

Homework Assignment 2

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Table des matières

| | | |
|----------|----------------------|----------|
| 1 | Exercise 1 | 1 |
| 1.1 | Question 1 | 1 |
| 1.2 | Question 2 | 1 |
| 1.3 | Question 3 | 1 |
| 1.4 | Question 4 | 1 |
| 1.5 | Question 5 | 2 |

1 Exercise 1

1.1 Question 1

For i, j in $[1, n]^2$, we have $AB_{i,j} = A_{i*}B_{*j} = \sum_{k=1}^n A_{i,k}B_{k,j}$, thus AB is computed by computed, for all i, j the product $A_{i,k}B_{k,j}$ and thus uses at most $\sum_{k=1}^n a_k b_k$ multiplications.

1.2 Question 2

The number of multiplications and additions is two times the number of multiplications. We just need to get a majoration of the number of multiplications. Yet, since $a_k \leq n$ for all k , $\sum_{k=1}^n a_k b_k \leq n \sum_{k=1}^n b_k = mn$. Then the number of multiplications and additions required is $\mathcal{O}(mn)$.

1.3 Question 3

Multiplying a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ can, by seeing the matrices as block matrices, be seen as multiplying two matrices in \mathcal{M}_p the number of times we need to compute the product of matrix in $\mathcal{M}_{a,b}$ by a matrix in $\mathcal{M}_{b,c}$:

$$\begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,bp} \\ \vdots & & \vdots \\ \alpha_{ap,1} & \dots & \alpha_{ap,bp} \end{pmatrix} = \begin{pmatrix} A_{1,1} & \dots & A_{1,b} \\ \vdots & & \vdots \\ A_{a,1} & \dots & A_{a,b} \end{pmatrix} \text{ where } A_{i,j} = \begin{pmatrix} \alpha_{ip+1,jp+1} & \dots & \alpha_{ip+1,j(p+1)} \\ \vdots & & \vdots \\ \alpha_{i(p+1),jp+1} & \dots & \alpha_{i(p+1),j(p+1)} \end{pmatrix} \in \mathcal{M}_p$$

Thus, we can multiply a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ in $M(a, b, c)M(p, p, p)$ multiplications. Thus :

$$M(ap, bp, cp) \leq M(a, b, c)M(p, p, p)$$

1.4 Question 4

- If $0 \leq r \leq \alpha$: $w(1, r, 1)$ is the smallest number k such that $M(n, n^r, n) = \mathcal{O}(n^{k+o(1)})$. But, again by seeing A an $n \times n^\alpha$ matrix as a $n \times n^r$ matrix next to a $n, n^\alpha - n^r$ matrix and same for B , we get $M(1, r, 1) \leq M(1, \alpha, 1)$ and thus $w(1, r, 1) \leq w(1, \alpha, 1) = 2$.

- If $\alpha \leq r \leq 1$: by seeing a $n \times n^r$ matrix A as a $n^{\frac{1-r}{1-\alpha}} \times n^{\frac{(1-r)\alpha}{1-\alpha}}$ bloc matrix with blocks of size $n^{\frac{r-\alpha}{1-\alpha}} \times n^{\frac{r-\alpha}{1-\alpha}}$ and applying the reasoning from 3. we get that :

$$\begin{aligned}
M(n, n^r, n) &= M\left(n^{\frac{1-r}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&\leq M\left(n^{\frac{1-r}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}}, n^{\frac{1-r}{1-\alpha}}\right) \\
&\quad \times M\left(n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&= \mathcal{O}\left(\left(n^{\frac{1-r}{1-\alpha}}\right)^{w(1,\alpha,1)} \left(n^{\frac{r-\alpha}{1-\alpha}}\right)^{\omega}\right) \\
&= \mathcal{O}\left(n^{\frac{2*(1-r)+(r-\alpha)\omega}{1-\alpha}}\right)
\end{aligned}$$

The first big O equality comes from a substitution in $M(n, n^\alpha, n) = \mathcal{O}(n^{w(1,\alpha,1)})$ of n by $n^{\frac{1-r}{1-\alpha}}$. We obtain :

$$\begin{aligned}
w(1, r, 1) &\leq \frac{2 \times (1-r) + (r-\alpha)\omega}{1-\alpha} \\
&= \frac{2 \times (1-\alpha) + 2 \times (\alpha-r) + (r-\alpha)\omega}{1-\alpha} \\
&= 2 + \frac{\omega \times (\alpha-r) - 2 \times (\alpha-r)}{1-\alpha} \\
&= 2 + \beta(r-\alpha)
\end{aligned}$$

1.5 Question 5

Let $1 \leq l \leq n$, a_k, b_k such that $a_k b_k$ is decreasing. If $l = 1$, the result is trivial.

Consider for $i \in \llbracket 1, l-1 \rrbracket$ the quantity $a_i b_j + a_j b_i$, we get :

- If $a_i > a_j$: $a_i b_j + a_j b_i > a_j b_j$
- Else : $a_i b_j + a_j b_i > a_i b_i > a_j b_j$ by hypothesis.

Then we get :

$$\sum_{j=i+1}^n a_i b_j + a_j b_i > \sum_{k=l}^n a_k b_k$$

since the a_k and b_k are positive. By summing :

$$\sum_{i=1}^{l-1} \sum_{j=i+1}^n a_i b_j + a_j b_i > l \sum_{k=l}^n a_k b_k$$