ART 1

17.5/20

Bei Dennis Loose

Kyra Klos

Max Pernklau

1 a)

(i) $\begin{pmatrix} -x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{diag}(-1,1,1,1) \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$?

Aufgabe: in $x^0, x^1, \ldots$ ausdrücken!

Das ist keine Matrixmultiplikation!

(ii) $\delta_\lambda{}^\lambda = 4$ ✓ $= \begin{bmatrix} -1 & 0 & & 0 \\ 0 & 1 & & \\ & & 1 & 0 \\ 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & & 0 \\ 0 & +1 & & \\ & & +1 & 0 \\ 0 & & 0 & +1 \end{bmatrix}$?

(iii) $\eta_{\alpha\beta} \eta^{\gamma\beta} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \delta_\alpha{}^\gamma$ (✓)

(iv) $\eta^{\mu\nu} x_\nu x_\mu = \underbrace{\begin{pmatrix} +x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mu}_{= x^\mu} \cdot x_\mu = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ (✓)

(v) $\eta^\mu{}_\alpha x_\sigma \eta^{\sigma\alpha} x_\mu = \eta^\mu{}_\alpha x_\mu \eta^{\sigma\alpha} x_\sigma \underset{\text{s.o.}}{=} \eta^\mu{}_\alpha x_\mu \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}^\alpha$

$= \begin{pmatrix} -x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}_\alpha \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}^\alpha = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ (✓)

b)

~~Transformationsverhalten der Ableitung?~~

$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \quad | \cdot \eta_{\nu\alpha}$

$x^{\mu'} \eta_{\nu\alpha} = \Lambda^{\mu'}{}_\nu x_\alpha \quad | \cdot \eta^{\nu\alpha}$

$x^{\mu'} \underbrace{\eta_{\nu\alpha} \eta^{\nu\alpha}}_{=1} = \Lambda^{\mu'\alpha} x_\alpha$

$x^{\mu'} = \Lambda^{\mu'\alpha} x_\alpha \quad | \cdot \eta_{\mu'\beta'}$

$x_{\beta'} = \Lambda_{\beta'}{}^\alpha x_\alpha$ ✓

zu Aufgabe 1b)

$\partial_\mu \equiv \partial/\partial x^\mu$ mit $f = f(x^\mu)$ und $df = \frac{\partial f}{\partial x^\mu} dx^\mu$

Konvention: Striche an Koordinaten Indizes

$x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow x^\mu = (\Lambda^{-1})^\nu{}_\mu x'^\mu$ einsetzen

$\Rightarrow df = \partial f/\partial x'^\mu \, dx'^\mu = \frac{\partial f}{\partial x^\nu} \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}}_{} dx'^\mu$

$= \underbrace{\boxed{\left((\Lambda^{-1})^\nu{}_\mu \, \partial/\partial x^\nu\right) f}}_{\text{Transformation}} dx'^\mu \left((\Lambda^{-1})^\nu{}_\mu\right)$ (✓)

Transformation erfolgt wie bei einem kovarianten Vierervektor ✓

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2 a)

(i)

$$S_{\mu\nu} T^{\mu\nu} \overset{!}{=} S_{\mu\nu} T^{(\mu\nu)} = S_{\mu\nu} \frac{1}{2} \left(T^{\mu\nu} + T^{\nu\mu} \right)$$

$$= \frac{1}{2} S_{\mu\nu} T^{\mu\nu} + \frac{1}{2} S_{\mu\nu} T^{\nu\mu}$$

$$\Leftrightarrow S_{\mu\nu} T^{\mu\nu} = S_{\mu\nu} T^{\nu\mu} \underset{\substack{\uparrow \\ S_{\mu\nu} \text{ is symmetric}}}{=} S_{\nu\mu} T^{\nu\mu} \underset{\substack{\uparrow \\ \text{renaming:} \\ \nu \to \mu \\ \mu \to \nu}}{=} S_{\mu\nu} T^{\mu\nu} \quad \square$$

nicht nötig. Namen von Dummy-Indizes sind irrelevant.

(ii)

$$A_{\mu\nu} T^{\mu\nu} = A_{\mu\nu} T^{[\mu\nu]} = A_{\mu\nu} \frac{1}{2} \left(T^{\mu\nu} - T^{\nu\mu} \right)$$

$$\Leftrightarrow A_{\mu\nu} T^{\mu\nu} = -A_{\mu\nu} T^{\nu\mu} \underset{\substack{\uparrow \\ A_{\mu\nu} \text{ is antisymmetric}}}{=} A_{\nu\mu} T^{\nu\mu} \underset{\substack{\uparrow \\ \text{renaming} \\ \nu \to \mu \\ \mu \to \nu}}{=} A_{\mu\nu} T^{\mu\nu} \quad \square$$

s.o.

(iii)

$$S_{\mu\nu} A^{\mu\nu} = 0$$

We know that $A^{\mu\nu} = -A^{\nu\mu}$ and $S_{\mu\nu} = S_{\nu\mu}$

so $-S_{\nu\mu} A^{\nu\mu} = S_{\mu\nu} A^{\mu\nu} = 0$ must be valid, too.

Renaming leads to $-S_{\mu\nu} A^{\mu\nu} = S_{\mu\nu} A^{\mu\nu}$ which can be true iff $S_{\mu\nu} A^{\mu\nu} = 0$ $\square$

s.o.

b)

$$T_{(\mu_1,\mu_2)} = \frac{1}{n!}\sum_P T_{\mu_1\mu_2} = \frac{1}{2}\left(T_{\mu_1\mu_2} + T_{\mu_2\mu_1}\right)$$

$$T_{[\mu_1,\mu_2]} = \frac{1}{2}\left(T_{\mu_1\mu_2} - T_{\mu_2\mu_1}\right)$$

$$T_{(\mu_1,\mu_2)} + T_{[\mu_1\mu_2]} = \frac{1}{2}\left(T_{\mu_1\mu_2} + T_{\mu_1\mu_2}\right) = T_{\mu_1\mu_2} \quad \square$$

$$T_{(\mu_1,\mu_2,\mu_3)} = \frac{1}{6}\left(T_{\mu_1\mu_2\mu_3} + T_{\mu_1\mu_3\mu_2} + T_{\mu_2\mu_1\mu_3}\right.$$
$$\left. T_{\mu_2\mu_3\mu_1} + T_{\mu_3\mu_1\mu_2} + T_{\mu_3\mu_2\mu_1}\right)$$

$$T_{[\mu_1\mu_2\mu_3]} = \frac{1}{6}\left(\overbrace{T_{\mu_1\mu_2\mu_3} + T_{\mu_3\mu_1\mu_2} + T_{\mu_2\mu_3\mu_1}}^{(2\text{ perm})}\right.$$
$$\left. - T_{\mu_3\mu_2\mu_1} - T_{\mu_1\mu_3\mu_2} - T_{\mu_2\mu_1\mu_3}\right)$$

$$T_{()} + T_{[]} = \frac{1}{3}\left(T_{\mu_1\mu_2\mu_3} + T_{\mu_2\mu_3\mu_1} + T_{\mu_3\mu_1\mu_2}\right) \overset{?}{=} T_{\mu_1\mu_2\mu_3} \quad ↯$$

Splitting a rank-3 tensor in this fashion is not possible.

c) $$T^{\mu}{}_{\nu} \overset{?}{=} T_{\nu}{}^{\mu} \quad | \cdot \eta_{\mu\alpha}$$

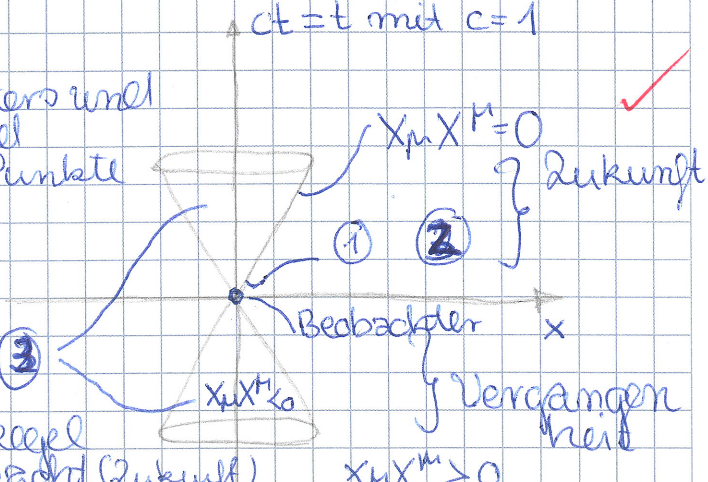
$$\Leftrightarrow T_{\alpha\nu} = T_{\nu\alpha} \Leftrightarrow T \text{ symmetric}$$

$\frac{6}{6}$

ART

A 3

a) $X_\mu X^\mu \equiv x^2$ } Norm eines Vierervektors und damit der Abstand zweier Punkte



i) (1) $X_\mu X^\mu = 0$ lichtartig ✓

↳ Ereignisse E werden direkt beobachtbar, da sie sich lichtartig verhalten. Sie liegen auf dem Lichtkegel von Ereignissen B die E verursacht (Zukunft) haben oder zu E führen (Vergangenheit) falls ~~dieser sich~~ a der Beobachter sich auf B befindet

ii) (2) $X_\mu X^\mu > 0$ ~~raumartig~~ ~~zeitartig~~ raumartig ✓

Was heißt das?

↳ Hier kann zwischen den Ereignissen keine kausale Korrelation stattfinden, da diese nicht mit Lichtgeschwindigkeit zusammenhängen (müsste Überlichtgeschwindigkeit sein (+)). Dies führt dazu das "Ursache" und "Wirkung" jederzeit durch Lorentztrafo vertauscht werden könnten (→ damit keine U. & W.) ✓
Die Ereignisse E liegen außerhalb des Lichtkegels

iii) (3) $X_\mu X^\mu < 0$ zeitartig

↳ Die Ereignisse können da sie via Lichtsignal überbrückbar sind als Ursache und Wirkung aufgefasst werden, die nicht durch eine Lorentz-Trafo vertauscht werden können (legitimiert) ✓
s.o.
Dennoch ist es möglich Ereignisse im selben Raum stattfinden zu lassen. Ereignis
s.o.
E liegt somit im Lichtkegel E, nachdem
? ob er vorher / nachher (nach B) stattgefunden hat und wären für den Beobachter auf B sichtbar.

b) für $x_\mu x^\mu < 0$ folgt $d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$

~~dies kann auch als~~ ~~$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ für $x_\mu x^\mu \le 0$~~ ~~aufgefasst werden~~

"$d\tau = dt \frac{1}{\gamma}$" → $d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu = -dx^\mu dx_\mu$

mit $-ds^2 = c^2 (dt)^2 - (d\vec{x})^2$ } Längenquadrat ist invariant unter Lorentz Trafo
c = 1 mit $ds^2 = dx_\mu dx^\mu$)

und mit Ruhesystem eines bewegten Teilchens gilt
1 $dx = dy = dz$ → $-ds^2 = c^2 d\tau^2$ und Lorentz-Invarianz

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ART - Blatt 1

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zu Aufgabe 3 b)

folgt aus

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$
$$= c^2 dt^2 - v^2(t) dt^2$$
$$= c^2\left(1 - \frac{v^2(t)}{c^2}\right) dt^2 \stackrel{!}{=} c^2 d\tau^2$$
$$\Rightarrow \left(1 - \frac{v^2}{c^2}\right) dt^2 = d\tau^2$$
$$\underbrace{\sqrt{1 - v^2/c^2}}_{1/\gamma}\, dt = d\tau$$

$\frac{1}{\gamma}$ liegt, da die maximale Geschwindigkeit v gegen c geht, bei ~~maximal c~~ minimal 0 und bei v=0 bei 1. $\rightarrow$ $1/\gamma \in [0, 1]$

c) $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ mit $x^\mu = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} c \\ \dot{\vec{x}} \end{pmatrix}}_{\begin{pmatrix} c \\ \vec{v} \end{pmatrix}} \underbrace{\frac{1}{\sqrt{1 - v^2/c^2}}}_{\gamma}$$

mit $|(u^\mu)^2| = \sqrt{u^\mu u^\nu \eta_{\mu\nu}}$
$= \sqrt{u_\mu u^\mu}$

folgt $\Rightarrow = \sqrt{\gamma^2 (c^2 - v^2)}$ $= \sqrt{(-1)^2 \gamma^2 (c^2 - v^2)} = -\sqrt{\gamma^2 (c^2 - v^2)}$ $= -c$

mit Normierung $-c = -1$ folgt

$|(u^\mu)^2| = 1$

f

$\sqrt{(-1)^2} \neq -1$

Norm: $u^\mu u_\mu = \ldots = -1$

siehe Gl. (g)

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A1 fehlt ☹

ART Blatt 2
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$\Sigma\ \frac{9.5}{20}$

2

$$f^\mu = \frac{dp^\mu}{d\tau} = q F^{\mu\nu} u_\nu \qquad F^{\mu\nu} \hat{=} \begin{bmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{bmatrix}$$

$$u_\nu \hat{=} (-\gamma, \gamma \dot{\vec{x}})$$

(a)

$$f^\mu = q F^{\mu\nu} u_\nu \hat{=} \gamma q \begin{bmatrix} 0 + \dot{x} E_x + E_y \dot{y} + E_z \dot{z} \\ E_x + 0 + B_z \dot{y} - B_y \dot{z} \\ E_y - B_z \dot{x} + 0 - B_x \dot{z} \\ E_z + B_y \dot{x} - B_x \dot{y} + 0 \end{bmatrix}$$

$\lceil f^0 = q F^{0i} u_i = \gamma q E^i v_i = \gamma q \vec{E}\cdot\vec{v} \rfloor$

First component: $[\cancel{+}\vec{v}\cdot\vec{E}\cdot q\gamma] = [P]$
Best guess: The first component is the power transfered ~~#~~ to a particle infulvenced by the EM field. ✓

For $\gamma \to 1$ $(v \to 0)$: $v \ll 1$

$$\lim_{\gamma\to 1} f^\mu e_{(\mu)} = +q \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q \begin{pmatrix} +B_z \dot{y} - B_y \dot{z} \\ +B_x \dot{z} - B_z \dot{x} \\ +B_y \dot{x} - B_x \dot{y} \end{pmatrix} = +q\vec{E} + q\dot{\vec{v}} \times \vec{B} \quad \square$$ ✓

(b)

$$f^\mu = q F^{\mu\nu} u_\nu \hat{=} q\gamma \begin{bmatrix} 0 \\ 0 \\ -B_z v \\ 0 \end{bmatrix} \Rightarrow f^\mu = \begin{cases} -\gamma q B_z v & , \mu = 2 \\ 0 & , \mu = 0,1,3 \end{cases}$$ ✓

(c)

$$\Lambda^{\mu'}{}_\mu \hat{=} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \Lambda^{\mu'}{}_\mu u^\mu e_{(\mu')} = \begin{pmatrix} +\gamma^2 - \gamma^2 v^2 \\ +\gamma v - \gamma v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$ ✓

$$\gamma^2(1-v^2) = \frac{1-v^2}{1-v^2} = 1$$

$$\boxed{\vec{E} = 0, \quad \vec{B} = B\vec{e}_z}$$

andere Vorzeichen als oben!

$$\Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu F^{\mu\nu} \hat{=} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= - \begin{bmatrix} -E_x v\gamma & -E_x\gamma & -E_y\gamma + B_z\gamma v & -E_z\gamma - B_y\gamma v \\ E_x\gamma & +E_x\gamma v & +E_y\gamma v - B_z\gamma & +E_z\gamma v + B_y\gamma \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

2(d) con't

$$\gamma^2 \cdot (1-v^2) = 1$$

$$= - \begin{bmatrix} 0 & +E_x v^2\gamma^2 - E_x\gamma^2 & -E_y\gamma + B_z\gamma v & -E_z\gamma - B_y\gamma v \\ \gamma^2 E_x - \gamma^2 E_x v^2 & 0 & E_y\gamma v - B_z\gamma & E_z\gamma v + B_y\gamma \\ \gamma E_y - B_z\gamma v & -\gamma v E_y + B_z\gamma & 0 & -B_x \\ E_z\gamma + \gamma v B_y & -E_z\gamma v - B_y\gamma & B_x & 0 \end{bmatrix}$$

Explizit für $\vec{E}=0,\ \vec{B}=B\vec{e}_z$

(v) $\Lambda^{\mu'}{}_{\mu} f^{\mu} e_{(\mu')} = \begin{pmatrix} 0 \\ 0 \\ -B_z v \\ 0 \end{pmatrix} \gamma q$?

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3

(a) There is no friction in a perfect fluid. The pressure is always isotropic.
Therfore its only parameters are its density and pressure:

$$\underbrace{T^{\mu\nu} \mathrel{\hat{=}} \begin{bmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}}_{\text{rest frame}} \quad \text{resp.} \quad \underbrace{T^{\mu\nu} = (p+\rho)U^{\mu}U^{\nu} + p\,\eta^{\mu\nu}}_{\text{any inertial frame}}$$

(ρ: Energy Density)

~~(b)~~

~~Energy and momentum is conserved~~

$$\cancel{\partial_\mu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} = (p+\rho)\frac{d(dx^\mu\, dx^\nu)}{dx^\mu\, d\tau\, d\tau} + \underbrace{\frac{d(\eta^{\mu\nu} p)}{dx^\mu}}_{=0\ \text{(isotropic pres.)}} = \frac{d}{d\tau}U^\nu (p+\rho) \overset{!}{=} 0}$$

$$\cancel{\hookrightarrow \frac{d\gamma}{d\tau}(p+\rho) = \underbrace{\left(\frac{d\gamma}{dt\gamma}\right)}_{=0}(p+\rho) + \gamma\gamma\underbrace{\left(\frac{dp}{dt} + \frac{d\rho}{dt}\right)}_{=0} = 0}$$

$$\cancel{\frac{d\gamma\vec{x}}{d\tau}(p+\rho) = \gamma\gamma\underbrace{\left(\frac{d}{dt}\dot{\vec{x}}\right)}_{\ddot{\vec{x}}\ =0\ (\text{SRT}:\ \ddot{x}=0)}(p+\rho) + \gamma\gamma\underbrace{\left(\frac{dp}{dt} + \frac{d\rho}{dt}\right)}_{=0}\dot{\vec{x}} = 0}$$

b) Energy, momentum is conserved. $\rightarrow$ $\partial_\mu T^{\mu\nu} = 0$

$T^{\mu\nu} = (p+\rho)\, u^\mu u^\nu + p\,\eta^{\mu\nu}$ $\qquad w = \rho + p$

particle flux: $n^\alpha = n u^\alpha$, where n is the particle density in rest frame

continuity equation: $\frac{\partial n^\alpha}{\partial x^\alpha} = 0$ no particles in the fluid should be destroied or created

$T^\mu_{\ \nu} = w u^\mu u_\nu + p\,\eta^\mu_{\ \nu}$

$\frac{\partial T^\mu_{\ \nu}}{\partial x^\mu} = u_\nu \frac{\partial (w u^\mu)}{\partial x^\mu} + w u^\mu \frac{\partial u_\nu}{\partial x^\mu} + \eta^\mu_{\ \nu} \frac{\partial p}{\partial x^\mu} = 0 \quad \Big| \cdot u^\nu$

Warum?

Ableitung wirkt nur auf p

$\Leftrightarrow \underbrace{u_\nu u^\nu}_{=-1} \frac{\partial (w u^\mu)}{\partial x^\mu} + \frac{1}{2} w u^\mu \underbrace{\frac{\partial (\overbrace{u_\nu u^\nu}^{=-1})}{\partial x^\mu}}_{=0} + u^\nu \eta^\mu_{\ \nu} \frac{\partial (p\, u^\nu)}{\partial x^\mu} = 0 = -\frac{\partial (w u^\mu)}{\partial x^\mu} + u^\mu \frac{\partial p}{\partial x^\mu}$ ($\checkmark$)

$u_\nu \partial_\mu u^\nu = \frac{1}{2} \partial_\mu u_\nu u^\nu$

$\Leftrightarrow$

particle flux:

$w u^\mu = n u^\mu \frac{w}{n}$

$n u^\mu \left(\frac{\partial}{\partial x^\mu} \frac{w}{n} - \frac{1}{n} \frac{\partial}{\partial x^\mu} p \right) = 0 = n u^\mu \left(\frac{\partial}{\partial x^\mu} \frac{\rho}{n} \right)$

$w = \rho + p$

~~Ableitung von u^μ?~~

~~(aus $\partial_\mu (w u^\mu)$)~~

ok. $\partial_\mu u^\mu = 0$

c)

offener Index

hier kann ν $\qquad$ hier σ und ν offen

$\frac{\partial T^\sigma_{\ \nu}}{\partial x^\sigma} - u_\nu u^\sigma \frac{\partial T^\mu_{\ \sigma}}{\partial x^\mu} \stackrel{!}{=} 0 = n u^\sigma \left(\frac{\partial}{\partial x^\sigma} \frac{\rho}{n} \right) - u_\nu u^\sigma n u^\mu \left(\frac{\partial}{\partial x^\mu} \frac{\rho}{n} \right)$

$= \frac{\partial}{\partial x^\nu} \rho - \underbrace{u_\nu}_{\begin{pmatrix} -\gamma \\ \vec{v}\gamma \end{pmatrix}} \underbrace{u^\mu}_{\begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix}} \underbrace{\frac{\partial}{\partial x^\mu} \rho}_{\begin{pmatrix} \dot{\rho} \\ \vec{\nabla}\rho \end{pmatrix}}$

$\Rightarrow \begin{pmatrix} \dot{\rho} \\ \vec{\nabla}\rho \end{pmatrix} = \begin{pmatrix} -\gamma \\ \gamma \vec{v} \end{pmatrix} \cdot \left(\gamma \dot{\rho} + \gamma\, \vec{v} \cdot \vec{\nabla} \rho \right)$

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ff

$\lim_{\gamma \to 1}: \begin{pmatrix} \dot{\rho} \\ \vec{\nabla}\rho \end{pmatrix} = \begin{pmatrix} -\dot{\rho} - \vec{v}\,\vec{\nabla}\rho \\ \dot{\rho}\vec{v} + v^2 \vec{\nabla}\rho \end{pmatrix} \Rightarrow$

$2\dot{\rho} = -\vec{v}\,\vec{\nabla}\rho \Leftrightarrow \dot{\rho} + \frac{1}{2}\vec{v}\,\vec{\nabla}\rho = 0$

$\vec{\nabla}\rho = \dot{\rho}\vec{v} + (\vec{\nabla}\rho)\, v^2 \Leftrightarrow \vec{\nabla}\rho = -\frac{1}{2}\vec{\nabla}\rho\, v^2 + \vec{\nabla}\rho \cdot v^2$

$\dot{\rho} = -\frac{1}{2}\vec{v}\,\vec{\nabla}\rho \Rightarrow \vec{\nabla}\rho = \frac{1}{2}\vec{\nabla}\rho\, v^2 \Rightarrow \vec{\nabla}\rho = 0$

A perfect fluid is incompressible!

ART Blatt 2

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A4

a) $S = \alpha \int d\tau = \int L\,dt$

mit $d\tau = \frac{1}{\gamma} dt$ (siehe Blatt 1)

$\alpha \int \frac{1}{\gamma} dt = \int L\,dt \Rightarrow L = \frac{\alpha}{\gamma} = \alpha\sqrt{1-v^2}$ ✓

b) $v << 1 \rightarrow \sqrt{1-v^2}$ via Taylorentwicklung vereinfachen.

$\sqrt{1-v^2} \approx (1 - \frac{1}{2}v^2 + O(v^3))$ höhere Terme werden vernachlässigt

$\hookrightarrow L = \alpha(1 - \frac{1}{2}v^2) = \alpha - \frac{\alpha}{2}v^2$

wähle hieraus da nicht relativistischer Fall

$-\alpha = m_0$:= Ruhemasse $\Rightarrow L = -m_0\sqrt{1-v^2}$

c) $\partial/\partial t \frac{\partial L}{\partial \dot{q}_1} - \frac{\partial L}{\partial q_1} = 0$ $\quad \frac{\partial L}{\partial \dot{x}} = \cancel{m_0 \dot{x}(1-\dot{x}^2)^{-1/2}}\ m_0 \dot{x}(1-\dot{x}^2)^{-1/2}$

$\cancel{\partial/\partial t\,(m_0\dot{x}(1-\dot{x}^2)^{-1/2}) = 0}$ $\quad \frac{\partial L}{\partial x} = 0$ $\quad \rightarrow m_0(\ddot{x}(1-\dot{x}^2)^{-1/2} + \dot{x}^2\ddot{x}(1-\dot{x})^{-3/2})$

$\partial/\partial t\,(m_0\dot{x}(1-\dot{x}^2)^{-1/2}) = 0$

$\hookrightarrow$ daraus folgt, dass $\dot{x}$ konstant sein muss und x eine gleichförmige Bewegung ohne Beschleunigung beschreibt ($\ddot{x} \neq 0$) ✓

$\vec{x}(t) = \vec{x}_0 \cdot t + \vec{s}_0$

3.5/4

$m_0\,\dot{x}\,(1-\dot{x}^2)^{-\frac{1}{2}} = m_0 v\gamma = p \quad \rightarrow \quad \frac{dp}{dt} = 0$

A1 $\quad \frac{17}{20}$ (4)

ART Blatt 3

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FLRW: $ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2(\theta)\, d\phi^2\right]$

$k \in [-1, 0, +1]$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta})$$

$$g_{\mu\nu} = \begin{array}{cccc} t & r & \theta & \phi \end{array} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t)\frac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2 \sin^2(\theta) \end{pmatrix}$$ folgt aus FLRW

mit $x_0 \; t$, $x_1 \; r$, $x_2 \; \theta$, $x_3 \; \phi$

$\underbrace{\Gamma^t_{tr} = \Gamma^t_{rt}}_{\text{Christoffel sym. bei unteren indizes (*)}} = \frac{1}{2} g^{t\lambda}(\partial_t g_{r\lambda} + \partial_r g_{\lambda t} - \partial_\lambda g_{tr}) \rightarrow \lambda = \mu$ wird (**) nun immer gewählt da sonst $g^{\mu\lambda} = 0 \rightarrow \Gamma = 0$ immer gilt

$= \frac{1}{2} g^{tt}(\underbrace{\partial_t g_{rt}}_{=0} + \underbrace{\partial_r g_{tt}}_{-1} - \underbrace{\partial_t g_{tr}}_{=0}) = 0$ ✓

$\Gamma^t_{t\theta} = \frac{1}{2} \underset{(**)}{g^{tt}}(\underbrace{\partial_t g_{\theta t}}_{0} + \underbrace{\partial_\theta g_{tt}}_{=-1} - \underbrace{\partial_t g_{t\theta}}_{0}) = 0 = \Gamma^t_{\theta t}$ ✓

$\Gamma^t_{t\phi} = \Gamma^t_{\phi t} = 0$ ✓ s.o. (da $\partial_\phi(-1) = 0$) ; XXX

$\Gamma^t_{tt} = \frac{1}{2} g^{tt}(\partial_t g_{tt} + \partial_t g_{tt} - \partial_t g_{tt}) = 0$ ✓ mit $g^{tt} = -1$

$\Gamma^t_{rr} = \frac{1}{2} g^{tt}(\underbrace{\partial_r g_{rt}}_{0} + \underbrace{\partial_r g_{tr}}_{0} - \partial_t g_{rr}) = \frac{1}{2}(-1)\cdot\left(-\partial_t \left(a^2(t)\frac{1}{1-kr^2}\right)\right) = a(t)\dot{a}(t)\frac{1}{(1-kr^2)}$ ✓

$\Gamma^t_{\theta\theta} = a(t)\dot{a}(t) r^2$ ✓ (vgl. Γ^t_{rr}) ; $\Gamma^t_{\phi\phi} = a(t)\dot{a}(t) r^2 \sin^2(\theta)$ ✓

XXX $\Rightarrow$ $\Gamma^t_{r\theta} = \Gamma^t_{\theta r} = \frac{1}{2} g^{tt}(\partial_\theta \overset{0}{g_{rt}} + \partial_r \overset{0}{g_{t\theta}} - \partial_t \overset{0}{g_{\theta r}}) = 0$ ✓

$\Gamma^t_{r\phi} = \Gamma^t_{\phi r} = \frac{1}{2} g^{tt}(\partial_\phi g_{rt} + \partial_r g_{t\phi} - \partial_t g_{\phi r}) = 0$ ✓

vgl.

$\Gamma^t_{\phi\theta} = \Gamma^t_{\theta\phi} = \frac{1}{2} g^{tt}(0) = 0$ [s.o.] ✓

$\Gamma^r_{rr} = \underset{(**)}{\frac{1}{2} g^{rr}}(\partial_r g_{rr} + \overbrace{\partial_r g_{rr} - \partial_r g_{rr}}^{0}) = \frac{kr}{(1-kr^2)}$ ✓ mit $g^{rr} = \frac{1-kr^2}{a^2(t)}$

$\underset{(*)}{\Gamma^r_{r\phi}} = \Gamma^r_{\phi r} = \frac{1}{2} g^{rr}(\partial_r \overset{0}{g_{\phi r}} + \partial_\phi \overset{0}{g_{rr}} - \partial_r \overset{0}{g_{r\phi}}) = 0$ ✓

$\Gamma^r_{rt} = \Gamma^r_{tr} = \frac{1}{2} g^{rr}(\partial_r \overset{0}{g_{tr}} + \partial_t g_{rr} - \partial_r \overset{0}{g_{rt}}) = \frac{1}{2} g^{rr}(\partial_t g^{rr}) = a^{-1}(t)\dot{a}(t)$ ✓

$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{1}{2} g^{rr}(\partial_r g_{\theta r} + \partial_\theta g_{rr} - \partial_r g_{r\theta}) = 0$ s.o. ✓

$\Gamma^r_{\theta\phi} = \Gamma^r_{\phi\theta} = 0$ [s.o.] ✓ ; $\Gamma^r_{t\phi} = \Gamma^r_{\phi t} = 0$ ✓ ; $\Gamma^r_{t\theta} = \Gamma^r_{\theta t} = 0$ ✓

$\partial_r g_{\phi\phi} = 2 a r \sin^2\theta$

$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr}(-\partial_r g_{\theta\theta}) = -r(1-kr^2)$ ✓ ; $\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr}(-\partial_r g_{\phi\phi}) = \frac{1}{2}\frac{1-kr^2}{a^2(t)}(-2\, a(t)^2 r \sin^2(\theta)) = -(1-kr^2) r \sin^2(\theta)$ f

$\Gamma^r_{tt} = \frac{1}{2} g^{rr}(-\partial_r g_{tt}) = 0$ ✓

mit $g^{\theta\theta} = \frac{1}{a(t)^2 r^2}$

$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta}(0) = 0$ ✓ ; $\Gamma^\theta_{tr} = \Gamma^\theta_{rt} = 0$; $\Gamma^\theta_{t\phi} = \Gamma^\theta_{\phi t} = 0$; $\Gamma^\theta_{r\phi} = \Gamma^\theta_{\phi r} = 0$ (***) ✓

$\Gamma^\theta_{\theta t} = \frac{1}{2} g^{\theta\theta}[\partial_\theta \overset{0}{g_{t\theta}} + \partial_t g_{\theta\theta} - \partial_\theta \overset{0}{g_{\theta t}}] = a^{-1}(t)\dot{a}(t)$ ✓ $= \Gamma^\theta_{t\theta}$

$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = r^{-1}$ ✓ [Rechnung s.o.] $\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \frac{1}{2} g^{\theta\theta}(\partial_\phi \overset{0}{g_{\theta\theta}}) = 0$ ✓

mit $(\partial_r g_{\theta\theta}) = 2 r a^2(t)$

$\Gamma^{\theta}_{tt} = \frac{1}{2} g^{\theta\theta}[-\partial_\theta g_{tt}] = 0$; $\Gamma^{\theta}_{rr} = 0$; $\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} [-\partial_\theta \sin^2\theta] = -\frac{1}{2} \sin(2\theta)$ ✓

$\Gamma^{\phi}_{\phi\phi} = 0$ (da $g_{\phi\phi}$ nicht von ϕ abhängt)

$\underbrace{\Gamma^{\phi}_{tr} = \Gamma^{\phi}_{rt} = \Gamma^{\phi}_{t\theta} = \Gamma^{\phi}_{\theta t} = \Gamma^{\phi}_{rt} = \Gamma^{\phi}_{tr}}_{=0 \quad [\text{s.o.}]}$ mit $g^{\phi\phi} = \frac{1}{a(t)^2 r^2 \sin^2(\theta)}$ ✓

$\Gamma^{\phi}_{\phi t} = \frac{1}{2} g^{\phi\phi}[\partial_t g_{\phi\phi}] = \frac{\dot{a}(t)}{a(t)} = \Gamma^{\phi}_{t\phi}$ ✓

$\Gamma^{\phi}_{\phi r} = \Gamma^{\phi}_{r\phi} = \frac{1}{2} g^{\phi\phi}[\partial_r g_{\phi\phi}] = r^{-1}$ ✓

$\Gamma^{\phi}_{tt} = \frac{1}{2} g^{\phi\phi}[-\underbrace{\partial_\phi g_{tt}}_{0}] = 0 = \Gamma^{\phi}_{rr} = \Gamma^{\phi}_{\theta\theta}$ da $g_{tt}, g_{rr}, g_{\theta\theta}$ nicht von ϕ abhängt.

$\Gamma^{\varphi}_{\theta\varphi} = ?$

8/9

A2

$$T^{\mu\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right] \longrightarrow T^{\mu}{}_{\mu}$$

~~gμν~~ allgemein $g_{\mu\nu}T^{\mu\nu} = T^{\mu}{}_{\mu}$ — funktioniert lokal immer (siehe VL17Mai)

$$= \frac{1}{4\pi}\left[g_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g_{\mu\nu}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right]$$

Wahl: $g_{\mu\nu} = \eta_{\mu\nu}$

mit $\eta_{\mu\nu} = \eta_{\nu\mu}$

$$= \frac{1}{4\pi}\left[\underbrace{\eta_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}}_{F^{\mu\alpha}F_{\mu\alpha}} - \frac{1}{4}\underbrace{\eta_{\nu\mu}\eta^{\mu\nu}}_{=4}F^{\alpha\beta}F_{\alpha\beta}\right]$$

$$= \frac{1}{4\pi}\left[F^{\mu\alpha}F_{\mu\alpha} - F^{\alpha\beta}F_{\alpha\beta}\right] \overset{(*)}{=} 0$$ ✓

~~somit passt mit (4) $\Box\phi = 0$~~

~~$T^{\mu\nu}$ ist im Gegensatz zu $T^{\mu}{}_{\mu}$ symmetrisch, was eine Voraussetzung damit in (4) ist~~

(*) Die Spur ist somit 0! Mit $\Box\phi = 0$ bekäme man somit eine klassische Wellengleichung, ~~was~~.

Dies, mit der Lorentzinvarianz von (4) sorgt nicht für eine geeignete Beschreibung von ~~der Masse und~~ Massereaktionen ? (Gravitation).

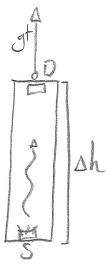
~~und ist somit nicht als Gravitations~~

Warum ?

3/5

3

a) This problem is equivalent to the building accelerating upwards at $a = g = 9{,}81\,\frac{m}{s^2}$ without a gravitational field.



Photon takes Δt to travel Δh (as observed from inside the building).

$$\underline{\Delta v} = g \cdot \Delta t = g \frac{\Delta h}{c} \doteq \underline{\underline{9{,}81 \times 10^{-7}\,\frac{m}{s}}}$$

The detector sees a photon emitted from a source that was moving Δv slower than itself. Thus the source must travel at Δv towards the detector to compensate.

b) $\Delta h = 10^{-2}\,m$

$$\Delta v = g \cdot \frac{\Delta h}{c} \doteq 3{,}27 \cdot 10^{-10}\,\frac{m}{s}$$

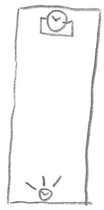
At 1cm the effect is extremly small.

c)

$\Delta t = \Delta h / c$

$\Delta v = g \Delta t = g \Delta h / c$

Frequency Shift: $f = f_0 \left(1 - \frac{\Delta v}{c}\right) = f_0 \left(1 - \frac{g \Delta h}{c^2}\right)$



The detector thus sees the bottom clock (S) ticking at a rate of f instead of f_0. As both clocks should tick at the same rate it can be deduced that at the bottom time passes slower by a factor of $\frac{f}{f_0} = \left(1 - \frac{g \Delta h}{c^2}\right) = \left(1 - \frac{\Delta \varphi}{c^2}\right)$.

$$\frac{f - f_0}{f} = \frac{g \Delta h}{c^2} \underset{\Delta h = 1cm}{\doteq} 1{,}09 \times 10^{-18}$$ ✓

To measure the difference in clock rates the rate of error of the clocks $\frac{\Delta \tau}{\tau}$ should be less than $1{,}09 \times 10^{-18}$. Modern atomic clocks can in fact measure such minuscule relativistic effects as their frequency uncertainty is as low as $2{,}1 \times 10^{-18}$.

d) An absolute potential can never be measured $\left(-\nabla \varphi = -\nabla(\varphi + c) = F \text{ with } c = \text{const.}\right)$. The detector clock can only measure frequency shifts relative to its own frequency. (✓)

"absolute" as in "relative to φ = const."

6/6

A1 $\frac{18}{20}$

Max Pernklau Kyra Kless ART Blatt 4

bei Dennis

(1) $R_{\mu\nu\rho\lambda} = -R_{\nu\mu\rho\lambda}$

$= g_{\mu\sigma} R^\sigma{}_{\nu\rho\lambda} = g_{\mu\sigma}(\underbrace{\partial_\rho \Gamma^\sigma_{\lambda\nu}}_{*} - \partial_\lambda \Gamma^\sigma_{\rho\nu})$ da $\Gamma^\mu_{\alpha\beta} = 0$ fällt der Rest weg

$* = \frac{1}{2}(\partial_\rho g^{\sigma i})(\partial_\nu g_{\lambda i} + \partial_\lambda g_{i\nu} - \partial_i g_{\nu\lambda})$ da $\partial_\mu g^{\alpha\beta} = 0$

$+ \frac{1}{2} g^{\sigma i}(\partial_\rho \partial_\nu g_{\lambda i} + \partial_\rho \partial_\lambda g_{i\nu} - \partial_\rho \partial_i g_{\nu\lambda})$

mit $g_{ab} g^{ca} = \delta^a_b$

$\Rightarrow g_{\sigma\mu} \partial_\rho \Gamma^\sigma_{\lambda\nu}$

$= \frac{1}{2} \delta^i_\mu (\partial_\rho \partial_\nu g_{\lambda i} + \partial_\rho \partial_\lambda g_{i\nu} - \partial_\rho \partial_i g_{\nu\lambda})$

$= \frac{1}{2}(\partial_\rho \partial_\nu g_{\lambda\mu} + \partial_\rho \partial_\lambda g_{\mu\nu} - \partial_\rho \partial_\mu g_{\nu\lambda})$

mitdem anderen Therm analog

$\Rightarrow R_{\mu\nu\rho\lambda} = \frac{1}{2}(\partial_\rho \partial_\nu g_{\lambda\mu} + \partial_\lambda \partial_\mu g_{\nu\rho} - \partial_\rho \partial_\mu g_{\nu\lambda} - \partial_\lambda \partial_\nu g_{\rho\mu})$

$-R_{\nu\mu\rho\lambda} = \frac{1}{2}(-\partial_\rho \partial_\mu g_{\lambda\nu} - \partial_\lambda \partial_\nu g_{\mu\rho} + \partial_\rho \partial_\nu g_{\mu\lambda} + \partial_\lambda \partial_\mu g_{\rho\nu})$

(2) $R^\mu{}_{\nu\rho\lambda} = -R^\mu{}_{\nu\lambda\rho} \Leftrightarrow -R^\mu{}_{\nu\rho\lambda} = R^\mu{}_{\nu\lambda\rho}$

$= ~~\ldots~~$

$(-\partial_\rho \Gamma^\mu_{\lambda\nu} + \partial_\lambda \Gamma^\mu_{\rho\nu}) + (-\Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\lambda\nu} + \Gamma^\mu_{\lambda\sigma}\Gamma^\sigma_{\rho\nu}) V^\nu$

$\overset{!}{=}$

$(\partial_\lambda \Gamma^\mu_{\rho\nu} - \partial_\rho \Gamma^\mu_{\lambda\nu}) + (\Gamma^\mu_{\lambda\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\lambda\nu}) V^\nu$

(3) $R_{\mu\nu\rho\lambda} = -R_{\rho\lambda\nu\mu}$ mit (1)

$-R_{\nu\mu\rho\lambda} \overset{!}{=} -R_{\rho\lambda\nu\mu}$

$\frac{1}{2}(-\partial_\rho \partial_\mu g_{\lambda\nu} - \partial_\lambda \partial_\nu g_{\mu\rho} + \partial_\rho \partial_\nu g_{\mu\lambda} + \partial_\lambda \partial_\mu g_{\rho\nu})$

$=$

$\frac{1}{2}(-\partial_\nu \partial_\lambda g_{\mu\rho} - \partial_\mu \partial_\rho g_{\lambda\nu} + \partial_\mu \partial_\lambda g_{\rho\nu} + \partial_\nu \partial_\rho g_{\mu\lambda})$

$\frac{6}{6}$

(4) $R^\mu{}_{\nu\rho\lambda} + R^\mu{}_{\lambda\nu\rho} + R^\mu{}_{\rho\lambda\nu} = 0$

$\Rightarrow \partial_\rho \Gamma^\mu_{\lambda\nu} - \partial_\lambda \Gamma^\mu_{\rho\nu} + \partial_\nu \Gamma^\mu_{\rho\lambda} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \partial_\lambda \Gamma^\mu_{\nu\rho} - \partial_\nu \Gamma^\mu_{\lambda\rho}$

$= 0$ erfüllt durch christ. sym. in unteren indizes.

~~ebenso bei dem $(\Gamma\Gamma - \Gamma\Gamma)V$ anteil~~

$\cancel{\ldots}\ \Gamma\Gamma + \Gamma\Gamma - \Gamma\Gamma + \Gamma\Gamma$

A2

mit Geodätengleichung

a) $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

$$g_{tt} = g^{tt} = -1 \qquad g_{ij} = a^2\delta_{ij} \; ; \; g^{ij} = \frac{1}{a^2}\delta_{ij} \qquad a = a(t)$$

$$\Gamma^t_{ij} = \frac{1}{2}g^{tt}(-\partial_t g_{ij}) = \dot a a \delta_{ij} \rightarrow \Gamma^t_{ii}$$

$d \neq \delta$

↳ Geodäte

$$\frac{d^2x^t}{d\lambda^2} + \Gamma^t_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0$$

$$\frac{dE}{d\lambda} + \dot a a \delta_{ij}\underbrace{\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}_{(\vec p)^2 = \frac{E^2}{a^2}} = 0 \qquad \text{mit } \frac{dx^0}{d\lambda}\frac{d}{dx^0} = \frac{d}{d\lambda}$$

$$\frac{d}{dx^0}E \cdot \frac{dx^0}{d\lambda} \qquad \text{mit } p^\alpha p_\alpha = 0$$

$$\cancel{\frac{E}{dt} E + E^2 (\dot a a \frac{dx^i}{dx^0}\frac{dx^j}{dx^0})\delta_{ij} = 0}$$

$$\cancel{\frac{E}{dt} + E \dot a a}$$

$$\frac{dE}{dt}E + E^2\left(\dot a a \cdot \frac{1}{a^2}\right) = 0 \qquad | : E$$

$$\frac{d}{dt}E + E\frac{\dot a}{a} = 0$$

b) $$\frac{dE}{dt}\frac{1}{E} = -\frac{\dot a}{a} \rightarrow \frac{\dot E}{E} = -\frac{\dot a}{a}$$

Substitution $\int \frac{\dot E}{E}dt = \int -\frac{\dot a}{a}dt = -\int\frac{\dot a}{a}dt$

mit $1/x$

$$\left[\int \frac{1}{x}dx\right]_{x=E} = \ln|x|\Big|_{x=E} = \ln|E| \stackrel{!}{=} -\ln(a)$$

$$\rightarrow \boxed{E \sim \frac{1}{a}} \qquad \rightarrow E \sim e^{-\ln(a)} = a^{-1}$$

Physical meaning ?

5/6

Max Perwitlau - Kyra Klos
bei Dennis

3

a) $g_{\mu\nu} = \sum_k \frac{\partial x^k}{\partial q^\mu} \frac{\partial x^k}{\partial q^\nu}$ $\qquad q^\nu \hat{=} \begin{pmatrix} \theta \\ \varphi \end{pmatrix}$, $\quad x^\mu \hat{=} \vec{r}(\theta, \varphi)$

$$g_{00} = \left(\frac{\partial x^0}{\partial q^0}\right)^2 + \left(\frac{\partial x^1}{\partial q^0}\right)^2 + \underbrace{\left(\frac{\partial x^2}{\partial q^0}\right)^2}_{=0} = \underbrace{((\cos)^2\theta + (-\sin)^2\theta)}_{=1} (a + r\cos\varphi)^2 = \underline{(a + r\cos\varphi)^2}$$

$$g_{11} = \left(\frac{\partial x^0}{\partial q^1}\right)^2 + \left(\frac{\partial x^1}{\partial q^1}\right)^2 + \left(\frac{\partial x^2}{\partial q^1}\right)^2 = \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} \underbrace{(-r\sin\varphi)^2 + (r^2\cos^2\varphi)}_{=r^2} = \underline{r^2}$$

$$g_{01} = \left(\frac{\partial x^0}{\partial q^0} \cdot \frac{\partial x^0}{\partial q^1}\right) + \left(\frac{\partial x^1}{\partial q^0} \cdot \frac{\partial x^1}{\partial q^1}\right) + \underbrace{\left(\underbrace{\frac{\partial x^2}{\partial q^0}}_{=0} \cdot \frac{\partial x^2}{\partial q^1}\right)}_{=0} = g_{10} = -\sin\theta(a + r\cos\varphi)\cos\theta(a - r\sin\varphi)$$
$$+ \cos\theta(a + r\cos\varphi)\sin\theta(a - r\sin\varphi)$$
$$= (a + r\cos\varphi)\underbrace{(\sin\theta\cos\theta - \sin\theta\cos\theta)}_{=0}(a - r\sin\varphi)$$
$$\underline{\underline{= 0}}$$

$$\Rightarrow g = \begin{bmatrix} (a + r\cos\varphi)^2 & 0 \\ 0 & r^2 \end{bmatrix} \quad \square$$

$g = J^T J$

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b)

$$\delta \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda = 0$$

Goal: $\frac{dx^\sigma}{d\lambda} \nabla_\sigma \left(\frac{dx^\mu}{d\lambda}\right) = 0 = \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda}$

$$= \int \delta \underbrace{\sqrt{g_{\mu\nu} \frac{dx^\mu dx^\nu}{d\lambda d\lambda}}}_{=\sqrt{-f}}\, d\lambda = -\frac{1}{2}\int \sqrt{\frac{1}{-f}}\, \delta f\, d\lambda = -\frac{1}{2}\int \frac{\delta f}{\sqrt{-f}}\, d\tau = -\frac{1}{2}\int \delta f\, d\tau = \delta I$$

λ is abitrary, we choose it to be τ

$$-f = g_{\mu\nu} \frac{dx^\mu}{d\tau} \underbrace{\frac{dx^\nu}{d\tau}}_{=u^\nu} = -1$$

$$\delta I = \delta \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}\, d\tau$$

$$x^\mu \to x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma$$

$\delta x^{\mu 2} \to 0$ (Taylor)

$$= \frac{1}{2}\int \Big(\underbrace{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau}}_{a} + \underbrace{g_{\mu\nu} \frac{d(\delta x^\mu)}{d\tau} \frac{dx^\nu}{d\tau}}_{b} + \underbrace{(\partial_\sigma g_{\mu\nu})\delta x^\sigma \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}_{c} + \mathcal{O}(\delta x^2)\Big) d\tau$$

(b): $$\frac{1}{2}\int_M \underbrace{g_{\mu\nu} \frac{dx^\nu}{d\tau}}_{u} \cdot \underbrace{\frac{d(\delta x^\mu)}{\partial \tau}}_{v'}\, d\tau = \underbrace{\left[\frac{1}{2} g_{\mu\nu} \frac{dx^\nu}{d\tau} \delta x^\mu\right]_M}_{\text{variation at endpoints is } 0} - \frac{1}{2}\int \delta x^\mu \left(\frac{d g_{\mu\nu}}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{d^2x^\nu}{d\tau^2}\right) d\tau$$

$$= -\frac{1}{2}\int \left(g_{\mu\nu} \frac{d^2x^\nu}{d\tau^2} + \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau}\right) \delta x^\mu\, d\tau$$

(a): analogue: $$-\frac{1}{2}\int \left(g_{\mu\nu} \frac{d^2x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\tau} \frac{dx^\nu}{d\tau}\right) \delta x^\nu\, d\tau$$

$$\delta I = -\frac{1}{2}\int \Big(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\sigma (-\partial_\sigma g_{\mu\nu}) + \frac{dx^\sigma}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\nu (\partial_\sigma g_{\mu\nu}) + \frac{dx^\sigma}{d\tau} \frac{dx^\mu}{d\tau} (\partial_\sigma g_{\mu\nu}) \delta x^\mu + g_{\mu\nu} \frac{d^2x^\mu}{d\tau^2} \delta x^\nu$$
$$+ g_{\mu\nu} \frac{d^2x^\nu}{d\tau^2} \delta x^\mu\Big)$$

$$\underset{\text{dummy indices}}{=} -\int \Big(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \frac{1}{2}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \delta x^\sigma + \frac{1}{2} \frac{d^2x^\mu}{d\tau^2} g_{\mu\sigma} \delta x^\sigma + \frac{1}{2} \frac{d^2x^\mu}{d\tau^2} g_{\sigma\mu} \delta x^\sigma\Big) d\tau$$

$$= - \int \underbrace{\left(\frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \frac{1}{2} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) + \frac{d^2 x^\mu}{d\tau^2} g_{\mu\sigma} \right)}_{=0 \text{ for any } \delta x^\sigma} \delta x^\sigma \, d\tau$$

$g_{\mu\nu} = g_{\nu\mu}$

$\left(\cdot g^{\alpha\sigma} \right.$

$$\frac{d^2 x^\alpha}{d\tau^2} + \underbrace{\frac{1}{2} g^{\alpha\sigma} \left(\overset{\mathrm{I}}{\partial_\mu g_{\sigma\nu}} + \overset{\mathrm{II}}{\partial_\nu g_{\mu\sigma}} - \overset{\mathrm{III}}{\partial_\sigma g_{\mu\nu}} \right)}_{= \Gamma^\alpha_{\mu\nu}} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

OK, aber die allgemeine Herleitung der Geodätengleichung war nicht gefragt.

I $\sigma = \nu = \alpha$

II $\mu = \sigma = \alpha$ $\qquad \alpha = 1$ as $\partial_\theta g_{00} = 0 = \partial_\varphi g_{11}$

III $\mu = \nu,\ \alpha = \sigma$

$\rightarrow \sigma = \alpha$

I $\nu = \alpha$

II $\mu = \alpha$ $\Big\} \mu = \nu \Rightarrow \Gamma^\alpha_{\mu\nu} = 0$ if $\mu \neq \nu$ as $g_{\mu\nu} = 0$ for $\mu \neq \nu$

III $\mu = \nu$

$$\Gamma^0_{00} = \frac{1}{2(a + r\cos\varphi)^2} \left(\overbrace{\partial_\theta (a + r\cos\varphi)^2}^{=0} + 0 \right) = 0$$

$$\Gamma^1_{00} = \frac{1}{2r^2} \left(-\partial_\varphi (a + r\cos\varphi)^2 \right) = \underline{\frac{(a + r\cos\varphi)}{r} \sin\varphi}$$

$$\Gamma^0_{11} = \frac{1}{2(a + r\cos\varphi)^2} \cdot \left(\underbrace{-\partial_\theta r^2}_{=0} \right) = 0$$

$$\Gamma^1_{11} = \frac{1}{2r^2} 0 = 0$$

$$\Gamma^0_{10} = \frac{1}{2(a + r\cos\varphi)^2} \left(\underbrace{\partial_\theta (a + r\cos\varphi)^2}_{=0} \right) = 0 = \Gamma^0_{01}$$

$$\Gamma^1_{10} = \frac{1}{2r^2} \left(\underbrace{\partial_\theta r^2}_{=0} \right) = 0 = \Gamma^1_{01}$$

Aufgabenstellung: $\delta \int ds = 0$ -> Geodätengl. (für den Torus)

-> Christoffelsymb.

$$\frac{1}{2} g^{\alpha\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) = \Gamma^\alpha_{\mu\nu}$$

a : first component
b : second "
l,m : any "

$$\partial_a g_{lm} = 0 \qquad \partial_b g_{lm} \hat{=} \begin{bmatrix} -(a+r\cos\varphi)\cdot 2r\sin\varphi & 0 \\ 0 & 0 \end{bmatrix}$$

$$= g_{lm,a} \qquad\qquad = g_{lm,b}$$

$$g^{lm} \hat{=} \begin{bmatrix} 1/(a+r\cos\varphi)^2 & 0 \\ 0 & 1/r^2 \end{bmatrix}$$

$$\Gamma^a_{aa} = \frac{1}{2} g^{aa}\left(g_{aa,a}\cdot 1\right) = 0$$

$$\boxed{\Gamma^a_{ba} = \Gamma^a_{ab}} = \frac{1}{2} g^{aa}\left(\underbrace{g_{ab,a}}_{=0} + g_{aa,b} - \underbrace{g_{ab,a}}_{=0}\right) = \frac{1}{2} g^{aa} g_{aa,b} = -\frac{r\sin\varphi}{a+r\cos\varphi}$$

$$\Gamma^a_{bb} = \frac{1}{2} g^{aa}\left(\underbrace{g_{ab,b}}_{=0} + \underbrace{g_{ba,b}}_{=0} - \underbrace{g_{bb,a}}_{=0}\right) = 0$$

$$\boxed{\Gamma^b_{aa}} = \frac{1}{2} g^{bb}\left(\underbrace{g_{ba,a}}_{=0} + \underbrace{g_{ab,a}}_{=0} - g_{aa,b}\right) = -\frac{1}{2} g^{bb} g_{aa,b} = \frac{(a+r\cos\varphi)}{r}\sin\varphi$$

$$\Gamma^b_{ba} = \Gamma^b_{ab} = \frac{1}{2} g^{bb}\left(g_{bb,a} + g_{ab,b} - g_{ab,b}\right) = 0$$

$$\Gamma^b_{bb} = \frac{1}{2} g^{bb}\left(\ldots\right) = 0$$

sorry for the mess!

c)

$$R^{\rho}{}_{\sigma\mu\nu} = \Gamma^{\rho}{}_{\nu\sigma,\mu} - \Gamma^{\rho}{}_{\mu\sigma,\nu} + \underbrace{\Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma}} - \underbrace{\Gamma^{\rho}{}_{\nu\lambda}}\underbrace{\Gamma^{\lambda}{}_{\mu\sigma}}$$

$$(\cancel{\Gamma^{a}{}_{ba}} \vee \Gamma^{a}{}_{ab}) \times (\Gamma^{b}{}_{aa})$$

$$\boxed{a=\rho=\nu=\mu=\sigma\neq\lambda \ \vee\ \nu=\lambda=\sigma\neq\mu=\rho=a \ \vee\ a=\lambda=\nu=\mu\neq\sigma=\rho=b}$$

$$\Gamma^{b}{}_{aa} \qquad \Gamma^{a}{}_{ba} \vee \Gamma^{a}{}_{ab}$$

derivatives

$$\Gamma^{a}{}_{ab,b} = \frac{-(r\sin\varphi)^2}{(a+r\cos\varphi)^2} - \frac{r\cos\varphi}{a+r\cos\varphi}$$

$$\Gamma^{b}{}_{aa,b} = \frac{1}{r}(a\cos\varphi + r\cos 2\varphi)$$

the only derivates that are $\neq 0$ ✓

$$\boxed{b=\lambda\neq\rho=\mu=\nu=\sigma=a \ \vee\ a=\lambda=\mu=\nu\neq\rho=\sigma=b \ \vee\ a=\lambda=\mu=\sigma\neq\nu=\rho=b}$$

$$\rightarrow R^{l}{}_{lll} = 0$$

$$R^{a}{}_{aab} \underset{(2)}{=} -R^{a}{}_{aba} = 0 \underset{(4)}{=} R^{a}{}_{baa}$$

$$R^{a}{}_{abb} = \Gamma^{a}{}_{ab,b} - \Gamma^{a}{}_{ab,b} + (\Gamma^{a}{}_{ab})^2 - (\Gamma^{a}{}_{ab})^2 = 0$$

$$R^{a}{}_{bab} = -R^{a}{}_{bba} = -\Gamma^{a}{}_{ba,b} - (\Gamma^{a}{}_{ba})^2 = \boxed{\frac{r\cos\varphi}{a+r\cos\varphi}}$$ ✓

$$R^{b}{}_{aab} = -R^{b}{}_{aba} = -\Gamma^{b}{}_{aa,b} + \Gamma^{b}{}_{aa}\Gamma^{b}{}_{ba} = \boxed{-\frac{1}{r}\cos\varphi(a+r\cos\varphi)}$$ ✓

$$R^{a}{}_{bbb} = R^{b}{}_{aaa} = 0$$

$$R^{b}{}_{bab} = 0$$

d)

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

$$\underline{R_{aa} = \frac{a + r\cos\varphi}{r}\cos\varphi}$$

$$\underline{R_{bb} = \frac{r\cos\varphi}{a + r\cos\varphi}}$$

$$R_{lm} = 0$$
$l \neq m$

$$R = R^{\mu}{}_{\mu} = g^{lm} R_{lm} = g^{aa} R_{aa} + g^{bb} R_{bb} = \underline{\underline{\frac{2\cos\varphi}{r(a + r\cos\varphi)}}}$$

e)

$$g_{00} = \sum_k \left(\frac{\partial x^k}{\partial q^0}\right)^2 = A^2(\sin^2 u + \cos^2 u) + 0 = 1 \cdot A^2$$

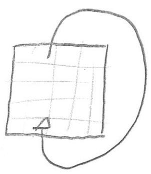
$$g_{11} = \sum_k \left(\frac{\partial x^k}{\partial q^1}\right)^2 = 0 + B^2(\sin^2 v + \cos^2 v) = 1 \cdot B^2$$

$$g_{01} = g_{10} = \sum_k \frac{\partial x^k}{\partial q^0}\frac{\partial x^k}{\partial q^1} = g_{00} \cdot 0 + g_{11} \cdot 0 = 0$$

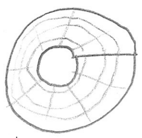
$$g = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}$$

Embedding a torus in 4D allows to choose a parametrisation that does not have a curvature. This was not possible in 3D, as seen in a).

A 2D→3D analogy could be a square where only top and bottom edge are associated:



When connecting these edges in 2D, curvature is unavoidable:

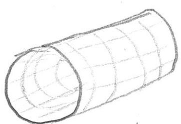


A ring

(circumference is greater on the outside of the "square").

gut!

By connecting these edges in 3D no curvature is introduced:



A cylinder

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