Second Homework Assignment

November 6, 2024

Please type down your answer using Latex. The deadline of this homework is the 21st of November, 2024.

You can discuss among yourselves, but everyone should write down his/her own answers. Please also indicate the persons with whom you have worked.

Question 1

Consider the following matching problem. Let $G = (A \cup B, E)$ be the given bipartite graph. People (A) are to be assigned to the hospitals (B). In particular, $a \in A$ can be assigned to $b \in B$ only if (a, b) is an edge in E.

A person can only be assigned to one hospital. However, each $b \in B$ can potentially take several persons, but under a matroid constraint. That is, each b has a matroid $\mathcal{M}_b$ over its neighbors on the A-sdie. The set of persons assigned to b must be an independent set in $\mathcal{M}_b$.

Prove that the following is a necessary and sufficient condition for the existence of a A-perfect matching (i.e. all people in A are assigned) is the following:

For each subset $A' \subseteq A$, $\sum_{b \in B} rank_{M_b}(N(b) \cap A') \geq |A'|$.

rado's independent transversal theorem

Note that here $N(b)$ means the neighbours of b. It should be easy to observe that this is a generalisation of Hall's marriage theorem.

I suggest that you use Edmonds' mini-max formula on matroid intersection to solve this exercise (but you don't have to).

Question 2

Let us consider a generalisation of the classic knapsack problem. Suppose that a set $\mathcal{I}$ of items are given, where each item $i \in \mathcal{I}$ comes with a profit $p(i)$ and a weight $w(i)$. We have k identical knapsacks, in each of which we can pack items with the total weight of at most W. The objective is to pack items into these k knapsacks so as to maximise the total profit.

Of course this problem is NP-hard even when $k = 1$. But remember that when $k = 1$, there is a FPTAS (fully-polynomial-time approximation scheme), namely, we can

obtain a $1-\epsilon$ approximation using $O(n^3/\epsilon)$ time. When $k > 1$, let us do something naive: we just apply the same FPTAS k times. Every time we pretend that we have only 1 knapsack and apply the same algorithm on the remaining items (those that are not already packed into previous knapsacks). Prove that this algorithm gives us the approximation ratio of $1 - \frac{1}{e} - O(\epsilon)$.

Question 3

In this exercise, we show a very cool application of Gomory-Hu tree.

Edmonds showed that the following linear program describes the convex hull of all perfect matchings in a general graph $G = (V, E)$.

$$\begin{aligned} x_e &\geq 0, \forall e \in E \\ \sum_{e \in \delta(v)} x_e &= 1, \forall v \in V \\ \sum_{e \in \delta(U)} x_e &\geq 1, \forall U \subseteq V, |U| \text{ is odd}, |U| \geq 3 \end{aligned}$$

This fact implies that to find a maximum weight perfect matching, given $w : E \to \mathbb{R}_{\geq 0}$, we just need to optimize over $\sum_{e \in E} w(e) x_e$. And this can be done by Ellipsoid algorithm—but there is a trouble: the number of inequalities is exponential. So we have to design a separation oracle. Given a fractional x, we need to find out which inequality is violated. The first two sets of inequalities are trivial to check. But the third set is problematic.

Let us proceed as follows: define the capacity of an edge $e \in E$ as x_e. The question then boils down to how to find an odd set $U \in V$ whose cut size $\sum_{e \in \delta(U)} x_e$ is minimized.

Let us do the following. First build a Gomory-Hu tree $H = (V, F)$ on all vertices. For each edge $f \in F$, define V_f as the set of vertices corresponding to either of the two components in H, after f is removed. Suppose that U is the odd set with the minimum cut-size. Let $f_1, f_2, \cdots, f_t \in F$ denote the edges connecting a vertex in U and a vertex in $V \backslash U$ in H.

- Prove that the symmetric differences of $V_{f_1}, V_{f_2}, \cdots, V_{f_t}$ is equivalent to either U or $V \backslash U$.
- Then use this fact to design an algorithm to check whether some odd set has its cut size less than 1.