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The Moebius strip and a social choice paradox

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Abstract

A remarkable topological property of the Moebius strip is equivalent to the Chichilnisky impossibility theorem in social choice theory, which states the impossibility of finding a continuous, anonymous and unanimous aggregation rule for two agents, defined on the unit circle.

Keywords: Continuous, unanimous, and anonymous aggregation rules in social choice; Moebius strip; Property of non-retractibility to the boundary

JEL classification: O25

1. Introduction

In the mathematical approach to social choice, topology is often used to deal with preference spaces [see, for example, Kelly (1971), Chichilnisky (1979, 1980, 1982a,b, 1991), McMannus (1982), Chichilnisky and Heal (1983), Le Breton et al. (1985), or Rasmussen (1992)]. Each element of the space represents the preference of, say, an economic agent, or a voter. A typical problem in this framework is concerned with the aggregation of preferences, i.e. how to find social choice rules that aggregate individual preferences.

There exists a wide range of models to catch the idea of common sense in the search for a rule that aggregates individual preferences. Perhaps the best known is the Arrowian one [Arrow (1951)], leading to the Arrow impossibility theorem. [For a more general view on this subject, consult the excellent books by Sen (1970), Fishburn (1973) and Kelly (1988).]

In this paper the aggregation problem is studied within a topological framework introduced by Chichilnisky (1979, 1980), in which the social choice rules are continuous, anonymous and respect

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unanimity. The mathematical formulation of this problem consists of looking for a map $F: X^n \rightarrow X$, X being the topological space of preferences, satisfying the following properties:

Continuity (endowing X^n with the product topology).

Anonymity: $F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)$, for every rearrangement $(y_1, \ldots, y_n)$ of the n-tuple $(x_1, \ldots, x_n)$.

Respect of unanimity: $F(x, \ldots, x) = x$, for every x in X.

The map F is said to be a *topological Chichilnisky aggregation n-rule* or *topological n-mean* defined on X [see Chichilnisky (1980) or Rasmussen (1992)].

The use of such models is motivated by the fact that they provide good aggregation results [see Chichilnisky and Heal (1983)]. However, there exists a class of spaces for which there is no Chichilnisky aggregation rule, even for the case of two agents. The term *social choice paradox* refers to some space of preferences X, and some values of n, such that an aggregation n-rule from X^n in X fails to exist. In this connection, Chichilnisky (1979) proved that for $X = S^1$ (the unit circle, with its usual topology), and for any $n \geq 2$, we have a social choice paradox.

The proof given in Chichilnisky (1979) is concerned with the case $n = 2$, and then a generalization for $n > 2$ follows. The main idea in the original Chichilnisky proof was the use of items related to fixed point theory.

Our aim in this paper is to present an alternative way to prove the paradox for $n = 2$.

This new proof leans on the properties of the *Moebius strip*, a counterintuitive geometrical example of a surface with one side only. *Perhaps our contribution is the first application to social choice theory of the properties of the Moebius strip.*

Remark. Several other proofs and generalizations of the paradox are known. For instance, in Chichilnisky (1980) a new proof was given using the theory of foliations, and the impossibility result was generalized to k-dimensional spheres (S^k).

In fact, the *Chichilnisky impossibility result* actually holds for any number of voters and on any Euclidean space of choices of any dimension, with or without linear preferences [see Chichilnisky (1982a)]. The case considered here $(n = 2,\ X = S^1)$ is only a special case.

Moreover, in Chichilnisky and Heal (1983) the following elegant and deep result has been proved: 'If the space of preferences X is a parafinite CW-complex, then the existence, for every $n \in \mathbb{N}$, of a topological n-Chichilnisky rule of aggregation is equivalent to the contractibility of X'. This result is known as the *resolution of the social choice paradox*.

2. Some mathematical background

If we take a narrow piece of paper and after a 180° twist we glue its shorter edges, we obtain a model of the *Moebius strip*, a counterintuitive surface with one side only.

This space has the *property of non-retractibility to the boundary*. This means that it is impossible to obtain a continuous deformation of the Moebius strip to its boundary, such that all the points in the boundary remain fixed. We shall come back to this fact, and prove it, in the next section. The property of non-retractibility is also shared by other topological spaces. A well-known example is the unit disk in the plane: a deformation of a disk onto its boundary, keeping every point in the boundary fixed, must break the disk. This fact is equivalent to Brouwer's fixed point theorem for the disk [see Massey (1989)], and it was the key to Chichilnisky's first proof of her theorem of impossibility [see Chichilnisky (1979)].

3. The two-agents social choice paradox for the space S^1

The one-dimensional sphere S^1 can be considered as being a space of preferences. For instance, in Chichilnisky (1982a,b) S^1 is identified with the set of non-trivial linear preferences defined on the real plane and endowed with the topology induced by the smooth preferences. [That topology was introduced in Debreu (1972).]

Let us prove now the non-existence of a topological Chichilnisky aggregation 2-rule from $S^1 \times S^1$ into S^1.

To do so, notice that the existence of a topological n-rule on a sphere of preferences X is equivalent to the existence of a continuous map $F : X^n/\mathscr{R} \to X$, such that F maps the class corresponding to $(x, \dots, x)$ on the element x, for every $x \in X$. Here, $\mathscr{R}$ is the equivalence relation defined on X^n such that two n-tuples are equivalent if and only if the second one is a rearrangement of the first one. This is an easy property of quotient topologies which, in the social choice context, was already pointed out in Baigent (1984).

In our situation (two factors $X = S^1$) it is enough to identify the quotient space $(S^1 \times S^1)/\mathscr{R}$ and study the existence of a map from that space into S^1, conserving the diagonal.

S^1 can be topologically understood as an interval I (e.g. $I = [0, 1]$) whose extreme points are identified (i.e. $0 = 1$).

$S^1 \times S^1$ is a *torus* (Fig. 1). We denote the diagonal (i.e. the set of points $\{(x, x); x \in I\}$) with the symbol $\triangle$. The side marked a is the set of points $\{(x, 0)$ or $(x, 1); x \in I\}$. The side marked b is the set of points $\{(0, x)$ or $(1, x); x \in I\}$. Thus our surface can be understood as a square on which we glue, following the arrows, the horizontal sides (marked a), and the vertical sides (marked b).

First we glue the sides marked b to obtain a cylinder (Fig. 2). Then we glue the circle bases (marked a) of the cylinder to obtain a torus. The diagonal $\triangle$ is a circle on the torical surface, and the point P belongs to $\triangle$ (Fig. 3).

But in the torical surface we must apply the equivalence $\mathscr{R}$, identifying points (x, y) $(x, y \in S^1)$ to points (y, x). Therefore the points in region (1) are glued to the corresponding points in region (2). Moreover the points $(x, 0)$ in side a are glued to points $(0, x)$ in side b, i.e. an identification appears on the boundary. The quotient space becomes Fig. 4.

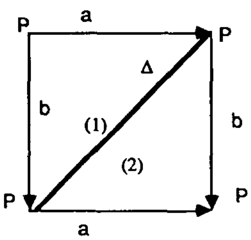

Fig. 1.

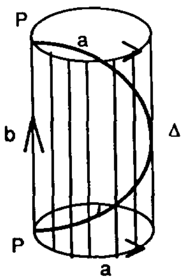

Fig. 2.

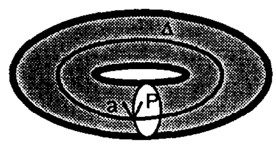

Fig. 3.

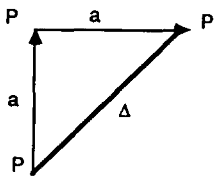

Fig. 4.

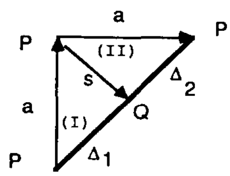

Fig. 5.

This quotient space is a *Moebius strip*. To see this we must glue the sides marked a following the arrows. To do so, first we consider an auxiliary cut (marked s) from P to a point Q in the diagonal $\triangle$. This additional cut divides the surface into two regions, marked (I) and (II). It also splits the diagonal into two parts, marked $\triangle_1$ and $\triangle_2$ (Fig. 5).

We separate the regions (I) and (II), and then glue the sides marked a (Fig. 6). Finally we glue, following the arrows, the sides marked s, and obtain a Moebius strip whose boundary is the diagonal $\triangle$ (Fig. 7).

We conclude the proof of the main result. Observe that a Chichilnisky 2-rule, f, in the quotient interpretation, would be a continuous map from the Moebius strip into itself, and such that $f(x, x) = x$ for every x in the diagonal $\triangle$ (the diagonal that corresponds to the boundary of the strip).

Identifying with S^1 the boundary of the strip through the homeomorphism that maps (x, x) onto x, the existence of a 2-rule would be equivalent to the existence of a continuous mapping g from the Moebius strip onto its boundary, whose restriction to the boundary is the identity map.

This would be equivalent to claiming that the boundary of the Moebius strip is a deformation retract of the whole strip, which is well known to be false.

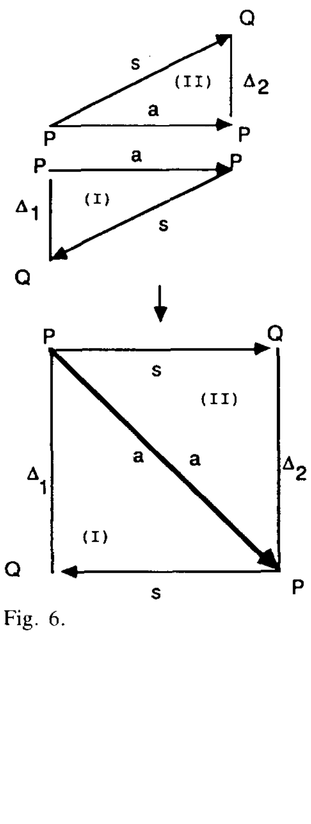

Fig. 6.

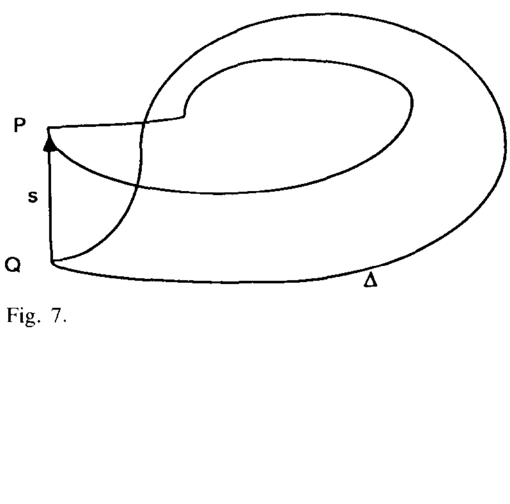

Fig. 7.

Let us prove this last assertion. Let M denote the Moebius strip, BM its boundary, and S the central line of the strip, which is a circle. The Moebius strip, M, clearly is continuously retractable on its central line. Denote by s the retraction from M to S, by i the embedding on S in M, by r the (possible) retraction from M onto BM, and by j the topological embedding of BM into M.

Let s^*, i^*, r^* and j^*, respectively, be the maps that s, i, r and j induce on the fundamental homotopy groups $\Pi(S)$, $\Pi(M)$ and $\Pi(BM)$. We then have that

(a) $\Pi(S)$ is isomorphic to $(\mathbb{Z}, +)$, the additive group of integers, because the fundamental group of a circle is infinitely cyclic [see section 5 of ch. 2 in Massey (1989)].

(b) $\Pi(M) = \mathbb{Z}$, because S is a deformation retract of M [see Massey (1989, pp. 66–67)].

(c) $\Pi(BM) = \mathbb{Z}$, because BM is topologically homeomorphic to a circle.

(d) s^* and i^* are isomorphisms, and $s^* = (i^*)^{-1}$ [see Massey (1989, p. 66)]. As an immediate consequence, either $i^*(1) = 1$ or $i^*(1) = -1$.

(e) r^*j^* is the identity map, r^* is an epimorphism and j^* is a monomorphism [see Massey (1989, p. 65)].

(f) $s^*j^*(1) = 2$. [To see this, it is enough to compute the number of loops of the boundary of the Moebius strip with respect to the central circle of the strip. Moreover, the pair (BM, S) is an example of a two-fold covering space of S^1 onto S^1. The projection map in this covering space is the composition $s \circ j$. Such a projection induces on the homotopy groups the homomorphism $\Pi(BM) = \mathbb{Z} \to \Pi(S) = \mathbb{Z}$ that sends 1 to 2. See Massey (1989, pp. 68–73 and p. 146).]

Now observe that an epimorphism from $\mathbb{Z}$ to $\mathbb{Z}$ must actually be an isomorphism. Consequently r^* is, in fact, an isomorphism.

Finally, we know that r^*j^* is the identity map, so $r^*(2) = r^*[i^*s^*(2)] = r^*i^*[s^*j^*(1)] = r^*i^*(1)$. Thus, either $r^*(2) = r^*(1)$ or $r^*(2) = r^*(-1)$.

Therefore r^* is not injective and cannot be an isomorphism. A contradiction. □

Remark. Actually, it is not difficult to prove that the particular case studied here of the Chichilnisky impossibility result (i.e. $n = 2$ and $X = S^1$) is *equivalent* to the theorem that asserts that the boundary of a Moebius strip is not a continuous deformation retract of the whole strip. Thus, we could reobtain a result in pure mathematics from a result in social choice. In a different context, the equivalence between mathematical theorems and results in economics (e.g. in General Equilibrium theory) was already pointed out in chs. 9 and 21 of Border (1985).

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