Differential Equations Assignment 03

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17.

We consider a mechanical system whose dynamic is modelled by the differential equation :

$$x'' + ax' + 2bx + 3x^2 = 0$$

where a, b are positive constants.
Let's determine the maximal region of asymptotic stability of the zero solution.
The system can be transformed into the following system :

$$\begin{cases} x' = y \\ y' = -ay - 2bx - 3x^2 \end{cases} \Leftrightarrow \begin{cases} x' = y \\ y' = -ay - g(x) \end{cases}$$

where $g(x) = 2bx + 3x^2$ for all $x \in \mathbb{R}$
Let E be the function defined by $E(x, y) := y^2 + 2\displaystyle\int_0^x g(u)du$.
g is locally Lipschitz and we have :

$$\begin{aligned}\frac{d}{dt}E(x,y) &= 2y'y + 2x'g(x) \\ &= 2\left(-ay^2 - yg(x) + yg(x)\right) \\ &= -2ay^2 \text{ because } a > 0 \\ &\leq 0\end{aligned}$$

Furthermore, $E(0, 0) = 0$.
Let's find the maximal open set, say D of $\mathbb{R}^2$ containing $(0, 0)$ and such that $E(x, y) > 0$ and $\frac{d}{dt}E(x, y) < 0$ for all $(x, y) \in D - \{(0, 0)\}$, ie E is a strict Lyapunov function of the system for $(0, 0)$ on D.
We have $\displaystyle\int_0^x g(u)du = \left[bt^2 + t^3\right]_0^x = bx^2 + x^3$
Thus $E(x, y) = y^2 + 2bx^2 + 2x^3 = y^2 + 2x^2(b + x)$.
Now we have with $b > 0$,

$$E(x,y) \leq 0 \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x \leq -b \\ y^2 + 2x^2(b+x) \leq 0 \end{cases}$$

$$\begin{cases} x \leq -b < 0 \\ y^2 + 2x^2(b+x) \leq 0 \end{cases} \Leftrightarrow \begin{cases} x \leq -b < 0 \\ y^2 \leq -2x^2(b+x) \end{cases}$$
$$\Leftrightarrow \begin{cases} x \leq -b < 0 \\ |y| \leq -x\sqrt{-2(b+x)} \end{cases}$$
$$\Leftrightarrow \begin{cases} x \leq -b \\ x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)} \end{cases}$$

Then $E(x,y) \leq 0$ if and only if :
$(x,y) \in \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 | x \leq -b,\ x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}$

Let $\boxed{D = \mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2 | x \leq -b,\ x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}}$

$\{(x,y) \in \mathbb{R}^2 | x \leq -b,\ x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}$ is closed.
So D is an open subset of $\mathbb{R}^2$ containing $(0,0)$ and such that:
$E(x,y) > 0$ and $\frac{d}{dt}E(x,y) < 0$ for all $(x,y) \in D - \{(0,0)\}$.
The open set D is the green region:

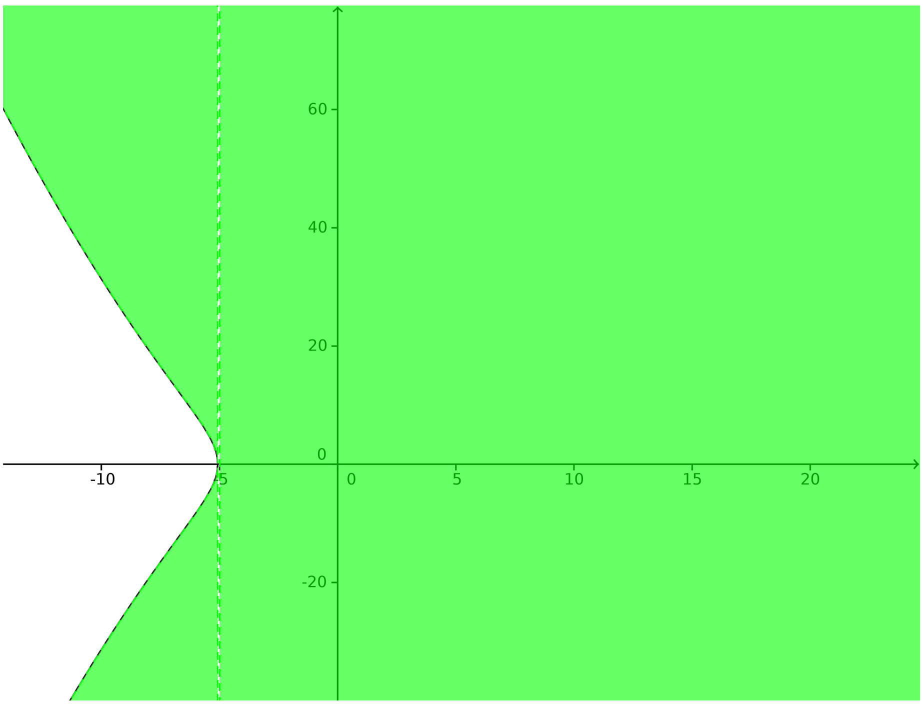

Figure 1: Region D

18

Let's find the limit sets of the orbits in **13.1** and **13.3**
13.1

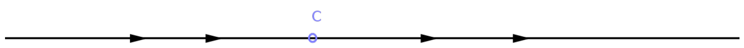

Figure 2:

with C=1
Let $\mathcal{O}_1$ be the orbit before C and $\mathcal{O}_2$ the orbit after C
We have :
$\alpha(\mathcal{O}_1) = \emptyset$, $\omega(\mathcal{O}_1) = \{\mathrm{C}\}$
$\alpha(\mathcal{O}_2) = \{\mathrm{C}\}$, $\omega(\mathcal{O}_2) = \emptyset$

13.3

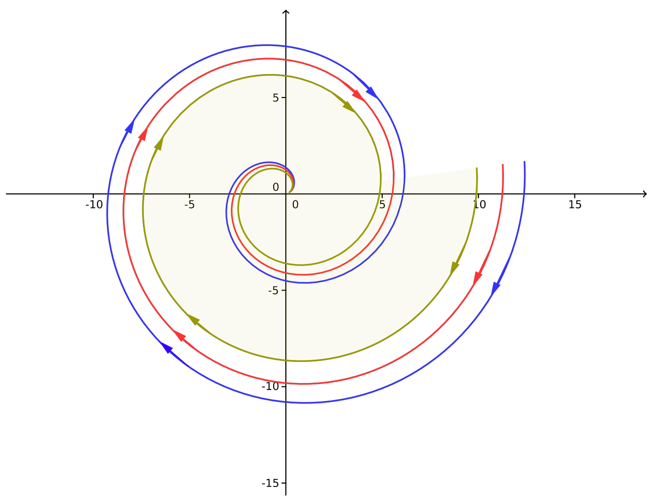

Figure 3: Domain D

For each orbit $\mathcal{O}$ here, we have :
$\alpha(\mathcal{O}) = \emptyset$, $\omega(\mathcal{O}) = \{(0,0)\}$

Sheet 10

19.

We consider the two dimensional system

$$\begin{cases} x' = x(1+3x-x^2-y^2) - y \\ y' = y(1+3x-x^2-y^2) + x \end{cases}$$

Let's prove that it has at least one periodic orbit.
We have :

$$\begin{cases} x' = x(1+3x-x^2-y^2) - y \\ y' = y(1+3x-x^2-y^2) + x \end{cases} \Rightarrow -x'y + xy' = x^2 + y^2$$

This shows that the only equilibrium point is $(0,0)$ (for $x,\ y$ constants we find $x^2+y^2 = 0$, then $x = y = 0$).

Change of variables : $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \Longrightarrow \begin{cases} x' = r'\cos(\theta) - r\theta'\sin(\theta) \\ y' = r'\sin(\theta) + r\theta'\cos(\theta) \\ \end{cases}$

$-x'y + xy' = r^2$ implies :

$$-rr'\cos(\theta)\sin(\theta) + r^2\theta'(\sin(\theta))^2 + rr'\cos(\theta)\sin(\theta) + r^2\theta'(\cos(\theta))^2 = r^2$$

$\Rightarrow r^2\theta' = r^2 \Rightarrow \theta' = 1$ Then using $x' = r'\cos(\theta) - r\theta'\sin(\theta)$, we find

$$r'\cos(\theta) - r\sin(\theta) = r\cos(\theta)\left[1 + 3r\cos(\theta) - r^2\right] - r\sin(\theta)$$

$$\Rightarrow r'\cos(\theta) = r\cos(\theta)\left[1 + 3r\cos(\theta) - r^2\right]$$

$$\Rightarrow (r' - r + r^3)\cos(\theta) = 3r^2(\cos(\theta))^2$$

$$\Rightarrow r' = r - r^3 + 3r^2(\cos(\theta))^2$$

We have : $-1 \leq \cos(\theta) \leq 1 \Rightarrow -3r^2 \leq 3r^2(\cos(\theta))^2 \leq 3r^2$

$$\Rightarrow r - r^3 - 3r^2 \leq r - r^3 + 3r^2(\cos(\theta))^2 \leq r - r^3 + 3r^2$$

We have : $r - r^3 - 3r^2 \geq 0$ if $0 < r \leq \dfrac{-3+\sqrt{13}}{2}$: this means that the radius r increases here
and $r - r^3 - 3r^2 < 0$ if $r > \dfrac{-3+\sqrt{13}}{2}$: this means that the radius r decreases here.

Also, we have : $r - r^3 + 3r^2 \geq 0$ if $0 < r \leq \dfrac{3+\sqrt{13}}{2}$ this means that the radius r decreases here
and $r - r^3 + 3r^2 < 0$ if $r > \dfrac{3+\sqrt{13}}{2}$: this means that the radius r increases here.
Then some orbits starting in the domain between the two circles $\mathcal{C}_1\left((0,0), \dfrac{-3+\sqrt{13}}{2}\right)$ and $\mathcal{C}_2\left((0,0), \dfrac{3+\sqrt{13}}{2}\right)$ stay there forever. But the domain between the two circles is closed and bounded, and doesn't contain the (only) equilibrium point $(0,0)$.
Therefore we can apply the theorem of **Poincarre-Bendixson** and conclude that **there exist at least one periodic orbit**.

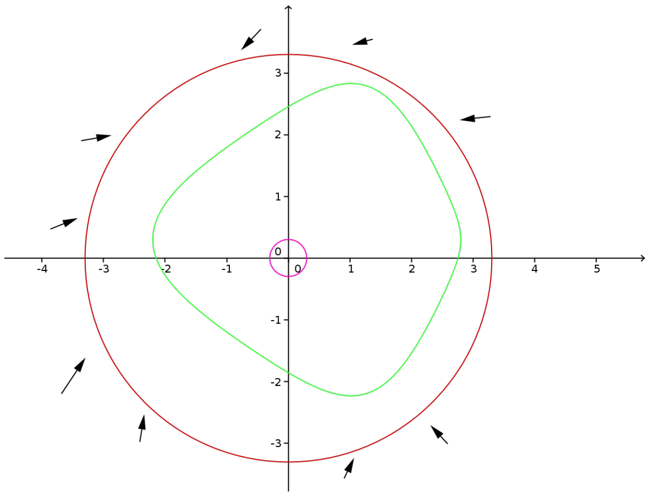

Figure 4: Example of periodic orbit (in green color)

In principle, there could be more than one periodic orbit.

Sheet 11

20.

Consider the two dimensional system of ordinary differential equations

$$\begin{cases} x' = -x & (1) \\ y' = -2y + x & (2) \end{cases}$$

Solve iteratively this system (first the equation in x and then the other) and write down a fundamental solution of it, $\Phi(t)$.
Solution: We know that the solution of equation (1) is $x(t) = c_1 e^{-t}, c_1 \in \mathbb{R}$
Now we use this solution in the equation (2)

$$\begin{aligned} y' &= -2y + x \\ &\Leftrightarrow y' = -2y + c_1 e^{-t},\ c_1 \in \mathbb{R} \\ &\Leftrightarrow y' + 2y = c_1 e^{-t},\ c_1 \in \mathbb{R} \end{aligned}$$

Let μ be a positive function of class C^1

$$\begin{aligned} y' + 2y &= c_1 e^{-t},\ c_1 \in \mathbb{R} \\ &\Leftrightarrow y'\mu + 2y\mu = c_1 \mu e^{-t},\ c_1 \in \mathbb{R}. \end{aligned}$$

We require μ to satisfy $\mu' = 2\mu$.
We can take $\mu(t) = e^{2t}$. Then

$$\begin{aligned} y'\mu + 2y\mu &= c_1 \mu e^{-t},\ c_1 \in \mathbb{R} \\ &\Leftrightarrow (y\mu)' = c_1 \mu e^{-t},\ c_1 \in \mathbb{R} \\ &\Leftrightarrow \int (y\mu)' dt = c_1 \int \mu e^{-t} dt\ ,\ c_1 \in \mathbb{R} \\ &\Leftrightarrow y\mu + c = c_1 \int e^{2t} e^{-t} dt\ ,\ c_1, c \in \mathbb{R} \\ &\Leftrightarrow y\mu = -c + c_1 \int e^{t} dt\ ,\ c_1, c \in \mathbb{R} \\ &\Leftrightarrow ye^{2t} = c_2 + c_1 e^{t},\ c_2 = -c \\ &\Leftrightarrow y(t) = c_1 e^{-t} + c_2 e^{-2t},\ c_1, c_2 \in \mathbb{R} \end{aligned}$$

$\boxed{y(t) = (c_1 e^{t} + c_2 e^{-2t}}, \qquad c_1, c_2 \in \mathbb{R}$

The solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}$
Fundamental matrix: $\Phi(t)$
We have $\begin{pmatrix} c_1 e^{-t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$
Thus $\boxed{\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix}}$

21.

Let X(t) be a fundamental solution of the system $x' = A(t)x$, where $A(.) : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ is continuous. Let's prove that if $b(.) : \mathbb{R} \to \mathbb{R}^n$ is also continuous, the function

$$x(t) = X(t)X(t_0)^{-1}x_0 + \int_{t_0}^{t} X(t)X(s)^{-1}b(s)ds$$

is the unique solution of $x' = A(t)x + b(t)$ with initial condition $x(t_0) = x_0$.

$x' = A(t)x + b(t) \quad (*)$
X(t) is the fundamental solution of $(*)$
$b(.) : \mathbb{R} \to \mathbb{R}^n$ is continuous
We have $x(t) = X(t)C$ where $C \in \mathbb{R}^n$
Then $C = X(t)X(t_0)^{-1}$ and
$x(t) = X(t)X(t_0)^{-1}x(t_0)$.
Every solution of $x' = A(t)x + b(t)$ is the sum of one solution of $x' = A(t)x$ and one solution of $y' = b(t)$.
$\int_{t_0}^{t} X(t)X(s)^{-1}b(s)ds$ is a solution of

$$\begin{cases} y' = b(t) \\ y(t_0) = O \end{cases}$$

because

$$\frac{d}{dt}\int_{t_0}^{t} X(t)X(s)^{-1}b(s)ds = X(t)X(t)^{-1}b(t)$$
$$= b(t)$$

and $x(t) = X(t)X(t_0)^{-1}x_0$ is a solution of

$$\begin{cases} x' = A(t)x \\ x(t_0) = x_0 \end{cases}$$

.
But $A(t,x) := A(t)x$ is locally Lipschitz in x.
Then the solution of

$$\begin{cases} x' = A(t)x \\ x(t_0) = x_0 \end{cases}$$

. is unique. **As a result,** $x(t) = X(t)X(t_0)^{-1}x_0 + \int_{t_0}^{t} X(t)X(s)^{-1}b(s)ds$ **is the unique solution of** $x' = A(t)x + b(t)$ **with the initial condition** $x(t_0) = x_0$

Exercise sheet 12

22. Let's prove that $e^A = \mathrm{diag}(e^{A_1}, \ldots, e^{A_k})$.

$A = \mathrm{diag}(A_1, \ldots, A_k)$, where the A_j are square matrices along the diagonal of A.

We have $e_A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

For all $n \in \mathbb{N}^*$, $A^n = \text{diag}(A_1^n, \ldots, A_k^n)$.

$$\begin{aligned}
\text{Thus,} \quad e^A &= \sum_{n=0}^{\infty} \frac{\text{diag}(A_1^n, \ldots, A_k^n)}{n!} \\
&= \sum_{n=0}^{\infty} \text{diag}\left(\frac{A_1^n}{n!}, \ldots, \frac{A_1^k}{n!}\right) \\
&= \text{diag}\left(\sum_{n=0}^{\infty} \frac{A_1^n}{n!}, \ldots, \sum_{n=0}^{\infty} \frac{A_1^k}{n!}\right) \\
e^A &= \text{diag}(e^{A_1}, \ldots, e^{A_k})
\end{aligned}$$

23.

We consider the following differential equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

a) Let's determine e^{At} where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We have $A = D + N$ where:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Also

$$DN = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \; ND = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $DN = ND$ and then we have $e^{D+N} = e^D e^N$.
As a result, $e^{At} = e^{tD} e^{tN}$.
But we have $N^2 = 0$ and then $e^{tN} = \sum_{k=0}^{\infty} t^k \frac{N^k}{k!} = I + tN$.
Using the exercise 22, we have $e^{Dt} = diag(e^t, e^{-t}, e^{-t}, e^{-t})$.

It follows that

$$\begin{aligned} e^{At} &= (I+tN)\ diag(e^t, e^{-t}, e^{-t}, e^{-t}) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \end{aligned}$$

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

b) We want the solutions to stay bounded when $t \to +\infty$.
The solutions are in the form :

$$x(t) = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & te{-}t & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} + c_3 t e^{-t} \\ c_3 e^{-t} \\ c_4 e^{-t} \end{bmatrix}, \ c_1, c_2, c_3, c_4 \ \in \mathbb{R}.$$

Since the only component which could go to ∞ is the first one ($c_1 e^t$), then we have to take $c_1 = 0$.
Therefore, we have to choose our initial conditions in the real vectorial space generated by : $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

This vectorial space is also defined by : $\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \mathbb{R}^4 \mid c_1 = 0 \right\}$.

Exercise sheet 13

24.

Let's draw the phase portraits.
a) $x' = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x$
Let $M = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.
The eigenvalues of M :

$$det(M-\lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} = (1+\lambda)^2$$

M has one eigenvalue $\lambda = -1$ with the algebraic multiplicity 2.
An eigenvector u of M is given by $(M+I)u = 0$.
Denoting u by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_2 = 0.$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of M.
We have $M = D + N$ with $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
$N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
$DN = ND$ and using the results in Exercise 22, we have:

$$e^{Mt} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

The solution of the system are given by:

$$\begin{aligned} x(t) &= e^{Mt}.C \\ &= \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= (c_1 e^{-t} + c_2 te^{-t}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ c_1, c_2 \ \in \mathbb{R} \end{aligned}$$

We can now draw the phase portrait:

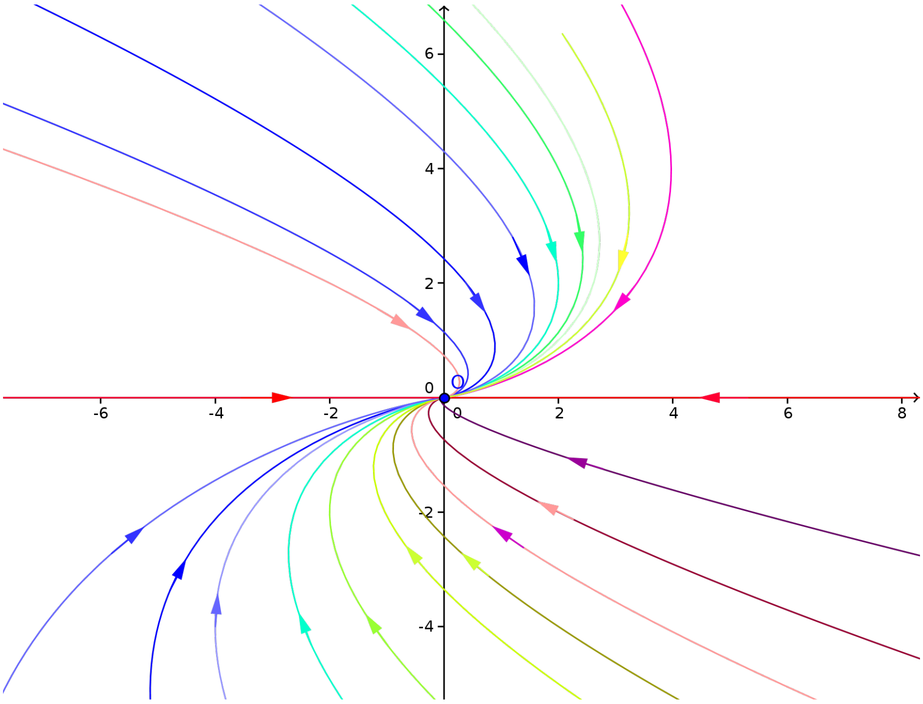

Figure 5:

b) $x' = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} x$
Let $M_1 = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$.
The eigenvalues of M :

$$\begin{aligned} det(M_1 - \lambda I) = 0 &\Leftrightarrow \begin{vmatrix} -2/3 - \lambda & 1/3 \\ 2/3 & -1/3 - \lambda \end{vmatrix} = 0 \\ &\Leftrightarrow \frac{2}{3}\lambda + \frac{1}{3}\lambda + \lambda^2 = 0 \\ &\Leftrightarrow \lambda = 0 \text{ or } \lambda = -1. \end{aligned}$$

M_1 has two different eigenvalues, thus M_1 is diagonalizable.
Eigenvectors:
Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$
For $\lambda = 0$, we have $M_1 u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$\Leftrightarrow -\frac{2}{3}u_1 + \frac{1}{3}u_2 = 0$
$\Leftrightarrow u_2 = 2u_1$
Thus $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of M_1.
For $\lambda = -1$, we have $(M_1 + I)u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \dfrac{1}{3}u_1 + \dfrac{1}{3}u_2 = 0$
$\Leftrightarrow u_2 = -u_1$
Thus, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of M_1.
$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.
The fundamental solution of the differential equation can be written in the form:

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; c_1, c_2 \; \in \mathbb{R}$$

$$\text{Then :} \quad \begin{cases} x_1(t) = c_1 + c_2 e^{-t} \\ x_2(t) = 2c_1 - c_2 e^{-t} \end{cases}$$

We have $x_2(t) = -x_1(t + 3c_1)$: this is the equation of straight lines parallel to the line $y = -x$.
Therefore the phase portrait is:

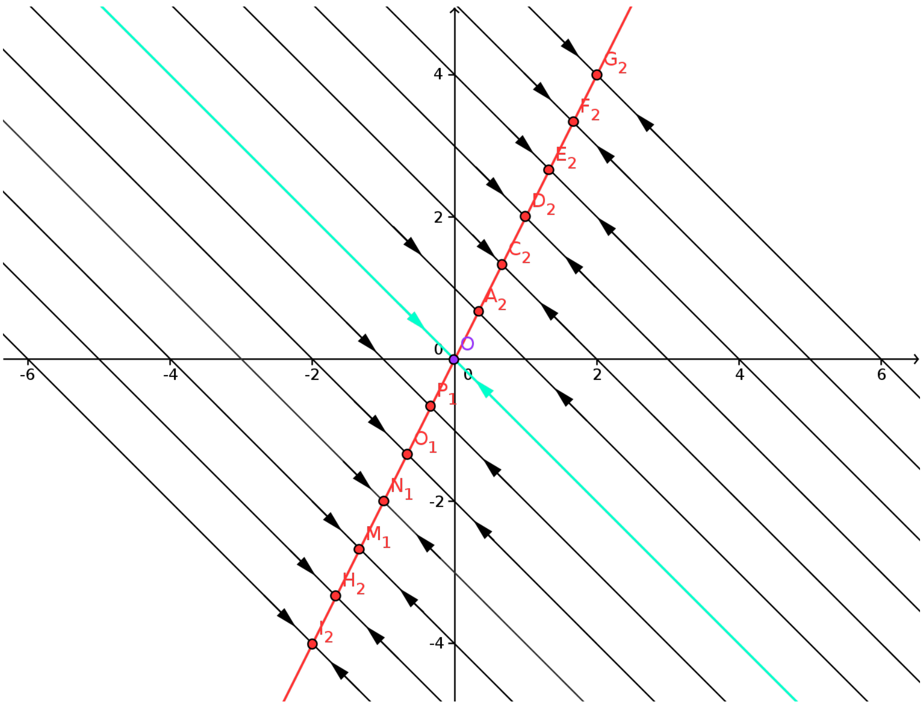

Figure 6:

Exercise sheet 14

25.

Verifying that $h(x,y) = (x, y + \frac{x^3}{4})$ is a differentiable conjugacy between the solutions of the systems

$$\begin{cases} x' = x \\ y' = -y \end{cases} \text{ and } \begin{cases} x' = x \\ y' = -y + x^3 \end{cases}$$

$S_1 \Leftrightarrow X' = F(X)$ with $F(X = (x,y)) = (x, -y)$
$S_2 \Leftrightarrow Y' = G(Y)$ with $G(Y = (x,y)) = (x, -y + x^3)$

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$
$$(x,y) \mapsto (x, y + \frac{x^3}{4})$$

is a differentiable function because $(x,y) \mapsto x$ and $(x,y) \mapsto y + \frac{x^3}{4}$ are differentiable

• Let's show that h is a bijection

• one-to-one

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

Suppose $h(x_1, y_1) = h(x_2, y_2)$

Then we have

$$\begin{cases} x_1 = x_2 \\ y_1 + \frac{x_1^3}{4} = y_2 + \frac{x_2^3}{4} \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \Leftrightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

h is one-to-one

•onto

Let $(u,v) \in \mathbb{R}^2$

We have $(u, v - \frac{u^3}{4}) \in \mathbb{R}^2$ and $h(u, v - \frac{u^3}{4}) = (u, v - \frac{u^3}{4} + \frac{u^3}{4}) = (u,v)$

We conclude that h is a bijection and $h^{-1}(u,v) = (u, v - \frac{u^3}{4}) \quad \forall\ (u,v) \in \mathbb{R}^2$.

• We can see also that h^{-1} is differentiable.

Now we have to show that:

$\frac{\partial h}{\partial X}(X)F(X) = G(h(X))$

We have $\frac{\partial h}{\partial X}(X) = \begin{pmatrix} 1 & 0 \\ \frac{3}{4}x^2 & 1 \end{pmatrix}$

$G(h(X)) = G(x, y + \frac{x^3}{4}) = (x, -y - \frac{x^3}{4} + x^3) = (x, -y + \frac{3x^3}{4})$

$\frac{\partial h}{\partial X}(X)F(X) = \begin{pmatrix} 1 & 0 \\ \frac{3}{4}x^2 & 1 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = (x, \frac{3x^3}{4} - y)$

Thus $\frac{\partial h}{\partial X}(X)F(X) = G(h(X))$

We conclude that h is a differentiable conjugacy between the solutions of the two systems.

26.

We consider the two dimensional Hamiltonian ordinary different equation

$$\begin{cases} q' &= \dfrac{\partial H}{\partial p} \quad (2) \\ p' &= \dfrac{-\partial H}{\partial q} \end{cases}$$

a) Let's prove that the Hamiltonian H is first interval of the Hamiltonian system. H is not constant in any open set of $\mathbb{R}^2$.
We have:

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix} \text{ is a solution } &\Longleftrightarrow \begin{cases} p' &= \dfrac{-\partial H}{\partial q} \\ q' &= \dfrac{\partial H}{\partial p} \end{cases} \\ &\Longleftrightarrow p'\frac{\partial H}{\partial p} = -q'\frac{\partial H}{\partial q} \\ &\Longleftrightarrow p'\frac{\partial H}{\partial p} + q'\frac{\partial H}{\partial q} = 0 \\ &\Longleftrightarrow \frac{\partial H}{dt}(p,q) = 0 \\ &\Longleftrightarrow \textbf{H is constant along the solutions of (2).} \end{aligned}$$

It follows that H is a first integral of the Hamiltonian system.

b)

$H(p,q) = q^3 - q^2 + p^2$

i) Equilibrium points :

We have $\dfrac{\partial H}{\partial p} = 2p$, $\dfrac{\partial H}{\partial q} = 3q^2 - 2q$

We solve : $\begin{cases} 2p = 0 \\ 3q^2 - 2q = 0 \end{cases} \Longleftrightarrow \begin{cases} p = 0 \\ q = 0 \end{cases}$ or $\begin{cases} p = 0 \\ q = \dfrac{2}{3} \end{cases}$

Thus, the equilibrium points are (0,0) and $(0,\frac{2}{3})$.

ii)

Let's the equilibrium point with which we can apply theorem of Hartman-Grobman. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(p,q) = (-\dfrac{\partial H}{\partial q}, \dfrac{\partial H}{\partial p})$ with $(p,q) \in \mathbb{R}$. So, we have $f(p,q) = (-3q^2 + 2q, 2p)$

$J_f(p,q) = \begin{pmatrix} 0 & -6q+2 \\ 2 & 0 \end{pmatrix}$

- $J_f(0,0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ Eigenvalues of $J_f(0,0)$:

$det(J_f(0,0) - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 2 & \lambda \end{vmatrix} = \lambda^2 - 4$ The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$

- $J_f(0,\frac{2}{3}) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

$det(J_f(0, \frac{2}{3}) - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 4$
The eigenvalues are : $\lambda_1 = 2i$, $\lambda_2 = -2i$.
So the eigenvalues of $J_f(0, \frac{2}{3})$ are zero-real part. Whereas the real parts of the eigenvalues of $J_f(0,0)$ are non-zero 0. Therefore we can apply the theorem of Hartman-Grobman for the equilibrium **point (0,0) only**.
iii)
Since the eigenvalues of $J_f(0,0)$ are non-zero real parts, the equilibrium point (0,0) is hyperbolic.
This means that in a neighbourhood of $(0,0)$, orbits should look like hyperbolas.
Using the theorem of Hartman-Grobman and Hadamard-Perre , we can say that the phase portrait of (2) close to (0,0) is almost the same as the one of $y' = \frac{\partial f}{\partial X}(0,0)y$ in a neighbourhood of $(0,0)$.

iv) Let's draw the phase portrait of (2):

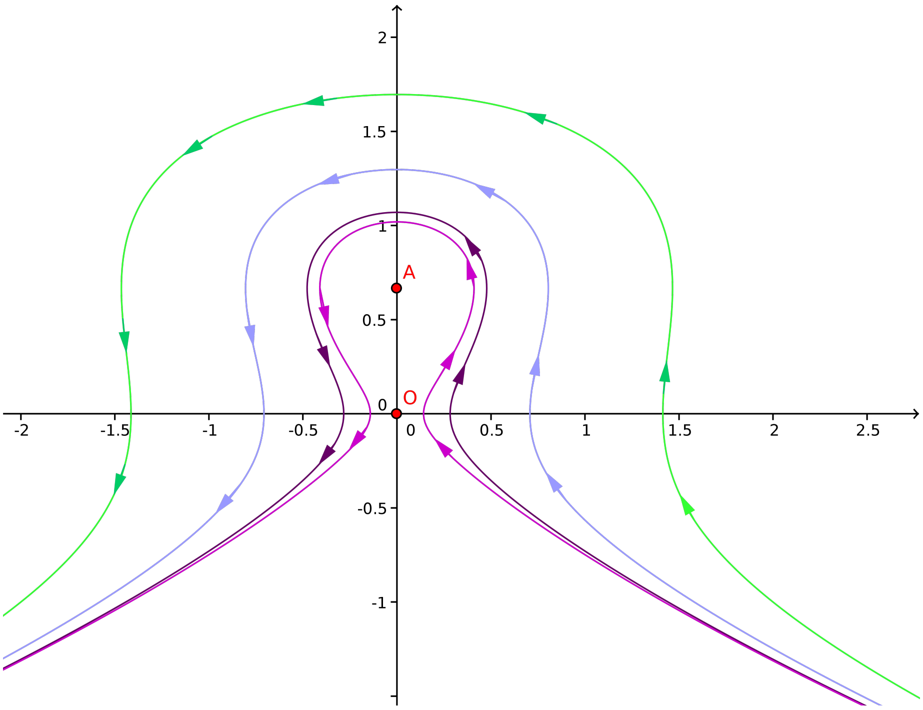

Figure 7: