Homework 2

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1 Question 1

■ **Notation 1.1** For $I \subseteq E$ and $b \in B$, we will denote $I(b) = \{a \in A \mid (a, b) \in I\}$ and by $I(X) = \{a \in X \mid \exists b \in B, (a, b) \in I\}$.

We then define the matroids $\mathbb{A} = (E, \mathcal{A})$, $\mathbb{B} = (E, \mathcal{B})$ where :

$$\begin{aligned}\mathcal{A} &= \{I \subseteq E \mid |I(a)| \leq 1 \forall a \in A\} \\ \mathcal{B} &= \{I \subseteq E \mid I(b) \in \mathcal{M}_b \forall b \in B\}\end{aligned}$$

We then see that $M \subseteq E$ is a A-perfect matching if and only if $|M| = |A|$ and M is an independent set of $\mathcal{A}$ and $\mathcal{B}$. Thus, we will call sets in $\mathcal{A} \cap \mathcal{B}$ independent matchings.

Then, since $|A| \geq \max_{I \in \mathcal{A}} |I|$, from Edmonds' mini-max formula on matroid intersection, we just need to have $\min_{I \subseteq E} r_{\mathcal{A}}(I) + r_{\mathcal{B}}(E \setminus I) \geq |A|$ to have the existence of a A-perfect matching.

We define $s : 2^E \to \mathbb{N}$ as :

$$s(I) = \sum_{b \in B} rank_{M_b}(I(b) \cap N(b)) \tag{1}$$

We see that the rank set in $\mathcal{B}$ can be seen as the ranks on each component (by separating edges on the $b \in \mathcal{B}$ they are connected to). Indeed, since $\mathcal{B}$ can be seen as a union of matroids (the M_b seen as matroids on the edges connected to b) we have, for $I \subseteq E$:

$$r_{\mathcal{B}}(I) = \min_{T \subseteq I} |I \setminus T| + s(T) = \min_{T \subseteq I} |I| - |T| + s(T)$$

Then plugging this into our main equation :

$$\begin{aligned}r_{\mathcal{A}}(E \setminus I) + r_{\mathcal{B}}(I) =& r_{\mathcal{A}}(E \setminus I) + \min_{T} |I| - |T| + s(T) \\ \geq& \min_{T} |I| - |T| + s(T) \\ =& \min_{T} |A| - |T(A)| + s(T)\end{aligned}$$

But since this should be greater than $|A|$ for all T and all I, it is equivalent to being true for all possible $A' = T(A)$ (and modifying the *type* of s accordingly, which doesn't change anything) and thus :

$$\boxed{\max_{I \in \mathcal{A} \cap \mathcal{B}} |I| = |A| \iff \forall A' \subseteq A, s(A') - |A'| \geq 0}$$

which is the wanted result.

2 Question 2

Let $F = 2^I$ and let us denote by $g : 2^{\mathcal{F}} \to \mathbb{R}^+$ the function that to a family of sets gives their combined profit. Clearly, g is submodular. Furthermore we denote by X_0 the empty set, and by X_i the set of items taken after i knapsacks were filled by our algorithm. Since we apply the FPTAS k times, and since g is submodular, we have :

$$g(X_i) - g(X_{i-1}) \geq (1-\varepsilon)\frac{OPT - g(X_{i-1})}{k} \tag{2}$$

for each i, where OPT is the weight of an optimal solution. Then, we have :

$$g(X_1) - g(X_0) = g(X_1) \geq (1-\varepsilon)\frac{OPT}{k} = OPT(1 - \left(1 - \frac{1}{k}\right) - \varepsilon) = OPT\left(1 - \left(1 - \frac{1}{k}\right) - \mathcal{O}(\varepsilon)\right) \tag{3}$$

and then :

$$\begin{aligned} g(X_2) \geq (1-\varepsilon)\frac{OPT - g(X_1)}{k} &= (1-\varepsilon)OPT\left(1 - \left(1 - \frac{1}{k}\right) - \varepsilon\right) \\ &= OPT\left(1 - \left(1 - \frac{1}{k}\right)^2 - \varepsilon\right) - OPT \times \varepsilon\left(1 - \left(1 - \frac{1}{k}\right) - \varepsilon\right) \\ &= OPT\left(1 - \left(1 - \frac{1}{k}\right)^2 - \mathcal{O}(\varepsilon)\right) \end{aligned}$$

By induction :

$$g(X_i i) \geq OPT\left(1 - \left(1 - \frac{1}{k}\right)^i - \mathcal{O}(\varepsilon)\right)$$

And thus :

$$g(X_k) \geq OPT\left(1 - \left(1 - \frac{1}{k}\right)^k - \mathcal{O}(\varepsilon)\right) \geq OPT\left(1 - \frac{1}{e} - \mathcal{O}(\varepsilon)\right)$$

3 Question 3

3.1 Part 1

Let us have two edge-cuts $U_1 = (A, V \setminus A = A^{\complement})$ and $V_2 = (B, B^{\complement})$. Then, $U_1 \Delta U_2$ is an edge-cut :

$$U_1 \Delta U_2 = \left(A \cup B \setminus (A \cap B), (A \cup B)^{\complement} \cup (A \cap B)\right) = \left(A \cup B \setminus (A \cap B), (A \cup B \setminus (A \cap B))^{\complement}\right) = \left(A \Delta B, (A \Delta B)^{\complement}\right)$$

This, in particular, means that $\Delta_{i \in [\![1,t]\!]} V_{f_i}$ is a cut set. Let us then see that the choice of V_f or it's complement $V^{\complement} = V \setminus V_f$ does not change the result. This is obvious when looking at the edge-cut equivalent to $(V, V^{\complement})$ in G since V and $V^{\complement}$ define the same edge-cut. Indeed, since an edge-cut can be seen as both $(A, A^{\complement})$ and $(A^{\complement}, A)$, we know the choice of V_f or $V \setminus V_f$ does not change anything.

3.2 Part 2

This algorithm takes :

$$\mathcal{O}\left(\underbrace{(n-1) \times \text{max-flow}}_{\text{Gomory-Hu algorithm}} + \underbrace{n^2}_{\text{Check Sizes}} + \underbrace{n}_{\text{Retrieve Cut-size}}\right)$$

Algorithme 1 Minimum Odd Size Cut

- First,we build the Gomory-Hu tree of our graph.
- Then, for each edge in the tree we consider both components formed by removing the edge.
- For every odd-sized such component, we retrieve the cut size (the label of the edge in the Gomory-Hu tree), if it's less than one we return True. If none are of cut size ≤ 1 then we return false.

For correctness we just need to show that one of the cuts determined by T is a minimum odd cut in G. To do so, we just need to see that if $\delta(U)$ is a min odd cut in G, then one of the V_f must be odd. Indeed, if there is only one V_f, then $V_f = U$ or $V \setminus V_f$ and we have the result. Else, if there are multiple edges in the path from u to v, then either $|V_{f_1}|$ or $|V \setminus V_{f_1}|$ is odd (and we have the result), or $|V_{f_2}|$ is odd. Indeed, if $f_1 = (a, b)$ and $f_2 = (b, c)$, then if $a \in V_{f_1}$ and $a \in V_{f_2}$, $|V_{f_2}| = |V_{f_1}| + 1$ since we add b to the part of the cut containing a and remove it from the other part. In the other naming cases, we could verify a similar equality with -1 if we lose b and $+1$ if we gain b. Finally, if $f \in F$ $\delta_G(V_f)$ is the minimum odd $u - v$ cut, since $\delta_G(U)$ is a $u - v$ cut, $\delta_G(V_f)$ is a minimum odd-cut.