$\mathbb{R} \cong 2^{\mathbb{N}}$?

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Outline

$\mathbb{R} \cong 2^{\mathbb{N}}$?

- **What's $\mathbb{R}$ and $2^{\mathbb{N}}$?**

1. In mathematics, a real number is a value of a continuous quantity that can represent a distance along a line. The real numbers include all the rational numbers such as the integer -5 and the fraction $\frac{4}{3}$ and all the irrational numbers such as $\sqrt{2} = 1.41421356...$
2. $2^{\mathbb{N}}$ is the set of all the functions from $\mathbb{N}$ to $\left\{0, 1\right\}$. So each element of that set is a sequence $\left(s_n\right)_{n\in\mathbb{N}}$ such that $s_n \in \left\{0, 1\right\}$ $\forall n \in \mathbb{N}$.

$$\mathbb{R} \cong 2^{\mathbb{N}} \ ?$$

Construction of φ_1

For all real number x, there exists a sequence $\left(a_n\right)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$ such that

$x = \pm \sum_{n\in\mathbb{N}} a_n 2^{\delta_n}$, where $\delta_n = \begin{cases} \frac{n}{2} \text{ if } n \in 2\mathbb{Z}, \\ \frac{-n-1}{2} \text{ if } n \notin 2\mathbb{Z}. \end{cases}$

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We define the concatenation operator $\mathcal{C}$ by

$\mathcal{C} : \Big\{0, 1\Big\} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$

$\Big(\epsilon, \Big(a_n\Big)_{n\in\mathbb{N}}\Big) \mapsto \epsilon\Big(a_n\Big)_{n\in\mathbb{N}} := \Big(b_n\Big)_{n\in\mathbb{N}}$,

where $b_0 = \epsilon$ and $b_n = a_{n-1}$ for $n \geq 1$.

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Now a real number $x = \pm \sum_{n\in\mathbb{N}} a_n 2^{\delta_n}$ will be represented by

$$\mathcal{C}\left(\left(\epsilon, \left(a_n\right)_{n\in\mathbb{N}}\right)\right) = \epsilon\left(a_n\right)_{n\in\mathbb{N}}$$

where $\epsilon = 0$ if $x \geq 0$ and $\epsilon = 1$ if $x < 0$.

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$$\varphi_1 : \mathbb{R} \to 2^{\mathbb{N}}$$
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φ_1 is clearly injective because if two real numbers x, y are represented by the same sequence $\epsilon\left(a_n\right)_{n\in\mathbb{N}}$ then they are equal since they would have the same sign ϵ and the same absolute value $\sum_{n\in\mathbb{N}} a_n 2^{\delta_n}$.

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So $\mathbb{R} \leq 2^{\mathbb{N}}$ (*).

Construction of φ_2

We define

$$\varphi_2 : 2^{\mathbb{N}} \to \mathbb{R}$$
$$\left(a_n\right)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} \frac{a_n}{(n+1)!}$$

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Let $\left(a_n\right)_{n\in\mathbb{N}}, \left(b_n\right)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$.

Suppose $N_0 = \left\{i \in \mathbb{N} : a_i \neq b_i\right\} \neq \emptyset$ and let $i_0 = min\left(N_0\right)$.

Suppose without loss of generality that $a_{i_0} = 1$ and $b_{i_0} = 0$.

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We have $\varphi_2\left(\left(a_n\right)_{n\in\mathbb{N}}\right) \geq \varphi_2\left(a_0, a_1, \ldots, a_{i_0-1}, 1, 0, 0, \ldots\right)$ and $\varphi_2\left(b_0, b_1, \ldots, b_{i_0-1}, 0, 1, 1, \ldots\right) \geq \varphi_2\left(\left(b_n\right)_{n\in\mathbb{N}}\right)$.

Construction of φ_2

So we just need to show that $\varphi_2\Big(a_0, a_1, \ldots, a_{i_0-1}, 1, 0, 0, \ldots\Big) > \varphi_2\Big(b_0, b_1, \ldots, b_{i_0-1}, 0, 1, 1, \ldots\Big)$, that is, $\varphi_2\Big(0, 0, \ldots, 0, 1, 0, 0, \ldots\Big) > \varphi_2\Big(0, 0, \ldots, 0, 0, 1, 1, \ldots\Big)$, because $a_i = b_i \ \forall\ 0 \leq i < i_0$.

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After computations, we find $\dfrac{\varphi_2\Big(0,0,\dots,0,0,1,1,\dots\Big)}{\varphi_2\Big(0,0,\dots,0,1,0,0,\dots\Big)} < 1$, that is,

$$\varphi_2\Big(0, 0, \dots, 0, 0, 1, 1, \dots\Big) < \varphi_2\Big(0, 0, \dots, 0, 1, 0, 0, \dots\Big).$$

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Hence $\varphi_2\Big(\big(a_n\big)_{n\in\mathbb{N}}\Big) > \varphi_2\Big(\big(b_n\big)_{n\in\mathbb{N}}\Big)$.

It follows that φ_2 is injective and we have $2^{\mathbb{N}} \le \mathbb{R}$ (**).

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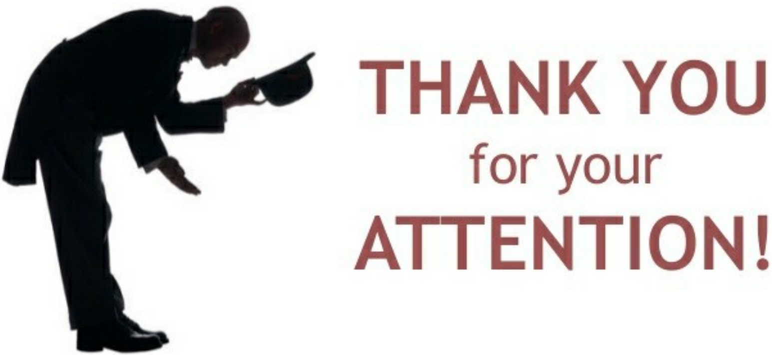
Conclusion

From (*) and (**) and using the **Theorem of Cantor-Bernstein-Schroeder** we conclude that $\mathbb{R} \cong 2^{\mathbb{N}}$


THANK YOU
for your
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