Homework 1

Matthieu Boyer

30 octobre 2024

1 Question 1

1.1 Question 1(a)

Initially, we have $d^* = 0$. Moreover, we always have $d^* \geq 0$ and an increase of d^* is caused by a relabeling. Thus, d^* can only increase $2n^2$ times (the maximum number of relabelings) and decrease as many times.

$\boxed{\text{There are thus at most } 4n^2 \text{ phases.}}$

1.2 Question 1(b)

- Relabeling v causes $\bar{d}(v)$ to increase but cannot cause $\bar{d}(w)$ to increase if $w \neq v$.
  $\boxed{\text{Thus, relabeling a node increases } \Phi \text{ by at most } \frac{n}{K}.}$
- A saturating push creates at most one new active node.
  $\boxed{\text{Thus, a saturating push increases } \Phi \text{ by at most } \frac{n}{K}.}$
- A nonsaturating push across the edge (u, v) deactivates node u and might activate node v. Then we have $\bar{d}(v) \leq \bar{d}(u)$, and hence a nonsaturating push does not increase Φ.

  During heavy phases, we execute $\rho > K$ nonsaturating pushes. Since d^* is constant during the phase, all ρ nonsaturating pushes must be from nodes at level d^*.
  Indeed, we choose nodes from the highest level, thus d^*.
  The phase terminates either when all nodes in level d^* are deactivated or when relabeling moves a node to level $d^* + 1$.
  Level d^* thus contains $\rho > K$ nodes (either active or inactive) throughout the phase.
  Hence, each nonsaturating push decreases Φ by at least one, since $\bar{d}(v) \leq \bar{d}(u) - 1$ for (u, v) with $|\{w \mid d(w) = d(u)\}| \geq K$.
  Finally, a heavy phase of non saturating push will decrease Φ by at least $\rho > K$.

  For light phases, the bound is easier : the number of nonsaturating pushes is bounded K.

1.3 Question 1(c)

The total increase of Φ is bounded by $\frac{(2n^2+2nm)n}{K}$ and so the total decrease cannot be more than that (since $\Phi \geq 0$). Therefore, the number of nonsaturating push cannot be more than $\frac{2n^3+2n^2m}{K}$. The number of non saturating pushes in both phases, is then bounded by :

$$\frac{2n^3 + 2n^2m}{K} + 4n^2K$$

since $4n^2$ is the number of phases (and thus more than the number of light phases).

Finally, since $n = \mathcal{O}(m)$ (the graph being connex $m \geq n-1$ and $n \leq m+1$), taking $K = \sqrt{m}$ we get a complexity in $\mathcal{O}(n^2\sqrt{m})$.

2 Question 2

We will use Ford-Fulkerson's theorem on an appropriate graph to prove this property. Let us write $U = \{a_1, \dots, a_n\}$. We define a set of vertices V by :

$$V = \left\{s, \bar{S}_1, \dots, \bar{S}_k, \bar{a_1}, \dots, \bar{a_n}, \tilde{a_1}, \dots \tilde{a_n}, \bar{T}_1, \dots, \bar{T}_n, t\right\}$$

That is, we have 4 families of vertices that we will connect :

1. $\mathcal{S} = \left\{\bar{S}_j \forall j \leq k\right\}$
2. $\bar{R} = \{\bar{a}_i \forall i \leq n\}$
3. $\tilde{R} = \{\tilde{a}_i \forall i \leq n\}$
4. $\mathcal{T} = \left\{\bar{T}_j \forall j \leq k\right\}$

Then, we add edges with a capacity function $c : E \to \mathbb{R}^+ \cup \{+\infty\}$ and a minimum flow function $l : E \to \mathbb{R}^+$:

- An edge $(s, \bar{S}_i)$ with capacity 1 for each $j \in [\![1, k]\!]$.
- An edge $(s, \tilde{a_i})$ with capacity 0 for each $i \in [\![1, n]\!]$.
- An edge $(\bar{S}_j, \bar{a_i})$ with infinite capacity for i, j such that $a_i \in S_j$.
- An edge $(\bar{a_i}, \tilde{a_i})$ with capacity 1 for each $i \in [\![1, n]\!]$.
- An edge $(\bar{a_i}, t)$ with capacity 0 for each $i \in [\![1, n]\!]$.
- An edge $(\tilde{a_i}, \bar{T}_j)$ with infinite capacity for i, j such that $a_i \in T_j$.
- An edge $(\bar{T}_j, t)$ with capacity 1 for each $j \in [\![1, k]\!]$.
- An edge (s, t) with infinite capacity and an edge (t, s) with minimum flow k and infinite capacity.

Then having a feasible flow on this graph is equivalent to having a common SDR for S and T, since necessarily we have 1 flow going in each S_i (from the fact that $\sum_{v \in \delta^-(s)} f(v, s) = f(t, s) \geq c(t, s) \geq k$) and thus need to choose an edge from S_i to a certain $\bar{a_j}$ (such that $a_j \in S_i$) and that means $\tilde{a_j}$ leaves its flow to a T_l such that $a_j \in T_l$.

Then, by Ford-Fulkerson's max-flow-min-cut theorem, this is possible if and only if, for all subset A of the vertices, we have the following inequality :

$$c(A, A^\complement) = \sum_{e \in E \cap \left(A \times A^\complement\right)} c(e) \geq \sum_{e \in E \cap \left(A^\complement \times A\right)} l(e) = l(A^\complement, A) \tag{1}$$

Here, we have multiple cases to more easily find a closed form for this inequality from the capacity and minimal flows :

1. Either both s and t are in A and then :

$$c(A, A^\complement) = c\left(s, \mathcal{S} \cap A^\complement\right) + c\left(\mathcal{S} \cap A, \bar{R} \cap A^\complement\right) + c\left(\bar{R} \cap A, \tilde{R} \cap A^\complement\right) + c\left(\tilde{R} \cap A, \mathcal{T} \cap A^\complement\right)$$
$$l\left(A^\complement, A\right) = l\left(\bar{R} \cap A^\complement, \tilde{R} \cap A\right) = 0$$

2. Either both s and t are in $A^\complement$ and then :

$$c\left(A, A^\complement\right) = c\left(\mathcal{S} \cap A, \bar{R} \cap A^\complement\right) + c\left(\bar{R} \cap A, \tilde{R} \cap A^\complement\right) + c\left(\tilde{R} \cap A, \mathcal{T} \cap A^\complement\right) + c(\mathcal{T} \cap A, t)$$
$$l\left(A^\complement, A\right) = l\left(\bar{R} \cap A^\complement, \tilde{R} \cap A\right) = 0$$

Both of these cases are symmetrical.

3. If $s \in A^{\complement}, t \in A$ then $c(A, A^{\complement}) \geq c(t,s) = \infty$ so the inequality is always verified.
4. Finally, if $s \in A$ and $t \in A^{\complement}$, we have :

$$c\left(A, A^{\complement}\right) = c\left(s, \mathcal{S} \cap A^{\complement}\right) + c\left(\mathcal{S} \cap A, \bar{R} \cap A^{\complement}\right) + c\left(\bar{R} \cap A, \tilde{R} \cap A^{\complement}\right) + c\left(\tilde{R} \cap A, \mathcal{T} \cap A^{\complement}\right) + c\left(\mathcal{T} \cap A, t\right)$$
$$l\left(A^{\complement}, A\right) = l\left(\bar{R} \cap A^{\complement}, \tilde{R} \cap \mathcal{A}\right) + k = k$$

However, the inequalities of the first two case are less strong than the equality of the last case since :

$$c\left(\mathcal{T} \cap A, t\right) = |\mathcal{T} \cap A| \leq k$$
$$c\left(s, \mathcal{S} \cap A^{\complement}\right) = \left|\mathcal{S} \cap A^{\complement}\right| \leq k$$

Therefore, the inequality 1 reduces to :

$$\left|\mathcal{S} \cap A^{\complement}\right| + c\left(\mathcal{S} \cap A, \bar{R} \cap A^{\complement}\right) + c\left(\bar{R} \cap A, \tilde{R} \cap A^{\complement}\right) + c\left(\tilde{R} \cap A, \mathcal{T} \cap A^{\complement}\right) + |\mathcal{T} \cap A| \geq k$$
$$\left|\mathcal{T} \cap A^{\complement}\right| + |\mathcal{S} \cap A| \leq k + c\left(\mathcal{S} \cap A, \bar{R} \cap A^{\complement}\right) + c\left(\bar{R} \cap A, \tilde{R} \cap A^{\complement}\right) + c\left(\tilde{R} \cap A, \mathcal{T} \cap A^{\complement}\right)$$

So there is a feasible flow if the above inequality stands for all $A \subseteq V$. This inequality is always true if either $\left(\mathcal{S} \cap A, \bar{R} \cap A^{\complement}\right)$ or $\left(\tilde{R} \cap A, \mathcal{T} \cap A^{\complement}\right)$ is not empty. If this is not true, then we know the outgoing edges of any node in $\mathcal{S} \cap A$ go to nodes in $\bar{R} \cap A$ and that all ingoing edges to nodes in $\mathcal{T} \cap A^{\complement}$ come from nodes in $\tilde{R} \cap A^{\complement}$. We then see that the right side is at its lowest point when taking $\bar{R} \cap A = \delta^{+}\left(\mathcal{S} \cap A\right)$ and $\tilde{R} \cap A^{\complement} = \delta^{-}\left(\mathcal{T} \cap A^{\complement}\right)$. Then, we get :

$$|\mathcal{S} \cap A| + \left|\mathcal{T} \cap A^{\complement}\right| \leq n + c\left(\delta^{+}\left(\mathcal{S} \cap A\right), \delta^{-}\left(\mathcal{T} \cap B\right)\right)$$

This being true for all $A \subseteq V$, we can replace $\mathcal{S} \cap A$ for any subset X of $\mathcal{S}$ or even $X \subseteq [\![1,k]\!]$, and similarly we replace $\mathcal{T} \cap A^{\complement}$ by any subset Y of $[\![1,k]\!]$. Then, the equation becomes :

$$|X| + |Y| \leq k + \sum_{I(X) \cap I(Y)} 1$$

which is exactly the result : There is a common set of distinct representatives for $S = \{S_1, \ldots, S_k\}$ and $T = \{T_1, \ldots, T_k\}$ over a same ground set U if and only if, for every $X, Y \subseteq [\![1,k]\!]$ the following inequality is verified :

$$|X| + |Y| \leq k + |I\left(X\right) \cap I\left(Y\right)|$$

---

3 Question 3

We will first prove that all edges in M_1 are in E''. Since by duality we know that one can get a minimum *integral* vertex cover from a maximum *cardinality* matching on a non weighted graph (which is the case of G' since all edges have the same positve non-zero weight), we know that a minimum integral vertex cover on G' will have exactly one vertex per edge of M_1 and edges for unconnected vertices in G'. Thus, E'' will contain all edges of M_1 plus the edges of G of weight 2 that are adjacent to an edge of M_1 and the edges of G of weight 1 that are not touched by the vertex cover, that is, edges that are at least one edge away from M_1.

Let M be a matching in G. Since M_1 is of maximum cardinality, M cannot have more than $|M_1|$ edges of weight 2. Moreover, when constructing M_2 by augmenting in G'', we maximise the matching size while respecting the vertex cover constraints and thus the matching constraints. We have thus a number as large as possible of edges of weight 1. Therefore, M cannot have have more edges than M_2 and thus $w(M) \leq w(M_2)$.

4 Question 4

First, we remind the Tutte-Berge's minimax formula on a network $\mathcal{G} = (V, E)$ in the formulation we will be using :

$$\max_{M \text{ matching}} |M| = \frac{1}{2} \min_{U \subseteq V} \left(|U| - \text{oc}\left(\mathcal{G} \setminus U\right) + |V|\right) \tag{2}$$

where oc(H) is the number of connex components of H with an odd number of vertices (which will be denoted as *odd components*). We know then that a graph $\mathcal{G}$ has a perfect matching if and only if there is a matching of size $\frac{|V|}{2}$, or, plugging it into the formula, if and only if the suppression of any subset U creates at most $|U|$ odd components, i.e., the following condition is true :

$$\forall U \subseteq V, \text{oc}\left(\mathcal{G} \setminus U\right) \leq |U| \tag{3}$$

Now, if the following condition is verified for our graph $G = (A \sqcup B, E)$ where $|A| = |B| = n = \frac{|V|}{2}$:

$$\forall X \subseteq A, |\Gamma\left(X\right)| \geq |X| \tag{4}$$

Consider the graph H built from G by adding edges to make B a clique (complete graph), then if H has a perfect matching, since edges from vertices in A considered in H must go to vertices in B considered as vertices in H, then G has a perfect matching. We will now show that if G satisfies 4 then H has a perfect matching, meaning G has a perfect matching. To do so, we will show that H verifies 3. Suppose there is a certain U such that $\text{oc}\left(H \setminus U\right) > |U|$. Since B seen in H forms a clique, there is a certain component C of $H \setminus U$ which contains all of $B \setminus U$, and the other components are k singletons from $A \setminus U$. There are then two cases : either $|C|$ is odd and $\text{oc}\left(H \setminus U\right) = k + 1$ or $|C|$ is even and $\text{oc}\left(H \setminus U\right) = k$. Since we know that G verifies 4, if we let :

$$S = \{x \in A \mid \Gamma(x) \subseteq B \cap U\}$$

we have

$$k = |S| \leq |B \cap U| \leq |U|$$

and thus $\text{oc}\left(H \setminus U\right) \leq |U| + 1$. However since $|H| = 2n$ is even, $\text{oc}\left(H \setminus T\right)$ has the same parity as $|T|$. Thus $k \leq |U|$ or $\text{oc}\left(H \setminus U\right) \leq |U|$ for any $U \subseteq V$. $\boxed{\text{Finally, } H \text{ has a perfect matching per 3 and thus } G \text{ has a perfect matching.}}$