Algebra Assignment 01

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1.
Let S be the set of all polynomials with real coefficients. If $f, g \in S$, we define $f \sim g$ if only if $f' = g'$ where f' is the derivative of f.
Then $\sim$ is an equivalence relation on S.
Let $f \in S$.
We have :

$$\boxed{[f] = \{f + c,\ c \in \mathbb{R}\ \textit{constant}\ \}}$$

2.
Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$.
• Let's decide whether the relation $\{(1, 4), (2, 4), (3, 6)\}$ is a function mapping A into B.
Each element in A is related at most one element in B.
Therefore, the relation maps A into B.
• One-to-one or onto
The function is **not one-to-one** cause 1 and 2 are two different elements of A which have the same functional value 4.
The function is **is not onto** cause the element 2 in B doesn't have a pre-image in A.

3.
Let's complete the table below so that * is a commutative binary operation on the set $S = \{a, b, c, d\}$.

*	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

4.

- Let's prove that the relation $\mathcal{R}$ defined in $\mathbb{R}$ by $x\mathcal{R}y$ if $|x| = |y|$ is an equivalence relation.

• **Reflexivity**:
We have : $|x| = |x| \; \forall \; x \; \in \; \mathbb{R}$
$\Rightarrow \; x\mathcal{R}x \; \forall \; x \; \in \; \mathbb{R}$

• **Symmetry**:

$$\begin{aligned} &\forall \; x, \; y \; \in \; \mathbb{R}, \\ & x\mathcal{R}y \Rightarrow |x| = |y| \\ &\qquad \Rightarrow |y| = |x| \\ &\qquad \Rightarrow y\mathcal{R}x. \end{aligned}$$

• **Transitivity**:

$$\begin{aligned} &\forall \; x, \; y, \; z \; \in \; \mathbb{R}, \\ &x\mathcal{R}y \text{ and } y\mathcal{R}z \Leftrightarrow |x| = |y| \text{ and } |y| = |z| \\ &\qquad \Rightarrow |x| = |z| \\ &\qquad \Rightarrow x\mathcal{R}z. \end{aligned}$$

We conclude that $\mathcal{R}$ is an equivalence relation in $\mathbb{R}$.

- Partition arising from the equivalence relation :
Each equivalence class contains two elements except the class of 0 which contains only 0 : $[0] = \{0\}$.
The class of every non zero real number is given by :
$[x] = \{x, -x\}$
The set of all the classes is a partition of $\mathbb{R}$ and we can write :
$\mathbb{R} = \{0\} \cup \bigcup_{x \in \mathbb{R}^+} \{x, -x\}$.

5.
Let $\mathcal{M}_2(\mathbb{R})$ be the set of 2×2 matrices with real entries.
Let $A, \; B \; \in \; \mathcal{M}_2(\mathbb{R})$.
We define the relation $\sim$ by : $A \sim B$ if only if there exists an invertible matrix P such that $PAP^{-1} = B$.

$i)$ For $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} -18 & 33 \\ -11 & 20 \end{pmatrix}$, let's find P such that $PAP^{-1} = B$.
Suppose $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
P must be invertible, thus : $ad - cb \neq 0$

We have :

$$
\begin{aligned}
PAP^{-1} = B &\Leftrightarrow PA = BP \\
&\Leftrightarrow PA = BP \\
&\Leftrightarrow \begin{pmatrix} a-b & 2a+b \\ c-d & 2c+d \end{pmatrix} = \begin{pmatrix} -18a+33c & -18b+33d \\ -11a+20c & -11b+20d \end{pmatrix} \\
&\Leftrightarrow \begin{cases} a-b = -18a+33c \\ 2a+b = -18b+33d \\ c-d = -11a+20 \\ 2c+d = -11b+20d \end{cases} \\
&\Leftrightarrow \begin{cases} 19a-33c-b = 0 \\ 19b-33d+2a = 0 \\ -19c+11a-d = 0 \\ -19d+2c+11b = 0 \end{cases} \\
&\Leftrightarrow \begin{cases} 19a-33c-b = 0 \\ 363a-627c-33d = 0 \quad \text{Row}_2 \longleftarrow \text{Row}_2 + 19\text{Row}_1 \\ -19c+11a-d = 0 \\ 209a-361c-19d = 0 \quad Row_4 \longleftarrow \text{Row}_4 + 11\text{Row}_1 \end{cases} \\
&\Leftrightarrow \begin{cases} 19a-33c-b = 0 \\ -19c+11a-d = 0 \quad \text{Row }_2, \text{ Row }_3 \text{ and } \text{ Row }_4 \text{ are equivalent} \end{cases} \\
&\Leftrightarrow \begin{cases} b = 19a-33c \quad a,\ c \ \in \ \mathbb{R} \\ d = 11a-19c \end{cases}
\end{aligned}
$$

Furthermore, $ad - cb \neq 0$

Then we can choose $a = 1,\ c = 0$ and find : $\begin{cases} a = 1 \\ b = 19 \\ c = 0 \\ d = 11 \end{cases}$

$$\boxed{P = \begin{pmatrix} 1 & 19 \\ 0 & 11 \end{pmatrix}}$$

$ii)$ Let's show that $\sim$ is an equivalence relation on $\mathcal{M}_2(\mathbb{R})$.

Reflexivity

Let $A \ \in \ \mathcal{M}_2(\mathbb{R})$

We have $A = I_2 A I_2^{-1}$ where I_2 is the 2×2 identity matrix. So, $A \sim A$.

*** Symmetry**

Let $A,\ B \ \in \ \mathcal{M}_2(\mathbb{R})$

$$
\begin{aligned}
A \sim B &\Rightarrow \exists \ P \ \in \ \mathcal{M}_2(\mathbb{R}) \ \text{ invertible such that } : \ PAP^{-1} = B \\
&\Rightarrow P^{-1}(PAP^{-1})P = P^{-1}BP \\
&\Rightarrow A = P^{-1}BP = P^{-1}B(P^{-1})^{-1} \\
&\Rightarrow B \sim A
\end{aligned}
$$

*** Transitivity**

Let $A,\ B,\ C\ \in\ \mathcal{M}_2(\mathbb{R})$

Suppose $A \sim B$ and $B \sim C$. Then we have :

$\exists P_1,\ P_2\ \in\ \mathcal{M}_2(\mathbb{R})$ invertible such that : $\begin{cases} P_1AP_1^{-1} = B \\ P_2BP_2^{-1} = C \end{cases}$

$$\Rightarrow P_2(P_1AP_1^{-1})P_2^{-1} = C$$
$$\Rightarrow (P_2P_1)A(P_1^{-1}P_2^{-1}) = C$$
$$\Rightarrow (P_2P_1)A(P_2P_1)^{-1} = C$$

cause P_2P_1 is invertible and $(P_2P_1)^{-1} = P_1^{-1}P_2^{-1}$

$$\Rightarrow A \sim C$$

We conclude that $\sim$ is an equivalence relation on the set of all 2×2 matrices with real entries.

6.

Let's prove that the set, say $\mathcal{H}$, of all 3 by 3 matrices with real entries of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ is a group under matrix multiplication.

All the elements of $\mathcal{H}$ are invertible matrices cause they all have the same determinant $det = 1$.

We can just show that $\mathcal{H}$ is a subgroup of the group $GL_3(\mathbb{R})$ of all 3 by 3 invertible matrices with real entries under matrix multiplication.

• For $a = b = c = 0$, we get the 3 by 3 identity matrix I_3. Then $I_3\ \in\ \mathcal{H}$ and we have $\mathcal{H} \neq \phi$.

• Let $A = \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & a_2 & b_2 \\ 0 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}\ \in\ \mathcal{H}$.

We want to show that $AB^{-1}\ \in\ \mathcal{H}$.

Let's find B^{-1} first.

Here, $com(M)$ denotes the co-matrix of the matrix M.

We have $det\ B = 1$ and :

$$com(B) = \begin{pmatrix} 1 & 0 & 0 \\ -a_2 & 1 & 0 \\ a_2c_2 - b_2 & -c_2 & 1 \end{pmatrix}$$

$$(com(B))^T = \begin{pmatrix} 1 & -a_2 & a_2c_2 - b_2 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that $B^{-1} = \begin{pmatrix} 1 & -a_2 & a_2c_2 - b_2 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}$

Then we have :

$$
\begin{aligned}
AB^{-1} &= \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -a_2 & a_2c_2 - b_2 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & a_1 - a_2 & a_2c_2 - b_2 - a_1c_2 + b_1 \\ 0 & 1 & c_1 - c_2 \\ 0 & 0 & 1 \end{pmatrix} \\
&\Rightarrow AB^{-1} \in \mathcal{H}
\end{aligned}
$$

Therefore, $\mathcal{H}$ is a subgroup of $GL_3(\mathbb{R})$, then a group.

7.

For a fixed point $(a, b) \in \mathbb{R}$, we define $T_{ab} : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \mapsto (x+a, y+b)$. Then $T(\mathbb{R}^2) = \{T_{a,b} | a, b, \in \mathbb{R}\}$ is a group under function composition.

$i)$ Let's show that $T_{a,b}T_{c,d} = T_{a+c,b+d}$ for all $(a, b),\ (c, d),\ \in \mathbb{R}^2$.
Let $(a, b),\ (c, d),\ \in \mathbb{R}^2$ and let $(x, y) \in \mathbb{R}^2$.
We have :

$$
\begin{aligned}
T_{a,b}T_{c,d}(x, y) &= T_{a,b}\left(T_{c,d}(x, y)\right) \\
&= T_{a,b}\left(x + c, y + d\right) \\
&= (a + (x + c), b + (y + d)) \\
&= (x + a + c, y + b + d) \qquad \text{cause } (\mathbb{R}, +) \text{ is commutative} \\
&= T_{a+c,b+d}(x, y)
\end{aligned}
$$

We have shown that $T_{a,b}T_{c,d}(x, y) = T_{a+c,b+d}(x, y)\ \forall\ (x, y) \in \mathbb{R}^2$.
Then we conclude that $T_{a,b}T_{c,d} = T_{a+c,b+d}$
$ii)$
The identity of the group is $T_{0,0}$ cause for all $(a, b) \in \mathbb{R}^2$, we have :
$T_{a,b}T_{0,0} = T_{a+0,b+0} = T_{a,b}$
$T_{0,0}T_{a,b} = T_{0+a,0+b} = T_{a,b}$
$iii)$
Let $T_{a,b} \in T(\mathbb{R}^2)$.
We have :
$T_{a,b}T_{-a,-b} = T_{a-a,b-b} = T_{0,0}$
$T_{-a,-b}T_{a,b} = T_{-a+a,-b+b} = T_{0,0}$
It follows that the inverse of $T_{a,b}$ is $T_{-a,-b}$