Differential Equations Assignment 1

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Exercise sheet 1

1. For the following differential equations, let's identify their order and classify them as *linear* or *non-linear*, and as *autonomous* or *non-autonomous*.

1.1 $x'' + \omega^2 \sin x = 1$

• Order:

The highest derivative is x''. So, the equation is a **second order** differential equation. Very Good!

• Linearity:

Let f_1 be the function defined by:

$f_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mapsto z + \omega^2 \sin(x) - 1$

We have $x'' + \omega^2 \sin(x) = 1 \Leftrightarrow f_1\left(\begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}\right) = 0.$

But f_1 is not linear cause because $f_1\left(\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}\right) = 0,$ Very good work!

$f_1\left(2 \times \begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}\right) = f_1\left(\begin{pmatrix} 2\pi \\ 0 \\ 2 \end{pmatrix}\right) = 1 \neq 2 \times f_1\left(\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}\right).$

Therefore, the differential equation (E_1) **is not linear**.

• Autonomous/Non-autonomous:

The equation doesn't depend explicitly on the variable t (variable of the unknown function x). We conclude that the equation is **autonomous.** Very Good!

1.2 $x'' + \omega^2 x = \cos(t) \quad (E_2)$

• **Order:**

The highest derivative is x''. So, the equation is a **second order** differential equation. Good!

• **Linearity**:

Let f_2 be the function defined by:
$f_2 : \left(t, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) \in \mathbb{R} \times \mathbb{R}^3 \mapsto z + \omega^2 x - \cos(t)$

We have $x'' + \omega^2 x = \cos(t) \Leftrightarrow f_2\left(t, \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}\right) = 0.$

We consider the function g defined by:

$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mapsto z + \omega^2 x$

We have :

$g\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = f_2\left(t, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) - f_2\left(t, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$ and g is linear cause:
because

Very very good!

$\forall \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in \mathbb{R}^3, \forall \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$ and $\forall \alpha \in \mathbb{R}$, we have:

$$\begin{aligned}
g\left(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) &= g\left(\begin{pmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \\ \alpha z_1 + z_2 \end{pmatrix}\right) \\
&= \alpha z_1 + z_2 + \omega^2(\alpha x_1 + x_2) \\
&= \alpha z_1 + \alpha\omega^2 x_1 + z_2 + \omega^2 x_2 \\
&= \alpha(z_1 + \omega^2 x_1) + z_2 + \omega^2 x_2 \\
&= \alpha g\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\right) + g\left(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right)
\end{aligned}$$

• **Autonomous/Non-autonomous:**

very good

The equation depends explicitly on t (cause of the term $\cos(t)$). As a result, the differential equation is **not autonomous.**

• **Solution of the equation:**

The equation is linear, so we can first solve the homogeneous equation:
$x'' + \omega^2 x = 0$

The characteristic equation of this differential equation is : $m^2 + w^2 = 0$ and has two complex solutions : $m_1 = -i\omega,\ m_2 = i\omega$

Then, the general solution of the homogeneous equation is :
$x_h(t) = A\cos(\omega t) + B\sin(\omega t),\ where\ A,\ B\ \in\ \mathbb{R}\ are\ constants.$

Now, let's find a particular solution of the equation (E_2)
The right hand side of (E_2) is $\cos(t)$.

* So, if $\omega^2 \neq 1$ (the particular solution of (E_2) should not look like the general solution of its homogeneous equation), we look for a particular solution of the form:

$x_p(t) = a\cos(t) + b\sin(t), where\ a,\ b\ \in\ \mathbb{R}\ are\ constants.$
$x'_p(t) = -a\sin(t) + b\cos(t);\ x''_p(t) = -a\cos(t) - b\sin(t)$

x_p satisfies (E_2), then we have:

$x''_p(t) + \omega^2 x_p(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow -a\cos(t) - b\sin(t) + a\omega^2\cos(t) + b\omega^2\sin(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow (a\omega^2 - a)\cos(t) + (b\omega^2 - b)\sin(t) = \cos(t)\ \forall\ t$
$a(\omega^2 - 1)\cos(t) + b(\omega^2 - 1)\sin(t) = \cos(t)\ \forall\ t$

$\omega^2 \neq 1$, then we have :

This is correct and okay. However, it was sufficient to only give the order and to show that it is linear or non-linear, autonomous or not autonomous

$$\begin{cases} a(\omega^2 - 1) = 1 \\ b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{\omega^2 - 1} \\ b = 0 \end{cases}$$

The particular solution is : $x_p(t) = \frac{1}{\omega^2-1}\cos(t)$

Therefore the general solution of (E_2) *with* $\omega^2\ \neq 1$ is

$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{\omega^2 - 1}\cos(t)\ \forall\ t\ \in\ \mathbb{R}}$$

where $A,\ B\ \in\ \mathbb{R}$ are constants.

* If $\omega^2 = 1$, then we have to look for a particular solution of the form:
$x_p(t) = at\cos(t) + bt\sin(t)$, where $a,\ b\ \in\ \mathbb{R}$ are constants.

$$\begin{aligned} x'_p(t) &= -at\sin(t) + a\cos(t) + b\sin(t) + bt\cos(t) \\ x''_p(t) &= -a\sin(t) - at\cos(t) - a\sin(t) + b\cos(t) - bt\sin(t) + b\cos(t) \\ &= (-2a - bt)\sin(t) + (2b - at)\cos(t) \end{aligned}$$

x_p is a solution of (E_2) thus :

$x_p''(t) + \omega^2 x_p(t) = \cos(t)]\ \forall\ t$
$\Leftrightarrow (a\omega^2 t + 2b - at)\cos(t) + (b\omega^2 t - 2a - bt)\sin(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow (at + 2b - at)\cos(t) + (bt - 2a - bt)\sin(t) = \cos(t)\ \forall\ t\ cause\ \omega^2 = 1$
$\Leftrightarrow 2b\cos(t) - 2a\sin(t) = \cos(t)\ \forall\ t$

$$\Leftrightarrow \begin{cases} a = 0 \\ b = \frac{1}{2} \end{cases}$$

$\Leftrightarrow x_p(t) = \frac{1}{2} t\sin(t).$

It follows that for $\omega^2 = 1$, the general solution of (E_2) is

$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{2} t\sin(t)\ \forall\ t\ \in\ \mathbb{R}}$, where$A,\ B\ \in\ \mathbb{R}$ are constants.

1.3 $(x^T)' = Ax^T$ (E_3), where A is a $n \times n$ real matrix

• Order

The highest derivative is $(x^T)'$. So, the equation is a **first order** differential equation. Very Good!

• Linearity

We consider the function f_3 defined by:
$f_3 : (x, y)\ \in\ \mathbb{R}^n \times \mathbb{R}^n \mapsto y^T - Ax^T$

We have :
$(x^T)' = Ax^T \Leftrightarrow f_3(x, x') = 0$

Let's show that f_3 is linear :

Let $(u_1, v_1), (u_2, v_2)\ \in\ \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha\ \in\ \mathbb{R}$.

We have

Very Good!

$$\begin{aligned} f_3(\alpha(u_1, v_1) + (u_2, v_2)) &= f_3(\alpha u_1 + u_2, \alpha v_1 + v_2) \\ &= (\alpha v_1 + v_2)^T - A(\alpha u_1 + u_2)^T \\ &= \alpha v_1^T + v_2^T - A[\alpha u_1^T + u_2^T] \\ &= \alpha v_1^T + v_2^T - A\alpha u_1^T - Au_2^T \\ &= \alpha v_1^T - A\alpha u_1^T + v_2^T - Au_2^T \\ &= \alpha(v_1^T - Au_1^T) + v_2^T - Au_2^T \\ &= \alpha f_3((u_1, v_1)) + f_3((u_2, v_2)) \end{aligned}$$

Then, f_3 is linear and we conclude that the equation (E_3) is linear.

• **Autonomous/Non-autonomous**

The equation doesn't depend explicitly on t, then it is an **autonomous** differential equation.

Very Good!

• **Solution of the differential equation :**
The solution of the equation is :

$\boxed{x(t) = x_0 e^{tA} = \sum_{i=0}^{+\infty} \frac{t^i A^i}{i!}}$, $x_0 \in \mathbb{R}^n$ is a constant row vector.

2. Let's transform each of the differential equations into an equivalent system of first order differential equations, and write them in vector form.

2.1 $x'' + \mu(t)x' + w^2 x = \sin(t)$
Transformation into an equivalent system of first order differential equations.

$$\begin{cases} x' = & y \\ y' = & -\mu(t)y - w^2 x + \sin(t) \end{cases}$$

Vector form:

Very Good!

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -w^2 & -\mu(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 x - \mu(t)y + \sin(t) \end{pmatrix}}$$

2.2 $x^{(5)} + x^{(3)} - x' + x = \sin(2\pi t)$
Transformation into an equivalent system of first order differential equations.

$$\begin{cases} x' = & p \\ y' = & -z + p - x + \sin(2\pi t) \\ z' = & y \\ k' = & z \\ p' = & k \end{cases}$$

Very good!

Vector form:

$$\begin{pmatrix} x \\ y \\ z \\ k \\ p \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ k \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(2\pi t) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \\ k' \\ p' \end{pmatrix} = \begin{pmatrix} p \\ -x + p - z + \sin(2\pi t) \\ y \\ z \\ k \end{pmatrix}}$$

Exercise sheet 2

3. Let's compute all solutions of the differential equations satisfying the given initial condition, and explicitly compute their maximal interval of definition I_{max}.

3.1 $x' = x \sin(t)$, satisfying $x(0) = 1$
$x(t) = 0 \; \forall \, t \in \mathbb{R}$ is a solution of the differential equation. But that solution doesn't satisfy $x(0) = 1$. Then we reject it.
Suppose there is another solution x. Because of the fact that we must have $x \in C^1$, there exists an interval I such that $x(t) \neq 0 \; \forall \, t \in I$.
$\forall \, t \in I$,

We integrate the left hand side with respect to t. We rather denote x by x(t) to avoid confusion

$$\begin{aligned}
x'(t) = x(t) \sin(t) &\Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = x(t) \sin(t) \\
&\Leftrightarrow \frac{\mathrm{d}x}{x} = \sin(t)\mathrm{d}t \\
&\Leftrightarrow \int \frac{1}{x}\mathrm{d}x = \int \sin(t)\mathrm{d}t \\
&\Leftrightarrow \ln|x| + a_1 = -\cos(t) + a_2 \\
&\Leftrightarrow \ln|x| = -\cos(t) + a_2 - a_1 \\
&\Leftrightarrow \ln|x| = -\cos(t) + a \quad \text{where} \quad a = a_2 - a_1 \\
&\Leftrightarrow e^{\ln|x|} = e^{-\cos(t)+a} \\
&\Leftrightarrow |x| = e^{-\cos(t)} \times e^{a} \\
x'(t) = x(t) \sin(t) &\Leftrightarrow |x| = ce^{-\cos(t)} \quad \text{where} \quad c = e^{a}.
\end{aligned}$$

So, x is either positive or negative.
However, we can write :

$x(t) = ce^{-\cos(t)}$, $c \in \mathbb{R}$ constant.

Initial condition: $x(0) = 1 = ce^{-\cos(0)} \Rightarrow c = e$

Therefore, the only solution satisfying the initial condition is :

$\boxed{x(t) = e^{1-\cos(t)}}$.

This solution is defined for all real number t and is of class C^∞ on $\mathbb{R}$. **Hence, the greatest interval on which the solution is defined is :** $I_{max} = \mathbb{R}$. This is correct!

3.2 $x' = \frac{1}{tx}$, satisfying $x(1) = 1$

This differential equation has no constant solution because for $x =$ constant, $x'(t) = 0 \neq \dfrac{1}{tx(t)}$.

We have :

$$\begin{aligned} x' = \frac{1}{tx} &\Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{tx} \\ &\Leftrightarrow x\mathrm{d}x = \frac{\mathrm{d}t}{t} \\ &\Leftrightarrow \int x\mathrm{d}x = \int \frac{1}{t}\mathrm{d}t \\ &\Leftrightarrow \frac{x^2}{2} + a_1 = \ln|t| + a_2 \\ &\Leftrightarrow x^2 = 2\ln|t| + 2(a_2 - a_1) \\ &\Leftrightarrow x^2 = 2\ln|t| + a \quad \text{where } a = 2(a_2 - a_1) \\ &\Leftrightarrow x(t) = \sqrt{2\ln|t| + a}, \ a \ \in \ \mathbb{R} \text{ constant.} \end{aligned}$$

Again, we integrate both sides with respect to t.

Initial condition: $x(1) = \sqrt{2\ln|1| + a} = 1 \Rightarrow 2\ln|1| + a = 1$

$$\Rightarrow a = 1.$$

$x(t) = \sqrt{2\ln|t| + 1}$.

$x(t)$ is defined if

$$2\ln|t| + 1 \geq 0$$
$$\ln|t| \geq -\frac{1}{2}$$
$$|t| \geq e^{-\frac{1}{2}}$$
$$t \in \left]-\infty, e^{-\frac{1}{2}}\right[\cup \left]e^{-\frac{1}{2}}, +\infty\right[$$

Good!

The initial condition value is contained in $\left]e^{-\frac{1}{2}}, +\infty\right[$, then:

$$\boxed{x(t) = \sqrt{2\ln|t| + 1}, \quad \forall\ t \in I_{max} = \left]e^{-\frac{1}{2}}, +\infty\right[}$$

3.3 $x' = x^{\frac{2}{3}}$, satisfying $x(0) = 0$

The null function $x(t) = 0\ \forall\ t\ \in\ \mathbb{R}$ is a solution of that differential equation and it satisfies $x(0) = 0$. Its maximal interval of definition is $\mathbb{R}$

Let's find the other solutions if they exist.
Let x be one another solution of the differential equation.

The solution must be of class C^1. So there exists an interval I such that $x(t) \neq 0 \; \forall \; t \; \in \; I$.

Then, for all $t \; \in \; I$:

$$\begin{aligned}
x' = x^{\frac{2}{3}} &\Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = x^{\frac{2}{3}} \\
&\Leftrightarrow \int x^{-\frac{2}{3}} \mathrm{d}x = \int \mathrm{d}t \\
&\Leftrightarrow 3x^{\frac{1}{3}} + a_1 = t + a_2 \\
&\Leftrightarrow 3x^{\frac{1}{3}} = t + a_2 - a_1 \\
&\Leftrightarrow 3x^{\frac{1}{3}} = t + a \quad \text{where} \quad a = a_2 - a_1 \\
&\Leftrightarrow x^{\frac{1}{3}} = \frac{1}{3}t + \frac{1}{3}a \\
&\Leftrightarrow x^{\frac{1}{3}} = \frac{1}{3}t + c \quad \text{where} \quad c = \frac{1}{3}a \\
&\Leftrightarrow \sqrt[3]{x} = \frac{1}{3}t + c \\
&\Leftrightarrow x(t) = \left(\frac{1}{3}t + c\right)^3, \; c \; \in \mathbb{R} \text{ constant.}
\end{aligned}$$

Initial condition: $x(0) = \left(\frac{1}{3}(0) + c\right)^3 = 0 \Rightarrow c = 0.$

$$\boxed{x(t) = \frac{1}{27}t^3, \quad \forall \; t \in I_{max} = \mathbb{R}.}$$

Very Good!

4. Let's obtain the solution of the Cauchy problem and determine its interval of definition I_{max}.

$$\begin{cases} y' &= 5y + e^{-2t}y^{-2} \\ y(0) &= 2. \end{cases}$$

Let's consider the change of variables $y \mapsto x := y^3$.

$$\begin{aligned}
x = y^3 &\Leftrightarrow x' = 3y^2y' \\
&\Leftrightarrow y' = \frac{x'}{3y^2}
\end{aligned}$$

Let's replace the value of y' in the main differential equation.

$$\begin{aligned}
\frac{x'}{3y^2} = 5y + e^{-2t}y^{-2} &\Leftrightarrow x' = 15y^3 + 3e^{-2t} \\
&\Leftrightarrow x' = 15x + 3e^{-2t} \quad \text{where} \quad x = y^3
\end{aligned}$$

Let $\mu : t \mapsto \mu(t)$ be a C^1 function.

We have $\mu(t)x' - 15\mu(t)x = 3\mu(t)e^{-2t}$. We are looking for a function μ such that $\mu'(t) = -15\mu(t)$.

The general solution of the differential equation $\mu' = -15\mu$ is $t \mapsto ke^{-15t}$ where $k \in \mathbb{R}$ is a constant.

We can take $\mu(t) = e^{-15t}$.

Then :

$$\begin{aligned}
\mu(t)x'(t) + \mu'(t)x(t) = 3e^{-2t}\mu(t) &\Leftrightarrow (\mu(t)x(t))' = 3e^{-2t}e^{15t} = 3e^{-2t}\mu(t) \\
&\Leftrightarrow \mu(t)x(t) + c_1 = \int 3e^{-2t}\mu(t)dt, \ c_1 \in \mathbb{R} \ constant \\
&\Leftrightarrow e^{-15t}x(t) + c_1 = \frac{-3}{17}e^{-17t} + c_2, c_1, \ c_2 \in \mathbb{R} \ constants \\
&\Leftrightarrow x(t) = \left(\frac{-3}{17}e^{-17t} + c\right)e^{15t}, \ c = c_2 - c_1 \in \mathbb{R} \ constant
\end{aligned}$$

Good work!

We have $y(0) = 2$, then $x(0) = 8$ and it follows that :

$\frac{-3}{17} + c = 8 \Leftrightarrow c = \frac{139}{17}$

Finally, $x(t) = \left(\frac{-3}{17}e^{-17t} + \frac{139}{17}\right)e^{15t}$ and

$$\boxed{y(t) = \left(\frac{-3}{17}e^{-17t} + \frac{139}{17}\right)^{1/3} e^{5t}}$$

The solution must be defined on an interval I such that $y(t) \neq 0$ for all $t \in I$.

We have :

$$\begin{aligned}
y(t) = 0 &\Leftrightarrow \frac{-3}{17}e^{-17t} + \frac{139}{17} = 0 \\
&\Leftrightarrow e^{-17t} = \frac{139}{3} \\
&\Leftrightarrow -17t = \ln\left(\frac{139}{3}\right) \\
&\Leftrightarrow t = \frac{-1}{17}\ln\left(\frac{139}{3}\right)
\end{aligned}$$

Then the maximal interval of definition of the solution is

$$\boxed{I_{max} = \left]\frac{-1}{17}\ln\left(\frac{139}{3}\right), +\infty\right[}$$

Good!

5. Let's show that if all solutions of $y' = ay$ are bounded when $t \to +\infty$, then the same is true for all solutions of $z' = (a + b(t))z$ if $\int_0^{+\infty} |b(t)\mathrm{d}t| < +\infty$.

Let a be a constant real number, $b(t)$ a continuous function defined in $\mathbb{R}_0^+$ and consider the two differential equations :

$$y' = ay \tag{1}$$
$$z' = (a + b(t))z \tag{2}$$

Suppose all the solutions of (1) are bounded when $t \to +\infty$ and $\int_0^{+\infty} |b(t)|dt < +\infty$.

We want to show that all the solutions of (2) are bounded when $t \to +\infty$. The differential equation (2) is linear and can be written as :

$z' = az + b(t)z$

Consider the differential equation :

$$u' = b(t)u$$

Then, a solution of (2) can be written as the sum of one solution of (1) and one solution of (3).

The general solution of (3) is :

$u(t) = ce^{\int_{t_0}^{t} b(s)ds}$, $c \in \mathbb{R}$ constant, $t_0 \in \mathbb{R}_0^+$ cause $b(t)$ is defined in $\mathbb{R}_0^+$. Then, the general solution of (2) can be written as :

$z(t) = y(t) + ce^{\int_{t_0}^{t} b(s)ds}$, where $c \in \mathbb{R}$ constant and $y(t)$is a solution of (1)

All the solutions of (1) are bounded when $t \to +\infty$. This implies that there exists $l_1 \in \mathbb{R}$ such that $\lim_{t \to +\infty} y(t) = l_1$.

We also have :

Very Good!

$\int_{t_0}^{+\infty} b(t)dt \leq |\int_{t_0}^{+\infty} b(t)dt| \leq \int_{t_0}^{+\infty} |b(t)|dt \leq \int_0^{+\infty} |b(t)|dt < +\infty$
$\Rightarrow ce^{\int_{t_0}^{+\infty} b(s)ds} < +\infty$

It follows that :

$$\boxed{\lim_{t \to +\infty} z(t) = l_1 + ce^{\int_{t_0}^{+\infty} b(s)ds} < +\infty}$$

Exercise sheet 3

6. For the Cauchy problems below, let's construct the sequence of iterates defined by $x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds$ and verify if it is convergent then, check that the limit is a solution of Cauchy problem.

6.1* $\begin{cases} x' = tx \\ x(0) = 1 \end{cases}$

Let's consider the function $x_1 : t \mapsto 1$

x_1 is not a solution of the problem but at least we have : $x_1(0) = 1$.
We begin constructing the sequence $(x_n(t))_n$:

- $$\begin{aligned} x_2(t) &= x_0 + \int_0^t f(s, x_1(s))ds \\ &= 1 + \int_0^t sds \\ x_2(t) &= 1 + \frac{1}{2}t^2 \end{aligned}$$

- $$\begin{aligned} x_3(t) &= x_0 + \int_0^t f(s, x_2(s))ds \\ &= 1 + \int_0^t s(1 + \frac{1}{2}s^2)ds \\ &= 1 + \int_0^t (s + \frac{1}{2}s^3)ds \\ &= 1 + \int_0^t sds + \int_0^t \frac{1}{2}s^3 ds \\ &= x_2(t) + \int_0^t \frac{1}{2}s^3 ds \\ &= 1 + \frac{1}{2}t^2 + \left[\frac{1}{8}s^4\right]_0^t \\ x_3(t) &= 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 \end{aligned}$$

Very Good!

- $$\begin{aligned} x_4(t) &= x_0 + \int_0^t f(s, x_3(s))ds \\ &= 1 + \int_0^t s(1 + \frac{1}{2}s^2 + \frac{1}{8}s^4)ds \\ &= 1 + \int_0^t s(1 + \frac{1}{2}s^2)ds + \int_0^t \frac{1}{8}s^5 ds \\ x_4(t) &= x_3(t) + \int_0^t \frac{1}{8}s^5 ds \end{aligned}$$

We can see that $\forall\ n \in \mathbb{N}^*,\ x_{n+1}(t) = x_n(t) + \int_0^t sR_n(s)ds$ where $R_n(t)$ is the last term of $x_n(t)$ (that is the one with the highest degree on t).

Let's find $R_n(t)$ for all n.

Good Work!

We have :

$$\begin{aligned}
R_1(t) &= 1 \\
R_2(t) &= \frac{1}{2}t^2 \\
R_3(t) &= \frac{1}{8}t^4 \\
&= \frac{1}{2!}\left(\frac{1}{2}t^2\right)^2 \\
R_4(t) &= \frac{1}{48}t^6 \\
&= \frac{1}{3!}\left(\frac{1}{2}t^2\right)^3
\end{aligned}$$

Then $R_n(t) = \dfrac{1}{(n-1)!}\left(\dfrac{1}{2}t^2\right)^{n-1}$ and we can integrate $\int_0^t sR_n(s)\mathrm{d}s$:

$$\begin{aligned}
\int_0^t sR_n(s)\mathrm{d}s &= \int_0^t \frac{s}{(n-1)!}\left(\frac{1}{2}s^2\right)^{n-1}\mathrm{d}s \\
&= \left[\frac{1}{n(n-1)!\left(\dfrac{s^2}{2}\right)^n}\right]_0^t \\
&= \left[\frac{1}{n!}\left(\frac{s^2}{2}\right)^n\right]_0^t \\
&= \frac{1}{n!}\left(\frac{t^2}{2}\right)^n
\end{aligned}$$

It follows that $x_{n+1}(t) = x_n(t) + S_n(t)$, where $S_n(t) = \frac{1}{n!}\left(\frac{t^2}{2}\right)^n$
Moreover,

$$\begin{aligned}
x_1(t) &= 1\\
x_2(t) &= x_1(t) + S_1(t)\\
&= 1 + S_1(t)
\end{aligned}$$

Veery Good!

$$\begin{aligned}
x_3(t) &= x_2(t) + S_2(t)\\
&= 1 + S_1(t) + S_2(t)\\
\\
x_4(t) &= x_3(t) + S_3(t)\\
&= 1 + S_1(t) + S_2(t) + S_3(t)\\
&\vdots\\
x_n(t) &= 1 + \sum_{k=1}^{n-1} S_k(t)\\
&= 1 + \sum_{k=1}^{n-1} \frac{1}{k!}\left(\frac{t^2}{2}\right)^k
\end{aligned}$$

As a result, for all $t \in \mathbb{R}$, the sequence $(x_n(t))_n$ converges to :

$$\begin{aligned}
1 + \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\frac{t^2}{2}\right)^k &= \sum_{k=0}^{+\infty} \frac{1}{k!}\left(\frac{t^2}{2}\right)^k\\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

The sequence converges to the function $x : t \in \mathbb{R} \mapsto e^{\frac{t^2}{2}}$ which is the solution of the problem because :

$$\begin{cases} x'(t) = te^{\frac{t^2}{2}} = tx(t)\\ x(0) = 1 \end{cases}$$

6.2* $\begin{cases} x' = x + 1\\ x(0) = 1 \end{cases}$

Here we consider again the function $x_1 : t \mapsto 1$

x_1 is not a solution of the problem but at least we have : $x_1(0) = 1$.

Using the previous method ($x_{n+1}(t) = x_n(t) + \int_0^t R_n(s)ds$), we begin constructing the sequence easily :

- $$\begin{aligned} x_2(t) &= x_0 + \int_0^t f(s, x_1(s))ds \\ &= 1 + \int_0^t 2ds \\ x_2(t) &= 1 + 2t \end{aligned}$$

- $$\begin{aligned} x_3(t) &= 1 + 2t + \int_0^t 2sds \\ &= 1 + 2t + t^2 \end{aligned}$$

- $$\begin{aligned} x_4(t) &= 1 + 2t + t^2 + \int_0^t s^2 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 \end{aligned}$$

- $$\begin{aligned} x_5(t) &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \int_0^t \frac{1}{3}s^3 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 \\ &= -1 + 2 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 \\ &= -1 + 2\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4\right) \\ &= -1 + 2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right) \end{aligned}$$

  Very Good!

- $$\begin{aligned} x_6(t) &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \int_0^t \frac{1}{12}s^4 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 \\ &- 1 + 2 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 \\ &= -1 + 2\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right) \\ &= -1 + 2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5\right) \end{aligned}$$

Then for all $n \in \mathbb{N}^*$,

$$x_n(t) = -1 + 2\left(\sum_{k=0}^{n-1} \frac{t^k}{k!}\right)$$

For all $t \in \mathbb{R}$,
the sequence $(x_n(t))_n$ converges to :

$$\begin{aligned} & -1 + 2\sum_{k=0}^{+\infty} \frac{t^k}{k!} \\ & = -1 + 2e^t \end{aligned}$$

Good Work!

The sequence converges to the function $x : t \in \mathbb{R} \mapsto -1 + 2e^t$ which is the solution of the problem because :

$$\begin{cases} x'(t) = 2e^t = 1 + x(t) \\ x(0) = 1 \end{cases}$$

7. Metric

Let's show that d defined in the book by Barreira & Valls is a distance in $X = C(I)$, the set of all bounded continuous functions in $I \subset \mathbb{R}^k$.

For all $x, y \in X$, we have : $d(x,y) = Sup_{t \in I}||x(t) - y(t)||$.
Then d is a positive function.

• *Symmetry*
Let $x, y \in X$. We have :

$$\forall\ t \in I,\ ||x(t) - y(t)|| = ||y(t) - x(t)|| \text{ because } ||.|| \text{ is a norm in } \mathbb{R}^n.$$

Hence $\sup_{t \in I}||x(t) - y(t)|| = \sup_{t \in I}||y(t) - x(t)|| \Leftrightarrow d(x,y) = d(y,x)$

• *Separation*

Let $x, y \in X$, we have :

Very good work!

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow \sup_{t \in I}||x(t) - y(t)|| = 0 \\ &\Leftrightarrow ||x(t) - y(t)|| = 0\ \forall\ t \in I \\ &\Leftrightarrow x(t) - y(t) = 0\ \forall\ t \in I \text{ cause } ||.|| \text{ is a norm in } \mathbb{R}^n \\ &\Leftrightarrow x(t) = y(t)\ \forall\ t \in I \\ &\Leftrightarrow x = y \end{aligned}$$

• *Triangular inequality*

Let $x,\ y,\ z\ \in\ X$, we have :

$||x(t)-z(t)|| \leq ||x(t)-y(t)|| + ||y(t)-z(t)||\ \forall\ t \in I,$ because $||.||$ is a norm in $\mathbb{R}^n$.

$\Rightarrow ||x(t)-z(t)|| \leq \sup_{s\in I}||x(s)-y(s)|| + \sup_{s\in I}||y(s)-z(s)||\ \forall\ t \in I$

$\Rightarrow \sup_{s\in I}||x(s)-z(s)|| \leq \sup_{s\in I}||x(s)-y(s)|| + \sup_{s\in I}||y(s)-z(s)||$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z).$ Very Good!

We conclude that d is a distance in X.

Exercise sheet 4

8.

We define φ on $\mathbb{R}$ by $\varphi(x) = x^2$

Let's show that φ is locally Lipschitz.

Let K be a compact of $\mathbb{R}$. Then there exists $a, b \in \mathbb{R}$ such that $K = [a, b]$

Let $x, y \in K$,we have :

$$\begin{aligned} |\varphi(x, y)| &= |x^2 - y^2| \\ &= |(x-y)||(x+y)| \end{aligned}$$

But :

$$\begin{cases} a \leq x \leq b \\ a \leq y \leq b \end{cases}$$

$$\Rightarrow 2a \leq x + y \leq 2b$$
$$\Rightarrow |x + y| \leq max\{2|a|,\ 2|b|\}$$

Then we take $L = max\{2|a|,\ 2|b|\}$ and : Very Good!

$\varphi(x, y)| \leq L|x - y|$ for all $x,\ y$ in $K = [a,\ b]$

Therefore φ is locally Lipschitz.

- Let's check if φ is globally Lipschitz.

Let's assume there exists $L \in \mathbb{R}^+$ such that for all compact $K = [a, b]$, $\forall\ x,\ y \in K$,
$|\varphi(x, y)| \leq L|x - y|$

We have :

$|\varphi(x, y)| = |x^2 - y^2| = |x - y||x + y|$ then :

$\varphi(x, y) \leq L|x - y|$ for all compact K, $\forall\ x,\ y \in K$

$\Rightarrow |x - y||x + y| \leq L|x - y|$ for all compact K $\forall\ x,\ y \in K$

$\Rightarrow |x + y| \leq L$ for all compact K, for all $x,\ y \in K :\ x \neq y$

$\Rightarrow |(L) + (L + 1)| \leq L$ for $K = [L - 2,\ L + 2]$, $x = L,\ y = L + 1$,

$\Rightarrow 2L + 1 \leq L$ Well Done! Good Work.

$\Rightarrow L \leq -1$ impossible because $L > 0$

Conclusion : φ is not globally Lipschitz

9. Barreira & Valls

Let $f \ :\mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $g \ :\mathbb{R} \to \mathbb{R}$ be continuous.

Let $(x_0, x_0, y_0,) \ \in \ \mathbb{R}^3$ be fixed arbitrarily.

We want to show that :

$$\begin{cases} x' = f(x) \\ y' = g(x)y \qquad x(t_0) = x_0, y(t_0) = y_0 \end{cases} \quad \text{has a unique solution.}$$

f is locally Lipschitz so the Cauchy problem $\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$ has a unique solution x_* defined on an interval I_*.

Is the Picard Londelöf Theorem applicable to this problem?

Now we consider $\begin{cases} y' = g(x_*)y \\ y(t_0) = y_0 \end{cases}$ and the function h defined by :
$h(t, y) = g(x_*)y, \ t \ \in \ I$ where $I = [a, b]$ is a compact of I_*.

We are going to show that h is locally Lipschitz in y.

Let $y_1, \ y_2 \ \in \ \mathbb{R}$ and $t \ \in \ I$. We have :

$$\begin{aligned} |h(t, y_1) - h(t, y_2)| &= |g(x_*)y_1 - g(x_*)y_2| \\ &= |g(x_*)||y_1 - y_2| \\ &\leq Sup_{s \in I} \ g(x_*(s))|y_1 - y_2| \text{ cause } g \text{ and } x_* \text{ are continuous on respectively } \mathbb{R} \text{ and } I. \end{aligned}$$

Good!

Thus h is locally *Lipschitz* in y.

Therefore, $\begin{cases} y' = g(x_*)y \\ y(t_0) = y_0 \end{cases}$ **has a unique solution y^* and we conclude that the initial problem of Cauchy :**

$$\begin{cases} x' = f(x) \\ y' = g(x)y \qquad x(t_0) = x_0, y(t_0) = y_0 \end{cases} \quad \textbf{has a unique solution.}$$

Exercise sheet 5

Exercise 10.

Let u, v, w be $[a,\ b] \mapsto \mathbb{R}$ with $w > 0$ and such that for all $t\ \in\ [a,\ b]$,
$u(t) \leq v(t) + \int_a^t w(s)u(s)ds$.

Let's show that for all $t\ \in\ [a,\ b]$,

$$u(t) \leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta \cdot} ds.$$

Let's define the function R by $R(t) = \int_a^t w(s)u(s)ds$ for all $t\ \in\ [a,\ b]$
We have :

$$\begin{aligned} R'(t) = w(t)u(t) &\leq w(t)\left(v(t) + \int_a^t w(s)u(s)ds\right)\ \forall\ t\ \in\ [a,\ b] \\ &= w(t)v(t) + w(t)R(t) \\ &\Rightarrow R'(t) - w(t)R(t) \leq\ w(t)v(t)\ \forall\ t\ \in\ [a,\ b] \end{aligned}$$

Let μ be a positive function defined on $[a,\ b]$.

We have :

$R'(t) - w(t)R(t) \leq w(t)v(t)\ \forall\ t\ \in\ [a,\ b] \Leftrightarrow R'(t)\mu(t) - \mu(t)w(t)R(t) \leq w(t)v(t)\mu(t)\ \forall\ t\ \in\ [a,\ b]$

We require μ to satisfy : $\mu'(t) = -w(t)\mu(t)\ \forall\ t\ \in\ [a,\ b]$

Then we can take $\mu(t) = e^{-\int_a^t w(s)ds}$ and find :

Very good! Well-presented solution.

$$R'(t)\mu(t) - w(t)R(t)\mu(t) \leq w(t)v(t)\mu(t) \ \forall\, t \in [a,\ b]$$
$$\Leftrightarrow (R(t)\mu(t))' \leq w(t)v(t)\mu(t) \ \forall\, t \in [a,\ b]$$
$$\Rightarrow \int_a^t (R(s)\mu(s))'\, ds \leq \int_a^t w(s)v(s)\mu(s)ds \ \forall\, t \in [a,\ b]$$
$$\Rightarrow R(t)e^{-\int_a^t w(s)ds} \leq \int_a^t w(s)v(s)e^{-\int_a^s w(\theta)d\theta}ds \ \forall\, t \in [a,\ b], \quad \text{because } R(a) = 0$$
$$\Rightarrow R(t) \leq e^{\int_a^t w(s)ds} \int_a^t w(s)v(s)e^{-\int_a^s w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]$$
$$\Rightarrow R(t) \leq \int_a^t w(s)v(s)e^{\int_a^t w(\theta)d\theta}e^{\int_s^a w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]$$
$$\Rightarrow R(t) \leq \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]$$

$$\Leftrightarrow \int_a^t w(s)u(s)ds \leq \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]$$
$$\Rightarrow v(t) + \int_a^t w(s)u(s)ds \leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]$$
$$\Rightarrow \boxed{u(t) \leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds \ \forall\, t \in [a,\ b]}$$
$$\text{using the hypothesis : } u(t) \leq v(t) + \int_a^t w(s)u(s)ds \ \forall\, t \in [a,\ b].$$

Exercise 11.

Let $f \ : \ D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \mapsto \mathbb{R}^n$ have all its first partial derivatives continuous.

Let $x(t, t_0, x_0, \lambda)$ be the solution of the Cauchy problem :

$$\begin{cases} x' = f(t, x, \lambda) \\ x(t_0) = x_0 \end{cases}$$

We want to prove that the Jacobian matrix $\dfrac{\partial x}{\partial \lambda}$ is a solution of Cauchy problem.

$$\begin{cases} y' = \dfrac{\partial f}{\partial x}(t, x(t), \lambda)y + \dfrac{\partial f}{\partial \lambda}(t, x(t), \lambda) \\ y(t_0) = O \end{cases}$$

Very Good!

We have :

$$\frac{dx}{dt} = f$$
$$\implies \frac{\partial}{\partial\lambda}\left(\frac{dx}{dt}\right) = \frac{\partial f}{\partial\lambda}$$

$\dfrac{\partial f}{\partial\lambda}$ is continuous then the second derivatives of x are continuous and we have :

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\left(\frac{dx}{dt}\right) &= \frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right)\\
\implies \frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right) &= \frac{\partial f}{\partial\lambda}(t, x(t, t_0, x_0, \lambda), \lambda)\\
&= \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial t}(t, x(t), \lambda)\frac{\partial t}{\partial\lambda} + \frac{\partial f}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda}\\
&\text{where } \quad \frac{\partial t}{\partial\lambda} = O\\
&= \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial\lambda}(t, x(t), \lambda)
\end{aligned}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right) = \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial\lambda}(t, x(t), \lambda)}$$

At $t = t_0,\ x(t) = x_0$ and we have $\dfrac{\partial x}{\partial\lambda}(t_0) = O$.

We conclude that $\dfrac{\partial x}{\partial\lambda}$ is a solution of the Cauchy problem :

$$\begin{cases} y' = \dfrac{\partial f}{\partial x}(t, x(t), \lambda)y + \dfrac{\partial f}{\partial\lambda}(t, x(t), \lambda)\\ y(t_0) = O \end{cases}$$