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Dynamic Stochastic Models for Time-Dependent Ordered Paired Comparison Systems

Ludwig FAHRMEIR and Gerhard TUTZ*

When paired comparisons are made sequentially over time as for example in chess competitions, it is natural to assume that the underlying abilities do change with time. Previous approaches are based on fixed updating schemes where the increments and decrements are fixed functions of the underlying abilities. The parameters that determine the functions have to be specified a priori and are based on rational reasoning. We suggest an alternative scheme for keeping track with the underlying abilities. Our approach is based on two components: a response model that specifies the connection between the observations and the underlying abilities and a transition model that specifies the variation of abilities over time. The response model is a very general paired comparison model allowing for ties and ordered responses. The transition model incorporates random walk models and local linear trend models. Taken together, these two components form a non-Gaussian state-space model. Based on recent results, recursive posterior mode estimation algorithms are given and the relation to previous approaches is worked out. The performance of the method is illustrated by simulation results and an application to soccer data of the German Bundesliga.

KEY WORDS: Kalman filter and smoother; Ordinal response; Posterior mode estimation; State space models; Time-dependent abilities.

1. INTRODUCTION

Statistical models for paired comparisons are useful in psychology, economy, and biometrics as well as for the evaluation of competitions. In fact, the first model for paired comparisons (Zermelo 1929) was designed for rating of chess skills. Since then, many extensions of the basic model have been given. (For an overview, see Bradley 1976, 1984, the bibliography of Davidson and Farquhar 1976, and David 1988.) Most of the publications consider the case where "objects," "items," "persons," "treatments," or "stimuli" are evaluated at a fixed time. But the evaluation of cognitive abilities in chess or of skills of sport professionals, which vary over time, requires dynamic models.

The example considered here is the competition of soccer teams in Germany. Because teams change over time, their ability and thus their performance will vary over time. In the German soccer league called the Bundesliga, teams meet twice in each season, giving each team the home court advantage once. Therefore, we have a sort of double round-robin tournament. The objective of dynamic paired-comparison models is to assess the varying ability of the teams based on their performance. Table 1 shows the results of six teams of the Bundesliga over 22 seasons (1966–1987). The results are given in three categories: win (1), draw (2), or loss (3). The first number gives the result from the view of the first team of the pair, the team with home court advantage. The second number gives the result from the view of the second team of the pair, which had home court advantage at this second meeting. For example, in the first season team 1 lost when meeting team 2 at home and lost (i.e., team 2 won) when meeting at the court of team 2. In seasons 11 and 12, no results for team 3 are given, because the team had to play in a lower league. But this does not affect the methods proposed.

The ranking of players or teams requires updating schemes that give the actual ability after a tournament or a season based on past performance. An updating scheme that is well accepted in the chess community is the Elo system (Elo 1978). Variations of the Elo system have been considered by Batchelder and Bershad (1979) and Batchelder and Simpson (1989). An axiomatic approach to a reward system, which gives a fixed updating rule given the previous ability, has been proposed by Batchelder, Bershad, and Simpson (1992).

The approach developed in this article is different from previous work in several respects. Here the model for time-dependent observations is accomplished by simple but flexible stochastic models for the variation of abilities and other parameters of interest over time. Consequently, updating is done with respect to the underlying "transition model" for abilities. This approach leads to inference on underlying abilities in a more general sense. Updating is not based merely on the actual observation and the previous estimated ability, but takes into consideration the preciseness of the previous estimates. It is based on the estimation error yielding updating algorithms with adaptively determined weights instead of fixed reward schemes. Moreover, if one is interested in the developments of abilities in time, updating as a forward strategy is not the best choice. The methods considered here provide smooth estimates of the development that make use of the total record of observations. The proposed method may be used for very general types of paired comparison systems allowing for ordered categories as well as order effects. Moreover, the explicit modeling of covariates and past observations is possible within the given framework.

Section 2 introduces a general static paired-comparison model with ordered response categories and explains how order effects are modeled within this framework. Binary versions of paired comparison systems turn out to be special cases. Then the binary updating schemes given by Batchelder, et al. (1992) are reviewed, setting the stage for comparing them to our method later on.

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Table 1. Results of German National Soccer League (Bundesliga) in the Years 1966–1987

	Competing Teams														
Season	1:2	1:3	1:4	1:5	1:6	2:3	2:4	2:5	2:6	3:4	3:5	3:6	4:5	4:6	5:6
1	3/1	3/3	1/3	1/3	1/2	1/3	1/1	1/2	1/2	1/3	3/1	2/1	1/1	1/1	3/1
2	1/3	2/3	1/1	1/1	3/1	1/2	1/2	2/3	3/1	3/2	3/2	1/1	1/1	2/2	3/3
3	3/2	1/1	1/2	1/1	1/3	2/1	1/1	1/1	1/3	3/1	1/2	1/1	2/2	2/1	3/2
4	1/2	1/1	1/1	1/2	1/2	1/1	1/1	1/1	1/3	1/3	1/1	1/1	3/1	2/2	3/2
5	3/3	3/3	2/2	1/3	1/1	1/3	1/1	1/3	3/2	1/1	2/3	1/1	2/1	1/1	1/2
6	1/3	1/2	1/1	1/3	1/3	1/3	3/2	1/1	2.2	1/3	2/1	1/1	1/1	1/1	1/2
7	2/3	2/3	1/3	1/3	1/1	1/2	1/1	1/2	2/2	1/1	3/3	2/1	1/1	2/1	1/1
8	2/1	1/3	1/1	1/3	1/1	1/1	1/1	1/2	1/1	1/1	1/1	2/1	2/2	3/1	1/1
9	1/1	1/2	2/1	1/3	2/2	1/1	1/3	3/1	2/1	3/1	1/1	1/1	3/3	3/1	1/1
10	1/1	2/3	3/3	3/1	1/1	1/1	1/2	1/1	2/1	3/1	3/1	3/2	1/1	2/1	1/3
11	3/1	—/—	3/1	1/3	2/1	—/—	2/2	2/1	2/2	—/—	—/—	—/—	1/1	1/2	1/1
12	1/1	—/—	1/2	1/1	3/1	—/—	1/1	2/1	1/1	—/—	—/—	—/—	1/1	2/1	1/1
13	3/1	1/2	1/1	1/2	1/1	1/1	1/3	1/1	3/2	1/3	3/1	1/1	1/1	1/3	2/3
14	1/2	2/1	1/1	3/3	1/1	3/3	2/2	3/1	3/3	1/1	1/2	1/3	1/1	1/2	1/2
15	3/3	1/3	1/2	2/1	1/1	2/1	1/1	3/1	2/1	1/1	2/1	1/1	1/1	3/3	1/1
16	2/3	2/3	1/1	1/2	1/2	1/1	2/1	3/1	1/1	1/1	1/3	2/1	2/1	1/1	1/2
17	2/1	1/3	1/1	3/1	1/1	1/2	3/2	2/1	1/1	1/1	3/2	1/1	2/1	1/2	1/1
18	3/1	1/2	3/1	2/2	1/1	3/1	1/1	2/1	2/1	2/3	3/1	1/1	2/2	1/2	1/2
19	1/1	2/1	1/3	1/1	1/2	2/1	3/2	3/2	1/3	1/2	3/3	2/3	3/1	1/1	3/2
20	1/3	1/3	1/3	2/1	1/2	2/1	1/1	1/1	1/3	1/1	2/1	1/1	2/1	1/2	1/1
21	1/2	1/2	1/3	1/2	1/2	1/1	2/1	2/2	2/2	1/2	1/1	1/2	3/1	2/2	1/1
22	1/2	1/3	1/2	1/3	1/2	2/1	2/1	2/1	2/3	2/1	2/1	1/1	3/1	1/2	1/3

NOTE: Teams are 1: 1.Fc Bayern Münchem; 2: 1.Fc Köln; 3: VfB Stuttgart; 4: 1.Fc Kaiserslautern; 5: Hamburger SV; 6: Eintracht Frankfurt.

Section 3 outlines the modeling approach. First, Section 3.1 gives basic stochastic trend models for time-varying abilities. Then, Section 3.2 introduces the general model in state-space form. The model is characterized by the *response* or *observation model,* which connects observations and underlying abilities as well as other parameters, and by the *transition model,* which specifies the variation of underlying abilities and parameters in time. It is shown how the basic models of Section 3.1 fit into the general model.

Section 4 gives the estimation theory for the general state-space model and introduces posterior mode estimation for the case of known and unknown hyperparameters. The method is based on an extension of Kalman filtering and smoothing for dynamic generalized linear models (Fahrmeir 1992; Fahrmeir and Kaufmann 1991). It is shown that in special cases, updating schemes may have a similar form as in the previous approach of Batchelder et al. (1992), but now are adaptive with respect to the estimation error. Finally, Section 5 gives some simulation results and applications of the method to the German soccer data.

2. MODELING OF PAIRED COMPARISONS

2.1 General Order-Effect Models for Ordered Response Categories

In paired-comparisons objects, treatments or teams $A = \{a_1, \ldots, a_n\}$ are compared with each other in pairs. Let y_{ij} denote the observed response when the pair (a_i, a_j) is presented. The simplest case is the dichotomous response where $y_{ij} = 1$ denotes dominance of a_i and $y_{ij} = 2$ denotes dominance of a_j. Dominance may mean preference when a pair of chocolate puddings is compared in a task test, whereas in our example of competing teams, dominance refers to winning the game. The model in common use in the simple dichotomous case is due to Bradley and Terry (1952) and has the logistic form

$$P(y_{ij} = 1) = \frac{\exp(\alpha_i - \alpha_j)}{1 + \exp(\alpha_i - \alpha_j)}, \tag{1}$$

where α_i and α_j are parameters that represent the strength of the stimuli a_i and a_j or the ability of the competing teams a_i and a_j.

To use the information of preference data or tournament results more effectively, two extensions are appropriate. First, instead of restricting the model to a dichotomous response, the model should allow for a more refined ordinal response. Second, the effect of presenting the pair in a specific order (a_i first, a_j second) should be incorporated. In our example for competing teams, the order effect is equivalent to the home court advantage, which is not to be neglected to get unbiased estimates of abilities.

Let the response y_{ij} for the comparison of the pair (a_i, a_j) take values from $\{1, \ldots, k\}$ where the categories represent a grading of dominance. Category 1 means that a_i dominates a_j most strongly whereas the highest category k means that a_j dominates a_i most strongly. A simple example is the trichotomous case, where the categories 1, 2, and 3 stand for "a_i wins," "draw," and "a_j wins." In the case of four response categories may refer to "a_i is strongly preferred" (category 1), "a_i is weakly preferred" (category 2), "a_j is weakly preferred" (category 3), or "a_j is strongly preferred" (category 4). It should be noted that in both cases the response categories are symmetric in the sense that the strength of dominance of a_i over a_j, represented by category r, has a counterpart in category $k + 1 - r$, which represents the same strength of dominance but now of a_j over a_i. This kind of symmetry is assumed throughout the article.

A common approach in choice theory as well as in ordinal modeling of responses is based on the consideration of latent variables. (See, for example, McFadden 1973, 1981 for choice models and McCullagh 1980 for ordinal models.) The basic assumption is that for each $a_i \in A$ there exists a latent random utility $U_i = \alpha_i + \varepsilon_i$, where $\alpha_i \in \mathbb{R}$ is a constant and ε_i is a random variable. The constant α_i represents the fixed utility of object a_i or the ability of team a_i. The connection between the latent random utility and the observable response y_{ij} when pair (a_i, a_j) is considered is modeled by the category boundaries approach (Edwards and Thurstone 1952; McCullagh 1980), which assumes that

$$y_{ij} = r \Leftrightarrow \theta_{r-1} < U_j - U_i \leq \theta_r,$$

where $-\infty = \theta_0 < \cdots < \theta_k = \infty$ are thresholds. Assuming a continuous distribution function F for the differences $\varepsilon_j - \varepsilon_i$ yields the general ordinal paired-comparison model

$$P(y_{ij} \leq r) = F(\theta_r + \alpha_i - \alpha_j), \tag{2}$$

$r = 1, \ldots, k$ or, equivalently,

$$P(y_{ij} = r) = F(\theta_r + \alpha_i - \alpha_j) - F(\theta_{r-1} + \alpha_i - \alpha_j). \tag{3}$$

For $k = 2$, $\theta_1 = 0$, model (3) is equivalent to the simple dichotomous model (1) when F is chosen as the logistic function. The logistic version of the ordinal model (2) was derived by Tutz (1986) and has been investigated further by Cox and Snell (1989) and more recently by Agresti (1992).

The soccer example may serve to illustrate the assumptions. Although α_i is the fixed ability of team a_i, the random utility U_i represents the ability that is actually present when meeting another team. The category boundaries approach is used here in a slightly different way than usual. Instead of considering one latent variable underlying the observable response (McCullagh 1980), the difference of utilities $U_j - U_i$ is considered as the latent variable underlying y_{ij}. The motivation lies in the meaning of the observable categories $1, \ldots, k$. The larger U_j (i.e., the actual ability of team a_j) compared to U_i (i.e., the actual ability of team a_j), the larger the difference $U_j - U_i$ will be and the larger the response category to be expected. This corresponds to the meaning of categories, because large values of y_{ij} stand for the dominance of a_j when (a_i, a_j) are compared. On the other hand, the probability of low categories $P(y_{ij} \leq r)$, standing for the dominance of a_i, increases with increasing ability α_i.

The role of thresholds refers to the order of presentation. For competing teams, the pair (a_i, a_j) implies that the game is played on the home court of a_i. The home court advantage is most obvious in the case where the abilities of teams are equal; that is, $\alpha_i = \alpha_j$. Then the probabilities $P(y_{ij} = r) = F(\theta_r) - F(\theta_{r-1})$ depend only on the thresholds. Because the teams have equal abilities, the probability of response categories reflects the home court advantage, which of course is specific for the game. For the soccer data from Table 1, the assumption of a logistic distribution F yields the thresholds $\hat{\theta}_1 = -.358$ and $\hat{\theta}_2 = 1.039$, which are stable over the years (see Sec. 5.2). For $\alpha_i = \alpha_j$, that means $P(y_{ij} = 1) = .411$, $P(y_{ij} = 2) = .328$, and $P(y_{ij} = 3) = .261$. Therefore, a soccer team will beat another team of equal ability on their home court with probability .411 and will be beaten only with probability .261.

2.2 The Reward System Approach to Paired Comparison Modeling With Time Dependence

If paired-comparison systems are considered over a wider range of time, it is not expected that abilities will remain constant over time. Time-dependent models have been considered mainly for the dichotomous order-independent case, where the basic form is given by

$$P(y_{ij}^{(t)} = 1) = F(\alpha_{ti} - \alpha_{tj}), \tag{4}$$

$y_{ij}^{(t)} \in \{1, 2\}$. In (4) the abilities α_{ti} as well as the observation $y_{ij}^{(t)}$ depend on discrete time t.

An elaborate approach to dynamic ratings of chess players was given recently by Batchelder et al. (1992). They did not specify a model for the change of the underlying abilities over time but defined a reasonable reward system for updating ratings that hopefully should keep track with the underlying variation of abilities whatever the variation. For two players (teams) in a competition (a_i, a_j) and $y_{ij}^{(t)} \in \{1, 2\}$, the reward system for model (2) is given by

$$\begin{aligned} \hat{\alpha}_{t+1,t} &= \hat{\alpha}_{ti} + W(\hat{\alpha}_{ti} - \hat{\alpha}_{tj}) && \text{if } y_{ij}^{(t)} = 1 \\ &= \hat{\alpha}_{ti} && \text{no comparison} \\ &= \hat{\alpha}_{ti} - L(\hat{\alpha}_{ti} - \hat{\alpha}_{tj}) && \text{if } y_{ij}^{(t)} = 2 \end{aligned}$$

and

$$\begin{aligned} \hat{\alpha}_{t+1,j} &= \hat{\alpha}_{tj} - L(\hat{\alpha}_{tj} - \hat{\alpha}_{ti}) && \text{if } y_{ij}^{(t)} = 1 \\ &= \hat{\alpha}_{tj} && \text{no comparison} \\ &= \hat{\alpha}_{tj} + W(\hat{\alpha}_{tj} - \hat{\alpha}_{ti}) && \text{if } y_{ij}^{(t)} = 2, \end{aligned}$$

where $L, W: \mathbb{R} \to \mathbb{R}_+$ are nonnegative real-valued functions. Moreover, Batchelder et al. (1992) assumed that W is monotonically decreasing and L is monotonically increasing on the support of the strictly increasing distribution function F. In a further step, reward function W and loss function L are restricted by reasonable assumptions: $W(x) = L(-x)$ for all $x \in \mathbb{R}$ (zero sum axiom), $W(x)F(x) = L(x)F(-x)$ for all $x \in \mathbb{R}$ (fair game axiom) and $F(x)W^2(x) + F(-x)W^2(-x) = C^2$ for a constant C (constant variance axiom). (For details, interpretation, and alternative versions see Batchelder et al. 1992.) For the case of two players, these assumptions yield the simple updating scheme

$$\begin{aligned} \hat{\alpha}_{t+1} &= \hat{\alpha}_t + 4C(1 - F(\hat{\alpha}_t)) && \text{if } y_{ij}^{(t)} = 1 \\ &= \hat{\alpha}_t - 4CF(\hat{\alpha}_t) && \text{if } y_{ij}^{(t)} = 2, \end{aligned}$$

where $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_t(a_1)$ and $\boldsymbol{\alpha}_t(a_2) = 0$ are the side constraints that guarantee identifiability and C is a constant. The scheme is essentially based on a weight C and $F(\hat{\alpha}_t)$ which is the probability that player one wins given that $\hat{\alpha}_t$ is the underlying ability. If the probability that player one wins is rather high, then the change $\hat{\alpha}_{t+1} - \hat{\alpha}_t$ will be low, because player one is expected to win. But if player one loses, the decrease $\hat{\alpha}_{t+1} - \hat{\alpha}_t = -4CF(\hat{\alpha}_t)$ is a strong one. Batchelder et al. (1992)

considered more thoroughly the case of underlying positive growth given by $\alpha^t = \alpha - (\alpha - \alpha_0)\gamma^t$ where $0 < \gamma < 1$. For the case of n players taking part in successive round-robin tournaments (binary response, no order effect) the axioms yield the updating scheme

$$\hat{\alpha}_{t+1,i} = \hat{\alpha}_{ti} + 2C \sum_{j \neq i} (2 - y_{ij}^{(t)}) - F(\hat{\alpha}_{ti} - \hat{\alpha}_{tj}).$$

It should be noted that now the side constraint for identifiability is given by $\sum_i \hat{\alpha}_{ti} = 0$.

The reward system approach is based on assumptions how a rational updating scheme may be constructed yielding a simple scheme which is determined by the constant C. The statistical problem how the scheme really works; that is, how the parameters of the underlying process are estimated is considered merely in the second step—it is not the basis on which the scheme is constructed.

3. DYNAMIC STOCHASTIC MODELS FOR TIME-DEPENDENT PAIRED COMPARISONS

We introduce dynamic paired comparison models that have two components: the modeling of the responses depending on underlying abilities and covariates and the modeling of the variation of abilities and other parameters of interest over time. The latter will be referred to as the transition model; the former, as the response or observation model. Beside the allowance for covariates, the main difference to the updating approach of Batchelder et al. (1992) lies in the explicit formulation of a rather general transition model and the development of estimation methods for the underlying parameters that are now specified by a model. First, in Section 3.1 simple models for the stochastic variation of the parameters over time are considered. Then the general form of response and transition model is introduced in Section 3.2.

3.1 Basic models

In a first step, we assume that the *observation model* is a time-dependent version of the general model (1). We consider an ordinal-response paired-comparison system with k response categories where the paired comparison function p_t depends on discrete time $t = 0, 1, 2, \ldots$. For each time t, the system is assumed to be a paired comparison system; that is, the probability of response r at time t is given by

$$P(y_{ij}^{(t)} = r | \theta_t, \alpha_{ti}, \alpha_{tj}) = F(\theta_{tr} + \alpha_{ti} - \alpha_{tj}) - F(\theta_{t,r-1} + \alpha_{ti} - \alpha_{tj}), \quad (5)$$

for $r = 1, \ldots, k$. Responses $y_{ij}^{(t)}$ at time t are assumed to be (conditionally) independent.

Abilities are assumed to follow *stochastic trend models,* which are well known in structural time-series analysis (e.g., Harvey 1989). Such models are rather flexible tools for tracking nonstationary time-varying abilities without presuming particular shapes or patterns (e.g., bell-shaped trends, peaks, seasonalities). For notational simplicity, let us drop index i. The simplest yet rather effective nonstationary trend model is the *first-order random walk model,* hereinafter denoted by RW(1):

$$\alpha_t = \alpha_{t-1} + u_t, \quad \text{or} \quad \nabla\alpha_t = \alpha_t - \alpha_{t-1} = u_t, \quad t = 1, 2, \ldots. \quad (6)$$

The sequence $\{u_t\}$ of errors is assumed to be Gaussian white noise—that is, $u_t \sim N(0, \sigma_\alpha^2)$—and u_t, u_s are mutually independent for all $t \neq s$. The initial ability α_0 is Gaussian, $\alpha_0 \sim N(\xi_0, \sigma_0^2)$, and is independent of $\{u_t\}$. For $\sigma_\alpha^2 = 0$, this model reduces to abilities $\alpha_1 = \cdots = \alpha_t$, which are constant in time. Higher-order random walks, which are appropriate for "smoother" tracking of trends, are obtained by further differencing. We consider only the *second-order random walk model,* RW(2):

$$\alpha_t = 2\alpha_{t-1} - \alpha_{t-2} + u_t, \quad \text{or} \quad \nabla^2\alpha_t = u_t, \quad t = 1, 2, \ldots, \quad (7)$$

with analogous assumptions on initial values and on errors. Another common model is the local linear trend model:

$$\alpha_t = \alpha_{t-1} + \lambda_{t-1} + u_{t\alpha}, \qquad \lambda_t = \lambda_{t-1} + u_{t\lambda}, \quad t = 1, 2, \ldots. \quad (8)$$

The error sequence $\{u_t\} = \{(u_{t\alpha}, u_{t\lambda})'\}$ is bivariate Gaussian white noise with independent components—that is, $u_t \sim N(0, \text{diag}(\sigma_\alpha^2, \sigma_\lambda^2))$. Although $u_{t\alpha}$ causes local shifts of the level α_t, $u_{t\lambda}$ allow for local changes of the slope λ_t.

To assure identifiability, the ability of team a_n is set to $\alpha_{tn} = 0$ in the sequel. Alternatively, and more natural in some applications, one may introduce the identifiability restriction $\sum_i \alpha_{ti} = 0$. To obtain simple and computationally tractable models for the vector $\alpha_t = (\alpha_{t1}, \ldots, \alpha_{t,m-1})'$ of abilities, we will make the natural assumption that individual abilities are mutually independent.

Threshold parameters may be constant or time-varying. For $k \geq 3$, the order restriction $\theta_{t1} \leq \cdots \leq \theta_{t,q}$, $q = k - 1$, has to be kept in mind. If thresholds vary according to one of the stochastic trend models, this ordering can be destroyed with positive probability. But this is not often a problem in practice, because we need only that the order restriction holds for the sequence of estimated thresholds. Moreover, the problem can be overcome by an appropriate reparameterization by using $\tilde{\theta}_1 = \theta_1$ and $\tilde{\theta}_r = \log(\theta_r - \theta_{r-1})$, $r = 2, \ldots, q$ instead of the original thresholds. For simplicity we will make the assumption that θ_t follows a q-dimensional RW(1) model $\theta_t = \theta_{t-1} + \mathbf{w}_t$, with independent components $\theta_{t1}, \ldots, \theta_{t,q}$; that is, $w_t \sim N(0, \boldsymbol{\Sigma}_\theta)$ with diagonal covariance matrix $\boldsymbol{\Sigma}_\theta$. The important special case of constant thresholds $\theta_t = \theta$ is obtained for $\boldsymbol{\Sigma}_\theta = 0$. In the soccer example (Sec. 5.2), thresholds were not restricted to be constant a priori. But the diagonal of $\boldsymbol{\Sigma}_\theta$ was estimated nearly as 0, so that thresholds can be assumed to be constant. Moreover, they were clearly separated and did not violate the order restriction.

3.2 A general state-space model for dynamic ordered comparisons

We introduce a general model that comprises the important special cases of Section 3.1 and allows various extensions—in particular, the incorporation of past comparisons or covariates in explicit form.

Response Model. Let A_t denote the set of pairs (a_i, a_j) compared at time or period t. Individual responses $y_{ij}^{(t)}$ are gathered in the response vector $\mathbf{y}_t$ (in lexicographical order). If covariates are present, $\mathbf{x}_t$ stands for the corresponding vector and $\boldsymbol{\gamma}_t$ for a (possibly time-varying) vector of covariate effects. Individual response probabilities are modeled by

$$P(y_{ij}^{(t)} = r) = F(\theta_{tr} + \alpha_{ti} - \alpha_{tj} + \mathbf{d}'_{tij}\boldsymbol{\gamma}_t) - F(\theta_{t,r-1} + \alpha_{ti} - \alpha_{tj} + \mathbf{d}'_{tij}\boldsymbol{\gamma}_t), \quad r = 1, \ldots, q = k-1. \tag{9}$$

The response probability $P(y_{ij}^{(t)} = k)$ for the reference category k is the complement of the sum of the q response probabilities in (9).

Thresholds θ_{tr}, $r = 1, \ldots, q$, must obey the same order restriction $\theta_{t1} < \cdots < \theta_{tq}$ as in Section 2.1. The vector $\mathbf{d}_{tij}$ may be any function of past response vectors $y_{t-1}, y_{t-2}, \ldots$ and of covariates $\mathbf{x}_t$ and has fixed dimension dim $\boldsymbol{\gamma}_t$. A very simple linear choice would be, say, $\mathbf{d}_{tij} = (\mathbf{y}_{t-1}, \mathbf{x}_t)$, but much more general forms are admissible.

To express the model as well as estimation algorithms in Section 4 in compact state-space form, we introduce a *state vector* $\boldsymbol{\beta}_t$ of unobservables. In the simplest case it comprises the vector $\boldsymbol{\theta}_t = (\theta_{t1}, \ldots, \theta_{tq})'$ of thresholds, the vector $\boldsymbol{\alpha}_t = (\alpha_{t1}, \ldots, \alpha_{t,n-1})'$ of abilities, and—if covariates are present—the vector $\boldsymbol{\gamma}_t$ of effects; that is,

$$\boldsymbol{\beta}_t = (\boldsymbol{\theta}'_t, \boldsymbol{\alpha}'_t, \boldsymbol{\gamma}'_t).$$

As in the examples of RW(2) or local linear trend models that follow, the state vector may be augmented by additional unobservable components such as slopes or past abilities.

The vector $\mathbf{p}_{ij}^{(t)} = (p_{ij1}^{(t)}, \ldots, p_{ijq}^{(t)})'$, $p_{ijr}^{(t)} = P(y_{ij}^{(t)} = r)$, of response probabilities can then be written as

$$\mathbf{p}_{ij}^{(t)} = h(\mathbf{Z}_{tij}\boldsymbol{\beta}_t), \tag{10}$$

where the *design matrix* has the structure

$$\mathbf{Z}_{tij} = \begin{bmatrix} 1 & & & c'_{ij} & \mathbf{d}'_{tij} \\ & \ddots & & \vdots & \vdots \\ & & 1 & c'_{ij} & \mathbf{d}'_{tij} \end{bmatrix}, \tag{11}$$

with appropriate indicator vectors $\mathbf{c}_{ij}$ as in the following examples, and the response function $h : \mathbb{R}^q \to [0, 1]^q$ is defined according to (9).

Transition Model. States $\boldsymbol{\beta}_t$ follow a Markovian transition model assumed to be linear and Gaussian for simplicity:

$$\boldsymbol{\beta}_{t+1} = \mathbf{T}_t\boldsymbol{\beta}_t = \mathbf{v}_{t+1}, \qquad t = 1, 2, \ldots$$
$$\boldsymbol{\beta}_0 = \boldsymbol{\xi}_0 + \mathbf{v}_0. \tag{12}$$

The transition matrices $\mathbf{T}_1, \mathbf{T}_2 \ldots$ and the initial value $\boldsymbol{\xi}_0$ are nonrandom. The error process $\{\mathbf{v}_t\}$ is Gaussian, $\mathbf{v}_t \sim N(0, \mathbf{Q}_t)$, with $\mathbf{v}_t$ independent of $\mathbf{v}_{t-1}, \ldots, \mathbf{v}_0$ and of past responses. Note that (12) is more general than a simple Markovian model for thresholds and abilities at time t, because the state vector $\boldsymbol{\beta}_t$ may contain past abilities, as in the RW(2) model, or trend and slope components, as in the local linear trend model. In some applications the initial vector ε_0 and the covariance matrices $\mathbf{Q}_t$ will be known. Where they are not known, they are considered nonrandom hyperparameters of the model and can be estimated jointly with the states as discussed in Section 4.3.

Taken together, the response model and the transition model form a state-space model for paired comparisons over time. It is important to note that this is not a common linear (Gaussian) state-space model, because the response model is nonlinear and non-Gaussian. Thus standard methodology for linear state-space models cannot be applied. Appropriate filtering and smoothing algorithms for tracking the state vector $\boldsymbol{\beta}_t$ and estimating hyperparameters are developed in Section 4. Further data-driven tools (e.g., identifying or checking models) still have to be developed.

Let us now consider how the trend models of Section 3.1 are obtained by appropriate specifications. Thresholds follow a random walk model or are constant ($\boldsymbol{\Sigma}_\theta = 0$), and abilities $\alpha_{t1}, \ldots, \alpha_{t,n-1}$ are assumed to be mutually independent. Because no covariates are present, $\mathbf{d}_{tij} = 0$.

The RW(1) Model. This is specified by

$$\boldsymbol{\beta}_t = \begin{bmatrix} \boldsymbol{\theta}_t \\ \boldsymbol{\alpha}_t \end{bmatrix}, \qquad \mathbf{T}_t = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$
$$\mathbf{v}_t = \begin{bmatrix} \mathbf{w}_t \\ \mathbf{u}_t \end{bmatrix}, \qquad \mathbf{Q}_t = \begin{bmatrix} \boldsymbol{\Sigma}_\theta & 0 \\ 0 & \boldsymbol{\Sigma}_\alpha \end{bmatrix}, \tag{13}$$

where $\boldsymbol{\Sigma}_\alpha = \mathrm{cov}(\boldsymbol{\alpha}_t) = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_{n-1}^2)$. The indicator vector is given by

$$\mathbf{c}_{ij} = (0, \ldots, 1, \ldots, -1, \ldots, 0);$$

that is, component i is 1, component j is -1, and all other components are 0. The random walk model has minimal dimension in the sense that the state vector $\boldsymbol{\beta}_t$ contains only thresholds $\boldsymbol{\theta}_t$ and abilities $\boldsymbol{\alpha}_t$ at time t.

The RW(2) Model. For this model, the state vector contains abilities at times t and $t-1$ and the transition model is given by

$$\boldsymbol{\beta}_t = \begin{bmatrix} \boldsymbol{\theta}_t \\ \alpha_{t1} \\ \alpha_{t-1,1} \\ \vdots \\ \alpha_{t,n-1} \\ \alpha_{t-1,n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & & & & & \\ & 2 & -1 & 0 & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 2 & -1 \\ & & & & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_{t-1} \\ \alpha_{t-1,1} \\ \alpha_{t-2,1} \\ \vdots \\ \alpha_{t-1,n-1} \\ \alpha_{t-2,n-1} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_t \\ u_{t1} \\ 0 \\ \vdots \\ u_{t,n-1} \\ 0 \end{bmatrix}, \tag{14}$$

where $\boldsymbol{\theta}'_t = (\theta_{t1}, \ldots, \theta_{tq})$ and $\mathbf{I}$ is the identity matrix. The indicator vector now has dimension $2n-2$ and is defined by

$$\mathbf{c}_{ij} = (0, \ldots, 1, 0, \ldots, -1, 0, \ldots, 0)',$$

with 1 as element $2i-1$, -1 as element $2j-1$, and 0 otherwise. Note that $\mathbf{Q}_t$ is singular in the state-space form (14) of the RW(2) model.

The Local Linear Trend Model. The state vector now includes slopes and the transition model is given by

$$\beta_t = \begin{bmatrix} \theta_t \\ \alpha_{t1} \\ \lambda_{t1} \\ \vdots \\ \alpha_{t,n-1} \\ \lambda_{t,n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & & & & \\ & 1 & 1 & 0 & \\ & 0 & 1 & & \\ & & & \ddots & \\ & 0 & & 1 & 1 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_{t-1} \\ \alpha_{t-1,1} \\ \lambda_{t-1,1} \\ \vdots \\ \alpha_{t-1,n-1} \\ \lambda_{t-1,n-1} \end{bmatrix} + \begin{bmatrix} \mathbf{w}_t \\ u_{t\alpha 1} \\ u_{t\lambda 1} \\ \vdots \\ u_{t\alpha,n-1} \\ u_{t\lambda,n-1} \end{bmatrix}. \quad (15)$$

The indicator vector is the same as for the RW(2) model.

To specify the model completely in terms of joint distributions, some additional basic assumptions are required. Let histories of states, covariates, and states up to time t be denoted by $\mathbf{y}_t^* = (\mathbf{y}_1', \ldots, \mathbf{y}_t')'$, $\mathbf{x}_t^* = (\mathbf{x}_1', \ldots, \mathbf{x}_t')$, and $\beta_t^* = (\beta_0' \ldots, \beta_t')'$. Let $A_t \subset A \times A$ denote the pairs of teams considered at time t

Assumption A1. Conditional on β_t and $\mathbf{y}_{t-1}^*, \mathbf{x}_t^*$, current comparisons $\mathbf{y}_{ij}^{(t)}$, $(a_i, a_j) \in A_t$, are independent; that is,

$$P(y_{ij}^{(t)} = r_{ij}, (a_i, a_j) \in A_t | \beta_t, \mathbf{y}_{t-1}^* \mathbf{x}_t^*) = \prod_{(a_i,a_j)\in A_t} P(\mathbf{y}_{ij}^{(t)} = r_{ij} | \beta_t, \mathbf{y}_{t-1}^*, \mathbf{x}_t^*),$$

$$r_{ij} \in 1, \ldots, q, \qquad t = 1, 2, \ldots.$$

Because independence is conditional, interaction among individuals may enter via the common "history" $\mathbf{y}_{t-1}^*, \mathbf{x}_t^*$.

Assumption A2. Conditional on β_t and $\mathbf{y}_{t-1}^*, \mathbf{x}_t^*$, current comparisons are independent of β_t^*; that is,

$$P(\mathbf{y}_{ij}^{(t)} = r | \beta_t^*, \mathbf{y}_{t-1}^* \mathbf{x}_t^*) = P(y_{ij}^{(t)} = r | \beta_t, \mathbf{y}_{t-1}^*, \mathbf{x}_t^*),$$

$$r = 1, \ldots, q, \qquad t = 1, 2, \ldots.$$

This conditional independence assumption states that, given $\mathbf{y}_{t-1}^*, \mathbf{x}_t^*$, the current state β_t contains the same information on $\mathbf{y}_t$ as the whole sequence β_t^*. Although this assumption is basic for state-space models, it is often not stated explicitly.

If covariates are stochastic, then a further conditional independence assumption must be introduced:

Assumption A3. Conditional on $\mathbf{y}_{t-1}^*, \mathbf{x}_{t-1}^*$, covariates are independent of β_{t-1}^*; that is,

$$f(\mathbf{x}_t | \beta_{t-1}^*, \mathbf{y}_{t-1}^*, \mathbf{x}_{t-1}^*) = f(\mathbf{x}_t | \mathbf{y}_{t-1}^*, \mathbf{x}_{t-1}^*), \qquad t = 1, 2, \ldots.$$

Loosely speaking, A3 means that the covariate process contains no information on the parameter process. It can be omitted for deterministic covariates.

4. CONDITIONAL MODE ESTIMATION OF TIME-VARYING ABILITIES AND PARAMETERS

The models of Section 3 contain two types of parameters: the sequence $\beta_t^* = (\beta_0', \ldots, \beta_t')$ of "states," including in particular abilities $\alpha_0, \ldots, \alpha_t$, and, so-called hyperparameters; for example, unknown variances or initial values in parameter transition models. Let us first consider estimation of β_t^* for known hyperparameters.

4.1 Estimation of states for known hyperparameters

Given the data, estimation of β_t^* will be based on the posterior density $p(\beta_t^* | \mathbf{y}_t^*, \mathbf{x}_t^*)$. A full Bayesian analysis, requiring numerical integration, becomes computationally critical even for a moderate number of categories and comparisons. Other approaches to state-space models for categorical observations (Harvey and Fernandes 1989; West, Harrison, and Migon 1985) cannot be applied to our situation. Therefore, we propose to estimate the parameter sequence $\{\beta_t\}$ by posterior modes. This approach is common in (empirical) Bayes estimation of nonnormal data. We will make use of posterior mode filtering and smoothing algorithms developed by Fahrmeir (1992) and Fahrmeir and Kaufmann (1991) for dynamic exponential family systems.

From the general model assumptions, it follows by repeated application of Bayes's theorem that

$$p(\beta_t^* | \mathbf{y}_t^*, \mathbf{x}_t^*) \propto \prod_{s=1}^{t} p(\mathbf{y}_s | \beta_s, \mathbf{y}_{s-1}^*, \mathbf{x}_s^*) \times \prod_{s=1}^{t} p(\beta_s | \beta_{s-1}) \cdot p(\beta_0),$$

where $p(\mathbf{y}_s | \beta_s, \mathbf{y}_{s-1}^*, \mathbf{x}_s^*)$ is determined by the observation model and the conditional independence assumption A2, and $p(\beta_s | \beta_{s-1})$, $p(\beta_0)$ are (conditionally) Gaussian according to the transition model (14). Assume now for the moment that covariance matrices $\mathbf{Q}_0, \mathbf{Q}_1, \ldots$ are positive definite. (This restriction, which would exclude the RW(2) model in the form (14) or constant thresholds, will be dropped later on.) Maximization of the posterior density is then equivalent to maximizing the penalized log-likelihood

$$L(\beta_t^*) = \sum_{s=1}^{t} l_s(\beta_s) - a(\beta_t^*),$$

where $l_s(\beta_s) = \log p(\mathbf{y}_s | \beta_s, \mathbf{y}_{s-1}^*, \mathbf{x}_s^*)$ is the log-likelihood contribution of comparisons $\mathbf{y}_s$ at time s and

$$a(\beta_t^*) = \frac{1}{2}\left[(\beta_0 - \xi_0)'\mathbf{Q}_0^{-1}(\beta_0 - \xi_0) + \sum_{s=1}^{t} (\beta_s - \mathbf{T}_s\beta_{s-1})\,\mathbf{Q}_s^{-1}(\beta_s - \mathbf{T}_s\beta_{s-1})\right]$$

is the penalty term. From a Bayesian viewpoint, $a(\beta_t^*)$ acts as a "smoothness prior" for time-varying abilities and other parameters. As an example, consider the case of only two alternatives, a_1, a_2 and a RW(2) model (6) for $\alpha_t := \alpha_{t1}$. Then the penalty term becomes

$$a(\beta_t^*) = \frac{1}{2}\left[(\alpha_0 - \xi_0)/\sigma_0^2 + \frac{1}{\sigma_\alpha^2}\sum_{s=1}^{t} \nabla^2\alpha_s\right].$$

The second term is the discrete time analog of the roughness penalty that leads to cubic spline smoothing (see O'Sullivan, Raynor, and Yandell 1986 or Green 1987 in the related context of univariate generalized linear models). This shows that our method may also be interpreted as a discrete-time nonparametric smoothing approach.

In state-space terminology, the posterior mode estimators (i.e., the maximizers of $L(\beta_t^*)$), are called "smoothers" for $s < t$ and "filters" for $s = t$. As for any maximization problem, various numerical algorithms, differing in computational complexity and approximation quality, can be designed. For the time series situation, which in our context corresponds to the case of only two alternatives, Fahrmeir (1992) proposed generalized extended Kalman filtering and smoothing as a recursive approximate posterior mode estimation algorithm for (conditionally) exponential family observations. This can be derived by generalizing the arguments of Sage and Melsa (1971) in their derivation of extended Kalman filtering for conditionally Gaussian observations, viewing the filter as a recursive gradient algorithm. Fahrmeir and Kaufmann (1991) developed a Gauss–Newton procedure for maximizing $L(\beta_t^*)$, making efficient use of the block-tridiagonal structure of second derivatives. Moreover, they showed that the computationally more attractive generalized extended Kalman algorithm of Fahrmeir (1992) (hereinafter denoted by GEK) is closely related to this Gauss–Newton procedure. For both algorithms, the restriction to positive definite covariance matrices made previously can be dropped in the following way: Without loss of generality, suppose that covariance matrices are block structured as

$$\mathbf{Q}_t = \begin{bmatrix} \tilde{Q}_t & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{Q}_t \text{ positive definite.}$$

Define

$$\mathbf{Q}_t(\delta) = \begin{bmatrix} \tilde{Q}_t & 0 \\ 0 & \delta\mathbf{I} \end{bmatrix}, \qquad \delta > 0,$$

and apply the algorithms, using $Q(\delta)$ instead of Q. Carrying out the limit operation $\delta \to 0$, algorithms for $\mathbf{Q}_t$ positive semidefinite can be obtained.

We will confine the presentation to the GEK algorithm, which in our experience has good estimation properties for categorical response models and is comparably fast. This is useful for joint estimation of hyperparameters. A further reason for considering only the GEK algorithm is that comparison to the reward system approach of Batchelder et al. (1992) can be presented more directly. The following additional notation will be used: Stressing dependence on β_t, we write $p_{ij}^{(t)}(\beta_t)$ for the vector $\mathbf{h}(\mathbf{Z}_{tij}\beta_t)$ of response probabilities in (12) and $\boldsymbol{\Sigma}_{tij}(\beta_t)$ for the corresponding (conditional) covariance matrix of the multinomial distribution; that is,

$$\boldsymbol{\Sigma}_{tij}(\beta_t) = \operatorname{diag}(\mathbf{p}_{ij}^{(t)}(\beta_t)) - \mathbf{p}_{ij}^{(t)}(\beta_t)(\mathbf{p}_{ij}^{(t)}(\beta_t))'.$$

Furthermore, $\mathbf{H}_{tij}(\beta_t)$ stands for the Jacobian $\partial\mathbf{h}/\partial\boldsymbol{\eta}$ of the response function $\mathbf{h} : \boldsymbol{\eta} \to \mathbf{h}(\boldsymbol{\eta})$ evaluated at $\boldsymbol{\eta} = \mathbf{Z}_{tij}\beta_t$. Then, up to constants, the conditional log-likelihood contribution of $y_{ij}^{(t)}$ is given by

$$l_{tij}(\beta_t) = \sum_{r=1}^{q+1} \log p_{ijr}^{(t)}(\beta_t), \qquad p_{ij,q+1}^{(t)}(\beta_t) = 1 - \sum_{r=1}^{q} p_{ijr}^{(t)}(\beta_t).$$

The score function has the form

$$\begin{aligned}\mathbf{r}_{tij}(\beta_t) &= \partial l_{tij}(\beta_t)/\partial\beta_t \\ &= \mathbf{Z}_{tij}\mathbf{H}_{tij}(\beta_t)\boldsymbol{\Sigma}_{tij}^{-1}(\beta_t)(\tilde{\mathbf{y}}_{ij}^{(t)} - \mathbf{p}_{ij}^{(t)}(\beta_t)),\end{aligned}$$

where $\tilde{\mathbf{y}}_{ij}^{(t)} = (y_{ij1}^{(t)}, \ldots, y_{ijq}^{(t)})$ is a vector of dummy variables defined by

$$\begin{aligned} y_{ijr}^{(t)} &= 1 \quad \text{if } \mathbf{y}_{ij}^{(t)} = r, \\ &= 0 \quad \text{otherwise.}\end{aligned}$$

The information matrix $E(-\partial^2 l_{tij}(\beta_t)/\partial\beta_t\partial\beta_t' \mid \beta_t, \mathbf{y}_{t-1}^*, \mathbf{x}_t^*)$ is given by

$$\mathbf{R}_{tij}(\beta_t) = \mathbf{Z}_{tij}'\mathbf{H}_{tij}(\beta_t)\boldsymbol{\Sigma}_{tij}^{-1}(\beta_t)\mathbf{H}_{tij}'(\beta_t)\mathbf{Z}_{tij}.$$

In addition to posterior mode filters $\hat{\beta}_t$ and smoothers $\hat{\beta}_{s|t}$, which estimate β_t and β_s, $s < t$, based on data up to t, the following algorithm provides estimates $\mathbf{V}_t$ and $\mathbf{V}_{s|t}$ of error covariance matrices as well as one-step predictions $\hat{\beta}_{t|t-1}$ together with $\mathbf{V}_{t|t-1}$.

Generalized Kalman Filter for Paired Comparisons

1. Initialization: $\hat{\beta}_0 = \xi_0$, $\mathbf{V}_0 = \mathbf{Q}_0$ for $t = 1, 2, \ldots$.
2. Prediction steps:

$$\hat{\beta}_{t|t-1} = \mathbf{T}_t\hat{\beta}_{t-1}, \qquad \mathbf{V}_{t|t-1} = \mathbf{T}_t\mathbf{V}_{t-1}\mathbf{T}_t' + \mathbf{Q}_t.$$

3. Correction steps:

$$\hat{\beta} = \hat{\beta}_{t|t-1} + \mathbf{V}_t\mathbf{r}_t, \qquad \mathbf{V}_t = (\mathbf{V}_{t|t-1}^{-1} + \mathbf{R}_t)^{-1},$$

with

$$\mathbf{r}_t = \sum_{(a_i,a_j)\in A_t} \mathbf{r}_{tij}(\hat{\beta}_{t|t-1}), \qquad \mathbf{R}_t = \sum_{(a_i,a_j)\in A_t} \mathbf{R}_{tij}(\hat{\beta}_{t|t-1}).$$

The algorithm can be easily derived by applying the filter of Fahrmeir (1992) to the "time series" of responses $\mathbf{y}_t = \{y_{ij}^{(t)}, (a_i, a_j) \in A_t\}$, $t = 1, 2 \ldots$, for paired comparisons $(a_i, a_j) \in A_t$. Due to the conditional independence assumption A2 for individual responses $y_{ij}^{(t)}$, the score function $\mathbf{r}_t$ and information matrix $\mathbf{R}_t$ in the correction steps are the sum of individual contributions $\mathbf{r}_{tij}$ and $\mathbf{R}_{tij}$. The correction steps can be interpreted as a recursive scoring algorithm (with only one iteration at t) for computing posterior moments $\hat{\beta}_t$ and V_t, using $\hat{\beta}_{t|t-1}$ and $\mathbf{V}_{t|t-1}$ as a prior information. The inverse $\mathbf{V}_{t|t-1}^{-1}$ is the (estimated) information on β_t given observations $\mathbf{y}_{t-1}^*$ up to $t - 1$, the matrix $\mathbf{R}_t$ is the information contributed by $\mathbf{y}_t$, and the sum $\mathbf{V}_{t|t-1}^{-1} + \mathbf{R}_t$ is the (estimated) information on β_t given $\mathbf{y}_t^* = (\mathbf{y}_{t-1}^*, \mathbf{y}_t)$. Inversion gives the estimated error covariance $\mathbf{V}_t$.

For fixed t, smoothers $\hat{\beta}_{s|t}$, $s = t - 1, \ldots, 0$ are available by the following step:

4. Backward smoother steps

$$\begin{aligned}\hat{\beta}_{s-1|t} &= \hat{\beta}_{s-1} + \beta_s(\hat{\beta}_{s|t} - \hat{\beta}_{s|s-1}), \\ \mathbf{V}_{s-1|t} &= \mathbf{V}_{s-1} + \mathbf{B}_s(\mathbf{V}_{s|t} - \mathbf{V}_{s|s-1})\mathbf{B}_s',\end{aligned}$$

where

$$\mathbf{B}_s = \mathbf{V}_{s-1}\mathbf{T}_s'\mathbf{V}_{s|s-1}^{-1}.$$

Whether one is interested in the sequence of filtering estimates $\hat{\beta}_t$, $t = 1, 2, \ldots$ alone or in the sequence $\{\hat{\beta}_{s|t}, s \leq t\}$ of smoothers may depend on the specific situation or application. Filters are updating algorithms for *current* states β_t given data up to t, whereas smoothers estimate the complete sequence $\beta_0, \ldots, \beta_t$ from a retrospective viewpoint. If one wants to analyze, for example, careers of players, then it seems natural to look at smoothed abilities. It should be noted that updating schemes like that of Batchelder et al. (1992) correspond to filtering and not to smoothing.

4.2 Special Cases

The simplest comparison system is that of only two alternatives, a_1, a_2 and binary comparisons ($q = 1$). Let us further assume an order-independent model; that is, $\theta = 0$. Suppressing indices 1 and 2, the observation model is

$$P(y^{(t)} = 1 \mid \alpha_t) = F(\alpha_t).$$

The simplest transition model is the RW(1) model (6). In this case the filtering algorithm reduces to a rather simple structure. For the logistic model, it is easily verified that $H_t(\alpha_t) = F_t(\alpha_t)[1 - F(\alpha_t)] = \Sigma_t(\alpha_t)$. Combining prediction and correction steps, one obtains

$$\hat{\alpha}_t = \hat{\alpha}_{t-1} + V_t(y^{(t)} - F(\hat{\alpha}_{t-1})) \tag{16}$$

and

$$V_t = \{(V_{t-1} + \sigma_\alpha^2)^{-1} + F(\hat{\alpha}_{t-1})[1 - F(\hat{\alpha}_{t-1})]\}^{-1}. \tag{17}$$

If there is no comparison at time t (i.e., $A_t = \phi$), then $r_t = 0$, $R_t = 0$. Setting $C_t = V_t/4$, equation (16) can be rewritten as

$$\begin{aligned} \hat{\alpha}_t &= \hat{\alpha}_{t-1} + 4C_t(1 - F(\hat{\alpha}_{t-1})), && \text{if } a_1 \text{ wins,} \\ &= \hat{\alpha}_{t-1} - 4C_t F(\hat{\alpha}_{t-1}), && \text{if } a_2 \text{ wins,} \\ &= \hat{\alpha}_{t-1}, && \text{if no comparison is made.} \end{aligned}$$

Thus we get the same form as the updating scheme of Batchelder et al. (1992); compare Section 2.2. But for the extended Kalman filter approach, the weights $4C_t$ are *not* constant but rather are functions of past comparisons, determined adaptively by (17). If the RW(1) model is replaced by other stochastic trend models, different yet similar filtering algorithms result. For the local trend model, (16) becomes

$$\hat{\alpha}_t = \hat{\alpha}_{t-1} + \hat{\lambda}_{t-1} + V_t(y^{(t)} - F(\hat{\alpha}_{t-1} + \hat{\lambda}_{t-1})),$$

with the corresponding modification for V_t in (17).

Let us next consider the two-player case with draws. Then $\tilde{\mathbf{y}}^{(t)} = (\tilde{y}_1^{(t)}, \tilde{y}_2^{(t)})$, with $\tilde{y}_1^{(t)} = 1$ if a_1 wins and $\tilde{y}_1^{(t)} = 0$ otherwise, and $\tilde{y}_2^{(t)} = 1$ for a draw and $\tilde{y}_2^{(t)} = 0$ otherwise. Assuming an order-independent model, the restriction $p(\mathbf{y}_{ij}^{(t)} = 1) = p(\mathbf{y}_{ji}^{(t)} = 3)$ gives $\theta_{t1} = -\theta_{t2}$ (see Sec. 2.1). Then, with $\theta_t = \theta_{t2}$, response probabilities are given by the response model

$$p_1^{(t)} = F(-\theta_t + \alpha_t), \qquad p_2^{(t)} = F(\theta_t + \alpha_t) - F(-\theta_t + \alpha_t).$$

For a RW(1) model, the resulting filtering steps are of the form

$$\hat{\theta}_t = \hat{\theta}_{t-1} + k_{11}^{(t)}(\tilde{y}_1^{(t)} - \hat{p}_1^{(t)}) + k_{12}^{(t)}(\tilde{y}_2^{(t)} - \hat{p}_2^{(t)})$$

and

$$\hat{\alpha}_t = \hat{\alpha}_{t-1} + k_{21}^{(t)}(\tilde{y}_1^{(t)} - \hat{p}_1^{(t)}) + k_{22}^{(t)}(\tilde{y}_2^{(t)} - \hat{p}_2^{(t)}),$$

with the weights determined by the elements of the (2×2) matrix $(k_{ij}^{(t)}) = \mathbf{V}_{tij}\mathbf{Z}_{tij}\mathbf{H}_{tij}\mathbf{\Sigma}_{tij}^{-1}$. Thresholds θ_t and abilities α_t are estimated simultaneously.

In the more general case of many alternatives and of multicategorical responses, the general structure remains the same. If a RW(1) model for $\beta_t = (\theta_t', \alpha_{t1}, \ldots, \alpha_{t,n-1})'$ is assumed, then the resulting filter steps for α_{tk}, $k = 1, \ldots, n - 1$, are of the form

$$\hat{\alpha}_{tk} = \hat{\alpha}_{t-1,k} + \sum_{(a_i, a_j) \in A_t} w_{tij}^k(\tilde{y}_{ij}^{(t)} - F(\hat{\theta}_{t-1} + \hat{\alpha}_{t-1,i} - \hat{\alpha}_{t-1,j})),$$

where the weights w_{tij}^k are determined by the correspondingly partitioned terms of $\mathbf{V}_{tij}\mathbf{Z}_{tij}'\mathbf{H}_{tij}\mathbf{\Sigma}_{tij}^{-1}$. For the special case of binary comparisons, the filter steps again have the same structure as in the reward system approach; however, weights are determined adaptively.

4.3 Estimation of Hyperparameters

Up to now we have assumed that hyperparameters are known. Let us consider the situation where ξ_0, $\mathbf{Q}_0$ and $\mathbf{Q}_t = \mathbf{Q}$ (independent of $t \geq 1$) are unknown. Following analogous proposals made in the related situation of empirical Bayes estimation in random effect models for categorical data (Stiratelli, Laird, and Ware 1984), we suggest using a variant of the EM algorithm for joint estimation of β_t^*, ξ_0, $\mathbf{Q}_0$, and $\mathbf{Q}$, thereby substituting posterior means and covariance matrices by posterior modes and curvatures obtained from the foregoing filtering smoothing algorithm. For the time series situation, the resulting estimation algorithm also was used by Fahrmeir (1992) and has been implemented and studied in detail by Goss (1990). Adapting it to the dynamic models for paired comparisons, the joint estimation procedure can be summarized as follows:

1. Choose appropriate starting values $\xi_0^{(0)}$, $\mathbf{Q}_0^{(0)}$, $\mathbf{Q}^{(0)}$ for $r = 0, 1, 2, \ldots$.
2. Smoothing: Compute $\hat{\beta}_{s|t}^{(r)}$, $V_{s|t}^{(r)}$, $s = 1, \ldots, t$ by generalized Kalman filtering and smoothing, with unknown parameters ξ_0, $\mathbf{Q}_0$, and $\mathbf{Q}$ replaced by $\xi_0^{(r)}$, $\mathbf{Q}_0^{(r)}$, and $\mathbf{Q}^{(r)}$.
3. EM step: Compute $\xi^{(r+1)}$, $\mathbf{Q}_0^{(r+1)}$, and $\mathbf{Q}^{(r+1)}$ by

$$\xi_0^{(r+1)} = \hat{\beta}_{0|t}^{(r)},$$

$$\mathbf{Q}_0^{(r+1)} = \mathbf{V}_{0|t}^{(r)},$$

and

$$\mathbf{Q}^{(r+1)} = \frac{1}{t}\sum_{s=1}^{t} [(\beta_{s|t}^{(r)} - \mathbf{T}_s\beta_{s-1|t}^{(r)})(\beta_{s|t}^{(r)} - \mathbf{T}_s\beta_{s-1|t}^{(r)})' + \mathbf{V}_{s|t}^{(r)} - \mathbf{T}_s\mathbf{B}_{s-1}\mathbf{V}_{s|t}^{(r)} - \mathbf{V}_{s|t}^{(r)\prime}\mathbf{B}_{s-1}'\mathbf{T}_t' + \mathbf{T}_t\mathbf{V}_{s-1|t}^{(r)}\mathbf{T}_t'],$$

where $\mathbf{B}_{s-1}$ is obtained by the smoother steps. Hyperparameters in the application of Section 5 have been estimated by this method.

5. APPLICATIONS

5.1 Simulations

For the special case of dichotomous order-independent paired comparisons, the estimation procedure considered here yields an updating scheme similar to the reward system of Batchelder et al. (1992); see Section 4.2. In a simulation study the two approaches are compared for $n = 3$ (see Batchelder et al. 1992 for a similar study with $n = 2$). Because the symmetric side constraint $\Sigma\alpha_{ti} = 0$ yields abilities that are much more easy to interpret, it is assumed that the underlying abilities are given by

$$\alpha_{t1} = \alpha - (\alpha - \alpha_0)\gamma^t, \qquad \alpha_{t2} = -\alpha_{t1}, \qquad \alpha_{t3} = 0,$$

where $\alpha = 1.5$, $\alpha_0 = 0$, and $\gamma = .95$. That means that the ability of a_1 is steadily growing, the ability of a_2 is decreasing, and the ability of a_3 remains constant over time.

For the reward system approach, the constant C must be specified a priori. The "appropriate" constant C depends strongly on the (unknown) variation of abilities and the type of model (here the logistic model is used). Figure 1 shows the mean over 50 trials with $t = 150$ together with the underlying abilities. It is assumed that the system at $t = 0$ knows the underlying abilities $\alpha_{11} = \alpha_{12} = \alpha_{13} = 0$. It is seen that as C increases, the bias increases.

But Figure 1 does not show how the system really works in applications. Because we have the mean over 50 trials, actually we are considering the case where 50 round-robin tournaments take place at each time t. Although Figure 1 shows only the global tendency of the system, Figure 2 shows the system at work in the case of just one round robin at time t. From Figure 2, it is seen how unstable the system actually is, in particular for larger C. Even for the case $C = .1$, where the global tendency seems appropriate, the fluctuation of the estimate is very high. In the more realistic case where C cannot be chosen by looking at simulations, the problem of unknown C (and, therefore, unknown bias) is added to the problem of fluctuation.

The generalized Kalman filter approach allows us to estimate the hyperparameters as well as values at $t = 0$. To see the system at work, we considered the case of *one* round-robin tournament at time t. For the first-order random-walk model, the estimation of hyperparameters was started with the least squares estimate $\xi_0^{(0)}$, the diagonal matrix $\mathbf{Q}_0^{(0)} = .05\mathbf{I}$, and $\mathbf{Q}^{(0)} = .01\mathbf{I}$, where $\mathbf{I}$ denotes the unit matrix. After 150 EM steps, the estimated $\mathbf{Q}$ matrix was given by

$$\mathbf{Q} = \begin{bmatrix} .008 & .0 \\ .0 & .009 \end{bmatrix}.$$

The values at $t = 0$ need not be specified a priori. They are estimated by the system. Figure 3 shows the updating (prediction and correction steps) that results from the estimated hyperparameters. Figure 4 shows the estimates when the backward smoother steps are included. As is seen from Figures 3 and 4, the estimates are quite near the underlying abilities. The data-based choice of hyperparameters yields values that are appropriate for keeping track of the variation of abilities. In particular, if backward smoothing is incorporated, the development of the players is represented rather well. The estimates have less bias and fluctuation than the reward system estimates. Moreover, they are not based on a priori knowledge of an appropriate constant but rather are purely data driven.

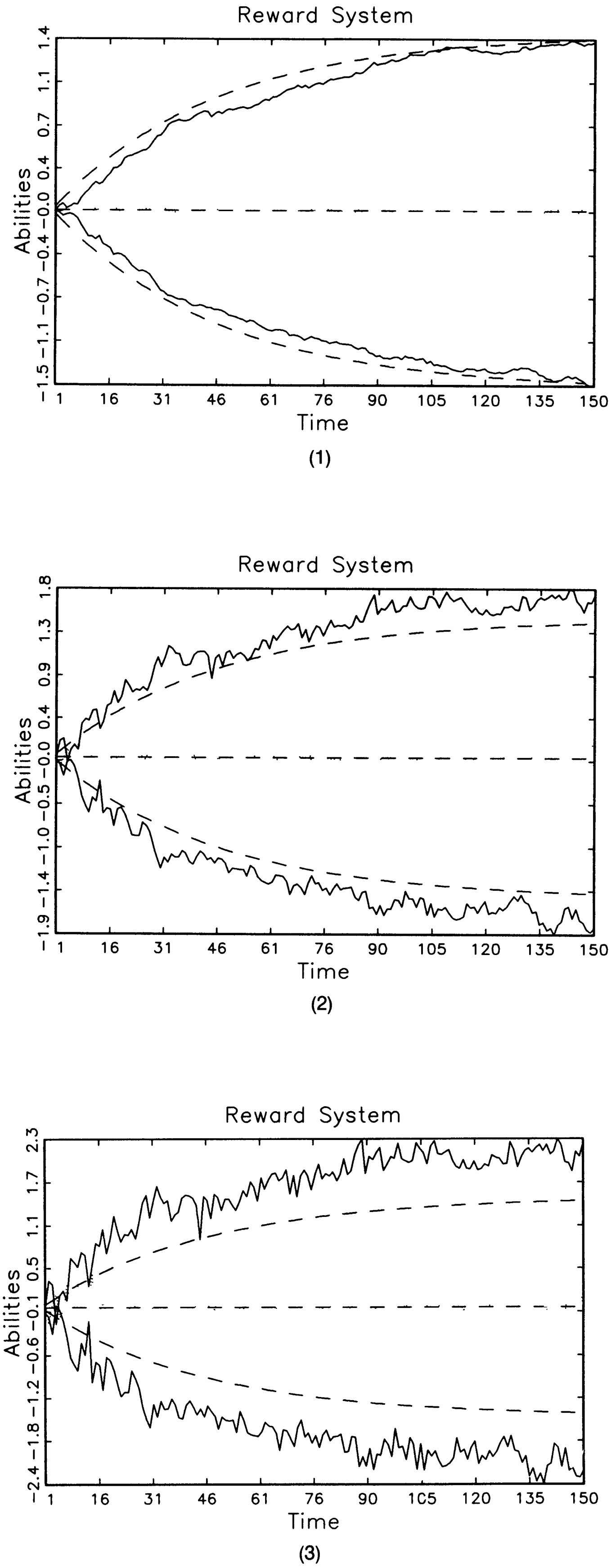

Figure 1. Average for the Reward System for t = 150, 50 Trials, and constants C = .1, .5, 1. Abilities are given by the dashed line; estimates of abilities 1 and 2 are represented by the solid line, and the estimate of ability 3 is represented by the dotted line. Ability 1 is the increasing curve, ability 2 is the decreasing curve, and ability 3 is the constant line.

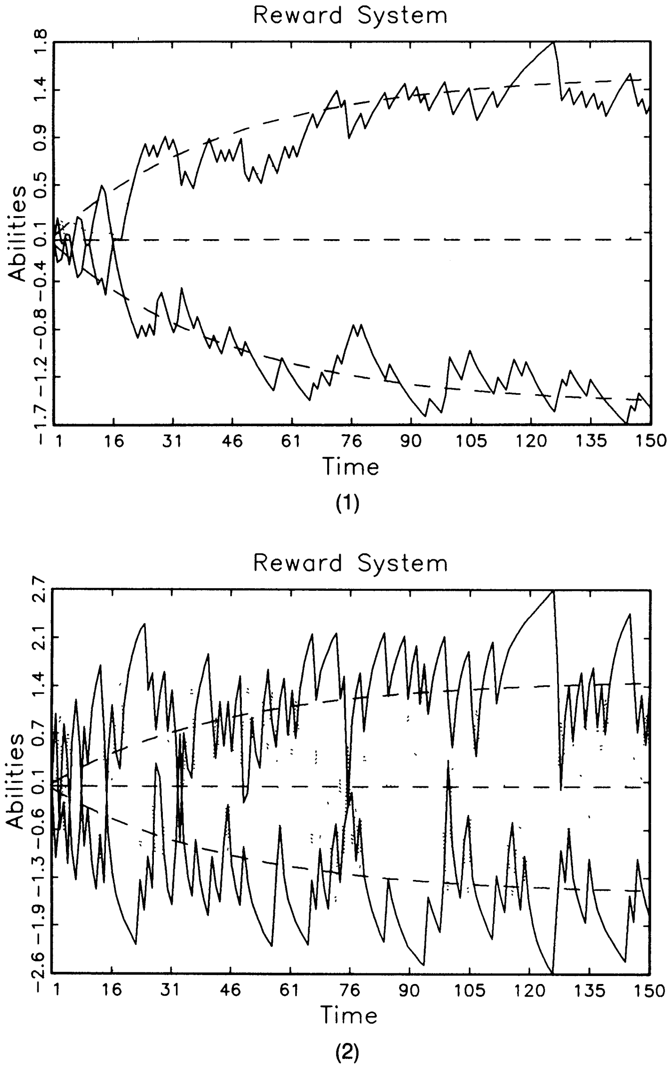

Figure 2. Reward System for t = 150 in the Case of a Round-Robin Tournament (One Trial at Time t) for Constant C = .1 and C = .5. Abilities are given by the dashed line; estimates of abilities 1 and 2 are represented by the solid line, and the estimate of ability 3 is represented by the dotted line. Ability 1 is the increasing curve, ability 2 is the decreasing curve, and ability 3 is the constant line.

5.2 Soccer Data

In the German National soccer league, teams meet twice within each season, giving each team the home court advantage once (see Table 1). Because German fans are very enthusiastic about their teams, the home court advantage may not be neglected. In the following, a logistic model is assumed that allows for home court advantage by including θ_1, θ_2 and three categories: win, draw, or loss. The RW(1) model has been fitted for threshold parameters as well as abilities. Although thresholds representing the home court advantage should be constant over years, they might be affected by factors, such as numbers of spectators, that vary over the years. Estimation of hyperstructure parameters yields a $\mathbf{Q}$ matrix that is essentially a diagonal matrix. All other values are below $.5 \cdot 10^{-4}$. For the thresholds, the estimated variances are .001 and .008. This means that the thresholds in fact remain rather stable over years. For the

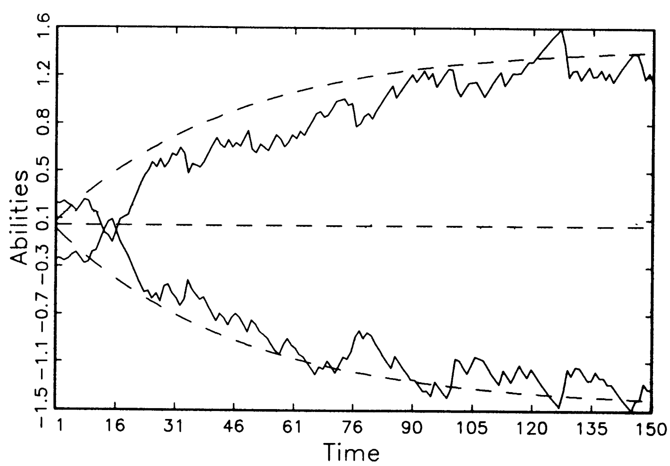

Figure 3. Kalman Filter for One Trial at Time t Based on a First-Order Random Walk. Abilities are given by the dashed line; estimates of abilities 1 and 2 are represented by the solid line, and the estimate of ability 3 is represented by the dotted line.

abilities, the estimated variances in $\mathbf{Q}$ are .124, .006, .005, .002, and .027. Figure 5 shows the smoothed abilities for the six teams based on these estimated hyperparameters. The large variance of the first team (.124) and the fifth team (.027) may also be seen from the picture, which shows strong fluctuation for Bayern Munich (team 1) and comparatively high fluctuation for Hamburger SV (team 5), whereas the other teams are quite constant. The highs and lows of Bayern Munich are in good agreement with the development, coming, and going of important players and coaches. For example, the peak about 1970–1972 coincides with the team's most successful years with Franz Beckenbauer as captain and the other important members of the national team at that time. While still successful in European Cup finals until 1974, success was steadily declining in the German national league, eventually leading to a distinct low when Franz Beckenbauer went to the New York Cosmos and others left the team. It took some time to form a new team, which

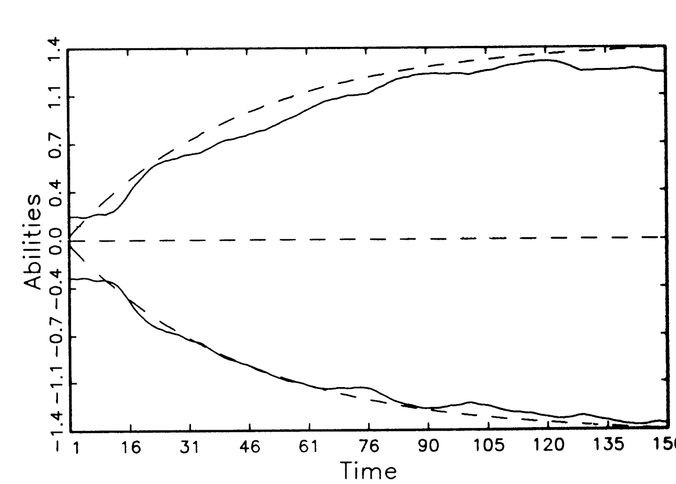

Figure 4. Extended Kalman Smoother for One Trial at Time t Based on a First-Order Random Walk. Abilities are given by the dashed line; estimates of abilities 1 and 2 are represented by the solid line, and the estimate of ability 3 is represented by the dotted line.

became better and eventually very successful again in the late 1980s. In this later period, Hamburger SV, which had become more and more powerful, and Bayern Munich were the dominating teams in the national soccer league.

The effect of home court advantage may be seen from the estimated thresholds. As already given in Section 2.1, the overall least squares estimates are $\hat{\theta}_1 = .358$ and $\hat{\theta}_2 = 1.039$.

Figure 6 shows the estimated abilities for the local linear trend model. The diagonal elements in **Q** are now .0, .012, .149, .001, .015, 0, .029, 0, .012, 0, .016, and 0. The first two values refer to the thresholds. The next values refer to the abilities α_{ti} and the trend λ_{ti} for $i = 1, \ldots, 5$. For α_{t2}, α_{t3}, and α_{t4}, the variances are larger than for the RW(1) model; for α_{t1}, the variance is comparable; and for α_{t5}, it is smaller. The variances of the trend components are very small; the only one greater than 10^{-3} is λ_{t1}. From Figure 6, it is seen that the abilities for teams 2, 3, 4, and 6 vary a little stronger in the local linear trend model. The abilities of team 1 and team 5 are almost the same as for the RW(1) model. In general, the results are very similar for the RW(1) and the local linear trend models. We also applied a grouped Cox model (i.e., a double exponential response function instead of the logistic one), to the data, leading to very similar conclusions. Because the reward system approach does not allow for home court advantage and ordered responses, it cannot be used for soccer data. Consequently, only our posterior mode approach has been applied.

6. CONCLUDING REMARKS

The basic ordinal paired-comparison model (2) is of rather general type. For logistic function F and three response categories, a special case is the model suggested by Rao and Kupper (1967). For $k = 2$, the model of Davidson and Beaver (1977) allowing for an effect of the ordering of the pair (a_i, a_j) is a special case. Model (2) may also be seen as a generalization of the various binary models proposed in the literature (see, for example, Stern 1990 or Yellott 1976). The

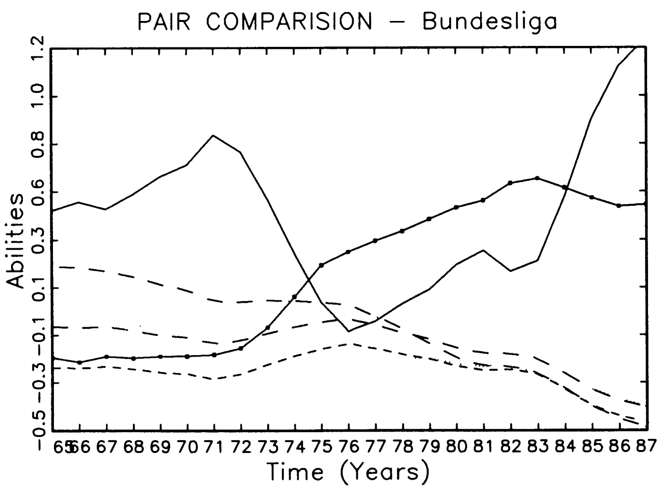

Figure 5. Kalman Filter and Smoother for Soccer Data Based on a Random Walk of First Order. Teams are Bayern München (——), 1 FC Köln (- · -), VFB Stuttgart (· · ·), Kaiserslautern (- - -), Hamburger SV (-•-), Eintracht Frankfurt (--).

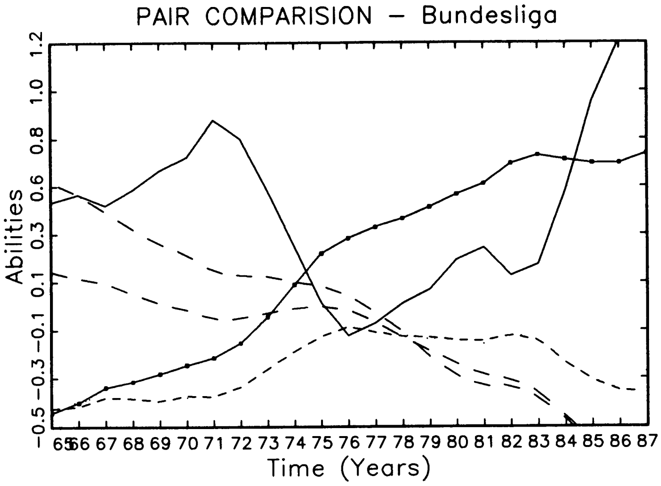

Figure 6. Kalman Filter and Smoother for Soccer Data Based on Local Linear Trend. Teams are Bayern München (——), 1 FC Köln (- · -), VFB Stuttgart (· · ·), Kaiserslautern (- - -), Hamburger SV (-•-), Eintracht Frankfurt (--).

logistic version used here has been investigated from a measurement theory viewpoint by Tutz (1986, 1989) and recently was compared to alternative models by Agresti (1992).

The strength of the proposed approach to time-dependent abilities lies in its versatility. The response model has a very general form; home court effects and ordered responses are easily modeled. The transition models may be chosen as very simple or very complex models. In both components, explanatory variables may be included. So the results of past comparisons (or, in the case of chess, the age of the player) may be included in the probability model. Quite different from previous updating systems, the reasoning is on statistical grounds. First, a model for the underlying process is specified, then estimation for this given model is investigated. In previous approaches, statistics and rational reasoning often are not well separated.

Selection and diagnosis of models is a topic needing further research. In the time-independent case considered in the papers mentioned earlier, the choice of models may be based on usual goodness-of-fit tests. For the time-dependent case, data-based tools, as they are available for linear Gaussian structural time series models (Harvey 1989), still must be developed for the more complex non-Gaussian models considered here.

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