Homework Assignment 2

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1 Exercise 1

1.1 Question 1

For i, j in $|1, n|^2$, we have $AB_{i,j} = A_{i*}B_{*j} = \sum_{k=1}^{n} A_{i,k}B_{k,j}$, thus AB is computed by computed, for all i, j the product $A_{i,k}B_{k,j}$ and thus uses at most $\sum_{k=1}^{n} a_k b_k$ multiplications.

1.2 Question 2

The number of multiplications and additions is two times the number of multiplications. We just need to get a majoration of the number of multiplications. Yet, since $a_k \leq n$ for all k, $\sum_{k=1}^{n} a_k b_k \leq n \sum_{k=1}^{n} b_k = mn$. Then the number of multiplications and additions required is $\mathcal{O}(mn)$.

1.3 Question 3

Multiplying a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ can, by seeing the matrices as block matrices, be seen as multiplying two matrices in $\mathcal{M}_p$ the number of times we need to compute the product of matrix in $\mathcal{M}_{a,b}$ by a matrix in $\mathcal{M}_{b,c}$:

$$\begin{pmatrix} \alpha_{1,1} & \dots & \alpha_{1,bp} \\ \vdots & & \vdots \\ \alpha_{ap,1} & \dots & \alpha_{ap,bp} \end{pmatrix} = \begin{pmatrix} A_{1,1} & \dots & A_{1,b} \\ \vdots & & \vdots \\ A_{a,1} & \dots & A_{a,b} \end{pmatrix} \text{ where } A_{i,j} = \begin{pmatrix} \alpha_{ip+1,jp+1} & \dots & \alpha_{ip+1,j(p+1)} \\ \vdots & & \vdots \\ \alpha_{i(p+1),jp+1} & \dots & \alpha_{i(p+1),j(p+1)} \end{pmatrix} \in \mathcal{M}_p$$

Thus, we can multiply a matrix in $\mathcal{M}_{ap,bp}$ by a matrix $\mathcal{M}_{bp,cp}$ in $M(a, b, c)M(p, p, p)$ multiplications. Thus :

$$M(ap, bp, cp) \leq M(a, b, c)M(p, p, p)$$

1.4 Question 4

— If $0 \leq r \leq \alpha$: $w(1, r, 1)$ is the smallest number k such that $M(n, n^r, n) = \mathcal{O}(n^{k+o(1)})$. But, again by seeing A an $n \times n^\alpha$ matrix as a $n \times n^r$ matrix next to a $n, n^\alpha - n^r$ matrix and same for B, we get $M(1, r, 1) \leq M(1, \alpha, 1)$ and thus $w(1, r, 1) \leq w(1, \alpha, 1) = 2$.

— If $\alpha \leq r \leq 1$: by seeing a $n \times n^r$ matrix A as a $n^{\frac{1-r}{1-\alpha}} \times n^{\frac{(1-r)\alpha}{1-\alpha}}$ bloc matrix with blocks of size $n^{\frac{r-\alpha}{1-\alpha}} \times n^{\frac{r-\alpha}{1-\alpha}}$ and applying the reasoning from 3. we get that :

$$\begin{aligned}
M(n, n^r, n) &= M\left(n^{\frac{1-r}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}}\right. \\
&\quad \left. \times n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{1-r}{1-\alpha}} \cdot n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&\leq M\left(n^{\frac{1-r}{1-\alpha}}, n^{\frac{(1-r)\alpha}{1-\alpha}}, n^{\frac{1-r}{1-\alpha}}\right) \\
&\quad \times M\left(n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}, n^{\frac{r-\alpha}{1-\alpha}}\right) \\
&= \mathcal{O}\left(\left(n^{\frac{1-r}{1-\alpha}} n^{w(1,\alpha,1)}\right)\left(n^{\frac{r-\alpha}{1-\alpha}}\right)^{\omega}\right) \\
&= \mathcal{O}\left(n^{\frac{2*(1-r)+(r-\alpha)\omega}{1-\alpha}}\right)
\end{aligned}$$

We obtain :

$$\begin{aligned}
w(1, r, 1) &\leq \frac{2 \times (1-r) + (r-\alpha)\,\omega}{1-\alpha} \\
&= \frac{2 \times (1-\alpha) + 2 \times (\alpha - r) + (r-\alpha)\,\omega}{1-\alpha} \\
&= 2 + \frac{\omega \times (\alpha - r) - 2 \times (\alpha - r)}{1-\alpha} \\
&= 2 + \beta(r-\alpha)
\end{aligned}$$

1.5 Question 5

Let $1 \leq l \leq n$: