by Elbasher: well done

Be careful with grammar, punctuation and spelling.

Differential Equations Assignment 2

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Exercise sheet 6

12. Let's prove that φ_t is a flow.
We have :

$$\varphi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} \varphi_0 \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos(0) & \sin(0) \\ -\sin(0) & \cos(0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

Then φ_0 is the identity of $\mathbb{R}^2$.

Moreover,

$$\varphi_{t+s} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(t+s) & \sin(t+s) \\ -\sin(t+s) & \cos(t+s) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos t \cos(s) - \sin t \sin(s) & \sin t \cos(s) + \sin(s) \cos t \\ -\sin t \cos(s) - \sin(s) \cos t & \cos t \cos(s) - \sin t \sin(s) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\varphi_{t+s} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(\cos t \cos(s) - \sin t \sin(s)) + y(\sin t \cos(s) + \sin(s) \cos t) \\ -x(\sin t \cos(s) + \sin(s) \cos t) + y(\cos t \cos(s) - \sin t \sin(s)) \end{pmatrix}$$

$$\varphi_s \begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} \cos s & \sin s \\ \\ -\sin s & \cos s \end{pmatrix}$$

$$\varphi_s \begin{pmatrix} x \\ \\ y \end{pmatrix} = \begin{pmatrix} x\cos s & y\sin s \\ \\ -x\sin s & y\cos s \end{pmatrix}$$

$$\varphi_t \circ \varphi_s = \begin{pmatrix} \cos t & \sin t \\ \\ -\sin s & \cos(s) \end{pmatrix} \begin{pmatrix} x\cos(s) & y\sin s \\ \\ -x\sin s & y\cos(s) \end{pmatrix}$$

$$\varphi_t \circ \varphi_s = \begin{pmatrix} x\cos t\cos(s) + y\sin s\cos t - x\sin t\sin(s) + y\sin t\cos(s) \\ \\ -x\sin t\cos(s) - y\sin t\sin s - x\sin(s)\cos t + y\cos t\cos(s) \end{pmatrix}$$

$$\varphi_t \circ \varphi_s = \begin{pmatrix} x(\cos t\cos(s) - \sin t\sin(s)) + y(\sin t\cos(s) + \sin(s)\cos t) \\ \\ -x(\sin t\cos(s) + \sin(s)\cos t) + y(\cos t\cos(s) - \sin t\sin(s)) \end{pmatrix}$$

We find that $\boxed{\varphi_{t+s} = \varphi_t \circ \varphi_s.}$

Therefore φ_t is a flow.

Exercise sheet 7

Exercise 13.1

- Constant solutions of (E_1).

  x is constant $\iff x' = 0$

  $\iff x = 1\ \forall t$

  The only constant solution of (E_1) is $x(t) = 1\ \forall t \in \mathbb{R}$.

- Let x not be constant and $x(t) \neq 1\ \forall t$.

  Such solutions are strictly increasing. Then we have the following phase portrait:

13.2 $\begin{cases} x_1' &= x_2(x_2^2 - x_1^2) \\ x_2' &= -x_1(x_2^2 - x_1^2) \end{cases}$

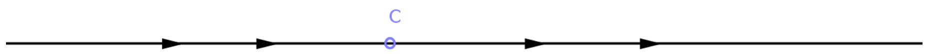


Figure 1:

- Constant solutions:

  $$\begin{cases} x_1 & = 0 \\ x_2 & = 0 \text{ or } x_1^2 = x_2^2 \end{cases}$$

  Or $x_1^2 = x_2^2 \Longleftrightarrow x_2 = x_1$ or $x_2 = -x_1$.

  The constant solutions are (0,0) , (x_1, x_1) , $(x_1, -x_1)$.

  We do the change of variables:

  $$\begin{cases} x_1 = r\cos\theta, r > 0 \\ x_2 = r\sin\theta \end{cases}$$

  We have : $\begin{cases} x_1' = r'\cos\theta - r\theta'\sin\theta \\ x_2 = r'\sin\theta + r\theta'\cos\theta \end{cases}$

  Then , $\begin{cases} r'\cos\theta - r\theta'\sin\theta = r^3(\sin\theta^2 - \cos\theta^2)\sin\theta \ (1) \\ r'\sin\theta + r\theta'\cos\theta = r^3(\sin\theta^2 - \cos\theta^2)\cos\theta \ (2) \end{cases}$

  So, $(1) \times (-\sin)\theta + (2) \times \cos\theta$

  $\Rightarrow r\theta'(\sin\theta)^2 + r\theta'(\cos\theta)^2 = r^3((\cos\theta)^2 - (\sin\theta)^2)$

  $\Rightarrow \theta' = r^2((\cos\theta)^2 - (\sin\theta)^2))$ $(*)$

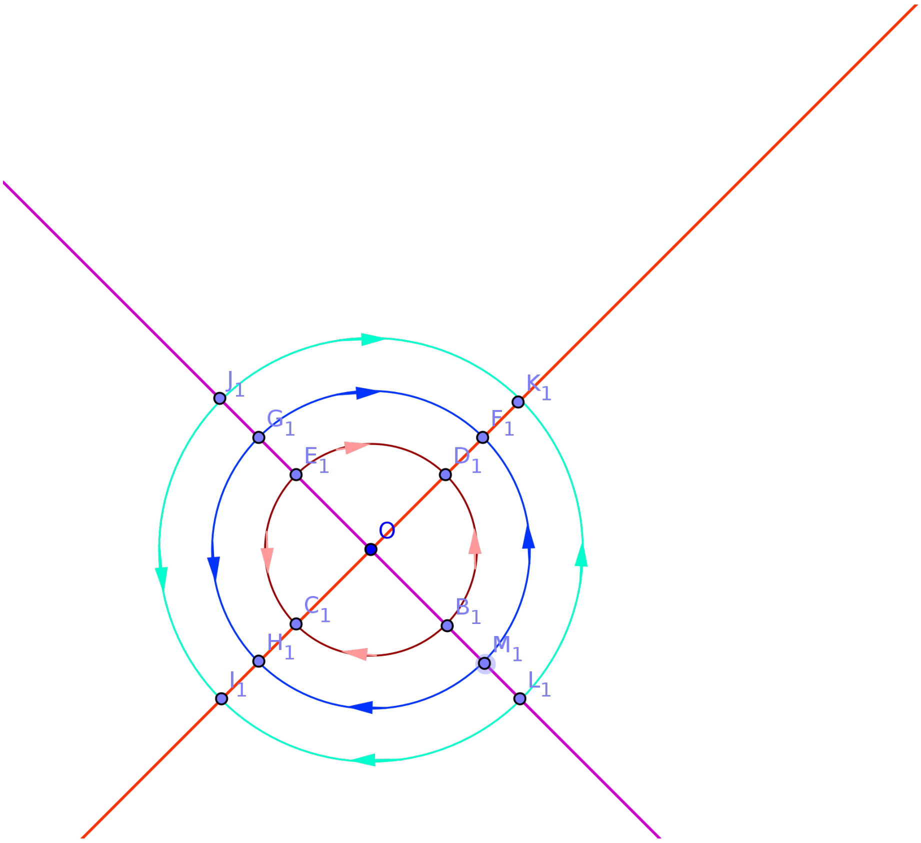
  So, $(1) \times \cos\theta + (2) \times \sin\theta$

$\Rightarrow r'(\cos\theta)^2 + r'(\sin\theta)^2 = 0$

$\Rightarrow r' = 0$

$\Rightarrow$ r is constant

But θ is not constant (see $(*)$). Then we have the phase portrait below:



13.3 $\begin{cases} u' &= \epsilon u - v - u(u^2+v^2) \\ v' &= u + \epsilon v - v(u^2+v^2) \end{cases}$

- Constant solution : (0,0)

  Change of variables:

  $\begin{cases} u &= r\cos\theta,\ r > 0 \\ v &= r\sin\theta \end{cases}$

  We have : $\begin{cases} u' = r'\cos\theta - r\theta'\sin\theta = r\epsilon\cos\theta - r\sin\theta - r^3\cos\theta \ (3) \\ v' = r'\sin\theta + r\theta'\cos\theta = r\cos\theta + r\epsilon\sin\theta - r^3\sin\theta \ (4) \end{cases}$

  $(3)\times\cos\theta + (4)\times\sin\theta \Rightarrow r'(\cos\theta)^2 + r'(\sin\theta)^2 = (r\epsilon - r^3)(\cos\theta)^2 - r\sin\theta\cos\theta + (r\epsilon - r^3)(\sin\theta)^2 + r\,\mathrm{co}$

  $r' = r(\epsilon - r^2),\ \epsilon < 0$

  We have $r > 0$ then $r' < 0$.

So r is decreasing function. Therefore we have the phase portrait below:

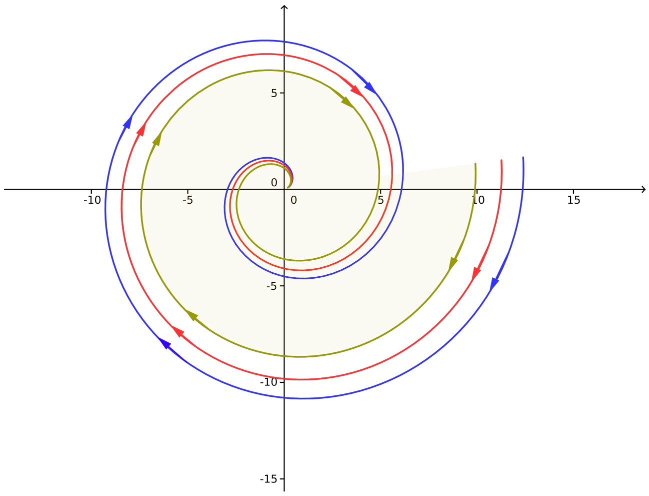


Figure 2:

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14) Let $H : \mathbb{R}^2 \Rightarrow \mathbb{R}$ be a C^2 function not constant in any open subset of $\mathbb{R}^2$. Let's prove that the system of ordinary differential equations

$$\begin{cases} x' = \frac{\partial H}{\partial y} \\ y' = -\frac{\partial H}{\partial x} \end{cases} \tag{1}$$

Proof:
we have :

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{is a solution of} \quad \begin{aligned} &\Leftrightarrow x'\frac{\partial H}{\partial x} = -y'\frac{\partial H}{\partial y} \\ &\Leftrightarrow x'\frac{\partial H}{\partial x} + y'\frac{\partial H}{\partial y} = 0 \\ &\Leftrightarrow \frac{dH}{dt}(x, y) = 0 \\ &\Leftrightarrow \quad \text{H is constant along solution.} \end{aligned}$$

how?

Moreover H is not constant in any open set. **Therefore, the system (1) is conservative and H is a first integral of it.**

15.1) Consider systems of the form

$$\begin{cases} x' = y \\ y' = -g(x) \end{cases} \tag{2}$$

where g is a locally Lipschitz function. Let's show that the function defined by $E(x, y) := y^2 + 2\int_0^x g(u)du$ is a first integral for (1).
Proof:

$E(x,y) := y^2 + 2\int_0^x g(u)du$ and g is locally Lipschitz then g is continuous and we have :

$$\begin{aligned}\frac{d}{dt}E(x,y) &= 2y'y + 2x'g(x)\\ &= 2(y'y + x'g(x))\\ &= 0 \qquad \text{for any } \begin{pmatrix} x\\ y\end{pmatrix} \text{ solution of (2)}\end{aligned}$$

Then E is constant along the solutions of (2). Let's prove that E is not constant in any open set.
We have :

$$\nabla E(x,y) = 2\begin{pmatrix} y\\ g(x)\end{pmatrix}.$$

$$\text{Suppose there exists an open set} \quad \emptyset \neq D \subset \mathbb{R}^2$$

$$\text{such that} \quad \nabla E(x,y) = \begin{pmatrix} 0\\ 0\end{pmatrix}, \forall (x,y) \in D.$$

$$\text{Then} \quad \forall (x,y) \in D, \quad \begin{cases} y = 0\\ g(x) = 0\end{cases}$$

$$\Rightarrow D \subset \mathbb{R} \times \{0\}$$

But none non-empty subset of $\mathbb{R} \times \{0\}$ can be open in $\mathbb{R}^2$ since there is no open ball of $\mathbb{R}^2$ contained in $\mathbb{R} \times \{0\}$. Therefore, E is not constant in any open subset of $\mathbb{R}^2$.

Conclusion : E is a first integral for (2)

15.2) For each of the following cases, let's draw the phase portrait of (1). Based on the obtained phase portrait we will classify all the equilibrium points (constant solutions) of (1) as **stable**, **asymptotically stable** or **unstable**.

a) $g(x) = x - 2x^3$

Answer: We have

$$Lipschitz \begin{cases} x' = y\\ y' = -g(x)\end{cases}$$

Constant solutions :

$$\begin{cases} x = 0\\ y = 0\end{cases}, \quad \begin{cases} 1 - 2x^2 = 0\\ y = 0\end{cases}$$

$$2x^2 - 1 = 0 \Leftrightarrow x = \pm\sqrt{\frac{1}{2}}$$

The constant solutions are :

$$\begin{cases} x = 0 \\ y = 0 \end{cases} , \quad \begin{cases} x = -\sqrt{\frac{1}{2}} \\ y = 0 \end{cases} , \quad \begin{cases} x = \sqrt{\frac{1}{2}} \\ y = 0 \end{cases}$$

• **Phase portrait :**
We study first the variations x and y.
When y is negative, x decreases and increases when y is positive. Using the sign of the function g we can see the variations of y with respect to x.
Then the phase portrait is :

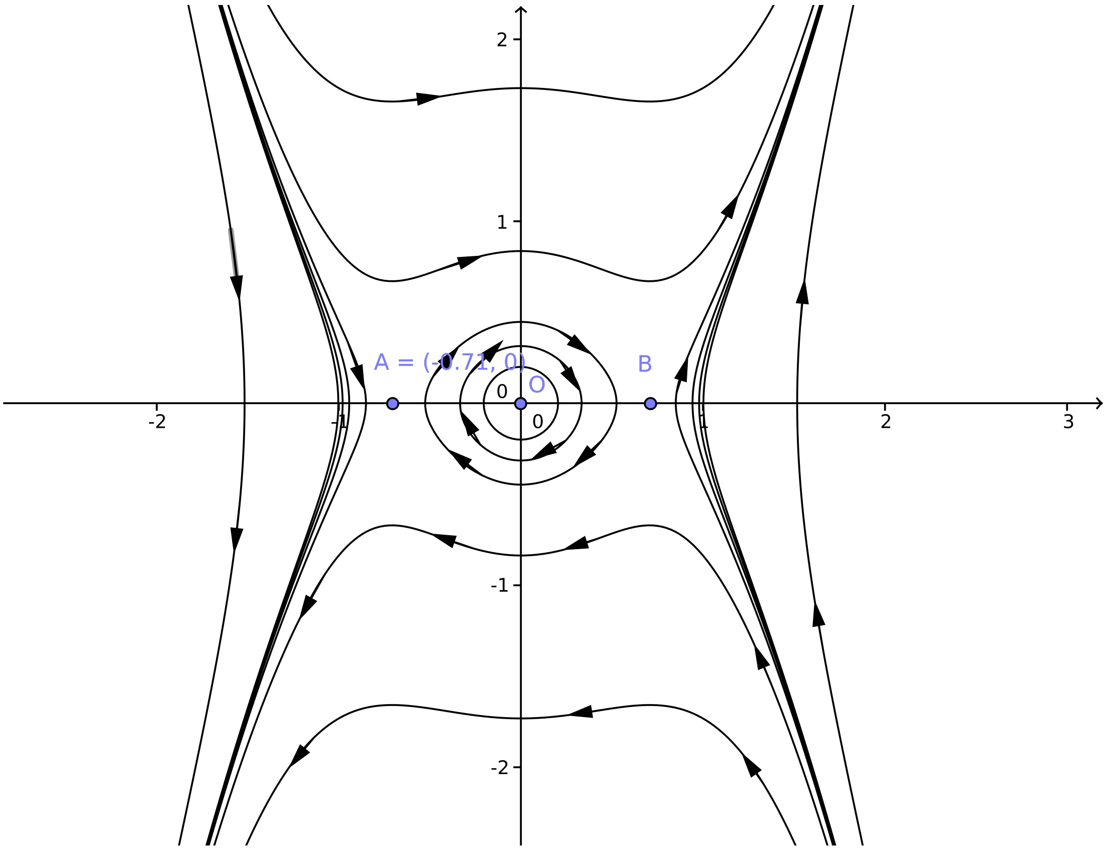

Figure 3:

Using the phase portrait, we can see that the solutions starting in a neighbourhood of $(0,0)$ don't go away after a while.This is not true for the two other equilibrium points.
In conclusion, $(0,0)$ is **stable** whereas $(-\sqrt{\frac{1}{2}},0)$ and $(\sqrt{\frac{1}{2}},0)$ are **unstable**.

b)

$\begin{cases} x' = y \\ y' = -g(x) \end{cases}$ with $g(x) = -2x + 3x^2$. The phase portrait in this case is

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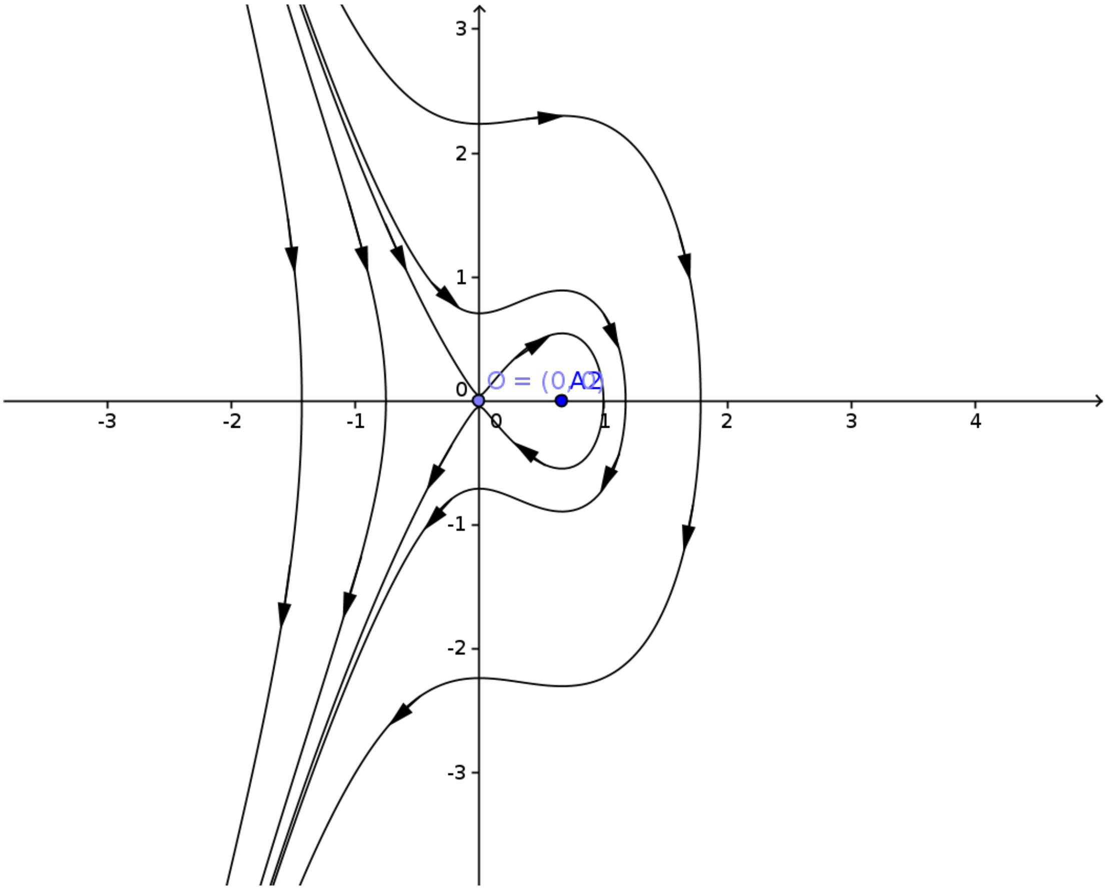

Here all the solution go away from the two equilibrium points after a while. So **the two points are unstable.**

Exercise sheet 8

16.

We discuss the stability of all the equilibrium points of the system below :

$$\begin{cases} x' = y - xf(x,y) \\ y' = -x - yf(x,y) \quad (2) \end{cases}$$

where f is a positive Lipschitz function.

Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be a solution of (2):

We have $\dfrac{\mathrm{d}}{\mathrm{d}t}V(x,y) = 2x'x + 2y'y$

If $\begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of (2), then we have:

$$\begin{aligned} 2x(y - xf(x,y)) + 2y(-x - yf(x,y)) &= -2f(x,y)(x^2+y^2) \\ &= -2V(x,y)f(x,y) \\ &\leq 0 \text{ because } f \text{ and } V \text{ are positive functions.} \end{aligned}$$

- Research of the equilibrium points:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \quad , \qquad \begin{cases} y = xf(x,y) \\ x = -yf(x,y) \\ x \neq 0, y \neq 0 \end{cases} \implies y = -y[f(x,y)]^2$$

please check!

$$\implies 1 = -[f(x,y)]^2 \qquad \textbf{impossible}$$

The only equilibrium point is $(0,0)$.

We have $V(0,0) = 0$ and $V\overset{x,y}{(0,0)} > 0$ if $(x,y) \neq (0,0)$.

$\dfrac{\mathrm{d}V}{\mathrm{d}t}(x,y) \leq 0$ along the solutions $\begin{pmatrix} x \\ y \end{pmatrix}$.

V is strictly decreasing if the orbit is not an equilibrium.

V is a Lyapunov function of the system for $(0,0)$.

Therefore $(0,0)$ is stable.