A Quantitative Analysis of Metrics on $\mathbb{R}^n$ with Almost Constant Positive Scalar Curvature

INTERNSHIP REPORT

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Abstract

In this paper, we adopt as a starting point the article published in 1984 by M. Struwe, in which he demonstrated the lack of compactness and the occurrence of a *bubbling phenomenon* for sequences satisfying the Palais-Smale condition. The aim of this report is to delve into a deeper comprehension of this phenomenon and to unravel its underlying ideas. Then, we will state and prove the main theorem obtained in the article of Ciraolo-Figalli-Maggi [3], which gives a quantitative bound for a Struwe decomposition with one bubble.

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1 Introduction

Inspired by the article of Ciraolo-Figalli-Maggi [3], we begin by analyzing the properties of positive H^1 solutions to the following elliptic equation:

$$-\Delta u = K u^p \qquad \text{on } \mathbb{R}^n, \quad n \geq 3 \tag{1.1}$$

where $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a given function, and $p = 2^* - 1, \quad 2^* = \frac{2n}{n-2}$.

If (M, g_0) is the n-dimensional round sphere endowed with the metric g_0, this equation can be linked to the problem of finding a metric g , conformal to g_0, whose scalar curvature is equal to a prescribed function R, such that if $F : \mathbb{R}^n \longrightarrow \mathbb{S}^n$ is inverse stereographic projection defined by

$$F(x) = \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{1+|x|^2} \right),$$

we have $R(F(x)) = K(x)$.

By [4, 5], in the case where K is constantly equal to $\kappa > 0$, we know that solutions to (1.1) are given explicitly by

$$\boxed{u(x) = \lambda^{\frac{n-2}{2}} w_\kappa(\lambda(x-z)) \qquad \forall x \in \mathbb{R}^n} \tag{1.2}$$

where $\lambda > 0$, $z \in \mathbb{R}^n$ and

$$w_\kappa(x) = \left(\frac{n(n-2)}{\kappa} \right)^{\frac{n-2}{4}} \frac{1}{(1+|x|^2)^{\frac{n-2}{2}}} \qquad \forall x \in \mathbb{R}^n. \tag{1.3}$$

Such solution u given by the above expression, is known as a “bubble” and is denoted by $v_\kappa[\lambda, z]$.

In addition, denote by $D^{1,2}(\mathbb{R}^n)$[1] the closure of $C_c^\infty(\mathbb{R}^n)$ for the norm $\|\nabla .\|_{L^2(\mathbb{R}^n)}$, and S the sharp Sobolev constant, given by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^n) - \{0\}} \frac{(\int_{\mathbb{R}^n} |\nabla u|^2)^{1/2}}{(\int_{\mathbb{R}^n} u^{2^*})^{1/2^*}} \tag{1.4}$$

Finally,the family of functions $\{v_\kappa[\lambda, z]\}_{\kappa,\lambda,z}$ corresponds to the minimizers in (1.4). (See *Theorem* in [7]). One can also show that

$$\int_{\mathbb{R}^n} |\nabla v_\kappa|^2 = S^2 \Big(\int_{\mathbb{R}^n} v_\kappa^{2^*} \Big)^{2/2^*} = \frac{S^n}{\kappa^{(n-2)/2}} \quad \text{and} \quad \int_{\mathbb{R}^n} v_\kappa^{2^*} = \frac{S^n}{\kappa^{n/2}} \tag{1.5}$$

[1] Equivalently, one can say that $D^{1,2}(\mathbb{R}^n) = \{u \in L^{2^*}(\mathbb{R}^n) \, : \, \nabla u \in L^2(\mathbb{R}^n)\}$

2 Preliminaries

In this section, we are interested in solutions u to the *prescribed scalar curvature equation* (1.1) in which K is close to a positive constant, from a standpoint that will be determined.

Let u be a solution of (1.1) and

$$K_0(u) = \frac{\int_{\mathbb{R}^n} K u^{2^*}}{\int_{\mathbb{R}^n} u^{2^*}} = \frac{\int_{\mathbb{R}^n} |\nabla u|^2}{\int_{\mathbb{R}^n} u^{2^*}} \tag{2.1}$$

The idea is to examine the norm of the difference $Ku^p - K_0 u^p$ in a space that will be identified by passing to the dual space:

$$-\Delta u = Ku^p \text{ in } H^{-1}(\mathbb{R}^n) \iff \forall \varphi \in D^{1,2}(\mathbb{R}^n), \int_{\mathbb{R}^n} \nabla u \cdot \nabla \varphi = \int_{\mathbb{R}^n} K u^p \varphi$$

where $H^{-1}(\mathbb{R}^n) = (D^{1,2}(\mathbb{R}^n))^*$.

However, the Sobolev embedding $D^{1,2}(\mathbb{R}^n) \longrightarrow L^{2^*}(\mathbb{R}^n)$ ensures thqt the right-hand side integral is defined only if $Ku^p \in L^{\frac{2n}{(n+2)}}$, which corresponds to the dual of L^{2^*}. (One can see it by applying Hölder's inequality). This allows us to define the following quantity, which in a certain sense, measures the distance of K to a constant:

$$\delta(u) := \|-\Delta u - K_0 u^p\|_{L^{\frac{2n}{(n+2)}}} = \left(\int_{\mathbb{R}^n} \left| \frac{\Delta u}{u^p} + K_0(u) \right|^{\frac{2n}{(n+2)}} u^{2^*} \, dx \right)^{\frac{n+2}{2n}}$$

2.1 Global compactness

Let $E : D^{1,2}(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u|^2 - \frac{1}{2^*} \int_{\mathbb{R}^n} u^{2^*}$$

Consider the prescribed scalar curvature equation with $K \equiv 1$,

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^n \tag{2.2}$$

Recall that u weakly solves (2.2) $\iff \langle DE(u), \varphi \rangle = 0, \ \forall \varphi \in C_c^\infty(\mathbb{R}^n)$, where DE denotes the Fréchet derivative of E.

Lemma 3.11

Any sequence (u_m) in $D^{1,2}(\mathbb{R}^n)$ that satisfies

$$E(u_m) \to \beta < \frac{1}{n} S^n \quad \text{and} \quad DE(u_m) \to 0 \text{ in } H^{-1}(\mathbb{R}^n) \tag{2.3}$$

as $m \to \infty$ is strongly relatively compact.

Proof. First, we show boundedness of (u_m) in $D^{1,2}(\mathbb{R}^n)$. Compute

$$\langle DE(u_m), u_m\rangle = \|u_m\|^2_{D^{1,2}} - \int_{\mathbb{R}^n} u_m^{2^*}$$

Moreover,

$$\int_{\mathbb{R}^n} u_m^{2^*} = \frac{2^*}{2}\|u_m\|^2_{D^{1,2}} - 2^* E(u_m)$$

So,

$$\begin{aligned}\|u_m\|^2_{D^{1,2}} &= \left(1-\frac{2^*}{2}\right)^{-1}\langle DE(u_m), u_m\rangle - 2^*\left(1-\frac{2^*}{2}\right)^{-1} E(u_m)\\ &= c\,[E(u_m) - \frac{1}{2^*}\langle DE(u_m), u_m\rangle] \qquad \text{where } c = \left(\frac{1}{2}-\frac{1}{2^*}\right)^{-1} = n.\end{aligned}$$

On the other hand, using Young's inequality, for all $\varepsilon > 0$, we have

$$\begin{aligned}|\langle DE(u_m), u_m\rangle_{H^{-1},D^{1,2}}| &\le \|DE(u_m)\|_{H^{-1}}\|u_m\|_{D^{1,2}(\mathbb{R}^n)}\\ &\le \frac{1}{2\varepsilon}\|DE(u_m)\|^2_{H^{-1}} + \frac{\varepsilon}{2}\|u_m\|^2_{D^{1,2}(\mathbb{R}^n)}\end{aligned}$$

One can choose $\varepsilon > 0$ such that this inequality holds:

$$\|u_m\|^2_{D^{1,2}} \le cE(u_m) + c'\|DE(u_m)\|^2_{H^{-1}} + \frac{1}{2}\|u_m\|^2_{D^{1,2}}, \qquad c' > 0.$$

Finally,

$$\frac{1}{2}\|u_m\|^2_{D^{1,2}} \le cE(u_m) + c'\|DE(u_m)\|^2_{H^{-1}}$$

Since the first term of the r.h.s. is bounded, and the second term converges to 0 (hence bounded), this implies the boundedness of (u_m). Therefore, we may assume that $u_m \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^n)$.

In addition, for all radii $R > 0$ the injection $D^{1,2}(B(0,R)) \longrightarrow L^2(B(0,R))$ is compact, this implies strong convergence of a subsequence in $L^p(B(0,R))$, for all $p < 2^*$, thus pointwise convergence a.e. in $B(0,R)$, for all $R > 0$. Therefore, we have a.e. convergence in $\mathbb{R}^n$.
Thus, u solves weakly (2.2). In fact, using weak convergence in $D^{1,2}(\mathbb{R}^n)$, and the fact that pointwise convergence implies convergence in the distribution sense, we have that $\forall\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle DE(u_m), \varphi\rangle = \int_{\mathbb{R}^n}(\nabla u_m\cdot\nabla\varphi - u_m^p\varphi) \to \int_{\mathbb{R}^n}(\nabla u\cdot\nabla\varphi - u^p\varphi) = \langle DE(u),\varphi\rangle = 0, \quad \text{by (2.3).}$$

Now by density,

$$\langle DE(u), u\rangle = 0 = \int_{\mathbb{R}^n}|\nabla u|^2 - \int_{\mathbb{R}^n} u^{2^*}$$

Hence,

$$E(u) = \left(\frac{1}{2}-\frac{1}{2^*}\right)\int_{\mathbb{R}^n} u^{2^*} \ge 0. \tag{2.4}$$

Furthermore,

$$\int_{\mathbb{R}^n} |\nabla u_m|^2 = \int_{\mathbb{R}^n} |\nabla(u_m - u + u)|^2 = \int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 \int_{\mathbb{R}^n} |\nabla u|^2 + \underbrace{2\int_{\mathbb{R}^n} \nabla(u_m - u) \cdot \nabla u}_{= \mathrm{o}(1)}$$

And by Brezis-Lieb lemma,

$$\int_{\mathbb{R}^n} u_m^{2^*} = \int_{\mathbb{R}^n} |u_m - u|^{2^*} + \int_{\mathbb{R}^n} u^{2^*} + o(1)$$

So,

$$E(u_m) = E(u) + E(u_m - u) + o(1)$$

By boundedness of (u_m), and (2.3),

$$\begin{aligned} o(1) = \langle DE(u_m), u_m - u\rangle &= \langle DE(u_m) - DE(u), u_m - u\rangle \\ &= \int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 - (u_m - u)(u_m^p - u^p) \\ &= \int_{\mathbb{R}^n} (|\nabla(u_m - u)|^2 - u_m^{2^*} - u^{2^*} + u_m u^p + u u_m^p) \\ &= \int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 - u_m^{2^*} - u^{2^*} + u^{2^*} + u^{2^*} + o(1) \\ &= \int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 - u_m^{2^*} + u^{2^*} + o(1) \\ &= \int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 - |u_m - u|^{2^*} + o(1) \end{aligned}$$

It follows that

$$E(u_m - u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u_m - u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^n} |u_m - u|^{2^*} = \frac{1}{n}\int_{\mathbb{R}^n} |\nabla(u_m - u)|^2 + o(1)$$

On the other hand,

$$\begin{aligned} E(u_m - u) &= E(u_m) - E(u) + o(1) \\ &\leq E(u_m) + o(1) \leq k < \frac{1}{n}S^n, \quad \text{for } m \text{ large enough.} \quad \text{(by (2.3), (2.4))} \end{aligned}$$

Hence,

$$\|u_m - u\|^2_{D^{1,2}} \leq k' < S^n, \quad \text{for } m \text{ large enough.}$$

Now by definition (1.4) of the Sobolev constant S, the Sobolev inequality $\|u_m - u\|^2_{D^{1,2}} \geq S^2 \|u_m - u\|^2_{L^{2^*}}$, which implies that $-S^{2^*} \|u_m - u\|^{2^*}_{D^{1,2}} \leq - \|u_m - u\|^{2^*}_{L^{2^*}}$ yields

$$\|u_m - u\|^2_{D^{1,2}} (1 - S^{2^*} \|u_m - u\|^{2^* - 2}_{D^{1,2}}) \leq \int_{\mathbb{R}^n} |\nabla u_m - u|^2 - \int_{\mathbb{R}^n} |u_m - u|^{2^*} = o(1).$$

By elevating to the power of $\frac{2}{n-2}$ in (31), we conclude that $1 - S^{2^*} \|u_m - u\|^{2^*-2}_{D^{1,2}} > 0$, and hence, that $u_m \to u$ strongly in $D^{1,2}(\mathbb{R}^n)$. $\square$

Remark 2.12

Contrary to the case of a bounded domain $\Omega \subset \mathbb{R}^n$, on the whole space $\mathbb{R}^n$, the limit of (u_m) must be 0:
Under the same assumptions of lemma, with $\Omega = \mathbb{R}^n$, suppose that $u_m \to u^0$ in $D^{1,2}(\mathbb{R}^n)$. Then, in particular, u^0 is a solution of (7) on $\mathbb{R}^n$ and is classified as a bubble.

On the other hand, $E(u_m) \to E(u^0) < \frac{1}{n}S^n < S^n$ = energy of one bubble.
Thus, u^0 must be identically 0.

Prior to stating the next theorem, we consider the following lemma:

Lemma 2.13

Let $f \in L^1(\mathbb{R}^n)$ be a positive function. Then, the following properties hold:

1. For all $r > 0$, the function $F_r(x) : \mathbb{R}^n \longrightarrow [0, +\infty[$ defined by

$$F_r(x) = \int_{B(x,r)} f \, dx$$

is continuous, and $\lim\limits_{\|x\| \to +\infty} F_r(x) = 0$.
Hence, F_r achieves its maximum.

2. Consider the function $Q : [0, +\infty[\longrightarrow [0, +\infty[$ defined by

$$Q(r) = \sup_{x \in \mathbb{R}^n} F_r(x) = \max_{x \in \mathbb{R}^n} F_r(x)$$

Then, Q is continuous. Furthermore, $Q(0) = 0, \quad Q(+\infty) = \int_{\mathbb{R}^n} f \, dx$.

Proof.

1. Let $r > 0$, and $(x_k)_k$ be a sequence such that $x_k \to x_* \in \mathbb{R}^n$. Since $f \in L^1$ and $|f \cdot \mathbb{1}_{B(x_k,r)}| = f \cdot \mathbb{1}_{B(x_k,r)} \leq f$, then, by the dominated convergence theorem,

$$\lim_{x \to x_*} \int_{B(x_k,r)} f \, dx = \int_{B(x_*,r)} f \, dx$$

which proves the continuity of F_r.
On the other hand, $\mathbb{1}_{B(x,r)} \to 0$ a.e as $|x| \to +\infty$. So again, by the dominated convergence theorem,

$$\lim_{\|x\| \to +\infty} F_r(x) = \lim_{\|x\| \to +\infty} \int_{B(x,r)} f \, dx = \int_{\mathbb{R}^n} 0 \, dx = 0.$$

Now, take any $\varepsilon > 0$, such that $\varepsilon < F_r(0)$. Since F_r is continuous and vanishes at infinity, there exists $\delta > 0$ where

$$\|x\| \geq \delta \implies F_r(x) \leq \varepsilon.$$

Hence,

$$\sup_{\|x\|\geq\delta} F_r(x) \leq F_r(0) \quad \leq \sup_{\|x\|\leq\delta} F_r(x)$$

We deduce that the supremum is reached on the ball $B(0,\delta)$. That is,

$$\sup_{x\in\mathbb{R}^n} F_r(x) = \sup_{\|x\|\leq\delta} F_r(x)$$

In addition, the function F_r is continous on the compact set $B(0,\delta)$. Therefore, the maximum is reached on $B(0,\delta)$. Which tells us that that maximum is then reached on $\mathbb{R}^n$, by the last equality.

2. This proof could be approached using various methods. Here, we follow the one presented in ([2] , *Lemma 3.1*) Since, for every $x \in \mathbb{R}^n$, the function

$$r \longrightarrow \int_{B(x,r)} f\,dx$$

is continuous, then, Q is lower semicontinuous[2], as a supremum of continuous functions. In addition, Q is also a non-decreasing function.
Now, suppose by contradiction that there exists $r_0 > 0$, such that

$$l_- := \lim_{r\to r_0^-} Q(r) \neq \lim_{r\to r_0^+} Q(r) := l_+$$

Since Q is lower semi-continuous, $l_- = Q(r_0)$, and $l_- < l_+$ by monotonicity.
Set $\varepsilon = l_+ - Q(r_0)$. Then, for all $r > r_0$, we have that $\varepsilon \leq Q(r) - Q(r_0)$. Moreover, by the property of the superior bound[3], there exists $x_0 \in \mathbb{R}^n$, depending on ε, r, such that

$$\varepsilon \leq \int_{B(x_0,r)} f\,dx - \int_{B(x_0,r_0)} f\,dx + \frac{\varepsilon}{2}$$

Hence,

$$\frac{\varepsilon}{2} \leq \int_{B(x_0,r)} f\,dx - \int_{B(x_0,r_0)} f\,dx = \int_{B(x_0,r)\setminus B(x_0,r_0)} f\,dx$$

Hence, we deduce that the latter term tends to 0 as $r \to r_0$, by the dominated convergence theorem. Which gives the desired contradiction. □

Now, we introduce *Palais-Smale* sequences with the following definition:

Definition

Let V be a Banach space, $E \in C^1(V)$, (u_m) be a sequence in V, and $\beta \in \mathbb{R}$.
We say that (u_m) is a *Palais-Smale* sequence relative to E, with value β, if

$$E(u_m) \to \beta \qquad \text{and} \qquad DE(u_m) \to 0 \qquad \text{as } m \to +\infty,$$

[2] A function f defined on a topological space X that takes values in $\mathbb{R}$ is said to be *lower semicontinuous* if and only if, for all $c \in \mathbb{R}$, the set $\{f > c\}$ is open.

[3] Before using the property of the supremum bound, we notice that if A and B are two arbitrary sets, then $\sup(A) - \sup(B) = \sup(A) + \inf(-B) \leq \sup(A - B)$

The following result, due to Struwe [6], demonstrates the lack of compactness on $\mathbb{R}^n$, and the emerging of a "bubbling phenomenon". In particular, whenever $\delta(u)$ is small, u is close to a sum of bubbles:

Theorem 2.14

Let (u_m) be a sequence in $D^{1,2}(\mathbb{R}^n)$ such that $E(u_m) \leq c$ and

$$\|DE(u_m)\|_{H^{-1}(\mathbb{R}^n)} \to 0$$

Then there exists a number $k \in \mathbb{N}$, sequences (R_m^j), (x_m^j), $1 \leq j \leq k$, of radii $R_m^j \to \infty$ as $m \to \infty$, and points $x_m^j \in \mathbb{R}^n$, solutions $u^j \in D^{1,2}(\mathbb{R}^n)$, $0 \leq j \leq k$, to (7), such that a subsequence of (u_m) satisfies

$$\left\| u_m - u^0 - \sum_{j=1}^{k} u_m^j \right\|_{D^{1,2}(\mathbb{R}^n)} \to 0$$

Here u_m^j denotes the rescaled function

$$u_m^j(x) = (R_m^j)^{\frac{(n-2)}{2}} u^j(R_m^j(x - x_m^j)), \ \ 1 \leq j \leq k.$$

Furthermore,

$$\|u_m\|^2_{D^{1,2}(\mathbb{R}^n)} \to \sum_{j=0}^{k} \left\|u^j\right\|^2_{D^{1,2}(\mathbb{R}^n)} \qquad \text{as } m \to \infty$$

and

$$E(u_m) \to \sum_{j=0}^{k} E(u^j) \qquad \text{as } m \to \infty$$

Proof of theorem. By the assumptions on (u_m), we proceed in a similar manner as in the previous lemma to prove that (u_m) is bounded in $D^{1,2}(\mathbb{R}^n)$. Therefore, there exists $u^0 \in D^{1,2}(\mathbb{R}^n)$, such that $u_m \rightharpoonup u^0$ in $D^{1,2}(\mathbb{R}^n)$. The same arguments provided in the proof of the previous lemma show that u^0 weakly solves (7), and

$$\int_{\mathbb{R}^n} |u_m - u|^{2^*} = \int_{\mathbb{R}^n} |u_m|^{2^*} - \int_{\mathbb{R}^n} |u^0|^{2^*} + o(1)$$
$$\int_{\mathbb{R}^n} |\nabla(u_m - u^0)|^2 = \int_{\mathbb{R}^n} |\nabla u_m|^2 + \int_{\mathbb{R}^n} |\nabla|u^0|^2 + o(1)$$

So,

$$E(u_m - u^0) = E(u_m) - E(u^0) + +o(1)$$

Furthermore, in $H^{-1}(\mathbb{R}^n)$, it is true that

$$DE(u_m - u^0) = DE(u_m) - DE(u^0) + +o(1)$$

where $o(1) \to 0$ in $H^{-1}(\mathbb{R}^n)$ as $m \to \infty$.

The following fundamental lemma illustrates the process of extracting bubbles upon having a weakly convergent subsequence satisfying the *Palais-Smale condition.* Applied each time for a different sequence, it allows us by induction to conclude the rest of the proof.

Lemma 2.15

Let (v_m) be a sequence in $D^{1,2}(\mathbb{R}^n)$ such that $v_m \rightharpoonup 0$ and

$$E(v_m) \to \beta \quad \text{ and } \quad \|DE(v_m)\|_{H^{-1}(\mathbb{R}^n)} \to 0$$

Then, there exists a sequence (x_m) of points in $\mathbb{R}^n$, a sequence (R_m) of radii $R_m \to \infty$, a non trivial solution v^0 to (7), and a sequence $(w_m) \in D^{1,2}(\mathbb{R}^n)$ satisfying the *Palais-Smale condition*, such that for a subsequence of (v_m),

$$w_m = v_m - R_m^{\frac{n-2}{2}} v^0(R_m(\,\cdot\, - x_m)) + o(1)$$

where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$ as $m \to \infty$. In particular, $w_n \rightharpoonup 0$.
Furthermore,

$$E(w_m) = E(v_m) - E(v^0) + o(1).$$

Proof of theorem (completed). Let $v_m^1 = u_m - u^0$. We know that $v_m^1 \rightharpoonup 0$ and by (43) - (44),

$$E(v_m^1) = E(u_m) - E(u^0) + o(1)$$
$$DE(v_m^1) = DE(u_m) - DE(u^0) + + o(1) = DE(u_m) + o(1).$$

This tells us that (v_m^1) satisfies the Palais-Smale condition. Now, apply lemma to this sequence (v_m^1) to extract a first bubble $u^1 = v_1^0$, whose rescaled sequence $u_m^1 = (R_m^1)^{\frac{n-2}{2}} v_1^0(R_m^1(\,\cdot\, - x_m^1))$.
Consider the new quantity $v_m^2 = v_m^1 - u_m^1$. . Notice that by (47), the process of extracting u_m^1 has consumed some energy.
However, lemma tells us that this sequence converges weakly to 0 and satisfies the Palais-Smale condition. This allows us to extract a bubble $u^2 = v_2^0$, whose rescaled sequence $u_m^2 = (R_m^2)^{\frac{n-2}{2}} v_2^0(R_m^2(\,\cdot\, - x_m^2))$.
We repeat this process for all sequences $v_m^j = v_m^1 - u_m^1 - u_m^2 - \ldots - u_m^{j-1}$, $j > 1$, where

$$u_m^i(x) = (R_m^i)^{\frac{n-2}{2}} u^i(R_m^i(\,\cdot\, - x_m^i))$$

and $u^i = v_i^0$.

Using the same convergence arguments, we prove by induction that

$$E(v_m^j) = E(u_m) - E(u^0) - \sum_{i=1}^{j-1} E(u^i) + o(1) \le E(v_m^1) - (j-1)S^n + o(1).$$

The latter will be negative. So, after a finite number of iterations k, we are left with a small amount of energy that is less than the energy of one bubble. Intuitively, this means that the process must stop after a finite number of iterations k, because we will run out of energy to extract a new bubble. Thus, by previous lemma, for this index k, we have

$$v_m^{k+1} = v_m^1 - u_m^1 - u_m^2 - \ldots - u_m^k \to 0$$

strongly in $D^{1,2}(\mathbb{R}^n)$ and

$$E(v_m^j) \quad = E(u_m) - E(u^0) - \textstyle\sum_{i=1}^k E(u^i)$$

This completes the proof of theorem. $\square$

<u>*Proof of lemma.*</u> First, since $v_m \rightharpoonup 0$, we may assume that $\beta \geq \frac{1}{n}S^n$. Otherwise, by previous lemma, $v_m \to 0$ and $\beta = 0$. In addition, since $DE(v_m) \to 0$. by boundedness of (v_m), we have that

$$\int_{\mathbb{R}^n} |\nabla v_m|^2 = nE(v_m) - \frac{n}{2^*}\langle DE(v_m), v_m\rangle \to n\beta \geq S^n$$

Hence,

$$\liminf_{m\to\infty} \int_{\mathbb{R}^n} |\nabla v_m|^2 \geq S^n$$

Denote

$$Q_m(r) = \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} |\nabla v_m|^2\,dx$$

the concentration function of v_m. Choose $x_m \in \mathbb{R}^n$, $R_m \geq 0$ such that the rescaled sequence

$$x \longrightarrow \tilde{v}_m(x) = R_m^{\frac{2-n}{2}} v_m(x/R_m + x_m)$$

satisfies

$$\tilde{Q}_m(1) = \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |\nabla \tilde{v}_m|^2\,dx = \int_{B(0,1)} |\nabla \tilde{v}_m|^2\,dx = \frac{1}{2L}S^n \tag{2.5}$$

where L is a number of balls $B(0,1)$ needed to cover $B(0,2)$.

For all $m \in \mathbb{N}$, this choice of (x_m, R_m) is indeed possible. In fact,

$$Q_m(0) = 0 \quad \text{and} \quad \lim_{r\to+\infty} Q_m(r) = \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla v_m|^2\,dx \geq S^n$$

By *Lemma 2.13*, Q_m is continuous. Hence, the mean value theorem tells us that there exists $R_m > 0$, such that $Q_m(R_m) = \frac{1}{2L}S^n$.

Furthermore, since $|\nabla v_m|^2 \in L^1$, then, by *Lemma 2.13*, the function $x \to \int_{B(x,R_m)} |\nabla v_m|^2\,dx$ reaches its supremum. Choose x_m such that

$$\int_{B(x_m,R_m)} |\nabla v_m|^2\,dx = \sup_{x\in\mathbb{R}^n} \int_{B(x,R_m)} |\nabla v_m|^2\,dx$$

In addition, notice that

$$\begin{aligned}\tilde{Q}_m(1) = \sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |\nabla \tilde{v}_m|^2\,dx = \sup_{x\in\mathbb{R}^n} \int_{B(x,R_m)} |\nabla v_m|^2\,dx &= \int_{B(x_m,R_m)} |\nabla v_m|^2\,dx = \frac{1}{2L}S^n \\ &= \int_{B(0,1)} |\nabla \tilde{v}_m|^2\,dx\end{aligned}$$

This justifies the choice of (x_m, R_m).

Since

$$\|\tilde{v}_m\|^2_{D^{1,2}(\mathbb{R}^n)} = \|v_m\|^2_{D^{1,2}(\mathbb{R}^n)} \to n\beta < \infty,$$

we may assume that $\tilde{v}_m \rightharpoonup v^0$ in $D^{1,2}(\mathbb{R}^n)$. By density of $C_c^\infty(\mathbb{R}^n)$ in $D^{1,2}(\mathbb{R}^n)$, there exists a sequence $(\tilde{v}_m^0) \subset H^1(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$, such that $v^0 = \lim_{m\to\infty} \tilde{v}_m^0$ in $D^{1,2}(\mathbb{R}^n)$.

Then, we prove that $\tilde{v}_m \to v^0$ strongly in $H^1(B(a,1))$, for any $a \in \mathbb{R}^n$.
For this, let $a \in \mathbb{R}^n$. We choose two cut-off functions $\varphi_1, \varphi_2 \in C_c^\infty(\mathbb{R}^n)$, such that $0 \le \varphi_i \le 1$ for $i = 1, 2$, and

$$\begin{cases} \varphi_1 = 1 \text{ in } B(a,1) \\ \varphi_1 = 0 \text{ outside } B(a,\frac{3}{2}) \end{cases} \qquad \begin{cases} \varphi_2 = 1 \text{ in } B(a,\frac{3}{2}) \\ \varphi_2 = 0 \text{ outside } B(a,2) \end{cases}$$

Now, let $\tilde{w}_m^i = (\tilde{v}_m - \tilde{v}_m^0)\,\varphi_i \in H^1(\mathbb{R}^n)$. Clearly, since $\varphi_i \in C_c^\infty(\mathbb{R}^n)$, $\nabla\varphi_i \in C_c^\infty(\mathbb{R}^n)$, we have

$$\left\|\tilde{w}_m^i\right\|_{H^1(\mathbb{R}^n)} \le c\left\|\tilde{v}_m - \tilde{v}_m^0\right\|_{H^1(\mathbb{R}^n)} < \infty.$$

Furthermore, $\tilde{w}_m^1$ is supported in $B(a, 3/2)$ and $\tilde{w}_m^2$ is supported in $B(a, 2)$.

Since $(\tilde{v}_m - \tilde{v}_m^0) \rightharpoonup 0$ in $D^{1,2}(\mathbb{R}^n)$, the restriction of these functions to $B(a,2)$ converges weakly to 0 in $H^1(B(a,2))$. In other words, $(\tilde{v}_m - \tilde{v}_m^0)_{|B(a,2)} \rightharpoonup 0$ in $H^1(B(a,2))$. Hence, $(\tilde{v}_m - \tilde{v}_m^0)_{|B(a,2)} \to 0$ in $L^p(B(a,2))$.

On the other hand, we repeat the same convergence arguments and use the Sobolev's inequality to establish

$$\begin{aligned} o(1) = \langle DE(\tilde{v}_m), \tilde{w}_m^1\varphi_1\rangle &= \int_{\mathbb{R}^n} (\nabla\tilde{v}_m \cdot \nabla\tilde{w}_m^1 - \tilde{v}_m^p\tilde{w}_m^1)\varphi_1 + o(1) \\ &= \int_{\mathbb{R}^n} (|\nabla(\tilde{v}_m - \tilde{v}_m^0)|^2 - |\tilde{v}_m - \tilde{v}_m^0|^{2^*})\varphi_1^2 + o(1) \\ &= \int_{\mathbb{R}^n} (|\nabla(\tilde{v}_m - \tilde{v}_m^0)|^2\varphi_1^2 - |\tilde{v}_m - \tilde{v}_m^0|^{2^*}\varphi_1^2\varphi_2^{2^*-2}) + o(1) \\ &= \int_{\mathbb{R}^n} (|\nabla\tilde{w}_m^1|^2 - |\tilde{w}_m^1|^2|\tilde{w}_m^2|^{2^*-2}) + o(1) \\ &\ge \left\|\tilde{w}_m^1\right\|^2_{D^{1,2}}\left(1 - S^{2^*}\left\|\tilde{w}_m^2\right\|^{2^*-2}_{D^{1,2}}\right) + o(1) \end{aligned}$$

Note that

$$\int_{\mathbb{R}^n} |\nabla\tilde{w}_m^2|^2 = \int_{\mathbb{R}^n} |\nabla(\tilde{v}_m - \tilde{v}_m^0)|^2\varphi_2^2 + o(1) \tag{2.6}$$

$$= \int_{\mathbb{R}^n} (|\nabla\tilde{v}_m|^2 - |\nabla v^0)|^2)\varphi_2^2 + o(1) \tag{2.7}$$

$$\le \int_{B(a,2)} |\nabla\tilde{v}_m|^2 + o(1) \ \le L\tilde{Q}_m(1) + o(1) \ = \frac{1}{2}S^n + o(1) \tag{2.8}$$

By elevating to the power of $\frac{2}{n-2}$ on the left and right-hand sides of the inequality, we conclude that $S^{2^*}\left\|\tilde{w}_m^2\right\|^{2^*-2}_{D^{1,2}} < 1$, for m sufficiently large. Then, from (2.8), we deduce that $\tilde{w}_m^1 \to 0$ in $D^{1,2}(\mathbb{R}^n)$. Thus, we have proved that $\tilde{v}_m \to v^0$ strongly locally $H^1(\mathbb{R}^n)$. By taking the limit in (2.5),

$$\int_{\mathbb{R}^n} |\nabla v^0|^2 = \frac{1}{2L}S^n > 0 \tag{2.9}$$

and $v^0 \not\equiv 0$. Hence, by definition of $\tilde{v}_m$, and the fact that $\tilde{v}_m \rightharpoonup v^0$ in $D^{1,2}(\mathbb{R}^n)$, in order for v_m to converge weakly to 0, we must have that $R_m \to \infty$.

Moreover, since we have local convergence of $\tilde{v}_m$ to v^0, in particular $\tilde{v}_m$ converges to v^0 on any compact in $\mathbb{R}^n$. This implies that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle DE(v^0), \varphi\rangle = \lim_{m\to\infty} \langle DE(\tilde{v}_m), \varphi\rangle \leq C \lim_{m\to\infty} \|DE(\tilde{v}_m)\|_{H^{-1}} = 0$$

Hence, v^0 weakly solves (2.2).
Lastly, define the sequence $(\overline{R}_m)$ such that $\tilde{R}_m := R_m/\overline{R}_m \to \infty$. Again, let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ in $B(0,1)$, $\varphi \equiv 1$ outside $B(0,2)$, and take

$$w_m(x) = v_m(x) - R_m^{\frac{n-2}{2}} v^0(R_m(x - x_m)) \cdot \varphi(\overline{R}_m(x - x_m)) \in D^{1,2}(\mathbb{R}^n).$$

Then by rescaling,

$$\tilde{w}_m(x) = R_m^{\frac{2-n}{2}} w_m(x) = \tilde{v}_m(x) - v^0(x)\varphi(x/\tilde{R}_m)$$

We claim that

$$\tilde{w}_m = \tilde{v}_m - v^0 + o(1) \tag{2.10}$$

where $o(1) \to 0$ in $D^{1,2}(\mathbb{R}^n)$.

By setting $\varphi_m(x) = \varphi(x/\tilde{R}_m)$, we have for all $\varepsilon > 0$, by Young's inequality, then Cacciopoli's inequality,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla(\tilde{w}_m - \tilde{v}_m + v^0)|^2 &= \int_{\mathbb{R}^n} |\nabla(v^0(\varphi_m - 1)|^2 \\
&= \int_{\mathbb{R}^n} |\nabla v^0|^2(\varphi_m - 1)^2 + \int_{\mathbb{R}^n} |\nabla(\varphi_m - 1)|^2|v^0|^2 + 2\int_{\mathbb{R}^n} \langle(\varphi_m - 1)\nabla v^0, v^0\nabla(\varphi_m - 1)\rangle \\
&\leq (1 + \frac{1}{\varepsilon}) \int_{\mathbb{R}^n} |\nabla v^0|^2(\varphi_m - 1)^2 + (1 + \varepsilon) \int_{\mathbb{R}^n} |\nabla(\varphi_m - 1)|^2|v^0|^2 \\
&\leq (1 + \frac{1}{\varepsilon}) \int_{\mathbb{R}^n\setminus B(0,\tilde{R}_m)} |\nabla v^0|^2 \; + \; (1 + \varepsilon)\, \frac{C_{cacc}}{\tilde{R}_m^2} \int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^2
\end{aligned} \tag{2.11}$$

However, since $\tilde{R}_m \to \infty$ the first must tend to 0, as $\nabla v^0 \in L^2$ by (2.9). Again, by Hölder's inequality, for the second term,

$$\begin{aligned}
\frac{1}{\tilde{R}_m^2} \int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^2 &\leq \frac{|B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)|^{2/n}}{\tilde{R}_m^2} \left(\int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^{2^*} \right)^{2/2^*} \\
&= C(n)\, \frac{(\,(2\tilde{R}_m)^n - (\tilde{R}_m)^n\,)^{2/n}}{\tilde{R}_m^2} \left(\int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^{2^*} \right)^{2/2^*} \\
&\leq 4\, C(n)(\tilde{R}_m)^{2-n} \left(\int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^{2^*} \right)^{2/2^*} \leq C \left(\int_{B(0,2\tilde{R}_m)\setminus B(0,\tilde{R}_m)} |v^0|^{2^*} \right)^{2/2^*}
\end{aligned}$$

where $C(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$.

But the last term of the inequality tends to 0 as $m \to \infty$, since $v^0 \in D^{1,2}(\mathbb{R}^n)$, and satisfies the L^{2^*} decay at infinity. Hence, both terms in (2.11) tend to 0. This is exactly what we wanted.

By invariance of the norms $D^{1,2}(\mathbb{R}^n)$, $L^{2^*}(\mathbb{R}^n)$ under translation and scaling, by repeating the same process as in the previous lemma we obtain

$$E(w_m) = E(\tilde{w}_m) = E(\tilde{v}_m) - E(v^0) + o(1) = E(v_m) - E(v^0) + o(1)$$

To sum up, again using (2.10), for all test functions $\psi \in D^{1,2}(\mathbb{R}^n)$ with $\|\psi\|_{D^{1,2}(\mathbb{R})^n} \leq 1$,

$$\begin{aligned}
\langle DE(\tilde{w}_m), \psi\rangle &= \int_{\mathbb{R}^n} \nabla \tilde{w}_m \cdot \nabla\psi - \tilde{w}_m|\tilde{w}_m|^{2^*-2}\psi \\
&= \int_{\mathbb{R}^n} (\nabla \tilde{v}_m \cdot \nabla\psi - \tilde{v}_m|\tilde{v}_m|^{2^*-2}\psi) - \int_{\mathbb{R}^n} (\nabla v^0 \cdot \nabla\psi - v^0|v^0|^{2^*-2}\psi) + o(1) \\
&= \langle DE(\tilde{v}_m), \psi\rangle - \langle DE(v^0), \psi\rangle + o(1)
\end{aligned}$$

Hence, by invariance of the norms $D^{1,2}(\mathbb{R}^n)$, $L^{2^*}(\mathbb{R}^n)$ under translation and scaling, compute

$$\begin{aligned}
\|DE(w_m)\|_{H^{-1}} &= \sup_{\substack{\psi\in D^{1,2}(\mathbb{R}^n) \\ \|\psi\|_{D^{1,2}}\leq 1}} |\langle DE(w_m), \psi\rangle| = \sup_{\substack{\psi\in D^{1,2}(\mathbb{R}^n) \\ \|\psi\|_{D^{1,2}}\leq 1}} |\langle DE(\tilde{w}_m), \psi\rangle| \\
&\leq |\langle DE(\tilde{v}_m), \psi\rangle| + |\langle DE(v^0), \psi\rangle| + o(1) \\
&\leq \|DE(v_m)\|_{H^{-1}} + \left\|DE(v^0)\right\|_{H^{-1}} + o(1) \\
&= \|DE(v_m)\|_{H^{-1}} + o(1) \to 0.
\end{aligned}$$

Therefore, the sequence (w_m) satisfies the Palais-Smale condition. This concludes the proof of lemma. □

Remark 2.16

Let $u \in H^1(\mathbb{R}^n)$ be a positive function on $\mathbb{R}^n$. Then, there exists $c > 0$, such that

$$\|DE(u)\|_{H^{-1}} \leq c\,\delta(u).$$

In fact, notice that

$$\delta(u) = \|-\Delta u - u^p\|_{L^{\frac{2n}{(n+2)}}} = \|DE(u)\|_{L^{\frac{2n}{(n+2)}}} = \|DE(u)\|_{(L^{2^*})^*}$$

However, the we have the following equivalence of continuous embeddings:

$$D^{1,2}(\mathbb{R}^n) \subset L^{2^*}(\mathbb{R}^n) \iff (L^{2^*}(\mathbb{R}^n))^* \subset (D^{1,2}(\mathbb{R}^n))^* = H^{-1}(\mathbb{R}^n).$$

Thus, there exists $c > 0$, such that

$$\|DE(u)\|_{H^{-1}} \leq c\,\|DE(u)\|_{(L^{2^*})^*} = c\,\delta(u)$$

3 Main result

We are now in position to state and prove our main theorem presented in the article of Ciraolo-Figalli-Maggi [3], asserting that in any dimension, for a Struwe decomposition of u with one bubble only $v_1[z, \lambda]$, whenever $\delta(u)$ is small, u is **quantitavely** close to $v_1[z, \lambda]$.

Let us briefly describe the structure of the proof: First, we show in *claim 3.2*, that for any ε_0, the $D^{1,2}$ - distance of u to any element of the manifold of bubbles is smaller than ε_0. Conveniently, we choose the element σ that minimizes the latter $D^{1,2}$ - distance. This will allow us to retrieve the orthogonality relations in (3.5), and then the positivity of a specific quadratic form in (3.6), is deduced by applying the Min-max principle to an operator A.
By then testing the equation $-\Delta u = u^p$ with ρ, and using the first orthogonality relation in (3.5), we have all the necessary tools to prove that $\|\nabla \rho\|_{L^2(\mathbb{R}^n)}$ is controlled by $\delta(u)$, as shown in (3.8).

Theorem 3.1

There exists a constant c_0, depending on n, with the following property:
Let $u \in D^{1,2}(\mathbb{R}^n)$ satisfying

$$K_0(u) = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla u|^2 \leq \frac{3}{2} S^n.$$

Then, there exist $z \in \mathbb{R}^n$, $\lambda \in]0, +\infty[$ such that

$$u = v_1[z, \lambda] + \rho$$

where

$$\|\nabla \rho\|_{L^2(\mathbb{R}^n)} \leq c_0\, \delta(u). \tag{3.1}$$

Proof. Let $u \in D^{1,2}(\mathbb{R}^n)$, such that u satisfies the assumptions above.

Indeed, it is enough to prove the theorem for functions u such that $\delta(u)$ is close to 0, in the sens that $\delta(u) < \delta_0$, where $\delta_0 = \delta_0(n)$ is suitably small constant that guaranties the non-triviality of this theorem. For instance, if $\delta(u) > \delta_0$, by choosing $z = 0$, $\lambda = 1$, setting

$$\rho = u - v_1[0, 1]$$

and c_0 large enough, the theorem is proved.

Claim 3.2

For any constant $\epsilon_0 = \epsilon_0(n) > 0$, one can choose a suitably small constant δ_0, in such a way that there exist $z \in \mathbb{R}^n$, $\lambda \in]0, +\infty[$, and $\alpha \in]0, +\infty[$ such that

$$\|\nabla u - \alpha \nabla v_1[z, \lambda]\, \|_{L^2(\mathbb{R}^n)} \leq \varepsilon_0$$
$$|\alpha - 1| \leq \varepsilon_0$$

Proof. By contradiction, assume that there exist $\varepsilon_0 > 0$, a sequence $(u_m) \subset D^1, 2(\mathbb{R}^n)$, with the following properties:

$$\begin{aligned} &\delta(u_m) \to 0 \quad \text{as } m \to \infty \\ &K_0(u_m) = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla u_m|^2 \leq \frac{3}{2} S^n, \quad \text{for all } m \in \mathbb{N} \end{aligned}$$

such that for all $z \in \mathbb{R}^n$, $\lambda \in]0, +\infty[$, and $\alpha \in]0, +\infty[$,

$$\begin{aligned} \| \nabla u_m - \alpha \nabla v_1[z, \lambda] \|_{L^2(\mathbb{R}^n)} &> \varepsilon_0 \\ |\alpha - 1| &\leq \varepsilon_0 \end{aligned} \tag{3.2}$$

By *Remark 2.16*,

$$\delta(u_m) \to 0 \implies \|DE(u_m)\|_{H^{-1}(\mathbb{R}^n)} \to 0$$

Thus, we deduce from *Theorem 2.14* that there exists a number $k \in \mathbb{N}$, sequences (R_m^j), (x_m^j), $1 \leq j \leq k$, of radii $R_m^j \to \infty$ as $m \to \infty$, and points $x_m^j \in \mathbb{R}^n$, solutions $u^j \in D^{1,2}(\mathbb{R}^n)$, $1 \leq j \leq k$, to (2.2), such that a subsequence of (u_m) satisfies

$$u_m \to u^0 + u_m^1 + \ldots + u_m^k \quad \text{in } D^{1,2}(\mathbb{R}^n).$$

where $u_m^i(x) = (R_m^i)^{\frac{n-2}{2}} u^i(R_m^i(\,\cdot\, - x_m^i))$.

Hence,

$$\left\| \nabla u_m - \sum_{j=0}^{k} \nabla u^j \left(\frac{x - x_m^j}{R_m^j}\right) \right\|_{L^2(\mathbb{R}^n)} \to 0$$

Furthermore,

$$\int_{\mathbb{R}^n} |\nabla u_m|^2 \, dx \to \sum_{j=1}^{k} \int_{\mathbb{R}^n} |\nabla u^j|^2 \, dx = kS^n$$

Since the latter left-hand side is strictly less than the energy of two bubbles, this means that $k \in \{0, 1\}$.

If $k = 1$, only one bubble can exist. For $z_0 \in \mathbb{R}^n, \lambda_0 \in]0, +\infty[$, let $v_1[z_0, \lambda_0]$ be this bubble. Thus,

$$\| \nabla u_m - \nabla v_1[z_0, \lambda_0] \|_{L^2(\mathbb{R}^n)} \to 0 \quad \text{in } L^2$$

But this is a contradiction with (3.2).

If $k = 0$, then

$$\int_{\mathbb{R}^n} |\nabla u_m|^2 \, dx \to 0. \tag{3.3}$$

On the other hand, for all $m \in \mathbb{N}$,

$$K_0(u_m) = \frac{\int_{\mathbb{R}^n} |\nabla u_m|^2}{\int_{\mathbb{R}^n} u_m^{2^*}} = 1$$

And using Sobolev's inequality, we obtain that

$$\|u_m\|_{D^{1,2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\nabla u_m|^2 = \int_{\mathbb{R}^n} u_m^{2^*} \leq S^{-2^*} \|u_m\|_{D^{1,2}(\mathbb{R}^n)}^{2^*}$$

Dividing by $\|u_m\|^2_{D^{1,2}(\mathbb{R}^n)}$, we have

$$\|u_m\|^{\frac{2^*}{2}}_{D^{1,2}(\mathbb{R}^n)} \geq S^{2^*} > 0.$$

which contradicts (3.3).
This concludes the proof of *claim 3.2*.

Now that this claim is proved, we conclude that there exist $z \in \mathbb{R}^n$, $\lambda \in]0,+\infty[$, and $\alpha \in]0,+\infty[$ such that

$$\begin{aligned} \|\nabla u - \alpha \nabla v_1[z,\lambda]\,\|_{L^2(\mathbb{R}^n)} &\leq \varepsilon_0 \\ |\alpha - 1| &\leq \varepsilon_0 \end{aligned}$$

Conveniently, we may assume that the parameters $z \in \mathbb{R}^n$, $\lambda \in]0,+\infty[$, and $\alpha \in]0,+\infty[$ are chosen in such a way that

$$\|\nabla u - \alpha \nabla v_1[z,\lambda]\,\|_{L^2} = \min_{\substack{a,\mu>0 \\ w\in\mathbb{R}^n}} \|\nabla u - a\nabla v_1[w,\mu]\,\|_{L^2} \tag{3.4}$$

Now, let

$$\rho = u - \sigma, \qquad \sigma = \alpha U$$

$$U = v_1[z,\lambda], \qquad V = \left.\frac{\partial v_1[w,\mu]}{\partial \lambda}\right|_{\substack{w=z\\ \mu=\lambda}} \qquad W^j = \left.\frac{\partial v_1[w,\mu]}{\partial w_j}\right|_{\substack{w=z\\ \mu=\lambda}}$$

By our previous choice of $z \in \mathbb{R}^n$ and $\lambda \in]0,+\infty[$, $\alpha \in]0,+\infty[$, we have that σ minimizes the $D^{1,2}$ - distance from u. Hence, ρ is $D^{1,2}$ - orthogonal to the manifold of bubbles. Then,

$$\int_{\mathbb{R}^n} \nabla U \cdot \nabla \rho \, dx = 0 \tag{3.5}$$

In addition, define the function of $n+2$ variables by

$$\varphi = \varphi(a,\mu,w) := u - a v_1[w,\mu] \qquad \forall a,\mu > 0, \text{ and } w = (w_1,\ldots,w_n) \in \mathbb{R}^n$$

In particular, $\varphi(\alpha,\lambda,z) = \rho$, and $\|\nabla\rho\|^2_{L^2}$ is a minimum of $\|\nabla\varphi\|^2_{L^2}$. It follows that (α,λ,z) is a critical point of the functional $\|\nabla\varphi\|^2_{L^2}$. Hence,

$$0 = \frac{\partial}{\partial\mu}(\|\nabla\varphi\|^2_{L^2})\,|_{(\alpha,\lambda,z)} = \int_{\mathbb{R}^n} \frac{\partial}{\partial\mu}(|\nabla\varphi|^2)\,|_{(\alpha,\lambda,z)}\, dx = 2\int_{\mathbb{R}^n} \nabla(\frac{\partial\varphi}{\partial\mu}\,|_{(\alpha,\lambda,z)}) \cdot \nabla\rho\, dx = 2\int_{\mathbb{R}^n} \nabla\rho\cdot\nabla V\, dx$$

Similarly, by differentiating $\|\nabla\varphi\|^2_{L^2}$ with respect to each variable w_j, we deduce that

$$0 = 2\int_{\mathbb{R}^n} \nabla\rho\cdot\nabla W^j\, dx \qquad \forall 1 \leq j \leq n.$$

To sum up, the following $n+2$ orthogonality conditions hold:

$$\int_{\mathbb{R}^n} \nabla U \cdot \nabla\rho\ dx = \int_{\mathbb{R}^n} \nabla\rho\cdot\nabla V\, dx \int_{\mathbb{R}^n} \nabla\rho\cdot\nabla W^j\, dx = 0 \qquad \forall 1 \leq j \leq n. \tag{3.6}$$

Thus, $\rho \in \operatorname{span}(U,V,W)^{\perp}$, where $W = (W^1,\ldots,W^n) = \nabla_w U$.
On the other hand, consider the operator $A = U^{2-2^*}\Delta$ defined on the weighted space $L^2_{U^{2^*-2}}(\mathbb{R}^n)$ endowed with the following norm and scalar product

$$f \mapsto \|f\,\|_{L^2_{U^{2^*-2}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} f^2(x)U^{2^*-2}(x)\, dx\right)^{1/2}$$

$$(f,g) \mapsto \langle f,g\rangle_{L^2_{U^{2^*-2}}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)g(x)U^{2^*-2}(x)\, dx$$

Using *Lemma A1* in [1], it is known that if λ_i, $i = 1, 2, 3, \ldots$ are the eigenvalues of A given in increasing order, then, the first eigenspace of this operator, corresponding the eigenvalue $\lambda_1 = S^{2^*}$ is spanned by U, and the second eigenspace, corresponding to the second eigenvalue $\lambda_2 = (2^* - 1)S^{2^*}$ is spanned by (V, W). In addition, since the embedding $D^{1,2} \to L^2_{U^{2^*-2}}(\mathbb{R}^n)$ is compact, A has a compact resolvent, and hence a purely discrete spectrum. Moreover, since $\rho \in \operatorname{span}(U, V, W)^\perp$, it follows from the Min-max principle that

$$\lambda_3 \leq \frac{\langle A\rho, \rho\rangle_{L^2_{U^{2^*-2}}(\mathbb{R}^n)}}{\|\rho\|^2_{L^2_{U^{2^*-2}}(\mathbb{R}^n)}} = \frac{\int_{\mathbb{R}^n} |\nabla \rho|^2}{\int_{\mathbb{R}^n} U^{2^*-2}\rho^2}$$

In conclusion, there exists $\Lambda := \lambda_3 > p$, such that

$$\int_{\mathbb{R}^n} |\nabla \rho|^2 \geq \Lambda \int_{\mathbb{R}^n} U^{2^*-2}\rho^2 \tag{3.7}$$

We will now take advantage of $\Lambda > p$, in order to prove that $\|\nabla \rho\|_{L^2(\mathbb{R}^n)}$. First we test the equation $-\Delta u = u^p$ with ρ, and use the identity $\int_{\mathbb{R}^n} \nabla U \cdot \nabla \rho \; dx = 0$, to obtain that

$$\int_{\mathbb{R}^n} |\nabla \rho|^2 = \int_{\mathbb{R}^n} |\nabla u - \alpha \nabla U|^2 = \int_{\mathbb{R}^n} K u^p \rho = \int_{\mathbb{R}^n} u^p \rho + \int_{\mathbb{R}^n} (K-1) u^p \rho$$

To control the first term of the right-hand side, we make use of the following inequality:

$$(a+b)^p b + a^p b \leq p a^{p-1} b^2 + c_n (a^{p-2} b^3 + b^{p+1})$$

where c_n is a dimensional constant. Moreover, the inequality holds without the term $a^{p-1}b$ if $n > 6$.
Then, if $3 \leq n \leq 6$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} u^p \rho &= \int_{\mathbb{R}^n} (\sigma + \rho)^p \rho \\
&\leq \int_{\mathbb{R}^n} \sigma^p \rho + p \int_{\mathbb{R}^n} \sigma^{p-1}\rho^2 + c_n \left(\int_{\mathbb{R}^n} \sigma^{p-2}\rho^3 + \int_{\mathbb{R}^n} \rho^{p+1} \right)
\end{aligned}$$

On the other hand, since $\sigma \in L^{2^*}$, we have by the Hölder inequality with exponents $p = \frac{2^*}{2^*-3}$ and $q = \frac{2^*}{3}$, and then the Sobolev inequality, that

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho^{p+1} + \int_{\mathbb{R}^n} \sigma^{p-2}\rho^3 &\leq \int_{\mathbb{R}^n} \rho^{p+1} + \left(\int_{\mathbb{R}^n} \sigma^{2^*} dx \right)^{\frac{2^*-3}{2^*}} \left(\int_{\mathbb{R}^n} \rho^{2^*} dx \right)^{\frac{3}{2^*}} \\
&\lesssim S^{2^*} \left(\int_{\mathbb{R}^n} |\nabla \rho|^2 \, dx \right)^{2^*/2} + \left(\int_{\mathbb{R}^n} |\nabla \rho|^2 \, dx \right)^{3/2}
\end{aligned}$$

And by (3.5),

$$\int_{\mathbb{R}^n} \sigma^{p-1}\rho^2 \leq \frac{\alpha^{2^*-2}}{\Lambda} \int_{\mathbb{R}^n} |\nabla \rho|^2$$

Lastly, using $\sigma^p = \alpha^p U^p = -\alpha^p \Delta U$ and using $\int_{\mathbb{R}^n} \nabla U \cdot \nabla \rho \; dx = 0$, we get

$$\int_{\mathbb{R}^n} \sigma^p \rho = 0$$

Hence, by rearranging all of the inequalities above, and by Hölder inequality with exponents $p = 2^*/(2^*)'$, and $q = (2^*)' = 2n/(n+2)$,

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla\rho|^2 &= \int_{\mathbb{R}^n} u^p\rho + \int_{\mathbb{R}^n} (K-1)u^p\rho \\ &\leq \frac{p\alpha^{2^*-2}}{\Lambda} \|\nabla\rho\|^2_{L^2(\mathbb{R}^n)} + S^{2^*}c_n \|\nabla\rho\|^{2^*}_{L^2(\mathbb{R}^n)} + c_n \|\nabla\rho\|^3_{L^2(\mathbb{R}^n)} + \delta(u)\|\rho\|_{L^{2^*}} \\ &\leq \frac{p\alpha^{2^*-2}}{\Lambda} \|\nabla\rho\|^2_{L^2(\mathbb{R}^n)} + S^{2^*}c_n \|\nabla\rho\|^{2^*}_{L^2(\mathbb{R}^n)} + c_n \|\nabla\rho\|^3_{L^2(\mathbb{R}^n)} + S\delta(u)\|\nabla\rho\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

So,

$$\left(1 - \frac{p\alpha^{2^*-2}}{\Lambda}\right)\int_{\mathbb{R}^n} |\nabla\rho|^2 \leq S^{2^*}c_n \|\nabla\rho\|^{2^*}_{L^2(\mathbb{R}^n)} + c_n \|\nabla\rho\|^3_{L^2(\mathbb{R}^n)} + S\delta(u)\|\nabla\rho\|_{L^2(\mathbb{R}^n)}$$

Since $\Lambda > p$, and $|\alpha - 1| \leq \varepsilon_0$, we now choose ε_0 sufficiently small, such that δ_0 is also sufficiently small, to guarantee that $1 - \frac{p\alpha^{2^*-2}}{\Lambda} \geq c_0 > 0$, for some dimensional constant c_0, and we get

$$c_0 \int_{\mathbb{R}^n} |\nabla\rho|^2 \leq S^{2^*}c_n \|\nabla\rho\|^{2^*}_{L^2(\mathbb{R}^n)} + c_n \|\nabla\rho\|^3_{L^2(\mathbb{R}^n)} + S\delta(u)\|\nabla\rho\|_{L^2(\mathbb{R}^n)}$$

Since $\|\nabla\rho\|^2_{L^2(\mathbb{R}^n)}$ is less than ε_0, we can conclude that

$$\|\nabla\rho\|_{L^2(\mathbb{R}^n)} \leq C\delta(u) \tag{3.8}$$

Now, if $n > 6$, the exponent p is less than 2, and therefore the inequality

$$\int_{\mathbb{R}^n} \sigma^{p-2}\rho^3 \leq \left(\int_{\mathbb{R}^n} \sigma^{2^*}\,dx\right)^{\frac{2^*-3}{2^*}} \left(\int_{\mathbb{R}^n} \rho^{2^*}\,dx\right)^{\frac{3}{2^*}}$$

is false. But as we mentioned, this inequality holds without the term $a^{p-2}b^2$. So the term $\int_{\mathbb{R}^n} \sigma^{p-2}\rho^3$ will not be present. So this result will be true in all dimensions.

We now quantitatively control $|\alpha - 1|$, by our last choice of ϵ_0. Now the objective is to show that $|\alpha - 1| \leq O(\delta^2)$, where $\delta := \delta(u)$.
Recall that our assumption $K_0(u) = 1$ is equivalent to

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} u^{2^*}$$

And since $\|\nabla\rho\|_{L^2(\mathbb{R}^n)} \leq C\delta(u)$, and by orthogonality,

$$\int_{\mathbb{R}^n} |\nabla u|^2 = \int_{\mathbb{R}^n} |\nabla\sigma|^2 + \int_{\mathbb{R}^n} |\nabla\rho|^2 = \alpha^2 S^n + O(\delta^2)$$

In addition, using also $\int_{\mathbb{R}^n} \sigma^p\rho = 0$, we get by using Taylor's expansion, that

$$\int_{\mathbb{R}^n} u^{2^*} = \int_{\mathbb{R}^n} \sigma^{2^*} + p\int_{\mathbb{R}^n} \sigma^p\rho + O(\delta^2) = \int_{\mathbb{R}^n} \sigma^{2^*} + O(\delta^2) = \alpha^{2^*}S^n + O(\delta^2)$$

Hence , by comparing these terms we have

$$\alpha^{2^*}S^n + O(\delta^2) = \alpha^2 S^n + O(\delta^2)$$

which implies that

$$\alpha^{p-1} - 1 = O(\delta^2)$$

Thus, $|\alpha - 1| \leq O(\delta^2)$. Now, by setting $\rho' = \rho + (\alpha - 1)U$, we proved that

$$u = U + \rho'$$

where $\|\nabla\rho'\|_{L^2(\mathbb{R}^n)} \leq \|\nabla\rho\|_{L^2(\mathbb{R}^n)} + |\alpha - 1|S^n \leq c_0\delta(u)$ □

This last corollary aims to show that the hypothesis $K_0(u) = 1$ in the last theorem was not restrictive, in the sense that for any $u \in D^{1,2}(\mathbb{R}^n)$, such that $K_0(u) = K$ with $\|\nabla u\|_{L^2(\mathbb{R}^n)}$ bounded by a certain constant depending on K, then, *Theorem 3.1* could also be applied to u via a rescaled function u_α satisfying $K_0(u_\alpha) = 1$ and $\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \leq \frac{3}{2}S^n$.

Corollary 3.3

For all $K > 0$, there exist

$$c(K) = \frac{3}{2}S^n \cdot K^{\frac{2-n}{2}}, \qquad c'(K) = c_0 K^{\frac{n}{4}}$$

where c_0 is the constant that gives the bound in (3.1) for a rescaled function u_α , such that if $u \in D^{1,2}(\mathbb{R}^n)$ satisfies

$$K_0(u) = K \quad \text{and} \quad \int_{\mathbb{R}^n} |\nabla u|^2 \leq c(K)$$

Then, there exist $z \in \mathbb{R}^n$, $\lambda \in]0, +\infty[$, such that

$$u = v_1[z, \lambda] + \rho$$

with

$$\|\nabla\rho\|_{L^2(\mathbb{R}^n)} \leq c'(K) \cdot \delta(u).$$

Proof. Let $u \in D^{1,2}(\mathbb{R}^n)$ and $K > 0$, such that $K_0(u) = K$. In order to apply the last theorem, we must verify that a rescaled function u_α can verify its hypotheses. Then, we deduce the quantitative closeness of $v_1[z, \lambda]$ to u, by using that of u_α.

We claim that there exists $\alpha > 0$, such that $u_\alpha(x) := u(\alpha x)$ satisfies

$$K_0(u_\alpha) = 1 \qquad \int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \leq \frac{3}{2}S^n$$

For this, compute

$$K_0(u_\alpha) = \frac{\int_{\mathbb{R}^n} |\nabla u_\alpha|^2\, dx}{\int_{\mathbb{R}^n} u_\alpha^{2^*}\, dx} = \alpha^2 \frac{\int_{\mathbb{R}^n} |\nabla u(\alpha x)|^2\, dx}{\int_{\mathbb{R}^n} u^{2^*}(\alpha x)\, dx} \overset{y=\alpha x}{=} \alpha^2 \frac{\int_{\mathbb{R}^n} |\nabla u(y)|^2\, dy}{\int_{\mathbb{R}^n} u^{2^*}(y)\, dy}$$

By choosing

$$\alpha = \frac{1}{\sqrt{K}}$$

we can establish that $K_0(u_\alpha) = 1$.

In addition,

$$\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \, dx = \alpha^2 \int_{\mathbb{R}^n} |\nabla u(\alpha x)|^2 \, dx = \alpha^{2-n} \int_{\mathbb{R}^n} |\nabla u(y)|^2 \, dy = K^{\frac{n-2}{2}} \int_{\mathbb{R}^n} |\nabla u(y)|^2 \, dy$$

By choosing $c(K) = \frac{3}{2} S^n \cdot K^{\frac{2-n}{2}}$, there holds

$$\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \leq \frac{3}{2} S^n.$$

Hence, we know by the last theorem that there exist $z_\alpha \in \mathbb{R}^n$, $\lambda_\alpha \in]0, +\infty[$ such that for all $x \in \mathbb{R}^n$,

$$u_\alpha(x) = v_1[z_\alpha, \lambda_\alpha](x) + \rho_\alpha(x)$$

where for a constant $c_0 > 0$,

$$\|\nabla \rho_\alpha\|_{L^2(\mathbb{R}^n)} \leq c_0 \, \delta(u_\alpha). \tag{3.9}$$

Then by a simple change of coordinates $y = \alpha x$, for all $y \in \mathbb{R}^n$,

$$u(y) = v_1[z_\alpha, \frac{\lambda_\alpha}{\alpha}](y) + \rho(y)$$

where $\rho(y) = \rho_\alpha(\frac{y}{\alpha})$, for all $y \in \mathbb{R}^n$.
On the other hand, one can show by direct computation, that

$$\begin{aligned}
\delta(u_\alpha) = \|-\Delta u_\alpha - u_\alpha^p\|_{L^{\frac{2n}{(n+2)}}} &= \Big(\int_{\mathbb{R}^n} \Big| \frac{\alpha^2 \Delta u(\alpha x)}{u^p(\alpha x)} \; + \; 1 \Big|^{\frac{2n}{(n+2)}} u^{2^*}(\alpha x) \, dx \Big)^{\frac{n+2}{2n}} \\
&= \Big(\int_{\mathbb{R}^n} \Big| \frac{\Delta u(\alpha x)}{K u^p(\alpha x)} \; + \; 1 \Big|^{\frac{2n}{(n+2)}} u^{2^*}(\alpha x) \, dx \Big)^{\frac{n+2}{2n}} \\
&= \Big(\int_{\mathbb{R}^n} \Big| \frac{\Delta u(\alpha x)}{u^p(\alpha x)} \; + \; K \Big|^{\frac{2n}{(n+2)}} \cdot \left(\frac{1}{K} \right)^{\frac{2n}{(n+2)}} \cdot u^{2^*}(\alpha x) \, dx \Big)^{\frac{n+2}{2n}} \\
&= \Big(\int_{\mathbb{R}^n} \Big| \frac{\Delta u(y)}{u^p(y)} \; + \; K \Big|^{\frac{2n}{(n+2)}} \cdot \left(\frac{1}{K} \right)^{\frac{2n}{(n+2)}} \cdot K^{\frac{n}{2}} \cdot u^{2^*}(y) \, dx \Big)^{\frac{n+2}{2n}} \\
&= K^{\frac{n-2}{4}} \Big(\int_{\mathbb{R}^n} \Big| \frac{\Delta u(y)}{u^p(y)} \; + \; K \Big|^{\frac{2n}{(n+2)}} u^{2^*}(y) \, dx \Big)^{\frac{n+2}{2n}} \\
&= K^{\frac{n-2}{4}} \cdot \delta(u).
\end{aligned}$$

Hence,

$$\|\nabla \rho\|_{L^2(\mathbb{R}^n)} = \frac{1}{\alpha} \|\nabla \rho_\alpha\|_{L^2(\mathbb{R}^n)} \leq \frac{c_0}{\alpha} \delta(u_\alpha) = c_0 K^{\frac{n}{4}} \delta(u) := c'(K) \delta(u).$$

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