Model Theory Assignment 01

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1.

Let A and B be sets and let $f : A \to B$. Let's prove each of the following assertions.

(a) For all $C \subseteq A,\ C \subseteq f^{-1}(f(C))$.

Let $C \subseteq A$ and $x \in C$. Then $f(x) \in f(C)$. But we have :

$$f(x) \in f(C) \Rightarrow x \in f^{-1}(f(C))$$

Therefore $C \subseteq f^{-1}(f(C))$ and we coclude that $\boxed{C \subseteq f^{-1}(f(C))}$ for all $C \subseteq A$

(b) For all $D \subseteq B,\ f(f^{-1}(D)) \subseteq D$

Let $D \subseteq B$ and $y \in f(f^{-1}(D))$. Then there exists $x \in f^{-1}(D) : f(x) = y$. But :

$$\begin{aligned} x \in f^{-1}(D) &\Rightarrow f(x) \in D \\ &\Rightarrow y \in D \text{ because } y = f(x) \end{aligned}$$

As a result $f(f^{-1}(D)) \subseteq D$ and we conclude that $\boxed{f(f^{-1}(D)) \subseteq D}$ for all $D \subseteq B$.

(c) The function f is injective if and only if, for all $C \subseteq A,\ C = f^{-1}(f(C))$

•Suppose f is injective and let $C \subseteq A$. We proved in **(a)** that $C \subseteq f^{-1}(f(C))$ even if f is not injective. So we just need to show that $f^{-1}(f(C)) \subseteq C$.
Let $x \in f^{-1}(f(C))$. Then $f(x) \in f(C)$. Since $f(x) \in f(C)$ then there exists $t \in C : f(t) = f(x)$. Thus $x = t$ because f is injective. It follows that $x \in C$ and then $f^{-1}(f(C)) \subseteq C$. As a result, $C = f^{-1}(f(C))$ for all $C \subseteq A$.

• Now suppose for all $C \subseteq A,\ C = f^{-1}(f(C))$. Let's show that f is injective.
Let $x, y \in A$ such that $f(x) = f(y)$.

Then $f(\{x\}) = f(\{y\})$ and $f^{-1}(f(\{x\})) = f^{-1}(f(\{y\}))$.
But $f^{-1}(f(\{x\})) = \{x\}$ and $f^{-1}(f(\{y\})) = \{y\}$ by assumption.
So $\{x\} = \{y\}$ and it follows that $x = y$. Hence f is injective.
Therefore, f is injective if and only if, for all $C \subseteq A,\ C = f^{-1}(f(C))$

(d) The function f is surjective if and only if, for all $D \subseteq B,\ f(f^{-1}(D)) = D$

• Suppose f is surjective and let $D \subseteq B$. We have shown in **(b)** that $f(f^{-1})(D) \subseteq D$. Let's prove that $D \subseteq f(f^{-1}(D))$.
Let $y \in D$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$.
We have :

$$
\begin{aligned}
\begin{cases} f(x) = y \\ y \in D \end{cases} \quad & \Rightarrow f(x) \in D \\
& \Rightarrow x \in f^{-1}(D) \\
& \Rightarrow f(x) \in f(f^{-1}(D)) \\
& \Rightarrow y \in f(f^{-1}(D)) \text{ because } y = f(x)
\end{aligned}
$$

So $D \subseteq f(f^{-1}(D))$ and we have :
if f is surjective then for all $D \subseteq B,\ f(f^{-1}(D)) = D$

• Suppose $D \subseteq B,\ f(f^{-1}(D)) = D$ for all. Let's show that f is surjective.
Let $y \in B$. By assumption we have $f(f^{-1}(\{y\})) = \{y\}$.
It follows that $f^{-1}(\{y\}) \neq \emptyset$, otherwise we would have $f(f^{-1}(\{y\})) = \emptyset$. So $\exists\, x \in A : f(x) = y$. Thus f is surjective.

2.

Let $J = \Big(1, +\infty\Big) \subseteq \mathbb{R}$. For $j \in J$, let $A_j = \left[1 + \dfrac{1}{j}, j^2 + 2j + 1\right] \subseteq \mathbb{R}$.
(a) Let's find $\bigcup\limits_{j \in J} A_j$.

$$\bigcup_{j \in J} A_j = \Big(1, +\infty\Big)$$

Proof :

• $\bigcup\limits_{j \in J} A_j \subseteq \Big(1, +\infty\Big)$

For all $j \in J,\ A_j \subseteq \Big(1, +\infty\Big)$. So $\bigcup_{j \in J} A_j \subseteq \Big(1, +\infty\Big)$.

- $\Big(1, +\infty\Big) \subseteq \bigcup_{j \in J} A_j$

Let $x \in \Big(1, +\infty\Big)$.

We want to find $j > 1$ such that $x \in A_j$, that is, $1 + \frac{1}{j} \leq x < j^2 + 2j + 1$.

For $j > 1$, since $x > 1$, we have :

$$\begin{aligned} 1 + \frac{1}{j} \leq x &\Leftrightarrow j + 1 \leq jx \\ &\Leftrightarrow j(x-1) \geq 1 \\ &\Leftrightarrow j \geq \frac{1}{x-1} \end{aligned}$$

Also,

$$x < j^2 + 2j + 1 \Leftrightarrow j^2 + 2j + 1 - x > 0$$

$$\Delta = 4 - 4(1 - x) = 4x$$

$$j_1 = \frac{-2 - 2\sqrt{x}}{2} = -1 - \sqrt{x}\ ; \quad j_2 = \frac{-2 + 2\sqrt{x}}{2} = \sqrt{x} - 1$$

$$\text{So, } \begin{cases} j^2 + 2j + 1 - x > 0 \\ j > 1 \end{cases} \quad \Leftrightarrow j > \sqrt{x} - 1$$

Then we can take $j = max\left\{\frac{1}{x-1}, \sqrt{x} - 1\right\} + 1$ and we have $x \in A_j$.

Therefore, $\Big(1, +\infty\Big) \subseteq \bigcup_{j \in J} A_j$ **and we conclude that :**

$$\boxed{\bigcup_{j \in J} A_j = \Big(1, +\infty\Big)}$$

(b) Let's find $\bigcap_{j \in J} A_j$.

$$\bigcap_{j \in J} A_j = \Big[2, 4\Big]$$

Proof :

• $\left[2,4\right] \subseteq \bigcap_{j\in J} A_j$

For all $j \in J,\ 1+\dfrac{1}{j} < 2 < 4 < j^2+2j+1$, that is, $\left[2,4\right] \subseteq A_j\ \forall j \in J$. So $\left[2,4\right] \subseteq \bigcap_{j\in J} A_j$.

• $\bigcap_{j\in J} A_j \subseteq \left[2,4\right]$

Let $x \in \bigcap_{j\in J} A_j$.

Then we have $x > 1$ and for all $j \in J,\ x \in A_j$.

$*$ Suppose by the contrary that $x < 2$. We are going to find $j_0 > 1$ such that $x \notin A_{j_0}$. We can take $1+\dfrac{1}{j_0} = \dfrac{x+2}{2}$, the center of $\left(x,2\right)$, that is, $j_0 = \dfrac{2}{x}$.

We have :

$$\begin{cases} j_0 > 1 \text{ because } 1 < x < 2 \\ \dfrac{1}{j_0}+1 = \dfrac{x+2}{2} = \dfrac{x}{2}+1 = x\left(\dfrac{1}{2}+\dfrac{1}{x}\right) > x \text{ because } \dfrac{1}{x} > \dfrac{1}{2} \text{ since } 1 < x < 2 \end{cases}$$

So $x \notin A_{j_0}$ and we have a contradiction because $x \in \bigcap_{j\in J} A_j$.

$*$ Suppose $x > 4$. Then $\sqrt{x}-1 > 1$ and taking $j_1 = \dfrac{\sqrt{x}}{2}$, we have :

$$\begin{cases} j_1 > 1 \\ j_1^2+2j_1+1 = \dfrac{x}{4}+\sqrt{x}+1 = x\left(\dfrac{1}{4}+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}\right) \end{cases}$$

$$\Rightarrow \begin{cases} j_1 > 1 \\ j_1^2+2j_1+1 = x\left(\dfrac{1}{4}+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}\right) < x\left(\dfrac{1}{4}+\dfrac{1}{2}+\dfrac{1}{4}\right) = x, \quad \text{because } x > 4 \end{cases}$$

So $x \notin A_{j_1}$, there is a contradiction because $x \in \bigcap_{j\in J} A_j$.

It follows that $\bigcap_{j\in J} A_j \subseteq \left[2,4\right]$ and we conclude that

$$\boxed{\bigcap_{j\in J} A_j = \left[2,4\right]}$$

(c) If we made $J = \Big[1, +\infty\Big)$ only $\bigcap_{j\in J} A_j$ would change and become

$$\bigcap_{j\in J} A_j = \Big[2, 4\Big)$$

3.

We define $\sim$ on $\mathbb{Z}\times\mathbb{R}$ as follows. For all $\Big(x_1, x_2\Big), \Big(y_1, y_2\Big) \in \mathbb{Z}\times\mathbb{R}$,

$$\Big(x_1, x_2\Big) \sim \Big(y_1, y_2\Big) \text{ if } x_1 = y_1.$$

(a) Let's prove that $\sim$ is an equivalent relation on $\mathbb{Z}\times\mathbb{R}$.

• **Reflexivity**

Let $\Big(x, y\Big) \in \mathbb{Z}\times\mathbb{R}$. We have $x = x$. So $\Big(x, y\Big) \sim \Big(x, y\Big)$. Hence, $\sim$ is reflexive.

• **Symmetry**

Let $\Big(x_1, x_2\Big), \Big(y_1, y_2\Big) \in \mathbb{Z}\times\mathbb{R}$. We have :

$$\begin{aligned}\Big(x_1, x_2\Big) \sim \Big(y_1, y_2\Big) &\Rightarrow x_1 = y_1\\ &\Rightarrow y_1 = x_1\\ &\Rightarrow \Big(y_1, y_2\Big) \sim \Big(x_1, x_2\Big).\end{aligned}$$

So $\sim$ is symmetric.

• **Transitivity**

Let $\Big(x_1, x_2\Big), \Big(y_1, y_2\Big), \Big(z_1, z_2\Big) \in \mathbb{Z}\times\mathbb{R}$

Suppose $\Big(x_1, x_2\Big) \sim \Big(y_1, y_2\Big)$ and $\Big(y_1, y_2\Big) \sim \Big(z_1, z_2\Big)$.

Then we have :

$$
\begin{aligned}
x_1 = y_1 \text{ and } y_1 = z_1 & \\
&\Rightarrow x_1 = z_1 \\
&\Rightarrow \Big(x_1, x_2\Big) \sim \Big(z_1, z_2\Big).
\end{aligned}
$$

Thus $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{R}$.

(b) Let's prove that $\mathbb{Z} \cong \Big(\mathbb{Z} \times \mathbb{R}\Big)\Big/\sim$. We use the notation $\Big[.\Big]$ for the equivalence classes.
Consider the function

$$
\begin{aligned}
\varphi : \Big(\mathbb{Z} \times \mathbb{R}\Big)\Big/ \sim &\longrightarrow \mathbb{Z} \\
\Big[\Big(x, y\Big)\Big] &\mapsto x
\end{aligned}
$$

• Let's show that φ is well defined.
Let $\Big[\Big(x_1, x_2\Big)\Big], \Big[\Big(y_1, y_2\Big)\Big] \in \Big(\mathbb{Z} \times \mathbb{R}\Big)\Big/ \sim$
Suppose $\Big[\Big(x_1, x_2\Big)\Big] = \Big[\Big(y_1, y_2\Big)\Big]$
Then we have :

$$
\begin{aligned}
x_1 = y_1 & \\
&\Rightarrow \varphi\Big(\Big[\Big(x_1, x_2\Big)\Big]\Big) = \varphi\Big(\Big[\Big(y_1, y_2\Big)\Big]\Big)
\end{aligned}
$$

So φ is well defined.

• Let's show that φ is one-to-one

Let $\Big[\Big(x_1, x_2\Big)\Big], \Big[\Big(y_1, y_2\Big)\Big] \in \Big(\mathbb{Z} \times \mathbb{R}\Big)\Big/ \sim$ such that

$\varphi\left(\left[\left(x_1, x_2\right)\right]\right) = \varphi\left(\left[\left(y_1, y_2\right)\right]\right)$. We have :

$$\begin{aligned}
\varphi\left(\left[\left(x_1, x_2\right)\right]\right) = \varphi\left(\left[\left(y_1, y_2\right)\right]\right) \\
&\Rightarrow x_1 = y_1 \\
&\Rightarrow \left(x_1, x_2\right) \sim \left(y_1, y_2\right) \\
&\Rightarrow \left[\left(x_1, x_2\right)\right] = \left[\left(y_1, y_2\right)\right]
\end{aligned}$$

It follows that φ **is one-to-one.**

• Let's show that φ is surjective.
Let $x \in \mathbb{Z}$. We have $\left[\left(x, x\right)\right] \in \mathbb{Z} \times \mathbb{R}$ and $\varphi\left(\left[\left(x, x\right)\right]\right) = x$. So φ is **surjective.**
As a result, φ is a bijection from $\left(\mathbb{Z} \times \mathbb{R}\right) \Big/ \sim$ to $\mathbb{Z}$ and we conclude that $\mathbb{Z} \cong \left(\mathbb{Z} \times \mathbb{R}\right) \Big/ \sim$.

(c) We define an addition operation on $\left(\mathbb{Z} \times \mathbb{R}\right) \Big/ \sim$ as follows.
For all $\left[\left(x_1, x_2\right)\right], \left[\left(y_1, y_2\right)\right] \in \left(\mathbb{Z} \times \mathbb{R}\right) \Big/ \sim$,
$\left[\left(x_1, x_2\right)\right] + \left[\left(y_1, y_2\right)\right] = \left[\left(x_1 + y_1, x_2 + y_2\right)\right]$
Let's prove that this operation is well defined.

Let $\left[\left(x_1, x_2\right)\right], \left[\left(y_1, y_2\right)\right], \left[\left(z_1, z_2\right)\right], \left[\left(t_1, t_2\right)\right] \in \left(\mathbb{Z} \times \mathbb{R}\right) \Big/ \sim$ such that $\left[\left(x_1, x_2\right)\right] = \left[\left(y_1, y_2\right)\right]$ and $\left[\left(z_1, z_2\right)\right] = \left[\left(t_1, t_2\right)\right]$.
Let's show that $\left[\left(x_1, x_2\right)\right] + \left[\left(z_1, z_2\right)\right] = \left[\left(y_1, y_2\right)\right] + \left[\left(t_1, t_2\right)\right]$.
We have $\left[\left(x_1, x_2\right)\right] + \left[\left(z_1, z_2\right)\right] = \left[\left(x_1 + z_1, x_2 + z_2\right)\right]$ and
$\left[\left(y_1, y_2\right)\right] + \left[\left(t_1, t_2\right)\right] = \left[\left(y_1 + t_1, y_2 + t_2\right)\right]$.
So we have to show that $\left[\left(x_1 + z_1, x_2 + z_2\right)\right] = \left[\left(y_1 + t_1, y_2 + t_2\right)\right]$.

We have :

$$\begin{cases}\Big[\Big(x_1,x_2\Big)\Big]=\Big[\Big(y_1,y_2\Big)\Big]\\ \Big[\Big(z_1,z_2\Big)\Big]=\Big[\Big(t_1,t_2\Big)\Big]\end{cases} \Rightarrow \begin{cases}\Big(x_1,x_2\Big)\sim\Big(y_1,y_2\Big)\\ \Big(z_1,z_2\Big)\sim\Big(t_1,t_2\Big)\end{cases}$$

$$\Rightarrow \begin{cases}x_1=y_1\\ z_1=t_1\end{cases}$$

$$\Rightarrow x_1+z_1=y_1+t_1$$

$$\Rightarrow \Big(x_1+z_1,x_2+z_2\Big)\sim\Big(y_1+t_1,y_2+t_2\Big)$$

$$\Rightarrow \Big[\Big(x_1+z_1,x_2+z_2\Big)\Big]=\Big[\Big(y_1+t_1,y_2+t_2\Big)\Big]$$

So that addition operation is well defined on $\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$.

4.

Let $1\leq n\in\mathbb{N}$.
We want to show that $\mathbb{N}\cong\mathbb{N}^n$.

Consider the map :

$$\begin{aligned}&\phi_1:\mathbb{N}\to\mathbb{N}^n\\ &m\mapsto\underbrace{(m,m,\ldots,m)}_{n\ times}.\end{aligned}$$

ϕ_1 is well defined because for all $x,y\in\mathbb{N}$, we have
$x=y\Rightarrow\underbrace{(x,x,\ldots,x)}_{n\ times}=\underbrace{(y,y,\ldots,y)}_{n\ times}\Rightarrow\phi_1(x)=\phi_2(y)$.
Let $m_1,m_2\in\mathbb{N}$ such that $\phi(m_1)=\phi(m_2)$. Then $\underbrace{(m_1,m_1,\ldots,m_1)}_{n\ times}=\underbrace{(m_2,m_2,\ldots,m_2)}_{n\ times}$

Thus $m_1=m_2$ and it follows that ϕ_1 is injective. (1)

Consider the map :

$$\begin{aligned}&\phi_2:\mathbb{N}^n\to\mathbb{N}\\ &(m_1,m_2,\ldots,m_n)\mapsto P_1^{m_1}\times P_2^{m_2}\times\cdots\times P_n^{m_n},\end{aligned}$$

where $P_1,P_2,...$ are consecutive (so distinct) prime numbers biginning by $P_1=2$.

ϕ_2 is well defined because for all $(m_1, m_2, \ldots, m_n), (m'_1, m'_2, \ldots, m'_n) \in \mathbb{N}^n$, we have
$(m_1, m_2, \ldots, m_n) = (m'_1, m'_2, \ldots, m'_n) \Rightarrow P_1^{m_1} \times P_2^{m_2} \times \cdots \times P_n^{m_n} = P_1^{m'_1} \times P_2^{m'_2} \times \cdots \times P_n^{m'_n}$
$\Rightarrow \phi_2((m_1, m_2, \ldots, m_n)) = \phi_2((m'_1, m'_2, \ldots, m'_n))$

We show that ϕ_2 is injective :
Let $(m_1, m_2, \ldots, m_n), (m'_1, m'_2, \ldots, m'_n) \in \mathbb{N}^n$.
Suppose $\phi_2((m_1, m_2, \ldots, m_n)) = \phi_2((m'_1, m'_2, \ldots, m'_n))$.
Then we have

$$\begin{aligned}
\phi_2((m_1, m_2, \ldots, m_n)) &= \phi_2((m'_1, m'_2, \ldots, m'_n)) \\
&\Leftrightarrow P_1^{m_1} \times P_2^{m_2} \times \cdots \times P_n^{m_n} = P_1^{m'_1} \times P_2^{m'_2} \times \cdots \times P_n^{m'_n} \\
&\Rightarrow P_1^{m_1} = P_1^{m'_1}, P_2^{m_2} = P_2^{m'_2}, \ldots, P_n^{m_n} = P_n^{m'_n}, \text{ because} \\
&\text{ the decomposition of a number} \\
&\text{into powers of distinct prime numbers is unique.} \\
&\Rightarrow m_1 = m'_1, m_2 = m'_2, \ldots, m_n = m'_n \\
&\Rightarrow (m_1, m_2, \ldots, m_n) = (m'_1, m'_2, \ldots, m'_n)
\end{aligned}$$

So ϕ_2 is injective. (2)
From (1) and (2) and using the theorem of **Cantor-Bernstein-Schroeder**, **we conclude that** $\mathbb{N} \cong \mathbb{N}^n$.

5.

Let $\mathbb{N}^{<\omega} = \bigcup\limits_{n \in \mathbb{N}} \mathbb{N}^n$
Let's prove that $\mathbb{N} \cong \mathbb{N}^{<\omega}$
Consider the map :

$$\begin{aligned}
&\varphi_1 : \mathbb{N} \to \mathbb{N}^{<\omega} \\
&m \mapsto m
\end{aligned}$$

φ_1 is well defined because for all $m_1, m_2 \in \mathbb{N}$ we have :
$m_1 = m_2 \Rightarrow \varphi_1(m_1) = \varphi_1(m_2)$.
We show that φ_1 is injective :
Let $m_1, m_2 \in \mathbb{N}$. Suppose $\varphi_1(m_1) = \varphi_1(m_2)$. Then we have $m_1 = m_2$ by definition of φ_1. So φ_1 is injective. (3)

Consider the map :

$$\varphi_2 : \mathbb{N}^{<\omega} \to \mathbb{N} \text{ defined by } : \begin{cases} \varphi_2(X) = 0 \text{ if } X = () \text{ the empty sequence} \\ \varphi_2(X) = P_1^{m_1} \times \cdots \times P_k^{m_k} \times P_{k+1}^{k} \text{ if } X = (m_1, \ldots, m_k) \end{cases}$$

where $1 \leq k \in \mathbb{N}$ and

$P_1, P_2, \ldots,$ are consecutive (so distinct) prime numbers beginning by $P_1 = 2$.

φ_2 is well defined because for all $(m_1, \ldots, m_{k_1}), (m'_1, \ldots, m'_{k_2}) \in \mathbb{N}^{<\omega}$ we have :
$(m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2}) \Rightarrow P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2} \Rightarrow \varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$

Of course, each element of $\mathbb{N}^{<\omega}$ is either $()$, the empty sequence or in the form $(m_1, \ldots, m_k)$ with $1 \leq k \in \mathbb{N}$ and $m_i \in \mathbb{N}$ for all $1 \leq i \leq k$.
Now we show that φ_2 is injective.
Let $(m_1, \ldots, m_{k_1}), (m'_1, \ldots, m'_{k_2}) \in \mathbb{N}^{<\omega}$.
Suppose $\varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$. If $k_1 = 0$, that is, $(m_1, \ldots, m_{k_1}) = ()$, then we have $(m'_1, \ldots, m'_{k_2}) = ()$ because by the definition of φ_2, only $()$ satisfies $\varphi(()) = 0$. Thus in this case we have $(m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2})$.

So suppose $k_1 \neq 0, k_2 \neq 0$ and suppose $k_1 \neq k_2$, say $k_1 < k_2$.
We have :

$$\varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$$
$$\Updownarrow$$
$$P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2}$$

But P_{k_2+1} divides $P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2}$ and does not divide $P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1}$. So there is a contradiction. It follows (by symmetry) that $k_1 = k_2$ and we have :

$$P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}}$$
$\Rightarrow m_1 = m'_1, \ldots, m_{k_1} = m'_{k_2},$ because
the decomposition of a number
into powers of distinct prime numbers is unique.
$\Rightarrow (m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2})$

Hence φ_2 is one-to-one (4).

From (3) and (4) and using the theorem of **Cantor-Bernstein-Schroeder**, **we conclude that** $\mathbb{N} \cong \mathbb{N}^{<\omega}$.

6.

For any set X the power of X is defined to be $\mathcal{P}(X) = \left\{ A : A \subseteq X \right\}$.
Let's prove that $X < \mathcal{P}(X)$ for every set X.
Let X be a set.
• If $X = \emptyset$ we have $|X| = 0$ and $|\mathcal{P}(X)| = 1$. So we have $X < \mathcal{P}(X)$.
• Suppose $X \neq \emptyset$.
$*$Consider the map :

$$\begin{aligned} \Phi : X &\to \mathcal{P}(X) \\ x &\mapsto \{x\} \end{aligned}$$

Φ is well defined because for all $x, y \in X$ we have :
$x = y \Rightarrow \{x\} = \{y\} \Rightarrow \Phi(x) = \Phi(y)$.
We show that Φ is injective :
Let $x_1, x_2 \in X$. Suppose $\Phi(x_1) = \Phi(x_2)$.
Then we have $\{x_1\} = \{x_2\} \Rightarrow x_1 = x_2$. So Φ **is one-to-one**.
$*$**Now we have to show that there is no surjective function from X to $\mathcal{P}(X)$.**
Suppose there exists such a function, say f.
Let $A = \left\{ x \in X : x \notin f(x) \right\}$.
Since f is surjective then there exists $y \in X$ such that $f(y) = A$.
But we have : $y \in A \Leftrightarrow y \notin f(y) = A$. There is a contradiction. Therefore, there is no surjective function from X to $\mathcal{P}(X)$.
we conclude that $X < \mathcal{P}(X)$.