Homework 1

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1 Question 1

1.1 Question 1(a)

Initially, we have $d^* = 0$. Moreover, we always have $d^* \geq 0$ and an increase of d^* is caused by a relabeling. Thus, d^* can only increase $2n^2$ times (the maximum number of relabelings) and decrease as many times.
$\boxed{\text{There are thus at most } 4n^2 \text{ phases.}}$

1.2 Question 1(b)

- Relabeling v causes $\bar{d}(v)$ to increase but cannot cause $\bar{d}(w)$ to increase if $w \neq v$.
  $\boxed{\text{Thus, relabeling a node increases } \Phi \text{ by at most } \dfrac{n}{K}.}$
- A saturating push creates at most one new active node.
  $\boxed{\text{Thus, a saturating push increases } \Phi \text{ by at most } \dfrac{n}{K}.}$
- A nonsaturating push across the edge (u, v) deactivates node u and might activate node v. Then we have $\bar{d}(v) \leq \bar{d}(u)$, and hence a nonsaturating push does not increase Φ. During heavy phases, we execute $\rho > K$ nonsaturating pushes. Since d^* is constant during the phase, all ρ nonsaturating pushes must be from nodes at level d^*. Indeed, we choose nodes from the highest level, thus d^*. The phase terminates when eiither when all nodes in level d^* are deactivated or when relabeling moves a node to level $d^* + 1$. Level d^* thus contains $\rho > K$ nodes (either active or inactive) throughout the phase. Hence, each nonsaturating push decreases Φ by at least one, since $\bar{d}(v) \leq \bar{d}(u) - 1$ for (u, v) with $|\{w \mid d(w) = d(u)\}| \geq K$.