Algebra Assignment 02

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Section A

In each of the following questions, let's write the answer in the block.

1.
The order of ab is :

$$\boxed{|ab| = 2}$$

2.
Let $G =< a >$ and $|a| = 24$.
Then the list of all the generators for a subgroup of order 8 is :

$$\boxed{a^3, \quad a^9, \quad a^{15}, \quad a^{21}}$$

3.
In the group $\mathbb{Z}_{12} \times \mathbb{Z}_8$, the order of $(4, 2)$ is :

$$\boxed{|(4, 2)| = 12}$$

4.
The elements of $\mathbb{Z}_2 \times \mathbb{Z}_4$ are :

$$\boxed{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)}$$

Let's verify if the group is cyclic.
Since 2 and 4 are not relatively prime, we conclude that $\mathbb{Z}_2 \times \mathbb{Z}_4$ is **not cyclic.**
Section B 1.
In $\mathbb{R} \oplus \mathbb{R}$ under componentwise, addition, let $H = \{(x, 3x)|x \in \mathbb{R}\}$
i) Let's prove that H is a subgroup of $\mathbb{R} \oplus \mathbb{R}$ and describe it geometrically.
First of all, H is a subset of $\mathbb{R} \oplus \mathbb{R}$.
• **We show that H$\neq \phi$**
We have for $x = 0$,

$$\begin{aligned}(0, 3 \times 0) &= (0, 0 + 0 + 0)\\ &= (0, 0) \text{ because } (\mathbb{R}, +) \text{ is a group and } 0 \in \mathbb{R}\\ &\Rightarrow (0, 0) \in H\\ &\Rightarrow H \neq \phi\end{aligned}$$

• Let $a, b \in H$
We need to show that $a \oplus b^{-1} \in H$.
There exists $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $a = (x, 3x),\ b = (y, 3y)$
The inverse of b is $(-y, -3y)$.
Thus we have :

$$\begin{aligned}a \oplus b^{-1} &= (x, 3x) \oplus (-y, -3y)\\ &= (x + (-y), 3x + (-3y))\\ &= (x - y, x + x + x + ((-y) + (-y) + (-y))\\ &= (x - y, (x + (-y)) + (x + (-y)) + (x + (-y))) \text{ using}\\ &\text{the commutativity and the associativity of } (\mathbb{R}, +)\\ &= (x - y, (x - y) + (x - y) + (x - y))\\ &= (x - y, 3(x - y)) \in H \text{ because } x - y \in \mathbb{R}\\ &\Rightarrow a \oplus b^{-1} \in H\end{aligned}$$

Therefore H is a subgroup of $\mathbb{R} \oplus \mathbb{R}$
• **Geometric description**
H is the straight line defined by : $y = 3x$. We can also say it is the graph of the function :

$$\begin{aligned}f\ &: \mathbb{R} \to \mathbb{R}\\ x &\mapsto 3x\end{aligned}$$

ii)
Let's show that $(2, 5) + H$ is a straight line passing through the point $(2, 5)$ and parallel to the line $y = 3x$. We have :

$$\begin{aligned}(2, 5) + H &= \{(2, 5) + (x, 3x)|x \in \mathbb{R}\}\\ &= \{(x + 2, 3x + 5)|x \in \mathbb{R}\}\\ &= \{(t, 3(t - 2) + 5)|t \in \mathbb{R}\} \text{ using the change of variables } t = x - 2\\ &= \{(t, 3t - 1)|t \in \mathbb{R}\} : \text{this is a line which rate of change is } 3, \text{ then parallel to the line } y = 3x.\end{aligned}$$

Moreover, $t=2$ gives the point $(2,5)$.

We deduce that the set $(2,5)+H$ is a straight line passing through the point $(2,5)$ and parallel to the line $y=3x$

2. H and K are subgroups of a group G and $|H|$ and $|K|$ are relatively prime.

Let's show that $H\cap K=\{e\}$ where e is the identity of the group G.

(First method) Let's show first that $H\cap K$ is a common subgroup of H and K.

• We have $e\in H$ and $e\in K$, thus $e\in H\cap K$ and we can write $H\cap K\neq\phi$.
• Now, let $x,y\in H\cap K$. y has an inverse y^{-1} and we have $y^{-1}\in K$ and $y^{-1}\in H$ because H and K are groups. Then, :

$$xy^{-1}\in H \text{ and } xy^{-1}\in K \text{ since } H \text{ and } K \text{ are groups}$$
$$\Longrightarrow xy^{-1}\in H\cap K$$

As a result, $H\cap K$ is both a subgroup of H and a subgroup of K.
It follows that the order of $H\cap K$ divides both $|H|$ and $|K|$.
And using the fact that $|H|$ and $|K|$ are relatively prime we get $|H\cap K|=1$ and **conclude that $H\cap K=\{e\}$**.

(Second method) H and K are subgroups of the group G, then $e\in H$ and $e\in K$, that is, $e\in H\cap K$, implying that$\{e\}\subseteq H\cap K$. Let $x\in H\cap K$. Then, $x\in H$ and $x\in K$. Thus $|x|$ divides both $|H|$ and $|K|$ and using the fact that $|H|$ and $|K|$ are relatively prime, we get $|x|=1\Rightarrow x=e$. As a result, $H\cap K\subseteq\{e\}$ **and we conclude that $H\cap K=\{e\}$**

3. Let p, q be prime numbers and G a group of order pq.

Let's show that any proper subgroup of G is cyclic.

Let H be a proper subgroup of G. Then $H\neq G$.
If $H=\{e\}$, **then $|H|$ is cyclic.**
If $H\neq\{e\}$, provided that the order $|H|$ of H divides the order $|G|$ of G and since $|H|\neq pq$, then we have $|H|=p$ or $|H|=q$.
p and q are prime numbers, **thus H is cyclic**.

Conclusion : Every proper subgroup of G is cyclic.

4.

Let's find all the left cosets of the subgroup $\{\rho_0,\mu_2\}$ of D_4.
The index of $\{\rho_0,\mu_2\}$ in D_4 is $\frac{|D_4|}{|H|}=\frac{8}{2}=4$. Then $\{\rho_0,\mu_2\}$ has 4 cosets.
These cosets are the following :

$$\{\rho_0,\mu_2\},\{\rho_1,\delta_2\},\{\rho_2,\mu_1\},\{\rho_3,\delta_1\}.$$