

# Homework 2

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## 1 Question 1

■ **Notation 1.1** For  $I \subseteq E$  and  $b \in B$ , we will denote  $I(b) = \{a \in A \mid (a, b) \in I\}$  and by  $I(X) = \{a \in X \mid \exists b \in B, (a, b) \in I\}$ .

We then define the matroids  $\mathbb{A} = (E, \mathcal{A})$ ,  $\mathbb{B} = (E, \mathcal{B})$  where :

$$\begin{aligned}\mathcal{A} &= \{I \subseteq E \mid |I(a)| \leq 1 \forall a \in A\} \\ \mathcal{B} &= \{I \subseteq E \mid I(b) \in \mathcal{M}_b \forall b \in B\}\end{aligned}$$

We then see that  $M \subseteq E$  is a  $A$ -perfect matching if and only if  $|M| = |A|$  and  $M$  is an independent set of  $\mathcal{A}$  and  $\mathcal{B}$ . Thus, we will call sets in  $\mathcal{A} \cap \mathcal{B}$  independent matchings.

Then, since  $|A| \geq \max_{I \in \mathcal{A}} |I|$ , from Edmonds' mini-max formula on matroid intersection, we just need to have  $\min_{I \subseteq E} r_{\mathcal{A}}(I) + r_{\mathcal{B}}(E \setminus I) \geq |A|$  to have the existence of a  $A$ -perfect matching.

We define  $s : 2^E \rightarrow \mathbb{N}$  as :

$$s(I) = \sum_{b \in B} \text{rank}_{M_b}(I(b) \cap N(b)) \quad (1)$$

We see that the rank set in  $\mathcal{B}$  can be seen as the ranks on each component (by separating edges on the  $b \in \mathcal{B}$  they are connected to). Indeed, since  $\mathcal{B}$  can be seen as a union of matroids (the  $M_b$  seen as matroids on the edges connected to  $b$ ) we have, for  $I \subseteq E$  :

$$r_{\mathcal{B}}(I) = \min_{T \subseteq I} |I \setminus T| + s(T) = \min_{T \subseteq I} |I| - |T| + s(T)$$

Then plugging this into our main equation :

$$\begin{aligned}r_{\mathcal{A}}(E \setminus I) + r_{\mathcal{B}}(I) &= r_{\mathcal{A}}(E \setminus I) + \min_T |I| - |T| + s(T) \\ &\geq \min_T |I| - |T| + s(T) \\ &= \min_T |A| - |T(A)| + s(T)\end{aligned}$$

But since this should be greater than  $|A|$  for all  $T$  and all  $I$ , it is equivalent to being true for all possible  $A' = T(A)$  (and modifying the *type* of  $s$  accordingly, which doesn't change anything) and thus :

$$\boxed{\max_{I \in \mathcal{A} \cap \mathcal{B}} |I| = |A| \iff \forall A' \subseteq A, s(A') - |A'| \geq 0}$$

which is the wanted result.

## 2 Question 2

Let  $F = 2^I$  and let us denote by  $g : 2^F \rightarrow \mathbb{R}^+$  the function that to a family of sets gives their combined profit. Clearly,  $g$  is submodular. Furthermore we denote by  $X_0$  the emptyset, and by  $X_i$  the set of items taken after  $i$  knapsacks were filled by our algorithm. Since we apply the FPTAS  $k$  times, and since  $g$  is submodular, we have :

$$g(X_i) - g(X_{i-1}) \geq \frac{OPT - g(X_{i-1})}{k} \quad (2)$$

for each  $i$ , where  $OPT$  is the weight of an optimal solution. Then, we have :

$$g(X_1) - g(X_0) = g(X_1) \geq \frac{OPT}{k} = OPT \left(1 - \left(1 - \frac{1}{k}\right)\right) \quad (3)$$

and then :

$$g(X_2) \geq OPT \left(1 - \left(1 - \frac{1}{k}\right)^2\right)$$

By induction :

$$g(X_i) \geq OPT \left(1 - \left(1 - \frac{1}{k}\right)^i\right)$$

And thus :

$$g(X_k) \geq OPT \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \geq OPT \left(1 - \frac{1}{e}\right)$$

## 3 Question 3

### 3.1 Part 1

### 3.2 Part 2

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#### Algorithm 1 Minimum Odd Size Cut

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- First, we build the Gomory-Hu tree of our graph.
  - Then, for each edge in the tree we consider both components formed by removing the edge.
  - For every odd-sized such component, we retrieve the cut size (the label of the edge in the Gomory-Hu tree), if it's less than one we return True. If none are of cut size  $\leq 1$  then we return false.
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This algorithm takes :

$$\mathcal{O} \left( \underbrace{(n-1) \times \text{max-flow}}_{\text{Gomory-Hu algorithm}} + \underbrace{n^2}_{\text{Check Sizes}} + \underbrace{n}_{\text{Retrieve Cut-size}} \right)$$

For correctness we just need to show that one of the cuts determined by  $T$  is a minimum odd cut in  $G$ . To do so, we just need to see that if  $\delta(U)$  is a min odd cut in  $G$ , then  $U$  or  $V \setminus U$  is the symmetric difference of the  $V_f$  for  $f \in \delta_F(U)$ . Hence,  $|V_f|$  must be odd for at least one of the  $f \in \delta_F(U)$  or  $|U|$  would be even. So  $\delta_G(V_f)$  is an odd cut. If  $f = (u, v)$ ,  $\delta_G(V_f)$  is a minimum  $u - v$  cut and as  $\delta_G(U)$  is  $u - v$  cut,  $\delta_G(V_f)$  is a minimum odd-cut.