Algebra Assignment 03

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1.

a) $S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$.
Let's show that S^1 is a group under the multiplication $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + i\theta_2)}$
S^1 is a subset of $\mathbb{C}$. Then we just need to show that S^1 is a subgroup of $\mathbb{C}$ under its natural multiplication.
• We have $1 = e^{i0}$, so $1 \in S^1$ and then $S^1 \neq \emptyset$.
• Let $x, y \in S^1$. Then there exists $\theta_1, \theta_2 \in \mathbb{R}$ such that $x = e^{i\theta_2}$ and $y = e^{i\theta_2}$.
y has an inverse in $\mathbb{C}$, say $y^{-1} = e^{-i\theta_2}$ and we have :
$xy^{-1} = e^{i\theta_1}e^{-i\theta_2} = e^{i(\theta_1 - \theta_2)} \in S^1$ because $\theta_1 - \theta_2 \in \mathbb{R}$.
We conclude that S^1 is a subgroup of $\mathbb{C}$, so a group.
b)
$\mathbb{Z}$ is a normal subgroup of $\mathbb{R}$ because $\mathbb{R}$ is commutative and $\mathbb{Z}$ is one of its subgroups.
c)
Let's use the first Isomorphism Theorem to prove that $\mathbb{R}/\mathbb{Z} \cong S^1$.
We consider the following map

$$\begin{aligned} \varphi : (\mathbb{R}, +) &\to S^1 \\ \theta &\mapsto e^{i2\pi\theta} \end{aligned}$$

• For all $\theta_1, \theta_2 \in \mathbb{R}$ we have $\varphi(\theta_1 + \theta_2) = e^{i(\theta_1 + i\theta_2)} = e^{i\theta_1}e^{i\theta_2} = \varphi(\theta_1)\varphi(\theta_2)$.
Thus φ is a homomorphism.
• Let $x \in S^1$. Then there exists $\theta \in \mathbb{R}$ such that $x = e^{i\theta}$. We want to find $\theta' \in \mathbb{R}$ such that $\varphi(\theta') = x$.
Let's take $\theta' = \frac{\theta}{2\pi}$. We have $\frac{\theta}{2\pi} \in \mathbb{R}$ and
$\varphi(\theta') = \varphi(\frac{\theta}{2\pi}) = e^{i(2\pi(\frac{\theta}{2\pi}))} = e^{i\theta} = x$.
It follows that φ is onto and then we have $Rg(\varphi) = S^1$.

- We have :

$$\begin{aligned} Ker(\varphi) &= \{\theta \in \mathbb{R} | \varphi(\theta) = 1\} \\ &= \{\theta \in \mathbb{R} | e^{i2\pi\theta} = 1\} \\ &= \{\theta \in \mathbb{R} | \theta \in \mathbb{Z}\} \text{ because } e^{i2\pi\theta} = 1 \Leftrightarrow \theta \in \mathbb{Z} \\ &= \mathbb{Z}. \end{aligned}$$

$$Ker(\varphi) = \mathbb{Z}.$$

Therefore, we can apply the First Theorem of Isomorphism :

$$\mathbb{R}/Ker(\varphi) \cong Rg(\varphi) \Leftrightarrow \boxed{\mathbb{R}/\mathbb{Z} \cong S^1}$$

2.

Let $T : \mathcal{M}_{22}(\mathbb{R}) \to \mathbb{R}$ be a linear transformation. We want to show that there exist $a, b, c, d \in \mathbb{R}$ such that

$$T \begin{bmatrix} w & x \\ y & z \end{bmatrix} = aw + bx + cy + dz$$

for all $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathcal{M}_{22}(\mathbb{R})$.

The set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis for $\mathcal{M}_{22}(\mathbb{R})$.

Then for all $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathcal{M}_{22}(\mathbb{R})$, we have

$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = wE_{11} + xE_{12} + yE_{21} + zE_{22}$.

Thus $T \begin{bmatrix} w & x \\ y & z \end{bmatrix} = wT(E_{11}) + xT(E_{12}) + yT(E_{21}) + zT(E_{22})$ because T is linear.

Let $a = T(E_{11}), b = T(E_{12}), c = T(E_{21})$ and $d = T(E_{22})$.

As a result, there exist $a, b, c, d \in \mathbb{R}$ such that

$$T \begin{bmatrix} w & x \\ y & z \end{bmatrix} = aw + bx + cy + dz$$

for all $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \in \mathcal{M}_{22}(\mathbb{R})$

3.

Let $\{v_1, \cdots, v_n\}$ be a basis for a vector space V and let $T : V \to V$ be a linear transformation. We want to prove that if $T(v_1) = v_1, ..., T(v_n) = v_n$, then T is the identity transformation on V.
Suppose V is a $K-$vectorial space.
Let $x \in V$, $\exists\ a_1, a_2, \cdots, a_n \in K$ such that $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$

$$\begin{aligned} \text{Then} \quad T(x) = T(\sum_{i=1}^{n} a_i v_i) &= \sum_{i=1}^{n} a_i T(v_i) \qquad \text{because T is linear} \\ &= \sum_{i=1}^{n} a_i v_i \text{ by assumption} \\ &= x \end{aligned}$$

Thus T is the identity transformation on V.

4.

Let $S : \mathcal{P} \to \mathbb{R}$ be the linear transformation defined by

$$S(p(x)) = \int_0^1 p(x)dx$$

• Let's find the **kernel** of S.

We have

$$Ker(S) = \left\{ p(x) \in \mathcal{P} | \int_0^1 p(x)dx = 0 \right\}$$

Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$.

$$\text{We have } \int_0^1 p(x)dx = \left[a_0x + \frac{1}{2}a_1x^2 + \cdots + \frac{1}{n+1}a_nx^{n+1} \right]_0^1$$
$$= a_0 + \frac{1}{2}a_1 + \cdots + \frac{1}{n+1}a_n$$
$$= \sum_{i=0}^{i=n} \frac{1}{i+1}a_i$$

Hence,

$$\boxed{Ker(S) = \left\{ p(x) = \sum_{i=0}^{i=n} a_ix^i \in \mathcal{P} \mid \sum_{i=0}^{i=n} \frac{1}{i+1}a_i = 0 \right\}}$$

• Let's find the **range** Rg of S.
We have :

$$\begin{aligned} Rg(S) &= \{S(p(x)) \mid p(x) \in \mathcal{P}\} \\ &= \left\{ \int_0^1 p(x)dx \mid p(x) \in \mathcal{P} \right\} \\ &\subseteq \mathbb{R} \end{aligned}$$

Let's show that $\mathbb{R} \subseteq Rg(S)$.
Let $\beta \in \mathbb{R}$.
Consider $p(x) = 2\beta x$. We have $p(x) \in \mathcal{P}$ and

$$\int_0^1 p(x)dx = \left[\beta x^2\right]_0^1 = \beta.$$

It follows that $\mathbb{R} \subseteq Rg(S)$ and we conclude that

$$\boxed{Rg(S) = \mathbb{R}}$$

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5.

Let's consider the subspace $W = span\left\{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\right\}$ of $\mathcal{D}$, the vector space of all differentiable functions over $\mathbb{R}$.

$a)$ Let's find the matrix of the differential operator D with respect to $\mathcal{B} = \{e^{2x}, e^{2x}\cos x, e^{2x}\sin x\}$.
We have :

$$D(e^{2x}) = 2e^{2x}, D(e^{2x}\cos x) = -e^{2x}\sin x + 2e^{2x}\cos x, D(e^{2x}\sin x) = e^{2x}\cos x + 2e^{2x}\sin x$$

In terms of coordinates in the basis $\mathcal{B}$, we have

$$D(e^{2x}) = \begin{bmatrix}2\\0\\0\end{bmatrix},\ D(e^{2x}\cos x) = \begin{bmatrix}0\\2\\-1\end{bmatrix},\ D(e^{2x}\sin x) = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

The matrix of the differential operator is then :

$$M = \begin{bmatrix}2 & 0 & 0\\0 & 2 & 1\\0 & -1 & 2\end{bmatrix}$$

$b)$
Let's compute the derivative of $f(x) = 3e^{2x} - e^{2x}\cos x + 2e^{2x}\sin x$ using the answer in $a)$.
In terms of coordinates in the basis $\mathcal{B}$ we have :

$$f(x) = \begin{bmatrix}3\\-1\\2\end{bmatrix}$$

Thus

$$\begin{aligned} D(f(x)) &= M \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} \end{aligned}$$

As a result,

$$\boxed{D(f(x)) = 6e^{2x} + 5e^{2x} \sin x}$$