Model Theory Assignment 01

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1.

Let A and B be sets and let $f : A \to B$. Let's prove each of the following assertions.

(a) For all $C \subseteq A$, $C \subseteq f^{-1}(f(C))$.

Let $C \subseteq A$ and $x \in C$. Then $f(x) \in f(C)$. But we have :

$$f(x) \in f(C) \Rightarrow x \in f^{-1}(f(C))$$

Therefore $C \subseteq f^{-1}(f(C))$ and we coclude that $\boxed{C \subseteq f^{-1}(f(C))}$ for all $C \subseteq A$

(b) For all $D \subseteq B$, $f(f^{-1}(D)) \subseteq D$

Let $D \subseteq B$ and $y \in f(f^{-1}(D))$. Then there exists $x \in f^{-1}(D) : f(x) = y$. But :

$$\begin{aligned} x \in f^{-1}(D) &\Rightarrow f(x) \in D \\ &\Rightarrow y \in D \text{ because } y = f(x) \end{aligned}$$

As a result $f(f^{-1}(D)) \subseteq D$ and we conclude that $\boxed{f(f^{-1}(D)) \subseteq D}$ for all $D \subseteq B$.

(c) The function f is injective if and only if, for all $C \subseteq A$, $C = f^{-1}(f(C))$

•Suppose f is injective and let $C \subseteq A$. We proved in **(a)** that $C \subseteq f^{-1}(f(C))$ even if f is not injective. So we just need to show that $f^{-1}(f(C)) \subseteq C$.
Let $x \in f^{-1}(f(C))$. Then $f(x) \in f(C)$. Since $f(x) \in f(C)$ then there exists $t \in C : f(t) = f(x)$. Thus $x = t$ because f is injective. It follows that $x \in C$ and then $f^{-1}(f(C)) \subseteq C$. As a result, $C = f^{-1}(f(C))$ for all $C \subseteq A$.

• Now suppose for all $C \subseteq A$, $C = f^{-1}(f(C))$. Let's show that f is injective. Let $x, y \in A$ such that $f(x) = f(y)$.

Then $f(\{x\}) = f(\{y\})$ and $f^{-1}(f(\{x\})) = f^{-1}(f(\{y\}))$.
But $f^{-1}(f(\{x\})) = \{x\}$ and $f^{-1}(f(\{y\})) = \{y\}$ by assumption.
So $\{x\} = \{y\}$ and it follows that $x = y$. Hence f is injective.
Therefore, f is injective if and only if, for all $C \subseteq A,\ C = f^{-1}(f(C))$

(d) The function f is surjective if and only if, for all $D \subseteq B,\ f(f^{-1}(D)) = D$

• Suppose f is surjective and let $D \subseteq B$. We have shown in **(b)** that $f(f^{-1})(D) \subseteq D$. Let's prove that $D \subseteq f(f^{-1}(D))$.
Let $y \in D$. Since f is surjective, there exists $x \in A$ such that $f(x) = y$.
We have :

$$\begin{aligned}
\begin{cases} f(x) = y \\ y \in D \end{cases} &\Rightarrow f(x) \in D \\
&\Rightarrow x \in f^{-1}(D) \\
&\Rightarrow f(x) \in f(f^{-1}(D)) \\
&\Rightarrow y \in f(f^{-1}(D)) \text{ because } y = f(x)
\end{aligned}$$

So $D \subseteq f(f^{-1}(D))$ and we have :
if f is surjective then for all $D \subseteq B,\ f(f^{-1}(D)) = D$

• Suppose $D \subseteq B,\ f(f^{-1}(D)) = D$ for all. Let's show that f is surjective.
Let $y \in B$. By assumption we have $f(f^{-1}(\{y\})) = \{y\}$.
It follows that $f^{-1}(\{y\}) \neq \emptyset$, otherwise we would have $f(f^{-1}(\{y\})) = \emptyset$. So $\exists\, x \in A : f(x) = y$. Thus f is surjective.

2.

Let $J = \Big(1, +\infty\Big) \subseteq \mathbb{R}$. For $j \in J$, let $A_j = \Big[1 + \frac{1}{j}, j^2 + 2j + 1\Big) \subseteq \mathbb{R}$.
(a) Let's find $\bigcup_{j \in J} A_j$.

$$\bigcup_{j \in J} A_j = \Big(1, +\infty\Big)$$

Proof :

• $\bigcup_{j \in J} A_j \subseteq \Big(1, +\infty\Big)$

For all $j \in J,\ A_j \subseteq \left(1, +\infty\right)$. So $\bigcup_{j \in J} A_j \subseteq \left(1, +\infty\right)$.

- $\left(1, +\infty\right) \subseteq \bigcup_{j \in J} A_j$

Let $x \in \left(1, +\infty\right)$.

We want to find $j > 1$ such that $x \in A_j$, that is, $1 + \frac{1}{j} \leq x < j^2 + 2j + 1$.

For $j > 1$, since $x > 1$, we have :

$$\begin{aligned} 1 + \frac{1}{j} \leq x &\Leftrightarrow j + 1 \leq jx \\ &\Leftrightarrow j(x-1) \geq 1 \\ &\Leftrightarrow j \geq \frac{1}{x-1} \end{aligned}$$

Also,

$$x < j^2 + 2j + 1 \Leftrightarrow j^2 + 2j + 1 - x > 0$$

$$\Delta = 4 - 4(1 - x) = 4x$$

$$j_1 = \frac{-2 - 2\sqrt{x}}{2} = -1 - \sqrt{x}\ ; \quad j_2 = \frac{-2 + 2\sqrt{x}}{2} = \sqrt{x} - 1$$

$$\text{So, } \begin{cases} j^2 + 2j + 1 - x > 0 \\ j > 1 \end{cases} \quad \Leftrightarrow j > \sqrt{x} - 1$$

Then we can take $j = max\left\{\frac{1}{x-1}, \sqrt{x} - 1\right\} + 1$ and we have $1 + \frac{1}{j} \leq x$ and $x < j^2 + 2j + 1$. So $x \in A_j$.

Therefore, $\left(1, +\infty\right) \subseteq \bigcup_{j \in J} A_j$ **and we conclude that :**

$$\boxed{\bigcup_{j \in J} A_j = \left(1, +\infty\right)}$$

(b) Let's find $\bigcap_{j \in J} A_j$.

$$\bigcap_{j \in J} A_j = \left[2, 4\right]$$

Proof :

- $\left[2,4\right] \subseteq \bigcap_{j\in J} A_j$

For all $j \in J,\ 1+\dfrac{1}{j} < 2 < 4 < j^2+2j+1$. Thus, $\left[2,4\right] \subseteq A_j\ \forall j \in J$.

So $\left[2,4\right] \subseteq \bigcap_{j\in J} A_j$.

- $\bigcap_{j\in J} A_j \subseteq \left[2,4\right]$

Let $x \in \bigcap_{j\in J} A_j$.

Then we have $x > 1$ and for all $j \in J,\ x \in A_j$.

$*$ Suppose by the contrary that $x < 2$. We are going to find $j_0 > 1$ such that $x \notin A_{j_0}$. We can take $1+\dfrac{1}{j_0} = \dfrac{x+2}{2}$, the center of $\left(x,2\right)$, that is, $j_0 = \dfrac{2}{x}$.

We have :

$$\begin{cases} j_0 > 1 \text{ because } 1 < x < 2 \\ \dfrac{1}{j_0}+1 = \dfrac{x+2}{2} = \dfrac{x}{2}+1 = x\left(\dfrac{1}{2}+\dfrac{1}{x}\right) > x \text{ because } \dfrac{1}{x} > \dfrac{1}{2} \text{ since } 1 < x < 2 \end{cases}$$

So $x \notin A_{j_0}$ and we have a contradiction because $x \in \bigcap_{j\in J} A_j$.

$*$ Suppose $x > 4$. Then $\sqrt{x}-1 > 1$ and taking $j_1 = \dfrac{\sqrt{x}}{2}$, we have :

$$\begin{cases} j_1 > 1 \\ j_1^2+2j_1+1 = \dfrac{x}{4}+\sqrt{x}+1 = x\left(\dfrac{1}{4}+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}\right) \end{cases}$$

$$\Rightarrow \begin{cases} j_1 > 1 \\ j_1^2+2j_1+1 = x\left(\dfrac{1}{4}+\dfrac{1}{\sqrt{x}}+\dfrac{1}{x}\right) < x\left(\dfrac{1}{4}+\dfrac{1}{2}+\dfrac{1}{4}\right) = x, \quad \text{because } x > 4 \end{cases}$$

So $x \notin A_{j_1}$, there is a contradiction because $x \in \bigcap_{j\in J} A_j$.

It follows that $\bigcap_{j\in J} A_j \subseteq \left[2,4\right]$ and we conclude that

$$\boxed{\bigcap_{j\in J} A_j = \left[2,4\right]}$$

nice!!

(c) If we made $J = \left[1, +\infty\right)$ only $\bigcap_{j \in J} A_j$ would change and become

$$\bigcap_{j \in J} A_j = \left[2, 4\right)$$

3.

We define $\sim$ on $\mathbb{Z} \times \mathbb{R}$ as follows. For all $\left(x_1, x_2\right), \left(y_1, y_2\right) \in \mathbb{Z} \times \mathbb{R}$,

$$\left(x_1, x_2\right) \sim \left(y_1, y_2\right) \text{ if } x_1 = y_1.$$

(a) Let's prove that $\sim$ is an equivalent relation on $\mathbb{Z} \times \mathbb{R}$.

• **Reflexivity**

Let $\left(x, y\right) \in \mathbb{Z} \times \mathbb{R}$. We have $x = x$. So $\left(x, y\right) \sim \left(x, y\right)$. Hence, $\sim$ is reflexive.

• **Symmetry**

Let $\left(x_1, x_2\right), \left(y_1, y_2\right) \in \mathbb{Z} \times \mathbb{R}$. We have :

$$\begin{aligned} \left(x_1, x_2\right) \sim \left(y_1, y_2\right) &\Rightarrow x_1 = y_1 \\ &\Rightarrow y_1 = x_1 \\ &\Rightarrow \left(y_1, y_2\right) \sim \left(x_1, x_2\right). \end{aligned}$$

So $\sim$ is symmetric.

• **Transitivity**

Let $\left(x_1, x_2\right), \left(y_1, y_2\right), \left(z_1, z_2\right) \in \mathbb{Z} \times \mathbb{R}$

Suppose $\left(x_1, x_2\right) \sim \left(y_1, y_2\right)$ and $\left(y_1, y_2\right) \sim \left(z_1, z_2\right)$.

Then we have :

$$\begin{aligned} x_1 = y_1 \text{ and } y_1 = z_1 & \\ & \Rightarrow x_1 = z_1 \\ & \Rightarrow \Big(x_1, x_2\Big) \sim \Big(z_1, z_2\Big). \end{aligned}$$

Thus $\sim$ is transitive.

Therefore, $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{R}$.

(b) Let's prove that $\mathbb{Z} \cong \Big(\mathbb{Z} \times \mathbb{R}\Big) \Big/ \sim$. We use the notation $\Big[.\Big]$ for the equivalence classes.

Consider the function

$$\begin{aligned} \varphi : \Big(\mathbb{Z} \times \mathbb{R}\Big) \Big/ \sim & \longrightarrow \mathbb{Z} \\ \Big[\Big(x, y\Big)\Big] & \mapsto x \end{aligned}$$

• Let's show that φ is well defined.

Let $\Big[\Big(x_1, x_2\Big)\Big], \Big[\Big(y_1, y_2\Big)\Big] \in \Big(\mathbb{Z} \times \mathbb{R}\Big) \Big/ \sim$

Suppose $\Big[\Big(x_1, x_2\Big)\Big] = \Big[\Big(y_1, y_2\Big)\Big]$

Then we have :

$$\begin{aligned} x_1 = y_1 & \\ & \Rightarrow \varphi\Big(\Big[\Big(x_1, x_2\Big)\Big]\Big) = \varphi\Big(\Big[\Big(y_1, y_2\Big)\Big]\Big) \end{aligned}$$

So φ is well defined.

• Let's show that φ is one-to-one

Let $\Big[\Big(x_1, x_2\Big)\Big], \Big[\Big(y_1, y_2\Big)\Big] \in \Big(\mathbb{Z} \times \mathbb{R}\Big) \Big/ \sim$ such that

$\varphi\Big(\Big[\Big(x_1,x_2\Big)\Big]\Big)=\varphi\Big(\Big[\Big(y_1,y_2\Big)\Big]\Big)$. We have :

$$\begin{aligned}\varphi\Big(\Big[\Big(x_1,x_2\Big)\Big]\Big)=\varphi\Big(\Big[\Big(y_1,y_2\Big)\Big]\Big)\\ &\Rightarrow x_1=y_1\\ &\Rightarrow \Big(x_1,x_2\Big)\sim\Big(y_1,y_2\Big)\\ &\Rightarrow \Big[\Big(x_1,x_2\Big)\Big]=\Big[\Big(y_1,y_2\Big)\Big]\end{aligned}$$

It follows that φ **is one-to-one.**

- Let's show that φ is surjective.

Let $x\in\mathbb{Z}$. We have $\Big[\Big(x,x\Big)\Big]\in\mathbb{Z}\times\mathbb{R}$ and $\varphi\Big(\Big[\Big(x,x\Big)\Big]\Big)=x$. So φ is **surjective.**

As a result, φ **is a bijection from** $\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$ **to** $\mathbb{Z}$ **and we conclude that** $\mathbb{Z}\cong\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$.

(c) We define an addition operation on $\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$ as follows.

For all $\Big[\Big(x_1,x_2\Big)\Big],\Big[\Big(y_1,y_2\Big)\Big]\in\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$,

$\Big[\Big(x_1,x_2\Big)\Big]+\Big[\Big(y_1,y_2\Big)\Big]=\Big[\Big(x_1+y_1,x_2+y_2\Big)\Big]$

Let's prove that this operation is well defined.

Let $\Big[\Big(x_1,x_2\Big)\Big],\Big[\Big(y_1,y_2\Big)\Big],\Big[\Big(z_1,z_2\Big)\Big],\Big[\Big(t_1,t_2\Big)\Big]\in\Big(\mathbb{Z}\times\mathbb{R}\Big)\Big/\sim$ such that $\Big[\Big(x_1,x_2\Big)\Big]=\Big[\Big(y_1,y_2\Big)\Big]$ and $\Big[\Big(z_1,z_2\Big)\Big]=\Big[\Big(t_1,t_2\Big)\Big]$.

Let's show that $\Big[\Big(x_1,x_2\Big)\Big]+\Big[\Big(z_1,z_2\Big)\Big]=\Big[\Big(y_1,y_2\Big)\Big]+\Big[\Big(t_1,t_2\Big)\Big]$.

We have $\Big[\Big(x_1,x_2\Big)\Big]+\Big[\Big(z_1,z_2\Big)\Big]=\Big[\Big(x_1+z_1,x_2+z_2\Big)\Big]$ and $\Big[\Big(y_1,y_2\Big)\Big]+\Big[\Big(t_1,t_2\Big)\Big]=\Big[\Big(y_1+t_1,y_2+t_2\Big)\Big]$.

So we have to show that $\Big[\Big(x_1+z_1,x_2+z_2\Big)\Big]=\Big[\Big(y_1+t_1,y_2+t_2\Big)\Big]$.

We have :

$$\begin{cases} \left[\left(x_1, x_2\right)\right] = \left[\left(y_1, y_2\right)\right] \\ \left[\left(z_1, z_2\right)\right] = \left[\left(t_1, t_2\right)\right] \end{cases} \Rightarrow \begin{cases} \left(x_1, x_2\right) \sim \left(y_1, y_2\right) \\ \left(z_1, z_2\right) \sim \left(t_1, t_2\right) \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = y_1 \\ z_1 = t_1 \end{cases}$$

$$\Rightarrow x_1 + z_1 = y_1 + t_1$$

$$\Rightarrow \left(x_1 + z_1, x_2 + z_2\right) \sim \left(y_1 + t_1, y_2 + t_2\right)$$

$$\Rightarrow \left[\left(x_1 + z_1, x_2 + z_2\right)\right] = \left[\left(y_1 + t_1, y_2 + t_2\right)\right]$$

So that addition operation is well defined on $\left(\mathbb{Z} \times \mathbb{R}\right) / \sim$.

4.

Let $1 \leq n \in \mathbb{N}$.
We want to show that $\mathbb{N} \cong \mathbb{N}^n$.

Consider the map :

$$\begin{aligned} \phi_1 &: \mathbb{N} \to \mathbb{N}^n \\ m &\mapsto \underbrace{(m, m, \ldots, m)}_{n \ times}. \end{aligned}$$

ϕ_1 is well defined because for all $x, y \in \mathbb{N}$, we have
$x = y \Rightarrow \underbrace{(x, x, \ldots, x)}_{n \ times} = \underbrace{(y, y, \ldots, y)}_{n \ times} \Rightarrow \phi_1(x) = \phi_2(y)$.
Let $m_1, m_2 \in \mathbb{N}$ such that $\phi(m_1) = \phi(m_2)$. Then $\underbrace{(m_1, m_1, \ldots, m_1)}_{n \ times} = \underbrace{(m_2, m_2, \ldots, m_2)}_{n \ times}$

Thus $m_1 = m_2$ and it follows that ϕ_1 is injective. (1)

Consider the map :

$$\begin{aligned} &\phi_2 : \mathbb{N}^n \to \mathbb{N} \\ &(m_1, m_2, \ldots, m_n) \mapsto P_1^{m_1} \times P_2^{m_2} \times \cdots \times P_n^{m_n}, \end{aligned}$$

where $P_1, P_2, \ldots$ are consecutive (so distinct) prime numbers biginning by $P_1 = 2$.

ϕ_2 is well defined because for all $(m_1, m_2, \ldots, m_n), (m'_1, m'_2, \ldots, m'_n) \in \mathbb{N}^n$, we have
$(m_1, m_2, \ldots, m_n) = (m'_1, m'_2, \ldots, m'_n) \Rightarrow P_1^{m_1} \times P_2^{m_2} \times \cdots \times P_n^{m_n} = P_1^{m'_1} \times P_2^{m'_2} \times \cdots \times P_n^{m'_n}$
$\Rightarrow \phi_2((m_1, m_2, \ldots, m_n)) = \phi_2((m'_1, m'_2, \ldots, m'_n))$

We show that ϕ_2 is injective :
Let $(m_1, m_2, \ldots, m_n), (m'_1, m'_2, \ldots, m'_n) \in \mathbb{N}^n$.
Suppose $\phi_2((m_1, m_2, \ldots, m_n)) = \phi_2((m'_1, m'_2, \ldots, m'_n))$.
Then we have

$$\begin{aligned}\phi_2((m_1, m_2, \ldots, m_n)) &= \phi_2((m'_1, m'_2, \ldots, m'_n)) \\ &\Leftrightarrow P_1^{m_1} \times P_2^{m_2} \times \cdots \times P_n^{m_n} = P_1^{m'_1} \times P_2^{m'_2} \times \cdots \times P_n^{m'_n} \\ &\Rightarrow P_1^{m_1} = P_1^{m'_1}, P_2^{m_2} = P_2^{m'_2}, \ldots, P_n^{m_n} = P_n^{m'_n}, \text{ because} \\ &\text{the decomposition of a number} \\ &\text{into powers of distinct prime numbers is unique.} \\ &\Rightarrow m_1 = m'_1, m_2 = m'_2, \ldots, m_n = m'_n \\ &\Rightarrow (m_1, m_2, \ldots, m_n) = (m'_1, m'_2, \ldots, m'_n)\end{aligned}$$

So ϕ_2 is injective. (2)
From (1) and (2) and using the theorem of **Cantor-Bernstein-Schroeder**, **we conclude that** $\mathbb{N} \cong \mathbb{N}^n$.

5.

Let $\mathbb{N}^{<\omega} = \bigcup\limits_{n \in \mathbb{N}} \mathbb{N}^n$
Let's prove that $\mathbb{N} \cong \mathbb{N}^{<\omega}$
Consider the map :

$$\begin{aligned}\varphi_1 : \mathbb{N} &\to \mathbb{N}^{<\omega} \\ m &\mapsto m\end{aligned}$$

φ_1 is well defined because for all $m_1, m_2 \in \mathbb{N}$ we have :
$m_1 = m_2 \Rightarrow \varphi_1(m_1) = \varphi_1(m_2)$.
We show that φ_1 is injective :
Let $m_1, m_2 \in \mathbb{N}$. Suppose $\varphi_1(m_1) = \varphi_1(m_2)$. Then we have $m_1 = m_2$ by definition of φ_1. So φ_1 is injective. (3)

Consider the map :

$$\varphi_2 : \mathbb{N}^{<\omega} \to \mathbb{N} \text{ defined by } : \begin{cases} \varphi_2(X) = 0 \text{ if } X = () \text{ the empty sequence} \\ \varphi_2(X) = P_1^{m_1} \times \cdots \times P_k^{m_k} \times P_{k+1}^{k} \text{ if } X = (m_1, \ldots, m_k) \end{cases}$$

where $1 \leq k \in \mathbb{N}$ and

$P_1, P_2, \ldots,$ are consecutive (so distinct) prime numbers beginning by $P_1 = 2$.

φ_2 is well defined because for all $(m_1, \ldots, m_{k_1}), (m'_1, \ldots, m'_{k_2}) \in \mathbb{N}^{<\omega}$ we have : $(m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2}) \Rightarrow P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2} \Rightarrow \varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$

Of course, each element of $\mathbb{N}^{<\omega}$ is either $()$, the empty sequence or in the form $(m_1, \ldots, m_k)$ with $1 \leq k \in \mathbb{N}$ and $m_i \in \mathbb{N}$ for all $1 \leq i \leq k$.
Now we show that φ_2 is injective.
Let $(m_1, \ldots, m_{k_1}), (m'_1, \ldots, m'_{k_2}) \in \mathbb{N}^{<\omega}$.
Suppose $\varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$. If $k_1 = 0$, that is, $(m_1, \ldots, m_{k_1}) = ()$, then we have $(m'_1, \ldots, m'_{k_2}) = ()$ because by the definition of φ_2, only $()$ satisfies $\varphi(()) = 0$. Thus in this case we have $(m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2})$.

So suppose $k_1 \neq 0, k_2 \neq 0$ and suppose $k_1 \neq k_2$, say $k_1 < k_2$.
We have :
$\varphi_2((m_1, \ldots, m_{k_1})) = \varphi_2((m'_1, \ldots, m'_{k_2}))$

$$\Updownarrow$$

$P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2}$
But P_{k_2+1} divides $P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}} \times P_{k_2+1}^{k_2}$ and does not divide $P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} \times P_{k_1+1}^{k_1}$. So there is a contradiction. It follows (by symmetry) that $k_1 = k_2$ and we have :

$$P_1^{m_1} \times \cdots \times P_{k_1}^{m_{k_1}} = P_1^{m'_1} \times \cdots \times P_{k_2}^{m'_{k_2}}$$

$\Rightarrow m_1 = m'_1, \ldots, m_{k_1} = m'_{k_2},$ because the decomposition of a number into powers of distinct prime numbers is unique.

$\Rightarrow (m_1, \ldots, m_{k_1}) = (m'_1, \ldots, m'_{k_2})$

Hence φ_2 is one-to-one (4).

From (3) and (4) and using the theorem of **Cantor-Bernstein-Schroeder**, **we conclude that** $\mathbb{N} \cong \mathbb{N}^{<\omega}$.

6.

For any set X the power of X is defined to be $\mathcal{P}(X) = \left\{ A : A \subseteq X \right\}$.
Let's prove that $X < \mathcal{P}(X)$ for every set X.
Let X be a set.
• If $X = \emptyset$ we have $|X| = 0$ and $|\mathcal{P}(X)| = 1$. So we have $X < \mathcal{P}(X)$.
• Suppose $X \neq \emptyset$.
$*$Consider the map :

$$\begin{aligned} \Phi : X &\to \mathcal{P}(X) \\ x &\mapsto \{x\} \end{aligned}$$

Φ is well defined because for all $x, y \in X$ we have :
$x = y \Rightarrow \{x\} = \{y\} \Rightarrow \Phi(x) = \Phi(y)$.
We show that Φ is injective :
Let $x_1, x_2 \in X$. Suppose $\Phi(x_1) = \Phi(x_2)$.
Then we have $\{x_1\} = \{x_2\} \Rightarrow x_1 = x_2$. So Φ **is one-to-one**.
$*$**Now we have to show that there is no surjective function from X to $\mathcal{P}(X)$.**
Suppose there exists such a function, say f.
Let $A = \left\{ x \in X : x \notin f(x) \right\}$.
Since f is surjective then there exists $y \in X$ such that $f(y) = A$.
But we have : $y \in A \Leftrightarrow y \notin f(y) = A$. There is a contradiction. Therefore, there is no surjective function from X to $\mathcal{P}(X)$.
we conclude that $X < \mathcal{P}(X)$.

nice job!!