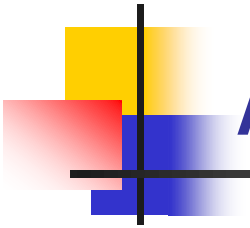
Gli alberi binari di ricerca (BST: Binary Search Trees)

Gianpiero Cabodi e Paolo Camurati

Dip. Automatica e Informatica
Politecnico di Torino

DEFINIRE CHIAVE MEDANA



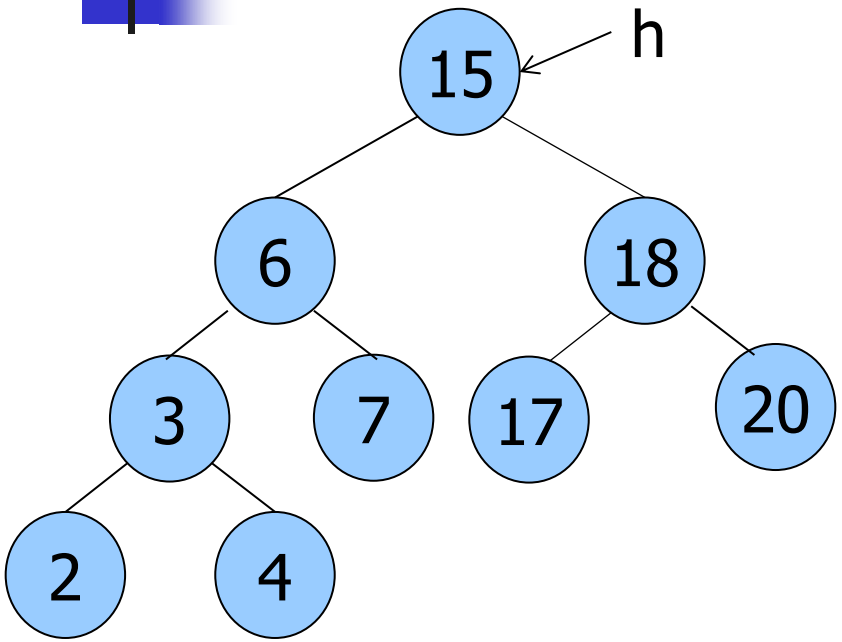
Attraversamenti di alberi binari

Attraversamento: elenco dei nodi secondo una strategia:

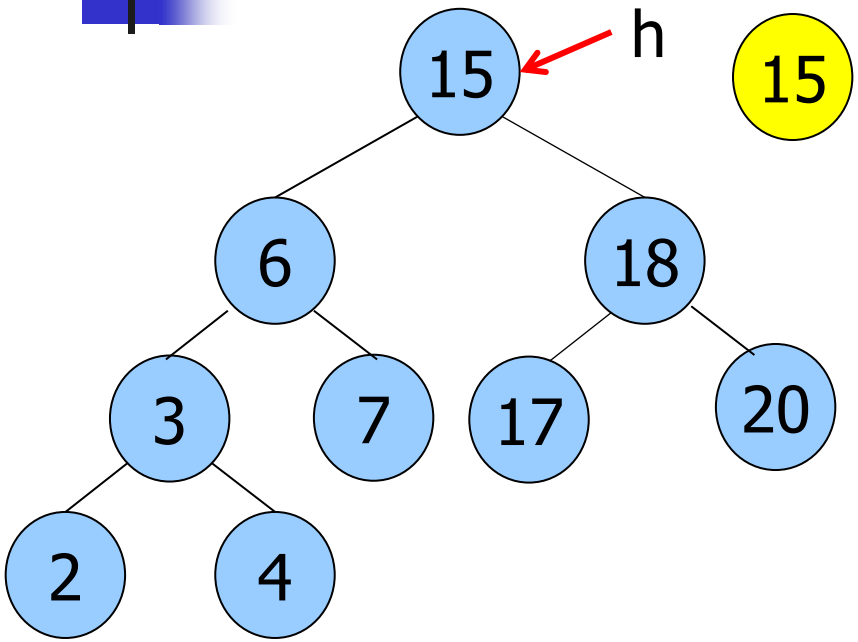
- Pre-ordine: h, Left(h), Right(h)
- In-ordine: Left(h), h, Right(h)
- Post-ordine: Left(h), Right(h), h

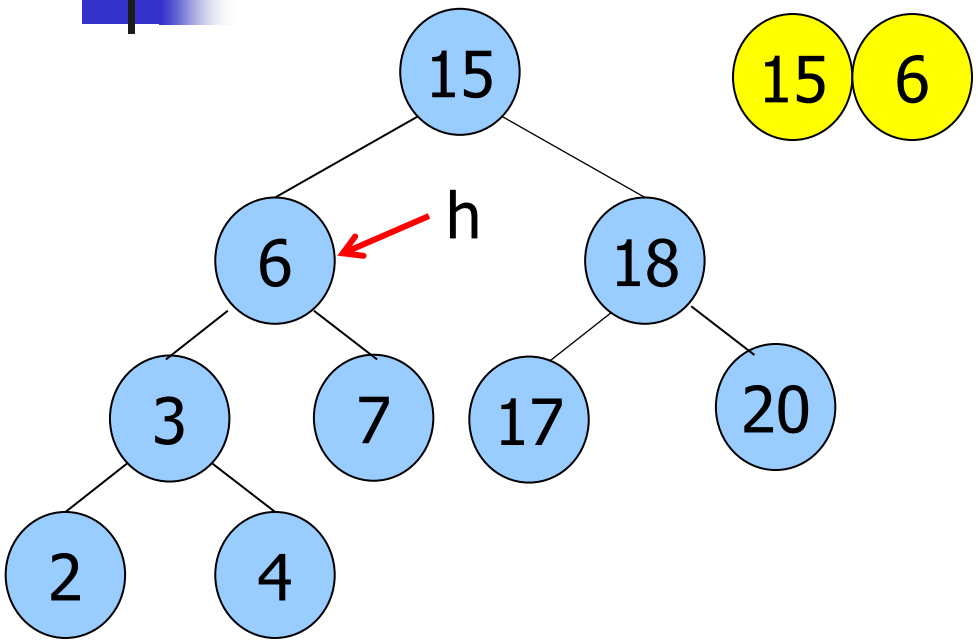
Complessità: $T(n) = \Theta(n)$.

Pre-ordine



Pre-ordine

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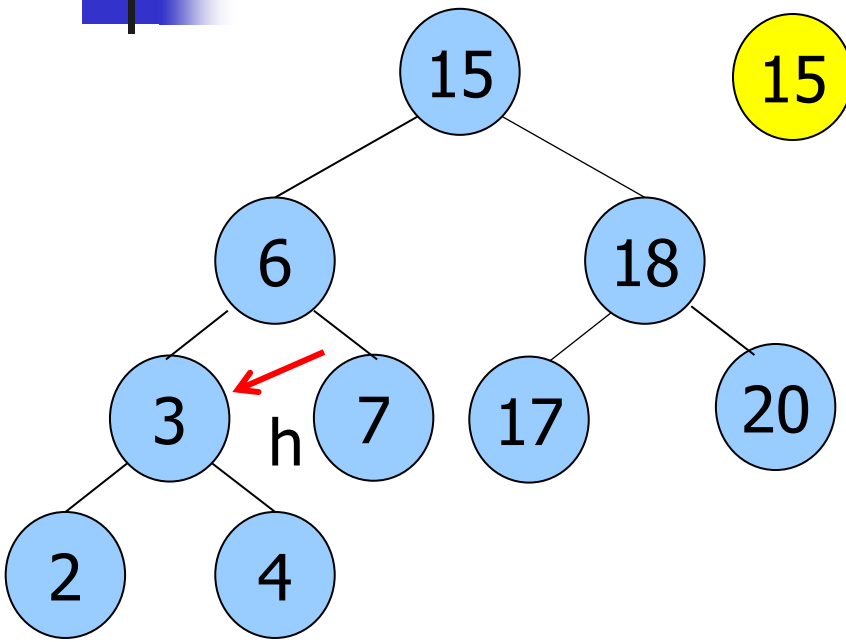
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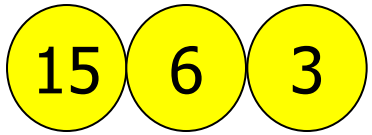
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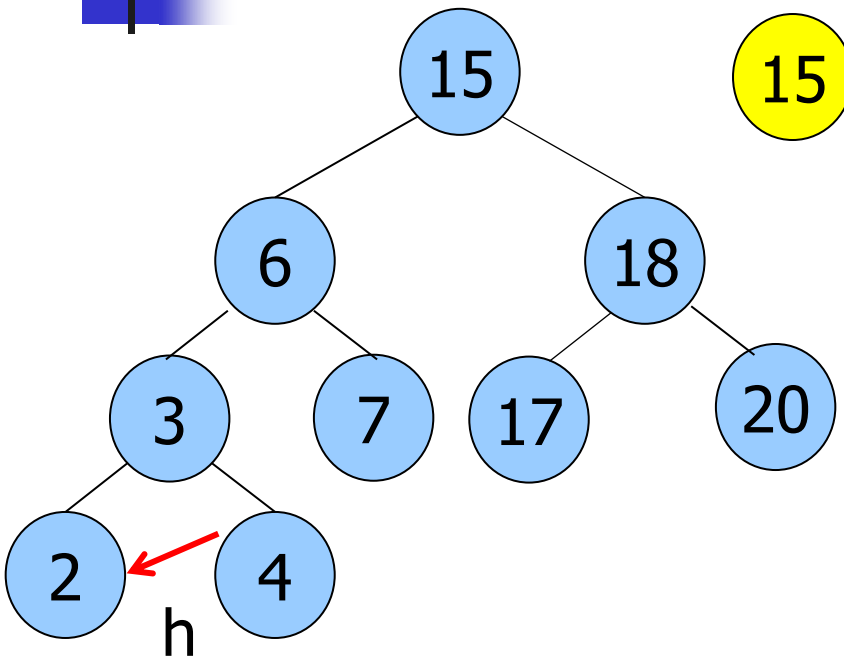
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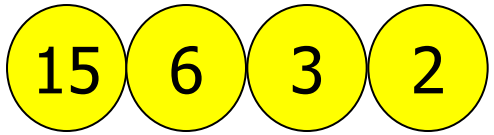
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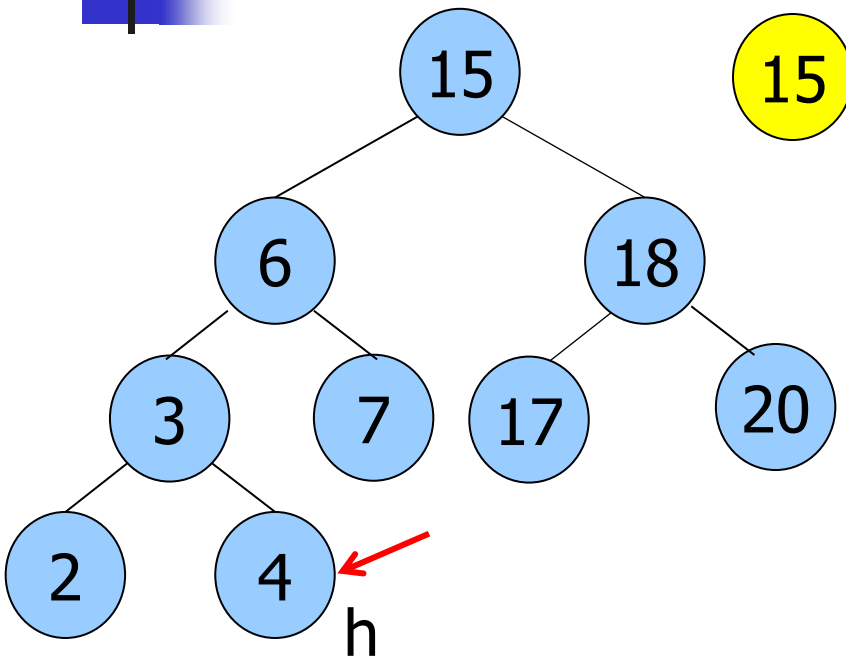
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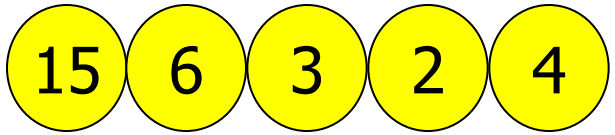
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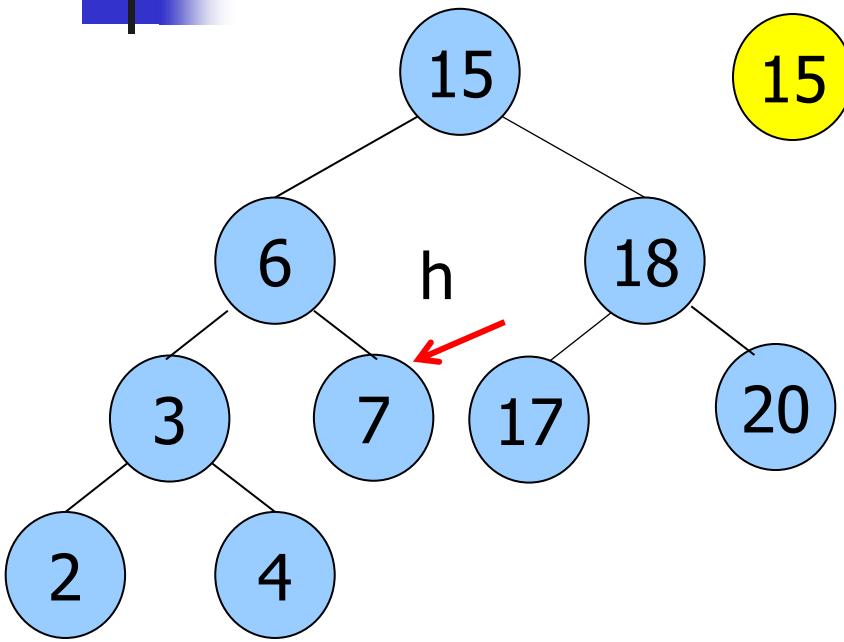
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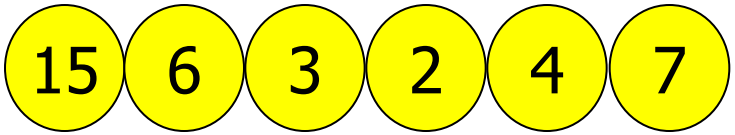
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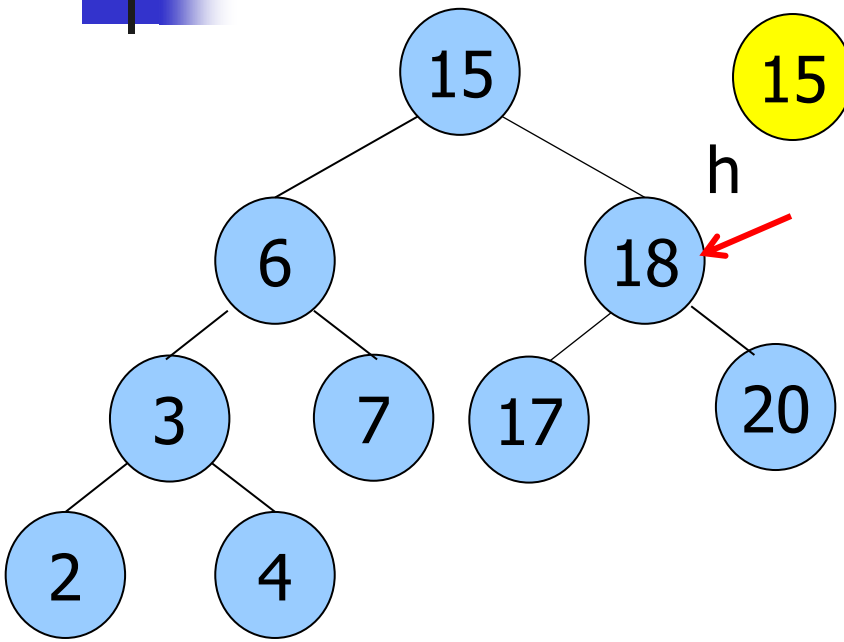
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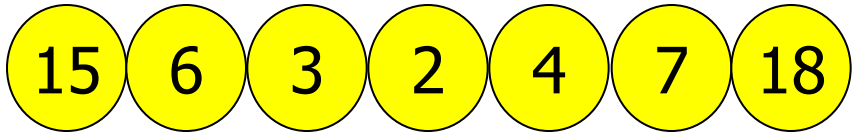
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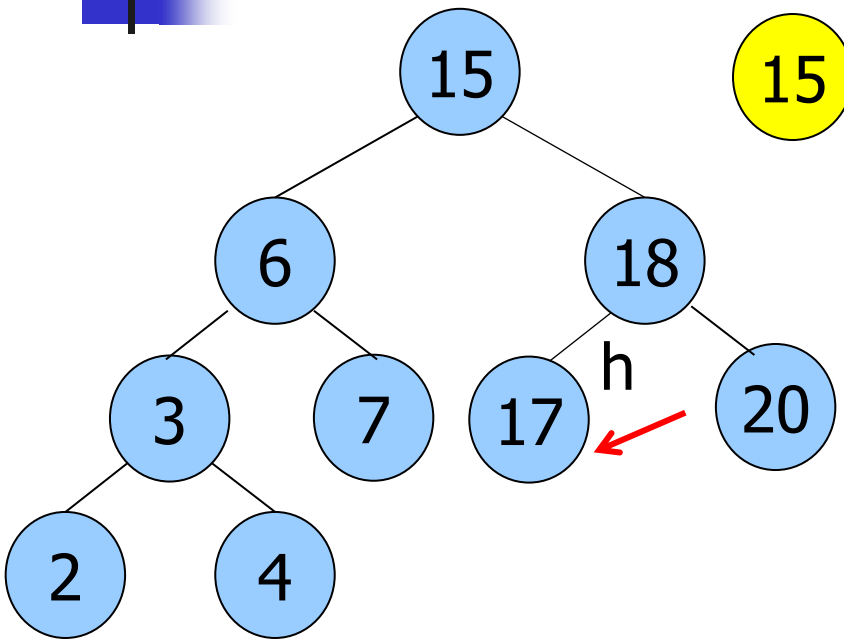
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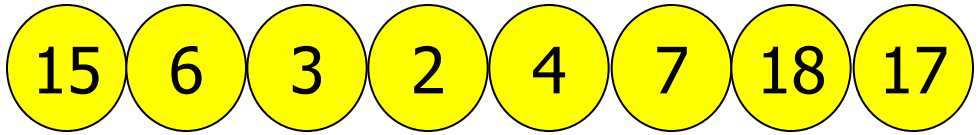
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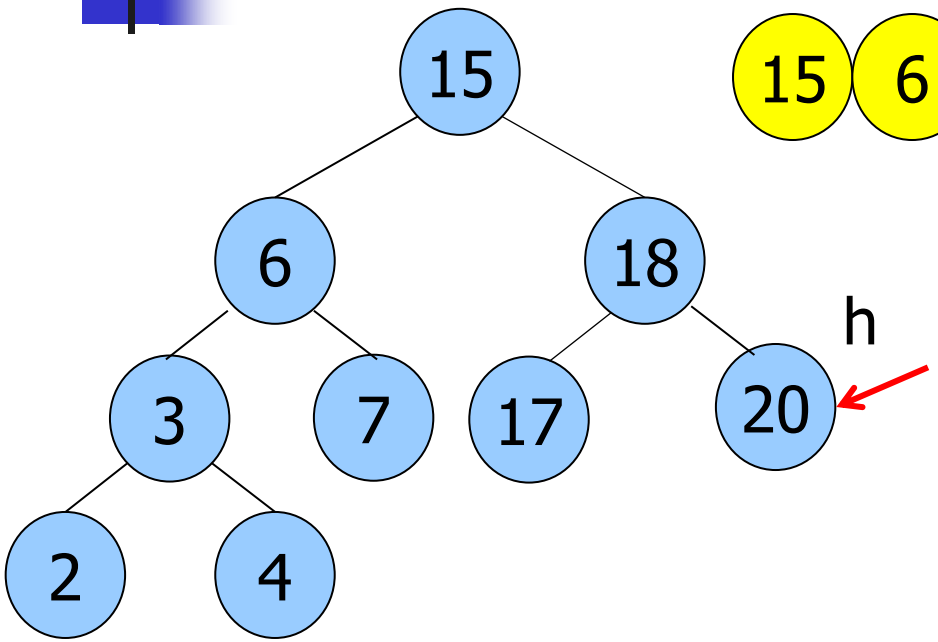
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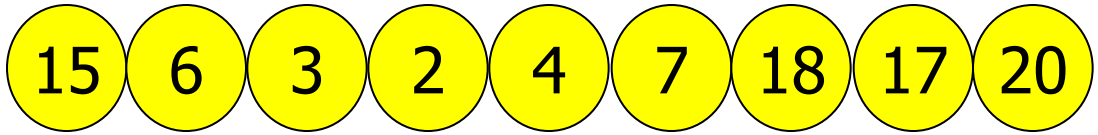
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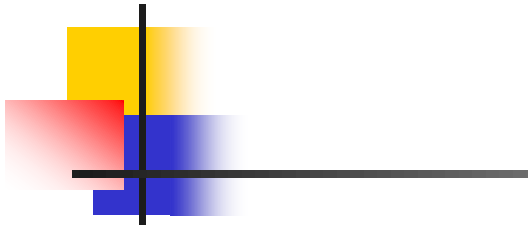

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void sortpreorderR(link h, void (*visit) (Item), link z){
  if (h == z)
    return;
  visit(h->item);
  sortpreorderR(h->l, visit, z);
  sortpreorderR(h->r, visit, z);
}

void BSTsortpreorder(BST bst, void (*visit)(Item)) {
  sortpreorderR(bst->head, visit, bst->z);
}
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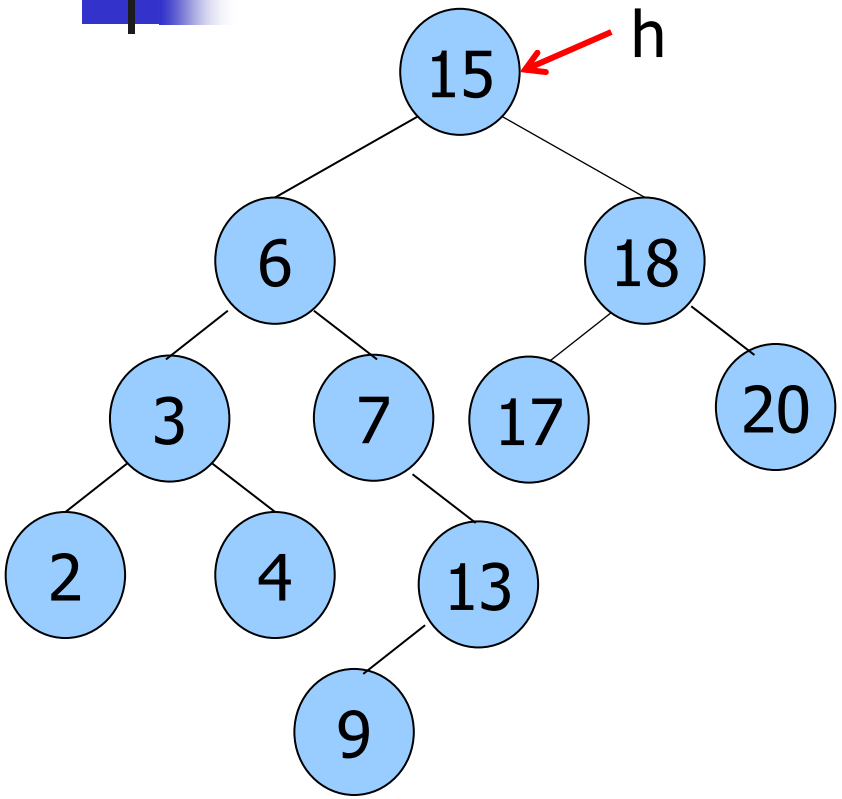
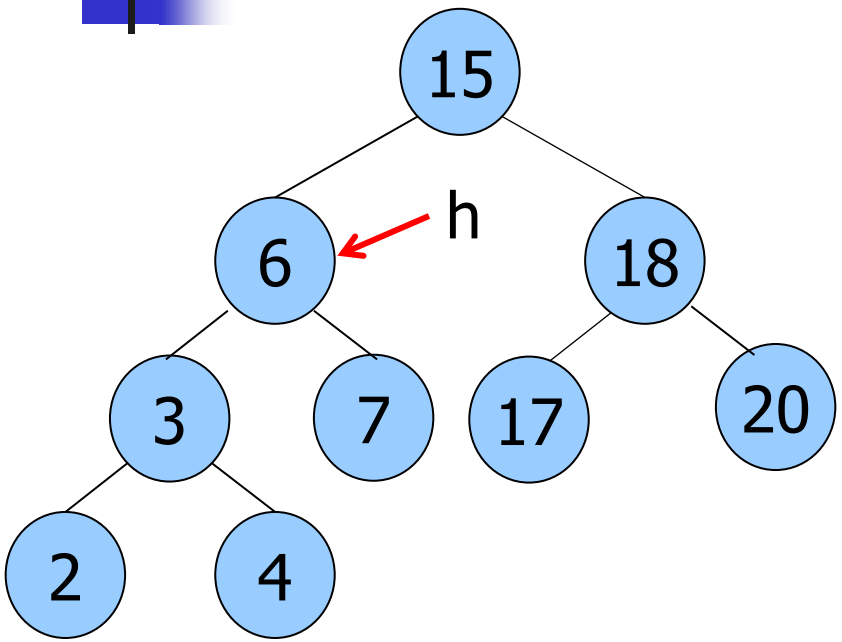
In-ordine

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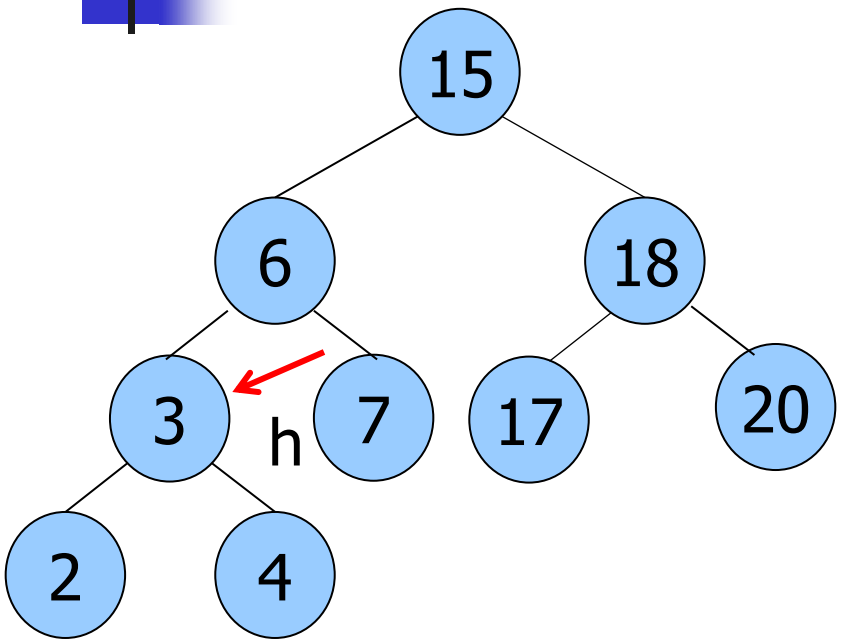
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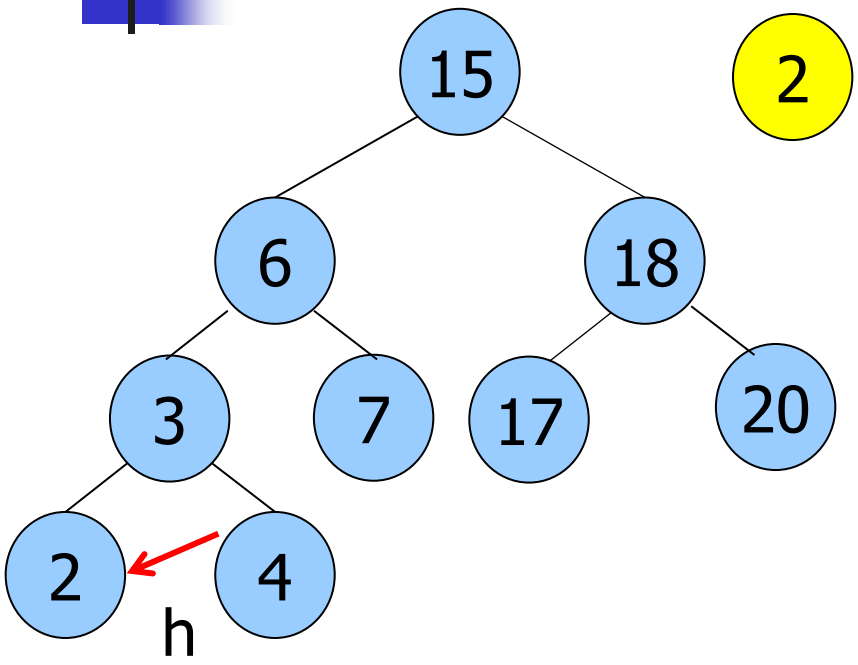
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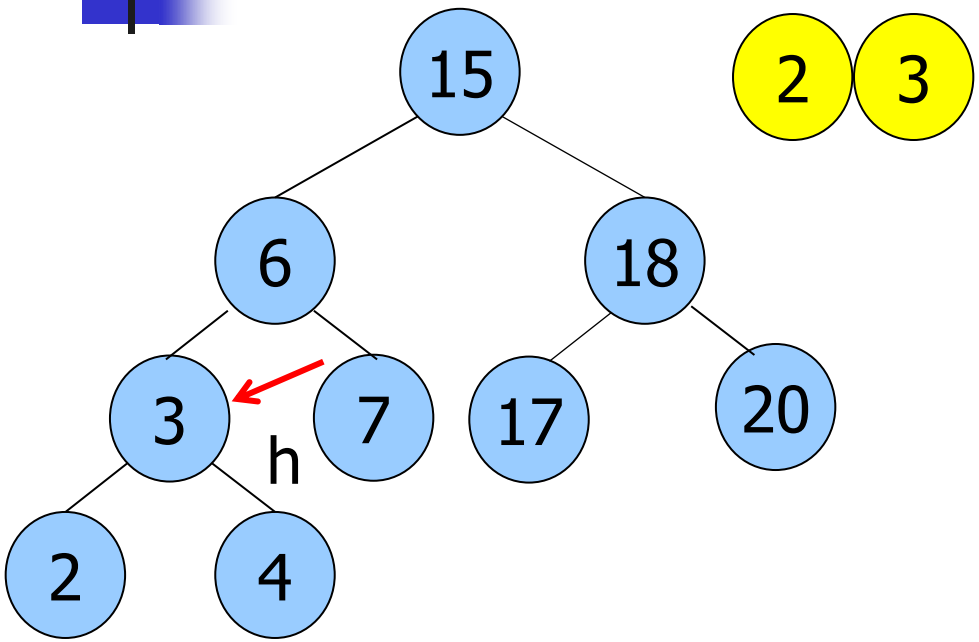
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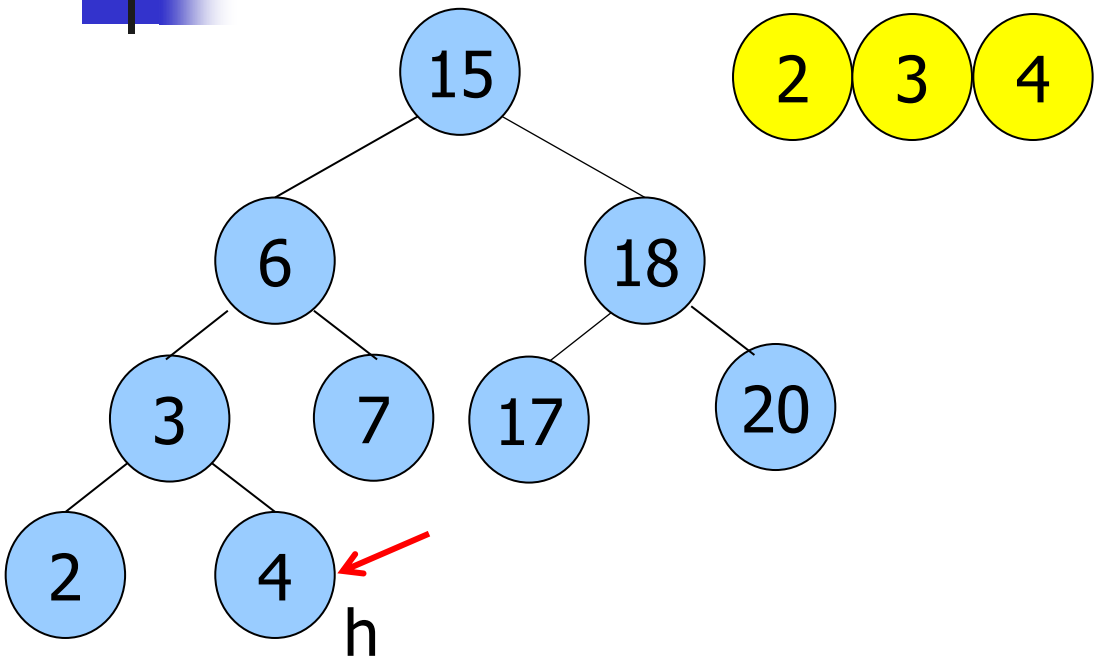
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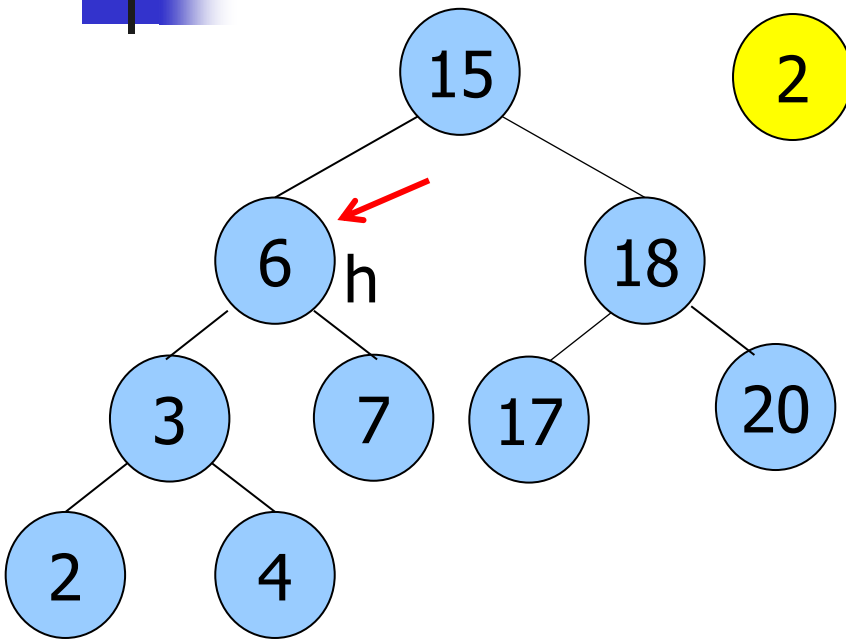
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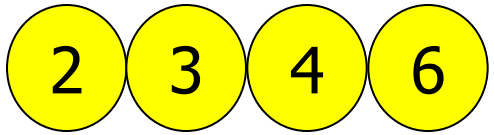
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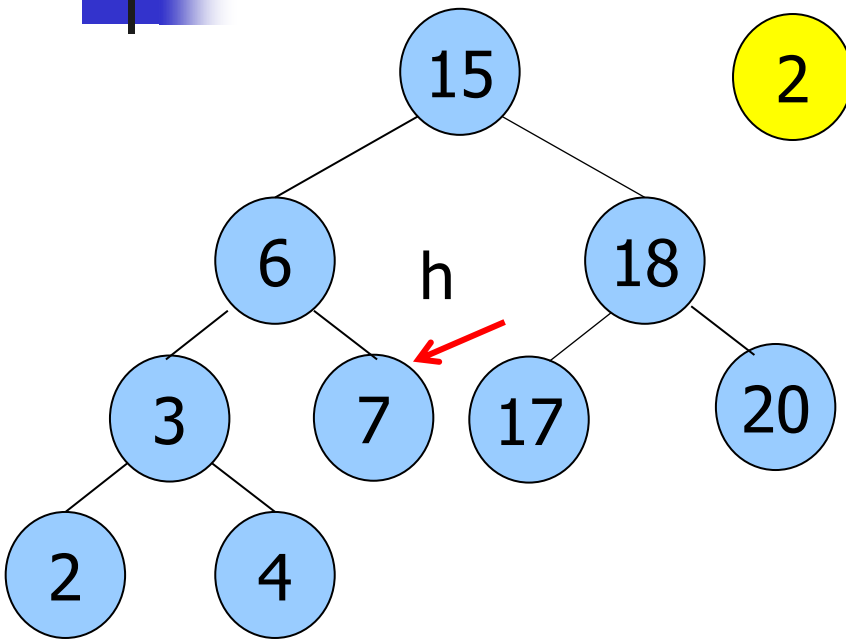
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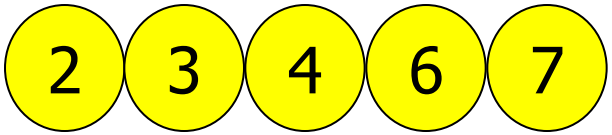
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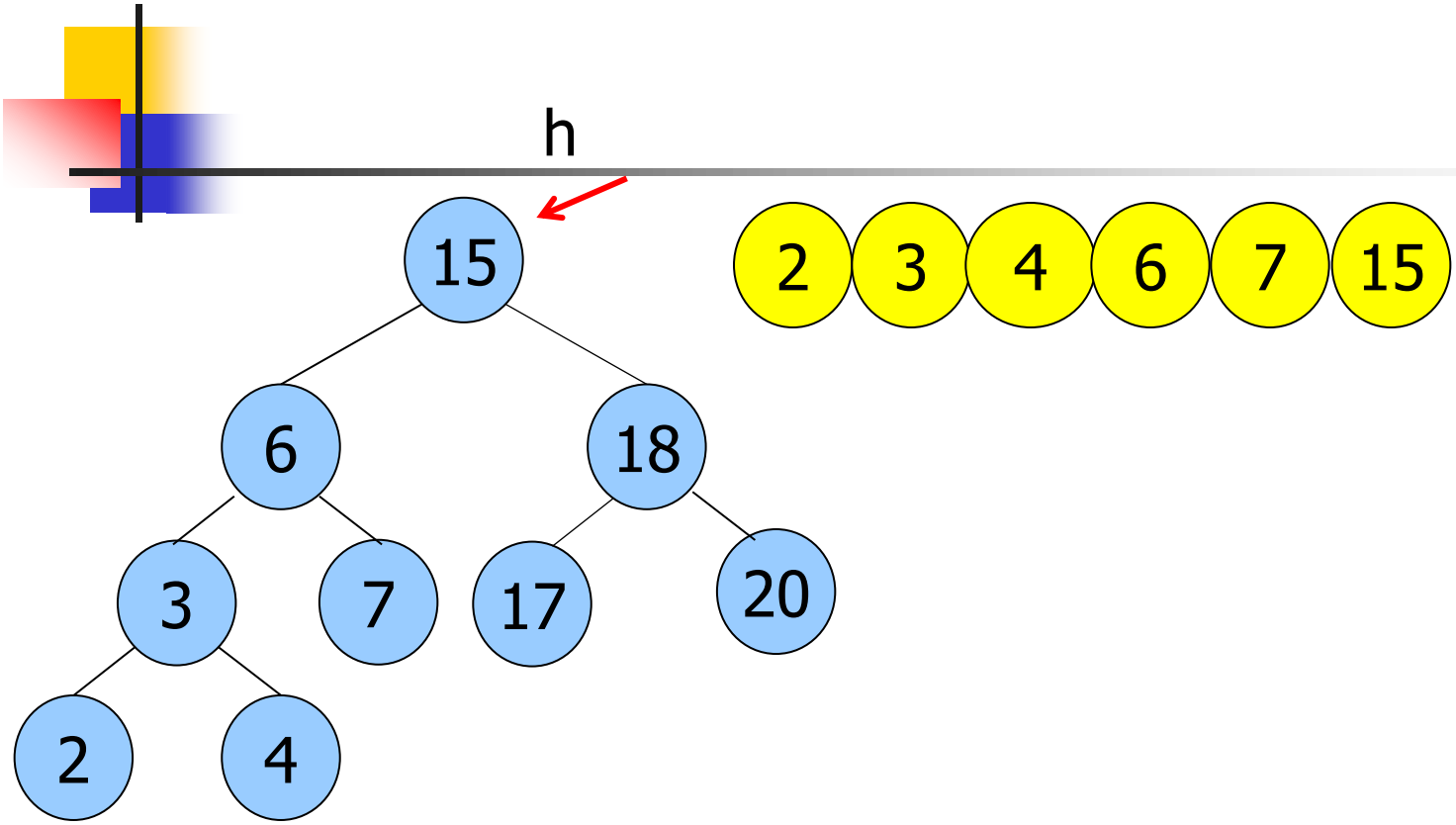
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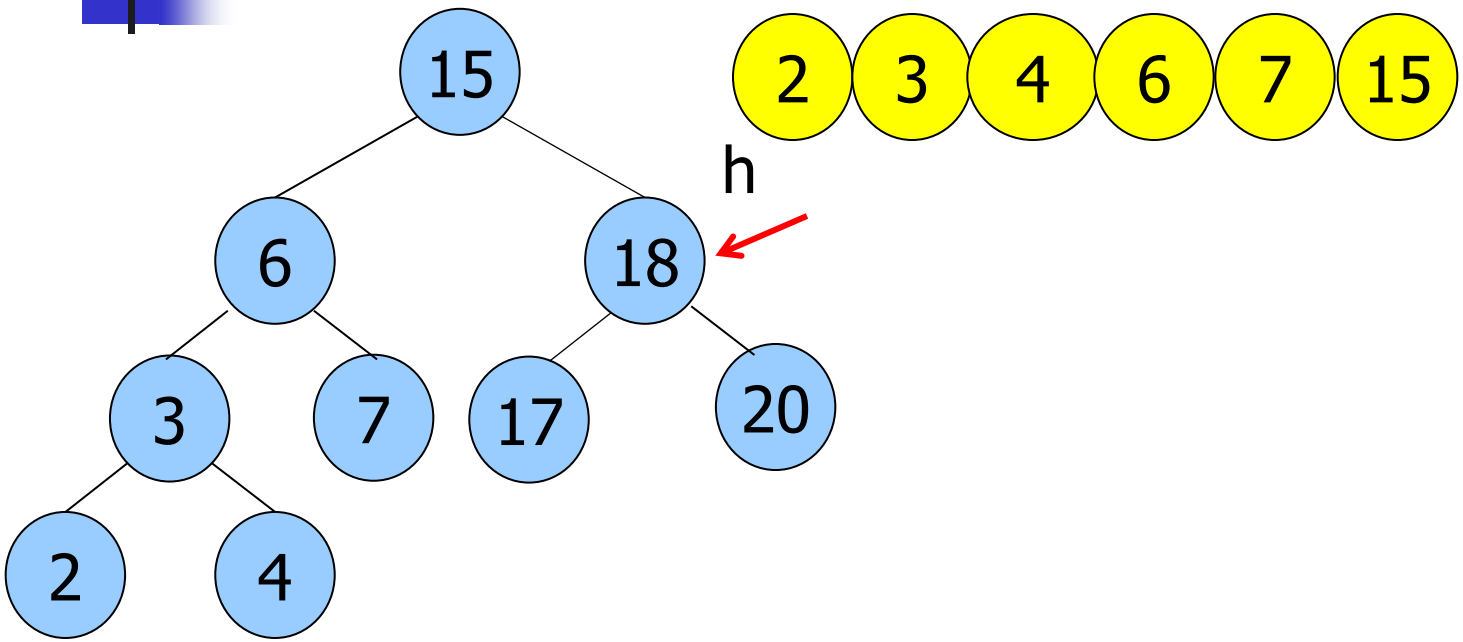
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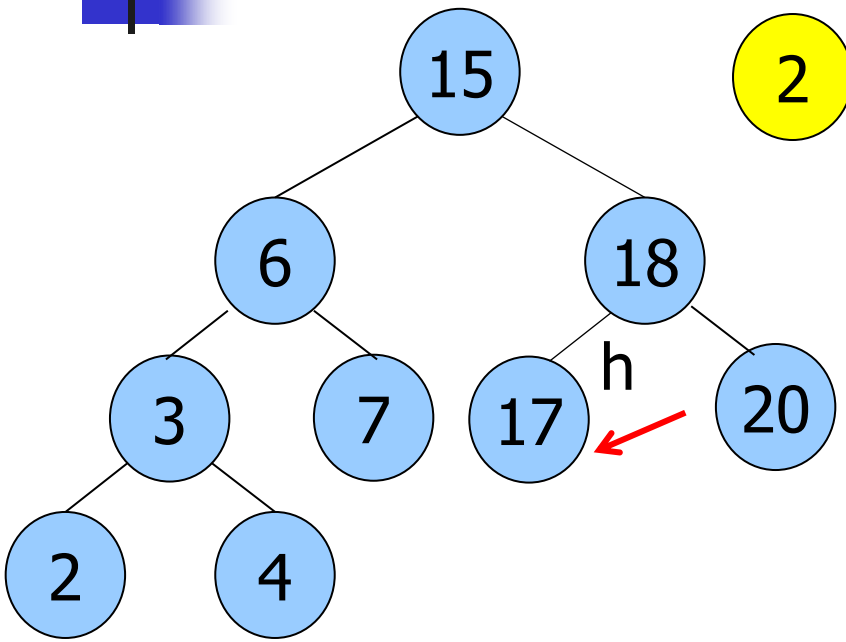
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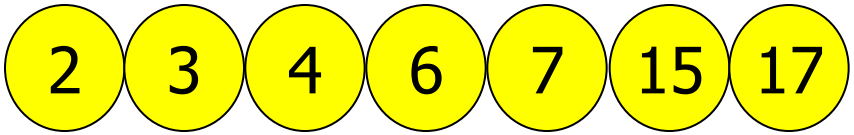
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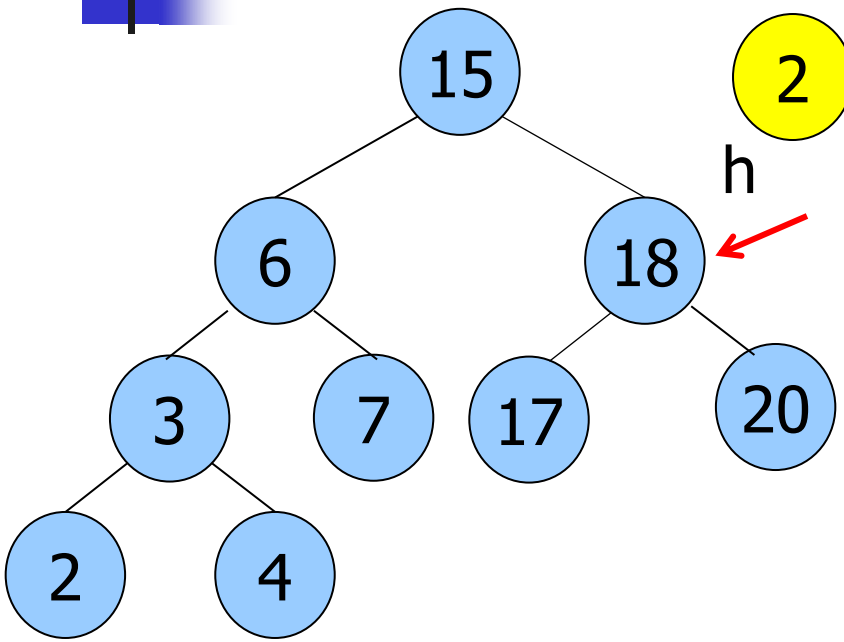
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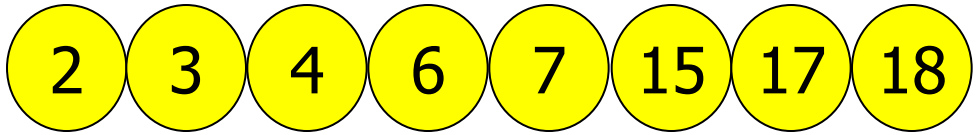
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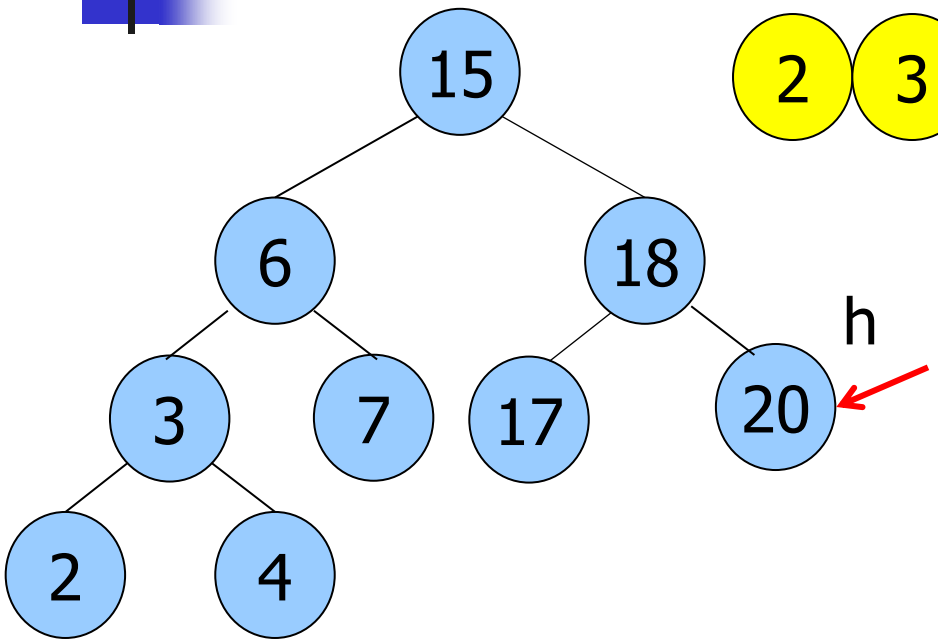
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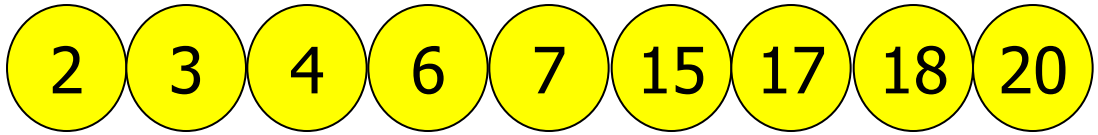
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void sortinorderR(link h, void (*visit) (Item), link z) {
  if (h == z) return;
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  visit(h->item);
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void BSTsortinorder(BST bst, void (*visit)(Item)) {
  sortinorderR(bst->head, visit, bst->z);
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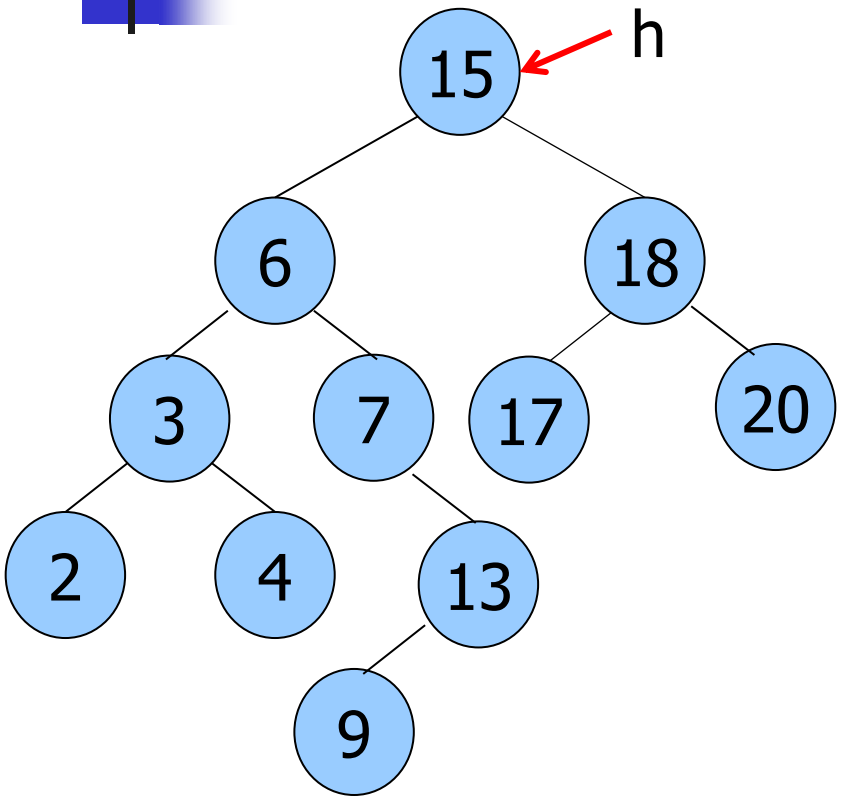
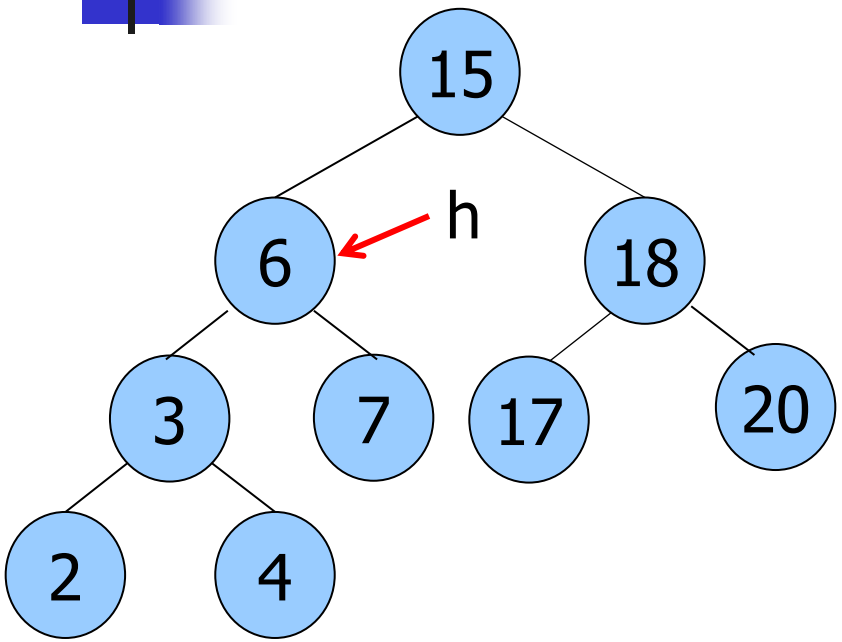
Post-ordine

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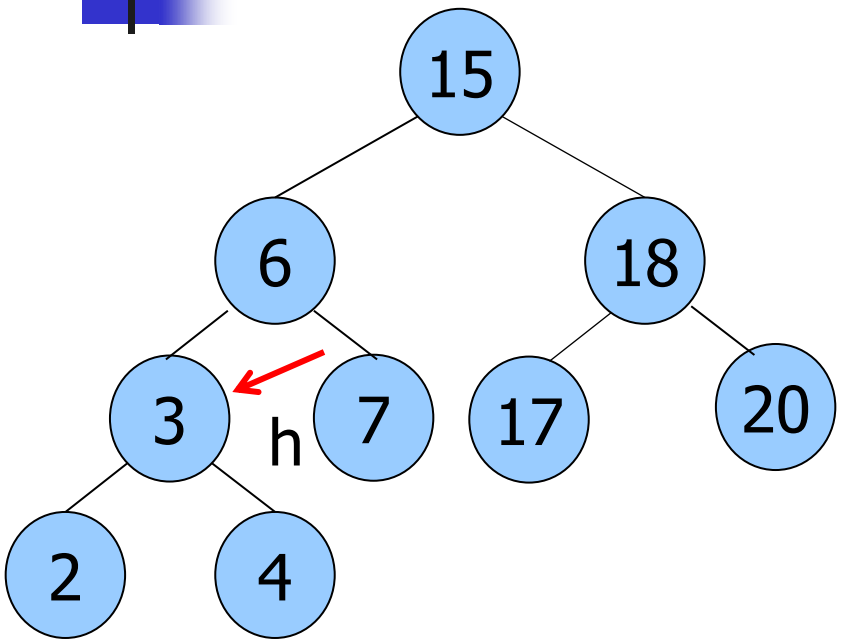
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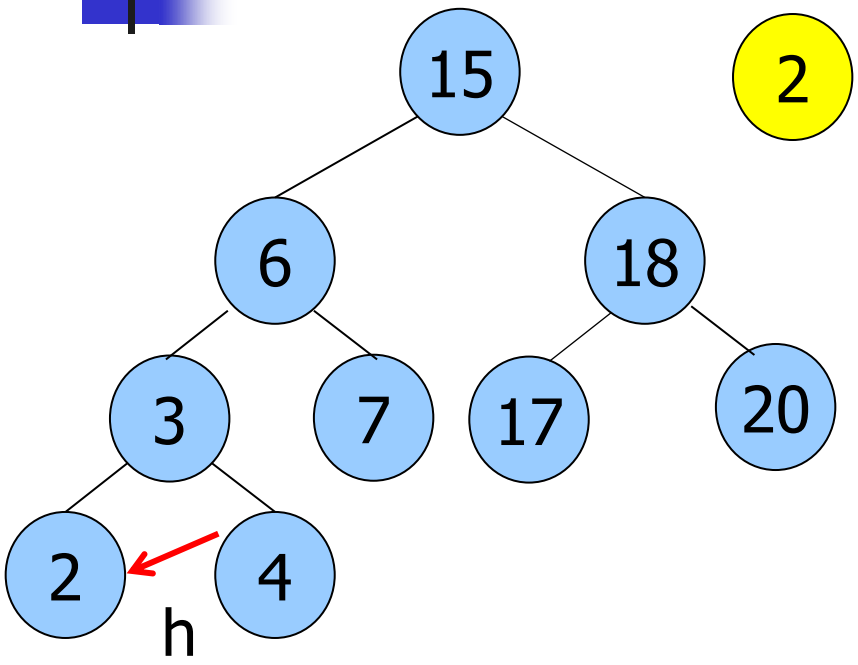
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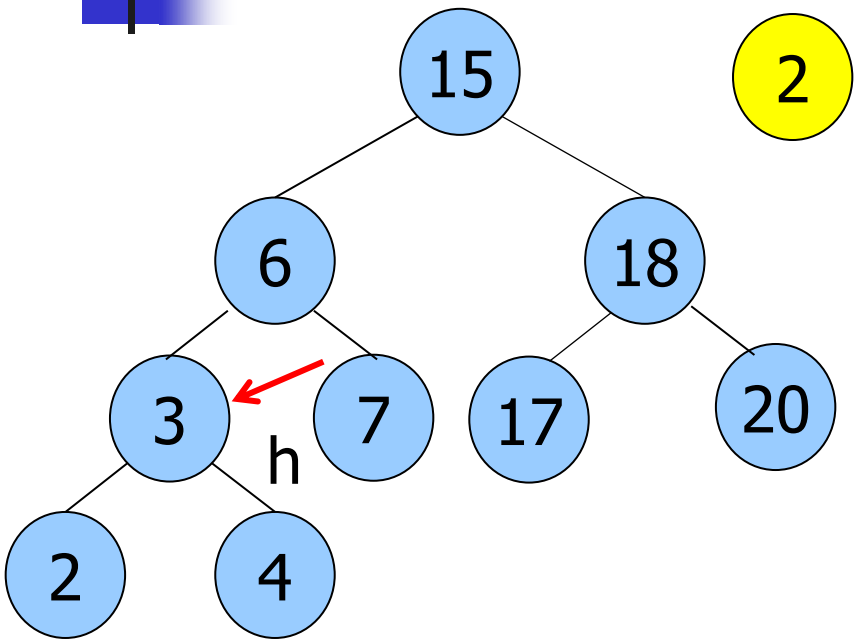
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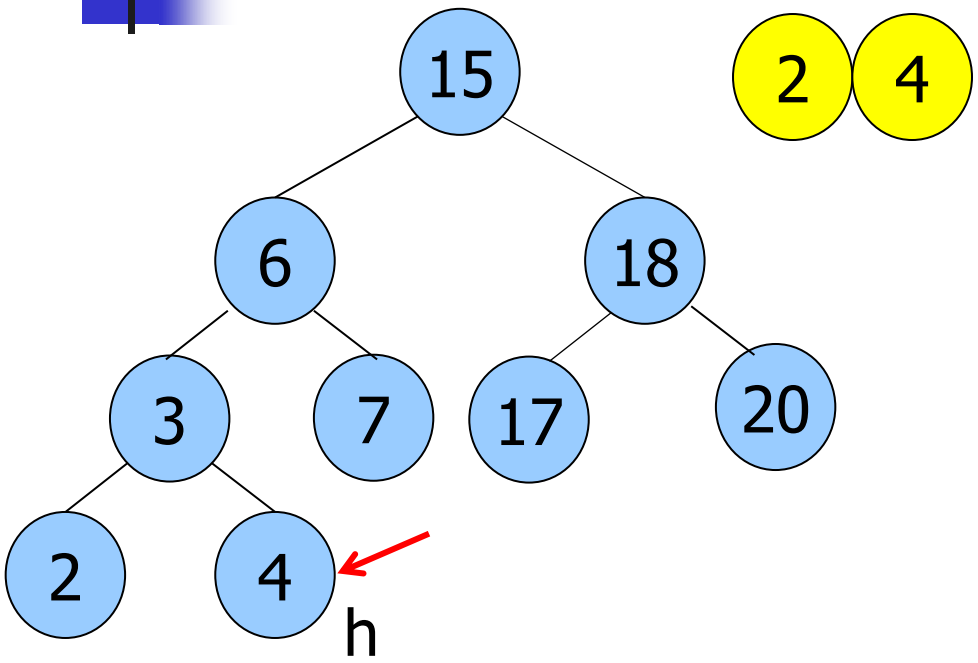
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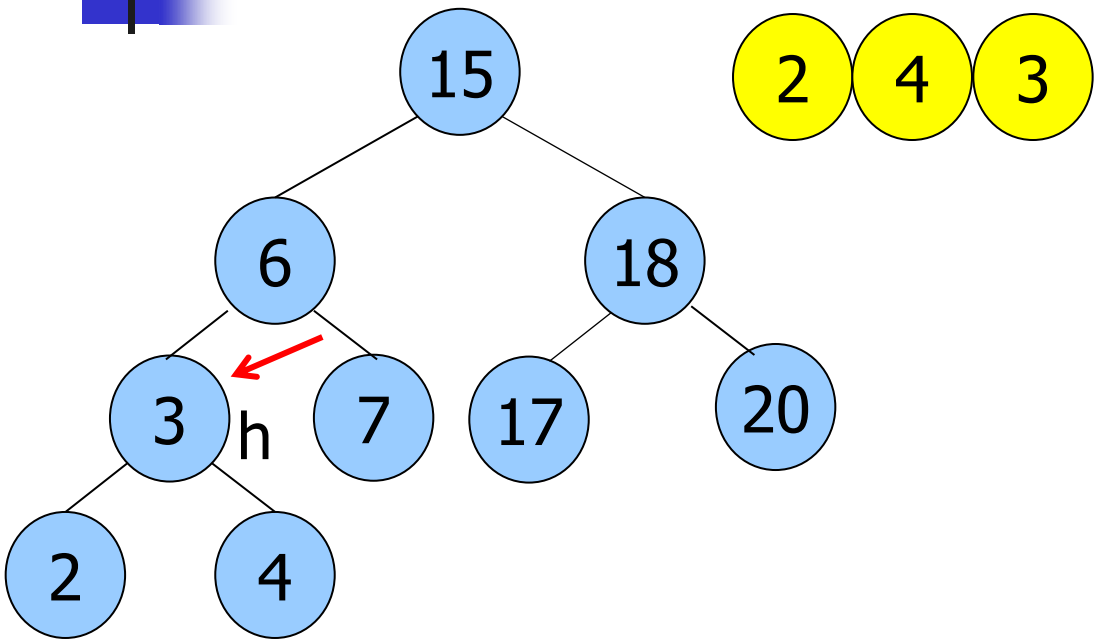
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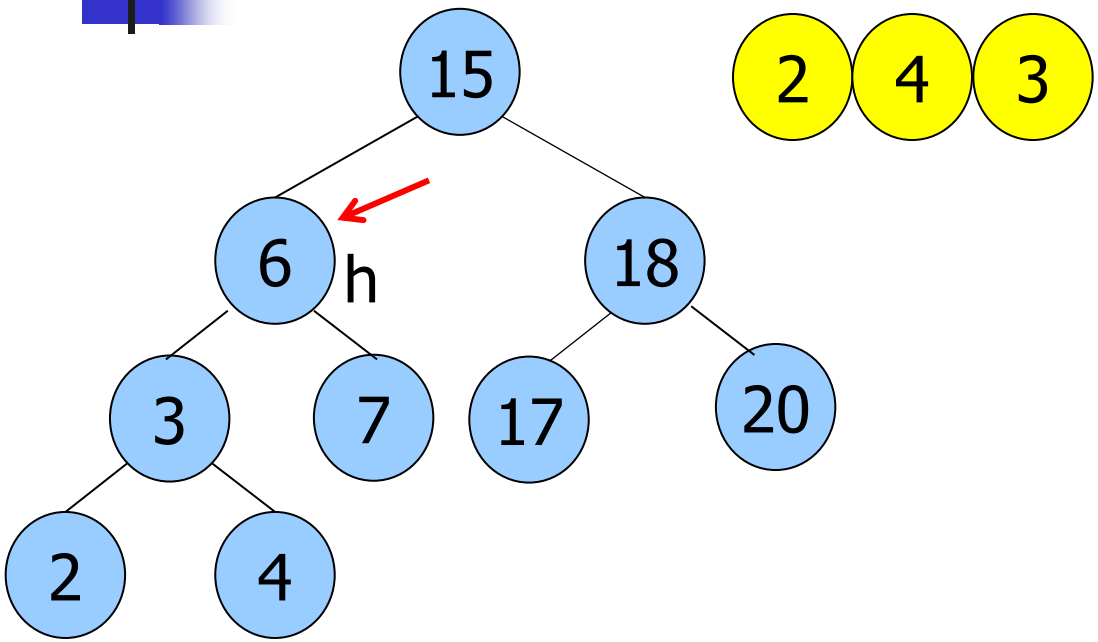
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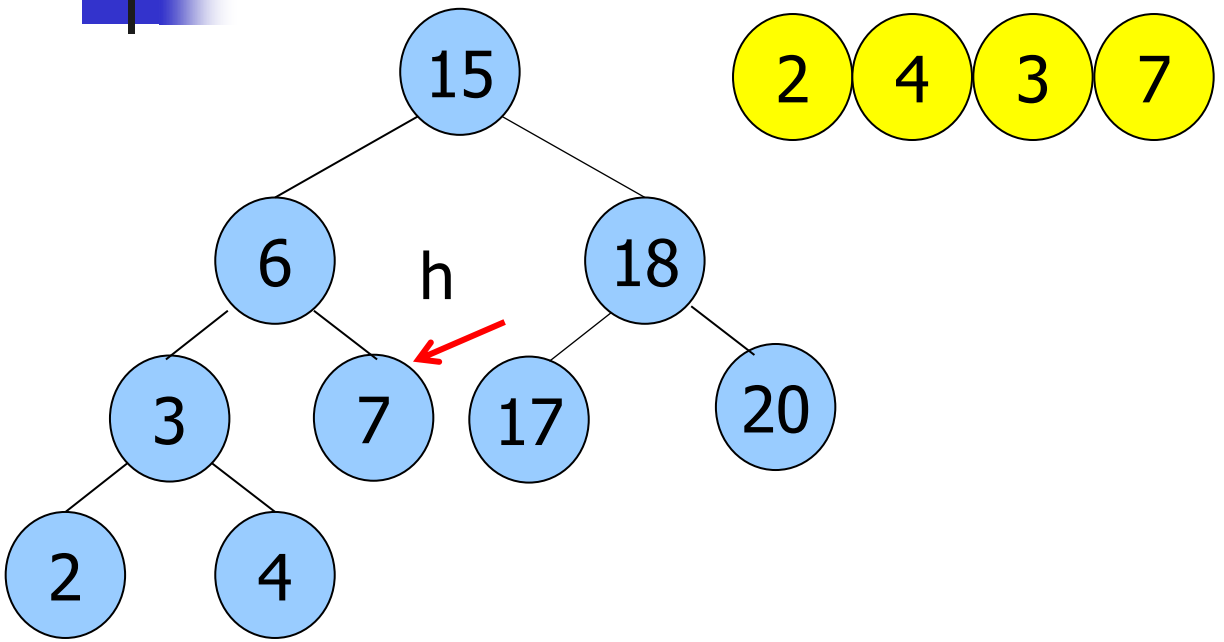
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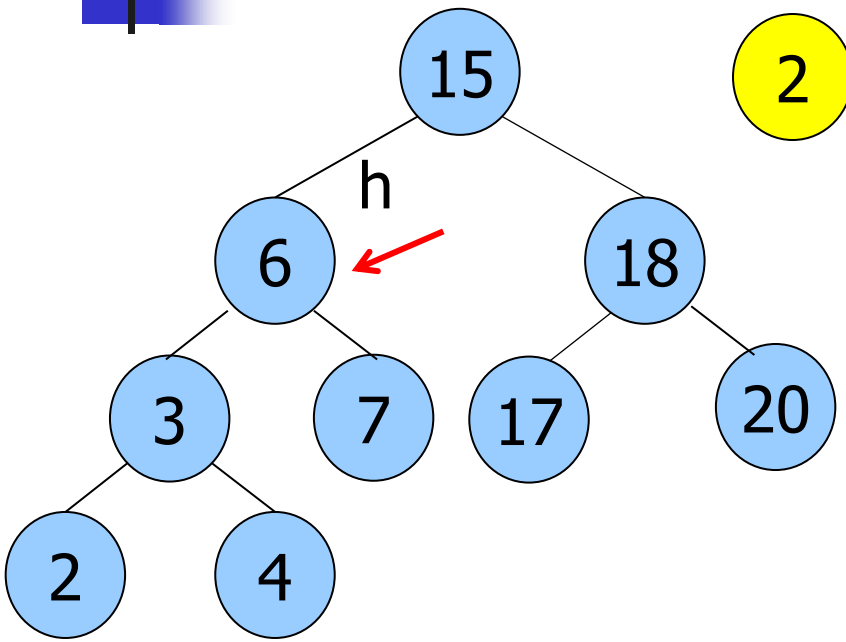
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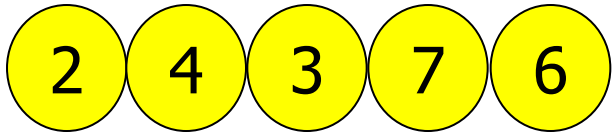
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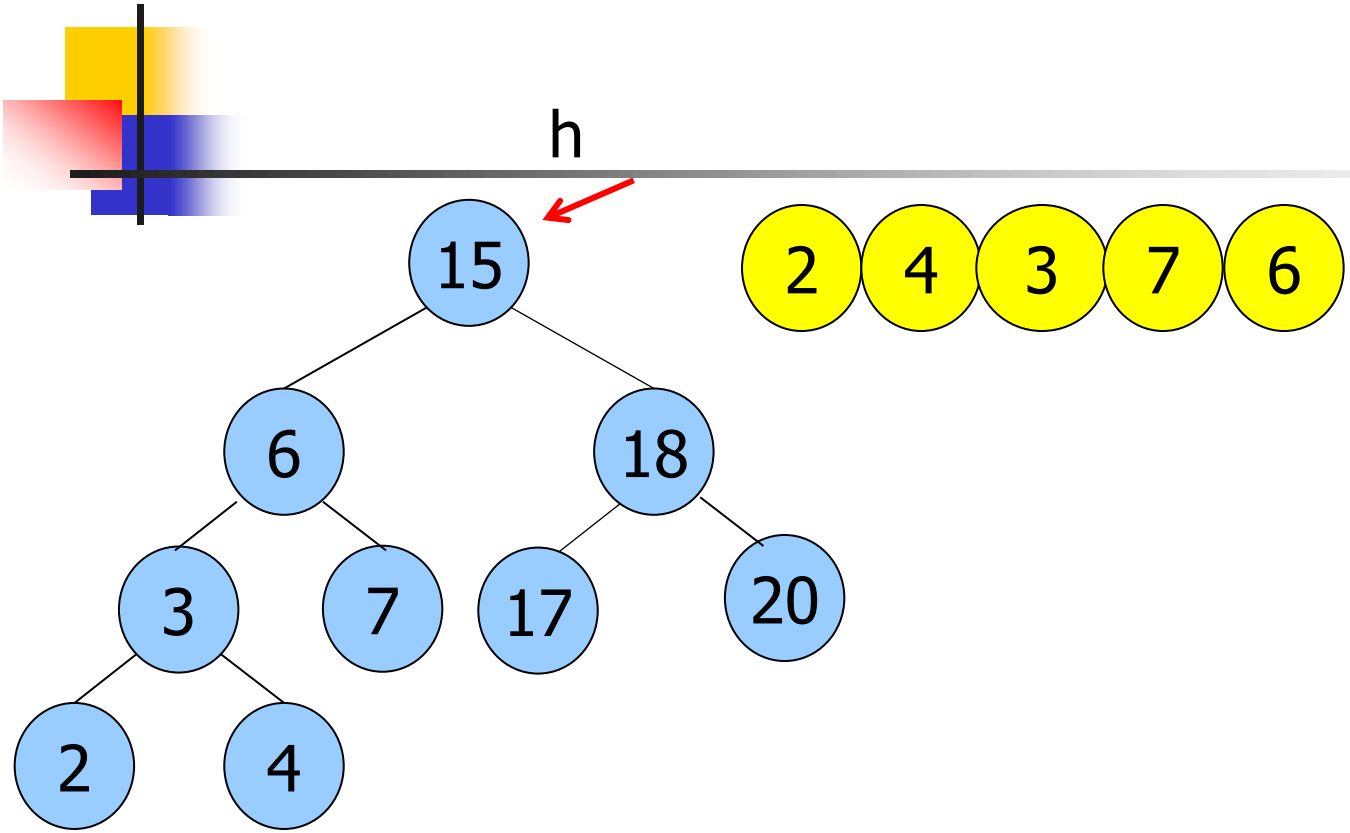
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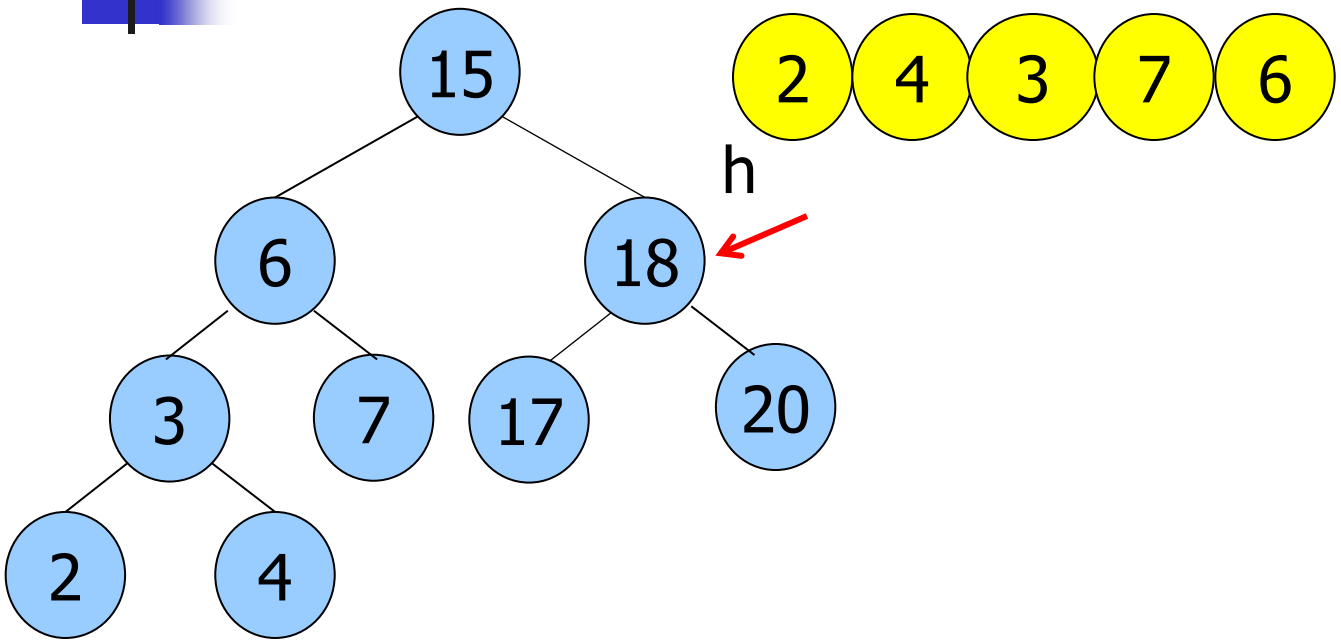
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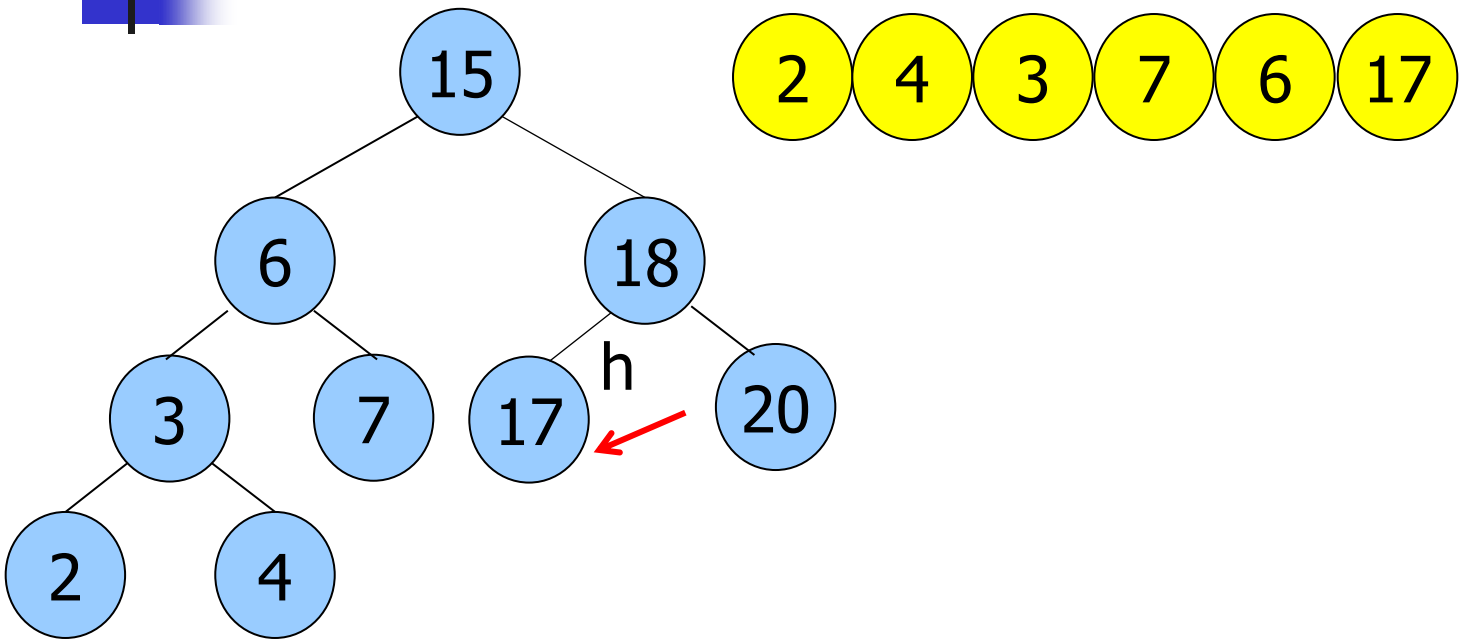
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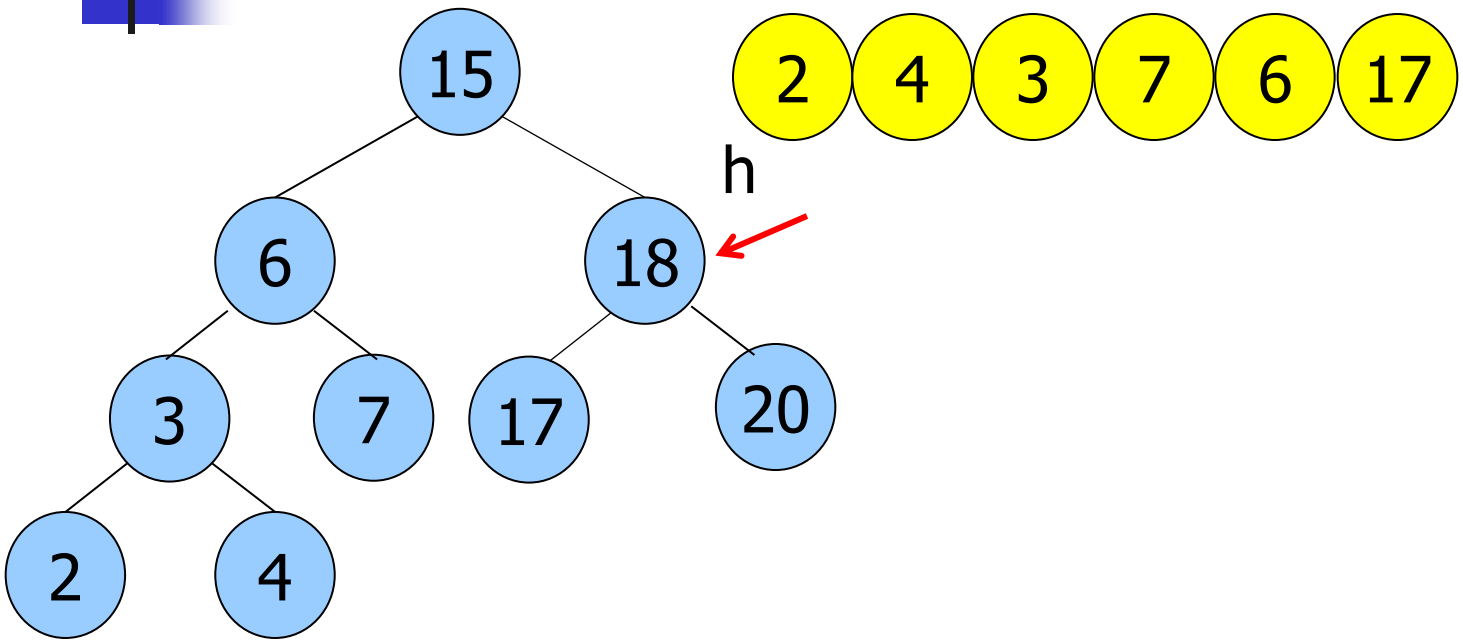
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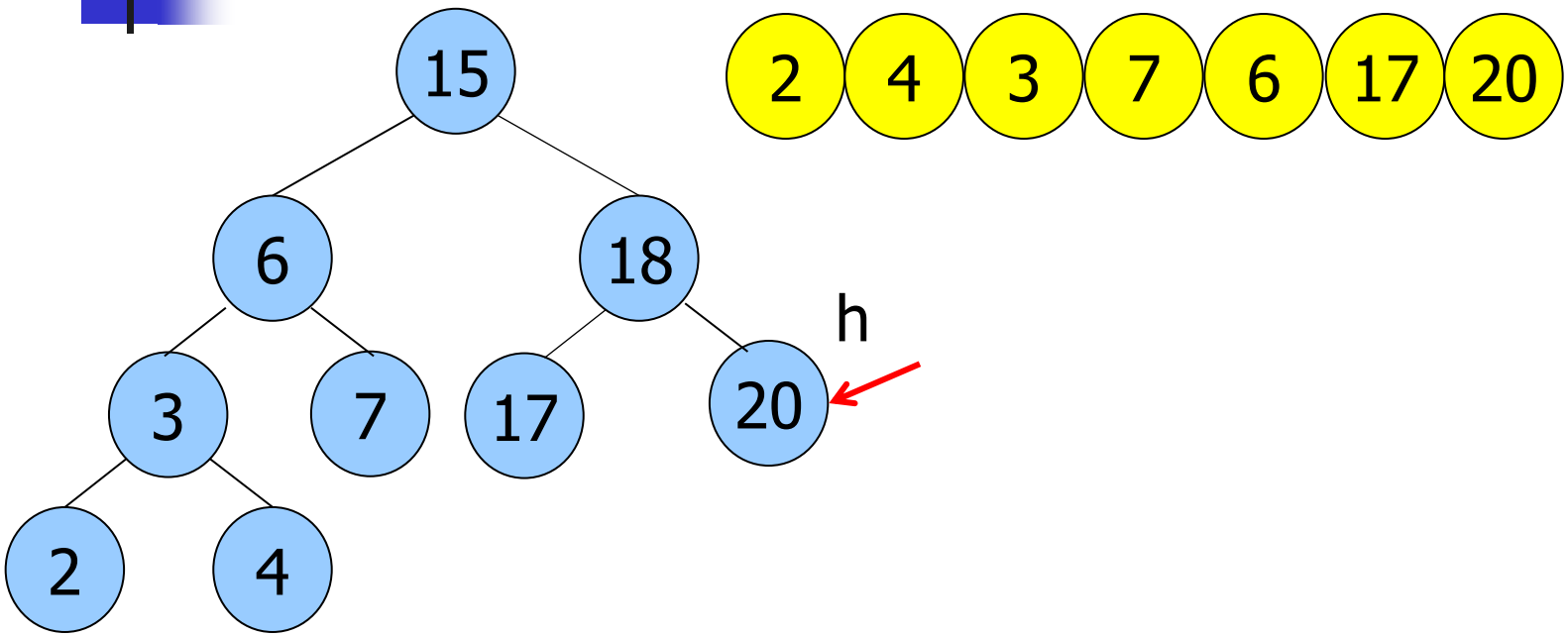
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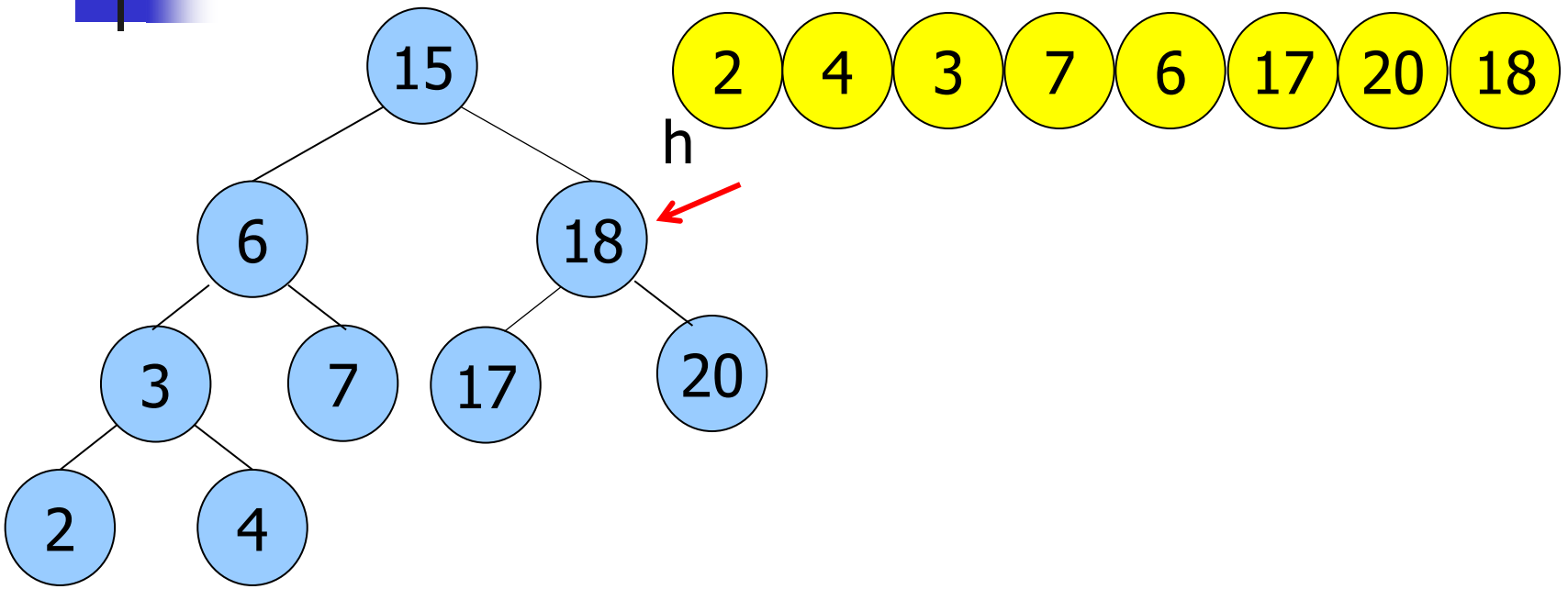
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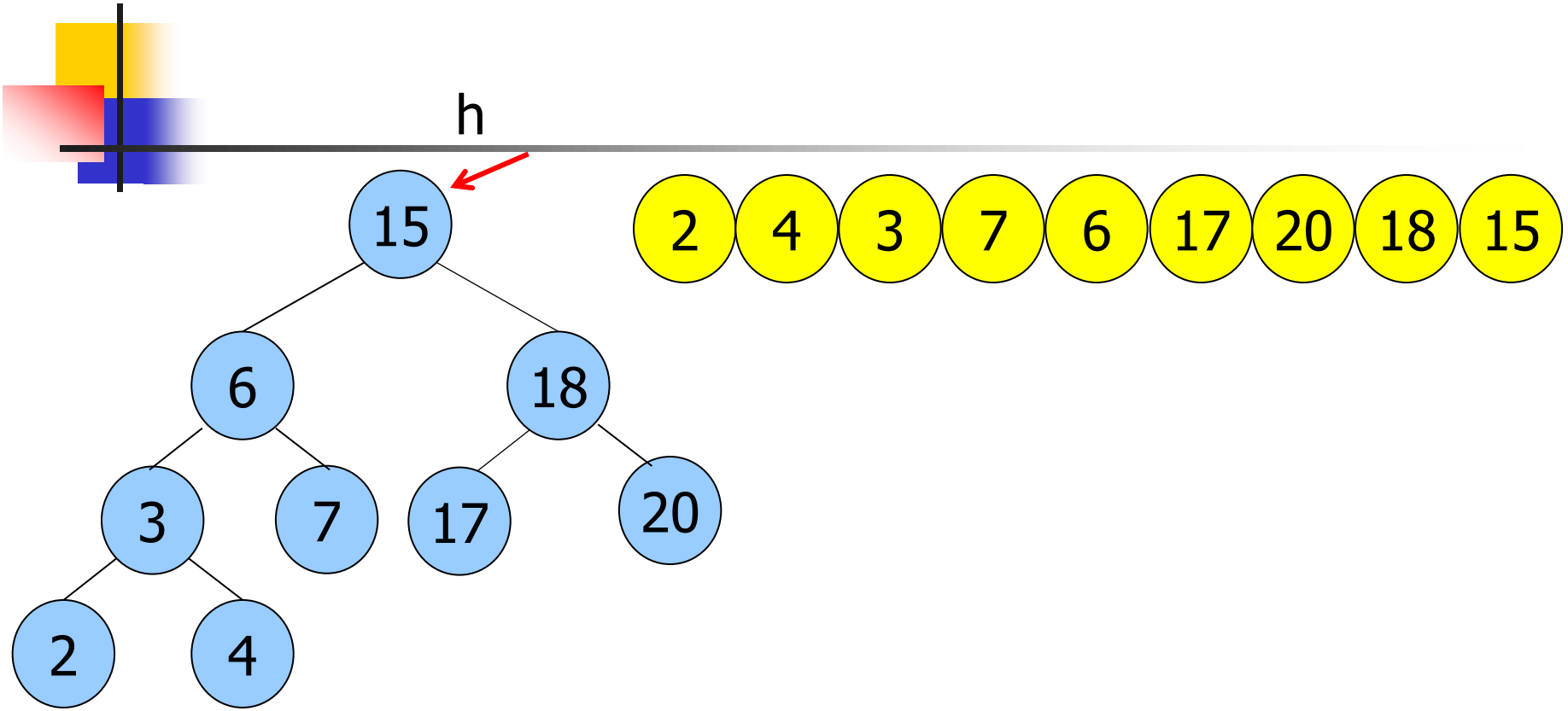
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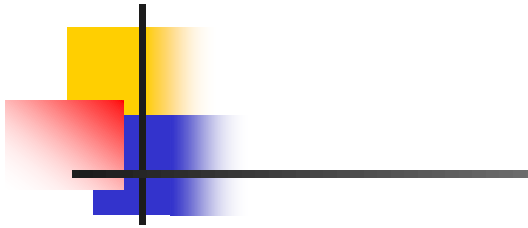
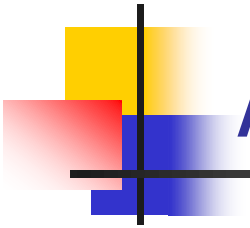


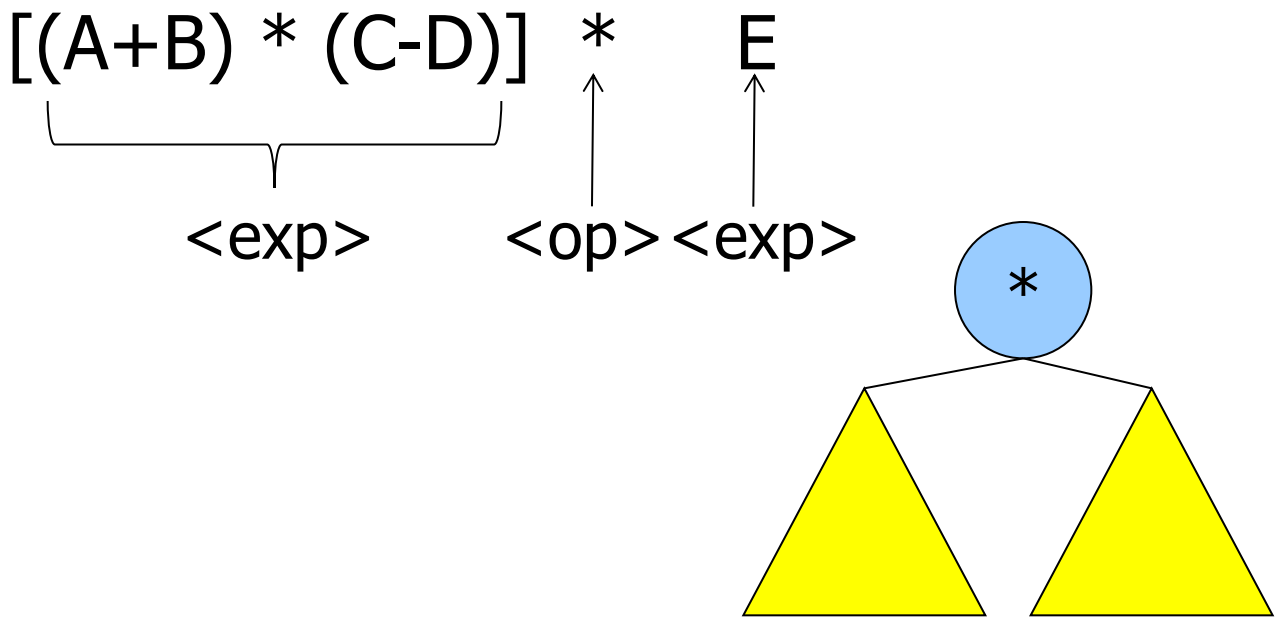
```
void sortpostorderR(link h, void (*visit) (Item), link z){
  if (h == z)
    return;
  sortpostorderR(h->l, visit, z);
  sortpostorderR(h->r, visit, z);
  visit(h->item);

}

void BSTsortpostorder(BST bst, void (*visit)(Item)) {
  sortpostorderR(bst->head, visit, bst->z);
}
```



Alberi binari ed espressioni

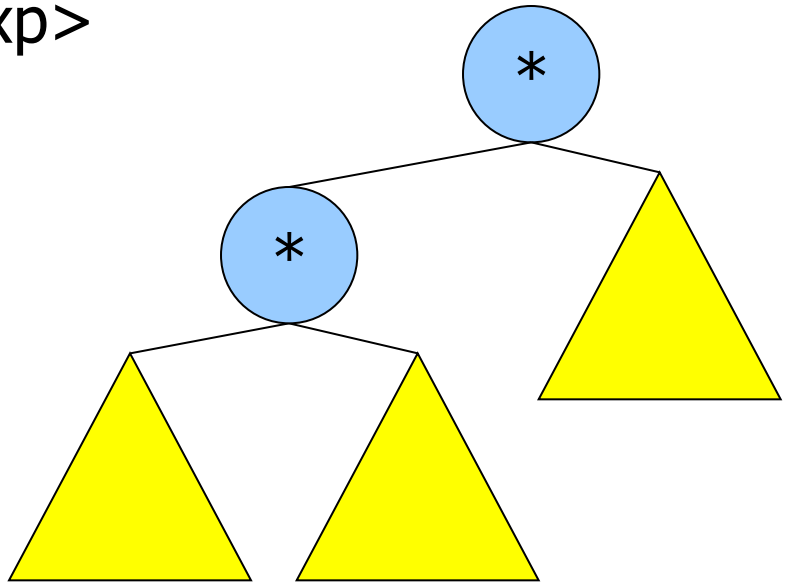
Data un'espressione algebrica in forma infissa (con eventuali parentesi), ricostruirne l'albero binario in base alla grammatica:

- <exp> = <operand> | <exp> <op> <exp>
- <operand> = A .. Z
- <op> = + | * | - | /


[(A+B) * (C-D)] * E
<exp>
<op>
<exp>
*

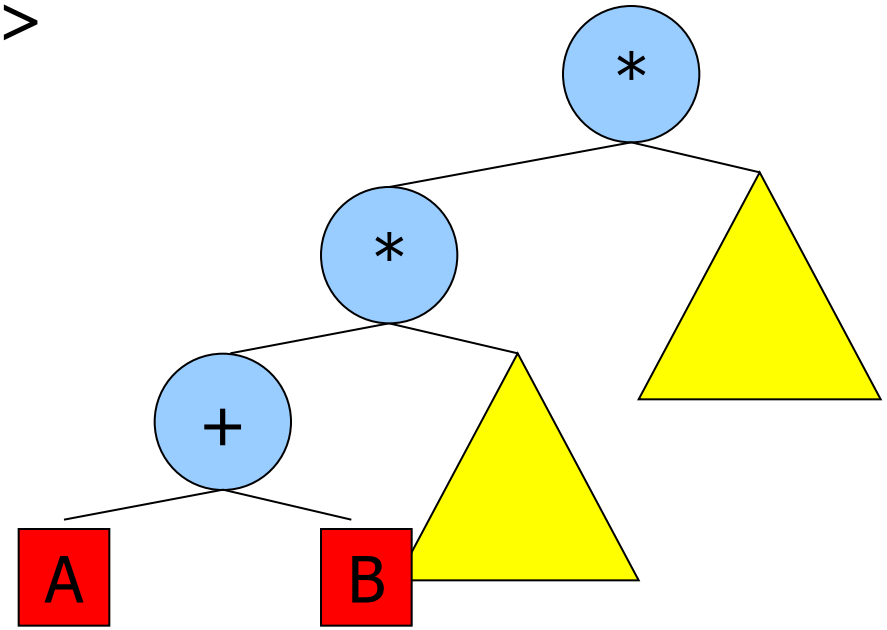
[(A+B) * (C-D)]* E

<exp><op> <exp>



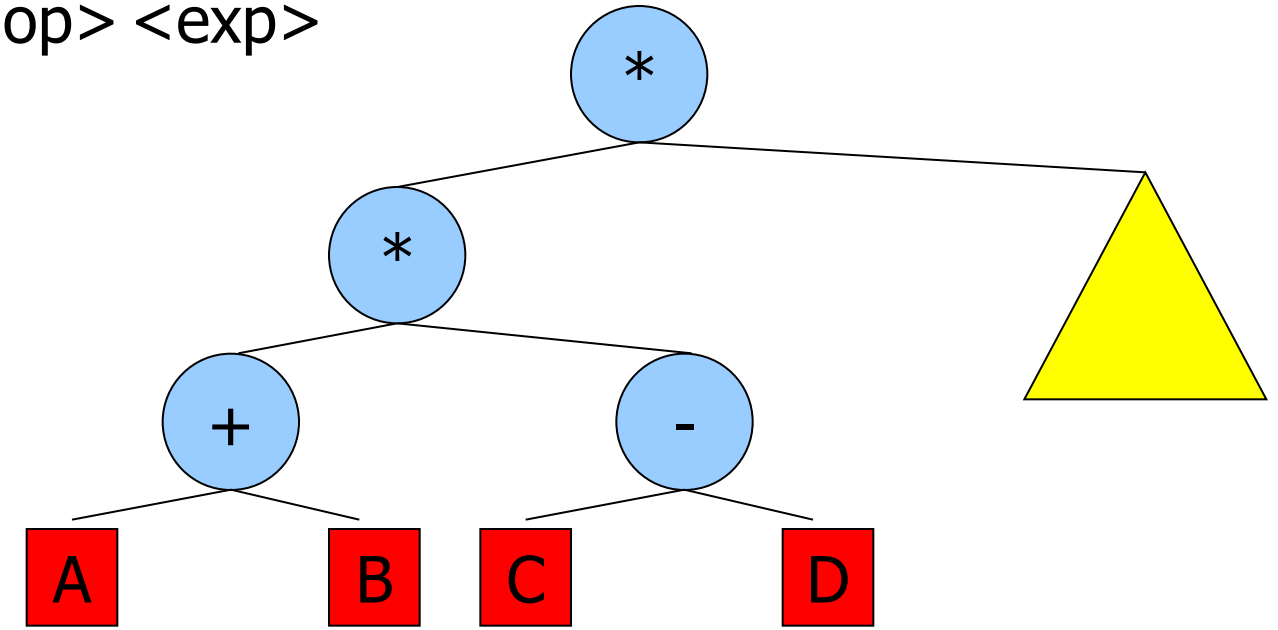
[(A+B) * (C-D)]* E

<exp><op> <exp>

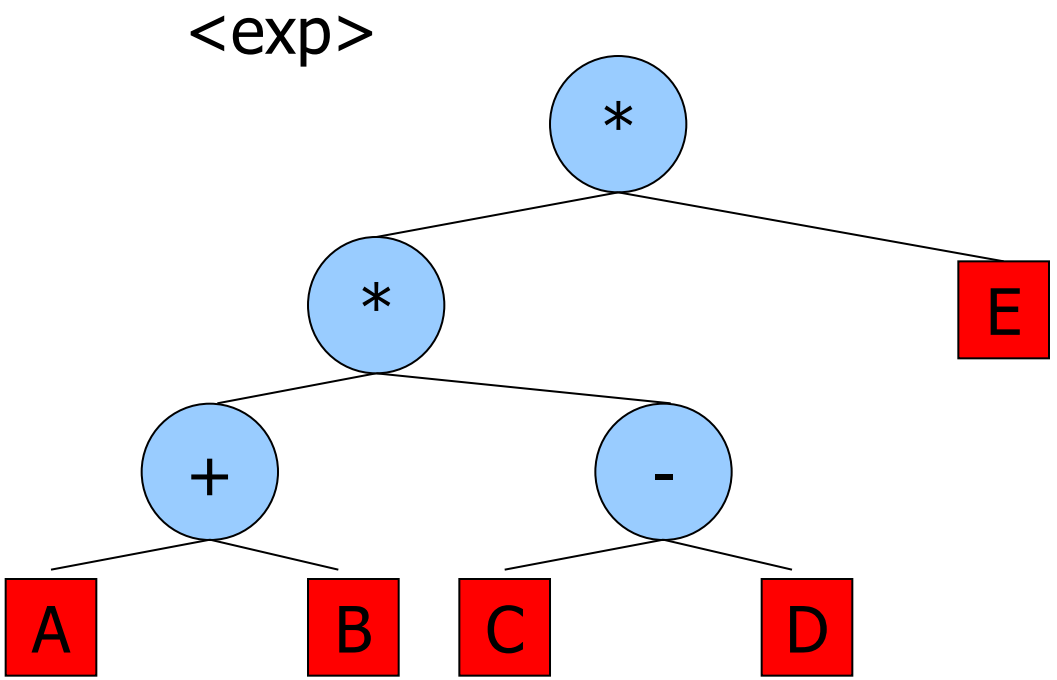


[(A+B) * (C-D)]* E

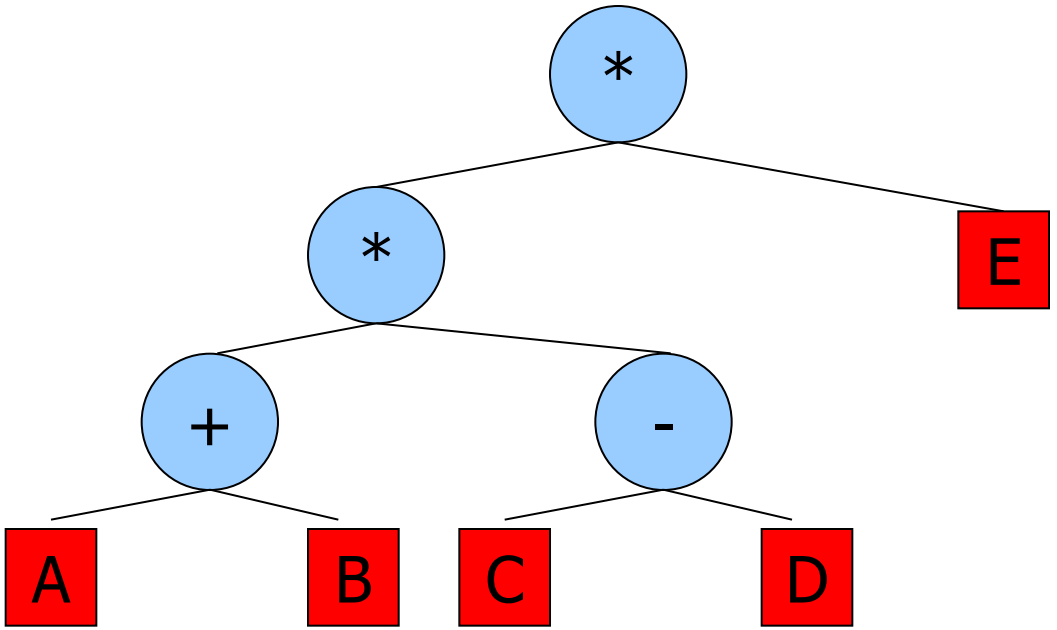
<exp><op> <exp>



[(A+B) * (C-D)]* E



L'attraversamento in post-ordine dell'albero dà la forma postfissa (Notazione Polacca Inversa o Rotate Polish Notation) dell'espressione



A B + C D - * E *

Calcolo ricorsivo di parametri

- Numero di nodi

```
int count(link h, link z) {
 if (h == z)
   return 0;
 return count(h->l, z) + count(h->r, z) + 1;
}
```

- Altezza

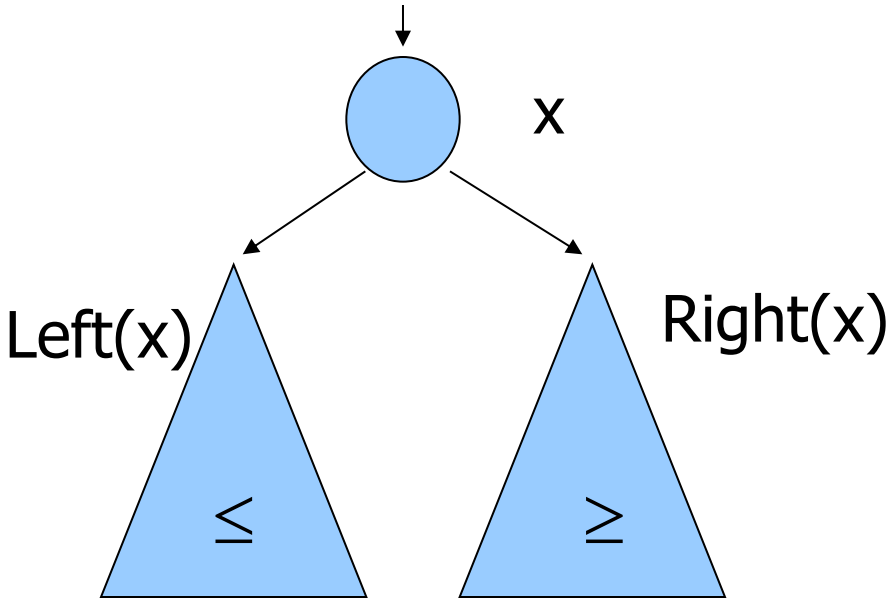
```
int height(link h, link z) {
 int u, v;
 if (h == z)
   return -1;
 u = height(h->l, z); v = height(h->r, z);
 if (u>v)
   return u+1;
 else
   return v+1;
}
```

Alberi binari di ricerca (BST)

ADT albero binario con proprietà:

$\forall$nodo x vale che:

- $\forall$ nodo y $\in$Left(x), key[y] $\leq$ key[x]
- $\forall$ nodo y $\in$Right(x), key[y] $\geq$ key[x]



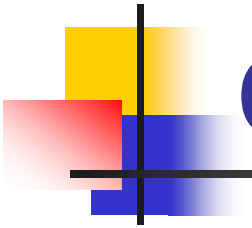
Item.h

confronti definiti sulle chiavi e non sugli item

```
#define neq(A, B) (key(A) != key(B))
#define eq(A, B) (key(A) == key(B))
#define less(A, B) (key(A) < key(B))
#define EMPTYitem { -1, -1 }
#define key(A) (A.x)
#define maxKey 100

typedef struct { int x; int y; } Item;

Item ITEMscan();
void ITEMshow(Item x);
Item ITEMrand();
```

Item.c

```
#include <stdlib.h>
#include "Item.h"
Item ITEMscan() {
  Item item;
  printf("x = "); scanf("%d", &item.x);
  printf("y = "); scanf("%d", &item.y);
  return item;
}
Item ITEMrand() {
  Item item;
  item.x = maxKey*(1.0 * rand()/RAND_MAX);
  item.y = maxKey*(1.0 * rand()/RAND_MAX);
  return item;
}
void ITEMshow(Item item) {
  printf("\n x = %d  y = %d \n", item.x, item.y);
}
```



Operazioni

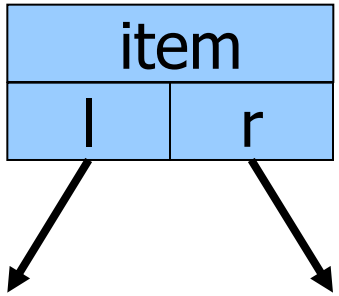
BST.h

```
typedef struct  binarysearchtree *BST;

BST   BSTinit() ;
Item  BSTmin(BST) ;
Item  BSTmax(BST) ;
void  BSTinsert_leafI(BST,Item) ;
void  BSTinsert_leafR(BST,Item) ;
void  BSTinsert_root(BST,Item) ;
Item  BSTsearch(BST, Key) ;
void  BSTsortinorder(BST,void (*visit) (Item)) ;
void  BSTsortpreorder(BST,void (*visit) (Item)) ;
void  BSTsortpostorder(BST,void (*visit) (Item)) ;
```

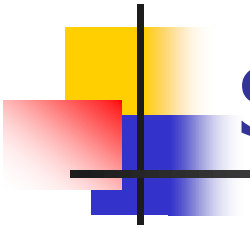
BST.c

```
typedef struct BSTnode* link;
struct BSTnode {Item item; link l; link  r; };
struct binarysearchtree { link head; int N; link z; };

Item NULLitem = EMPTYitem;

link NEW(Item item, link l, link r) {
  link x = malloc(sizeof *x);
  x->item = item; x->l = l; x->r = r;
  return x;
}
BST BSTinit( ) {
  BST bst = malloc(sizeof *bst) ;
  bst->N = 0;
  bst->head = (bst->z = NEW(NULLitem, NULL, NULL));
  return bst;
}
```

BSTnode



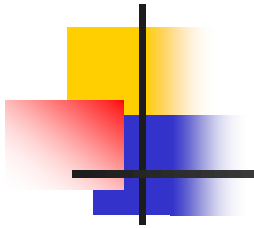
nodo sentinella



Search

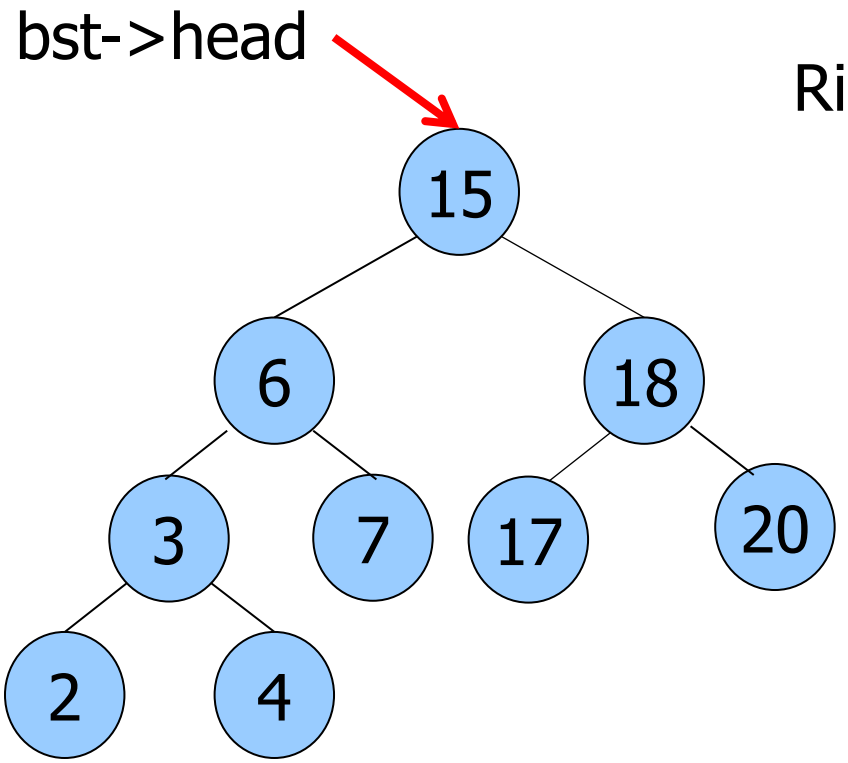
Ricerca ricorsiva di un nodo che contiene un item dato:

- percorrimento dell'albero dalla radice
- terminazione: la chiave dell'item cercato è uguale alla chiave del nodo corrente (search hit) oppure si è giunti ad un albero vuoto (search miss)
- ricorsione: dal nodo corrente
  - su sottoalbero sinistro se la chiave dell'item cercato < della chiave del nodo corrente
  - su sottoalbero destro altrimenti



```
Item searchR(link h, Key k, link z) {
  if (h == z)
    return NULLitem;
  if (eq(k, key(h->item)))
    return h->item;
  if (less(k, key(h->item)))
    return searchR(h->l, k, z);
  else
    return searchR(h->r, k, z);
}


Item BSTsearch(BST bst, Key k) {
  return searchR(bst->head, k, bst->z);
}
```

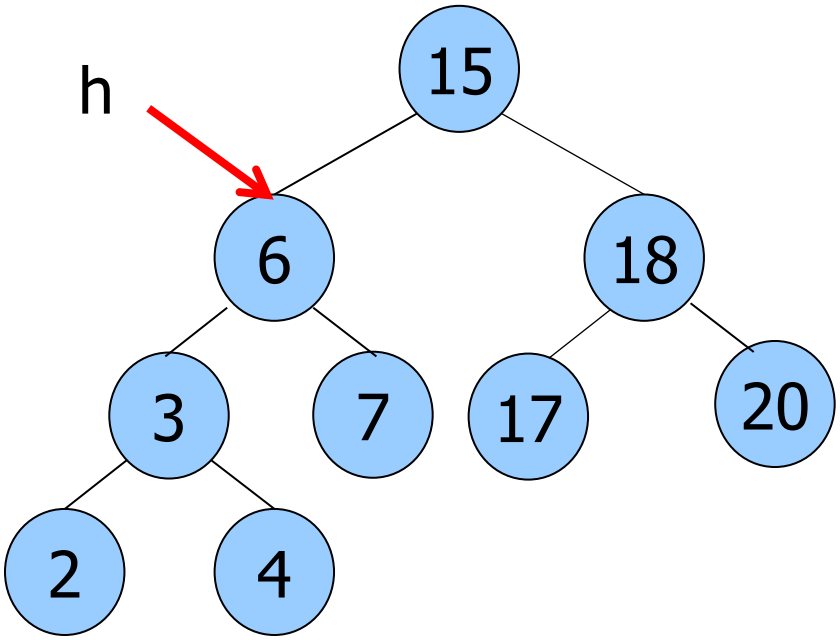
Esempio

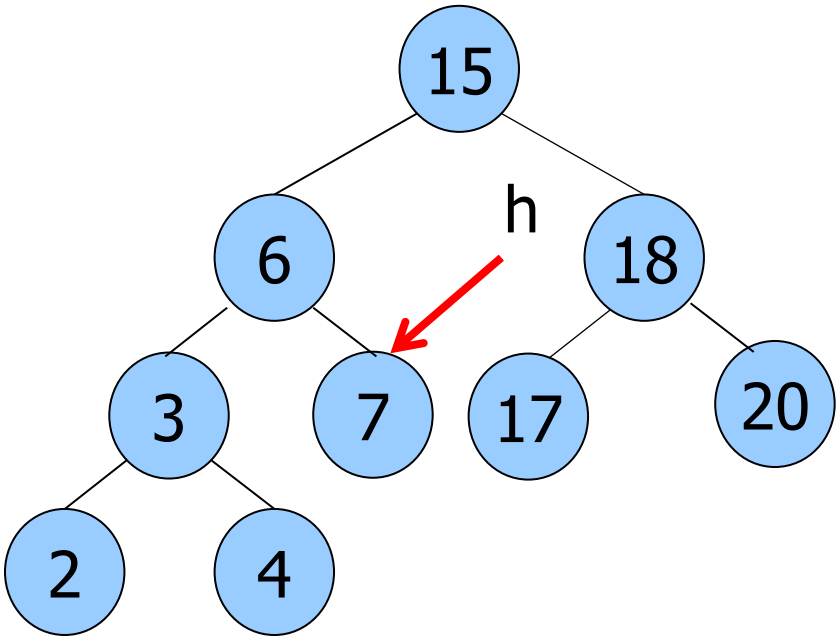


Ricerca dell'item
con chiave 7

NB: per semplicità si riporta solo il campo chiave dell'item

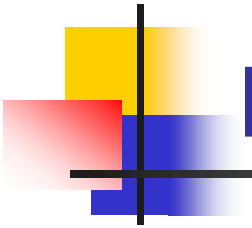
Ricerca dell'item con chiave 7





Ricerca dell'item con chiave 7

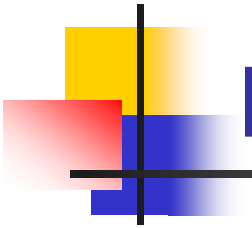
Hit!



Min

- seguire il puntatore al sottoalbero sinistro finchè esiste

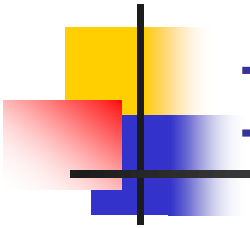
```
Item minR(link h, link z) {
  if (h == z)
    return NULLitem;
  if (h->l == z)
    return (h->item);
  return minR(h->l, z);
}

Item BSTmin(BST bst) {
  return minR(bst->head, bst->z);
}
```



Max

- seguire il puntatore al sottoalbero destro finchè esiste

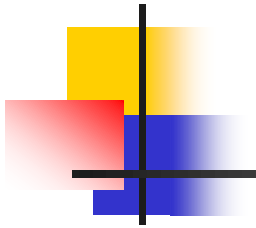
```
Item maxR(link h, link z) {
  if (h == z)
    return NULLitem;
  if (h->r == z)
    return (h->item);
  return maxR(h->r, z);
}

Item BSTmax(BST bst) {
  return maxR(bst->head, bst->z);
}
```



Insert (in foglia)

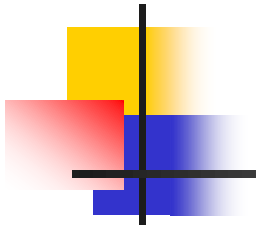
Inserire in un albero binario di ricerca un nodo che contiene un item $\Rightarrow$ mantenimento della proprietà:

- se il BST è vuoto, creazione del nuovo albero
- inserimento ricorsivo nel sottoalbero sinistro o destro a seconda del confronto tra la chiave dell'item e quella del nodo corrente
- inserimento iterativo: prima si ricerca la posizione, poi si appende il nuovo nodo.



RICORSIVO

```
link insertR(link h, Item x, link z) {
  if (h == z)
    return NEW(x, z, z);
  if (less(key(x), key(h->item)))
    h->l = insertR(h->l, x, z);
  else
    h->r = insertR(h->r, x, z);
  return h;
}

void BSTinsert_leafR(BST bst, Item x) {
  bst->head = insertR(bst->head, x, bst->z);
}
```

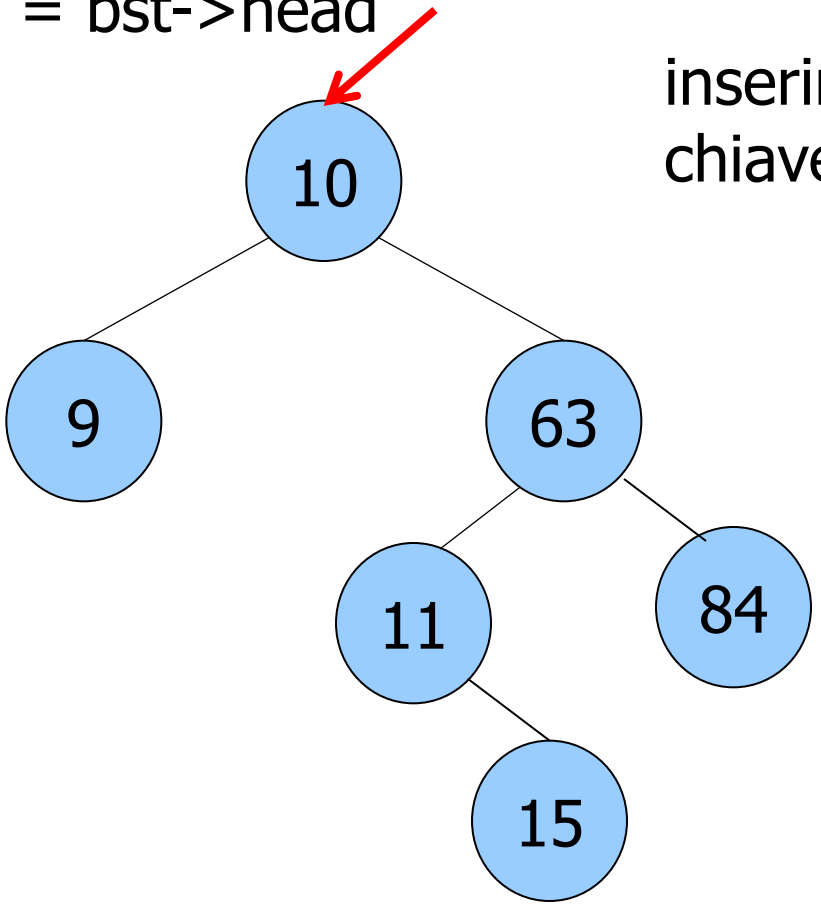


ITERATIVO

```
void BSTinsert_leafI(BST bst, Item x) {
  link p = bst->head, h = p;
  if (bst->head == bst->z) {
    bst->head = NEW(x, bst->z, bst->z);
    return;
  }
  while (h != bst->z) {
    p = h;
    h = less(key(x), key(h->item))? h->l : h->r;
  }
  h = NEW(x, bst->z, bst->z);
  if (less(key(x), key(p->item)))
    p->l = h;
  else
    p->r = h;
}
```

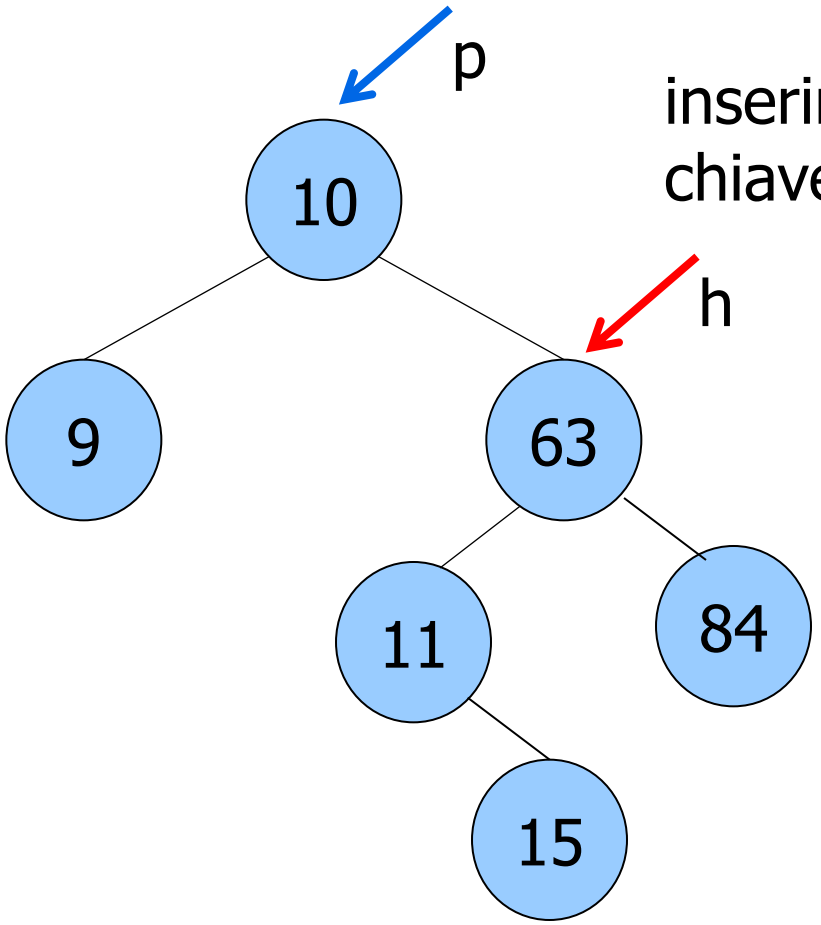
Esempio

h =p = bst->head



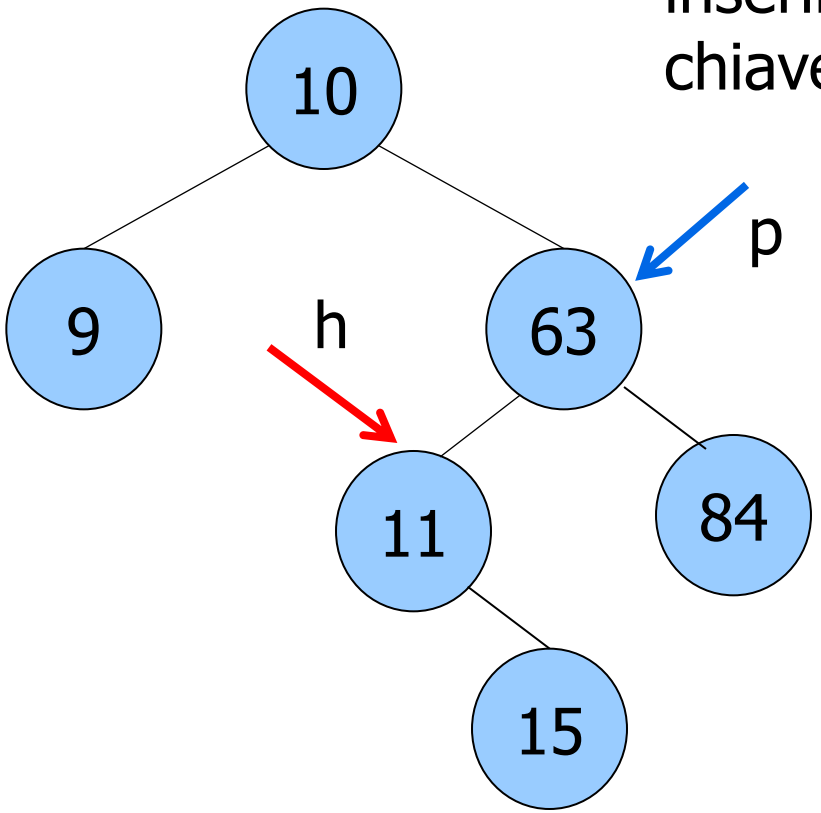
inserimento dell'item con chiave 62

NB: per semplicità si riporta solo il campo chiave dell'item

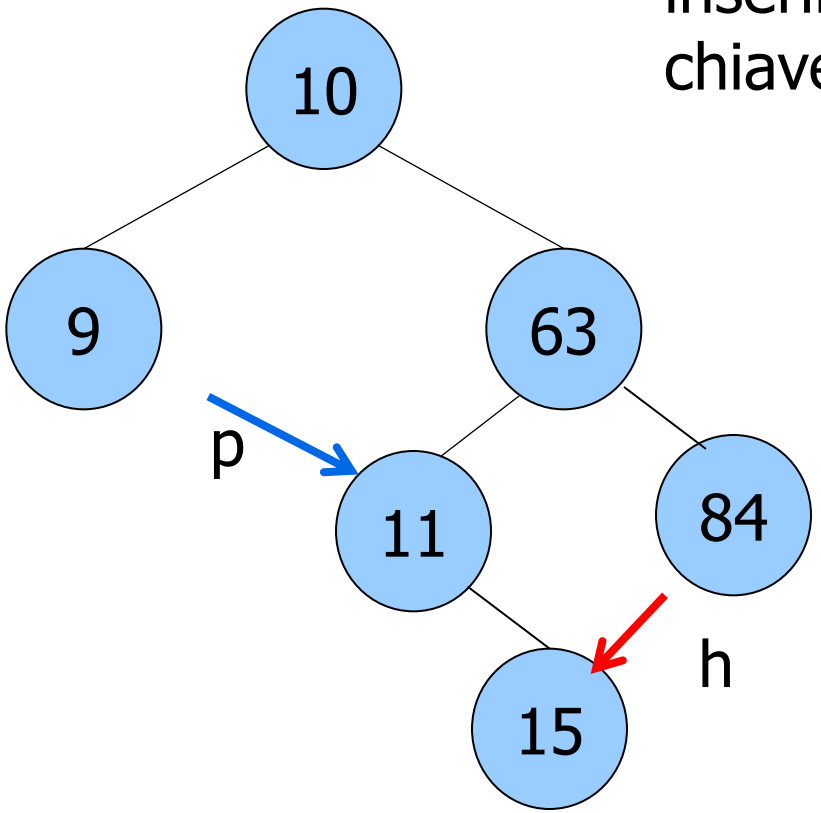


inserimento dell'item con chiave 62

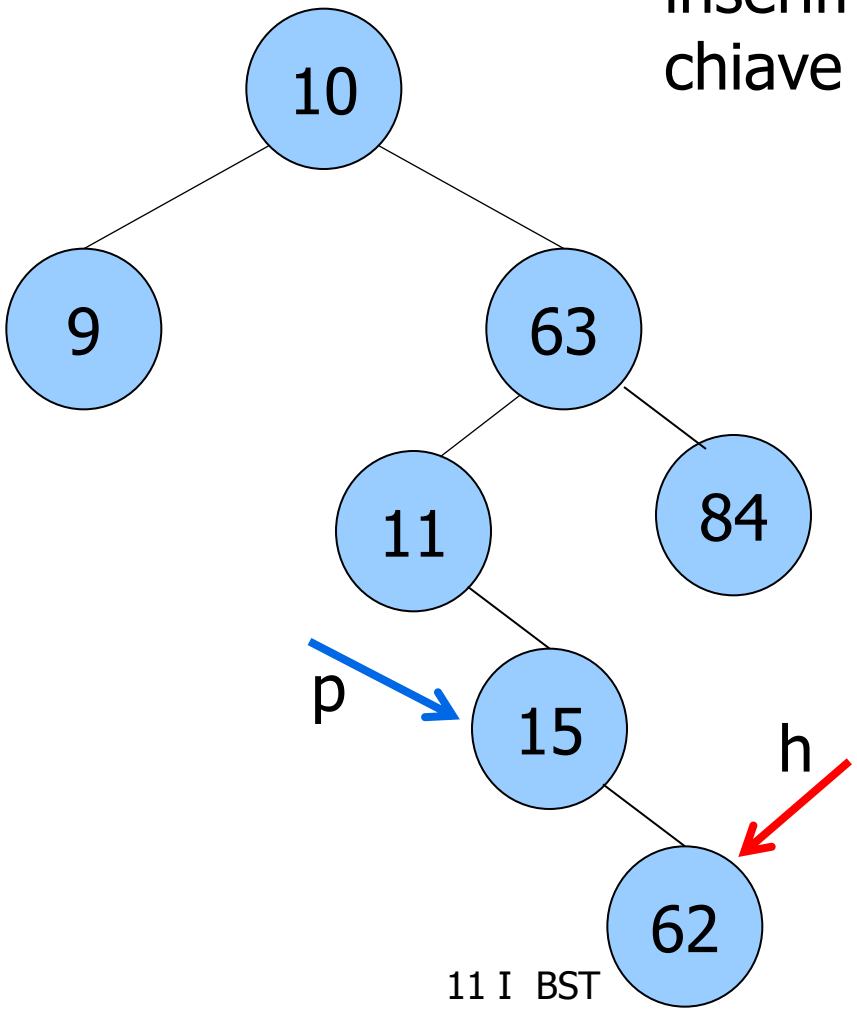
inserimento dell'item con chiave 62



inserimento dell'item con chiave 62

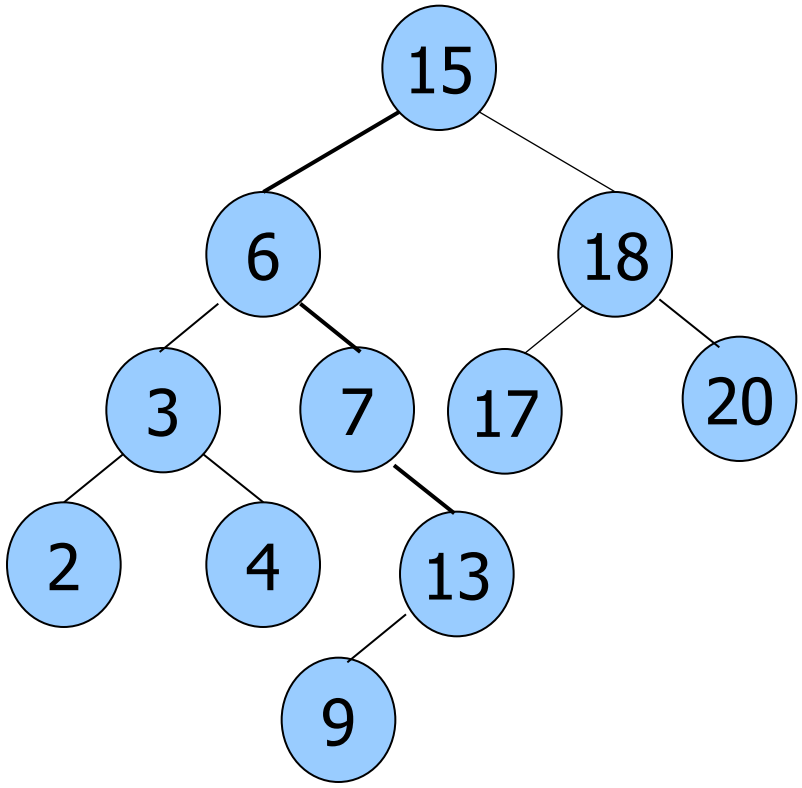


inserimento dell'item con chiave 62



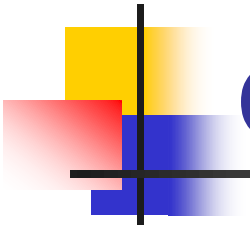
Sort

Attraversamento in-ordine: ordinamento crescente delle chiavi.



2 3 4 6 7 9 13 15 17 18 20

Chiave mediana: chiave di mezzo nell'ordinamento.

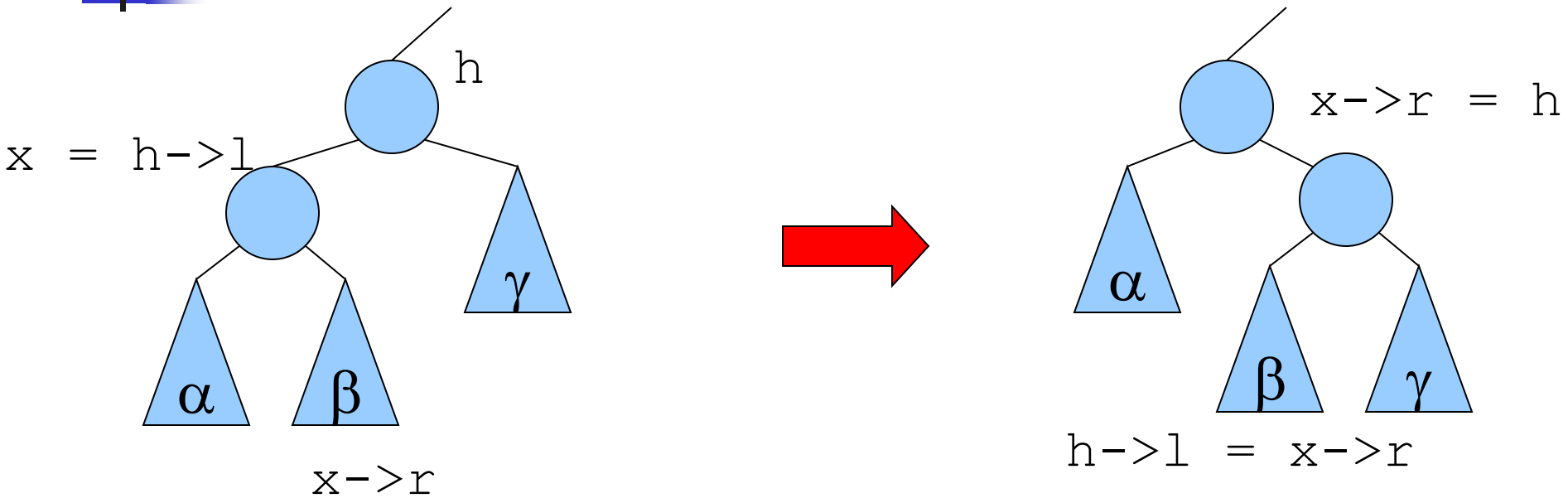


Complessità

Le operazioni hanno complessità $T(n) = O(h)$:

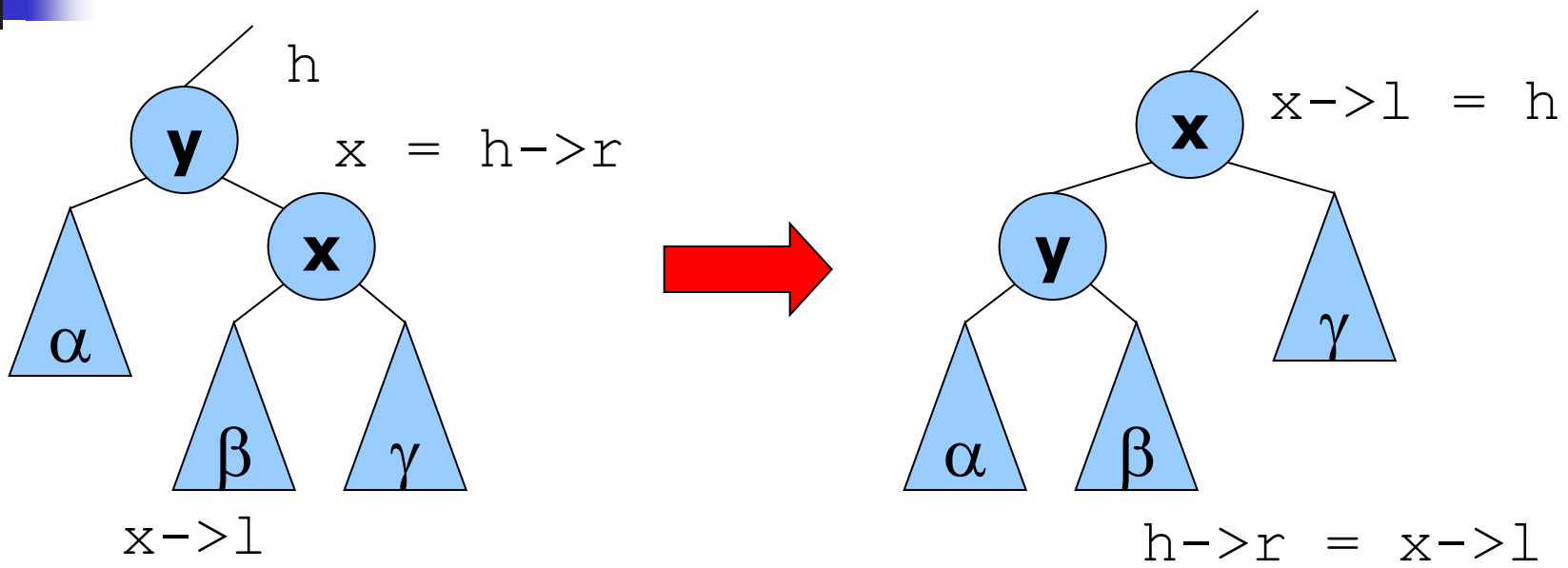
- albero con n nodi completamente bilanciato
  - altezza $h = \log_2 n$
- albero con n nodi completamente sbilanciato ha
  - altezza $h = n$
- $O(\log n) \leq T(n) \leq O(n)$

Rotazione a destra di BST



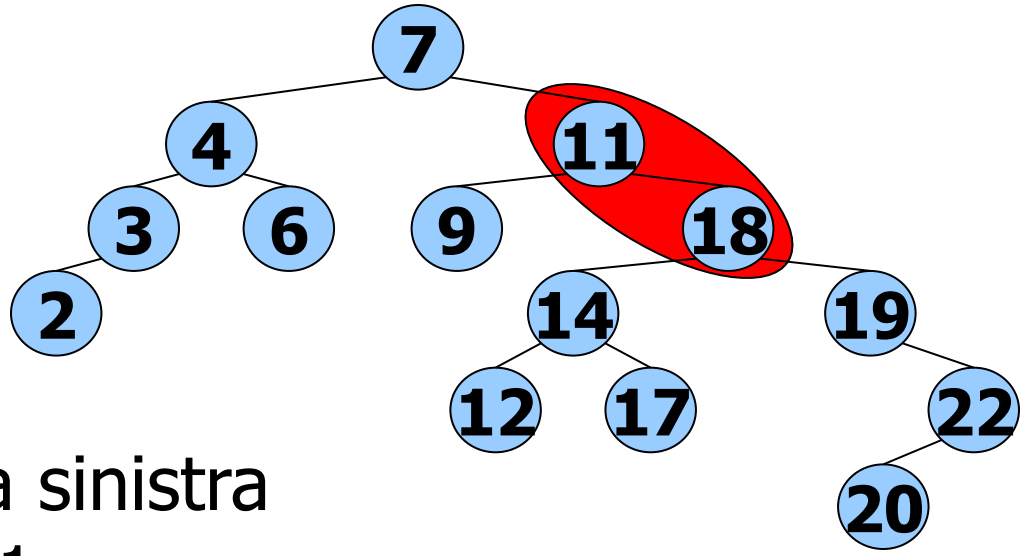

```
link rotR(link h)
{
   link x = h->l;
   h->l = x->r;
   x->r = h;
   return x;
}
```

Rotazione a sinistra di BST

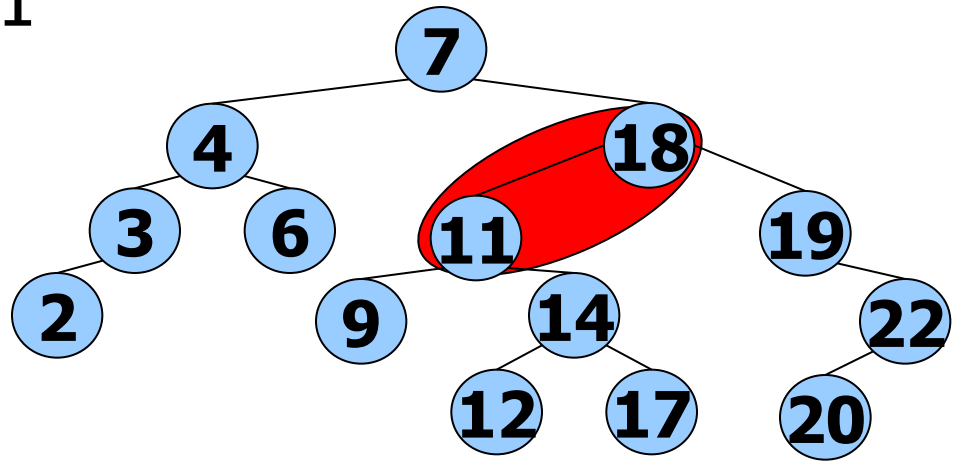


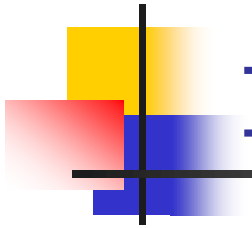

```
link rotL(link h)
{
  link x = h->r;
  h->r = x->l;
  x->l = h;
  return x;
}
```

Esempio



Rotazione a sinistra attorno a 11



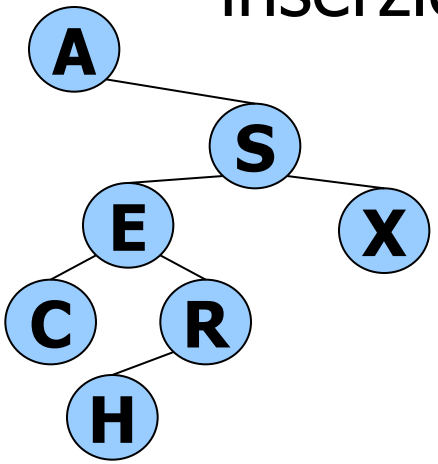


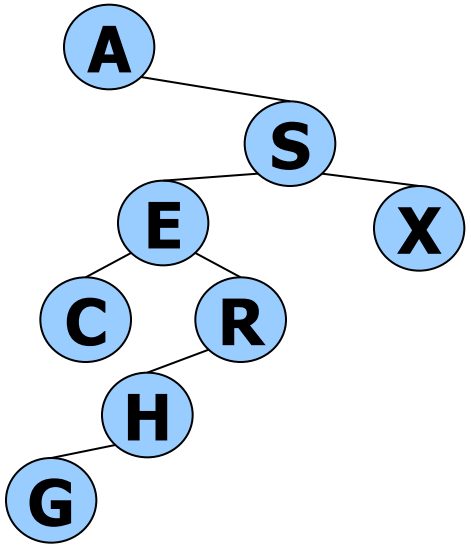
Inserimento alla radice

- Inserimento dalle foglie a scelta e non obbligatorio
- Nodi più recenti nella parte alta del BST
- Inserimento ricorsivo alla radice:
  - inserimento nel sottoalbero appropriato
  - rotazione per farlo diventare radice dell'albero principale.

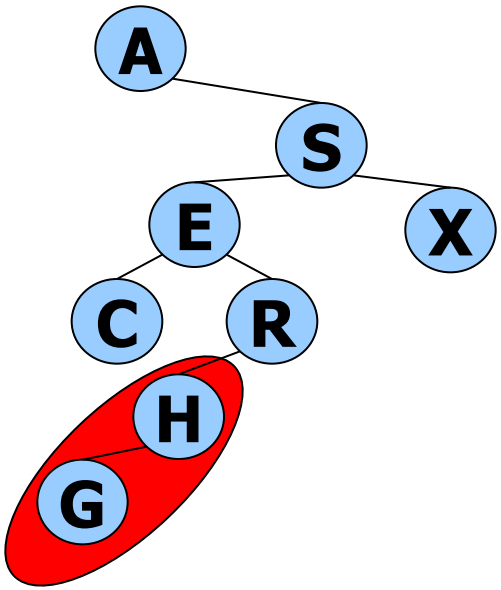
Esempio

inserzione di G

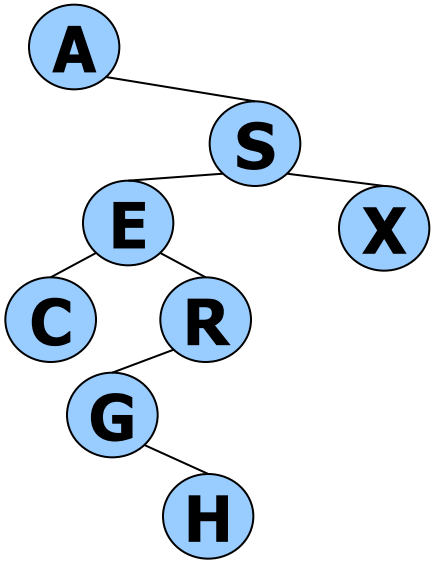


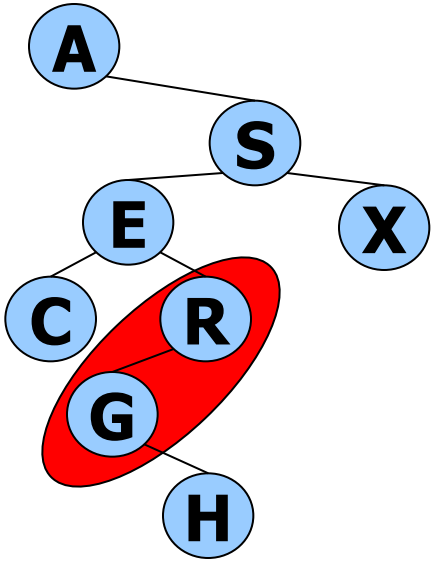


inserzione di G alla radice del sottoalbero opportuno

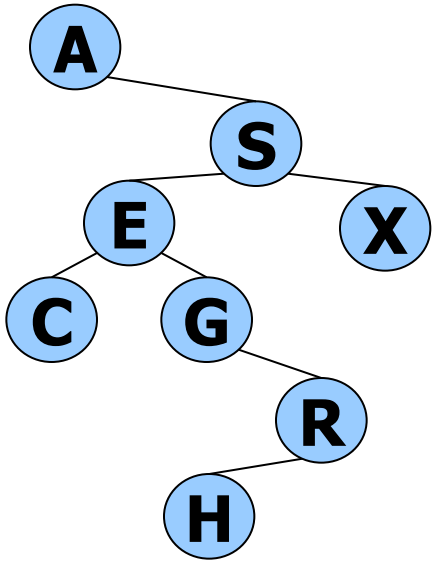


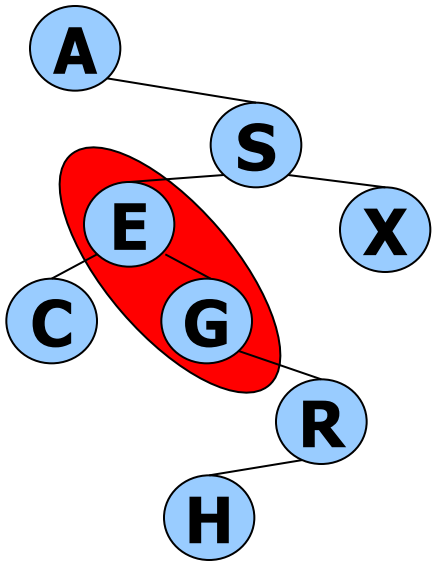
rotazione a DX



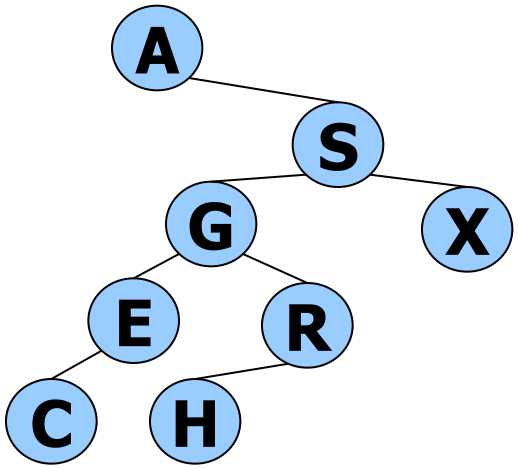


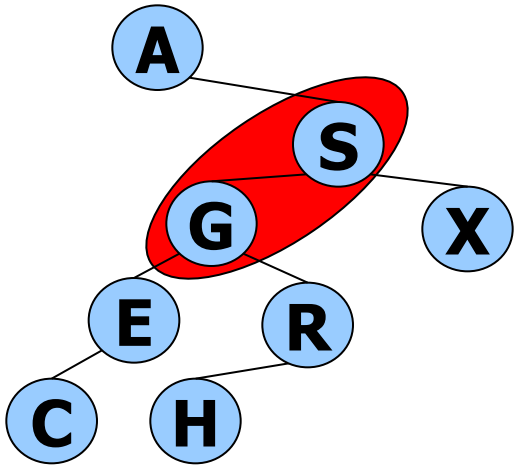
rotazione a DX



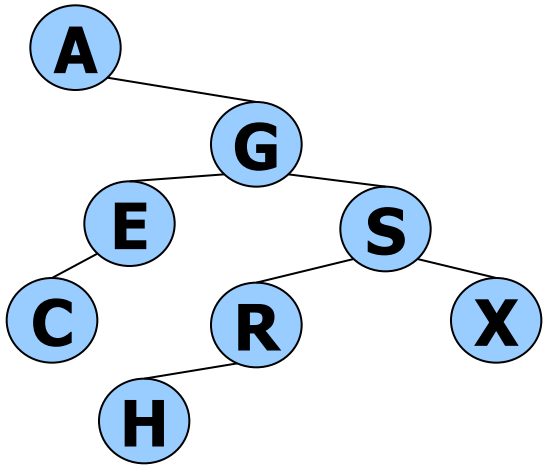


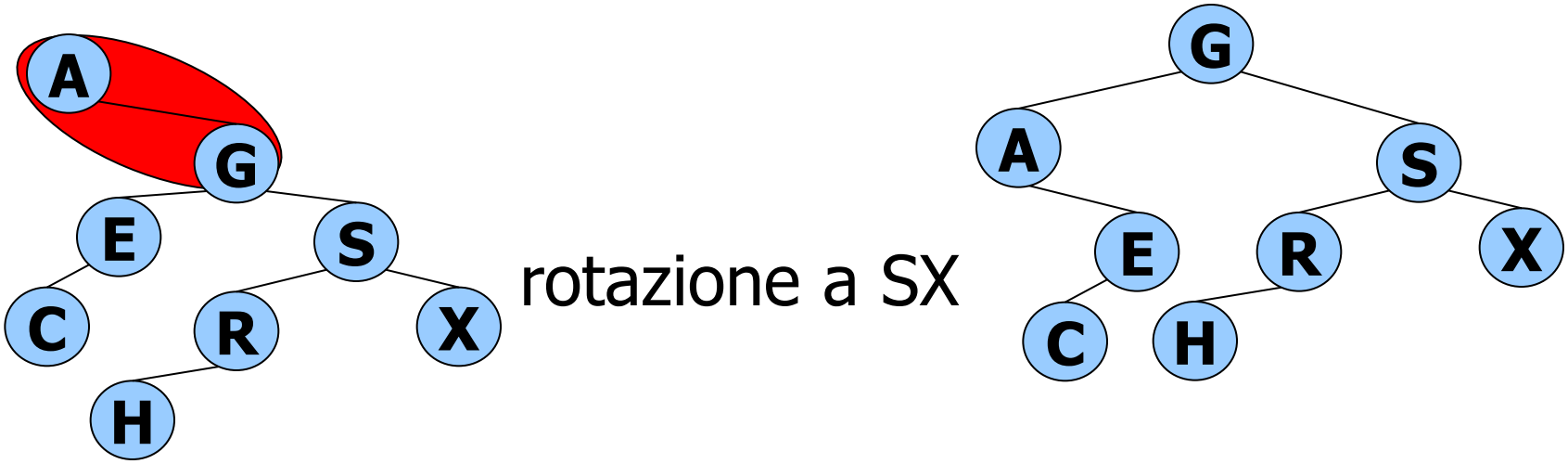
rotazione a SX

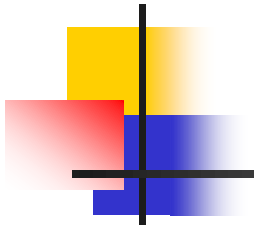




rotazione a DX

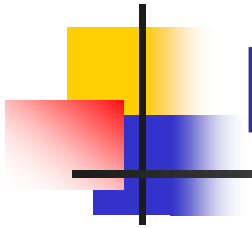



A
G
E
S
C
R
X
H
rotazione a SX
G
A
S
E
R
X
C
H



```
link insertT(link h, Item x, link z) {
  if ( h == z)
    return NEW(x, z, z );
  if (less(key(x), key(h->item))) {
    h->l = insertT(h->l, x, z);
    h = rotR(h);
  }
  else  {
    h->r = insertT(h->r, x, z);
    h = rotL(h);
  }
  return h;
}

void BSTinsert_root(BST bst, Item x) {
    bst->head = insertT(bst->head, x, bst->z);
}
```

Estensioni dei BST elementari



Al nodo elementare si possono aggiungere informazioni che permettono lo sviluppo semplice di nuove funzioni:

- puntatore al padre
- numero di nodi dell'albero radicato nel nodo corrente.

Queste informazioni devono ovviamente essere gestite (quando necessario) da tutte le funzioni già viste.

Operazioni

BST.h

```
typedef struct  binarysearchtree *BST;

BST   BSTinit() ;
int   BSTcount(BST) ;
int   BSTempty(BST) ;
Item  BSTmin(BST) ;
Item  BSTmax(BST) ;
void  BSTinsert_leafI(BST,Item);
void  BSTinsert_leafR(BST,Item);
void  BSTinsert_root(BST,Item);
Item  BSTsearch(BST,Key) ;
void  BSTdelete(BST,Item) ;
Item  BSTselect(BST,int) ;
void  BSTsortinorder(BST,void (*visit) (Item)) ;
void  BSTsortpreorder(BST,void (*visit) (Item)) ;
void  BSTsortpostorder(BST,void (*visit) (Item)) ;
Item  BSTsucc(BST,Item) ;
Item  BSTpred(BST,Item) ;
```

nuove funzioni
funzioni modificate

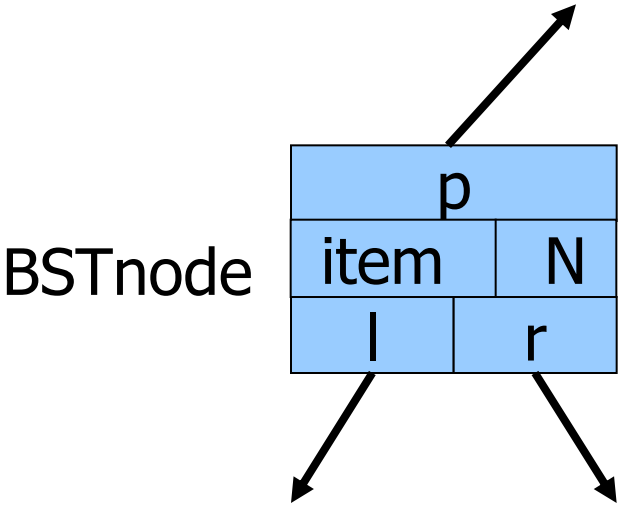
Order-Statistic BST

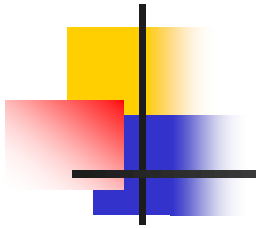
BST.c

puntatore al padre

dimensione sottoalbero

```
typedef struct BSTnode* link;
struct BSTnode {Item item;link p;link l;link  r;int N;};
struct binarysearchtree { link head; int N; link z; };


link NEW(Item item, link p, link l, link r, int N){
  link x = malloc(sizeof *x);
  x->item = item;
  x->p = p;
  x->l = l;
  x->r = r;
  x->N = N;
  return x;
}
```





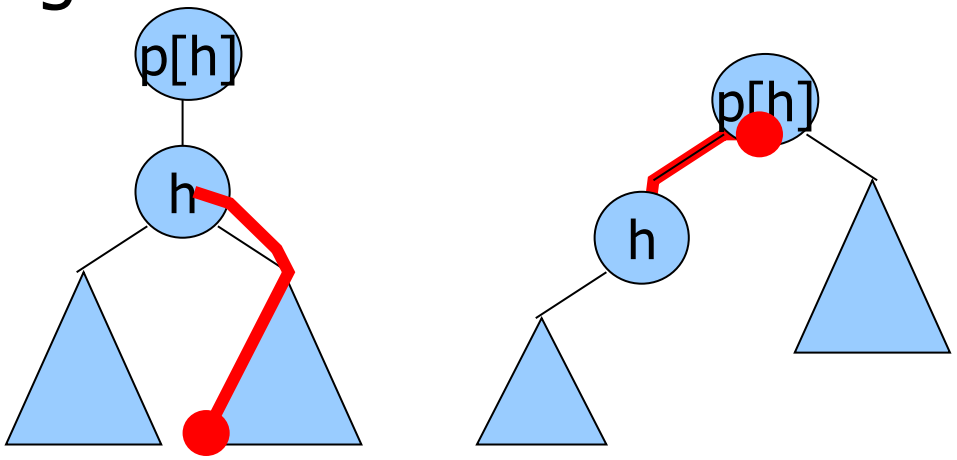
```
BST BSTinit( ) {
  BST bst = malloc(sizeof *bst) ;
  bst->N = 0;
  bst->head =(bst->z=NEW(NULLitem, NULL, NULL, NULL, 0));
  return bst;
}

int BSTcount(BST bst) {
  return bst->N;
}

int BSTempty(BST bst) {
  if ( BSTcount(bst) == 0)
    return 1;
  else
    return 0;
}
```

Successor

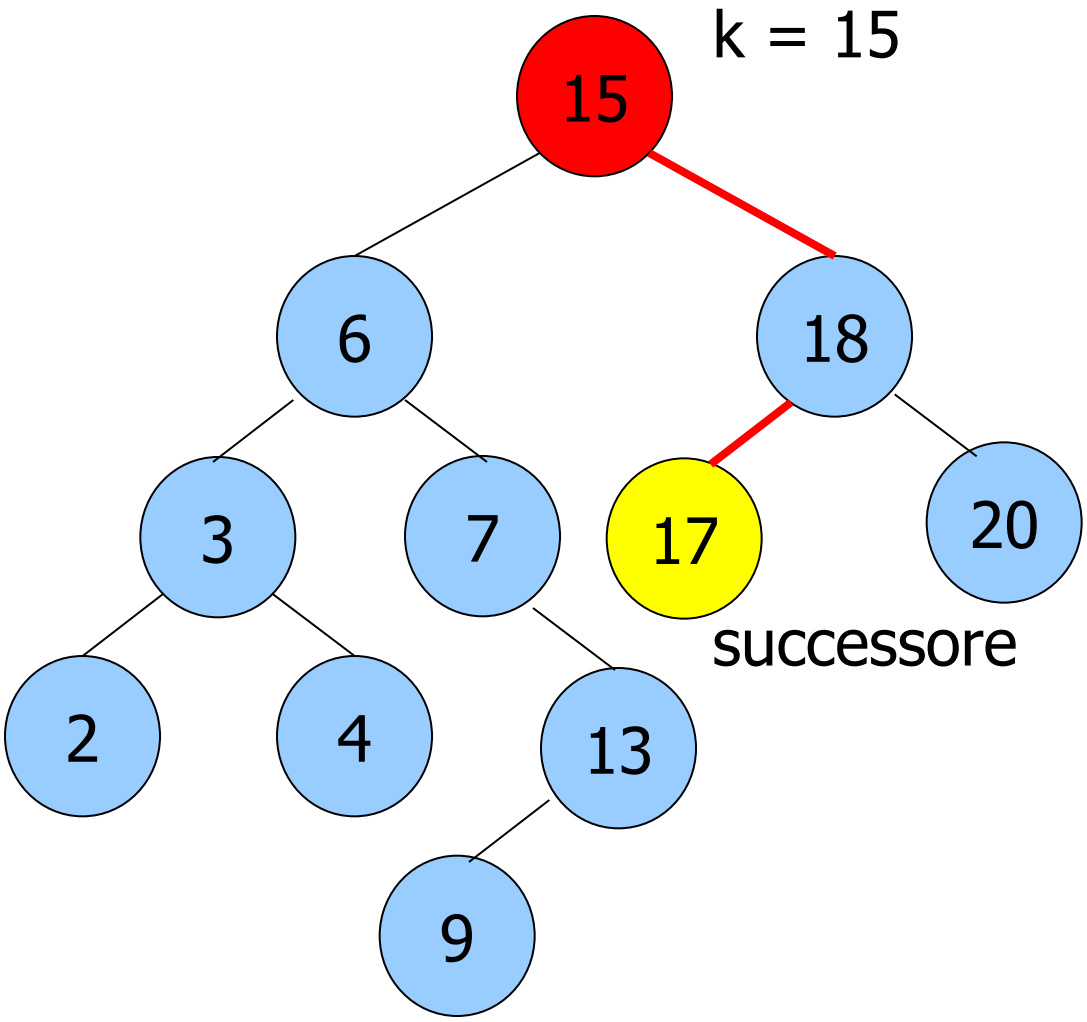
Successore di un item: nodo h con un item con la più piccola chiave > della chiave di item.

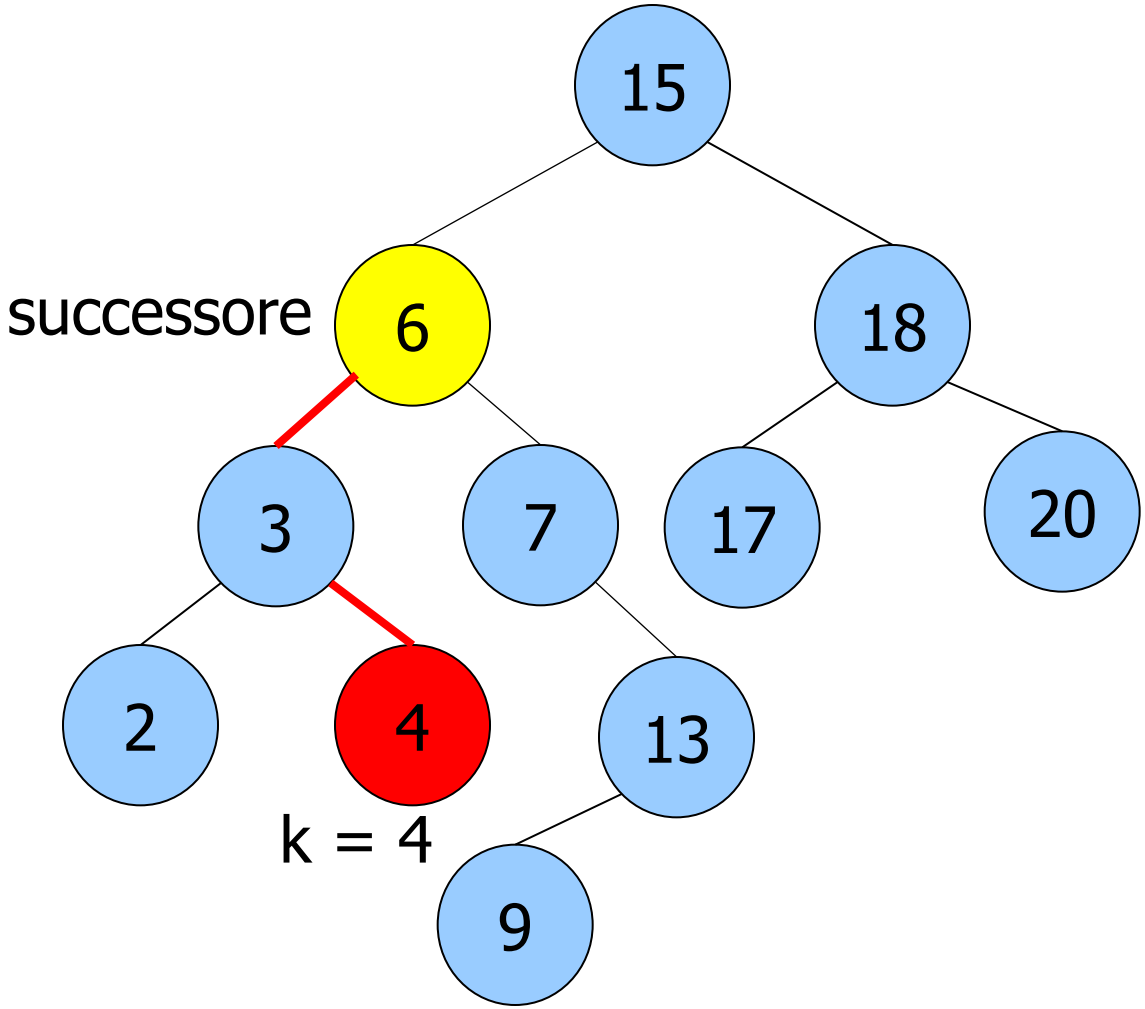
Due casi:

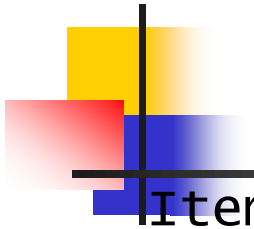
- ∃ Right(h): succ(key(h)) = min(Right(h))
- ∄ Right(h): succ(key(h)) = primo antenato di h il cui figlio sinistro è anche un antenato di h.



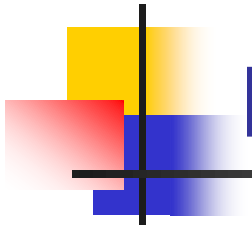
Esempio




15
successore
6
18
3
7
17
20
2
4
13
k = 4
9



```
Item searchSucc(link h, Item x, link z) {
  link p;
  if (h == z) return NULLitem;
  if (eq(key(x), key(h->item)) {
    if (h->r != z) return minR(h->r, z);
    else {
      p = h->p;
      while (p != z && h == p->r) {
        h = p; p = p->p;
      }
      return p->item;
    }
  }
  if (less(key(x), key(h->item)))
    return searchSucc(h->l, x, z);
  else return searchSucc(h->r, x, z);
}
Item BSTsucc(BST bst, Item x) {
  return searchSucc(bst->head, x, bst->z);
}
```



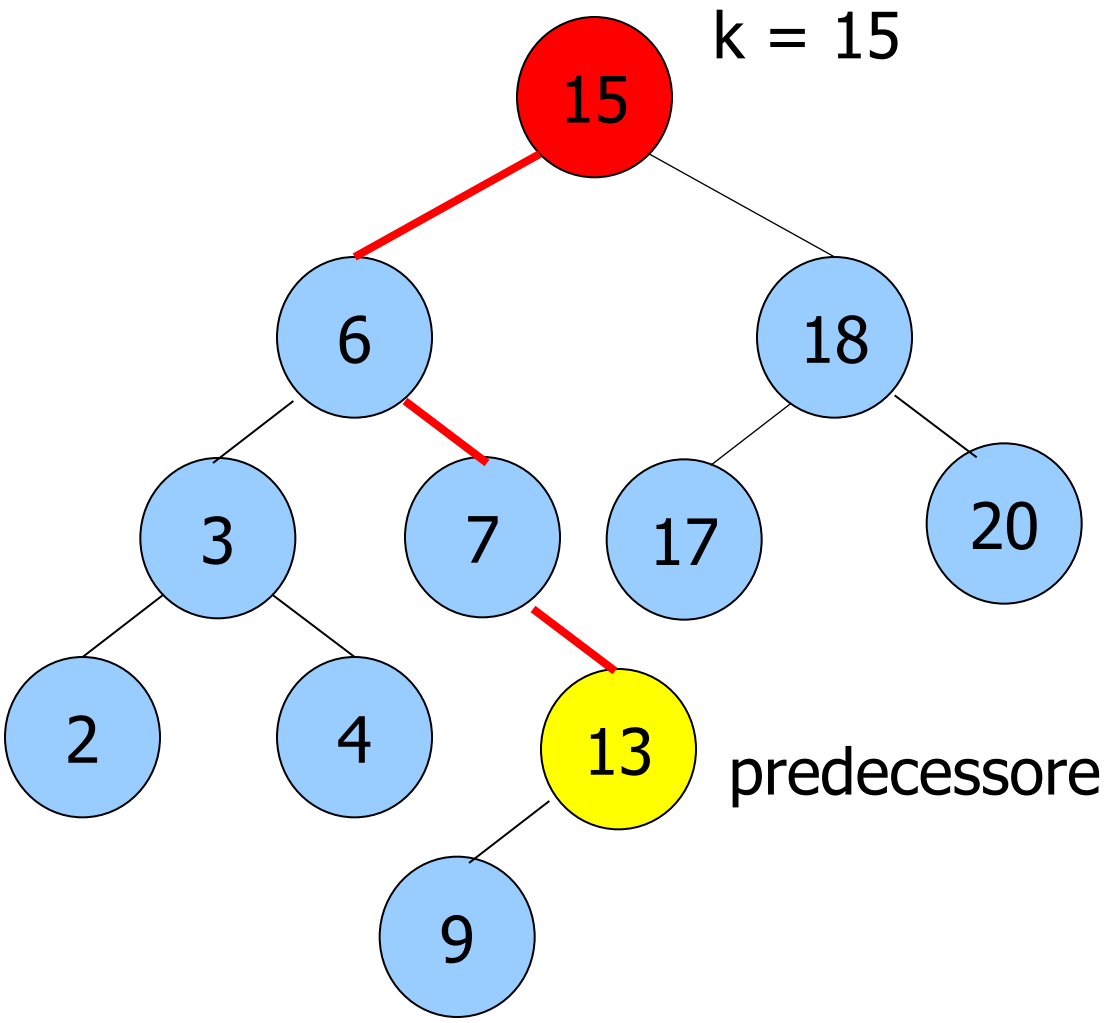
Predecessor

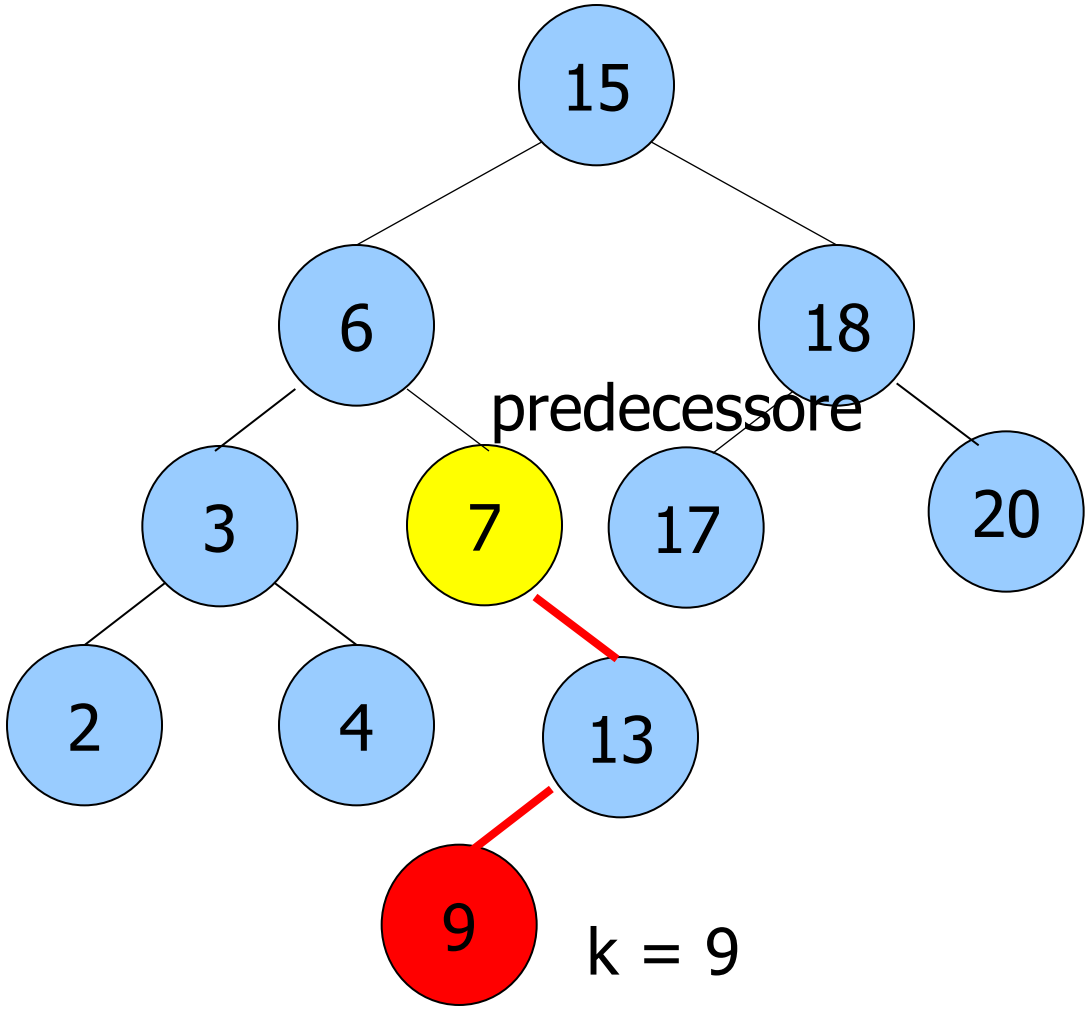
Predecessore di un item: nodo h con item con la più grande chiave < della chiave di item.

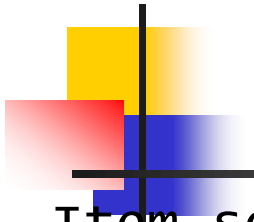
Due casi:

- ∃ Left(h): pred(key(h)) = max(Left(h))
- ∄ Left(h): pred(key(h)) = primo antenato di h il cui figlio destro è anche un antenato di h.

Esempio




15
6
18
predecessore
3
7
17
20
2
4
13
9
k = 9



```
Item searchPred(link h, Item x, link z) {
  link p;
  if (h == z) return NULLitem;
  if (eq(key(x), key(h->item))) {
    if (h->l != z) return maxR(h->l, z);
    else {
      p = h->p;
      while (p != z && h == p->l) {
        h = p; p = p->p;
      }
      return p->item;
    }
  }
  if (less(key(x), key(h->item)))
    return searchPred(h->l, x, z);
  else return searchPred(h->r, x, z);
}
Item BSTpred(BST bst, Item x) {
  return searchPred(bst->head, x, bst->z);
}
```

Insert (in foglia)

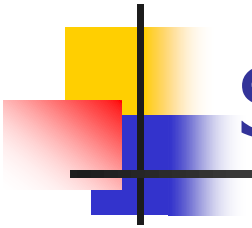
RICORSIVO

```
link insertR(link h, Item x, link z) {
  if (h == z)
    return NEW(x, z, z, z, 1);
  if (less(key(x), key(h->item))) {
    h->l = insertR(h->l, x, z);
    h->l->p = h;
  }
  else  {
    h->r = insertR(h->r, x, z);
    h->r->p = h;
  }
  (h->N)++;
  return h;
}
void BSTinsert_leafR(BST bst, Item x) {
  bst->head = insertR(bst->head, x, bst->z);
  bst->N++;
}
```

Insert (in foglia)

ITERATIVO

```
void BSTinsert_leafI(BST bst, Item x) {
  link p = bst->head, h = p;
  if (bst->head == bst->z) {
    bst->head = NEW(x, bst->z, bst->z, bst->z, 1);
    bst->N++;
    return;
  }
  while (h != bst->z) {
    p = h; h->N++;
    h = less(key(x), key(h->item))? h->l : h->r;
  }
  h = NEW(x, p, bst->z, bst->z, 1);
  bst->N++;
  if (less(key(x), key(p->item)))  p->l = h;
  else  p->r = h;
}
```

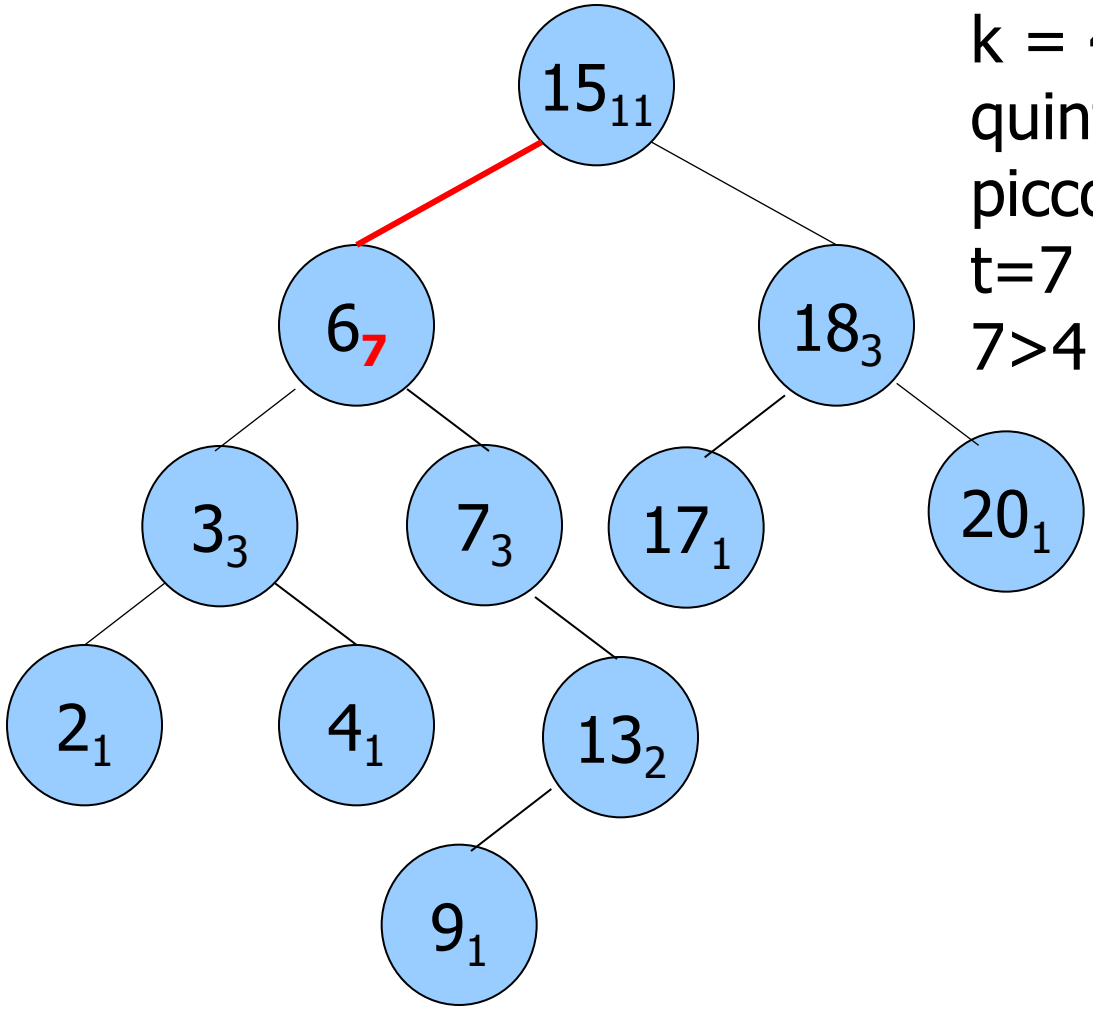


Select

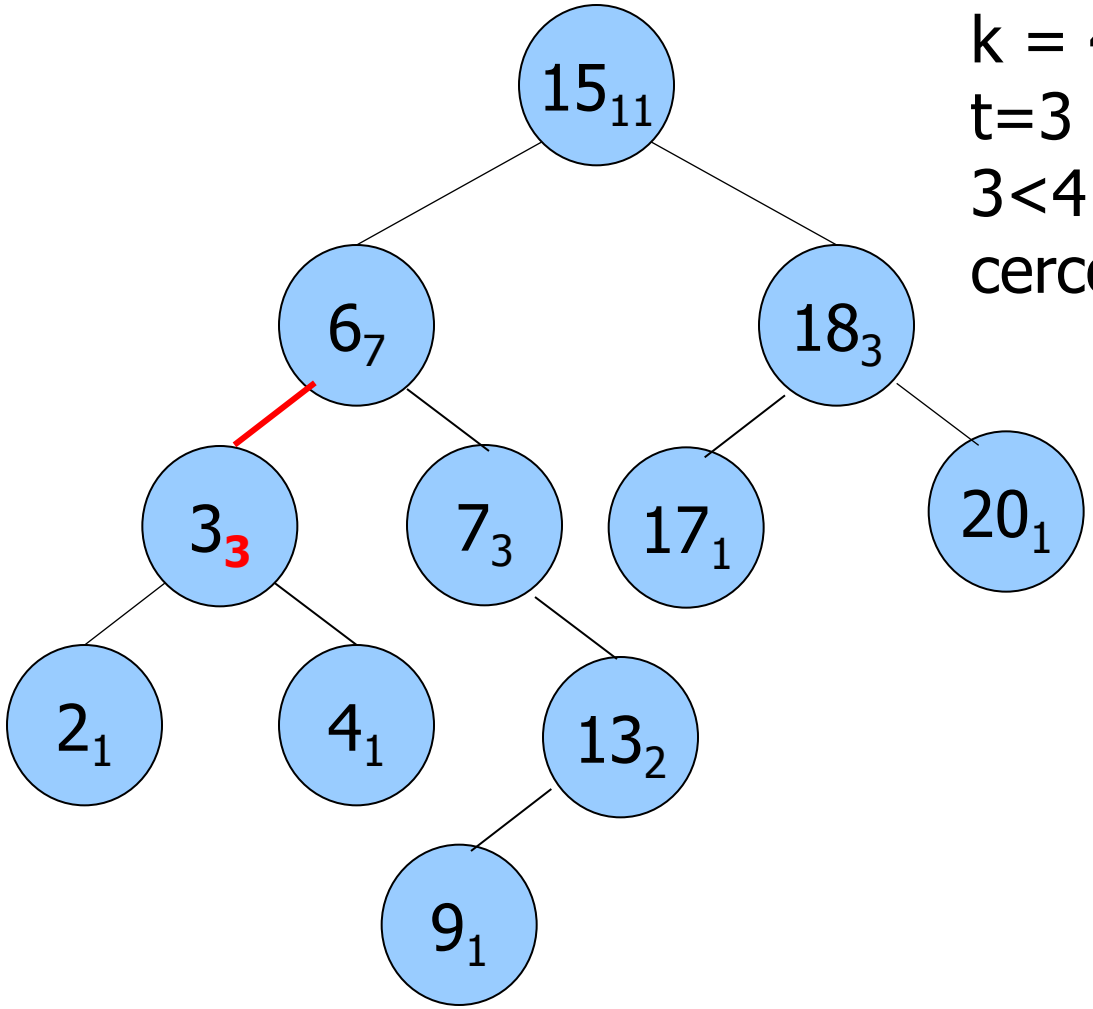
Selezione dell'item con a k-esima chiave più piccola (k=0 item con chiave minima): t è il numero di nodi del sottoalbero sinistro:

- t = k: ritorno la radice del sottoalbero
- t > k: ricorsione nel sottoalbero sinistro alla ricerca della k-esima chiave più piccola
- t < k: ricorsione nel sottoalbero destro alla ricerca della (k-t-1)-esima chiave più piccola

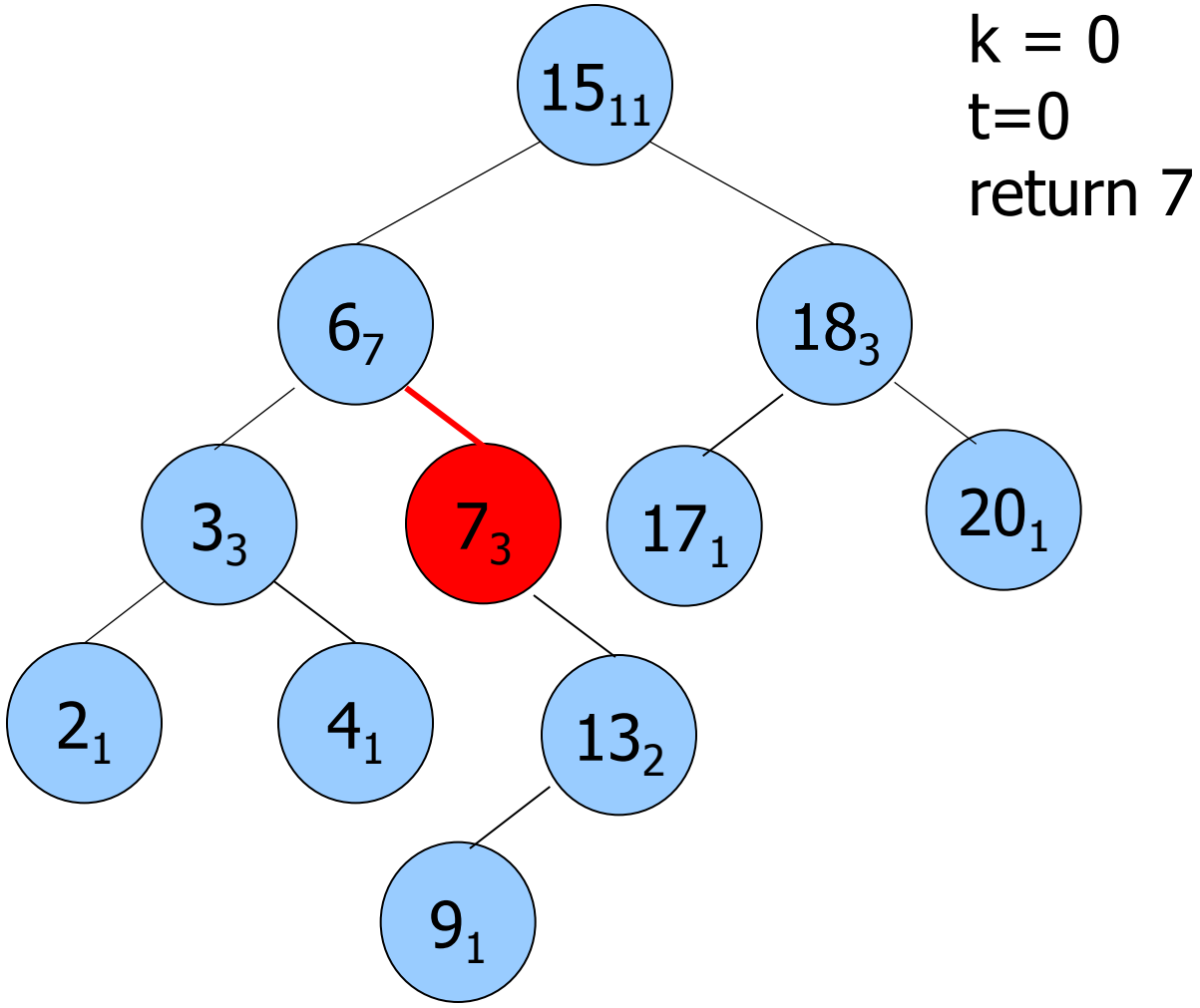
Esempio

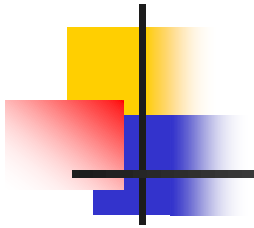


k = 4
quinta più
piccola chiave
t=7
7>4 scendo a SX



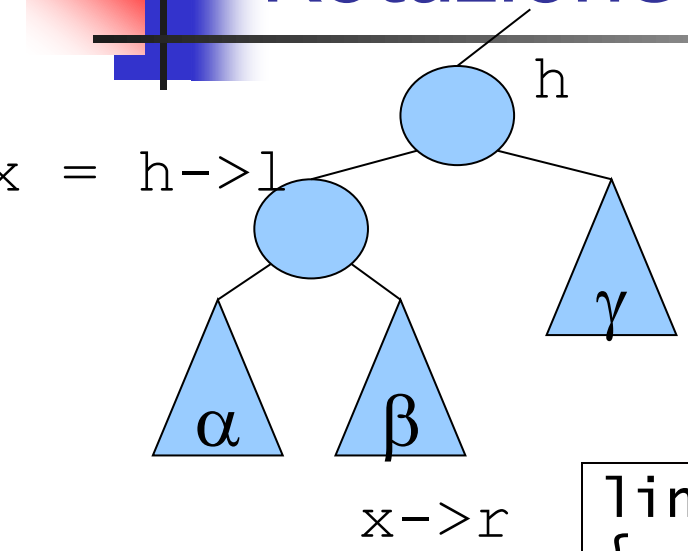
k = 4
t=3
3<4 scendo a DX
cerco k=4-3-1=0

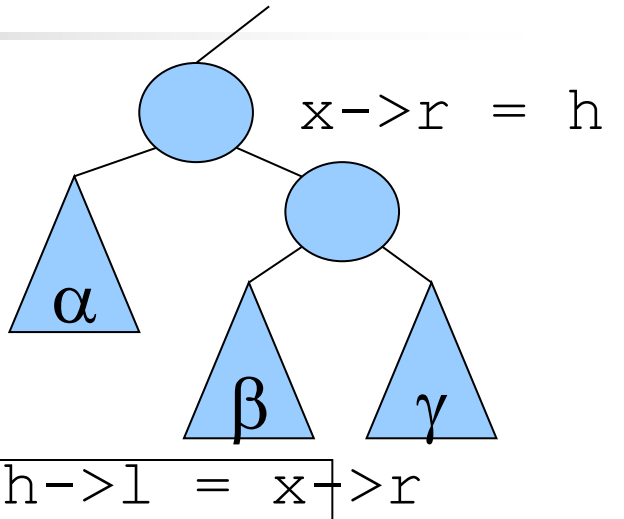

15_{11}
6_7
18_3
3_3
7_3
17_1
20_1
2_1
4_1
13_2
9_1
k = 0
t=0
return 7



```
Item selectR(link h, int k, link z) {
  int t;
  if (h == z)
    return NULLitem;
  t = (h->l == z) ? 0 : h->l->N;
  if (t > k)
    return selectR(h->l, k, z);
  if (t < k)
    return selectR(h->r, k-t-1, z);
  return h->item;
}

Item BSTselect(BST bst, int k) {
  return selectR(bst->head, k, bst->z);
}
```

Rotazione a destra di BST



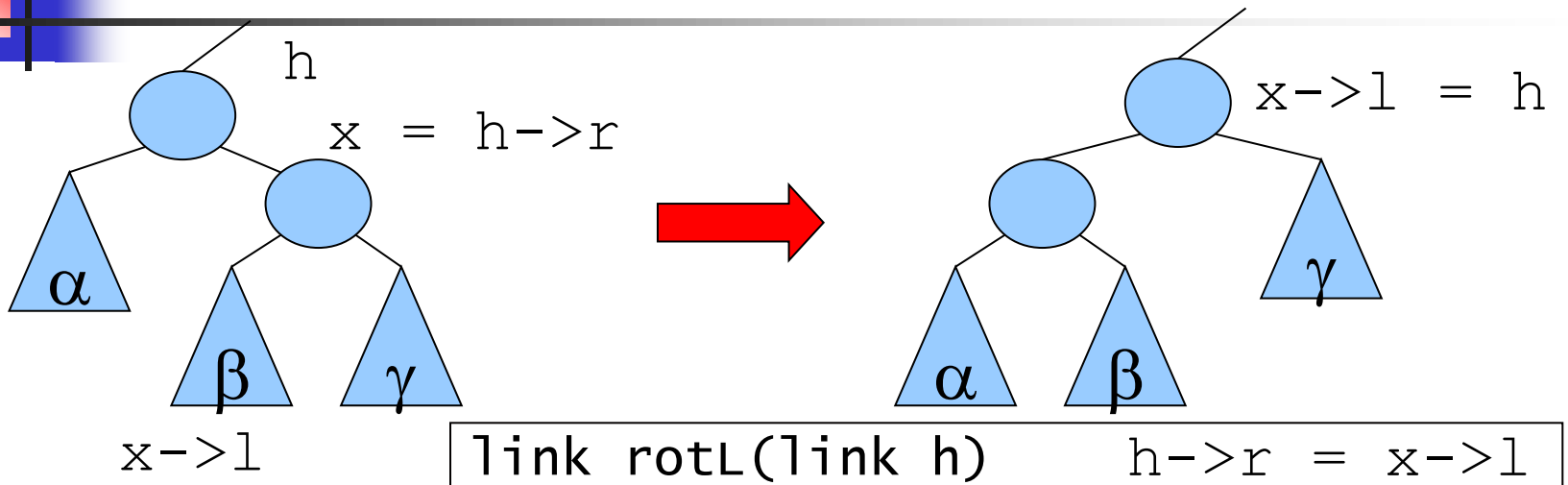



```
link rotR(link h)
{
  link x = h->l;
  h->l = x->r;
  x->r->p = h;
  x->r = h;
  x->p = h->p;
  h->p = x;
  x->N = h->N;
  h->N = h->r->N + h->l->N +1;
  return x;
}
```

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

Rotazione a sinistra di BST




```
link rotL(link h)
{
  link x = h->r;
  h->r = x->l;
  x->l->p = h;
  x->l = h;
  x->p = h->p;
  h->p = x;
  x->N = h->N;
  h->N = h->l->N + h->r->N +1;
  return x;
}
```

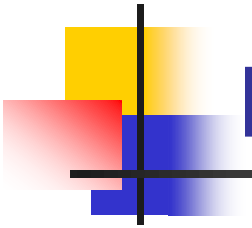
h->r = x->l

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

Inserimento alla radice

```
link insertT(link h, Item x, link z) {
  if ( h == z)
    return NEW(x, z, z, z, 1);
  if (less(key(x), key(h->item))) {
    h->l = insertT(h->l, x, z);
    h = rotR(h); h->N++;
  }
  else {
    h->r = insertT(h->r, x, z);
    h = rotL(h); h->N++;
  }
  return h;
}

void BSTinsert_root(BST bst, Item x) {
  bst->head = insertT(bst->head, x, bst->z);
  bst->N++;
}
```

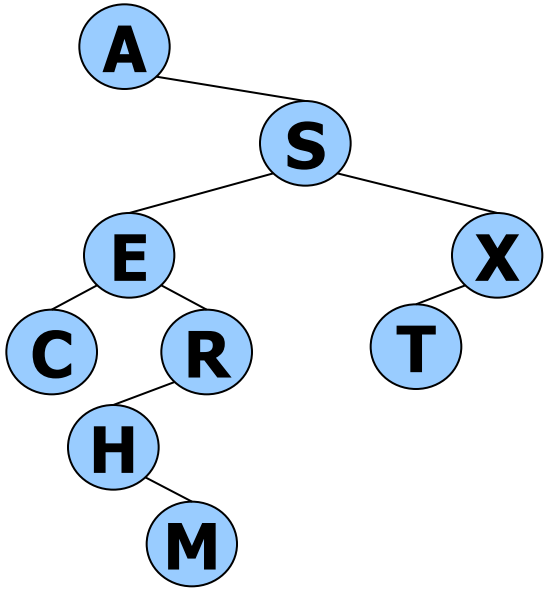


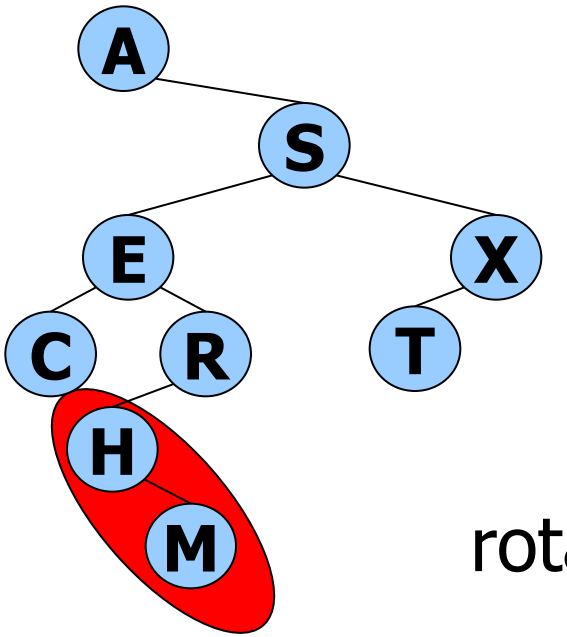
Partition

- Riorganizza l'albero avendo l'item con la k-esima chiave più piccola nella radice:
  - poni il nodo come radice di un sottoalbero:
    - t > k: ricorsione nel sottoalbero sinistro, partizionamento rispetto alla k-esima chiave più piccola, al termine rotazione a destra
    - t < k: ricorsione nel sottoalbero destro, partizionamento rispetto alla (k-t-1)-esima chiave più piccola , al termine rotazione a sinsitra.

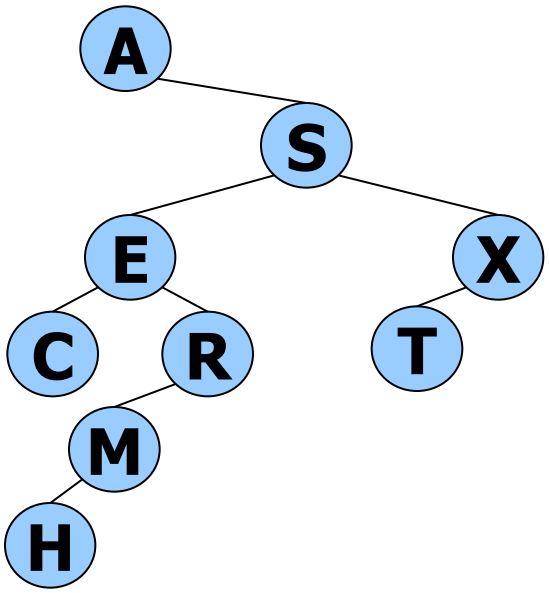
Esempio

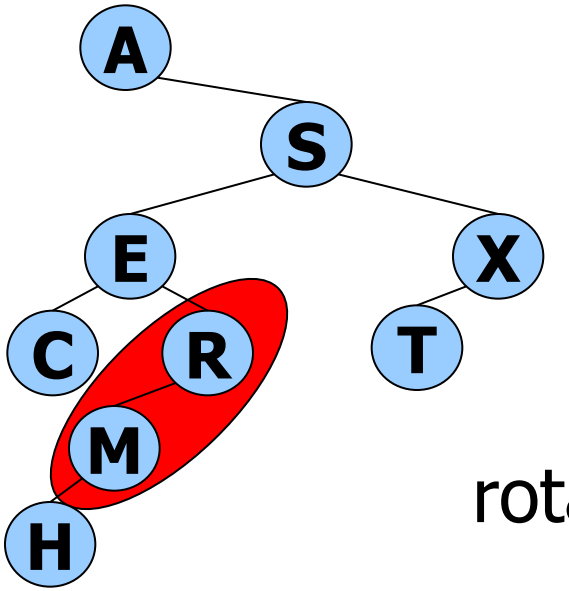
Partizionamento rispetto alla
5a chiave più piccola (M, k=4)



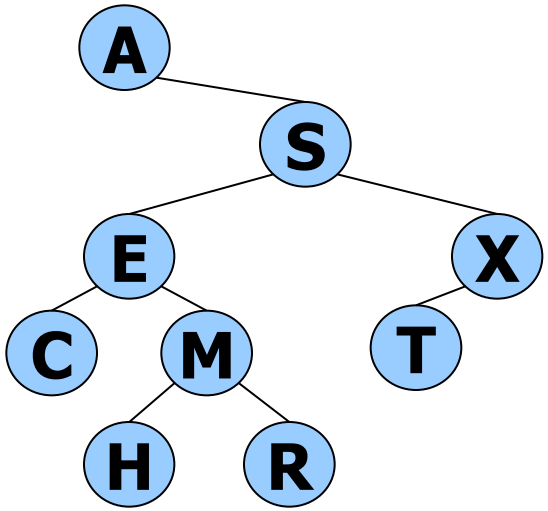


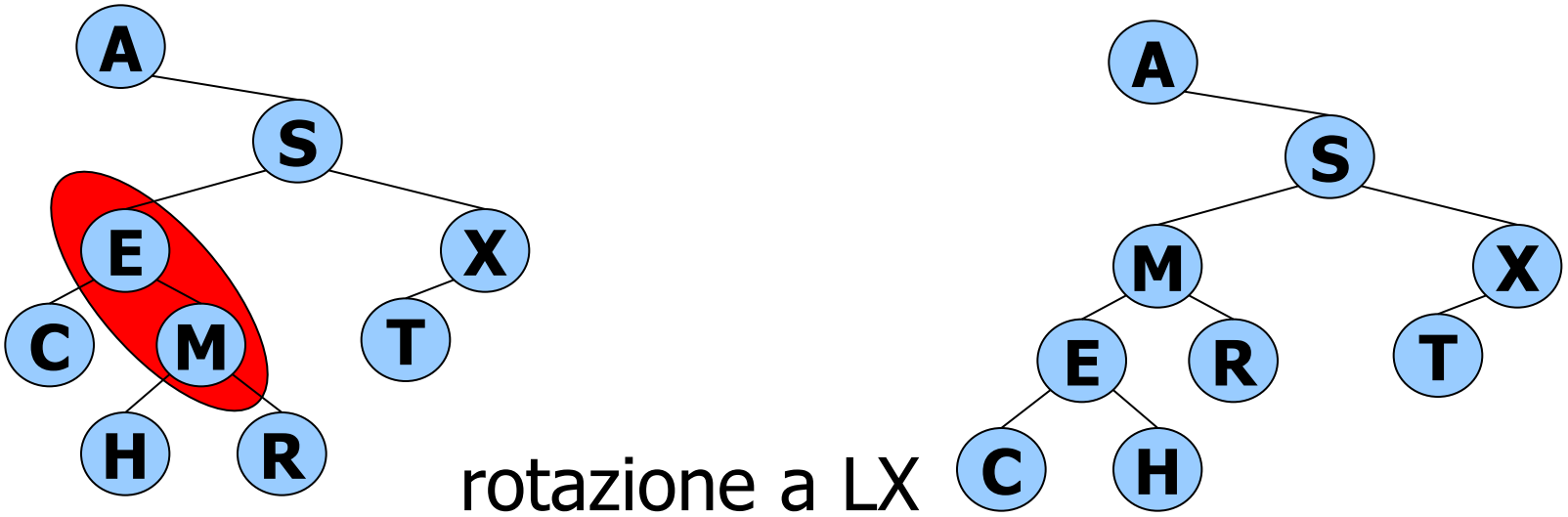
rotazione a SX





rotazione a DX




A
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rotazione a LX
A
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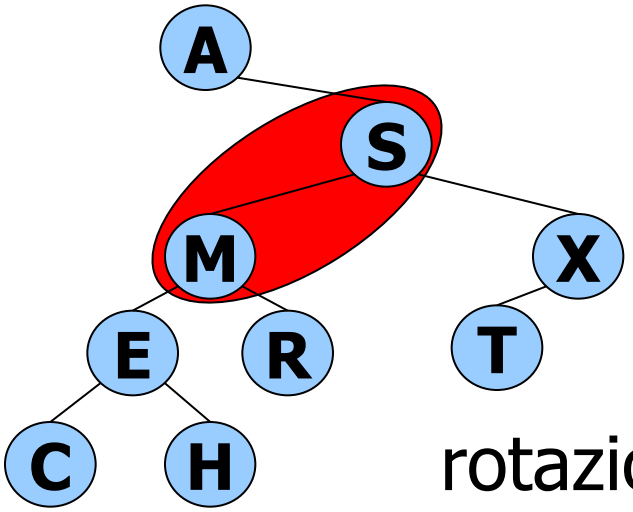
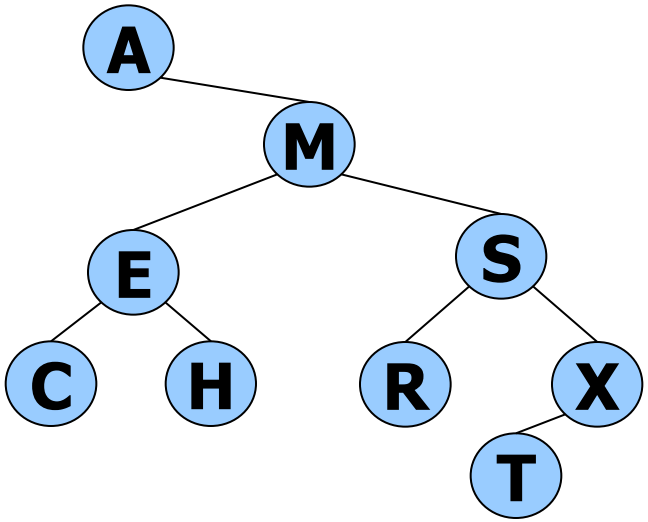
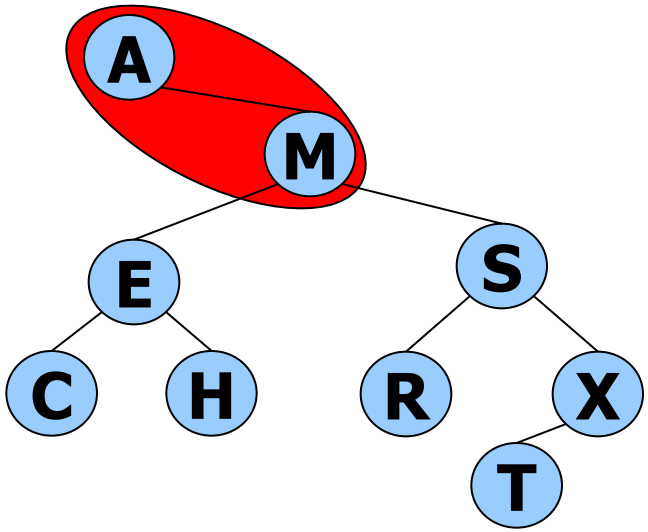
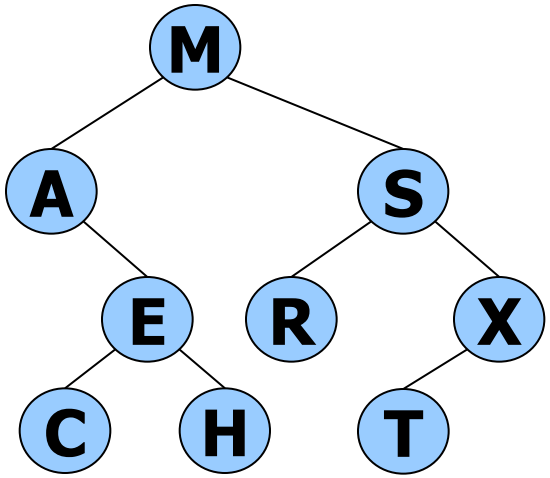


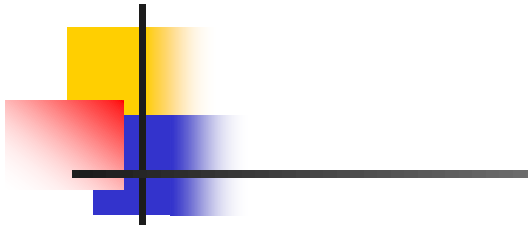
rotazione a DX



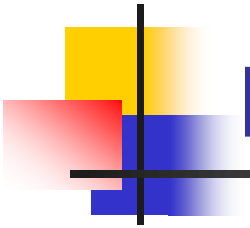




rotazione a SX



```
link partR(link h, int k) {
  int t  = h->l->N;
  if ( t > k) {
    h->l = partR(h->l, k);
    h = rotR(h);
  }
  if ( t < k) {
    h->r = partR(h->r, k-t-1);
    h = rotL(h);
  }
  return h;
}
```



Delete

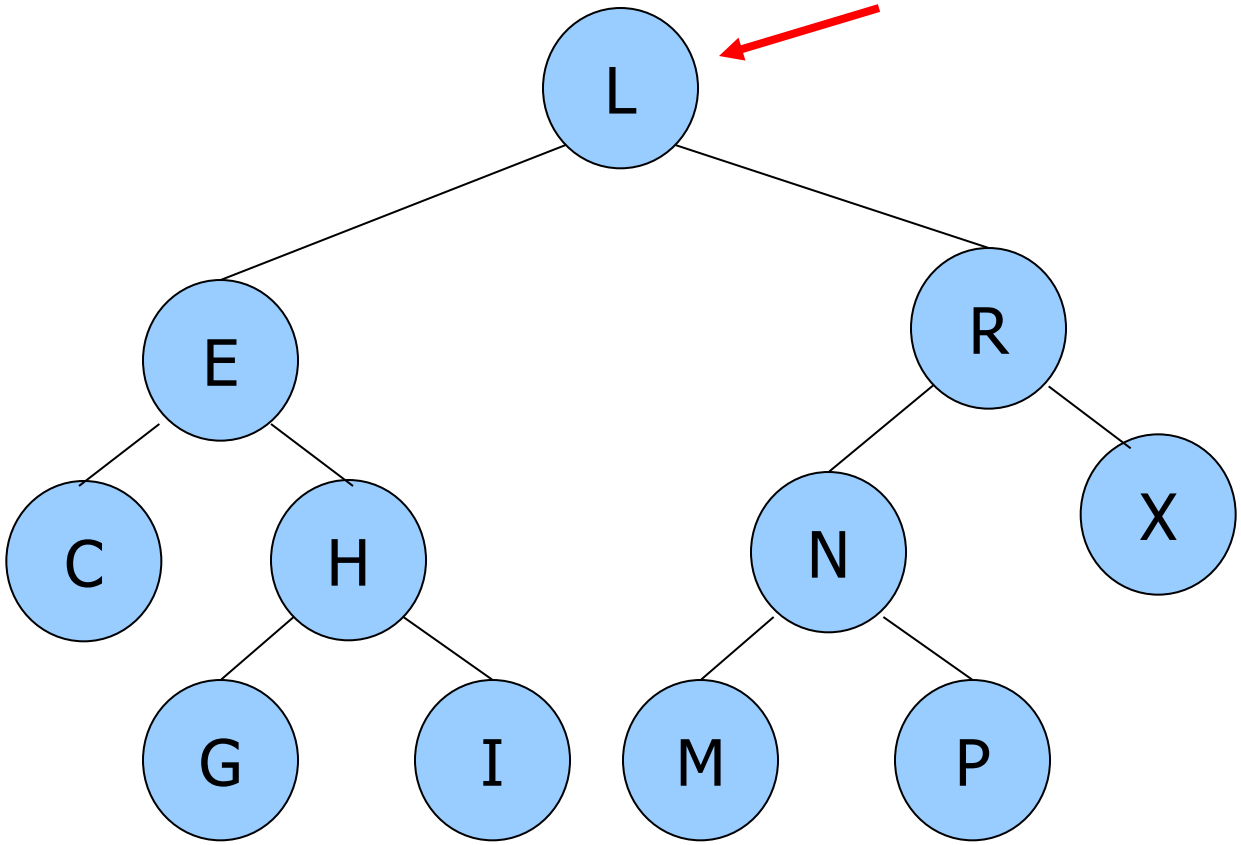
Per cancellare da un albero binario di ricerca un nodo con item con chiave k bisogna mantenere:

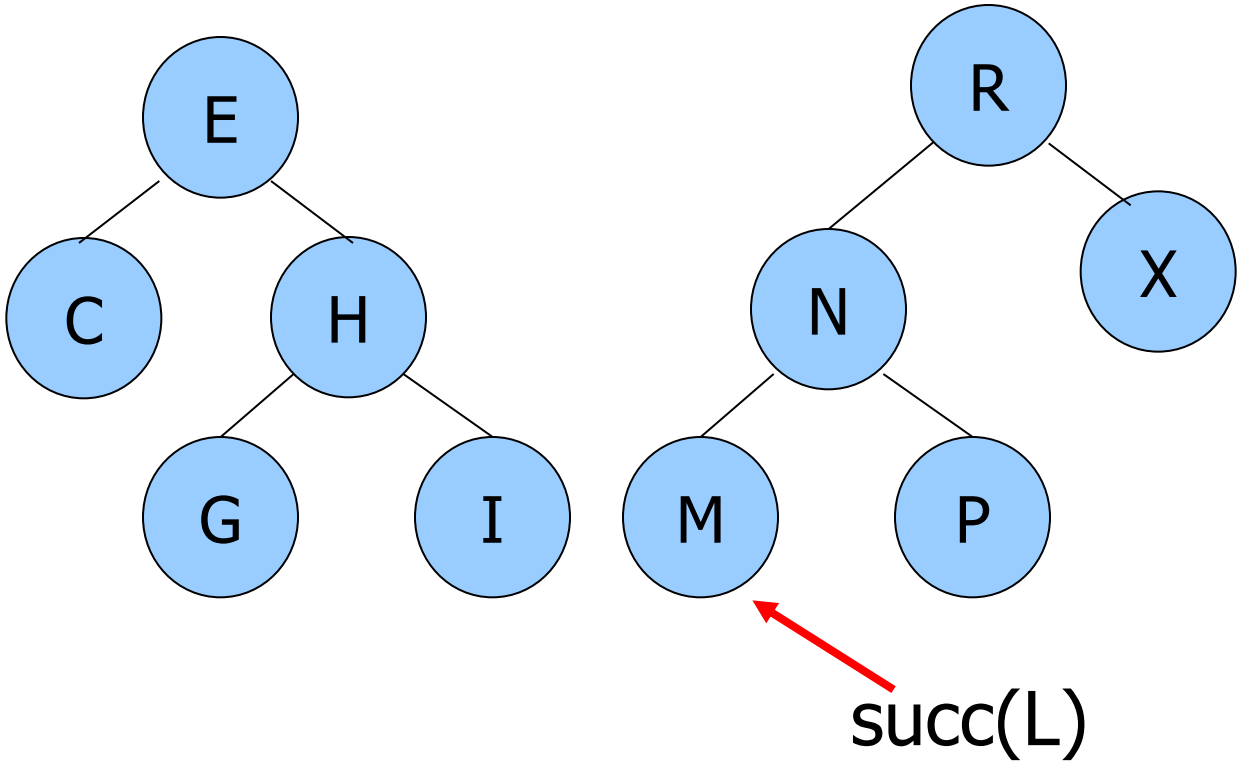
- la proprietà dei BST
- la struttura ad albero binario

Passi:

- controllare se il nodo con l'item da cancellare è in uno dei sottoalberi. Se sì, cancellazione ricorsiva nel sottoalbero
- se è la radice, eliminarlo e ricombinare i 2 sottoalberi. La nuova radice è il succ o il pred dell'item cancellato.

Cancellazione di una radice




E
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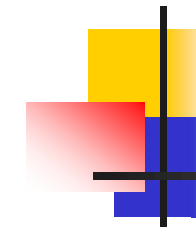
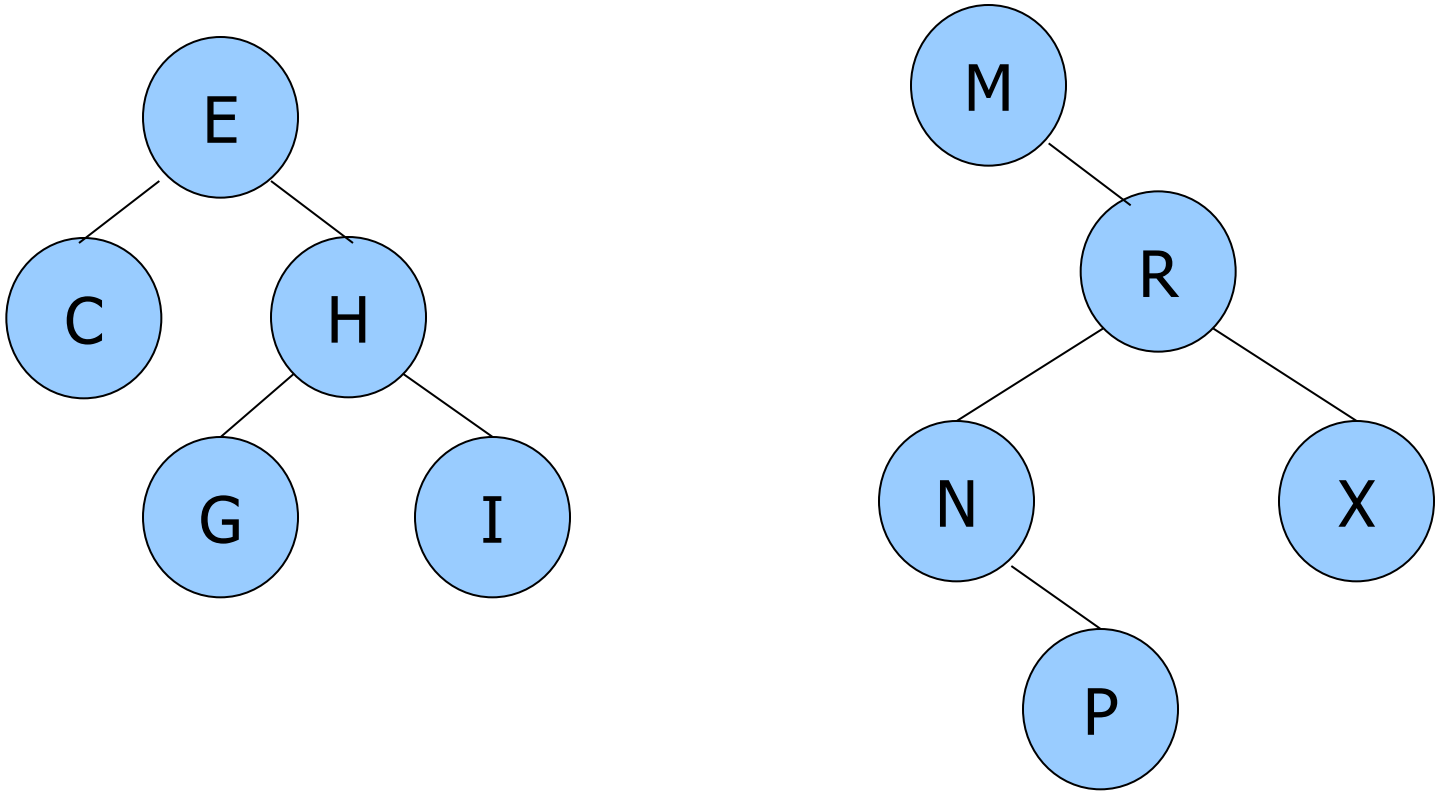
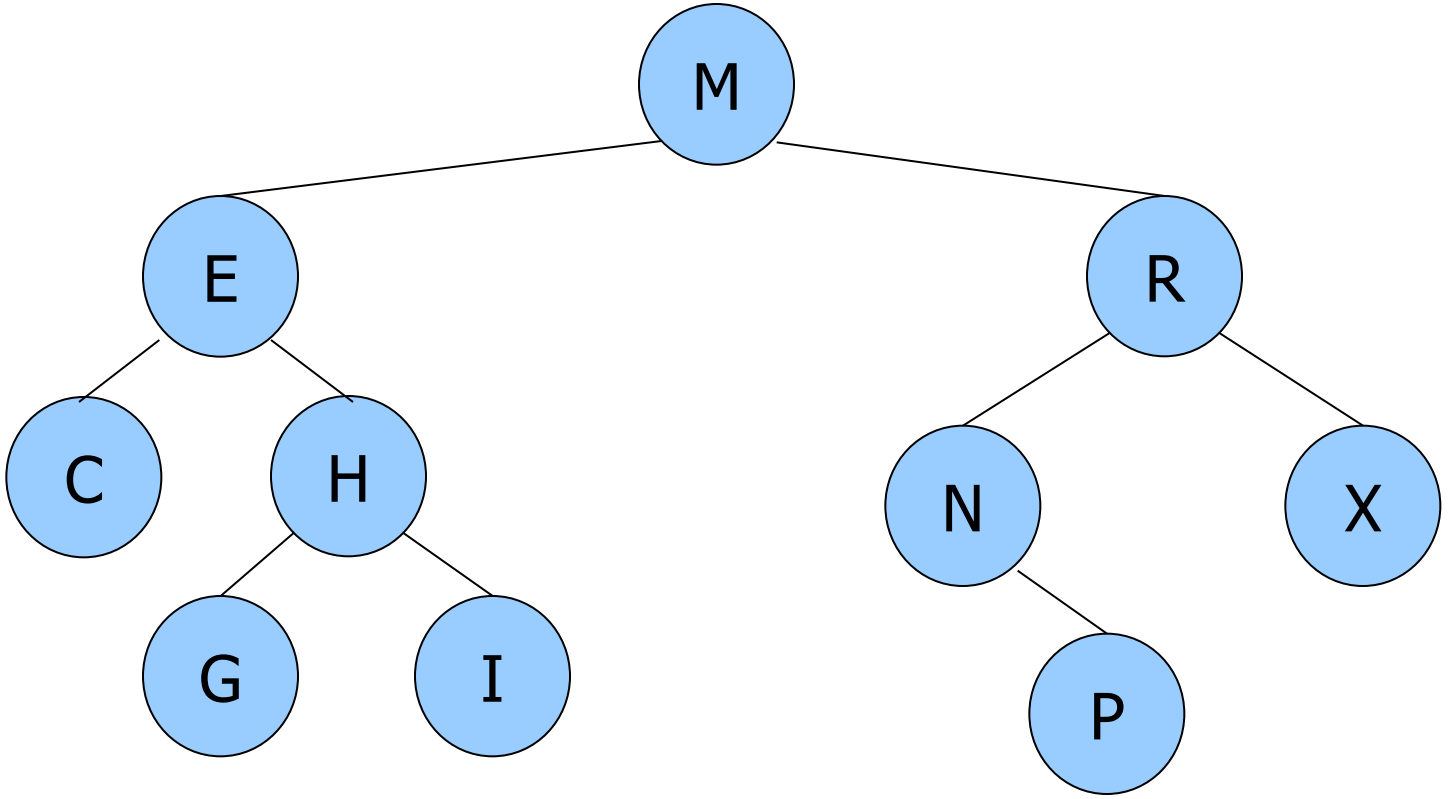
M
P
succ(L)



partition rispetto a M

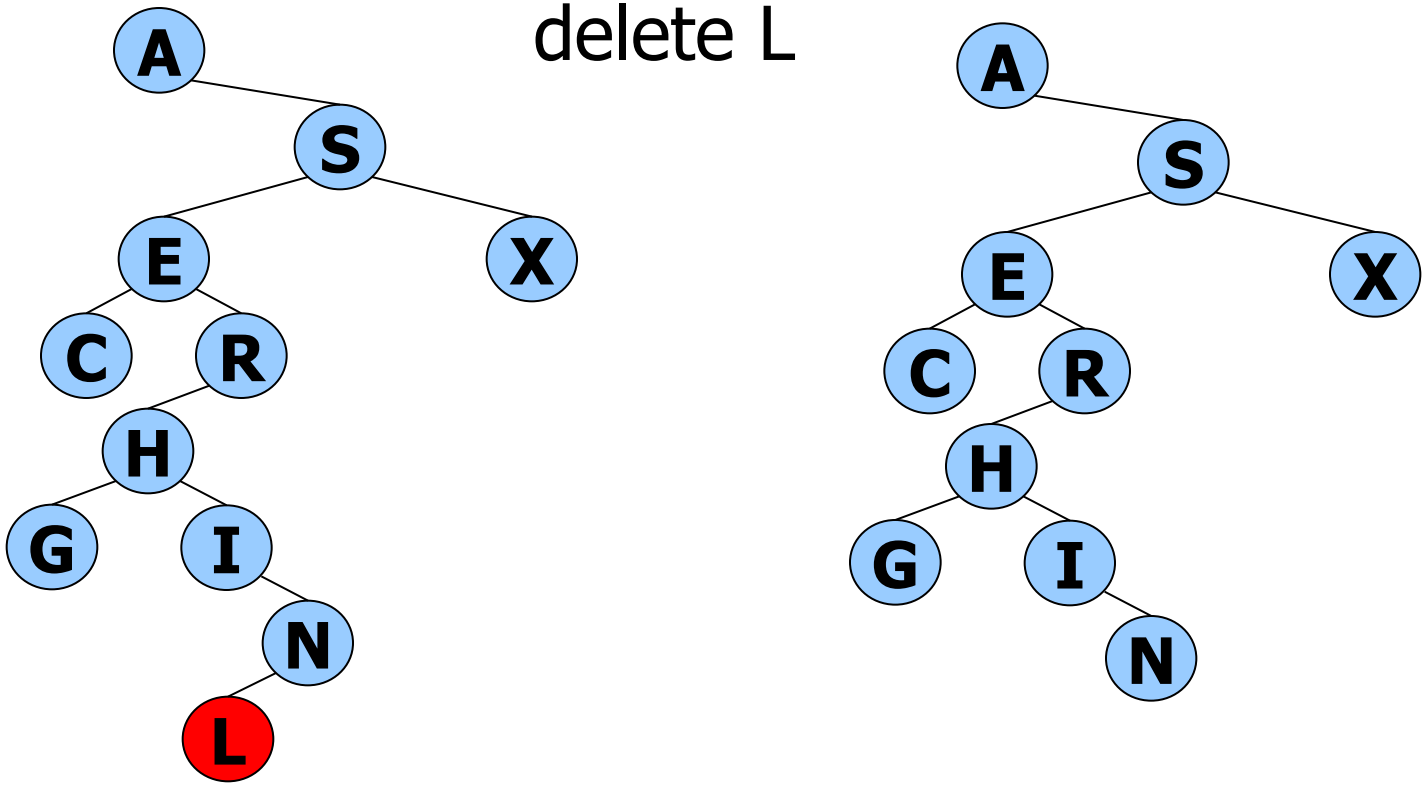


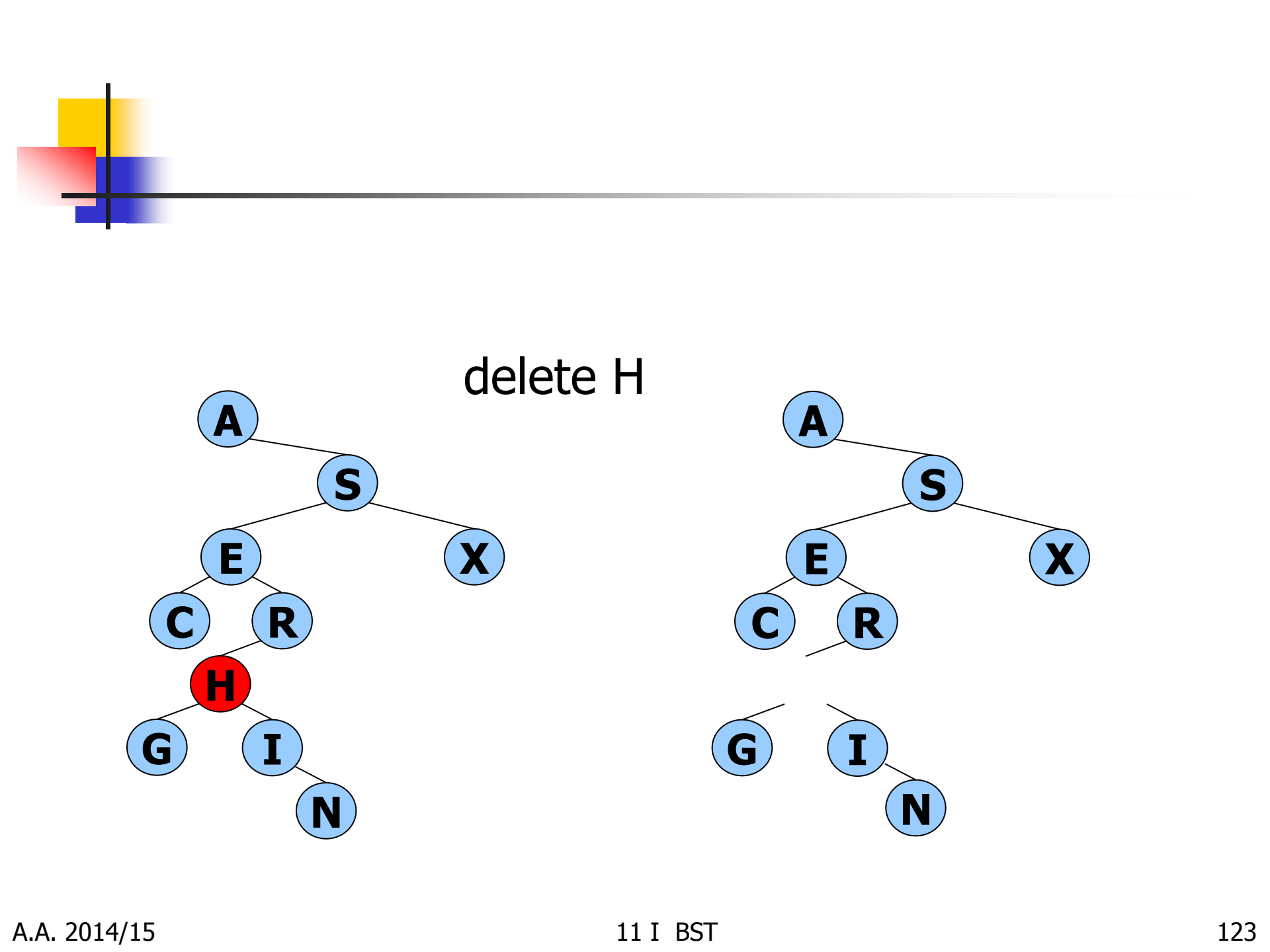
ricostruzione dell'albero con radice M



Esempio

cancellazione in sequenza di L, H, E




delete H
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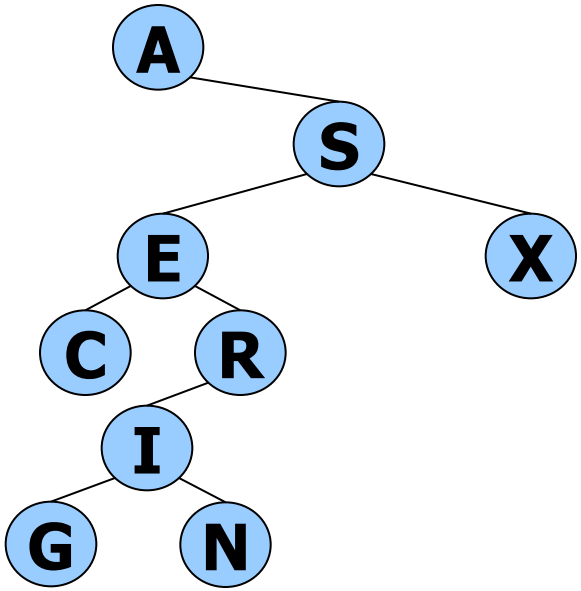
A
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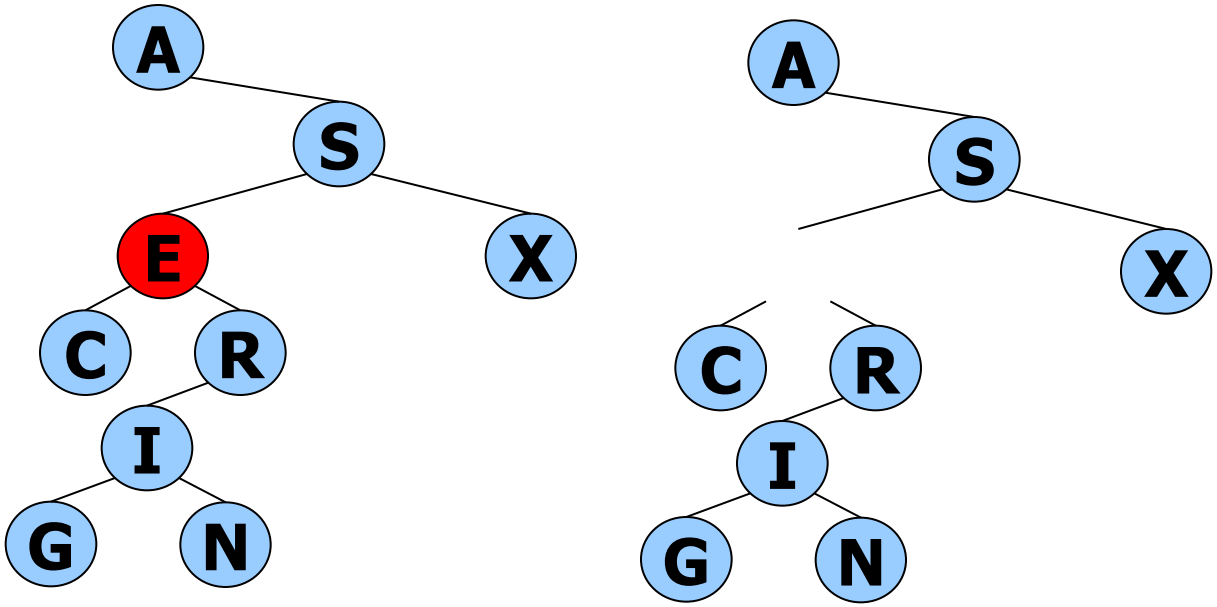
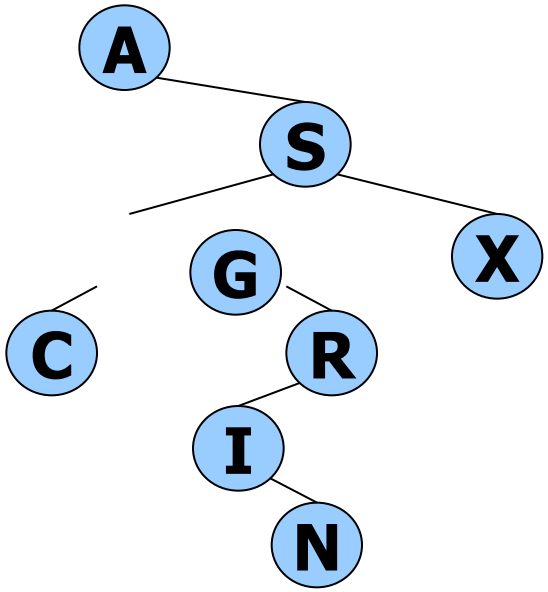
delete E




A
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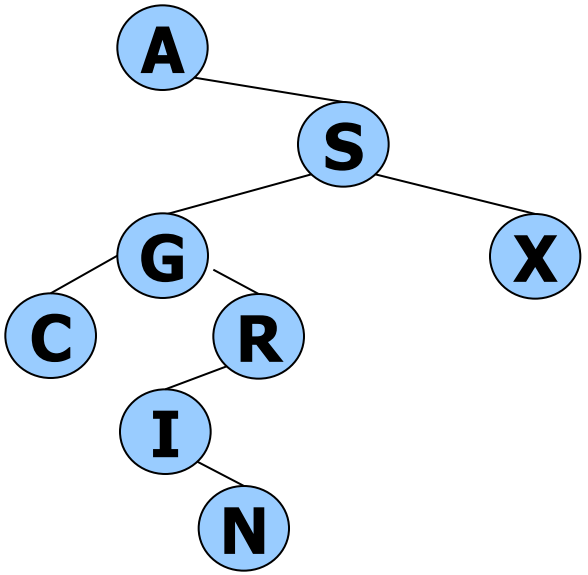
A
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G
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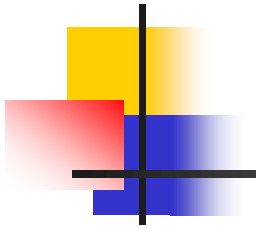


```
link joinLR(link a, link b) {
  if (b == z)
    return a;
  b = partR(b, 0);
  b->l = a;
  a->p = b;
  b->N = a->N + b->r->N +1;
  return b;
}
```

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

```
link deleteR(link h, Item x, link z) {
  link y, p;
  if (h == z) return z;
  if (less(key(x), key(h->item))) {
    h->l = deleteR(h->l, x, z);
  if (less(key(h->item), key(x)))
    h->r = deleteR(h->r, x, z);
  (h->N)--;
  if (eq(key(x), key(h->item))) {
    y = h; p = h->p; h = joinLR(h->l, h->r, z);
    h->p = p; free(y);
  }
  return h;
}

void BSTdelete(BST bst, Item x) {
  bst->head = deleteR(bst->head, x, bst->z);
  bst->N--;
}
```

aggiornamento puntatore al padre

Estensioni delle strutture dati

Prima di sviluppare una nuova struttura dati è bene valutare se si possano «estendere» strutture esistenti con informazioni opportune.

Procedura:

1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

Esempio: Order-Statistic BST

Procedura:

1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

dimensione del sottoalbero

BST

O(1)

```
Item BSTselect(BST, int);
```

Interval BST

Intevallo chiuso: coppia ordinata di reali $[t_1, t_2]$, dove $t_1 \leq t_2$ e $[t_1, t_2] = \{t \in \in \Re : t_1 \leq t \leq t_2\}$.

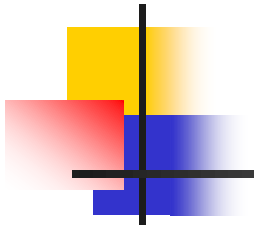
L'intervallo $[t_1, t_2]$ può essere rappresentato da una `struct` con campi `low` = t_1 e `high` = t_2.

Gli intervalli i e i' hanno intersezione se e solo se:

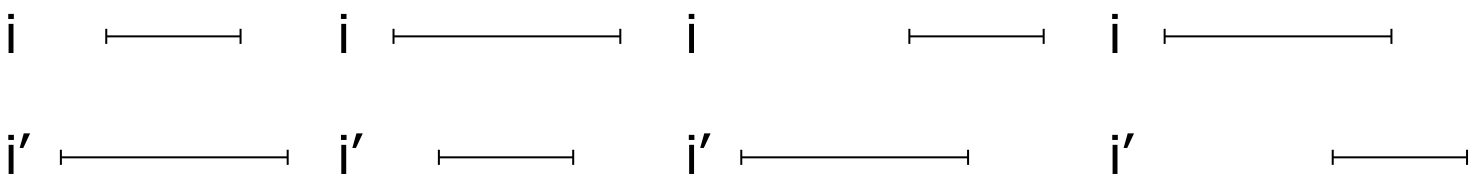
`low[i]` $\leq$ `high[i']` && `low[i']` $\leq$ `high[i]`.

$\forall$ i, i' vale la seguente tricotomia:

a. i e i' hanno intersezione
b. `high[i]` $\leq$ `low[i']`
c. `high[i']` $\leq$ `low[i]`

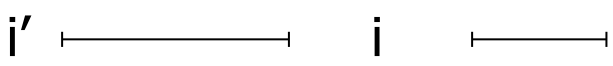


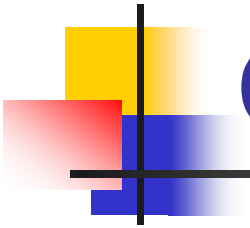
caso a



caso b

caso c





Operazioni

```
Item IBSTsearch(IBST, Item) ;
```

cerca un item (intervallo) nell'Interval BST

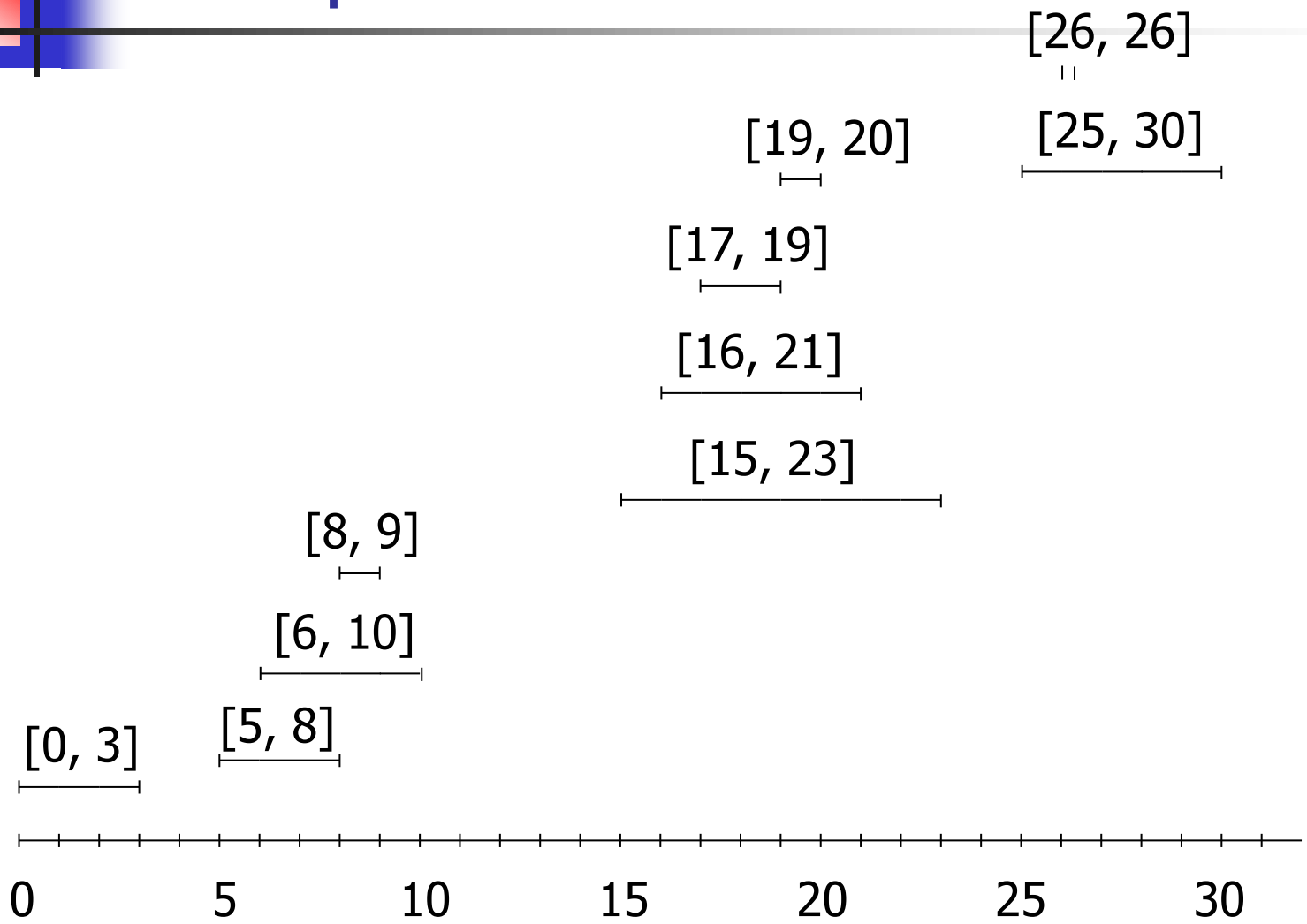
```
void     IBSTinsert(IBST, Item) ;
```

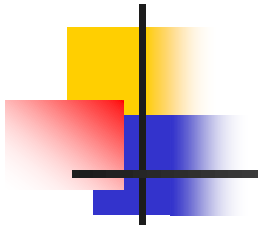
inserisci un item (intervallo) nell'Interval BST

```
void     IBSTdelete(IBST, Item) ;
```

cancella un item (intervallo) dall'Interval BST

Esempio





Proced

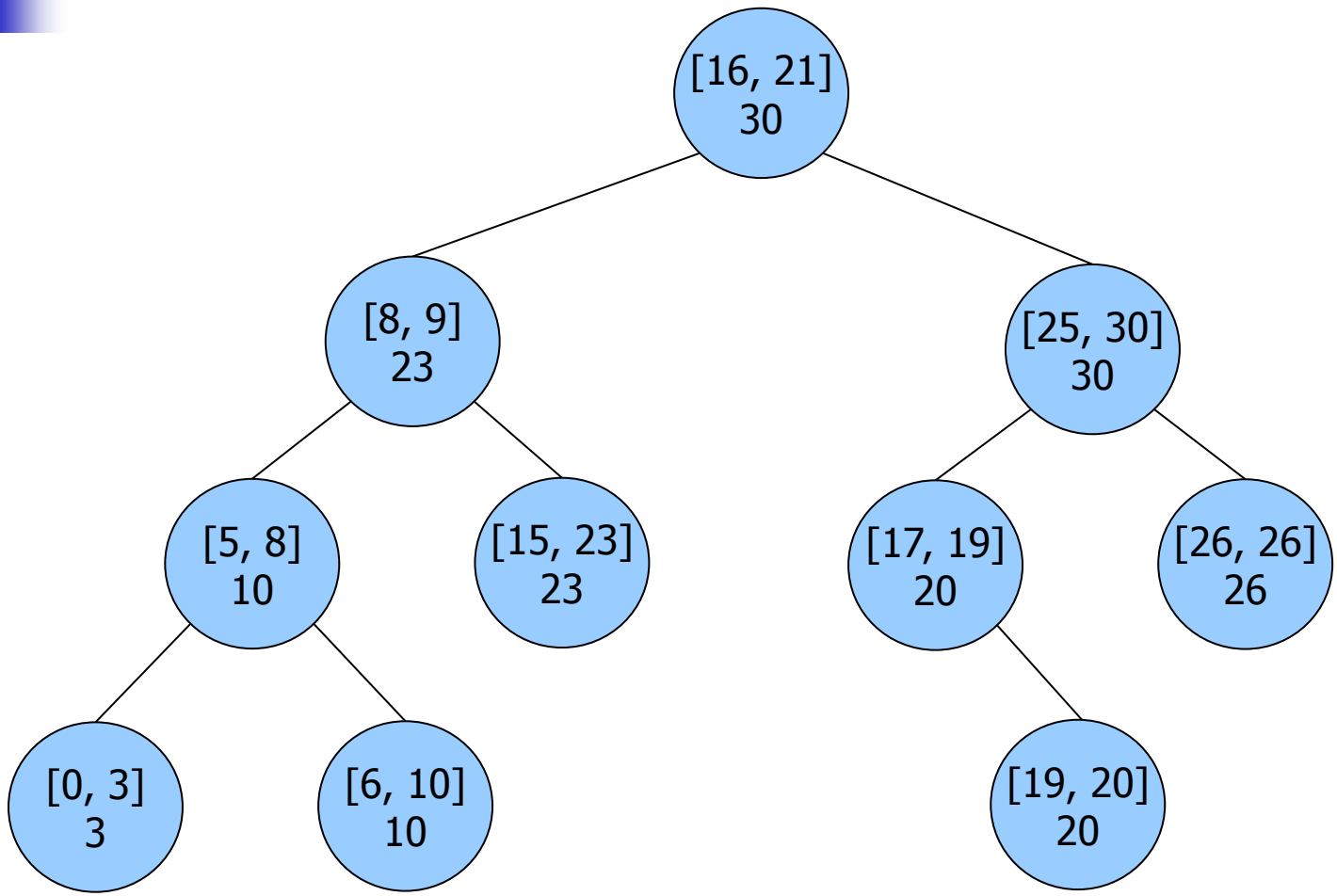
1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

`max`: massimo `high` del sottoalbero

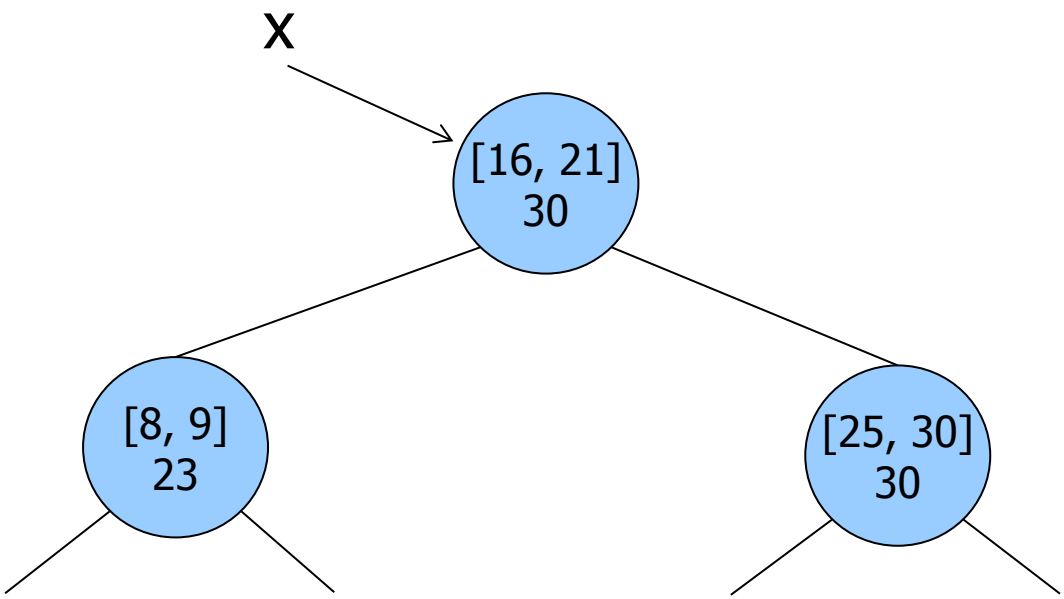
BST con chiave `low`

O(1)

`Item IBSTsearch(IBST, Item);`


[16, 21]
30
[8, 9]
23
[25, 30]
30
[5, 8]
10
[15, 23]
23
[17, 19]
20
[26, 26]
26
[0, 3]
3
[6, 10]
10
[19, 20]
20

x->max = max (high(x), x->left->max, x->right->max)



Operazioni

IBST.h

```
typedef struct  intervalbinarysearchtree *IBST;

void  IBSTinit(IBST) ;
void  IBSTinsert(IBST, Item) ;
void  IBSTdelete(IBST, Item) ;
Item  IBSTsearch(IBST, Item) ;
void  IBSTsortinorder(IBST,void (*visit) (Item)) ;
void  IBSTsortpreorder(IBST, void (*visit) (Item)) ;
void  IBSTsortpostorder(IBST, void (*visit) (Item)) ;
```

nuove funzioni

funzioni modificate

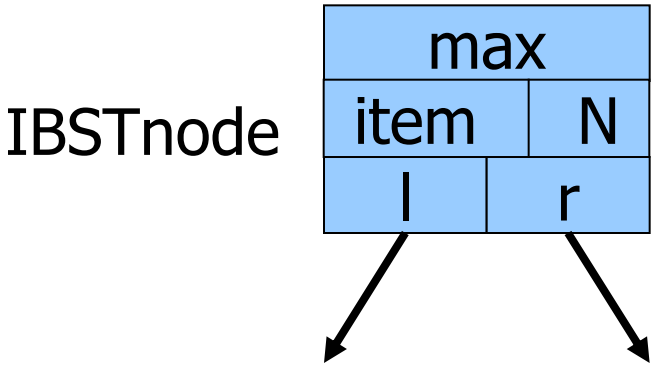
IBST.c

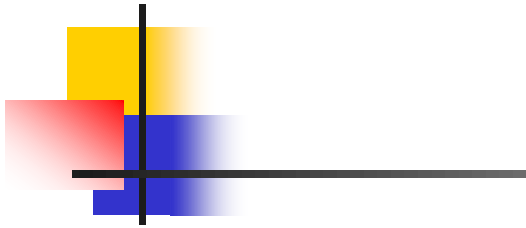
massimo high del sottoalbero

```
typedef struct IBSTnode *link;
struct IBSTnode {Item item; link l, r; int N; int max;} ;

struct intervalbinarysearchtree {link head;int N;link z;};

link NEW(Item item, link l, link r, int N, int max)
{
    link x = malloc(sizeof *x);
    x->item = item;
    x->l = l;
    x->r = r;
    x->N = N;
    x->max = max;
    return x;
}
```



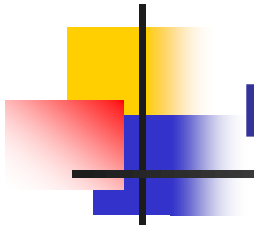


```
IBST IBSTinit( ) {
  IBST ibst = malloc(sizeof *ibst) ;
  ibst->N = 0;
  ibst->head=(ibst->z=NEW(NULLitem,NULL,NULL,0,-1));
  return ibst;
}

int max (int a, int b, int c) {
  int m = a;
  if (b > m) m = b;
  if (c > m) m = c;
  return m;
}
```

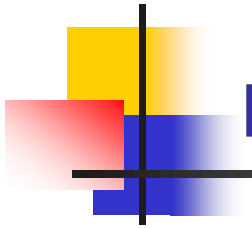
Insert

```
link insertR(link h, Item item, link z) {
  if (h == z)
    return NEW(item, z, z, 1, high(item));
  if (less(key(item), key(h->item))) {
    h->l = insertR(h->l, item, z);
    h->max = max(h->max, h->l->max, h->r->max);
  }
  else {
    h->r = insertR(h->r, item, z);
    h->max = max(h->max, h->l->max, h->r->max);
  }
  (h->N)++;
  return h;
}
void IBSTinsert(IBST ibst, Item item) {
  ibst->head = insertR(ibst->head, item, ibst->z);
  ibst->N++;
}
```



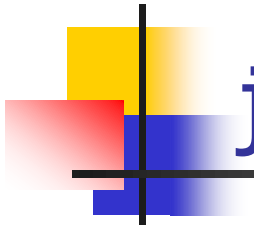
rotL

```
link rotL(link h) {
  link x = h->r;
  h->r = x->l;
  x->l = h;
  x->N = h->N;
  h->N = h->l->N + h->r->N +1;
  h->max = max(high(h->item), h->l->max, h->r->max);
  x->max = max(high(x->item), x->l->max, x->r->max);
  return x;
}
```



rotR

```
link rotR(link h) {
  link x = h->l;
  h->l = x->r;
  x->r = h;
  x->N = h->N;
  h->N = h->r->N + h->l->N +1;
  h->max = max(high(h->item), h->l->max, h->r->max);
  x->max = max(high(x->item), x->l->max, x->r->max);
  return x;
}
```

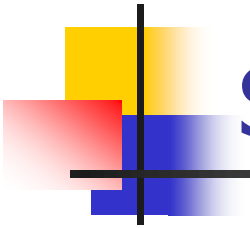


joinLR

```
link joinLR(link a, link b, link z) {
  if (b == z)
    return a;
  b = partR(b, 0);
  b->l = a;
  b->N = a->N + b->r->N +1;
  b->max = max(high(b->item), a->max, b->r->max);
  return b;
}
```

deleteR

```
link deleteR(link h, Item item, link z) {
  link x;
  if (h == z) return z;
  if (less(key(item), key(h->item))) {
    h->l = deleteR(h->l, item, z);
    h->max = max(high(h->item), h->l->max, h->r->max);
  }
  if (less(key(h->item), key(item))) {
    h->r = deleteR(h->r, item, z);
    h->max = max(high(h->item), h->l->max, h->r->max);
  }
  (h->N)--;
  if (eq(item, h->item)) {
    x = h; h = joinLR(h->l, h->r, z); free(x);
  }
  return h;
}
void IBSTdelete(IBST ibst, Item item) {
  ibst->head = deleteR(ibst->head, item, ibst->z);
  ibst->N--;
}
```

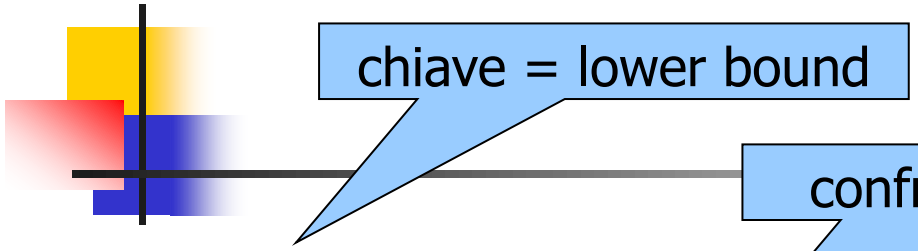


Search

Ricerca iterativa di un nodo h con intervallo che interseca l'intervallo i:

- percorrimento dell'albero dalla radice
- terminazione: trovato intervallo che interseca i oppure si è giunti ad un albero vuoto
- ricorsione: dal nodo h
  - su sottoalbero sinistro se h->l->max ≥ low[i]
  - su sottoalbero destro se h->l->max < low[i]

Item.h



chiave = lower bound

confronto tra chiavi

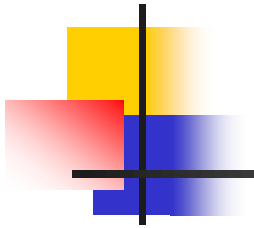
```
#define key(A) (A.x)
#define less(A, B) (A < B)
#define low(A) (A.x)
#define high(A) (A.y)
#define overlap(A,B) ((low(A)<=high(B))&&(low(B)<= high(A)))
#define neq(A, B) (low(A) != low(B) || high(A) != high(B))
#define eq(A, B) (low(A) == low(B) && high(A) == high(B))
#define less_int(A, B) (low(A) < h->l->max)
```

estrazione lower/upper bound

intersezione tra intervalli

condizione di ricorsione a DX o SX

confronto tra intervalli

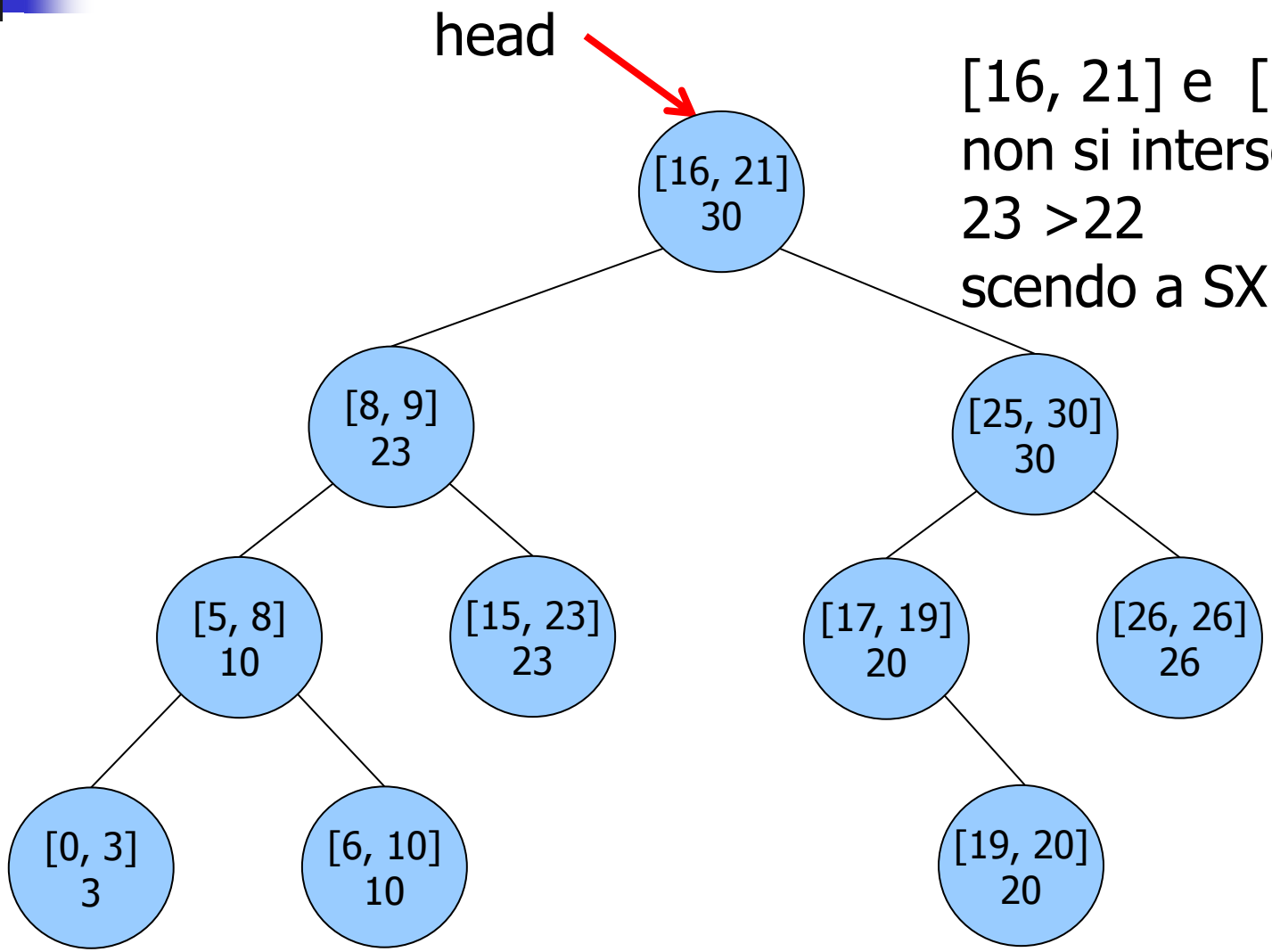


```
Item searchR(link h, Item item, link z) {
if (h == z)
  return NULLitem;
if (overlap(item, h->item))
  return h->item;
if (less_int(item, h->item))
  return searchR(h->l, item, z);
else
  return searchR(h->r, item, z);
}

Item IBSTsearch(IBST ibst, Item item) {
  return searchR(ibst->head, item, ibst->z);
}
```

Esempio

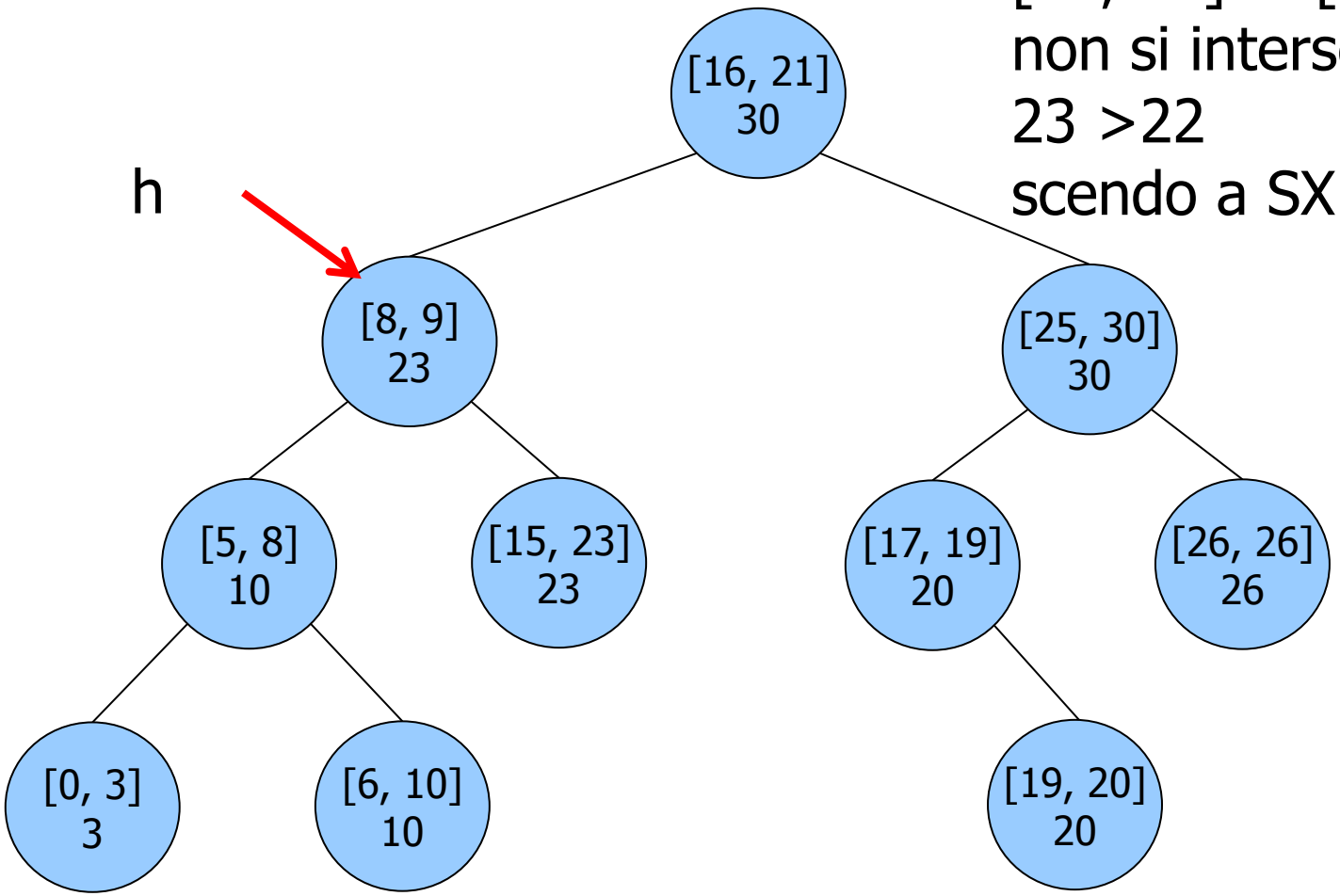
Ricerca di un intervallo che interseca [22, 25]



[16, 21] e [22, 25]
non si intersecano
23 >22
scendo a SX

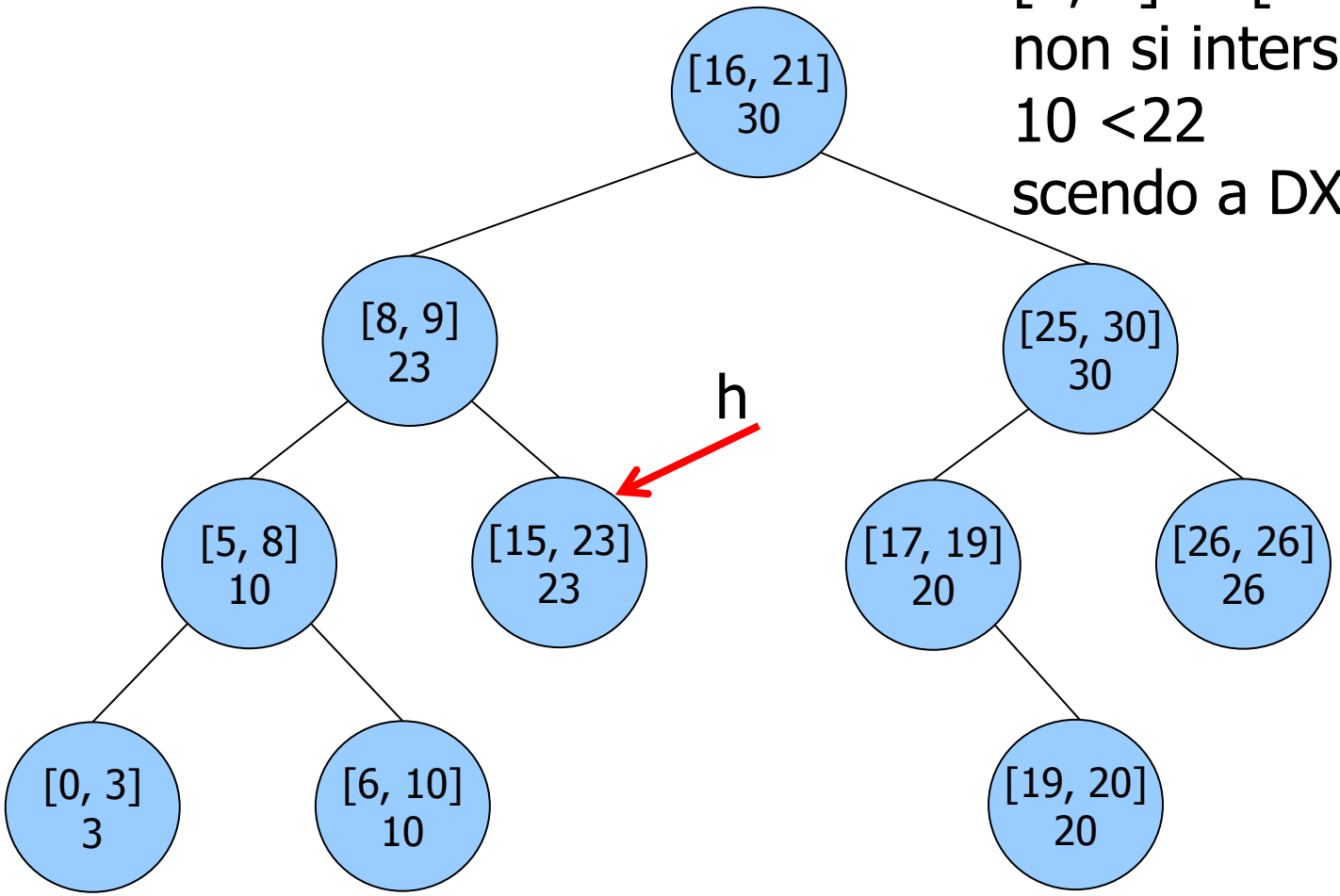
Ricerca di un intervallo che interseca [22, 25]

[16, 21] e [22, 25] non si intersecano
23 >22
scendo a SX



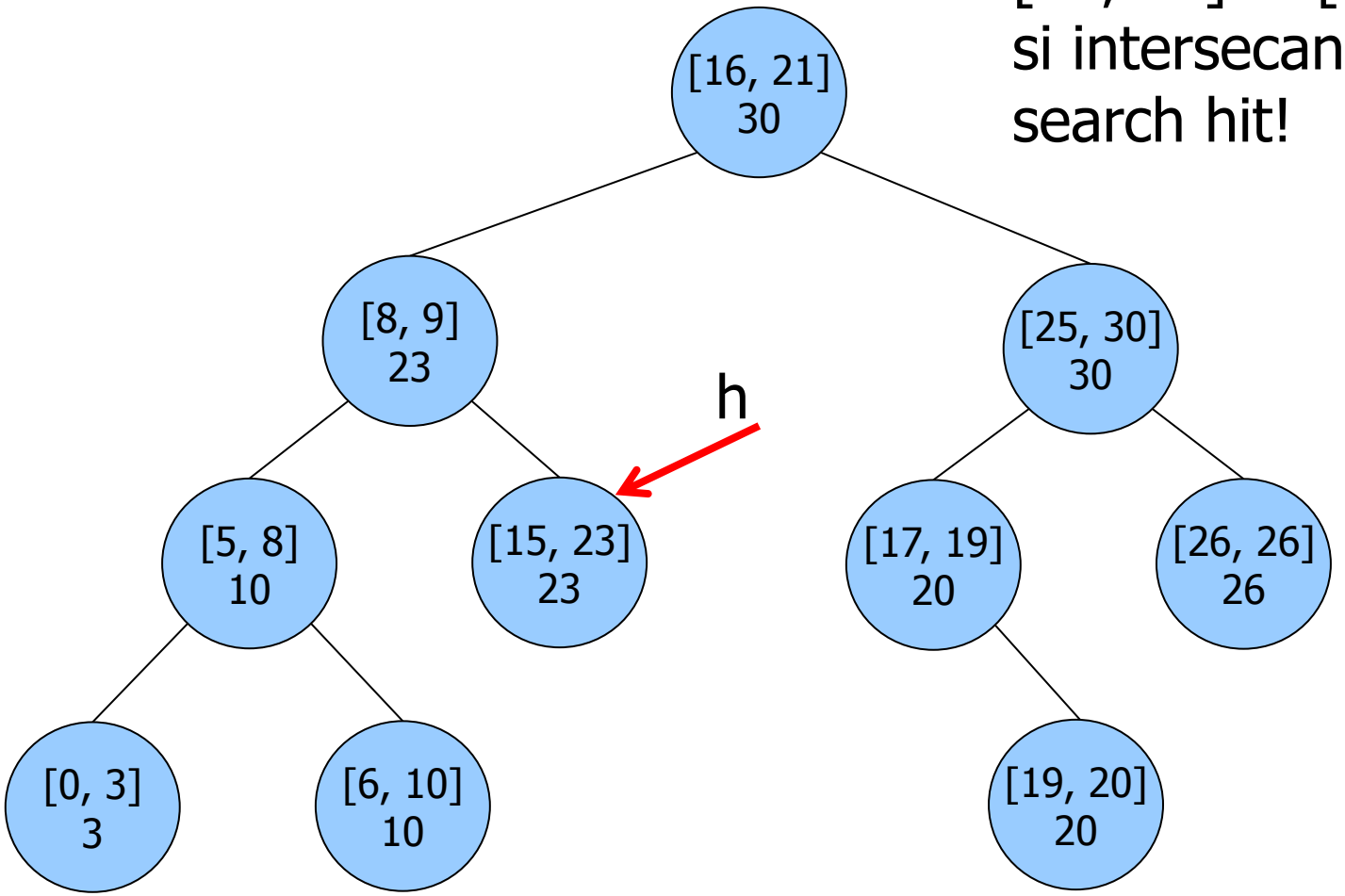
Ricerca di un intervallo che interseca [22, 25]

[8, 9] e [22, 25]
non si intersecano
10 <22
scendo a DX



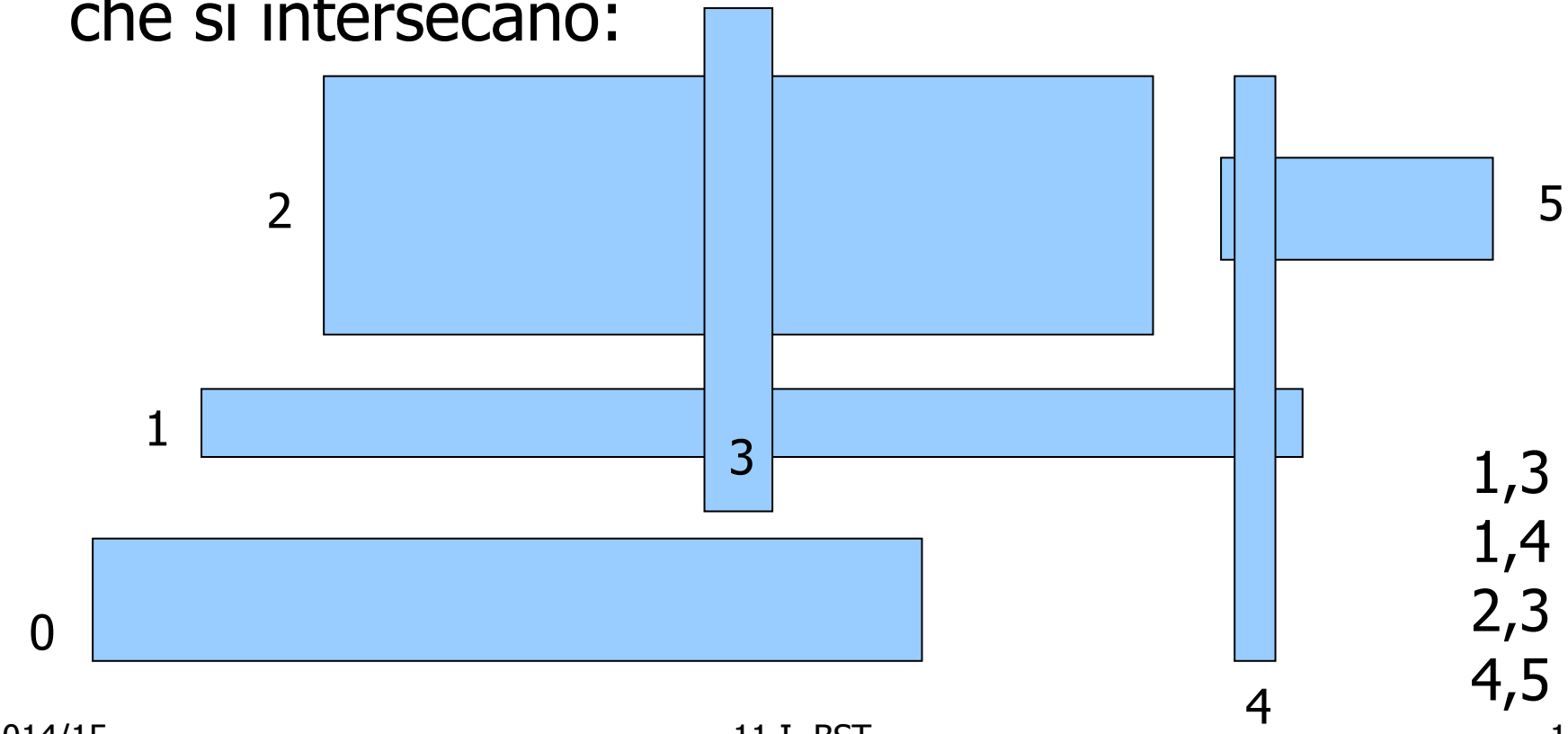
Ricerca di un intervallo che interseca [22, 25]

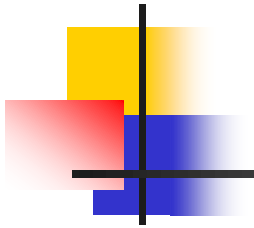
[15, 23] e [22, 25] si intersecano search hit!



Applicazioni degli I-BST

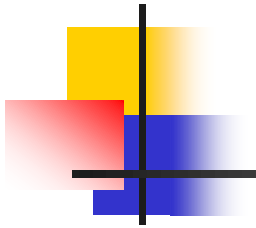
Dati N rettangoli disposti parallelamente agli assi ortogonali, determinare tutte le coppie che si intersecano:





Applicazione al CAD elettronico: verificare se le piste si intersecano in un circuito elettronico.

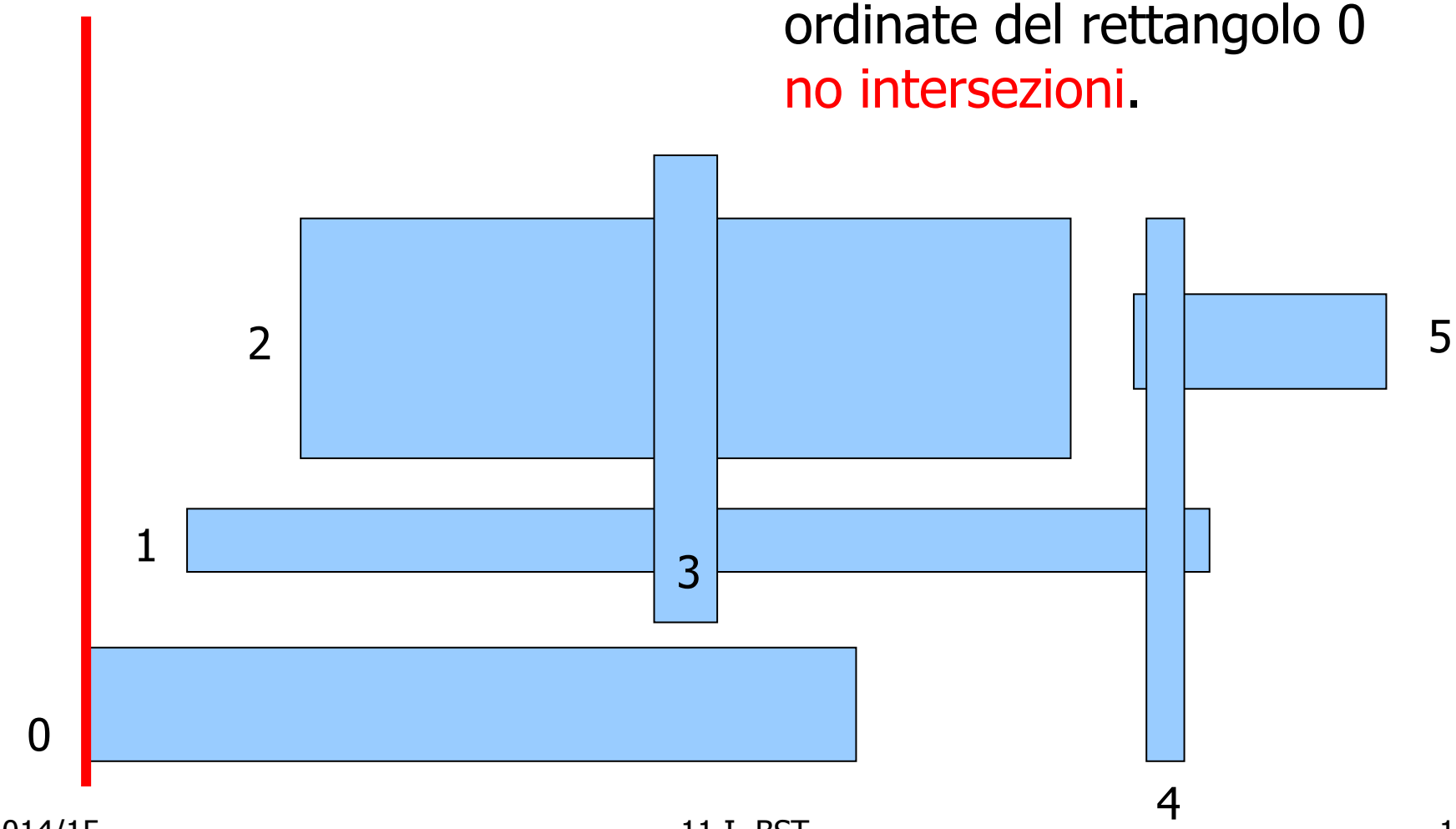
Algoritmo banale: controllare l'intersezione tra tutte le coppie di rettangoli, complessità $O(N^2)$.



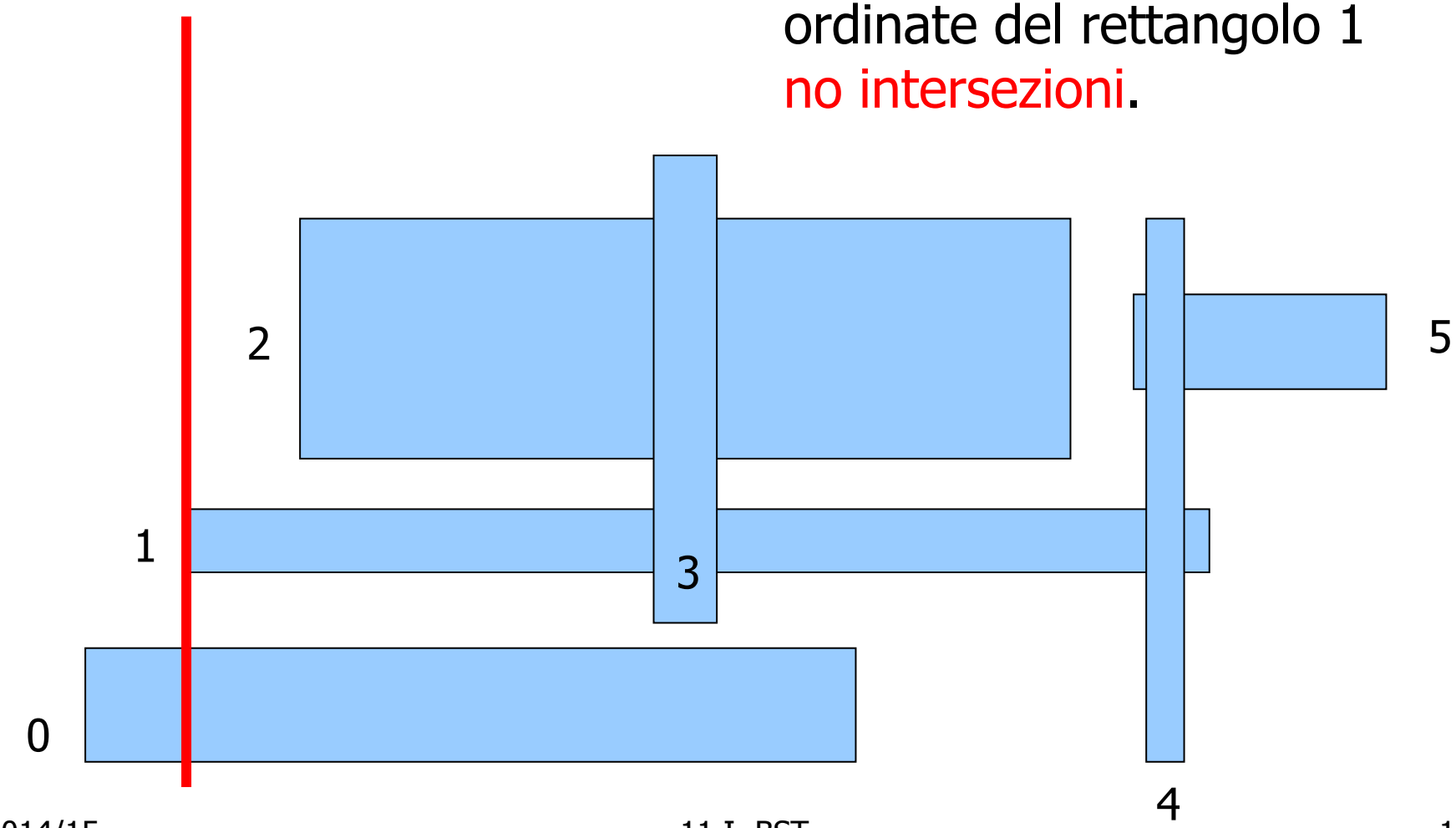
Algoritmo efficiente: complessità O(NlogN), applicabilità a VLSI e oltre:

- ordina i rettangoli per ascisse dell'estremo sinistro crescenti
- itera sui rettangoli per ascisse crescenti:
  - quando incontri l'estremo sinistro, inserisci in un I-BST l'intervallo delle ordinate e controlla l'intersezione
  - quando incontri l'estremo destro, rimuovi l'intervallo delle ordinate dall'I-BST.

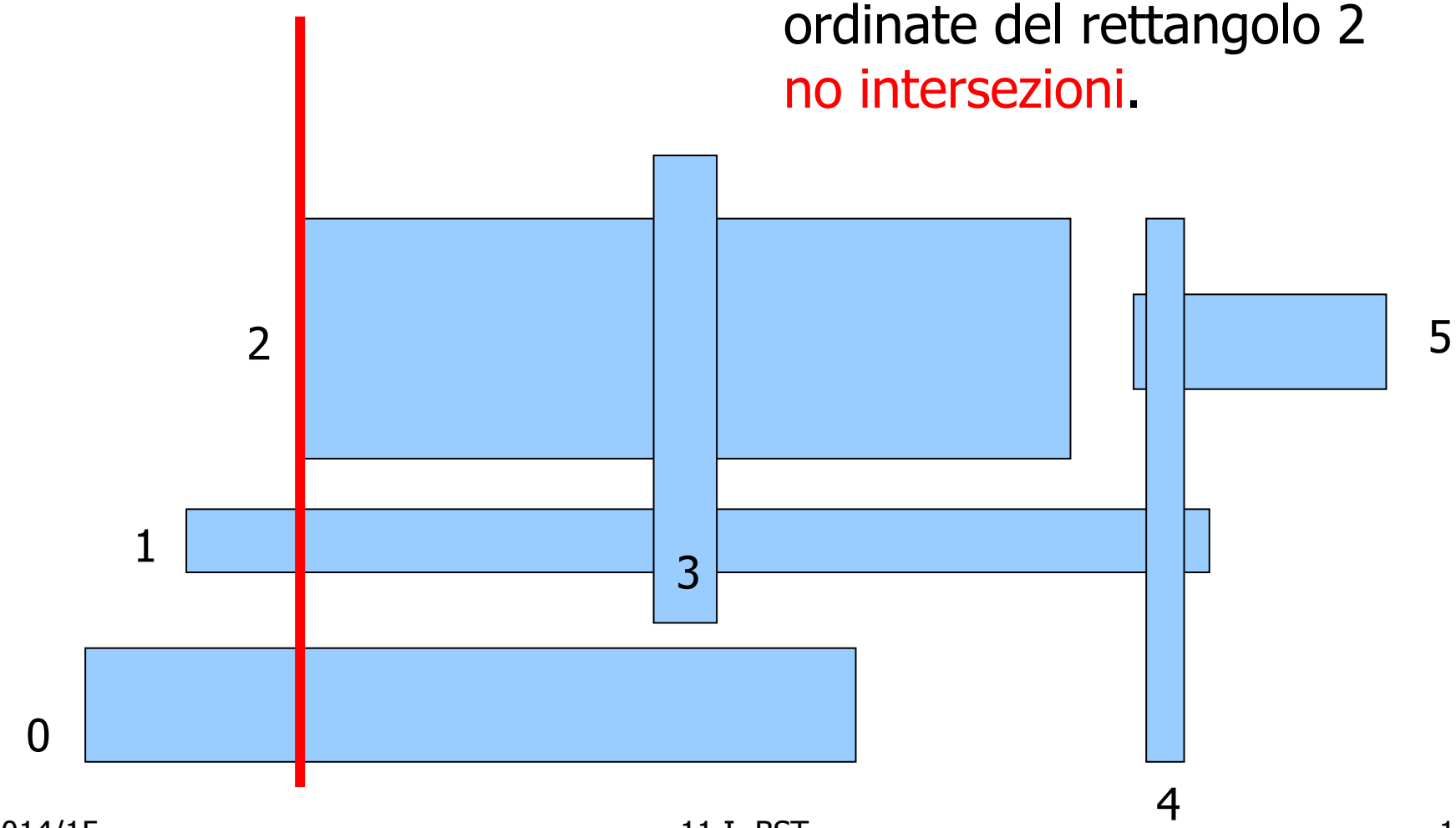
inserisci intervallo ordinate del rettangolo 0 no intersezioni.



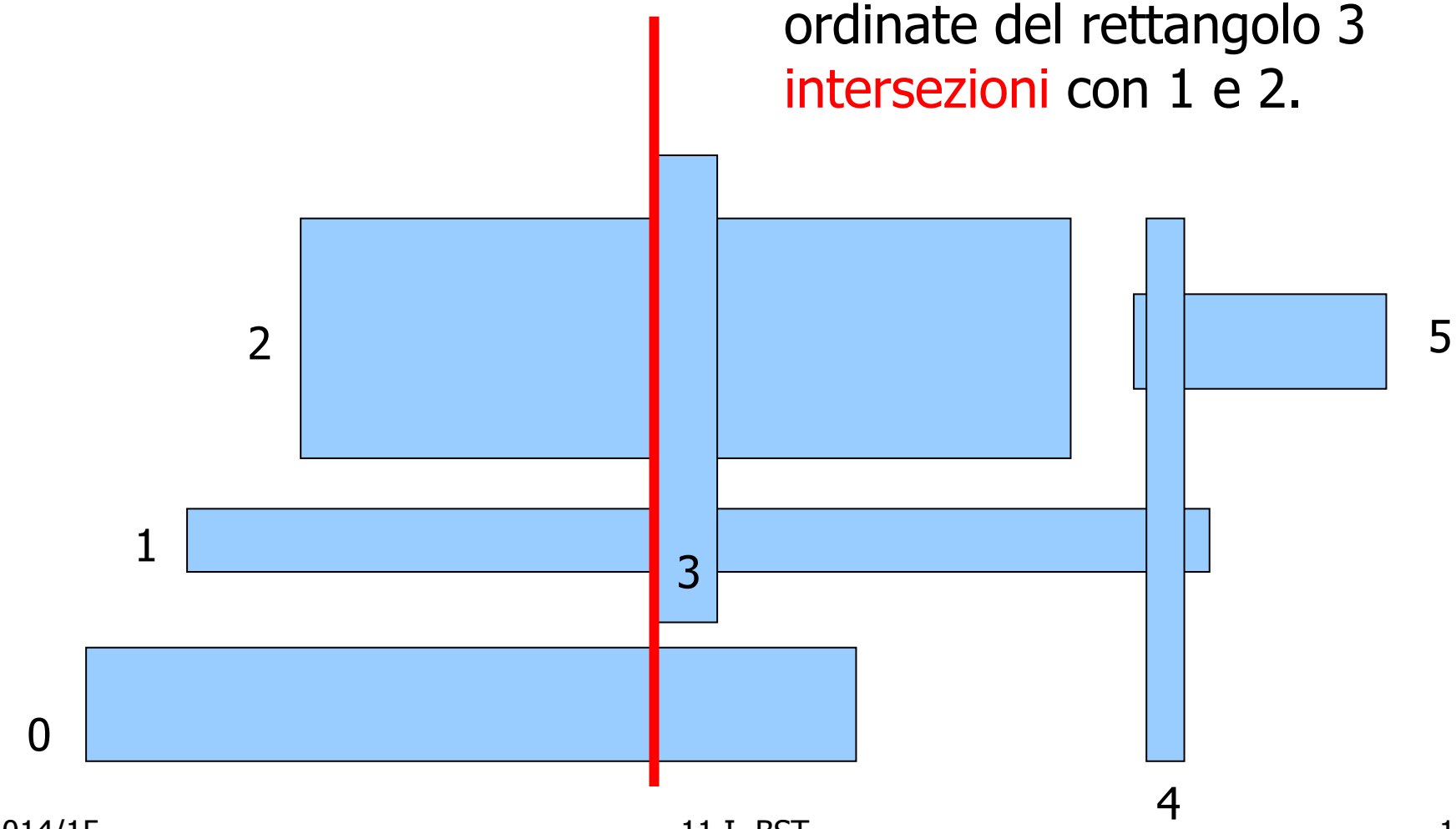
inserisci intervallo ordinate del rettangolo 1
no intersezioni.



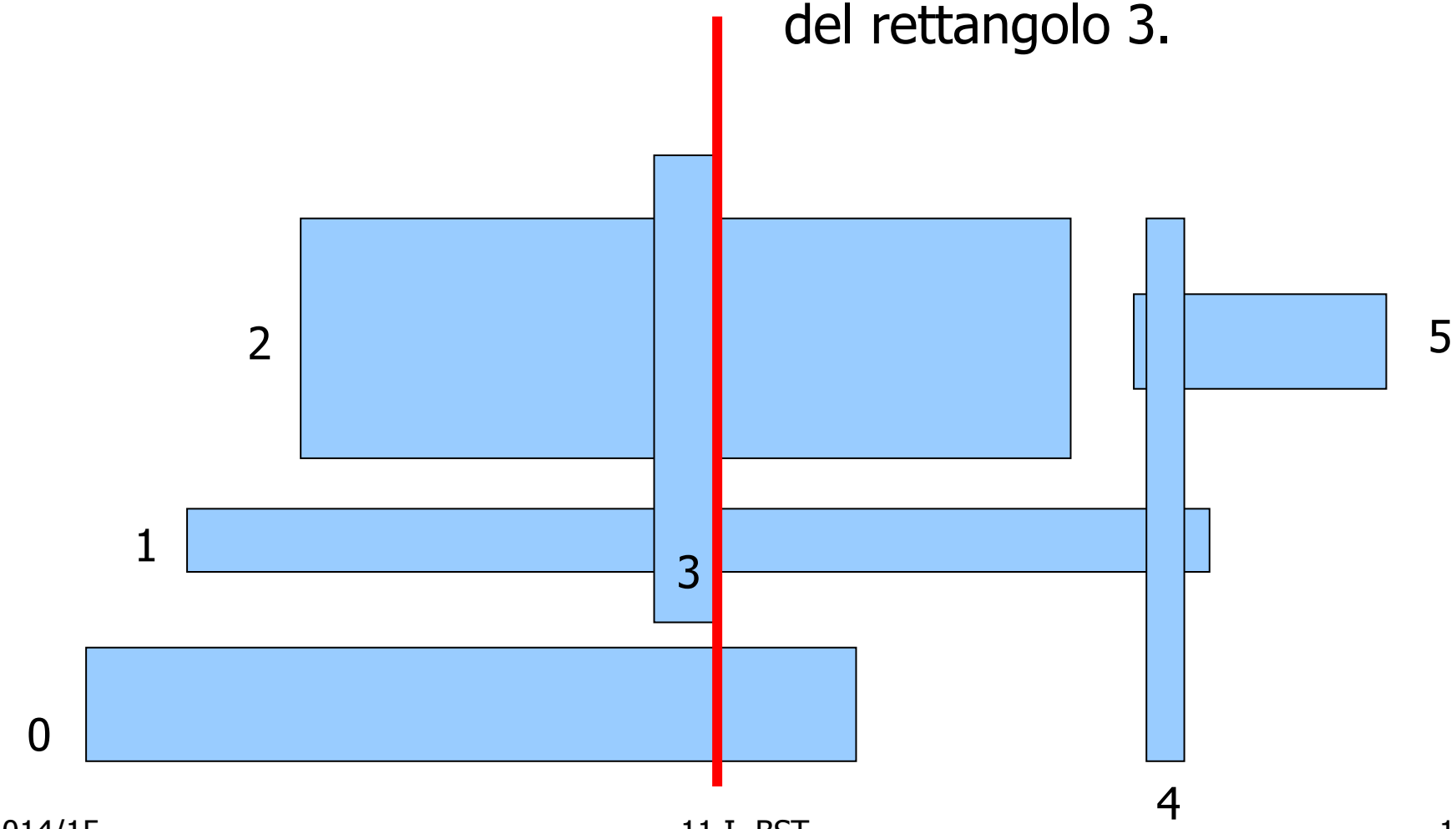
inserisci intervallo ordinate del rettangolo 2 no intersezioni.



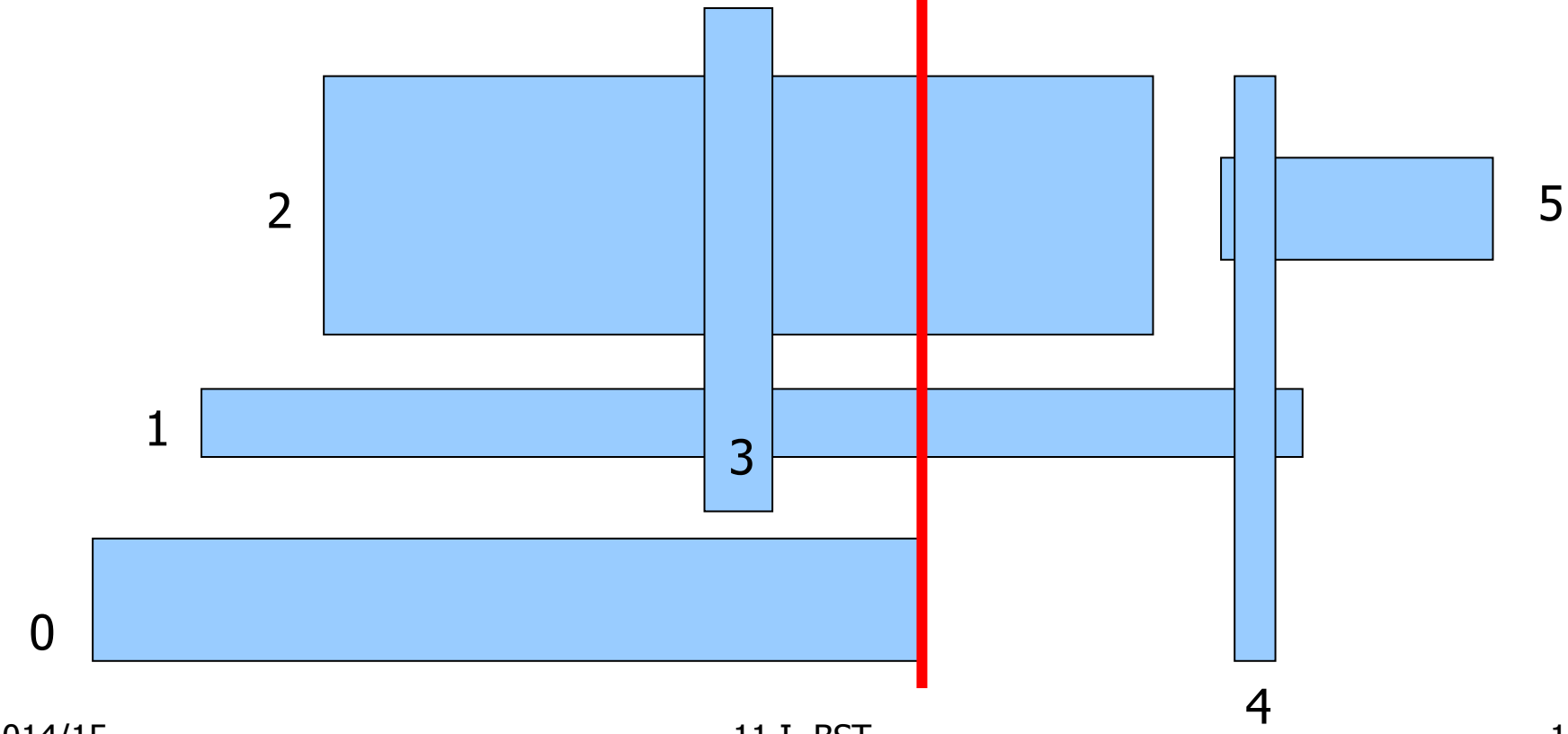
inserisci intervallo
ordinate del rettangolo 3
intersezioni con 1 e 2.



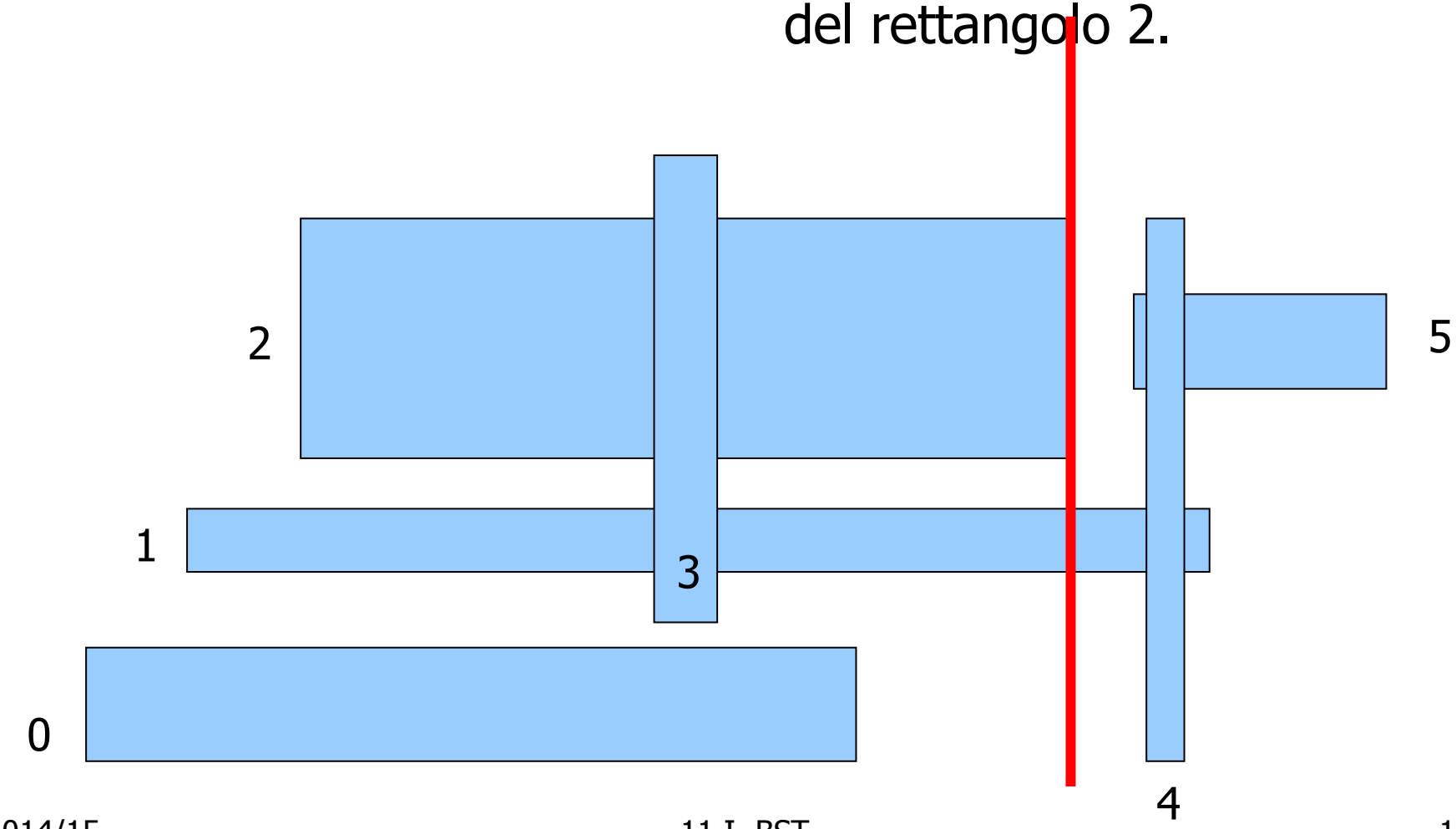
rimuovi intervallo ordinate del rettangolo 3.



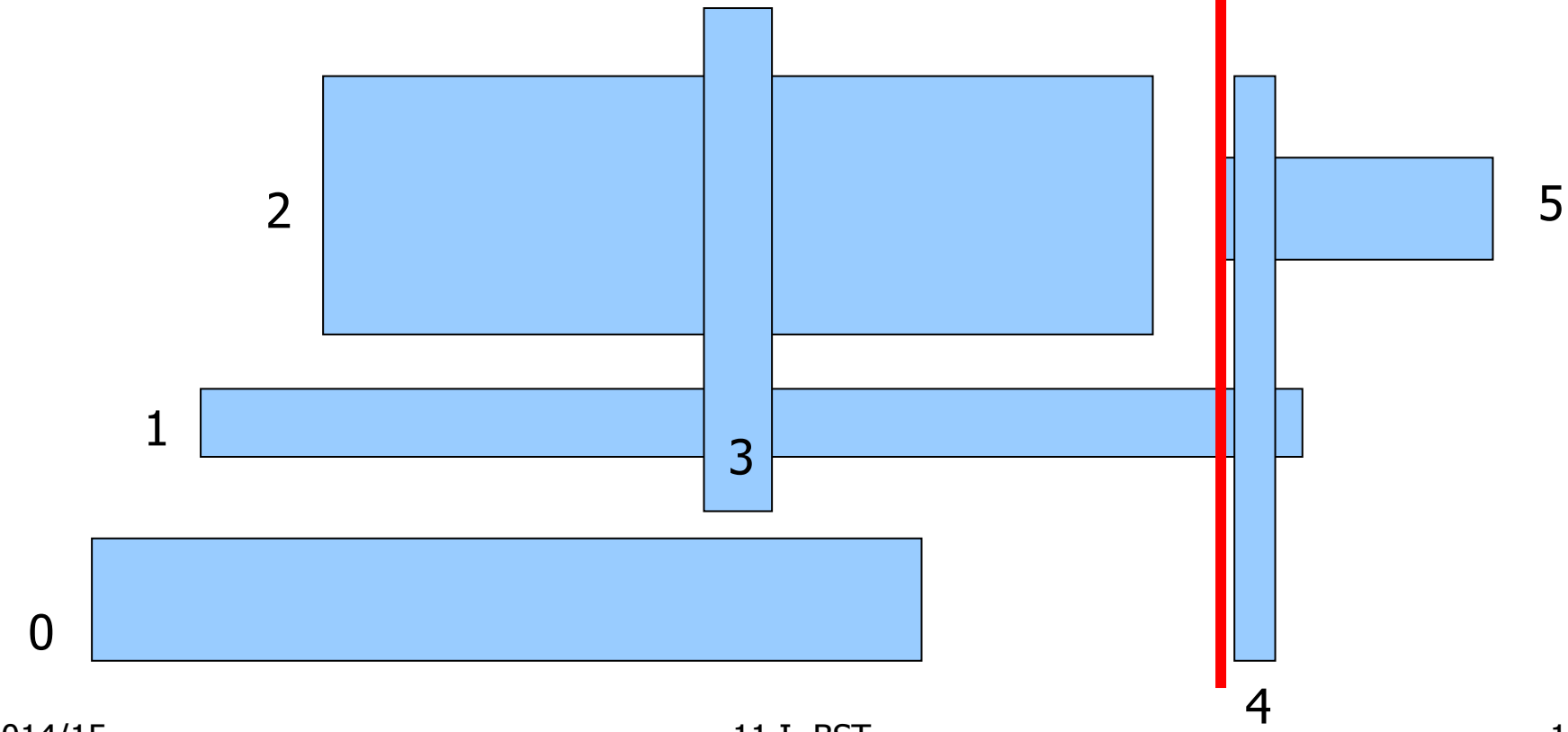
rimuovi intervallo ordinate del rettangolo 0.



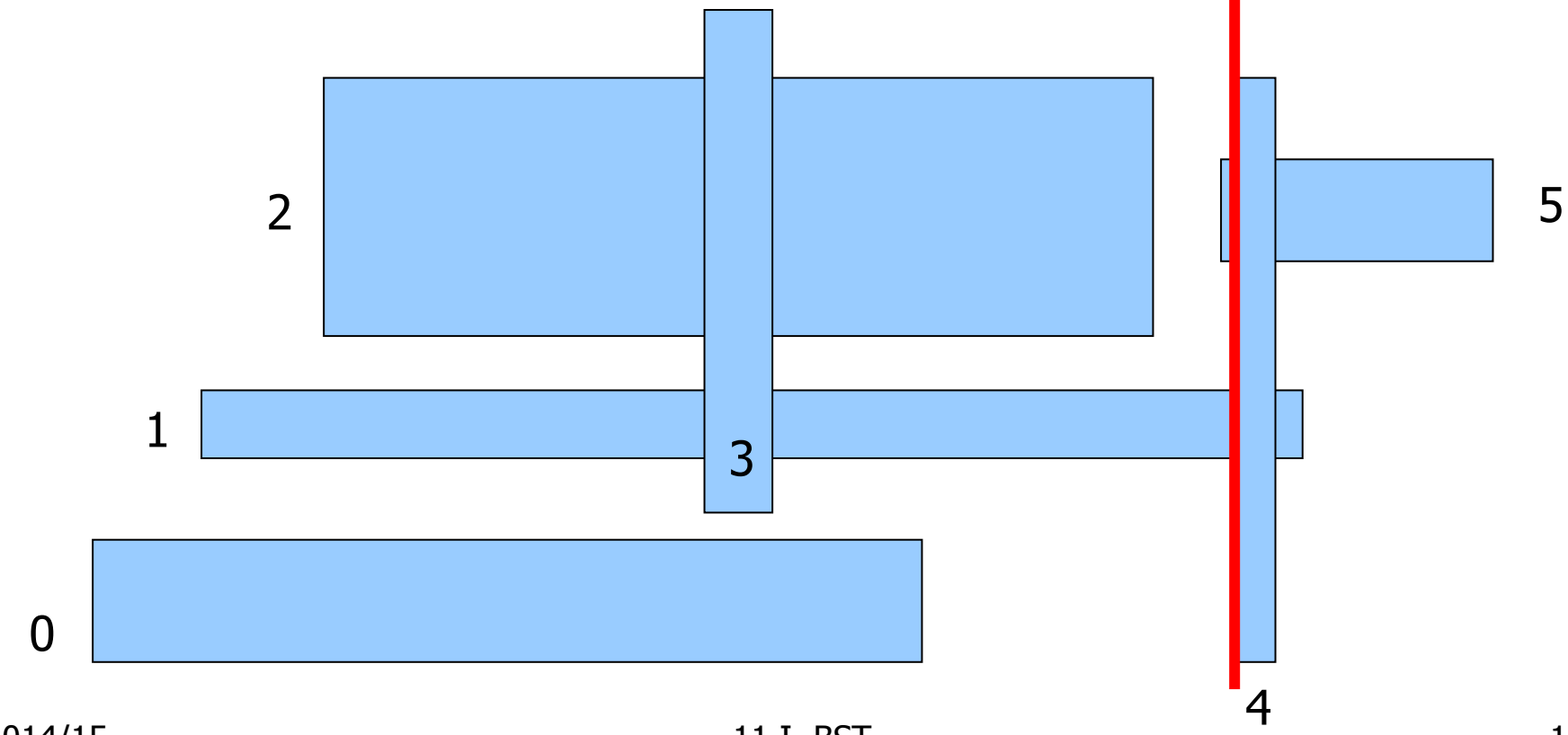
rimuovi intervallo ordinate del rettangolo 2.



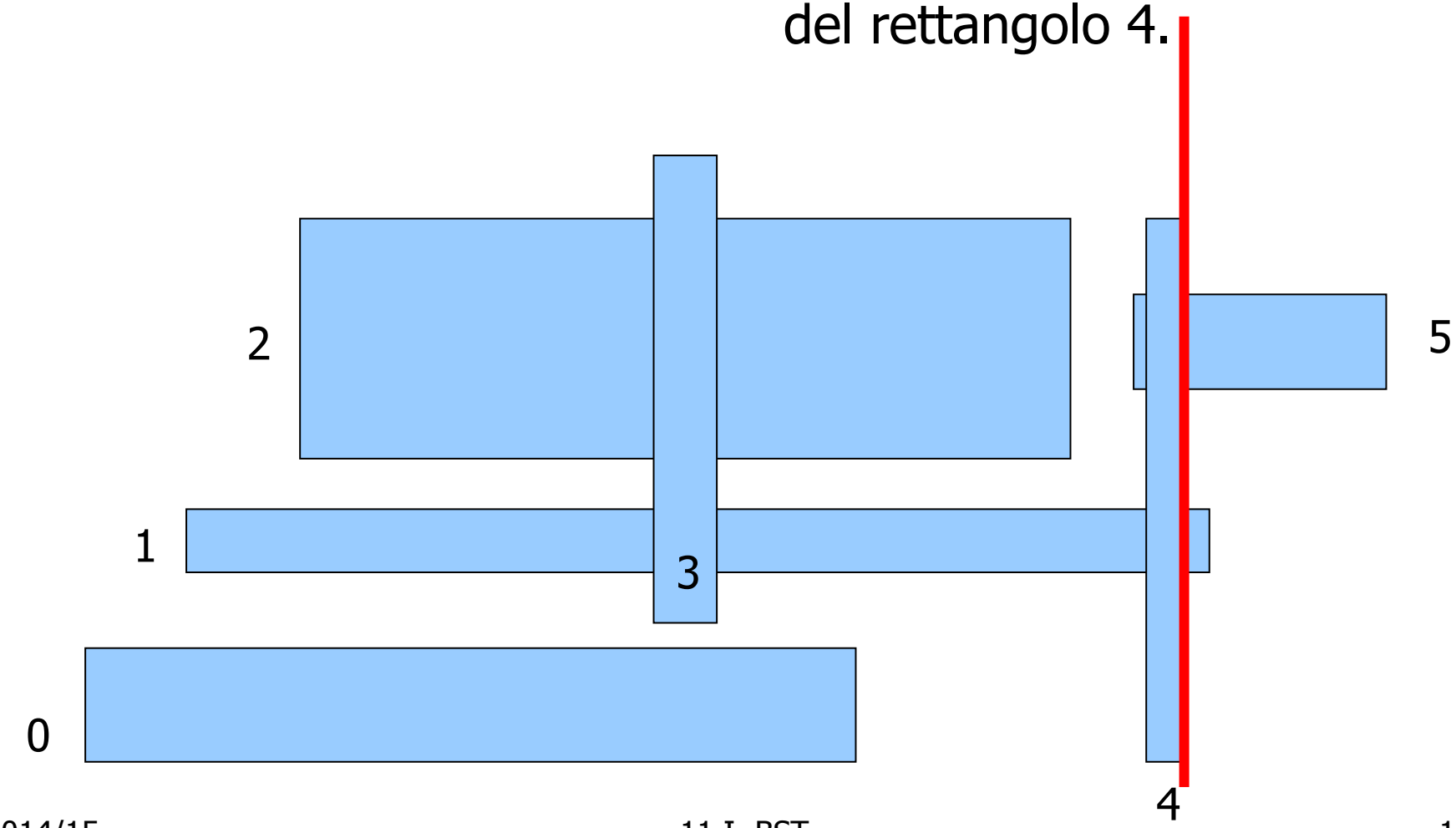
inserisci intervallo
ordinate del rettangolo 5
no intersezioni.



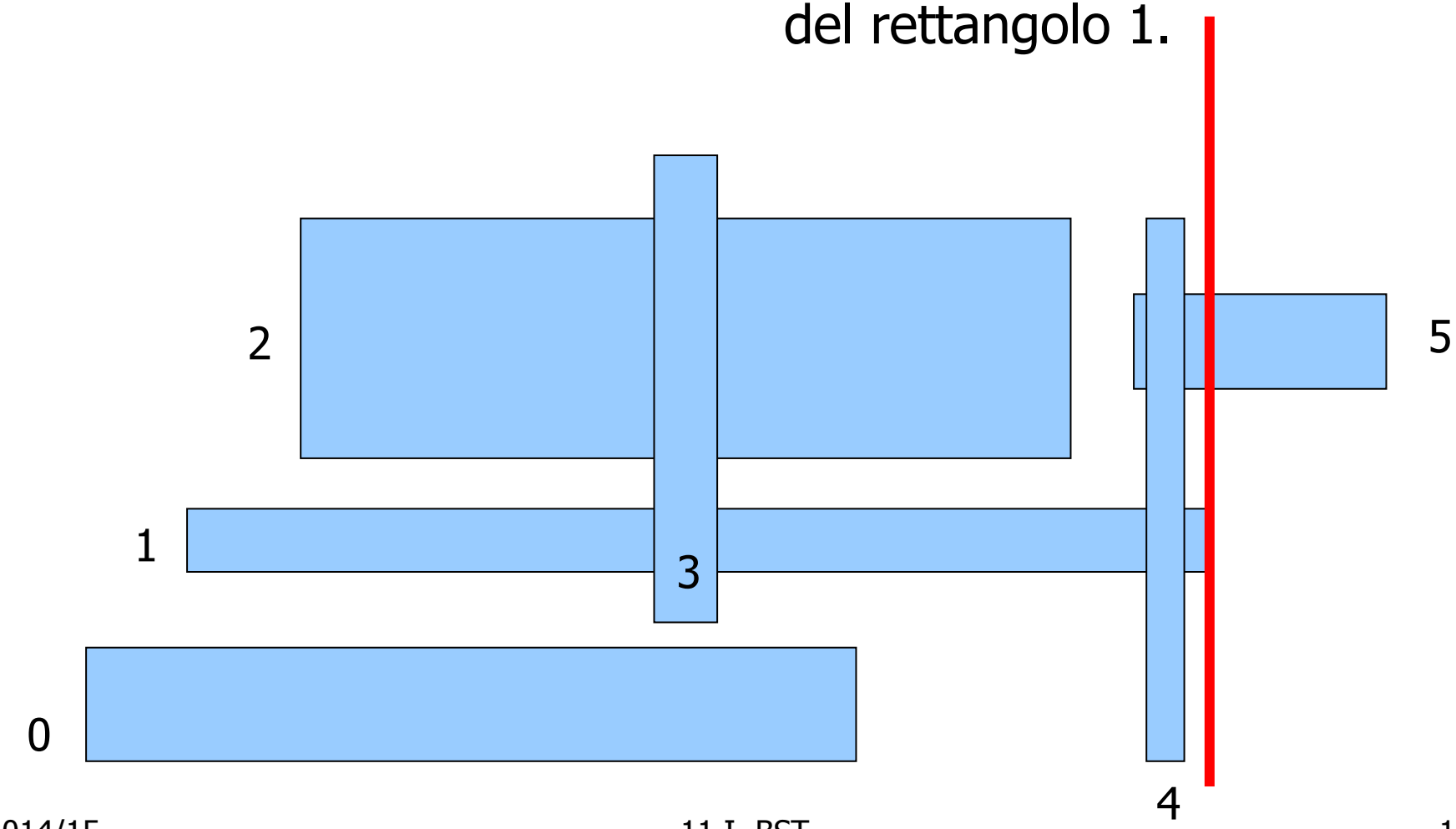
inserisci intervallo
ordinate del rettangolo 4
intersezioni con 1 e 5.



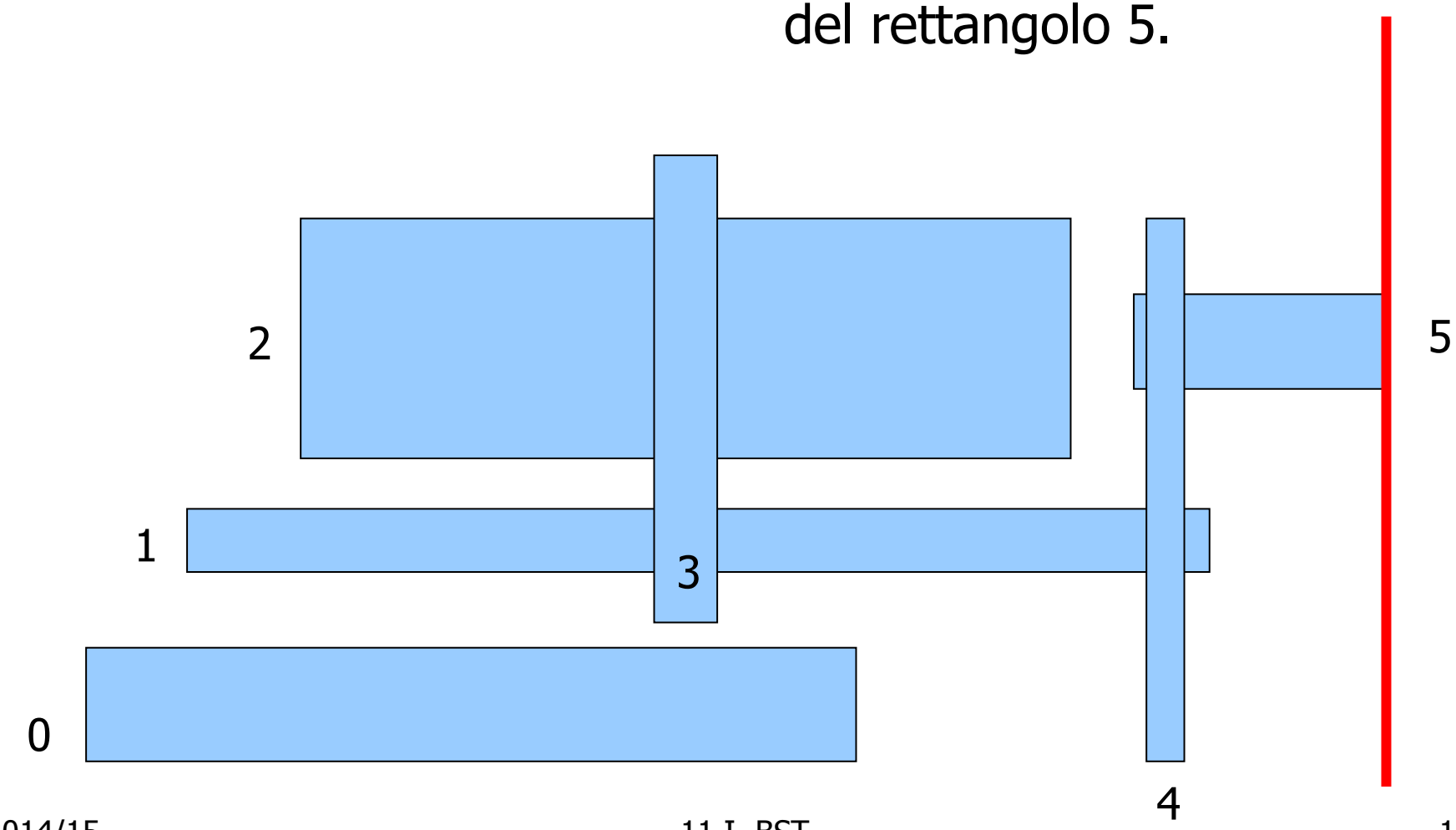
rimuovi intervallo ordinate del rettangolo 4.

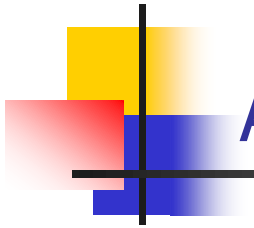


rimuovi intervallo ordinate del rettangolo 1.



rimuovi intervallo ordinate del rettangolo 5.



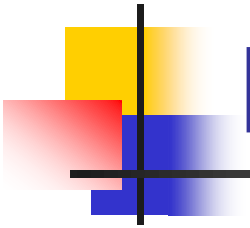


Analisi

Ordinamento: O(NlogN)

Se l'IBST è bilanciato:

- ogni inserzione/cancellazione costa O(logN), quindi per N rettangoli O(NlogN)
- N ricerche con R intersezioni costano O(NlogN + RlogN).



Riferimenti

- Alberi binari
  - Sedgewick 5.6, 5.7
- Binary Search Tree
  - Cormen 13.1, 13.2, 13.3
  - Sedgewick 12.5, 12.8, 12.9
- Order-statistic BST
  - Cormen 15.1
- Interval BST
  - Cormen 15.3