Homework Assignment 2

Matthieu Boyer

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Table des matières

1 Exercise 1

1.1 Question 1

For i, j in $|1, n|^2$, we have $AB_{i,j} = A_{i*}B_{*j} = \sum_{k=1}^{n} A_{i,k}B_{k,j}$, thus AB is computed by computed, for all i, j the product $A_{i,k}B_{k,j}$ and thus uses at most $\sum_{k=1}^{n} a_k b_k$ multiplications.

1.2 Question 2

The number of multiplications and additions is two times the number of multiplications. We just need to get a majoration of the number of multiplications. Yet, since $a_k \leq n$ for all k, $\sum_{k=1}^{n} a_k b_k \leq n \sum_{k=1}^{n} b_k = mn$. Then the number of multiplications and additions required is $\mathcal{O}(mn)$.

1.3 Question 3

Multiplying a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ can, by seeing the matrices as block matrices, be seen as multiplying two matrices in $\mathcal{M}_p$ the number of times we need to compute the product of matrix in $\mathcal{M}_{a,b}$ by a matrix in $\mathcal{M}_{b,c}$:

$$\begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,bp} \\ \vdots & & \vdots \\ \alpha_{ap,1} & \dots & \alpha_{ap,bp} \end{pmatrix} = \begin{pmatrix} A_{1,1} & \dots & A_{1,b} \\ \vdots & & \vdots \\ A_{a,1} & \dots & A_{a,b} \end{pmatrix} \text{ where } A_{i,j} = \begin{pmatrix} \alpha_{ip+1,jp+1} & \dots & \alpha_{ip+1,j(p+1)} \\ \vdots & & \vdots \\ \alpha_{i(p+1),jp+1} & \dots & \alpha_{i(p+1),j(p+1)} \end{pmatrix} \in \mathcal{M}_p$$

Thus, we can multiply a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ in $M(a,b,c)M(p,p,p)$ multiplications. Thus :

$$M(ap, bp, cp) \leq M(a,b,c)M(p,p,p)$$

1.4 Question 4

— If $0 \leq r \leq \alpha$: $w(1,r,1)$ is the smallest number k such that $M(n, n^r, n) = \mathcal{O}(n^{k+o(1)})$. But, again by seeing A an $n \times n^\alpha$ matrix as a $n \times n^r$ matrix next to a $n, n^\alpha - n^r$ matrix and same for B, we get $M(1,r,1) \leq M(1,\alpha,1)$ and thus $w(1,r,1) \leq w(1,\alpha,1) = 2$.

— If $\alpha \leq r \leq 1$: by seeing a $n \times n^r$ matrix A as a $n^{\frac{1-r}{1-\alpha}} \times n^{\frac{(1-r)\alpha}{1-\alpha}}$ bloc matrix with blocks of size $n^{\frac{r-\alpha}{1-\alpha}} \times n^{\frac{r-\alpha}{1-\alpha}}$ and applying the reasoning from 3. we get that :

$$\begin{aligned}
M(n, n^r, n) &= M\left(n^{\frac{1-r}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}}\right. \\
&\qquad \left. \times n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{1-r}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&\leq M\left(n^{\frac{1-r}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}}, n^{\frac{1-r}{1-\alpha}}\right) \\
&\qquad \times M\left(n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&= \mathcal{O}\left(\left(n^{\frac{1-r}{1-\alpha}}\right)^{w(1,\alpha,1)} \left(n^{\frac{r-\alpha}{1-\alpha}}\right)^{\omega}\right) \\
&= \mathcal{O}\left(n^{\frac{2*(1-r)+(r-\alpha)\omega}{1-\alpha}}\right)
\end{aligned}$$

The first big O equality comes from a substitution in $M(n, n^\alpha, n) = \mathcal{O}(n^{w(1,\alpha,1)})$ of n by $n^{\frac{1-r}{1-\alpha}}$. We obtain :

$$\begin{aligned}
w(1, r, 1) &\leq \frac{2 \times (1-r) + (r-\alpha)\,\omega}{1-\alpha} \\
&= \frac{2 \times (1-\alpha) + 2 \times (\alpha - r) + (r-\alpha)\,\omega}{1-\alpha} \\
&= 2 + \frac{\omega \times (\alpha - r) - 2 \times (\alpha - r)}{1-\alpha} \\
&= 2 + \beta(r - \alpha)
\end{aligned}$$

1.5 Question 5

Let $1 \leq l \leq n$, a_k, b_k such that $a_k b_k$ is decreasing. If $l = 1$, the result is trivial.
Consider for $i \in [\![1, l-1]\!]$ the quantity $a_i b_j + a_j b_i$, we get :

— If $a_i > a_j$: $a_i b_j + a_j b_i > a_j b_j$

— Else : $a_i b_j + a_j b_i > a_i b_i > a_j b_j$ by hypothesis.

Then we get :

$$\sum_{j=i+1}^{n} a_i b_j + a_j b_i > \sum_{k=l}^{n} a_k b_k$$

since the a_k and b_k are positive. By summing :

$$\sum_{i=1}^{l-1} \sum_{j=i+1}^{n} a_i b_j + a_j b_i > l \sum_{k=l}^{n} a_k b_k$$

But :

$$m_1 m_2 = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j = \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i b_l > \sum_{i=1}^{l-1} \sum_{j=i+1}^{n} a_i b_j$$

Thus :

$$\sum_{k=l}^{n} a_k b_k < \frac{m_1 m_2}{l}$$

1.6 Question 6

We will first prove correctness by induction :
Initialization : If $m^2 \leq n^2$, it is clear.

Heredity : Since π is bijective, $I \sqcup J = [\![1, n]\!]$, we get that no two columns in A_{*I} and A_{*J} have an element in the same spot. Then, if $i, j \in [\![1, n]\!]$, we get :

$$[AB]_{i,j} = \sum_{k \in [\![1,n]\!]} a_{i,k} b_{k,j} = \sum_{k \in I} a_{i,k} b_{k,j} + \sum_{k \in J} a_{i,k} b_{k,j} = [A_{*I} B_{I*}]_{i,j} + [A_{*J} B_{J*}]_{i,j}$$

Hence, the algorithm is correct.

Then, for the complexity we get that :

- If $m \leq n$, the algorithm runs in $\mathcal{O}(mn)$.
- Else, its complexity is the sum of three steps :
  - Calculating $A_{*I} B_{I*}$ takes $M(n, l, n) = n^{w(1,r,1)+o(1)}$ multplications for $r = \frac{\ln l}{\ln n} < 1$.
  - From question 5), calculating $A_{*J} B_{J*}$ takes at most $\frac{m^2}{l}$ multiplications.
  - Summing the results takes $\mathcal{O}(n^2)$ operations.

We obtain a time complexity in the worst case in $\mathcal{O}(n^{w(1,r,1)+o(1)} + \frac{m^2}{l} + n^2)$