Projection and inverse projection as a method of reformulating linear and integer

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Example

(Maximise **z**)

Subject to:

$$4x_1 - 5x_2 - 3x_3 + z \leq 0 \qquad C\ 0$$

$$-x_1 + x_2 - x_3 \leq 2 \qquad C\ 1$$

$$x_1 + x_2 + 2x_3 \leq 3 \qquad C\ 2$$

$$-x_1 \leq 0 \qquad C\ 3$$

$$-x_2 \leq 0 \qquad C\ 4$$

$$-x_3 \leq 0 \qquad C\ 5$$

Project out x_1

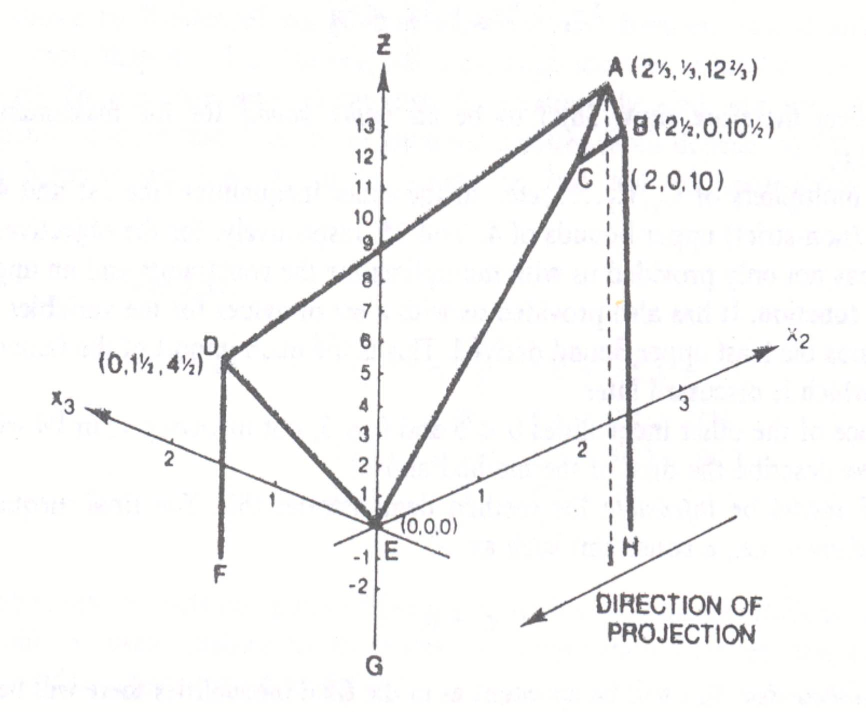


FIG. 1(i)

$\rightarrow$

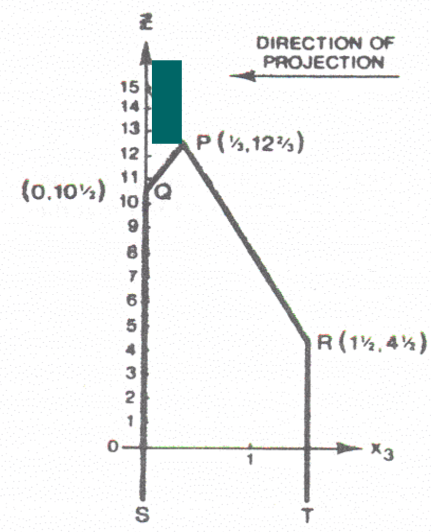


FIG. 1(ii)

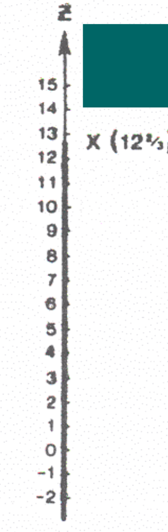


FIG. 1(iii)

$$\begin{array}{lr} -x_2 - 7x_3 + z & \le 8 \\ -5x_2 - 3x_3 + z & \le 0 \\ 2x_2 + x_3 & \le 5 \\ x_2 + 2x_3 & \le 3 \\ x_2, x_3 \le 0 & \end{array} \quad \rightarrow \quad \begin{array}{lr} -13x_3 + 2z & \le 21 \\ x_3 & \le 5 \\ 7x_3 + z & \le 15 \\ 2x_3 & \le 3 \\ -x_3 \le 0 & \end{array} \quad \rightarrow \quad \begin{array}{l} z \le 43 \\ z \le 15 \\ z \le \frac{38}{3} \end{array}$$

How to carry out Projection

The following statements are equivalent:

$$\exists x[\, f_i \le x \le g_j\,] \qquad \text{all } i, j : x \in \Re$$

$$f_i \le g_j \qquad \text{all } i, j$$

Proof $\quad \rightarrow \quad$ Immediate

$\leftarrow \quad$ Take $\quad \boldsymbol{x = \underset{i}{Max}\{f_i\}}$

$\quad$ **(or** $\boldsymbol{x = \underset{j}{Min}\{g_j\}}$**)**

(Decision Procedure of Langford for Theory of Dense Linear Order)

We can take f_i and g_i as linear expressions in the other variables (apart from x)

The constraints of any linear programme can be put in this form and x eliminated (projected out)

But need to combine **every** inequality of form "$\boldsymbol{x \le g_j}$"

With **every** inequality of form "$\boldsymbol{x \ge f_i}$"

Can lead to combinatorial explosion in number of inequalities.

Equations and associated variables can be eliminated prior to this by Gaussian Elimination.

Example
(Maximise **z**)

Subject to:

$$
\begin{array}{llll}
4x_1 - 5x_2 - 3x_3 + z & \le 0 & & C\ 0 \\
-x_1 + x_2 - x_3 & \le 2 & & C\ 1 \\
x_1 + x_2 + 2x_3 & \le 3 & & C\ 2 \\
-x_1 & \le 0 & & C\ 3 \\
-x_2 & \le 0 & & C\ 4 \\
-x_3 & \le 0 & & C\ 5
\end{array}
$$

Write in form

$$\left.\begin{aligned} -2+x_2-x_3 \le \\ 0 \le \end{aligned}\right\} x_1 \left\{\begin{aligned} &\le \frac{1}{4}\left(5x_2+3x_3-z\right) \\ &\le 3-x_2-2x_3 \end{aligned}\right.$$

$$-x_2 \le 0$$

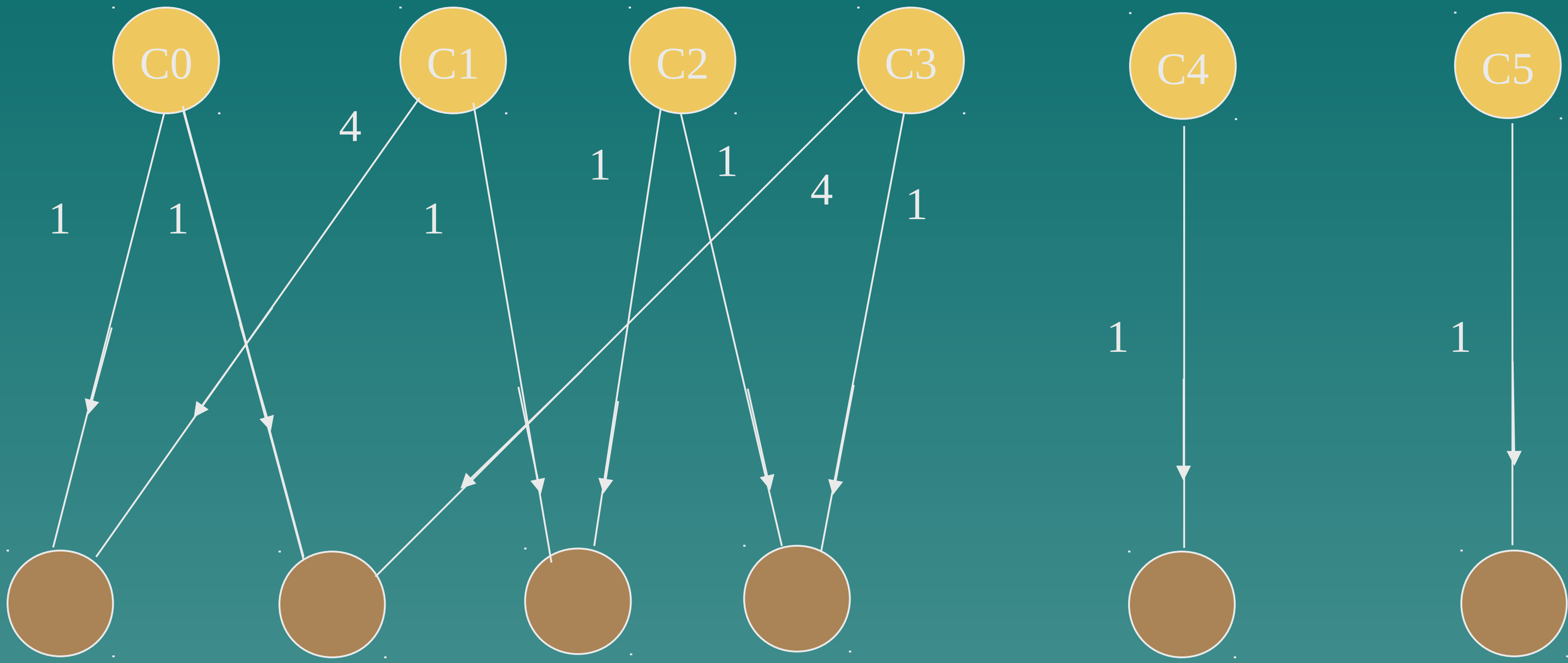
$$-x_3 \le 0$$

Eliminate x_1

$$
\begin{aligned}
-2 + x_2 - x_3 &\le \frac{1}{4}(5x_2 + 3x_3 - z) \\
-2 + x_2 - x_3 &\le 3 - x_2 - 2x_3 \\
0 &\le \frac{1}{4}(5x_2 + 3x_3 - z) \\
0 &\le 3 - x_2 - 2x_3 \\
-x_2 &\le 0 \\
-x_3 &\le 0
\end{aligned}
$$

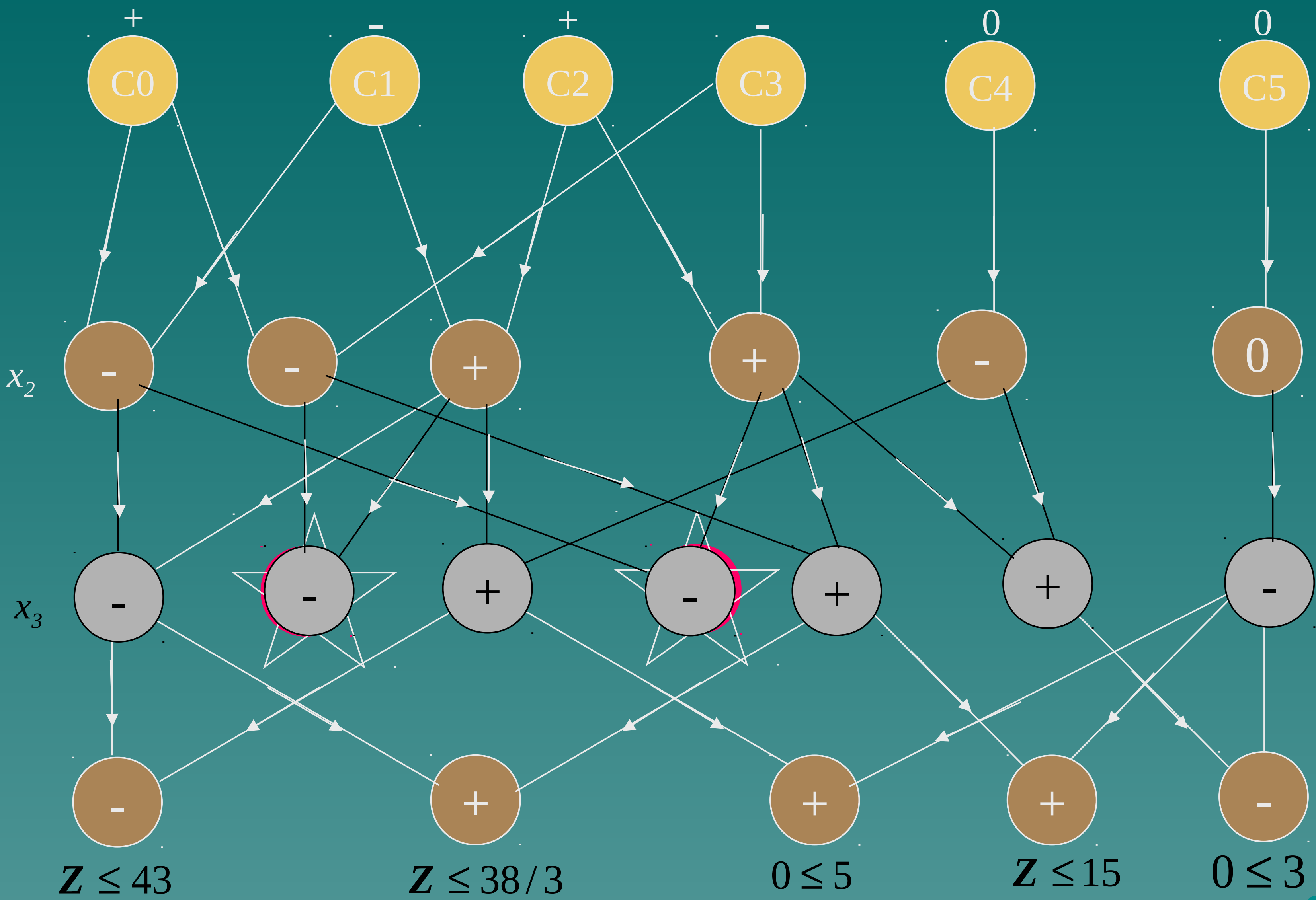
i.e.
$$
\begin{array}{lll}
-x_2 + 7x_3 + z & \le 8 & C0 + 4C1 \\
-5x_2 - 3x_3 + z & \le 0 & C0 + 4C3 \\
2x_2 + x_3 & \le 5 & C1 + C2 \\
x_2 + 2x_3 & \le 3 & C2 + C3 \\
-x_2 & \le 0 & C4 \\
-x_3 & \le 0 & C5
\end{array}
$$

Have **added** (in suitable multiples) every inequality in which x_1 has a positive coefficient to every inequality in which x_1 has a negative coefficient



Can continue process to eliminate $\boldsymbol{x_2}$ and $\boldsymbol{x_3}$

This is Fourier-Motzkin.

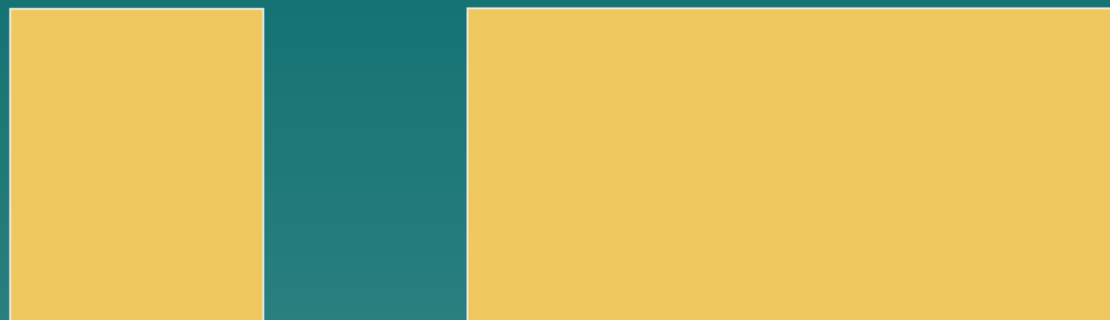


Can be shown (Kohler) that if, after *n* variables have been eliminated, an inequality depends on more than *n+1* of original inequalities it is redundant.

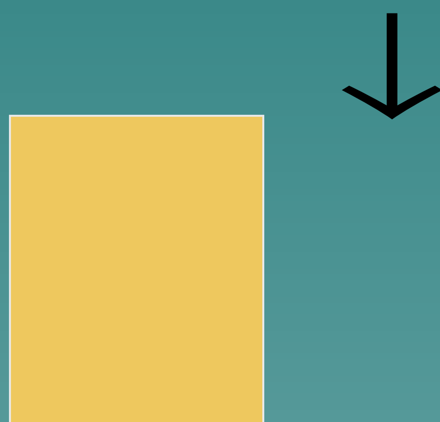
Can choose which variables to project out and order in which to do so.

EXAMPLE OF PARTIAL PROJECTION

Benders' Decomposition



Partially project out some variables (for MIP, continuous variables) to give bound on objective (Benders Cut)



Potentially very large increase in constraints.

Performed partially and iteratively.

Application of Projection to Non-Exponential Formulations of the Travelling Salesman Problem

Example

Sequential Formulation (Miller, Tucker and Zemlin)

$$x_{ij} = 1 \ \ iff \ \ i \rightarrow j \text{ part of tour}$$

$$u_i = \text{sequence number in which city is visited}$$

$$(i = 2,3,\ldots,n)$$

$$\sum_i x_{ij} = 1 \qquad \forall j$$

$$\sum_j x_{ij} = 1 \qquad \forall i$$

$$u_i - u_j + n\, x_{ij} \le n - 1 \qquad \forall i, j \ne 1$$

$0(\boldsymbol{n}^2)$ Constraints and Variables

Project out u_i variables

Gives $\sum_{i,j \in S} x_{ij} \le |S| - \frac{|S|}{n}$ all <u>directed cycles</u>

— $S \subset \{1,2\ldots,n\}$

A relaxation of Conventional Formulation

$\sum_{i,j \in s} x_{ij} \le |S| - 1$ all <u>proper subsets</u>

$S \subset \{1,2\ldots,n\}$

Other non-exponential formulations give modifications of subtour elimination constraints of (exponential) conventional formulation

Single Commodity Flow Formulation gives

$$\sum_{i,j\in S} x_{ij} \le |S| - \frac{|S|}{n-1} \quad \text{all } S \subset \{1,2,\ldots,n\}$$

Modified Single Commodity Flow Formulation

$$\frac{1}{n-1}\sum_{\substack{i\in \bar{S}-\{1\}\\ j\in S}} x_{ij} + \sum_{\substack{i\in S\\ j\in \bar{S}-(1)}} x_{ij} \le |S| - \frac{|S|}{n-1}$$

Multi Commodity Flow Formulation

$$\sum_{i,j \in S} x_{ij} \leq |S| - 1$$

i.e. of equal strength to Conventional Flow Formulation

Time Staged Formulation

$$\frac{1}{n-1} \sum_{\substack{i \in \bar{S} - \{1\} \\ j \in S}} x_{ij} + \frac{1}{n-1} \sum_{\substack{i \in S \\ j \in \bar{S} - (1)}} x_{ij} + \sum_{i,j \in S} x_{ij} \leq |S| - \frac{|S|}{n-1}$$

INVERSE PROJECTION

Apply the dual procedures to eliminate constraints (as opposed to variables).

-

$$\text{Minimise} \quad 2y_1 + 3y_2$$

$$\text{subject to:} \quad \begin{aligned} -y_1 &+ y_2 \geq -4 \\ y_1 &+ y_2 \geq 5 \\ -y_1 &+ 2y_2 \geq 3 \end{aligned}$$

$$y_1, y_2 \geq 0$$

Write in form

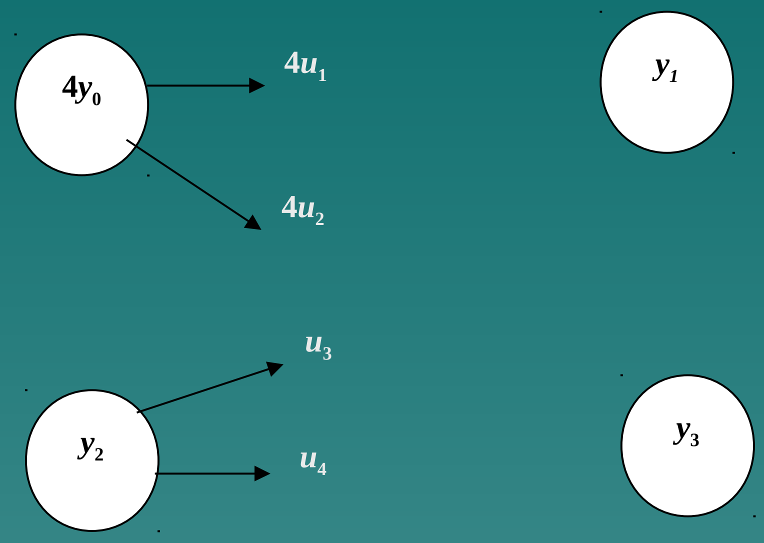
$$\text{Minimise} \quad 2y_1 + 3y_2$$

$$\begin{aligned}
\text{subject to:} \quad & 4y_0 - y_1 + y_2 - y_3 = 0 \\
& -5y_0 + y_1 + y_2 - y_4 = 0 \\
& -3y_0 - y_1 + 2y_2 - y_5 = 0 \\
& y_0 = 1
\end{aligned}$$

$$y_{1,}\ y_2,\ y_{3,}\ y_{4,}\ y_5 \geq 0$$

Eliminate <u>homogeneous</u> constraints by adding columns (in suitable multiples) where coefficients have opposite sign.

Implemented by a transformation of variables.



First homogeneous constraint vanishes.

Substitute

$$
\begin{aligned}
4y_0 &= 4u_1 + 4u_2 \\
y_1 &= 4u_1 + u_3 \\
y_2 &= u_3 + u_4 \\
y_3 &= 4u_2 + u_4
\end{aligned}
$$

Model becomes

Minimise $8u_1 + 5u_3 + 3u_4$

subject to

$$-u_1 + 5u_2 + 2u_3 + u_4 - y_4 = 0$$

$$-7u_1 - 3u_2 + u_3 + 2u_4 - y_5 = 0$$

$$u_1 + u_2 = 1$$

$u_1, u_2, u_3, u_4, y_4, y_5 \geq 0$

Repeat with transformations to eliminate second and third homogeneous constraints.

Results in

Minimise $43w_1 + 5w_2 + {}^{38}/_3 w_3 + 15w_4 + 3w_5$

$$w_1 + w_3 + w_4 = 1$$

$$w_1, w_2, w_3, w_4, w_5 \geq 0$$

Applying corresponding transformations to identity matrix gives relation between final and original variables

	w_1	w_2	w_3	w_4	w_5
y_0	1	0	1	1	0
y_1	11	1	$7/3$	0	0
y_2	7	1	$8/3$	5	1
y_3	0	0	$13/3$	9	1
y_4	13	2	0	0	1
y_5	0	1	0	7	2

Gives all <u>vertices</u> and <u>extreme rays</u> of model

		y_1	y_2	
C:	$w_1 = 1$ gives	(11	7)	Objective = 43
	w_2 gives extreme ray	(1	1)	
B:	$w_3 = 1$ gives	($^7/_3$	$^8/_3$)	Objective = $^{38}/_3$
A:	$w_4 = 1$ gives	(0	5)	Objective = 15
	w_5 gives extreme ray	(0	1)	

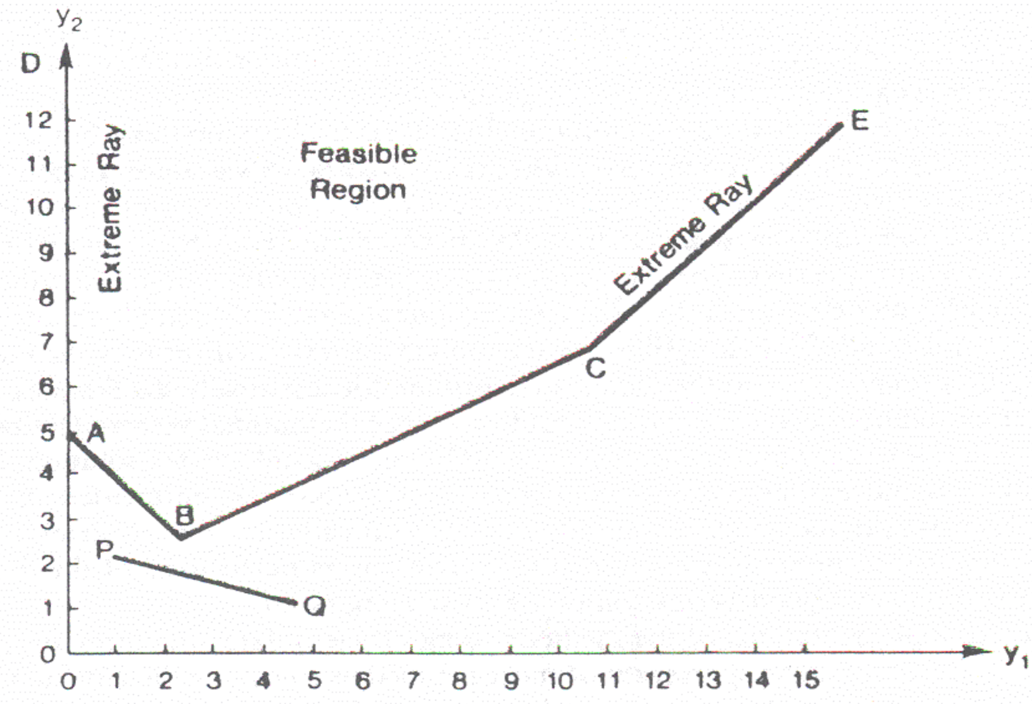


FIG. 3

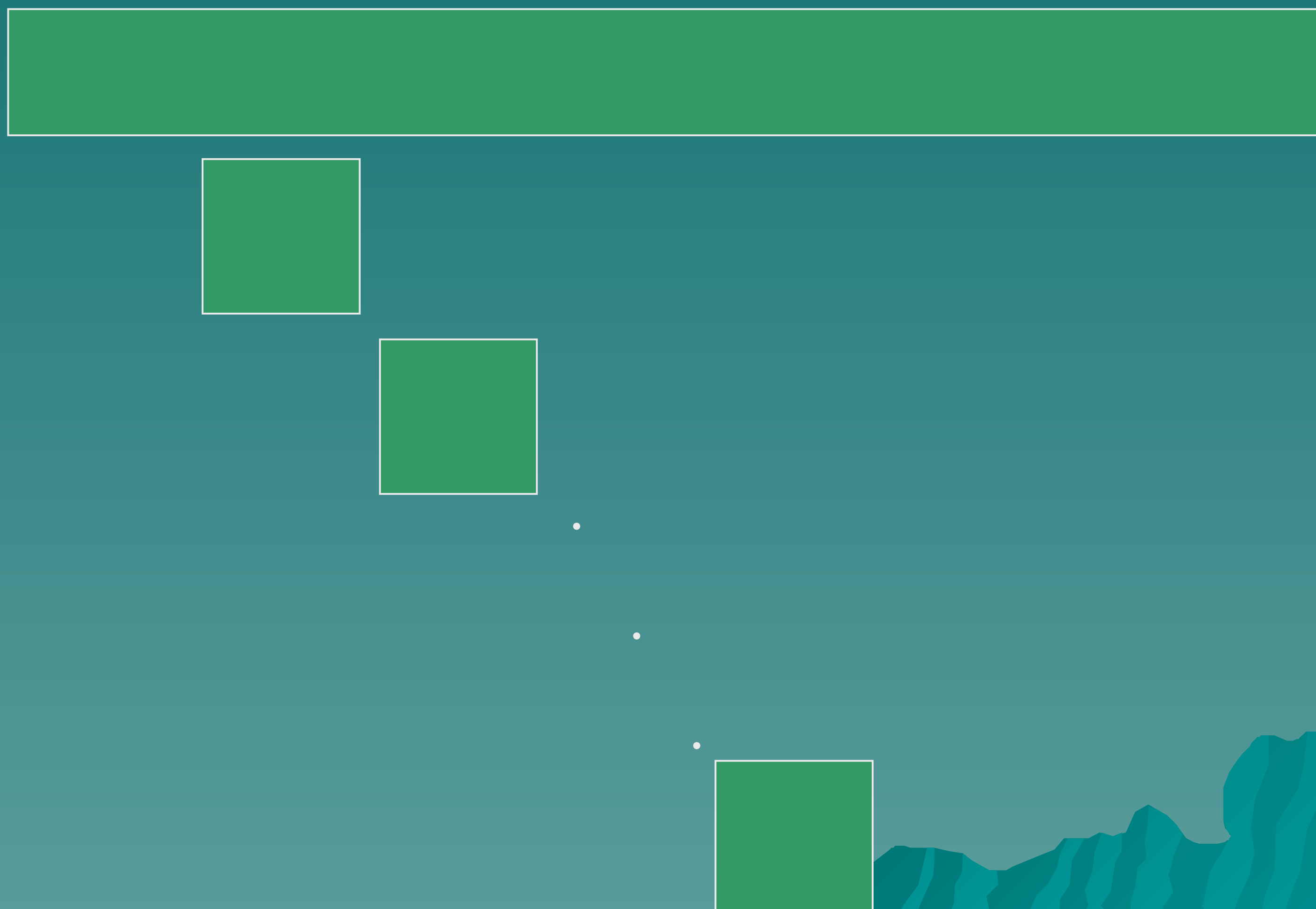
NB: Transformations mirror those of (Primal) Projection.

Redundant transformations if variable depends on more than ***n*+1** of original variables after ***n*** constraints eliminated.

Example of Inverse Projection

(On some constraints)

Dantzig-Wolfe Decomposition



Project out subproblems into single (convexity) constraints

1111….1

1111….

.

.

.

1111….1

Also **Modal Formulations**

i.e. Variables represent **extreme modes** of processes rather than quantities

NB Inverse Projection is **not** the same as Reverse Projection (not well defined).

INTEGER PROJECTION

The Elimination of Integer Variables

$$-ax + f \le 0$$

$$bx + g \le 0$$

a, b positive integers

x an integer variable

f and ***g*** linear expressions

$$\updownarrow$$

$$bf \le abx \le -ag$$

$$\downarrow \not\uparrow$$

$$bf \le -ag \qquad i.e. \qquad bf + ag \le 0$$

Need to state condition “a multiple of ***ab*** lies between ***bf*** and ***–ag***”

Can be done in a **finite** way by

$$bf + bs \le -ag, \quad f + s \equiv 0 \pmod{a}$$
$$s \in \{0,1,2,\ .\ .\ .\ ,a-1\}$$

i.e.

$$bf + ag + bs \le 0, \ f + s \equiv 0 \pmod{a}$$
$$s \in \{0,1,2,\ .\ .\ .\ ,a-1\}$$

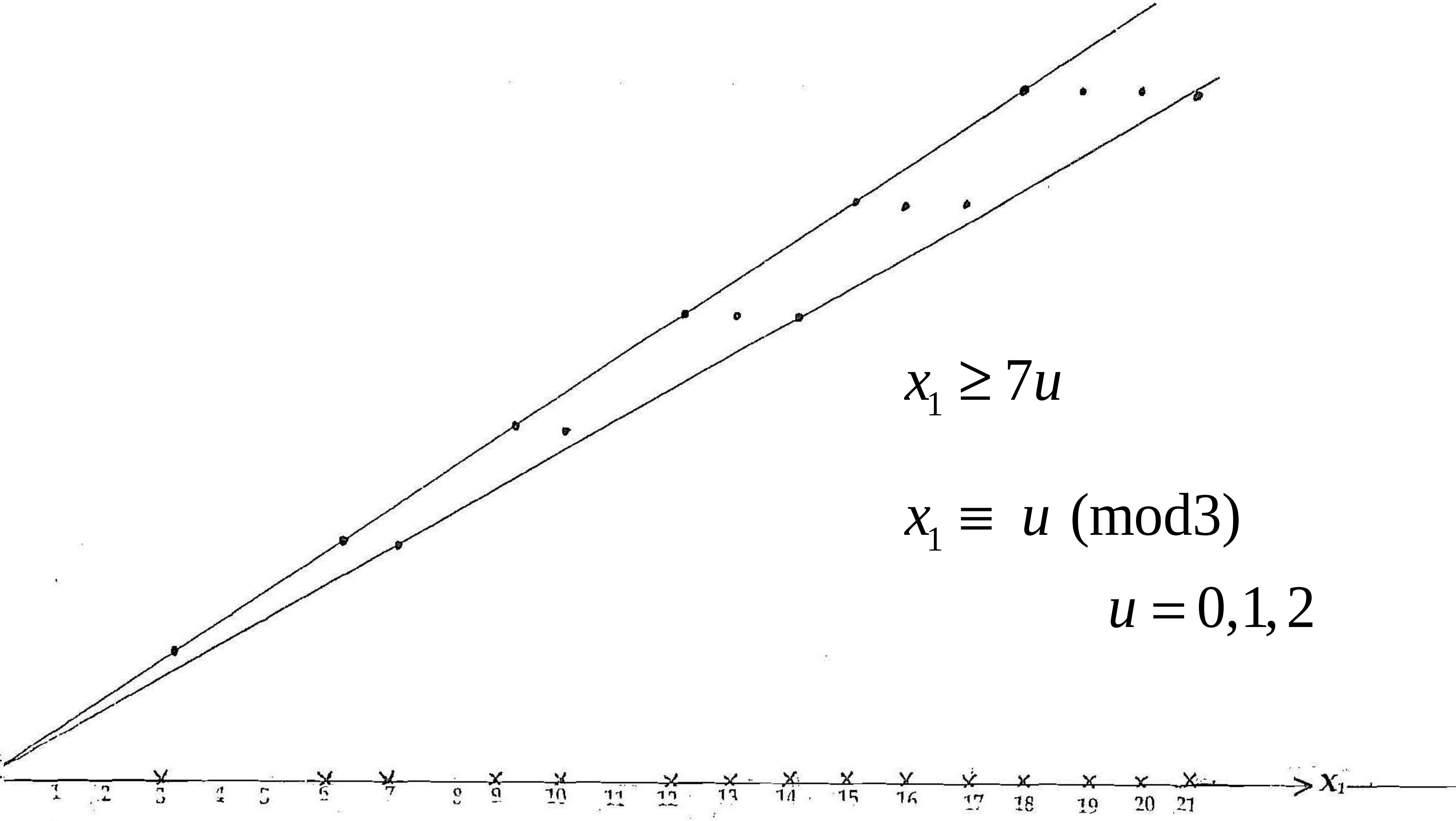
Alternatively

$$bf + ag + at \le 0, g + t \equiv 0 \pmod{b}$$
$$t \in \{0,1,2,\ .\ .\ .\ ,t-1\}]$$

Presburger Arithmetic I.e. Arithmetic “without multiplication”)

Projection of an IP may not produce an IP in lower dimension

e.g. Project out x_2



$$x_1 \geq 7u$$

$$x_1 \equiv u \pmod{3}$$

$$u = 0, 1, 2$$

But if coefficient of ***x*** is unity in one of inequalities can apply F-M elimination

$$\exists x \in Z : \quad f \leq x \qquad ax \leq g \qquad f, g \text{ integer expressions}$$

$$\updownarrow$$

$$\exists x \in Z : \quad af \leq ax \leq g$$

$$\downarrow\uparrow$$

$$af \leq g$$

NB The projection of an IP generally does not result in an IP in a lower dimension

INTEGER PROJECTION

Example **Minimize** z

$$z - 5x_1 - 2x_2 \geq 0$$

$$2x_1 + 4x_2 \geq 5$$

$$6x_1 + 2x_2 \geq 5$$

$$-6x_1 + 2x_2 \geq -15$$

$$9x_1 - 3x_2 \geq -10$$

i.e.

$$5(-4x_2 + 5) \leq 10x_1 \leq 2(z - 2x_2)$$

$$5(-2x_2 + 5) \leq 30x_1 \leq 6(z - 2x_2)$$

$$5(3x_2 - 10) \leq 45x_1 \leq 9(z - 2x_2)$$

$$3(-4x_2 + 5) \leq 6x_1 \leq 2x_2 + 15$$

$$-2x_2 + 5 \leq 6x_1 \leq 2x_2 + 15$$

$$2(3x_2 - 10) \leq 18x_1 \leq 3(2x_2 + 15)$$

Eliminate x_1

$$
\begin{aligned}
2z + 16x_2 &\geq 25 + 2u \\
6z - 2x_2 &\geq 25 + 6u \\
9z - 33x_2 &\geq -50 + 9u \\
14x_2 &\geq 0 + v \\
4x_2 &\geq -10 + v \\
0 &\geq -65 + v
\end{aligned}
$$

$$
\begin{aligned}
z - 2x_2 &\equiv u \pmod{5} \qquad & u \in \{0, \ldots, 4\} \\
2x_2 &\equiv 3 + v \pmod{6} \qquad & v \in \{0, \ldots, 5\}
\end{aligned}
$$

Eliminate x_2 (taking into account congruence relations)

$$50z \geq 225 + 50u + 8w$$

$$210z \geq 25 + 210u + 16s$$

$$126z \geq -700 + 126u + 33v + 14s$$

$$12z \geq 40 + 12u + v + 2w$$

$$36z \geq -530 + 36u + 33v + 4s$$

$$42z \geq 175 + 42u + v + 7w$$

$$0 \geq -65 + v$$

$$v \equiv 1 \pmod{2}$$

$$v + w \equiv 0 \pmod{2}$$

$$v + w \equiv 2 \pmod{3} \qquad u \in \{0,\ldots,4\}$$

$$w \equiv 0 \pmod{5} \qquad v \in \{0,\ldots,5\}$$

$$s \equiv 2 \pmod{3} \qquad w \in \{0,\ldots,29\}$$

$$s \equiv 0 \pmod{5} \qquad s \in \{0,\ldots,989\}$$

$$6v + 5s \equiv 7 \pmod{9}$$

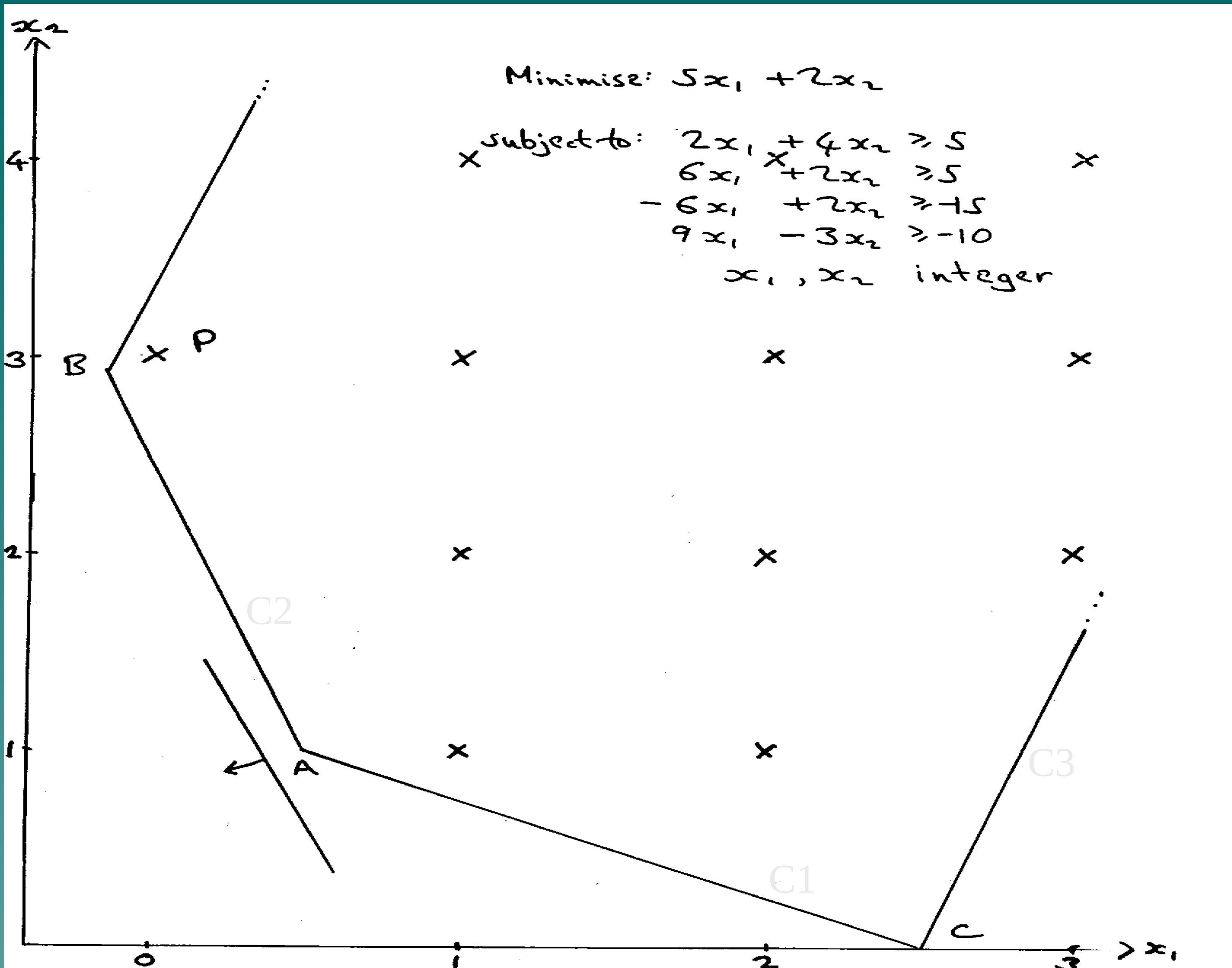
$$4z + 7u + 2s \equiv 1 \pmod{11}$$

Optimal Solution

$$\boldsymbol{z} = 6, \ \boldsymbol{x}_1 = 0, \boldsymbol{x}_2 = 3$$

$$\boldsymbol{u} = 0, \ \boldsymbol{v} = 3, \boldsymbol{w} = 5, \ \boldsymbol{s} = 5$$

Optimal Solution a *Lattice* Point within Polytope. Reduces problem from an infinite number to a *finite* number of solutions.


Minimise: 5x1 + 2x2
subject to: 2x1 + 4x2 ≥ 5
6x1 + 2x2 ≥ 5
−6x1 + 2x2 ≥ −15
9x1 − 3x2 ≥ −10
x1 , x2 integer
x2
x1
0
1
2
3
4
B
P
A
C
C1
C2
C3

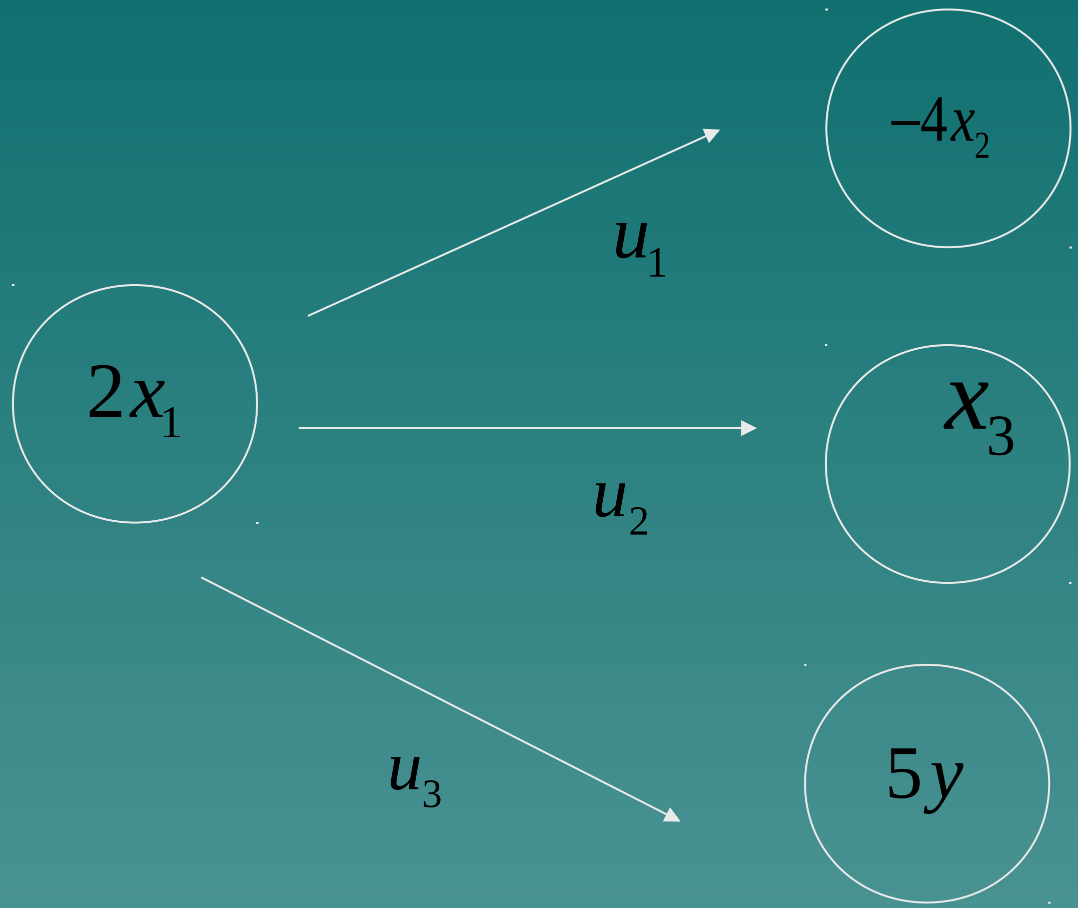
Inverse Integer Projection

Example **Minimize** z $5\boldsymbol{x}_1 + 2\boldsymbol{x}_2$

Subject to:

$$2x_1 + 4x_2 - x_3 - 5y = 0$$
$$6x_1 + 2x_2 - x_4 - 5y = 0$$
$$-6x_1 + 2x_2 - x_5 + 15y = 0$$
$$9x_1 - 3x_2 - x_6 + 10y = 0$$
$$y = 1$$

$$x_1, x_2 \text{ integer}, x_3 x_4 x_5 x_6 \geq 0$$


$2x_1$
u_1
u_2
u_3
$-4x_2$
x_3
$5y$

Eliminate constraint 1 by a transformation of variables

$$x_1 = \frac{1}{2}\left(u_1 + u_2 + u_3\right)$$

$$x_2 = \frac{1}{4}u_1 \qquad \text{(NB: } x_1, x_2, y \text{ not sign constrained)}$$

$$x_3 = u_2 \qquad u_2 \geq 0 \; (u_1 u_3 \text{ not sign constrained)}$$

$$y = \frac{1}{2}u_3$$

$$u_1 + u_2 + u_3 \equiv 0 \pmod{2}$$

$$u_1 \equiv 0 \pmod{4}$$

$$u_3 \equiv 0 \pmod{5}$$

Leads to:

Minimise $$\frac{1}{2}\left(4\boldsymbol{u}_1 + 5\boldsymbol{u}_2 + 5\boldsymbol{u}_3\right)$$

Subject to:
$$\begin{aligned}
5u_1 + 6u_2 + 4u_3 - 2x_4 &= 0 \\
7u_1 + 6u_2 + 2x_5 &= 0 \\
21u_1 + 18u_2 + 26u_3 - 4x_6 &= 0 \\
u_3 &= 5
\end{aligned}$$

$$\begin{aligned}
u_1 + u_2 + u_3 &\equiv 0 \ (\mathrm{mod}2) \\
u_1 &\equiv 0 \ (\mathrm{mod}4)
\end{aligned}$$

$$x_4, x_5, x_6, \geq 0 \qquad (u_1, u_3 \ \text{ not sign constrained})$$

Eliminate constraints 2,3,4

Leads to:

Minimise $$\frac{1}{251160}\left(253w_1 + 259w_2 + 345w_3 + 351w_4\right)$$

Subject to:

$$42\boldsymbol{w}_2 + 23\boldsymbol{w}_3 + 65\boldsymbol{w}_4 = 209300$$

$$23\boldsymbol{w}_1 - 7\boldsymbol{w}_2 + 69\boldsymbol{w}_3 + 39\boldsymbol{w}_4 \equiv 0 \ (\mathbf{mod\ 251160})$$

$$23\boldsymbol{w}_1 + 49\boldsymbol{w}_2 \qquad\qquad 26\boldsymbol{w}_4 \equiv 0 \ (\mathbf{mod\ 83720})$$

$$\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4 \geq 0$$

Optimal LP Solution. $w_4 = 3220$ Objective = 4.5

Optimal IP Solution. $w_1 = 630,\ w_2 = 4830,\ w_3 = 280$ objective 6

giving $x_1 = 0,\ x_2 = 3$ (at P) giving $x_1 = 0,\ x_2 = 3$ (At P)

N.B. Could replace congruence conditions by Mixed Integer Constraints.

Feasible Solutions defined by ***Convexity Constraint + Congruence Conditions***

Illustrates Hilbert Basis result

Alternatively we could scale variables and write in form

Minimise $$\frac{1}{9041760}\left(3036\boldsymbol{w}_1{}^1+185\boldsymbol{w}_2{}^1+450\boldsymbol{w}_3{}^1+162\boldsymbol{w}_4{}^1\right)$$

Subject to: $w_2{}^1+w_3{}^1+w_4{}^1 \ =1$

$$w_1{}^1-4550w_2{}^1+16380w_3{}^1+16380w_4{}^1 \qquad \equiv 0 \ (\text{mod } 32760)$$
$$w_1{}^1+15925w_2{}^1 \qquad \equiv 0 \ (\text{mod } 10920)$$

$$w_1{}^1, w_2{}^1, w_3{}^1, w_4{}^1 \geq 0$$

$w_1{}^1$ corresponds to the extreme ray

and $w_2{}^1, w_3{}^1, w_4{}^1$ to vertices

Optimal Solution: $w_1{}^1=1890$

$w_2{}^1=0.9692$

(Objective = 6)

$w_3{}^1=0.0307$

$w_4{}^1=0$

Projection and

Sign Patterns

+		-
+		-
:		:
+	or	-
0		0
0		0
:		:
0		0

Can remove columns and all constraints in which non-negative entry.

+ + × × + 0 0 × 0 × ≤

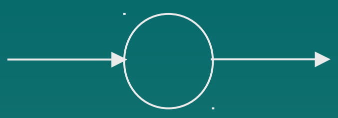
- - × × - 0 0 × 0 × ≥

Can remove constraints and all variables in which non-negative entry

+ 1 0 0 M M 0

Can add constraints in suitable multiple and remove variable

Eg Network Flow



can remove node

Integer Variable (and other variables integer)

1		-1
1		-1
M		M
1		-1
–		+
–	or	+
:		:
–		+
0		0
0		0
:		:
0		0

Can regard variable as continuous

And generalisations of above

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