Differential Equations Assignment 01

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Exercise sheet 1

1.1 $x'' + \omega^2 \sin(x) = 1 \quad (E_1)$

• **Order:**
The highest derivative is x''. So, the equation is a **second order** differential equation.
• **Linearity**:
Let f_1 be the function defined by:
$f_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mapsto z + \omega^2 \sin(x) - 1$

We have $x'' + \omega^2 \sin(x) = 1 \Leftrightarrow f_1(\begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}) = 0.$

But f_1 is not linear cause $f_1(\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}) = 0,$

$f_1(2 \times \begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}) = f_1(\begin{pmatrix} 2\pi \\ 0 \\ 2 \end{pmatrix}) = 1 \neq 2 \times f_1(\begin{pmatrix} \pi \\ 0 \\ 1 \end{pmatrix}).$
Therefore, the differential equation (E_1) is not linear.
• **Autonomous/Non-autonomous:**
The equation doesn't depend explicitly on the variable t (variable of the unknown function x). We conclude that the equation is **autonomous.**

1.2 $x'' + \omega^2 x = \cos(t) \quad (E_2)$

• **Order:**
The highest derivative is x''. So, the equation is a **second order** differential equation.
• **Linearity**:
Let f_2 be the function defined by:
$f_2 : (t, \begin{pmatrix} x \\ y \\ z \end{pmatrix}) \in \mathbb{R} \times \mathbb{R}^3 \mapsto z + \omega^2 x - \cos(t)$

We have $x'' + \omega^2 x = \cos(t) \Leftrightarrow f_2(t, \begin{pmatrix} x \\ x' \\ x'' \end{pmatrix}) = 0.$

We consider the function g defined by:
$g : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mapsto z + \omega^2 x$
We have :
$g(\begin{pmatrix} x \\ y \\ z \end{pmatrix}) = f_2(t, \begin{pmatrix} x \\ y \\ z \end{pmatrix}) - f_2(t, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix})$ and g is linear cause:
$\forall \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \in \mathbb{R}^3, \forall \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in \mathbb{R}^3$ and $\forall \alpha \in \mathbb{R},$ we have:

$$\begin{aligned} g(\alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) &= g(\begin{pmatrix} \alpha x_1 + x_2 \\ \alpha y_1 + y_2 \\ \alpha z_1 + z_2 \end{pmatrix}) \\ &= \alpha z_1 + z_2 + \omega^2(\alpha x_1 + x_2) \\ &= \alpha z_1 + \alpha \omega^2 x_1 + z_2 + \omega^2 x_2 \\ &= \alpha(z_1 + \omega^2 x_1) + z_2 + \omega^2 x_2 \\ &= \alpha g(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + g(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) \end{aligned}$$

• **Autonomous/Non-autonomous:**
The equation depends explicitly on t (cause of the term $\cos(t)$). As a result, the differential equation is **not autonomous.**
• **Solution of the equation:**
The equation is linear, so we can first solve the homogeneous equation:
$x'' + \omega^2 x = 0$
The characteristic equation of this differential equation is : $m^2 + w^2 = 0$ and has two complex solutions : $m_1 = -i\omega,\ m_2 = i\omega$
Then, the general solution of the homogeneous equation is :
$x_h(t) = A\cos(\omega t) + B\sin(\omega t),\ \textit{where}\ A,\ B\ \in\ \mathbb{R}\ \textit{are constants.}$
Now, we find a particular solution of the equation (E_2)
The right hand side of (E_2) is $\cos(t)$.
*So, if $\omega^2 \neq 1$ (the particular solution of (E_2) should not look like the general solution of its homogeneous equation), we look for a particular solution of the form:
$x_p(t) = a\cos(t) + b\sin(t), \textit{where}\ a,\ b\ \in\ \mathbb{R}\ \textit{are constants.}$
$x'_p(t) = -a\sin(t) + b\cos(t);\ x''_p(t) = -a\cos(t) - b\sin(t)$
x_p satisfies (E_2), then we have:
$x''_p(t) + \omega^2 x_p(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow -a\cos(t) - b\sin(t) + a\omega^2\cos(t) + b\omega^2\sin(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow (a\omega^2 - a)\cos(t) + (b\omega^2 - b)\sin(t) = \cos(t)\ \forall\ t$
$a(\omega^2 - 1)\cos(t) + b(\omega^2 - 1)\sin(t) = \cos(t)\ \forall\ t$
$\omega^2 \neq 1$, then we have :
$\begin{cases} a(\omega^2 - 1) = 1 \\ b = 0 \end{cases} \Leftrightarrow \begin{cases} a = \frac{1}{\omega^2 - 1} \\ b = 0 \end{cases}$

The particular solution is : $x_p(t) = \frac{1}{\omega^2-1}\cos(t)$
Therefore the general solution of (E_2) *with* $\omega^2 \neq 1$ is

$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{\omega^2-1}\cos(t)\ \forall\ t\ \in\ \mathbb{R}}$$

,where $A,\ B\ \in\ \mathbb{R}$ *are constants.*
* If $\omega^2 = 1$ then we look for a particular solution of the form:
$x_p(t) = at\cos(t) + bt\sin(t), where\ a,\ b\ \in\ \mathbb{R}\ are\ constants.$

$$\begin{aligned}
x_p'(t) &= -at\sin(t) + a\cos(t) + b\sin(t) + bt\cos(t)\\
x_p''(t) &= -a\sin(t) - at\cos(t) - a\sin(t) + b\cos(t) - bt\sin(t) + b\cos(t)\\
&= (-2a - bt)\sin(t) + (2b - at)\cos(t)
\end{aligned}$$

x_p is a solution of (E_2) thus :
$x_p''(t) + \omega^2 x_p(t) = \cos(t)]\ \forall\ t$
$\Leftrightarrow (a\omega^2 t + 2b - at)\cos(t) + (b\omega^2 t - 2a - bt)\sin(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow (at + 2b - at)\cos(t) + (bt - 2a - bt)\sin(t) = \cos(t)\ \forall\ t\ cause\ \omega^2 = 1$
$\Leftrightarrow 2b\cos(t) - 2a\sin(t) = \cos(t)\ \forall\ t$
$\Leftrightarrow \begin{cases} a = 0\\ b = \frac{1}{2}\end{cases}$
$\Leftrightarrow x_p(t) = \frac{1}{2}t\sin(t).$
It follows that for $\omega^2 = 1$**, the general solution of** (E_2) **is**
$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t) + \dfrac{1}{2}t\sin(t)\ \forall\ t\ \in\ \mathbb{R}}$ *, where* $A,\ B\ \in\ \mathbb{R}$ *are constants.*

1.3 $(x^T)' = Ax^T$ (E_3), where A is a $n \times n$ real matrix

• Order
The highest derivative is $(x^T)'$. So, the equation is a **first order** differential equation.
• Linearity
We consider the function f_3 defined by:
$f_3 : (x, y)\ \in\ \mathbb{R}^n \times \mathbb{R}^n \mapsto y^T - Ax^T$
We have :
$(x^T)' = Ax^T \Leftrightarrow f_3(x, x') = 0$
Let's show that f_3 is linear :
Let $(u_1, v_1), (u_2, v_2)\ \in\ \mathbb{R}^n \times \mathbb{R}^n$ and $\alpha\ \in\ \mathbb{R}$
We have

$$\begin{aligned}
f_3(\alpha(u_1, v_1) + (u_2, v_2)) &= f_3(\alpha u_1 + u_2, \alpha v_1 + v_2)\\
&= (\alpha v_1 + v_2)^T - A(\alpha u_1 + u_2)^T\\
&= \alpha v_1^T + v_2^T - A[\alpha u_1^T + u_2^T]\\
&= \alpha v_1^T + v_2^T - A\alpha u_1^T - Au_2^T\\
&= \alpha v_1^T - A\alpha u_1^T + v_2^T - Au_2^T\\
&= \alpha(v_1^T - Au_1^T) + v_2^T - Au_2^T\\
&= \alpha f_3((u_1, v_1)) + f_3((u_2, v_2))
\end{aligned}$$

Then, f_3 is linear and we conclude that the equation (E_3) is linear.
• **Atonomous/Non-autonomous**
The equation doesn't depend explicitly on t, then it is an **autonomous** differential equation.
• **Solution of the differential equation :**
The solution of the equation is :
$\boxed{x(t) = x_0 e^{tA} = \sum_{i=0}^{+\infty} \frac{t^i A^i}{i!}}$, $x_0 \in \mathbb{R}^n$ is a constant row vector. **2. Writing differential in vector form**

2.1 $x'' + \mu(t)x' + \omega^2 x = \sin(t)$
We do the following transformations :
$x'' = (x')' = y'$
$y = x'$
$y' = -\mu(t)y - \omega^2 x + \sin(t)$
Then we can write :

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\mu(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}}$$
$$\boxed{\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -\omega^2 x - \mu(t)y + \sin(t) \end{pmatrix}}$$

2.2 $x^{(5)} + x^{(3)} - x' + x = \sin(2\pi t)$
Transformations :
$x^{(5)} = (x^{(4)})' = y'$
$x^{(4)} = (x^{(3)})'$=z'=y
$x^{(3)} = (x'')'$=u'=z
$x'' = (x')' = v' = u$
$x' = v$
$y' = -z + v - x + \sin(2\pi t)$
Now we can write :

$$\boxed{\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(2\pi t) \\ 0 \\ 0 \\ 0 \end{pmatrix}}$$
$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \\ u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -x + v - z + \sin(2\pi t) \\ y \\ z \\ u \end{pmatrix}}$$

Exercise sheet 2

3.
Let's find all the solutions of the following ordinary differential equations and compute their maximal interval of definition:
3.1 $x' = x\sin(t),\ x(0) = 1$
$x(t) = 0\ \forall\ t\ \in\ \mathbb{R}$ is a solution of the differential equation. But that solution doesn't satisfy $x(0) = 1$. Then we reject it.
Suppose there is another solution x. Because of the fact that we must have $x\ \in\ C^1$, there exists an interval I such that $x(t) \neq 0\ \forall\ t\ \in\ I$.
$\forall\ t\ \in\ I$,

$$\begin{aligned}
x'(t) = x(t)\sin(t) &\Leftrightarrow \frac{x'(t)}{x(t)} = \sin(t) \\
&\Leftrightarrow \int \frac{x'(t)}{x(t)} dt = -\cos(t) + c_1,\ c_1\ \in\ \mathbb{R}\ \mathit{constant} \\
&\Leftrightarrow \ln(|x(t)|) + c_2 = -\cos(t) + c_1,\ c_1,\ c_2\ \in\ \mathbb{R}\ \mathit{constants} \\
&\Leftrightarrow |x(t)| = ke^{-\cos(t)},\ k = e^{c_1 - c_2}\ \in\ \mathbb{R}^+\ \mathit{constant} \\
&\Leftrightarrow x(t) = \pm ke^{-\cos(t)},\ k\ \in\ \mathbb{R}^+\ \mathit{constant}
\end{aligned}$$

So, x is either positive or negative.
However, we can write :
$x(t) = ke^{-\cos(t)},\ k\ \in\ \mathbb{R}\ \mathit{constant}.$
With the initial condition $x(0) = 1$, we have :
$ke^{-1} = 1\ \Rightarrow k = e^1$
Therefore, the only solution satisfying the initial condition is :
$\boxed{x(t) = e^{1-\cos(t)}}$. This solution is defined for all real number t and is of class C^∞ on $\mathbb{R}$. **Hence, the greatest interval on which the solution is defined is : $I_{max} = \mathbb{R}$.**

3.2 $x' = \frac{1}{tx},\ x(1) = 1$
This differential equation has no constant solution cause for $x = \mathit{constant},\ x'(t) = 0 \neq \frac{1}{tx(t)}$.
We have :

$$\begin{aligned}
x'(t) = \frac{1}{tx} &\Leftrightarrow x'(t)x(t) = \frac{1}{t} \\
&\Leftrightarrow \frac{1}{2}\frac{d}{dt}x^2(t) = \frac{1}{t} \\
&\Leftrightarrow \frac{d}{dt}x^2(t) = \frac{2}{t} \\
&\Leftrightarrow x^2(t) = 2\ln(|t|) + c,\ c\ \in\ \mathbb{R}\ \mathit{constant} \\
&\Leftrightarrow 2\ln(|t|) + c \geq 0\ \mathit{and}\ x(t) = \pm\sqrt{2\ln(|t|) + c},\ c\ \in\ \mathbb{R}\ \mathit{constant}
\end{aligned}$$

We have $x(1) = 1$, then the solution is a positive function and we can write: $x(t) = \sqrt{2\ln(|t|) + c},\ c\ \in\ \mathbb{R}\ \mathit{constant}$

It follows that :
$\sqrt{0+c} = 1 \Leftrightarrow c = 1$
The solution of the differential equation satisfying $x(1) = 1$ is :
$\boxed{x(t) = \sqrt{1 + 2\ln(|t|)}}$ $with\ 1 + 2\ln(|t|) \geq 0$.

We have :

$$\begin{aligned} 1 + 2\ln(|t|) \geq 0 &\Leftrightarrow \ln(|t|) \geq \frac{-1}{2} \\ &\Leftrightarrow |t| \geq e^{-\frac{1}{2}} \\ &\Leftrightarrow t \ \in \ \left]-\infty, -e^{-\frac{1}{2}}\right[\cup \left]e^{-\frac{1}{2}}, +\infty\right[\end{aligned}$$

It's domain of definition is : $\left]-\infty, -e^{-\frac{1}{2}}\right[\cup \left]e^{-\frac{1}{2}}, +\infty\right[$.
But that domain is not an interval.
The initial condition is $x(1) = 1$.
Therefore, the maximal interval on which the solution is defined is :
$\boxed{I_{max} = \left]e^{-\frac{1}{2}}, +\infty\right[}$.

3.3 $\quad x' = x^{2/3},\ x(0) = 0$
The null function $x(t) = 0\ \forall\ t\ \in\ \mathbb{R}$ is a solution of that differential equation and it satisfies $x(0) = 0$. Its maximal interval of definition is $\mathbb{R}$
Let's find the other solutions if they exist.
Let x be one another solution of the differential equation.
The solution must be of class C^1. So there exists an interval I such that $x(t) \neq 0\ \forall\ t\ \in\ I$.
Then, for all $t\ \in\ I$,

$$\begin{aligned} x'(t) = x^{2/3}(t) & \\ &\Leftrightarrow x'(t)x^{\frac{-2}{3}}(t) = 1 \\ &\Leftrightarrow \frac{1}{\frac{-2}{3}+1}x^{\frac{-2}{3}+1}(t) = t + c,\ c\ \in\ \mathbb{R}\ constant \\ &\Leftrightarrow 3x^{\frac{1}{3}}(t) = t + c,\ c\ \in\ \mathbb{R}\ constant \\ &\Leftrightarrow x(t) = \left(\frac{1}{3}(t+c)\right)^3,\ c\ \in\ \mathbb{R}\ constant \end{aligned}$$

We have $x(0) = 0 \Rightarrow c = 0$.
The second solution of the differential equation satisfying $x(0) = 0$ is :
$\boxed{x(t) = (\frac{t}{3})^3}$.
It's maximal interval of definition is : $\boxed{I_{max} = \mathbb{R}}$

4. $\quad y' = 5y + e^{-2t}y^{-2}$, y(0)=2
We do the change of variables : $x = y^3$

$x' = 3y'y^2$
We have :

$$\begin{aligned} y' = 5y + e^{-2t}y^{-2} &\Leftrightarrow y'y^2 = 5y^3 + e^{-2t} \\ &\Leftrightarrow \frac{1}{3}x' = 5x + e^{-2t} \\ &\Leftrightarrow x' - 15x = 3e^{-2t} \end{aligned}$$

Let $\mu : t \mapsto \mu(t)$ be a C^1 function.
We have $\mu(t)x' - 15\mu(t)x = 3\mu(t)e^{-2t}$. We look for a function μ such that $\mu'(t) = -15\mu(t)$.
The general solution of the differential equation $\mu' = -15\mu$ is $t \mapsto ke^{-15t}$ where $k \in \mathbb{R}$ is a constant.
We can take $\mu(t) = e^{-15t}$.
Then :

$$\begin{aligned} &\mu(t)x'(t) + \mu'(t)x(t) = 3e^{-2t}\mu(t) \\ &\qquad \Leftrightarrow (\mu(t)x(t))' = 3e^{-2t}e^{15t} = 3e^{-2t}\mu(t) \\ &\qquad \Leftrightarrow \mu(t)x(t) + c_1 = \int 3e^{-2t}\mu(t)dt,\ c_1 \in \mathbb{R}\ constant \\ &\qquad \Leftrightarrow e^{-15t}x(t) + c_1 = \frac{-3}{17}e^{-17t} + c_2, c_1,\ \ c_2 \in \mathbb{R}\ constants \\ &\qquad \Leftrightarrow x(t) = \left(\frac{-3}{17}e^{-17t} + c\right)e^{15t},\ c = c_2 - c_1 \in \mathbb{R}\ constant \end{aligned}$$

We have $y(0) = 2$, then $x(0) = 8$ and it follows that :
$\frac{-3}{17} + c = 8 \Leftrightarrow c = \frac{139}{17}$

Finally, $x(t) = \left(\frac{-3}{17}e^{-17t} + \frac{139}{17}\right)e^{15t}$ and

$$\boxed{y(t) = \left(\frac{-3}{17}e^{-17t} + \frac{139}{17}\right)^{1/3} e^{5t}}$$

The solution must be defined on an interval I such that $y(t) \neq 0$ for all $t \in I$.
We have :

$$\begin{aligned} y(t) = 0 &\Leftrightarrow \frac{-3}{17}e^{-17t} + \frac{139}{17} = 0 \\ &\Leftrightarrow e^{-17t} = \frac{139}{3} \\ &\Leftrightarrow -17t = \ln\left(\frac{139}{3}\right) \\ &\Leftrightarrow t = \frac{-1}{17}\ln\left(\frac{139}{3}\right) \end{aligned}$$

Then the maximal interval of definition of the solution is

$$\boxed{I_{max} = \left]\frac{-1}{17}\ln\left(\frac{139}{3}\right), +\infty\right[}$$

5.
Let a be a constant real number, $b(t)$ a continuous function defined in $\mathbb{R}_0^+$ and consider the two differential equations :

$$y' = ay \tag{1}$$
$$z' = (a + b(t))z \tag{2}$$

Suppose all the solutions of (1) are bounded when $t \to +\infty$ and
$\int_0^{+\infty} |b(t)|dt < +\infty$.
We want to show that all the solutions of (2) are bounded when $t \to +\infty$
The differential equation (2) is linear and can be written as :
$z' = az + b(t)z$
Consider the differential equation :

$$u' = b(t)u \tag{3}$$

Then, a solution of (2) can be written as the sum of one solution of (1) and one solution of (3).
The general solution of (3) is :
$u(t) = ce^{\int_{t_0}^{t} b(s)ds},\ c \in \mathbb{R}\ constant,\ t_0 \in \mathbb{R}_0^+\ cause\ b(t)\ is\ defined\ in\ \mathbb{R}_0^+$
Then, the general solution of (2) can be written as :
$z(t) = y(t) + ce^{\int_{t_0}^{t} b(s)ds},\ where\ \ c \in \mathbb{R}\ constant\ and\ y(t)\ is\ a\ solution\ of\ (1)$
All the solutions of (1) are bounded when $t \to +\infty$. This implies that there exists $l_1 \in \mathbb{R}$ such that $\lim\limits_{t\to+\infty} y(t) = l_1$
We also have :
$\int_{t_0}^{+\infty} b(t)dt \leq |\int_{t_0}^{+\infty} b(t)dt| \leq \int_{t_0}^{+\infty} |b(t)|dt \leq \int_0^{+\infty} |b(t)|dt < +\infty$
$\Rightarrow ce^{\int_{t_0}^{+\infty} b(s)ds} < +\infty$
It follows that :

$$\boxed{\lim_{t\to+\infty} z(t) = l_1 + ce^{\int_{t_0}^{+\infty} b(s)ds} < +\infty}$$

Exercise sheet 3
6. For the Cauchy problems below, let's construct the sequence of iterates defined by $x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds$ and verify if it is convergent and check that the limit is a solution of Cauchy problem.

6.1* $\begin{cases} x' = tx \\ x(0) = 1 \end{cases}$
Let's consider the function $x_1 : t \mapsto 1$
x_1 is not a solution of the problem but at least we have : $x_1(0) = 1$.
We begin constructing the sequence $(x_n(t))_n$:

$$\begin{aligned}
\bullet\ x_2(t) &= x_0 + \int_0^t f(s, x_1(s))ds \\
&= 1 + \int_0^t s ds \\
x_2(t) &= 1 + \frac{1}{2}t^2 \\
\bullet\ x_3(t) &= x_0 + \int_0^t f(s, x_2(s))ds \\
&= 1 + \int_0^t s(1 + \frac{1}{2}s^2)ds \\
&= 1 + \int_0^t (s + \frac{1}{2}s^3)ds \\
&= 1 + \int_0^t s ds + \int_0^t \frac{1}{2}s^3 ds \\
&= x_2(t) + \int_0^t \frac{1}{2}s^3 ds \\
&= 1 + \frac{1}{2}t^2 + \left[\frac{1}{8}s^4\right]_0^t \\
x_3(t) &= 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 \\
\bullet\ x_4(t) &= x_0 + \int_0^t f(s, x_3(s))ds \\
&= 1 + \int_0^t s(1 + \frac{1}{2}s^2 + \frac{1}{8}s^4)ds \\
&= 1 + \int_0^t s(1 + \frac{1}{2}s^2)ds + \int_0^t \frac{1}{8}s^5 ds \\
x_4(t) &= x_3(t) + \int_0^t \frac{1}{8}s^5 ds
\end{aligned}$$

We can see that $\forall\ n \in \mathbb{N}^*,\ x_{n+1}(t) = x_n(t) + \int_0^t sR_n(s)ds$ where $R_n(t)$ is the last term of $x_n(t)$ (that is the one with the highest degree on t).
Let's find $R_n(t)$ for all n.

We have :

$$R_1(t) = 1$$

$$R_2(t) = \frac{1}{2}t^2$$

$$\begin{aligned} R_3(t) &= \frac{1}{8}t^4 \\ &= \frac{1}{2!}\left(\frac{1}{2}t^2\right)^2 \end{aligned}$$

$$\begin{aligned} R_4(t) &= \frac{1}{48}t^6 \\ &= \frac{1}{3!}\left(\frac{1}{2}t^2\right)^3 \end{aligned}$$

Then $R_n(t) = \frac{1}{(n-1)!}\left(\frac{1}{2}t^2\right)^{n-1}$ and we can integrate $\int_0^t sR_n(s)ds$:

$$\begin{aligned} \int_0^t sR_n(s)ds &= \int_0^t \frac{s}{(n-1)!}\left(\frac{1}{2}s^2\right)^{n-1} ds \\ &= \left[\frac{1}{n(n-1)!(\frac{s^2}{2})^n}\right]_0^t \\ &= \left[\frac{1}{n!}(\frac{s^2}{2})^n\right]_0^t \\ &= \frac{1}{n!}(\frac{t^2}{2})^n \end{aligned}$$

It follows that $x_{n+1}(t) = x_n(t) + S_n(t)$, where $S_n(t) = \frac{1}{n!}(\frac{t^2}{2})^n$

Moreover,

$$\begin{aligned}
x_1(t) &= 1 \\
x_2(t) &= x_1(t) + S_1(t) \\
&= 1 + S_1(t) \\
\\
x_3(t) &= x_2(t) + S_2(t) \\
&= 1 + S_1(t) + S_2(t) \\
\\
x_4(t) &= x_3(t) + S_3(t) \\
&= 1 + S_1(t) + S_2(t) + S_3(t) \\
&\vdots \\
x_n(t) &= 1 + \sum_{k=1}^{n-1} S_k(t) \\
&= 1 + \sum_{k=1}^{n-1} \frac{1}{k!}(\frac{t^2}{2})^k
\end{aligned}$$

As a result, for all $t \in \mathbb{R}$,
the sequence $(x_n(t))_n$ converges to :

$$\begin{aligned}
&1 + \sum_{k=1}^{+\infty} \frac{1}{k!}(\frac{t^2}{2})^k \\
&= \sum_{k=0}^{+\infty} \frac{1}{k!}(\frac{t^2}{2})^k \\
&= e^{\frac{t^2}{2}}
\end{aligned}$$

The sequence converges to the function $x : t \in \mathbb{R} \mapsto e^{\frac{t^2}{2}}$ which is the solution of the problem cause :
$\begin{cases} x'(t) = te^{\frac{t^2}{2}} = tx(t) \\ x(0) = 1 \end{cases}$

6.1* $\begin{cases} x' = x + 1 \\ x(0) = 1 \end{cases}$

Here we consider again the function $x_1 : t \mapsto 1$
x_1 is not a solution of the problem but at least we have : $x_1(0) = 1$.
Using the previous method ($x_{n+1}(t) = x_n(t) + \int_0^t R_n(s)ds$) we begin constructing the sequence easily :

- $$\begin{aligned} x_2(t) &= x_0 + \int_0^t f(s, x_1(s))ds \\ &= 1 + \int_0^t 2ds \\ x_2(t) &= 1 + 2t \end{aligned}$$
- $$\begin{aligned} x_3(t) &= 1 + 2t + \int_0^t 2sds \\ &= 1 + 2t + t^2 \end{aligned}$$
- $$\begin{aligned} x_4(t) &= 1 + 2t + t^2 + \int_0^t s^2 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 \end{aligned}$$
- $$\begin{aligned} x_5(t) &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \int_0^t \frac{1}{3}s^3 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 \\ &= -1 + 2 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 \\ &= -1 + 2\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 \right) \\ &= -1 + 2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right) \end{aligned}$$
- $$\begin{aligned} x_6(t) &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \int_0^t \frac{1}{12}s^4 ds \\ &= 1 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 \\ &- 1 + 2 + 2t + t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{60}t^5 \\ &= -1 + 2\left(1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5\right) \\ &= -1 + 2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5\right) \end{aligned}$$

Then for all $n \in \mathbb{N}^*$, $x_n(t) = -1 + 2\left(\sum_{k=0}^{n-1} \frac{t^k}{k!}\right)$
For all $t \in \mathbb{R}$,
the sequence $(x_n(t))_n$ converges to :

$$\begin{aligned} &- 1 + 2\sum_{k=0}^{+\infty} \frac{t^k}{k!} \\ &= -1 + 2e^t \end{aligned}$$

The sequence converges to the function $x : t \in \mathbb{R} \mapsto -1 + 2e^t$ which is the solution of the problem cause :
$$\begin{cases} x'(t) = 2e^t = 1 + x(t) \\ x(0) = 1 \end{cases}$$

7. Metric

Let's show that d defined in the book by Barreira & Valls is a distance in $X = C(I)$, the set of all bounded continuous functions in $I \subset \mathbb{R}^k$.
For all $x, y \in X$, we have :
$d(x,y) = Sup_{t\in I}||x(t) - y(t)||$.
Then d is a positive function.
• *Symmetry*
Let $x, y \in X$. We have :

$$\forall\ t \in I,\ ||x(t) - y(t)|| = ||y(t) - x(t)|| \text{ cause } ||.|| \text{ is a norm in } \mathbb{R}^n$$

Hence $Sup_{t\in I}||x(t) - y(t)|| = Sup_{t\in I}||y(t) - x(t)|| \Leftrightarrow d(x,y) = d(y,x)$
• *Separation*
Let $x, y \in X$ We have :

$$\begin{aligned} d(x,y) = 0 \Leftrightarrow & \\ & \Leftrightarrow Sup_{t\in I}||x(t) - y(t)|| = 0 \\ & \Leftrightarrow ||x(t) - y(t)|| = 0\ \forall\ t \in I \\ & \Leftrightarrow x(t) - y(t) = 0\ \forall\ t \in I \text{ cause } ||.|| \text{ is a norm in } \mathbb{R}^n \\ & \Leftrightarrow x(t) = y(t)\ \forall\ t \in I \\ & \Leftrightarrow x = y \end{aligned}$$

• *Triangular inequality*
Let $x, y, z \in X$ We have :

$||x(t) - z(t)|| \leq ||x(t) - y(t)|| + ||y(t) - z(t)||\ \forall\ t \in I,$ cause $||.||$ is a norm in $\mathbb{R}^n$
$\Rightarrow ||x(t) - z(t)|| \leq Sup_{s\in I}||x(s) - y(s)|| + Sup_{s\in I}||y(s) - z(s)||\ \forall\ t \in I$
$\Rightarrow Sup_{s\in I}||x(s) - z(s)|| \leq Sup_{s\in I}||x(s) - y(s)|| + Sup_{s\in I}||y(s) - z(s)||$
$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$

We conclude that d is a distance in X.

Exercise sheet 4

8.

We define on $\mathbb{R}$ φ by $\varphi(x) = x^2$
Let's show that φ is locally Lipschitz.
Let K be a compact of $\mathbb{R}$. Then there exists $a, b \in \mathbb{R}$ such that $K = [a, b]$

Let $x, y \in K$. We have :

$$|\varphi(x,y)| = |x^2 - y^2|$$
$$= |(x-y)||(x+y)|$$

But :

$$\begin{cases} a \leq x \leq b \\ a \leq y \leq b \end{cases}$$
$$\Rightarrow 2a \leq x + y \leq 2b$$
$$\Rightarrow |x+y| \leq max\{2|a|,\ 2|b|\}$$

Then we take $L = max\{2|a|,\ 2|b|\}$ and :
$\varphi(x,y)| \leq L|x-y|$ for all $x,\ y$ in $K = [a,\ b]$
Therefore φ is locally Lipschitz.
- Let's check if φ is globally Lipschitz.
Suppose there exists $L \in \mathbb{R}^+$ such that for all compact $K = [a,b]$, $\forall\ x,\ y \in K$,
$|\varphi(x,y)| \leq L|x-y|$

We have :

$|\varphi(x,y)| = |x^2 - y^2| = |x-y||x+y|$ then :
$\varphi(x,y) \leq L|x-y|$ for all compact K,
$\forall\ x,\ y \in K$
$\Rightarrow |x-y||x+y| \leq L|x-y|$ for all compact $K\ \forall\ x,\ y \in K$
$\Rightarrow |x+y| \leq L$ for all compact K, for all $x,\ y \in K :\ x \neq y$
$\Rightarrow |(L) + (L+1)| \leq L$ for $K = [L-2,\ L+2],\ x = L,\ y = L+1,$
$\Rightarrow 2L + 1 \leq L$
$\Rightarrow L \leq -1$ impossible cause $L > 0$

Conclusion : φ is not globally Lipschitz

9. Barreira & Valls
Let $f\ : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $g\ : \mathbb{R} \to \mathbb{R}$ be continuous.
Let $(x_0, x_0, y_0,) \in \mathbb{R}^3$ be fixed arbitrarily.
We want to show that :
$\begin{cases} x' = f(x) \\ y' = g(x)y \qquad x(t_0) = x_0, y(t_0) = y_0 \end{cases}$ has a unique solution.

f is locally Lipschitz so the Cauchy problem $\begin{cases} x' = f(x) \\ x(t_0) = x_0 \end{cases}$ has a unique solution x_* defined on an interval I_*
Now we consider $\begin{cases} y' = g(x_*)y \\ y(t_0) = y_0 \end{cases}$ and the function h defined by :

$h(t,y) = g(x_*)y,\ t \in I$ where $I=[a,b]$ is a compact of I_*.
We are going to show that h is locally Lipschitz in y.

Let $y_1,\ y_2 \in \mathbb{R}$ and $t \in I$. We have :

$$\begin{aligned}|h(t,y_1)-h(t,y_2)| &= |g(x_*)y_1 - g(x_*)y_2| \\ &= |g(x_*)||y_1-y_2| \\ &\leq Sup_{s\in I}\ g(x_*(s))|y_1-y_2| \text{ cause } g \text{ and } x_* \text{ are continuous on respectively } \mathbb{R} \text{ and } I\end{aligned}$$

Thus h is locally *Lipschitz* in y.
Therefore, $\begin{cases} y' = g(x_*)y \\ y(t_0)=y_0 \end{cases}$ **has a unique solution** y^* **and we conclude that the initial problem of Cauchy :**

$$\begin{cases} x' = f(x) \\ y' = g(x)y \qquad x(t_0)=x_0, y(t_0)=y_0 \end{cases}$$ **has a unique solution.**

Exercise sheet 5.

Exercise 10.
Let u,v,w be $[a,\ b]\mapsto \mathbb{R}$ with $w>0$ and such that for all $t \in [a,\ b]$,
$u(t) \leq v(t) + \int_a^t w(s)u(s)ds.$

Let's show that for all $t \in [a,\ b]$,

$$u(t) \leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta\cdot}ds.$$

Let's define the function R by $R(t) = \int_a^t w(s)u(s)ds$ for all $t \in [a,\ b]$
We have :

$$\begin{aligned}R'(t) &= w(t)u(t) \\ &\leq w(t)\left(v(t)+\int_a^t w(s)u(s)ds\right)\ \forall\ t \in [a,\ b] \\ &= w(t)v(t)+w(t)R(t) \\ \Rightarrow R(t) - w(t)R(t) &\leq\ w(t)v(t) \forall\ t \in [a,\ b]\end{aligned}$$

Let μ be a positive function defined on $[a,\ b]$.

We have :

$$\begin{aligned} R'(t)-w(t)R(t) &\leq w(t)v(t)\ \forall\ t \in [a,\ b] \\ \Leftrightarrow R'(t)\mu(t) - \mu(t)w(t)R(t) &\leq w(t)v(t)\mu(t)\ \forall\ t \in [a,\ b]\end{aligned}$$

We require μ to satisfy : $\mu'(t) = -w(t)\mu(t)\ \forall\ t\ \in\ [a,\ b]$
Then we can take $\mu(t) = e^{-\int_a^t w(s)ds}$ and find :

$$
\begin{aligned}
R'(t)\mu(t) - w(t)R(t)\mu(t) &\leq w(t)v(t)\mu(t)\ \forall\ t\ \in\ [a,\ b] \\
\Leftrightarrow (R(t)\mu(t))' &\leq w(t)v(t)\mu(t)\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow \int_a^t (R(s)\mu(s))'\,ds &\leq \int_a^t w(s)v(s)\mu(s)ds\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow R(t)e^{-\int_a^t w(s)ds} &\leq \int_a^t w(s)v(s)e^{-\int_a^s w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b], \\
&\qquad \text{because } R(a) = 0 \\
\Rightarrow R(t) &\leq e^{\int_a^t w(s)ds}\int_a^t w(s)v(s)e^{-\int_a^s w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow R(t) &\leq \int_a^t w(s)v(s)e^{\int_a^t w(\theta)d\theta}e^{\int_s^a w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow R(t) &\leq \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b] \\
\Leftrightarrow \int_a^t w(s)u(s)ds &\leq \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow v(t) + \int_a^t w(s)u(s)ds &\leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b] \\
\Rightarrow &\boxed{u(t) \leq v(t) + \int_a^t w(s)v(s)e^{\int_s^t w(\theta)d\theta}ds\ \forall\ t\ \in\ [a,\ b]} \\
\text{using the hypothesis : } u(t) &\leq v(t) + \int_a^t w(s)u(s)ds\ \forall\ t\ \in\ [a,\ b]
\end{aligned}
$$

Exercise 11.

Let $f\ :\ D \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \mapsto \mathbb{R}^n$ have all its first partial derivatives continuous.
Let $x(t, t_0, x_0, \lambda)$ be the solution of the Cauchy problem :

$$
\begin{cases} x' = f(t, x, \lambda) \\ x(t_0) = x_0 \end{cases}
$$

.
We want to prove that the jacobian matrix $\frac{\partial x}{\partial \lambda}$ is a solution of Cauchy problem
:

$$
\begin{cases} y' = \frac{\partial f}{\partial x}(t, x(t), \lambda)y + \frac{\partial f}{\partial \lambda}(t, x(t), \lambda) \\ y(t_0) = O \end{cases}
$$

.
We have :

$$\frac{dx}{dt} = f$$
$$\Longrightarrow \frac{\partial}{\partial\lambda}\left(\frac{dx}{dt}\right) = \frac{\partial f}{\partial\lambda}$$

$\frac{\partial f}{\partial\lambda}$ is continuous then the second derivatives of x are continuous and we have :

$$\begin{aligned} \frac{\partial}{\partial\lambda}\left(\frac{dx}{dt}\right) &= \frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right) \\ \Longrightarrow \frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right) &= \frac{\partial f}{\partial\lambda}(t, x(t, t_0, x_0, \lambda), \lambda) \\ &= \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda} \\ &= \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial\lambda}(t, x(t), \lambda) \end{aligned}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial x}{\partial\lambda}\right) = \frac{\partial f}{\partial x}(t, x(t), \lambda)\frac{\partial x}{\partial\lambda} + \frac{\partial f}{\partial\lambda}(t, x(t), \lambda)}$$

At $t = t_0,\ x(t) = x_0$ and we have $\frac{\partial x}{\partial\lambda}(t_0) = O$.

We conclude that $\frac{\partial x}{\partial\lambda}$ is a solution of Cauchy problem :

$$\begin{cases} y' = \frac{\partial f}{\partial x}(t, x(t), \lambda)y + \frac{\partial f}{\partial\lambda}(t, x(t), \lambda) \\ y(t_0) = O \end{cases}$$