rho U^2 a\hat{x} + \rho U^2 a_1 \vec{i} - \rho U^2 a_2 \vec{i}$$
$$\vec{F} = \rho U^2 \left(-a\hat{x} + (a_1 - a_2)\vec{i}\right)$$

Since the force $\vec{F}$ is normal to the plate (so parallel to $\vec{j}$), then its $\vec{i}$ component is 0.
But we have $\hat{x} = \sin(\theta)\vec{i} + \cos(\theta)\vec{j}$.
Then:

$$\vec{F} = \rho U^2 \left(-a\left(\sin(\theta)\vec{i} + \cos(\theta)\vec{j}\right) + (a_1 - a_2)\vec{i}\right)$$
$$\vec{F} = \rho U^2 \left[\left(-a\sin(\theta) + a_1 - a_2\right)\vec{i} - a\cos(\theta)\vec{j} \right] \quad (**)$$

From $(**)$ it follows that

$$\vec{F} = -\rho a U^2 \cos(\theta)\vec{j}$$

and

$$a_1 - a_2 - a\sin(\theta) = 0 \quad (***)$$

So the force normal to the wall is

$$\boxed{\vec{F} = -\rho a U^2 \cos(\theta)\vec{j}}$$

5.

In this question, $A_1 = a_1,\ A_2 = a_2,\ A = a$.
• The pressure force parallel to the wall is zero because the pressure at both the inlet and the outlet to the control volume are atmospheric.
• Let's solve for a_1 and a_2 in terms of a
We use Bernoulli theorem:

$$\frac{1}{2}\rho U^2 + p_{atm} = \frac{1}{2}\rho u_1^2 + p_{atm} \quad \text{(streamline } B\text{)},$$
$$\frac{1}{2}\rho U^2 + p_{atm} = \frac{1}{2}\rho u_2^2 + p_{atm} \quad \text{(streamline } AF\text{)}.$$

From these two equations we have: $u_1^2 = U^2 = u_2^2 \Rightarrow u_1 = u_2 = U$.
But also from the mass conservation, we have $\rho U a = \rho u_1 a_1 + \rho u_2 a_2$
It follows that $a = a_1 + a_2$ $(****)$.

$$\begin{aligned} a_1 - a_2 - a\sin(\theta) = 0 \text{ and } (****) &\Rightarrow a_1 - (a - a_1) - a\sin(\theta) = 0 \\ &\Rightarrow 2a_1 - a - a\sin(\theta) = 0 \\ &\Rightarrow 2a_1 = a(1 + \sin(\theta)) \\ &\Rightarrow a_1 = \frac{1 + \sin(\theta)}{2}a \end{aligned}$$

$$\begin{aligned} a = a_1 + a_2 \Rightarrow a_2 &= a - a_1 \\ &= a(1 - \frac{1 + \sin(\theta)}{2}) \\ &= \frac{1 - \sin(\theta)}{2}a \end{aligned}$$

Therefore,

$$\boxed{a_1 = \frac{1 + \sin(\theta)}{2} a}$$

$$\boxed{a_2 = \frac{1 - \sin(\theta)}{2} a}$$