Fluid dynamics Assignment 02

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1. u=(u(y,t), 0) is unsteady and there is a body force f=(f_x, f_y). Let's show that :

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x$$
$$0 = -\frac{\partial p}{\partial y} + f_y$$

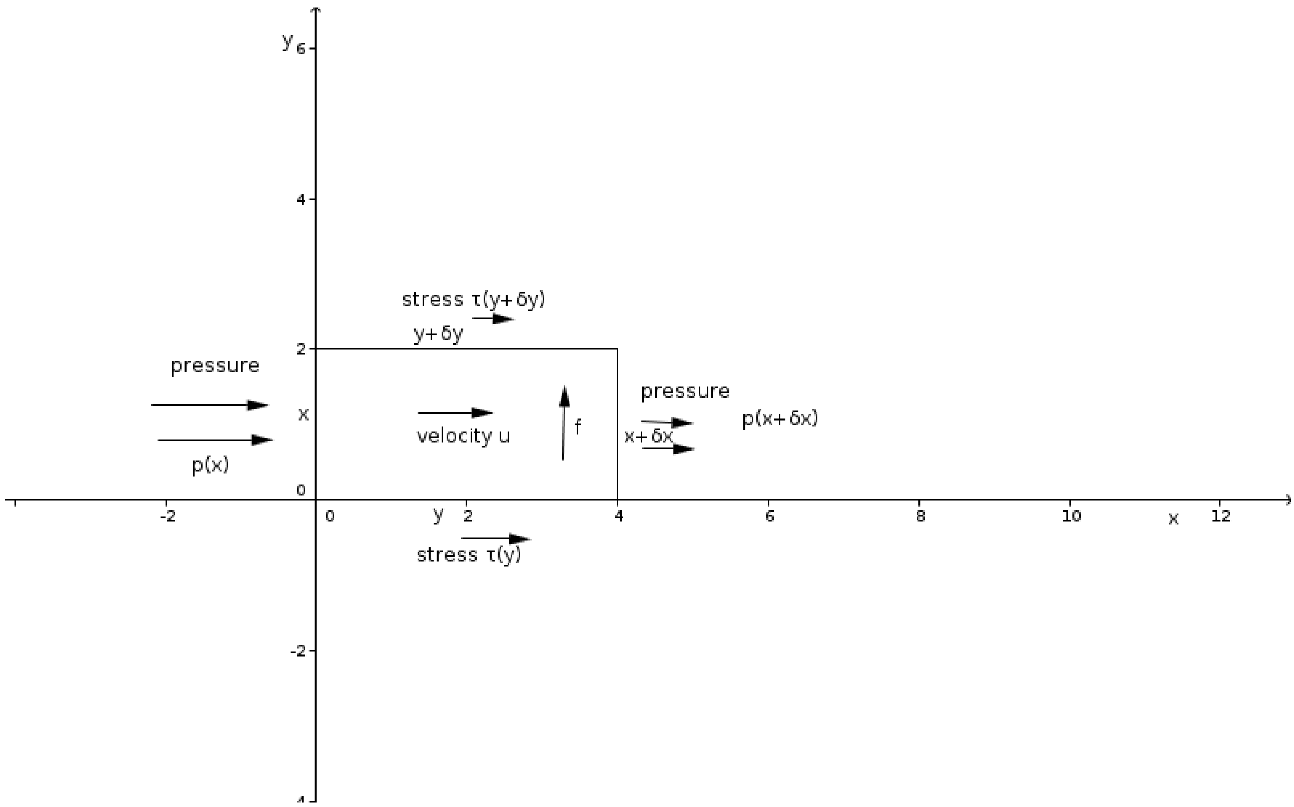

Figure 1: Unsteady flow

Here we denote $\tau = \tau_s$
The vertical sides of the slab experience pressure forces in the x direction, while the horizontal sides of the slab experience tangential shear stresses in the x direction from the surrounding fluid.
Newton law: $F = ma$ and $a = \frac{\partial u}{\partial t}$. So for small mass $\delta m = \rho\delta x\delta y$ we have: $dF = \delta m\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}\rho\delta x\delta y$.

Since the flow is unsteady and there is a body force (f_x, f_y) we have:

$$
\begin{aligned}
p(x)\delta y - p(x+\delta x)\delta y + \tau_s(y+\delta y)\delta x + \tau_s(y)\delta x + f_x\delta x\delta y &= \delta m\frac{\partial u}{\partial t} \\
\Leftrightarrow p(x)\delta y - p(x+\delta x)\delta y + \tau_s(y+\delta y)\delta x + \tau_s(y)\delta x + f_x\delta x\delta y &= \rho\frac{\partial u}{\partial t}\delta x\delta y \\
\Leftrightarrow -(p(x+\delta x)\delta y - p(x))\delta y + \mu\frac{\partial u}{\partial y}(y+\delta y)\delta x - \mu\frac{\partial u}{\partial y}(y)\delta x + f_x\delta x\delta y &= \rho\frac{\partial u}{\partial t}\delta x\delta y \\
&\text{because the normal to the lower surface of the slab pointing into} \\
&\text{the surrounding fluid is in the negative y direction.} \\
\Leftrightarrow -\frac{p(x+\delta x) - p(x)}{\delta x} + \mu\frac{\frac{\partial u}{\partial y}(y+\delta y) - \frac{\partial u}{\partial y}(y)}{\delta y} + f_x &= \rho\frac{\partial u}{\partial t} \text{ dividing by}
\end{aligned}
$$

Taking $\delta x \to 0$ and $\delta y \to 0$ we have $\frac{p(x+\delta x)\delta y - p(x)}{\delta x} \to \frac{\partial p(x)}{\partial x}$ and $\frac{\frac{\partial u}{\partial y}(y+\delta y) - \frac{\partial u}{\partial y}(y)}{\delta y} \to \frac{\partial^2 u(y)}{\partial y^2}$
Therefore, we have:

$$
\boxed{\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} + f_x}
$$

• For the second relation $-\frac{\partial p}{\partial y} + f_y$ we consider the vertical motion.
Since in y direction the flow is steady, then we have:

$$
\begin{aligned}
p(y)\delta y - p(y+\delta y)\delta x + f_y\delta x\delta y &= 0 \\
\Leftrightarrow \frac{p(y) - p(y+\delta y)}{\delta y} + f_y &= 0 \\
\Leftrightarrow -\frac{p(y+\delta y) - p(y)}{\delta y} + f_y &= 0
\end{aligned}
$$

Taking $\delta y \to 0$ we have $\frac{p(y) - p(y+\delta y)}{\delta y} \to \frac{\partial p(y)}{\partial y}$. As a result:

$$
\boxed{-\frac{\partial p}{\partial y} + fy = 0.}
$$

2.

Consider the steady flow of a layer of fluid of uni- form thickness h down a rigid vertical plane. Assume that the surrounding air exerts no stress on the fluid.

2.a Picture of the situation

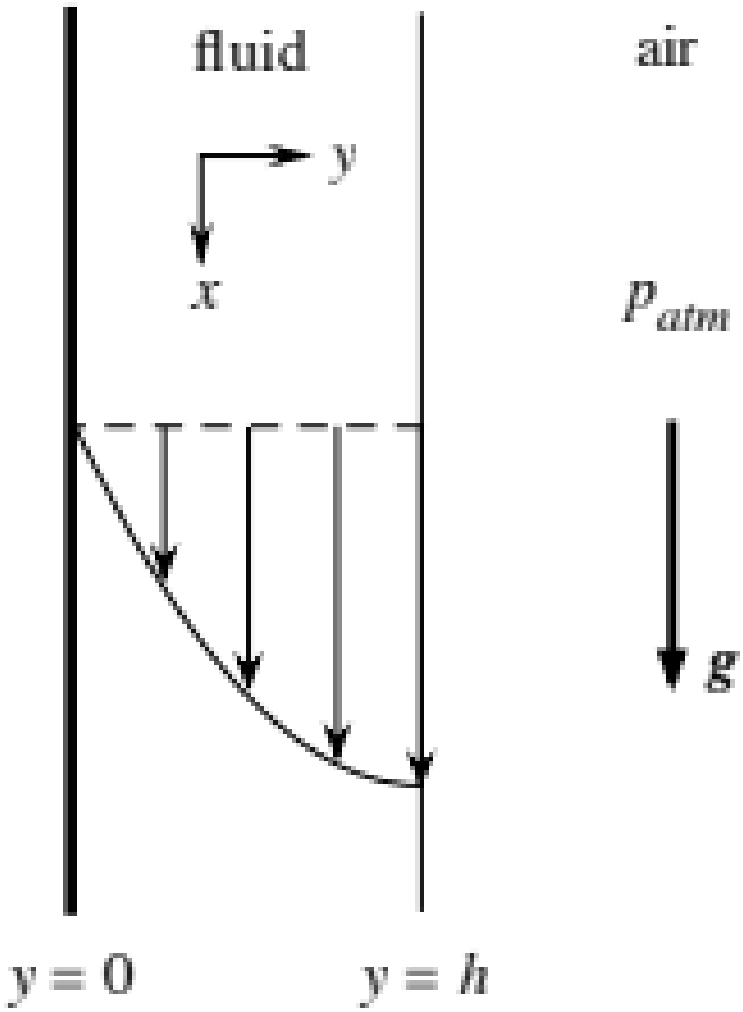

Figure 2: A film of viscous fluid of thickness h flowing down a vertical, rigid wall at y = 0.

2.b Dynamical equation governing the vertical velocity of the fluid

Navier Stokes equation in x direction :
$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \rho g_x$ with $g_x = g$ because the flow is vertical.
Pressure does not depend on y. So $\frac{\partial p}{\partial y} = 0$ and $p = p(x)$.
Also pressure is continuous across a fluid boundary, so the pressure is equal to the atmospheric pressure P_a, which we assume to be constant on the scale of the flow.
So $\frac{\partial p}{\partial x} = 0$. Since the flow is steady, we have $\frac{\partial u}{\partial t} = 0$.
Therefore the equation of motion in the x direction is given by:

$$\mu\frac{\partial^2 u}{\partial y^2} + \rho g = 0$$

.

2.c Boundary conditions

The boundary conditions are:
The velocity $u(y)$ is subject to no slip at the solid boundary ($u(0) = 0$) and no stress at the solid boundary with the air ($\frac{\partial u}{\partial y}(h) = 0$).

2.d Velocity profile and volume flow rate per unit width

• For the velocity profile, we solve the differential equation $\mu\frac{\partial^2 u}{\partial y^2} + \rho g = 0$ $(*)$
Using the method of separation of variables, we can write $u = X(t)Y(y)$. Then:
$\frac{\partial u}{\partial y} = X(t)Y'(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(t)Y''(y) \qquad \frac{\partial u}{\partial t} = X'(t)Y(y)$.
(*) becomes $\mu X(t)Y''(y) = -\rho g$ and we have:

$$\begin{aligned}
\mu X(t)Y''(y) = -\rho g &\Rightarrow X(t) = c_1 \in \mathbb{R},\ \text{a constant.}\\
&\Rightarrow Y''(y) = \frac{-\rho g}{\mu c_1}\\
&\Rightarrow Y'(y) = \frac{-\rho g}{\mu c_1}y + c_2,\ c_2 \in \mathbb{R} \text{ a constant} \qquad (1)\\
&\Rightarrow Y(y) = \frac{-1}{2}\frac{\rho g}{\mu c_1}y^2 + c_2 y + c_3,\ c_3 \in \mathbb{R} \text{ a constant .}
\end{aligned}$$

We use the boundary conditions:

$$\begin{aligned}
u(0) = 0 &\Rightarrow X(t)Y(0) = c_1\left(\frac{-1}{2}\frac{\rho g}{\mu c_1}0^2 + c_2 \times 0 + c_3\right) = 0\\
&\Rightarrow c_3 = 0. \text{ So } Y(y) = \frac{-1}{2}\frac{\rho g}{\mu c_1}y^2 + c_2 y\\
\text{Also } \frac{\partial u}{\partial y}(h) = 0 &\Rightarrow X(t)Y'(h) = c_1\left(\frac{-\rho g}{\mu c_1}h + c_2\right) = 0 \text{ using (1)}\\
&\Rightarrow \frac{-\rho g}{\mu}h + c_1 c_2 = 0\\
&\Rightarrow c_1 c_2 = \frac{\rho g}{\mu}h.
\end{aligned}$$

$$\begin{aligned}
\text{It follows that } u = c_1\left(\frac{-1}{2}\frac{\rho g}{\mu c_1}y^2 + c_2 y\right) &= \frac{-1}{2}\frac{\rho g}{\mu}y^2 + c_1 c_2 y\\
&= \frac{-1}{2}\frac{\rho g}{\mu}y^2 + \frac{\rho g}{\mu}hy\\
&= \frac{\rho g}{2\mu}y\left(2h - y\right)
\end{aligned}$$

$$\boxed{u = \frac{\rho g}{2\mu}y\left(2h - y\right)}$$

• **Volume flow rate:**
Let Q be the volume flow rate per unit width.
We have :

$$\begin{aligned}
Q &= \int_0^h u\,dy\\
&= \int_0^h \frac{\rho g}{2\mu}y\left(2h - y\right)dy\\
&= \frac{\rho g}{2\mu}\int_0^h \left(2yh - y^2\right)dy\\
&= \frac{\rho g}{2\mu}\left[hy^2 - \frac{1}{3}y^3\right]_0^h\\
&= \frac{\rho g}{2\mu}\left(h^3 - \frac{1}{3}h^3\right)\\
&= \frac{\rho g}{2\mu}\left(\frac{2}{3}h^3\right)\\
&= \frac{\rho g h^3}{3\mu}
\end{aligned}$$

Therefore $\boxed{Q = \dfrac{\rho g h^3}{3\mu}}$

3.

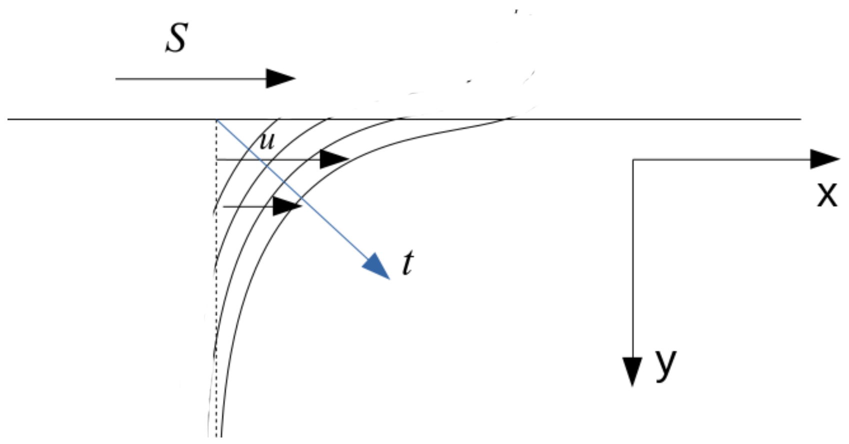

Figure 3:

• Surface velocity for $\nu t << d^2$

We want to solve $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ $\quad (PDE)$ with the boundary conditions: $\mu \frac{\partial u}{\partial y} = -S$ and $\lim\limits_{y \to +\infty} u(t, y) = 0$

Suppose there exists a function F such that $u = SF(\eta)$ with $\eta = \frac{y}{\sqrt{\nu t}} = y\,(\nu t)^{-\frac{1}{2}}$.

We have $\frac{\partial \nu}{\partial t} = -\frac{1}{2} y \nu\,(\nu t)^{-\frac{3}{2}} = -\frac{1}{2t}\frac{y}{\sqrt{\nu t}} = -\frac{\eta}{2t}$ and $\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{\nu t}}$

Then:

$$
\begin{aligned}
\frac{\partial u}{\partial y} &= S\frac{\partial \eta}{\partial y}F'(\eta) \\
&= \frac{S}{\sqrt{\nu t}}F'(\eta) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{S}{\sqrt{\nu t}}\frac{\partial \eta}{\partial y}F''(\eta) \\
&= \frac{S}{\nu t}F''(\eta) \\
\bullet\ \nu\frac{\partial^2 u}{\partial y^2} &= \frac{S}{t}F''(\eta) \\
\frac{\partial u}{\partial t} &= S\frac{\partial \eta}{\partial t}F'(\eta) \\
&= \frac{-Sy}{2t\sqrt{\nu t}}F'(\eta) \\
\bullet\ \frac{\partial u}{\partial t} &= \frac{-S\eta}{2t}F'(\eta).
\end{aligned}
$$

So from (PDE) we find: $\dfrac{-S\nu}{2t}F'(\eta) = \dfrac{S}{t}F''(\eta) \Leftrightarrow \dfrac{F''(\eta)}{F'(\eta)} = -\dfrac{\eta}{2} \quad (ODE)$

$$
\begin{aligned}
(ODE) \Leftrightarrow \ln(F'(\eta)) &= -\frac{\eta^2}{4} + c,\ c \text{ constant of } \eta \\
\Leftrightarrow F'(\eta) &= \lambda \exp\left(-\frac{\eta^2}{4}\right)
\end{aligned}
$$

We have $u(t,y) \to 0$ when $y \to +\infty$, that is, $\eta \to +\infty$ so $F(\eta) \to 0$ when $\eta \to +\infty. \exp\left(-\frac{y^2}{4}\right)$ So

$$F(\eta) = \lambda \int_{\eta}^{+\infty} \exp\left(-\frac{\eta^2}{4}\right) d\eta$$

At the surface, $y \to 0$, and $\eta \to 0$.

$$\begin{aligned}
F(0) &= \lambda \int_{0}^{+\infty} \exp\left(-\frac{r^2}{4}\right) dr \\
&= \lambda \sqrt{\left(\int_{0}^{+\infty} \exp\left(-\frac{r^2}{4}\right) dr\right)^2} \\
&= \lambda \sqrt{\left(\int_{0}^{+\infty} \exp\left(-\frac{r^2}{4}\right) dr\right)} \\
&= \lambda \sqrt{\left(\int_{0}^{+\infty} \exp\left(-\frac{x^2}{4}\right) dx\right)\left(\int_{0}^{+\infty} \exp\left(-\frac{y^2}{4}\right) dy\right)} \\
&= \lambda \sqrt{\int_{0}^{+\infty} \exp\left(-\frac{x^2+y^2}{4}\right) dxdy}
\end{aligned}$$

Using the change of variable $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$ we find

$F(0) = \sqrt{\pi}\lambda$. So $u(t,0) = S\lambda\sqrt{\pi}$

Using the boundary conditions, we have: $\frac{\partial u}{\partial y}(0) = \frac{S}{\sqrt{\nu t}} F'(0) = \lambda \frac{S}{\sqrt{\nu t}} \exp\left(-\frac{0^2}{4}\right) = -\frac{S}{\mu} \Rightarrow \lambda = -\frac{\sqrt{\nu t}}{\mu}$.

Therefore $\boxed{u_{surface} = -\frac{\sqrt{\pi\nu t}}{\mu} S.}$

• Let's assume that the wind has been blowing for sufficiently long time to establish a steady state. The lake has a constant depth d before the wind blows. The wind will tend to push water towards the downwind end of the lake and thus establish a small variation in depth h(x). If we also assume that the flow in the lake is parallel (except near the ends) then $\frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x} = 0$ (1) and $-\frac{\partial p}{\partial y} - \rho g = 0$ (2)
We integrate (2) to find $p = -\rho g y + \rho g h(x) + p_{atm}$. given that the pressure is equal to atmospheric pressure at the surface of the lake. Using this expression in (1) yields: $\frac{\partial^2 u}{\partial y^2} = \rho g h'(x)$
This equation is subject to the boundary conditions: no slip at the bed of the lake and the imposed stress at the surface.
$u(0) = 0$; $\mu \frac{\partial u}{\partial y}(d) = S$, with d the depth of the lake. Using exactly the same method we used at 4.d, we find $u = -\frac{\rho g h'(x) y}{2\mu}(2d - y) + \frac{Sy}{\mu}$.
Now we calculate the volume flux per unit width:

$$q = \int_0^h u dy = \frac{\rho g d^3 h'(x)}{3\mu} + \frac{Sd^2}{2\mu}$$

Since the flow is steady the volume flux across any vertical cross section is equal to zero. But we have: $q = 0$ if and only if $h'(x) = \frac{3S}{2\rho g d}$
So the velocity profile

$$\begin{aligned}
u &= -\frac{\rho g}{2\mu} \frac{3S}{2\rho g d} y(2d - y) + \frac{S}{\mu} y \\
&= -\frac{3S}{4\mu d} y(2d - y) + \frac{S}{\mu} y
\end{aligned}$$

Therefore, taking $y = d$ at the surface, we find the velocity at the surface:

$$\left(\int_0^{+\infty} \exp\left(-\frac{x^2}{4}\right) dx\right)$$

$$\begin{aligned}
u_{surf} &= -\frac{3S}{4\mu d} d(2d - d) + \frac{S}{\mu} d \\
&= \frac{Sd}{4\mu}
\end{aligned}$$

$$\boxed{u_{surf} = \frac{Sd}{4\mu}}$$

4.

• **The equation of motion is**:

$$\mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\theta = 0 \tag{1}$$

• **The boundary conditions**:

$$u(0) = 0$$
$$\frac{\partial u}{\partial y}(h) = 0$$

. • **Pressure:**

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta$$

So $p = -\rho g y \cos\theta + c,\ c$ constant
Using the boundary conditions; at $y = h$, $p = p_{atm}$ we have:

$$f(x) = p_{atm} + \rho g h \cos\theta$$

Therefore $p = p_{atm} + \rho g h \cos\theta - \rho y \cos\theta$
• **Velocity profile** Here again, we compute the velocity profile analogously as in 2.d and we obtain:

$$\boxed{u = \frac{\rho g \sin\theta}{2\mu} y(2h - y)}$$

Volume flux Then volume flux per unit width is given by:

$$\begin{aligned}
Q &= \int_0^h u dy \\
Q &= \int_0^h \frac{\rho g \sin\theta}{2\mu} y(2h - y)\partial y \\
&= \frac{\rho g \sin\theta}{2\mu} \int_0^h (2hy - y^2)\partial y \\
&= \frac{\rho g \sin\theta}{2\mu} \left[\frac{2hy^2}{2} - \frac{y^3}{3}\right]_0^h \\
&= \frac{\rho g \sin\theta}{\mu} \left(\left(\frac{h^3}{2} - \frac{h^3}{6}\right) - 0\right) \\
&= \frac{\rho g h^3 \sin\theta}{3\mu}
\end{aligned}$$

$$\boxed{Q = \frac{\rho g h^3 \sin\theta}{3\mu}}$$

We consider a second layer of fluid flowing steadily on top of the first layer.
We denote by u_1, p_1 respectively the velocity and pressure of the first fluid and u_2, p_2 the ones of the second.
• **Boundary conditions**
At the interface: $u_1 = u_2$, $\mu\frac{\partial u_1}{\partial y} = \beta\mu\frac{\partial u_2}{\partial y}$ and $p_1 = p_2$ (by continuity).
• **Pressure**

$$\boxed{p_1 = \rho g \cos\theta(\alpha h + h - y) + p_{atm}}$$

$$\boxed{p_2 = \rho g \cos\theta(\alpha h + h - y) + p_{atm}}$$

• **Velocities**

$$\boxed{u_1 = \frac{\rho g}{\mu} \sin\theta(\alpha h y + h y - \frac{1}{2}y^2)}$$

$$\boxed{u_2 = \frac{\rho g}{\mu} \sin\theta(\alpha h + h)}$$