

Differential Equations Assignment 03

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17.

We consider a mechanical system whose dynamic is modelled by the differential equation :

$$x'' + ax' + 2bx + 3x^2 = 0$$

where a, b are positive constants.

Let's determine the maximal region of asymptotic stability of the zero solution.

The system can be transformed into the following system :

$$\begin{cases} x' = y \\ y' = -ay - 2bx - 3x^2 \end{cases} \Leftrightarrow \begin{cases} x' = y \\ y' = -ay - g(x) \end{cases}$$

where $g(x) = 2bx + 3x^2$ for all $x \in \mathbb{R}$

Let E be the function defined by $E(x, y) := y^2 + 2 \int_0^x g(u) du$.

g is locally Lipschitz and we have :

$$\begin{aligned} \frac{d}{dt} E(x, y) &= 2y'y + 2x'g(x) \\ &= 2(-ay^2 - yg(x) + yg(x)) \\ &= -2ay^2 \text{ because } a > 0 \text{ ok} \\ &\leq 0 \end{aligned}$$

Furthermore, $E(0, 0) = 0$. ok

Let's find the maximal open set, say D of \mathbb{R}^2 containing $(0, 0)$ and such that $E(x, y) > 0$ and $\frac{d}{dt} E(x, y) < 0$ for all $(x, y) \in D - \{(0, 0)\}$, ie E is a strict Lyapunov function of the system for $(0, 0)$ on D . ok

We have $\int_0^x g(u) du = [bt^2 + t^3]_0^x = bx^2 + x^3$

Thus $E(x, y) = y^2 + 2bx^2 + 2x^3 = y^2 + 2x^2(b + x)$.

Now we have with $b > 0$,

$$E(x, y) \leq 0 \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x \leq -b \\ y^2 + 2x^2(b + x) \leq 0 \end{cases}$$

there are two equilibria for the system; (0,0) and (-2b/3,0), check!!

$$\begin{aligned} \begin{cases} x \leq -b < 0 \\ y^2 + 2x^2(b+x) \leq 0 \end{cases} &\Leftrightarrow \begin{cases} x \leq -b < 0 \\ y^2 \leq -2x^2(b+x) \end{cases} \\ &\Leftrightarrow \begin{cases} x \leq -b < 0 \\ |y| \leq -x\sqrt{-2(b+x)} \end{cases} \\ &\Leftrightarrow \begin{cases} x \leq -b \\ x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)} \end{cases} \end{aligned}$$

Then $E(x, y) \leq 0$ if and only if :

$$(x, y) \in \{(0, 0)\} \cup \{(x, y) \in \mathbb{R}^2 | x \leq -b, x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}$$

Let $D = \mathbb{R}^2 - \{(x, y) \in \mathbb{R}^2 | x \leq -b, x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}$

$\{(x, y) \in \mathbb{R}^2 | x \leq -b, x\sqrt{-2(b+x)} \leq y \leq -x\sqrt{-2(b+x)}\}$ is closed.

So D is an open subset of \mathbb{R}^2 containing (0, 0) and such that:

$E(x, y) > 0$ and $\frac{d}{dt}E(x, y) < 0$ for all $(x, y) \in D - \{(0, 0)\}$.

The open set D is the green region:

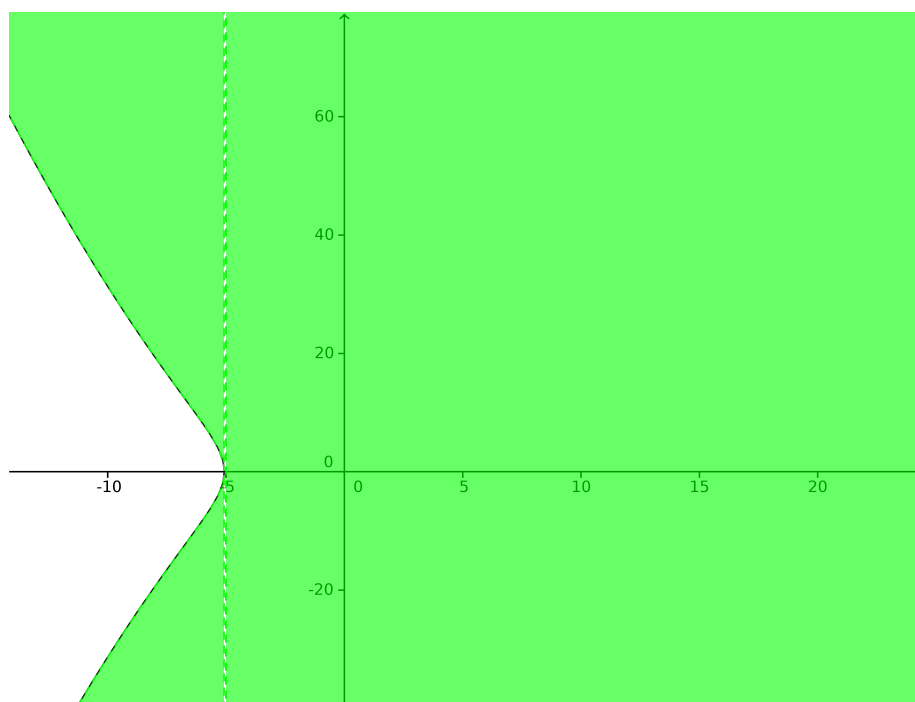


Figure 1: Region D

the plot does not give any information, no labels, etc...

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Let's find the limit sets of the orbits in **13.1** and **13.3**

13.1

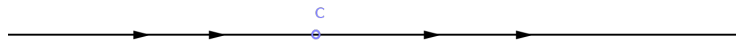


Figure 2:

with $C=1$

Let \mathcal{O}_1 be the orbit before C and \mathcal{O}_2 the orbit after C

We have :

$$\alpha(\mathcal{O}_1) = \emptyset, \omega(\mathcal{O}_1) = \{C\} \quad \text{ok}$$

$$\alpha(\mathcal{O}_2) = \{C\}, \omega(\mathcal{O}_2) = \emptyset \quad \text{ok}$$

what about the orbit at C ?

13.3

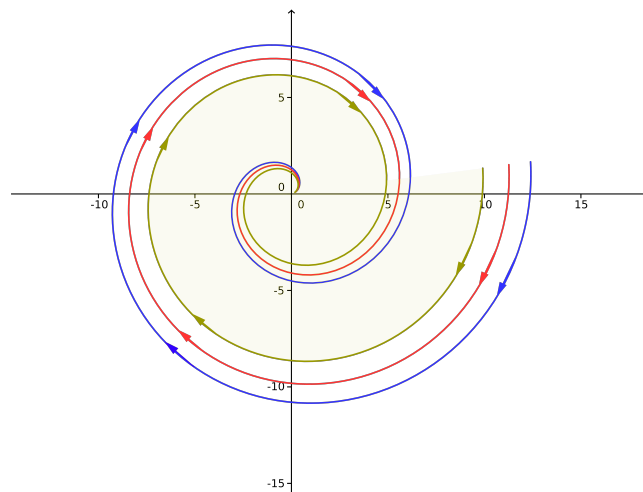


Figure 3: Domain D

all other orbits

For each orbit \mathcal{O} here, we have :

$$\alpha(\mathcal{O}) = \emptyset, \omega(\mathcal{O}) = \{(0, 0)\}$$

for the equilibrium point $O = (0, 0)$: $\alpha(O) = O = \omega(O)$;

Sheet 10

19.

We consider the two dimensional system

$$\begin{cases} x' = x(1 + 3x - x^2 - y^2) - y \\ y' = y(1 + 3x - x^2 - y^2) + x \end{cases}$$

Let's prove that it has at least one periodic orbit.

We have :

$$\begin{cases} x' = x(1 + 3x - x^2 - y^2) - y \\ y' = y(1 + 3x - x^2 - y^2) + x \end{cases} \Rightarrow -x'y + xy' = x^2 + y^2$$

This shows that the only equilibrium point is $(0, 0)$ (for x, y constants we find $x^2 + y^2 = 0$, then $x = y = 0$). **ok**

$$\text{Change of variables : } \begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases} \Rightarrow \begin{cases} x' = r' \cos(\theta) - r\theta' \sin(\theta) \\ y' = r' \sin(\theta) + r\theta' \cos(\theta) \end{cases}$$

$-x'y + xy' = r^2$ implies :

$$-rr' \cos(\theta) \sin(\theta) + r^2 \theta' (\sin(\theta))^2 + rr' \cos(\theta) \sin(\theta) + r^2 \theta' (\cos(\theta))^2 = r^2$$

$\Rightarrow r^2 \theta' = r^2 \Rightarrow \theta' = 1$ Then using $x' = r' \cos(\theta) - r\theta' \sin(\theta)$, we find

$$r' \cos(\theta) - r \sin(\theta) = r \cos(\theta) [1 + 3r \cos(\theta) - r^2] - r \sin(\theta)$$

$$\Rightarrow r' \cos(\theta) = r \cos(\theta) [1 + 3r \cos(\theta) - r^2]$$

$$\Rightarrow (r' - r + r^3) \cos(\theta) = 3r^2 (\cos(\theta))^2$$

$$\Rightarrow r' = r - r^3 + 3r^2 (\cos(\theta))^2 \quad \text{what about } \theta'? \cdot$$

We have : $-1 \leq \cos(\theta) \leq 1 \Rightarrow -3r^2 \leq 3r^2 (\cos(\theta))^2 \leq 3r^2$

$$\Rightarrow r - r^3 - 3r^2 \leq r - r^3 + 3r^2 (\cos(\theta))^2 \leq r - r^3 + 3r^2$$

We have : $r - r^3 - 3r^2 \geq 0$ if $0 < r \leq \frac{-3 + \sqrt{13}}{2}$: this means that the radius r increases here

and $r - r^3 - 3r^2 < 0$ if $r > \frac{-3 + \sqrt{13}}{2}$: this means that the radius r decreases here.

Also, we have : $r - r^3 + 3r^2 \geq 0$ if $0 < r \leq \frac{3 + \sqrt{13}}{2}$ this means that the radius r decreases here

and $r - r^3 + 3r^2 < 0$ if $r > \frac{3 + \sqrt{13}}{2}$: this means that the radius r increases here.

Then some orbits starting in the domain between the two circles

$\mathcal{C}_1 \left((0, 0), \frac{-3 + \sqrt{13}}{2} \right)$ and $\mathcal{C}_2 \left((0, 0), \frac{3 + \sqrt{13}}{2} \right)$ stay there forever. But the domain between the two circles is closed and bounded, and doesn't contain the (only) equilibrium point $(0, 0)$.

Therefore we can apply the theorem of **Poincarre-Bendixson** and conclude that **there exist at least one periodic orbit**.

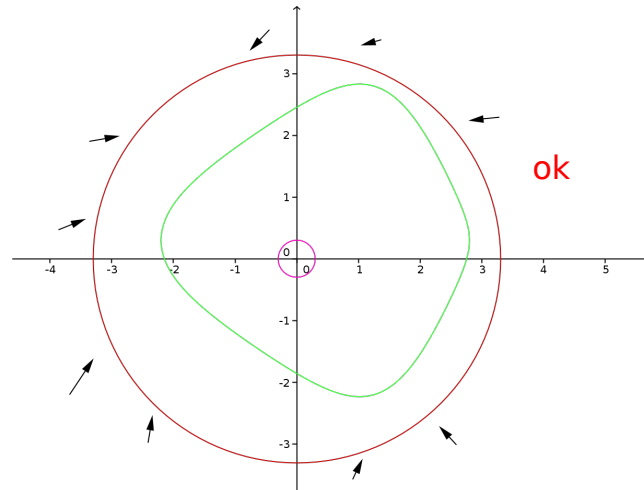


Figure 4: Example of periodic orbit (in green color)

In principle, there could be more than one periodic orbit.

Sheet 11

20.

Consider the two dimensional system of ordinary differential equations

$$\begin{cases} x' = -x & (1) \\ y' = -2y + x & (2) \end{cases}$$

Solve iteratively this system (first the equation in x and then the other) and write down a fundamental solution of it, $\Phi(t)$.

Solution: We know that the solution of equation (1) is $x(t) = c_1 e^{-t}$, $c_1 \in \mathbb{R}$
Now we use this solution in the equation (2)

$$\begin{aligned}
y' &= -2y + x \\
&\Leftrightarrow y' = -2y + c_1 e^{-t}, \quad c_1 \in \mathbb{R} \\
&\Leftrightarrow y' + 2y = c_1 e^{-t}, \quad c_1 \in \mathbb{R}
\end{aligned}$$

Let μ be a positive function of class C^1

$$\begin{aligned}
y' + 2y &= c_1 e^{-t}, \quad c_1 \in \mathbb{R} \\
&\Leftrightarrow y'\mu + 2y\mu = c_1 \mu e^{-t}, \quad c_1 \in \mathbb{R}.
\end{aligned}$$

We require μ to satisfy $\mu' = 2\mu$.

We can take $\mu(t) = e^{2t}$. Then

$$\begin{aligned}
y'\mu + 2y\mu &= c_1 \mu e^{-t}, \quad c_1 \in \mathbb{R} \\
&\Leftrightarrow (y\mu)' = c_1 \mu e^{-t}, \quad c_1 \in \mathbb{R} \\
&\Leftrightarrow \int (y\mu)' dt = c_1 \int \mu e^{-t} dt, \quad c_1 \in \mathbb{R} \\
&\Leftrightarrow y\mu + c = c_1 \int e^{2t} e^{-t} dt, \quad c_1, c \in \mathbb{R} \\
&\Leftrightarrow y\mu = -c + c_1 \int e^t dt, \quad c_1, c \in \mathbb{R} \\
&\Leftrightarrow y e^{2t} = c_2 + c_1 e^t, \quad c_2 = -c \\
&\Leftrightarrow y(t) = c_1 e^{-t} + c_2 e^{-2t}, \quad c_1, c_2 \in \mathbb{R}
\end{aligned}$$

$$\boxed{y(t) = (c_1 e^t + c_2 e^{-2t})}, \quad c_1, c_2 \in \mathbb{R}$$

The solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$

Fundamental matrix: $\Phi(t)$

We have $\begin{pmatrix} c_1 e^{-t} \\ c_1 e^{-t} + c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Thus $\boxed{\Phi(t) = \begin{pmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{pmatrix}}$ **ok**

21.

Let $X(t)$ be a fundamental solution of the system $x' = A(t)x$, where $A(\cdot) : \mathbb{R} \rightarrow \mathcal{M}_n(\mathbb{R})$ is continuous. Let's prove that if $b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ is also continuous, the function

$$x(t) = X(t)X(t_0)^{-1}x_0 + \int_{t_0}^t X(t)X(s)^{-1}b(s)ds$$

is the unique solution of $x' = A(t)x + b(t)$ with initial condition $x(t_0) = x_0$.

try to make logical arguments in a flow of points

$$x' = A(t)x + b(t) \quad (*)$$

$X(t)$ is the fundamental solution of $(*)$

$b(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ is continuous

We have $x(t) = X(t)C$ where $C \in \mathbb{R}^n$

Then $C = X(t)X(t_0)^{-1}$ and

$$x(t) = X(t)X(t_0)^{-1}x(t_0).$$

Every solution of $x' = A(t)x + b(t)$ is the sum of one solution of $x' = A(t)x$ and one solution of $y' = b(t)$.

$\int_{t_0}^t X(t)X(s)^{-1}b(s)ds$ is a solution of

$$\begin{cases} y' = b(t) \\ y(t_0) = 0 \end{cases}$$

because

$$\begin{aligned} \frac{d}{dt} \int_{t_0}^t X(t)X(s)^{-1}b(s)ds &= X(t)X(t)^{-1}b(t) \\ &= b(t) \end{aligned}$$

and $x(t) = X(t)X(t_0)^{-1}x_0$ is a solution of

$$\begin{cases} x' = A(t)x \\ x(t_0) = x_0 \end{cases}$$

.

But $A(t, x) := A(t)x$ is locally Lipschitz in x .

Then the solution of

$$\begin{cases} x' = A(t)x \\ x(t_0) = x_0 \end{cases}$$

. is unique. **As a result,** $x(t) = X(t)X(t_0)^{-1}x_0 + \int_{t_0}^t X(t)X(s)^{-1}b(s)ds$ is the unique solution of $x' = A(t)x + b(t)$ with the initial condition $x(t_0) = x_0$

Exercise sheet 12

22. Let's prove that $e^A = \text{diag}(e^{A_1}, \dots, e^{A_k})$.

$A = \text{diag}(A_1, \dots, A_k)$, where the A_j are square matrices along the diagonal of A .

We have $e_A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$

For all $n \in \mathbb{N}^*$, $A^n = \text{diag}(A_1^n, \dots, A_k^n)$.

$$\begin{aligned} \text{Thus, } e^A &= \sum_{n=0}^{\infty} \frac{\text{diag}(A_1^n, \dots, A_k^n)}{n!} \\ &= \sum_{n=0}^{\infty} \text{diag}\left(\frac{A_1^n}{n!}, \dots, \frac{A_k^n}{n!}\right) \\ &= \text{diag}\left(\sum_{n=0}^{\infty} \frac{A_1^n}{n!}, \dots, \sum_{n=0}^{\infty} \frac{A_k^n}{n!}\right) \quad \text{ok} \\ e^A &= \text{diag}(e^{A_1}, \dots, e^{A_k}) \end{aligned}$$

23.

We consider the following differential equation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

a) Let's determine e^{At} where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

We have $A = D + N$ where:

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Also

$$DN = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad ND = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $DN = ND$ and then we have $e^{D+N} = e^D e^N$.

As a result, $e^{At} = e^{tD} e^{tN}$.

But we have $N^2 = 0$ and then $e^{tN} = \sum_{k=0}^{\infty} t^k \frac{N^k}{k!} = I + tN$.

Using the exercise 22, we have $e^{Dt} = \text{diag}(e^t, e^{-t}, e^{-t}, e^{-t})$.

It follows that

$$e^{At} = (I + tN) \operatorname{diag}(e^t, e^{-t}, e^{-t}, e^{-t})$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

b) We want the solutions to stay bounded when $t \rightarrow +\infty$.

The solutions are in the form :

$$x(t) = \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & te^{-t} & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad \text{ok}$$

$$x(t) = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} + c_3 t e^{-t} \\ c_3 e^{-t} \\ c_4 e^{-t} \end{bmatrix}, \quad c_1, c_2, c_3, c_4 \in \mathbb{R}.$$

Since the only component which could go to ∞ is the first one ($c_1 e^t$), then we have to take $c_1 = 0$. **ok**

Therefore, we have to choose our initial conditions in the real vectorial space

generated by : $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

This vectorial space is also defined by : $\left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \in \mathbb{R}^4 \mid c_1 = 0 \right\}.$

Exercise sheet 13

24.

Let's draw the phase portraits.

a) $x' = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x$

Let $M = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$

The eigenvalues of M :

$$\det(M - \lambda I) = \begin{vmatrix} -1 - \lambda & 1 \\ 0 & -1 - \lambda \end{vmatrix} = (1 + \lambda)^2$$

M has one eigenvalue $\lambda = -1$ with the algebraic multiplicity 2. **good**
 An eigenvector u of M is given by $(M + I)u = 0$.

Denoting u by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we have

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow u_2 = 0.$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of M . **ok**

We have $M = D + N$ with $D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$N^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$DN = ND$ and using the results in Exercise 22, we have:

$$e^{Mt} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

The solution of the system are given by:

$$\begin{aligned} x(t) &= e^{Mt}.C \\ &= \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= (c_1 e^{-t} + c_2 t e^{-t}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R} \quad \text{ok} \end{aligned}$$

We can now draw the phase portrait:

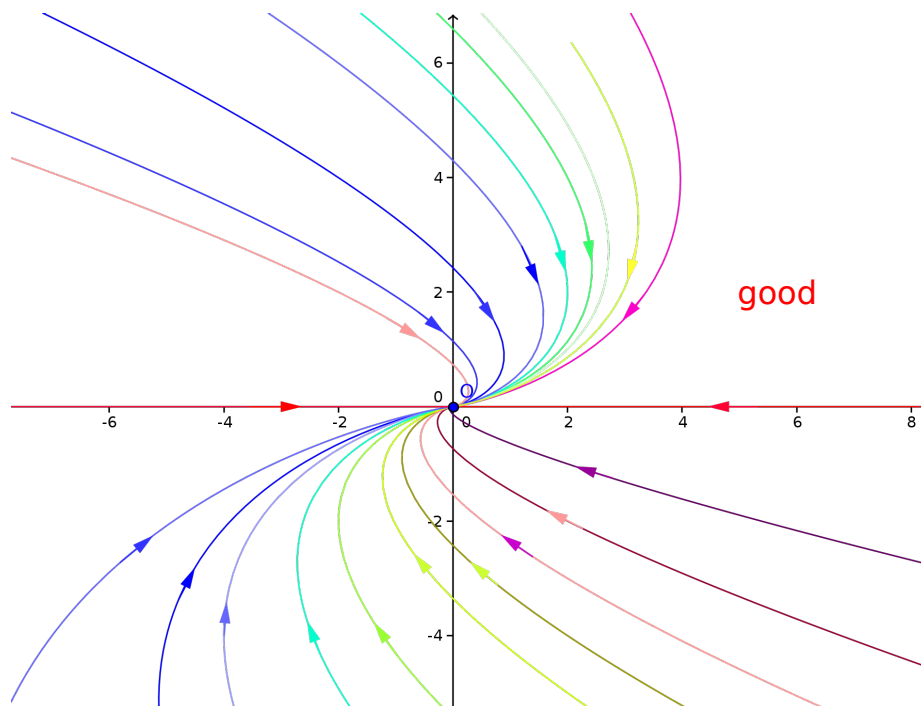


Figure 5: **caption?**

b) $x' = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} x$
 Let $M_1 = \begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$.
 The eigenvalues of M :

$$\begin{aligned} \det(M_1 - \lambda I) = 0 &\Leftrightarrow \begin{vmatrix} -2/3 - \lambda & 1/3 \\ 2/3 & -1/3 - \lambda \end{vmatrix} = 0 \\ &\Leftrightarrow \frac{2}{3}\lambda + \frac{1}{3}\lambda + \lambda^2 = 0 \\ &\Leftrightarrow \lambda = 0 \text{ or } \lambda = -1. \quad \text{ok} \end{aligned}$$

M_1 has two different eigenvalues, thus M_1 is diagonalizable.

Eigenvectors:

Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

For $\lambda = 0$, we have $M_1 u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow -\frac{2}{3}u_1 + \frac{1}{3}u_2 = 0$$

$$\Leftrightarrow u_2 = 2u_1$$

Thus $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of M_1 . **ok**

For $\lambda = -1$, we have $(M_1 + I)u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \frac{1}{3}u_1 + \frac{1}{3}u_2 = 0$$

$$\Leftrightarrow u_2 = -u_1$$

Thus, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of M_1 . **ok**

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent.

The fundamental solution of the differential equation can be written in the form:

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R} \quad \text{good}$$

$$\text{Then : } \begin{cases} x_1(t) = c_1 + c_2 e^{-t} \\ x_2(t) = 2c_1 - c_2 e^{-t} \end{cases}$$

We have $x_2(t) = -x_1(t + 3c_1)$: this is the equation of straight lines parallel to the line $y = -x$.

Therefore the phase portrait is:

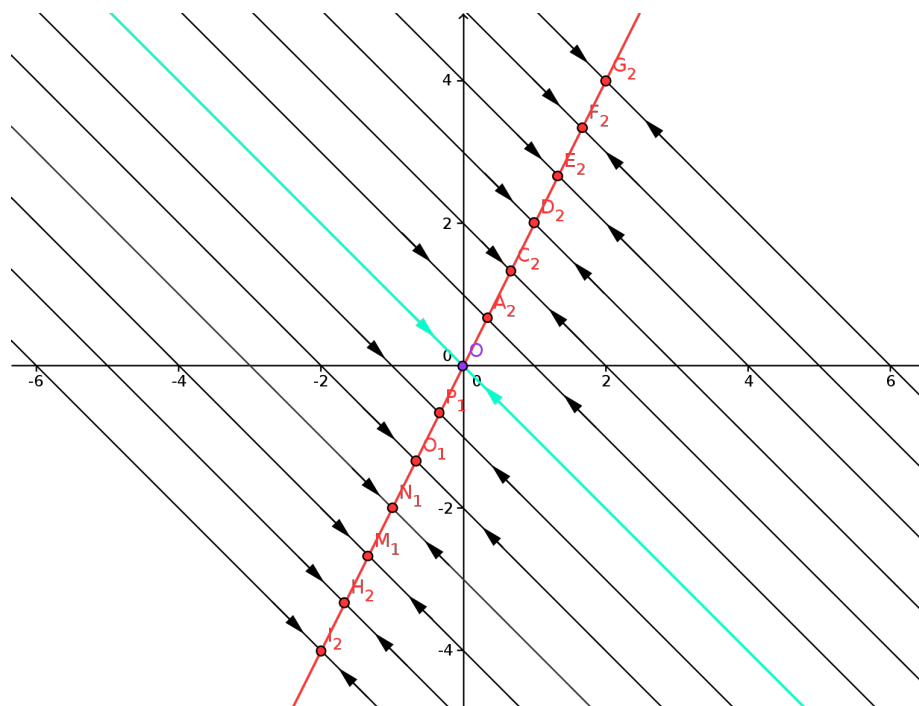


Figure 6: **caption?**

for Q.25, you start first by solving the system of linear and non-linear ODEs, check that $h \circ \phi t = \psi t \circ h$ before checking that h^{-1} exists and is a differentiable function.

Exercise sheet 14

25.

Verifying that $h(x, y) = (x, y + \frac{x^3}{4})$ is a differentiable conjugacy between the solutions of the systems

$$\begin{cases} x' = x \\ y' = -y \end{cases} \quad \text{and} \quad \begin{cases} x' = x \\ y' = -y + x^3 \end{cases}$$

$S_1 \Leftrightarrow X' = F(X)$ with $F(X = (x, y)) = (x, -y)$

$S_2 \Leftrightarrow Y' = G(Y)$ with $G(Y = (x, y)) = (x, -y + x^3)$

$$h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto (x, y + \frac{x^3}{4})$$

is a differentiable function because $(x, y) \mapsto x$ and $(x, y) \mapsto y + \frac{x^3}{4}$ are differentiable

• Let's show that h is a bijection

• one-to-one

Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$

Suppose $h(x_1, y_1) = h(x_2, y_2)$

Then we have

$$\begin{cases} x_1 = x_2 \\ y_1 + \frac{x_1^3}{4} = y_2 + \frac{x_2^3}{4} \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases} \Leftrightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

h is one-to-one

• onto

Let $(u, v) \in \mathbb{R}^2$

We have $(u, v - \frac{u^3}{4}) \in \mathbb{R}^2$ and $h(u, v - \frac{u^3}{4}) = (u, v - \frac{u^3}{4} + \frac{u^3}{4}) = (u, v)$

We conclude that h is a bijection and $h^{-1}(u, v) = (u, v - \frac{u^3}{4}) \quad \forall (u, v) \in \mathbb{R}^2$.

• We can see also that h^{-1} is differentiable. **ok**

Now we have to show that:

$$\frac{\partial h}{\partial X}(X)F(X) = G(h(X))$$

$$\text{We have } \frac{\partial h}{\partial X}(X) = \begin{pmatrix} 1 & 0 \\ \frac{3}{4}x^2 & 1 \end{pmatrix}$$

$$G(h(X)) = G(x, y + \frac{x^3}{4}) = (x, -y - \frac{x^3}{4} + x^3) = (x, -y + \frac{3x^3}{4})$$

$$\frac{\partial h}{\partial X}(X)F(X) = \begin{pmatrix} 1 & 0 \\ \frac{3}{4}x^2 & 1 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = (x, \frac{3x^3}{4} - y)$$

$$\text{Thus } \frac{\partial h}{\partial X}(X)F(X) = G(h(X))$$

We conclude that h is a differentiable conjugacy between the solutions of the two systems.

26.

We consider the two dimensional Hamiltonian ordinary different equation

$$\begin{cases} q' = \frac{\partial H}{\partial p} \\ p' = -\frac{\partial H}{\partial q} \end{cases} \quad (2)$$

a) Let's prove that the Hamiltonian H is first interval of the Hamiltonian system. H is not constant in any open set of \mathbb{R}^2 .

We have:

$$\begin{aligned} \begin{pmatrix} p \\ q \end{pmatrix} \text{ is a solution} &\iff \begin{cases} p' = \frac{-\partial H}{\partial q} \\ q' = \frac{\partial H}{\partial p} \end{cases} \\ &\iff p' \frac{\partial H}{\partial p} = -q' \frac{\partial H}{\partial q} \\ &\iff p' \frac{\partial H}{\partial p} + q' \frac{\partial H}{\partial q} = 0 \\ &\iff \frac{\partial H}{dt}(p, q) = 0 \quad \text{ok} \\ &\iff \mathbf{H \text{ is constant along the solutions of (2).} \end{aligned}$$

It follows that H is a first integral of the Hamiltonian system.

b)

$$H(p, q) = q^3 - q^2 + p^2$$

i) Equilibrium points :

$$\text{We have } \frac{\partial H}{\partial p} = 2p, \frac{\partial H}{\partial q} = 3q^2 - 2q$$

$$\text{We solve : } \begin{cases} 2p = 0 \\ 3q^2 - 2q = 0 \end{cases} \iff \begin{cases} p = 0 \\ q = 0 \end{cases} \quad \text{or} \quad \begin{cases} p = 0 \\ q = \frac{2}{3} \end{cases} \quad \text{ok}$$

Thus, the equilibrium points are $(0,0)$ and $(0, \frac{2}{3})$.

ii)

Let's the equilibrium point with which we can apply theorem of Hartman-Grobman. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(p, q) = (-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p})$ with

$(p, q) \in \mathbb{R}$. So, we have $f(p, q) = (-3q^2 + 2q, 2p)$

$$J_f(p, q) = \begin{pmatrix} 0 & -6q + 2 \\ 2 & 0 \end{pmatrix}$$

• $J_f(0, 0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ Eigenvalues of $J_f(0, 0)$:

$$\det(J_f(0, 0) - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ 2 & \lambda \end{vmatrix} = \lambda^2 - 4 \quad \text{The eigenvalues are } \lambda_1 = -2, \lambda_2 = 2$$

• $J_f(0, \frac{2}{3}) = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

$$\det(J_f(0, \frac{2}{3}) - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 4$$

The eigenvalues are : $\lambda_1 = 2i$, $\lambda_2 = -2i$.

So the eigenvalues of $J_f(0, \frac{2}{3})$ are zero-real part. Whereas the real parts of the eigenvalues of $J_f(0, 0)$ are non-zero 0. Therefore we can apply the theorem of Hartman-Grobman for the equilibrium **point (0,0) only**. **good**

iii)

Since the eigenvalues of $J_f(0, 0)$ are non-zero real parts, the equilibrium point (0,0) is hyperbolic. **ok**

This means that in a neighbourhood of (0,0), orbits should look like hyperbolas.

Using the theorem of Hartman-Grobman and Hadamard-Perre , we can say that the phase portrait of (2) close to (0,0) is almost the same as the one of $y' = \frac{\partial f}{\partial X}(0,0)y$ in a neighbourhood of (0,0).

iv) Let's draw the phase portrait of (2):

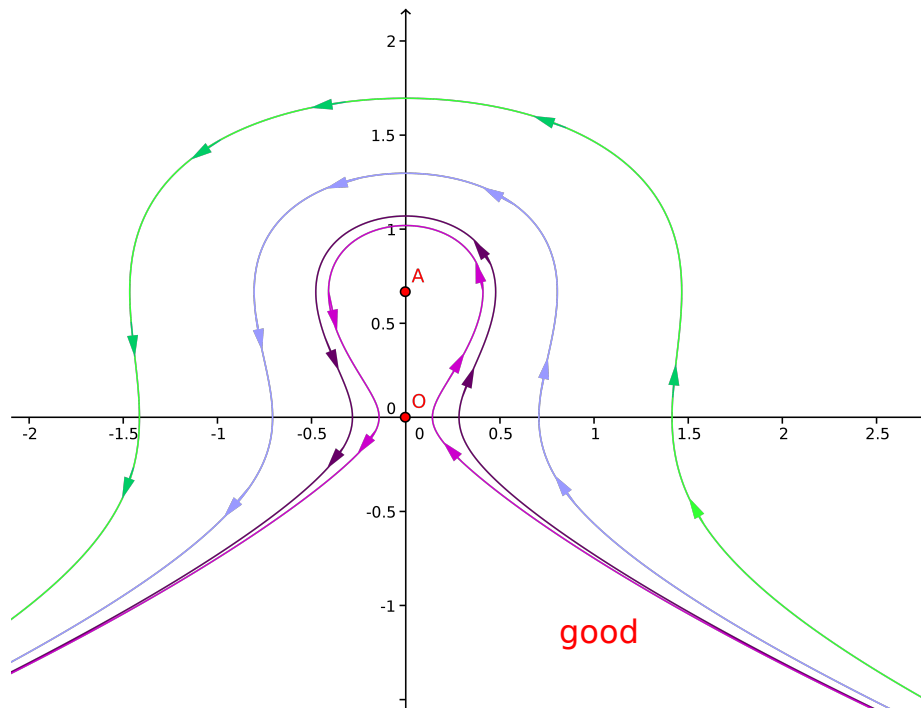


Figure 7: