Algorithmique

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Part I
Lecture 1 - 28/09

Contents

1 Organisation

Mail Tatiana : `starikovskaya@di.ens.fr` Homeworks are 30% of the final grade, final (theory from lecture) Textbooks :

- *Introduction to Algorithms* - Cormen, Leiserson, Riverst, Stein
- *Algorithms on strings, trees, and sequences* - Gusfield
- *Approximation Algorithms* - Vazirani
- *Parametrized Algorithms* - Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Saurabh

2 Introduction

Algorithm take Inputs and give an output.

Open Problem 1 (Mersenne Prime)**.** *Find a new prime of form* $2^n - 1$

Algorithms do not depend on the language. Algorithms should be simple, fast to write and efficient. Word RAM model : Two Parts : one with a constant number of registers of w bits with direct access, and one with any number of registers, only with indirect access (pointers). Allows for elementary operations: basic arithmetic and bitwise operations on registers, conditionals, goto, copying registers, halt and malloc. To index the memory storing input of size n with n words, we need register length to verify $w \geq \log n$ Algorithms can always be rewritten using only elementary operations. Complexity :

- $Space(n)$ is the maximum number of memory words used for input of size n
- $Time(n)$ is the maximum number of *elementary* operations used for input of size n

Complexity Notations :

- $f \in \mathcal{O}(g)$ if $\exists n_0 \in \mathbb{N}, c \in \mathbb{R}_+,\ f(n) \leq c \cdot g(n),\ \forall n \geq n_0$
- $f \in \Omega(g)$ if $\exists n_0 \in \mathbb{N}, c \in \mathbb{R}_+,\ f(n) \geq c \cdot g(n),\ \forall n \geq n_0$
- $f \in \Theta(g)$ if $\exists n_0 \in \mathbb{N}, c_1, c_2 \in \mathbb{R}_+,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n),\ \forall n \geq n_0$

3 Data Structures

3.1 Introduction

Way to store elements of a data base that is created to answer frequently asked queries using pre-processing. We care about space used, construction, query and update time. Can be viewed as an algorithm, which analysed on basics. Containers are basic Data Structures, maintaining the following operations :

1. Random Access : given i, access e_i
2. Access first/last element
3. Insert an element anywher
4. Delete any element

3.2 Array

An array is a pre-allocated contiguous memory area of a *fixed* size. It has random access in $\mathcal{O}(1)$, but doesn't allow insertion nor deletion.

Linear Search : given an integer x return 1 if $e_i = x$ else 0.

Algorithme 1 Linear Search in an Array.
Complexity : Time = $\mathcal{O}(n)$ | Space = $\mathcal{O}(n)$

Input$x$

3.3 Doubly Linked List

Memory area that does not have to be contiguous and consists of registers containing a value and two pointers to the previous and next elements. It has random access in $\mathcal{O}(n)$, access/insertion/deletion at head/tail in $\mathcal{O}(1)$.

Algorithme 2 Insertion in a Doubly Linked List
Complexity : $\mathcal{O}(1)$

InputL, x
$x.next \leftarrow L.head$
if $L.head \neq NIL$ **then**
  $L.head.prev \leftarrow x$
end if
$L.head \leftarrow x$
$x.prev = Nil$

3.4 Stack and Queue

Stack : Last-In-First-Out data structure, abstract data structure. Access/insertion/deletion to top in $\mathcal{O}(1)$.

Open Problem 2 (Optimum Stack Generation)**.** *Given a finite alphabet Σ and $X \in \Sigma^n$. Find a shortest sequence of stack operations push, pop, emit that prints out X. You must start and finish with an empty stack. Current best solution is in $\tilde{\mathcal{O}}(n^{2.8603})$.*

Queue : First-In-First-Out abstract data structure. Access to front, back in $\mathcal{O}(1)$, deletion and insertion at front and back in $\mathcal{O}(1)$.

4 Approaches to algorithm design

Solve smalle sub-problems to solve a large one.

4.1 Dynamic Programming

Break the problem into many closely related sub-problems, memorize the result of the sub-problems to avoid repeated computation

Examples :

Levenshtein Distance between two strings can be computed in $\mathcal{O}(mn)$ instead of exponential time. Based on `https://arxiv.org/pdf/1412.0348.pdf`, this is the best one can do. RNA folding : retrieving the 3D shape of RNA based on their representation as strings. Currently, we know it is possible to find $\mathcal{O}(n^3)$, in $\tilde{\mathcal{O}}(n^{2.8606})$ and if *SETH* is true, there is no $\mathcal{O}(n^{\omega-\varepsilon})$. We know $\omega \in [2, 2.3703)$

Open Problem 3. *Is there a better Complexity for RNA folding ? What is the true value of ω ?*

Knapsack problem : An optimization problem with bruteforce complexity $\mathcal{O}(2^n)$.

Algorithme 3 Recursive Fibonacci Numbers
Complexity: Exponential

RFibo(n) :
Input$n$
if $n \leq 1$ **then**
  return n
end if
return RFibo($n-1$) + RFibo($n-2$)

Algorithme 4 Dynamic Programming Fibonacci Numbers
Time = $\mathcal{O}(n)$ | Space = $\mathcal{O}(n)$

Input$n$
$Tab \leftarrow zeros(n)$ ▷ $zeros(n)$ returns a n-array of zeros.
$Tab[0] \leftarrow 0$
$Tab[1] \leftarrow 1$
for $i \leftarrow 2$ to n **do**
  $Tab[i] = Tab[i-1] + Tab[i-2]$
end for
return Tab[n]

Algorithme 5 Knapsack : Dynamic Programming
Time = $\mathcal{O}(nW)$ | Space = $\mathcal{O}(nW)$

Input W, w, v ▷ Capacity, weight and values vectors.
$KP = zeros(n, W)$
for $i \leftarrow 0$ to n **do**
  $KP[i, 0] = 0$
end for
for $w \leftarrow 0$ to W **do**
  $KP[0, w] = 0$
end for
for $i \leftarrow 0$ to n **do**
  for $w \leftarrow 0$ to W **do**
    if $w < w_i$ **then**
      $KP[i, w] \leftarrow KP[i-1, w]$
    else
      $$KP[i,w] = \max \begin{cases} KP[i-1, w] \\ KP[i-1, w-w_i] + v_i \end{cases}.$$
    end if
  end for
end for
return $KP[n, W]$

4.2 Greedy Techniques

Problems solvable with the greedy technique form a subset of those solvable with DP. Problems must have the optimal substrcture property. Principle : choosing the best at the moment.

Example : The Fractional Knapsack Problem

Algorithm : Iteratively select the greatest value-per-weight ratio.

Théorème 4.2.1. *This algorithm returns the best solution, in time* $\mathcal{O}(n \log n)$

By contradiction. Suppose we have $\frac{v_1}{w_1} \geq \ldots \geq \frac{v_n}{w_n}$. Let $ALG = p = (p_1, \ldots, p_n)$ be the output by the algorithm and $OPT = q = (q_1, \ldots, q_n)$ be optimal.
Assume that $OPT \neq ALG$, let i be the smallesst index such $p_i \neq q_i$. There is $p_i > q_i$ by construct. Thus, there exists $j > i$ such that $p_j < q_j$. We set $q' = (q_1^{'}, \ldots, q_n^{'}) = (q_1, \ldots, q_{i-1}, q_i + \varepsilon, q_{i+1}, \ldots, q_j - \varepsilon \frac{w_i}{w_j}, \ldots, q_n)$
$q^{'}$ is a feasible solution : $\sum_{i=1}^{n} q_i^{'} \cdot w_i = \sum_{i=1}^{n} q_i w_i \leq W$
Yet, $\sum_{i=1}^{n} q_i^{'} \cdot v_i > \sum_{i=1}^{n} q_i \cdot v_i$, ce qui contredit la ■

Part II

Lecture 2 - 5/10

Contents

5 Divide and Conquer

Divide a problem into smaller ones to solve those, then combine the solutions to get a solution of the bigger problem.
Example : *Merge Sort* : Its complexity verifies $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \mathcal{O}(n)$. From that we will derive that $T(n) = \mathcal{O}(n \log n)$

6 Analysis of Recursive Algorithms

We have recurrences we want to solve. We have three methods :

6.1 Substitution Method

The method :

1. Guess the asymptotic of $T(n)$
2. Show the answer via induction

For *Merge Sort*: we guess $T(n) \leq c \cdot n \log_2 n, \forall n \geq 2$. We choose c that verifies that property until $n = 6$.
Substituting in the recurrence equation :

$$T(n) \leq cn \log_2 \frac{n}{2} + c \log_2 \frac{n}{2} + c\frac{n+2}{2} + a \cdot n = c\dot{n} \log_2 n + a \cdot n + c \cdot \log_2 n - c\frac{n}{2}$$

If we then choose c so that it is bigger than $20a$ we get :

$$T(n) \leq cn \log_2 n + a \cdot n - c \cdot n/20 \leq cn \log_2 n$$

6.2 Recursion-tree Method

1. Simplify the equation :
   - Delete floors and ceils
   - Suppose n is of a good form
2. Draw a tree, rooted with the added term and the recursive calls
3. Start again, and recursively fill the tree

We get a tree of depth $\log_k n$ if n is divided by k in recursive calls. We can now sum the values of the nodes, to get an approximation, and start verifying.

7 Master Theorem

7.1 The Theorem

Théorème 7.1.1 (Master Theorem)**.** *If we have recurrence equation $T(n) = aT(n/b) + f(n)$ where $a \geq 1, b > 1$ are integers, $f(n)$ is asymptotically positive. Let $r = \log_b a$, we have :*

1. *If $f(n) = \mathcal{O}(n^{r-\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^r)$*
2. *If $f(n) = \Theta(n^r)$ then $T(n) = \Theta(n^r \log n)$*
3. *If $f(n) = \Omega(n^{r+\varepsilon})$ for some $\varepsilon > 0$, and $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n (regularity condition) then $T(n) = \Theta(f(n))$.*

Remarque 7.1.1.1. *The Master Theorem 7.1.1 does not cover all possible cases for $f(n)$. Example : $f(h) = h^r / \log h$*

Remarque 7.1.1.2. *The Master Theorem 7.1.1 is sometimes called* Théorème sur les Récurrences de Partition

8 The Proof

Plan :

- Analyse the recurrence as if T is defined over reals (continuous version)
- Prove the discrete version

8.1 Continuous Master Theorem

Proof.

Lemme 8.1.1. *Define* $T(n) = \begin{cases} \Theta(1) & \text{if } n \leq \hat{n} \\ aT(n/b) + f(n) & \text{if } n > \hat{n} \end{cases}$ *Then*

$$T(n) = \Theta(n^r) + \sum_{k=0}^{\lceil \log_b(n/\hat{n}) \rceil - 1} a^k f(n/b^k)$$

Proof. In the Recursion-Tree, stopped when the argument of T is smaller than $\hat{n}$ which is when depth is $\lceil \log_b(n/\hat{n}) \rceil - 1$, we get :

$$\begin{aligned} T(n) &\leq \sum_{k=0}^{\lceil \log_b(n/\hat{n}) \rceil - 1} a^k f(n/b^k) + \Theta(a^{log_b(n/\hat{n})}) \\ &= \sum_{k=0}^{\lceil \log_b(n/\hat{n}) \rceil - 1} a^k f(n/b^k) + \Theta(a^{\log_b(n)}) \\ &= \sum_{k=0}^{\lceil \log_b(n/\hat{n}) \rceil - 1} a^k f(n/b^k) + \Theta(n^{\log_b(a)}) \end{aligned}$$

∎

Back to the proof :

Lemme 8.1.2. *Define* $g(n) = \Theta(n^r) + \sum_{k=0}^{q} a^k f(n/b^k)$ *Then :*

1. *If* $f(n) = \mathcal{O}(n^{r-\varepsilon})$ *then* $g(n) = \Theta(n^r)$
2. *If* $f(n) = \Theta(n^r)$ *then* $g(n) = \Theta(n^r \log n)$
3. *If* $f(n) = \Omega(n^{r+\varepsilon})$ *and we have the regularity condition then* $g = \Theta(f)$

Proof. 1. Case 1 :

$$g(n) \begin{aligned} &= \Theta(n^r) + \sum_{k=0}^{q} a^k f(n/b^k) \\ &= \Theta(n^r) + \mathcal{O}\left(\sum_{k=0}^{q} a^k (n/b^k)^{r-\varepsilon}\right) \end{aligned}$$

However :

$$\begin{aligned} \sum_{k=0}^{q} a^k (n/b^k)^{r-\varepsilon} &= n^{r-\varepsilon} \sum_{k=0}^{q} (ab^\varepsilon/b^r)^k \\ &= n^{r-\varepsilon} \sum_{k=0}^{\lceil \log_b(n/\hat{n}) \rceil - 1} (b^\varepsilon)^k = \mathcal{O}(n^{r-\varepsilon}(n/\hat{n})^{epsilon}) \end{aligned}$$

Thus : $g(n) = \Theta(n^r)$

2. Case 2 : We have :

$$\begin{aligned} g(n) &= \Theta(n^r) + \sum_{k=0}^{q} a^k f(n/b^k) \\ &= \Theta(n^r) + \Theta\left(\sum_{k=0}^{q} a^k (n/b^k)^r\right) \end{aligned}$$

However :

$$\sum_{k=0}^{q} a^k (n/b^k)^r = n^r \sum_{k=0}^{q} (a/b^r)^k$$

$$= n^r \sum_{k=0}^{\lceil \log_b (n/\hat{n}) \rceil - 1} 1 = n^r \Theta(\log_b n/\hat{n})$$

3. Case 3 : By induction on k : $a^k f(n/b^k) \leq c^k f(n)$. Thus :

$$\sum_{k=0}^{q} a^k f(n/b^k) \leq \sum_{k=0}^{q} c^k f(n) = f(n) \sum_{k=0}^{q} c^k = \Theta(f(n))$$

∎

We thus have proved the continuous Master Theorem. ∎

8.2 Discrete Master Theorem

We have now showed the continuous Master Theorem, following William Kuszmaul, Charles E. Leiserson, *Floors and Ceilings in Divide-and-Conquer Recurrences*, Symposium on Simplicity in Algorithms 2021.

Proof. See slides below

Why not to follow CLRS textbook?

Floors and Ceilings in Divide-and-Conquer Recurrences*

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Abstract
The master theorem is a core tool for algorithm analysis. Many applications use the discrete version of the theorem, in which floors and ceilings may appear within the recursion. Several of the known proofs of the discrete master theorem include substantial errors, however, and other known proofs employ sophisticated mathematics. We present an elementary and approachable proof that applies generally to Akra-Bazzi-style recurrences.

include the claim that the theorem holds in the presence of floors and ceilings.

To distinguish the two situations, we call the master theorem without floors and ceilings the ***continuous master theorem***[1] and the master theorem with floors and ceilings the ***discrete master theorem***. When we speak only of the master theorem, we mean the discrete master theorem, but we usually include the term "discrete" in this paper for clarity in distinguishing the two cases.

proved the theorem for exact powers of b. Cormen, Leiserson, and Rivest [5, Section 4.3] presented the discrete master theorem, extending Bentley, Haken, and Saxe's earlier treatment to include floors and ceilings, but their proof is at best a sketch, not a rigorous argument, and it leaves key issues unaddressed. These problems have persisted through two subsequent editions [6, 7] with the additional coauthor Stein.

Why not to follow CLRS textbook?

- Aho, Hopcroft, Ullman offered one of the first treatments of divide-and-conquer recurrences, giving three cases for recurrences of the form T(n) = aT (n/b) + cn (1974)
- Bentley, Haken, and Saxe introduced the master theorem in modern form, but proved it for $n = b^k$ only (1980)
- CLRS extended the proof to the discrete version, but gave only a sketch of the proof (1990)
- Akra and Bazzi considered $T(n) = \sum_{i=1}^{t} a_i T(n/b_i) + f(n)$ (1998)
- Leighton simplifies the proof of Akra and Bazzi and extends is to the discrete version (1996)
- Campbell spots several flows in the proof of Leighton and devotes more than 300 pages to carefully correct the issues (2020)
- More generalizations by Drmota and Szpankowski(2013), Roura (2001), Yap (2011)

Definitions

Discrete recurrences

$$T(n) = f(n) + \sum_{i \in S} a_i T(\lfloor n/b_i \rfloor) + \sum_{i \notin S} a_i T(\lceil n/b_i \rceil)$$

$a_i \in \mathbb{R}^+, b_i \in \mathbb{R}^+, n \geq \hat{n}$

For $1 \leq n < \hat{n}$, there exist c_1, c_2: $c_1 \leq T(n) \leq c_2$

Polynomial-growth condition

$\exists \hat{n} > 0$ such that $\forall \Phi \geq 1 \, \exists d > 1 : d^{-1} f(n) \leq f(\varphi n) \leq d f(n)$

for all $1 \leq \varphi \leq \Phi$ and $n \geq \hat{n}$

6 technical slides ahead!

KEEP CALM AND CARRY ON

Lemma 1. For $\beta > 1, n \in \mathbb{N}$ let $L = \Pi_{i=1}^{n}(1 - \dfrac{1}{\beta^i + 1})^2$, $U = \Pi_{i=1}^{n}(1 + \dfrac{1}{\beta^i - 1})^2$

We have $L = \Omega(1)$ and $U = O(1)$.

Proof.

$$\beta > 1 \Rightarrow \frac{1}{\beta^i} < \frac{1}{\beta^i - 1} \Rightarrow 1/L = \Pi_{1=1}^{n}(1 + \frac{1}{\beta^i})^2 < \Pi_{i=1}^{n}(1 + \frac{1}{\beta^i - 1})^2 = U$$

$$U = \Pi_{i=1}^{n}(1 + \frac{1}{\beta^i - 1})^2 \leq \Pi_{i=1}^{\infty}(1 + \frac{1}{\beta^i - 1})^2 \leq \Pi_{i=1}^{\infty}(e^{1/(\beta^i - 1)})^2 =$$

(Here we use $1 + 1/x \leq e^{1/x}$ for $x \neq 0$)

$$= \exp(\sum_{i=1}^{\infty} \frac{2}{\beta^i - 1}) \leq \exp(\sum_{i=1}^{\infty} \frac{4}{\beta^i}) + O(1) = O(1)$$

Lemma 2. Let $\beta > 1; \beta_i \geq \beta, 1 \leq i \leq k; B := \Pi_{i=1}^{k}\beta_i$

There exists $c = c(\beta) > 0$ such that for all $n_1, n_2, \ldots, n_k$ where $n_i > \max(\beta, 1 + 1/(\sqrt{\beta} - 1))$ and $\lfloor n_{i-1}/\beta_i \rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i \rceil$, we have $c^{-1/4}(n_0/B) \leq n_k \leq c^{1/4}(n_0/B)$.

Proof. Let $\rho_i := \dfrac{n_i}{n_{i-1}/\beta_i}$.

$$(n_0/B)\Pi_{i=1}^{k}\rho_i = \frac{n_0\Pi_{i=1}^{k}\rho_i}{\Pi_{i=1}^{k}\beta_i} = n_0\Pi_{i=1}^{k}\frac{\rho_i}{\beta_i} = n_0\Pi_{i=1}^{k}\frac{n_i}{n_{i-1}} = n_k$$

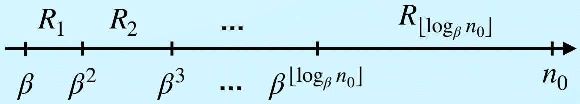
It is enough to show that $c^{-1/4} \leq \Pi_{i=1}^{k}\rho_i \leq c^{1/4}$ for some $c = c(\beta)$

$$n_{i-1}/\beta_i - 1 \leq \lfloor n_{i-1}/\beta_i \rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i \rceil \leq n_{i-1}/\beta_i + 1 \Rightarrow$$

$$n_i - 1 \leq n_{i-1}/\beta_i \leq n_i + 1 \Rightarrow \boxed{\underbrace{\frac{n_i}{n_i + 1}}_{1 - \frac{1}{n_i + 1}} \leq \rho_i \leq \underbrace{\frac{n_i}{n_i - 1}}_{1 + \frac{1}{n_i - 1}} \; (*)}$$

Proof of Lemma 2 (continued).

$$\boxed{\underbrace{\frac{n_i}{n_i+1}}_{1-\frac{1}{n_i+1}} \le \rho_i \le \underbrace{\frac{n_i}{n_i-1}}_{1+\frac{1}{n_i-1}} \; (*)}$$



From $(*)$: $\rho_i \le 1 + \dfrac{1}{n_i - 1} \le 1 + \dfrac{1}{1/(\sqrt{\beta}-1)} = \sqrt{\beta}$

$n_{i+2} = \dfrac{n_i \rho_{i+1} \rho_{i+2}}{\beta_{i+1}\beta_{i+2}} \le n_i/\beta \Rightarrow$ every range R_j contains at most two n_i's

From $(*)$ again: $n_i \in R_j \Rightarrow 1 - \dfrac{1}{\beta^j + 1} \le \rho_i \le 1 + \dfrac{1}{\beta^j - 1} (n_i > \beta^j)$

Therefore, $\Pi_{i=1}^k \rho_i = \Pi_{j=1}^{\lfloor \log_\beta n_0 \rfloor} (\Pi_{n_i \in R_j} \rho_i) \le \Pi_{j=1}^{\lfloor \log_\beta n_0 \rfloor} (1 + \dfrac{1}{\beta^j - 1})^2 \le c^{1/4}$ **(Lemma 1)**

$$\Pi_{i=1}^k \rho_i \ge \Pi_{j=1}^{\lfloor \log_\beta n_0 \rfloor} (1 - \frac{1}{\beta^j + 1})^2 \ge c^{-1/4} \text{ **(Lemma 1)**}$$

Lemma 3. $\beta_{\min}, \beta_{\max} > 1$. Assume that for all $1 \le i \le k$, $\beta_{\min} \le \beta_i \le \beta_{\max}$, and let $B = \Pi_i \beta_i$.

There exists $c = c(\beta_{\min}, \beta_{\max})$ such that for any $n_1, n_2, \ldots, n_k$ with $n_0 \ge cB$ and $\lfloor n_{i-1}/\beta_i \rfloor \le n_i \le \lceil n_{i-1}/\beta_i \rceil$, we have $c^{-1}(n_0/B) \le n_k \le c(n_0/B)$.

Proof.

Let $c = c(\beta_{\min})$ be the constant from Lemma 3. W.l.o.g. $\sqrt{c} > \max\{\dfrac{1}{\sqrt{\beta_{\min}} - 1} + 1, \beta_{\min}\} \; (*)$ and $c^{1/4} > 2\beta_i$

If $n_j \ge \sqrt{c}$ for all j, then **Lemma 3** follows from **Lemma 2** and $(*)$. Let j be the smallest value such that $n_j \le \sqrt{c}$. We have $j \ge 1$ as $n_0 \ge cB \ge \sqrt{c}$.

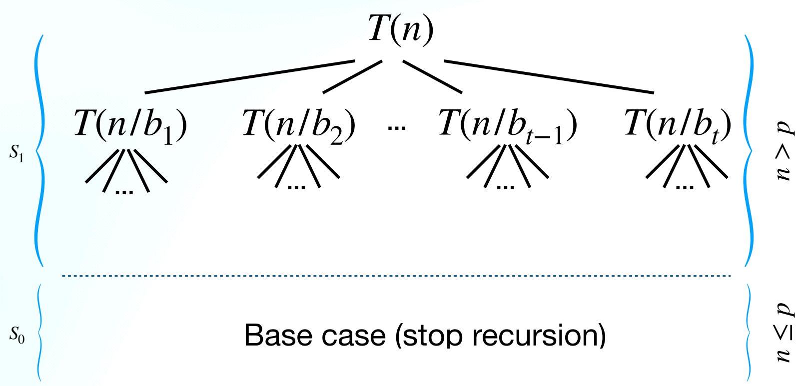
- If $j = 1$, then $n_{j-1} = n_0 \ge c^{-1/4}(n_0/B)$ (trivial).
- If $j > 1$, we apply **Lemma 2** to $\beta_1, \beta_2, \ldots, \beta_{j-1}$ and $n_0, n_1, \ldots, n_{j-1}$ and $\beta = \beta_{\min}$ (all conditions are satisfied) to obtain that $n_{j-1} \ge c^{-1/4}(n_0/B)$

In both cases, $n_{j-1} \ge c^{-1/4}(n_0/B) \ge c^{3/4}$. Therefore, $n_j \ge \lfloor \underbrace{n_{j-1}}_{\ge c^{1/4} > 2\beta_j} / \beta_j \rfloor \ge n_{j-1}/(2\beta_j) > n_{j-1}/c^{1/4} \ge \sqrt{c}$

Lemma 4. Let $a_1, a_2, \ldots, a_t > 0$ and $b_1, b_2, \ldots, b_t > 1$ be constants, $f(n) : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies the polynomial-growth condition. Consider $T(n) = f(n) + \sum_{i=1}^{t} a_i T(n/b_i)$ defined for $n \in \mathbb{R}^+$ ($*$). Assume that $T'(n)$ defined on $\mathbb{N}$ also satisfies ($*$), but each n/b_i is replaced with $\lfloor n/b_i \rfloor$ or $\lceil n/b_i \rceil$. Then $T'(n) = \Theta(T(n))$.

Proof.

Let c be the constant from Lemma 3 for $\beta_{\min} = \min_i b_i$ and $\beta_{\max} = \max_i b_i$. Let $\hat{n}$ be a sufficiently large constant. Define $p := \max\{\hat{n}, c \cdot \max_i b_i\}$. For $T(n)$, the base case is $n \le p$.



Subproblem q at level j:
$q = < q(1), q(2), \ldots, q(j) >$
$q(i) \in \{1,2,\ldots,t\}, |q| = j$
$q(i)$ - choice of the recursion brunch at level i
$n_q = n/\Pi_{i=1}^{|q|} b_{q(i)}$

Proof.

$$T(n) = \sum_{q \in S_1} f(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + \sum_{q \in S_0} T(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + f(n) = \sum_{q \in S_1} f(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + \Theta(\sum_{q \in S_0} \Pi_{i=1}^{|q|} a_{q(i)}) + f(n)$$

When computing $T'(n)$ for a subproblem q:

$$\lfloor \frac{n'_{<q(1),q(2),\ldots,q(j-1)>}}{q(j)} \rfloor \le n'_q \le \lceil \frac{n'_{<q(1),q(2),\ldots,q(j-1)>}}{q(j)} \rceil$$

$$T'(n) = \sum_{q \in S_1} f(n'_q)\Pi_{i=1}^{|q|} a_{q(i)} + \sum_{q \in S_0} T'(n'_q)\Pi_{i=1}^{|q|} a_{q(i)} + f(n) \ (*)$$

As $n_q > p$ for $q \in S_1$, $n_q > p/\max_i b_i \ge c$ for all $q \in S$. By Lemma 3 with $\beta_i = b_{q(i)}$, for all q we have $n'_q = \Theta(n_q)$. It follows that $\exists \Phi > 1$ such that $n'_q \in [\Phi^{-1} n_q, \Phi n_q]$. Therefore, $n'_q \ge n_q/\Phi \ge \hat{n}/\Phi$ and we can choose $\hat{n}$ so that ($*$) is defined correctly.

By the polynomial-growth condition, $f(n'_q) = \Theta(f(n_q))$ for all $q \in S$. For $q \in S_0$, $n'_q = \Theta(1)$ and therefore $T'(n'_q) = \Theta(1)$. It follows:

$$T'(n') = \sum_{q \in S_1} \Theta(f(n_q))\Pi_{i=1}^{|q|} a_{q(i)} + \Theta(\sum_{q \in S_0} \Pi_{i=1}^{|q|} a_{q(i)}) + f(n) = \Theta(T(n))$$

Proof.

$$T(n) = \sum_{q\in S_1} f(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + \sum_{q\in S_0} T(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + f(n) = \sum_{q\in S_1} f(n_q)\Pi_{i=1}^{|q|} a_{q(i)} + \Theta(\sum_{q\in S_0} \Pi_{i=1}^{|q|} a_{q(i)}) + f(n)$$

When computing $T'(n)$ for a subproblem q:

$$\lfloor \frac{n'_{<q(1),q(2),\ldots,q(j-1)>}}{q(j)} \rfloor \le n'_q \le \lceil \frac{n'_{<q(1),q(2),\ldots,q(j-1)>}}{q(j)} \rceil$$

$$T'(n) = \sum_{q\in S_1} f(n'_q)\Pi_{i=1}^{|q|} a_{q(i)} + \sum_{q\in S_0} T'(n'_q)\Pi_{i=1}^{|q|} a_{q(i)} + f(n) \ (*)$$

As $n_q > p$ for $q \in S_1$, $n_q > p/\max_i b_i \ge c$ for all $q \in S$. By Lemma 3 with $\beta_i = b_{q(i)}$, for all q we have $n'_q = \Theta(n_q)$, and hence $\exists \Phi > 1$ such that $n'_q \in [\Phi^{-1} n_q, \Phi n_q]$. Therefore, $n'_q \ge n_q/\Phi \ge \hat{n}/\Phi$ and we can choose $\hat{n}$ so that $(*)$ is defined correctly.

By the polynomial-growth condition, $f(n'_q) = \Theta(f(n_q))$ for all $q \in S$. For $q \in S_0$, $n'_q = \Theta(1)$ and therefore $T'(n'_q) = \Theta(1)$. It follows:

$$T'(n') = \sum_{q\in S_1} \Theta(f(n_q))\Pi_{i=1}^{|q|} a_{q(i)} + \Theta(\sum_{q\in S_0} \Pi_{i=1}^{|q|} a_{q(i)}) + f(n) = \Theta(T(n))$$

Discrete Master theorem

$T(n) = a_1 T(\lfloor n/b \rfloor) + a_2 T(\lceil n/b \rceil) + f(n)$, where
$a := a_1 + a_2 \ge 1$, $b > 1$, $f(n)$ - asymptotically positive.

Define $r := \log_b a$.

Case 1. If $f(n) = O(n^{r-\varepsilon})$ for some $\varepsilon > 0$, then $T(n) = \Theta(n^r)$.

Case 2. If $f(n) = \Theta(n^r)$, then $T(n) = \Theta(n^r \log n)$.

Case 3. If $f(n) = \Omega(n^{r+\varepsilon})$ for some $\varepsilon > 0$, and if $a_1 f(\lfloor n/b \rfloor) + a_2 f(\lceil n/b \rceil) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Discrete Master theorem

Case 1.

Fact. Replacing $f(n)$ with a function $f'(n)$ satisfying $f'(n) \leq f(n)$ (resp. $f'(n) \geq f(n)$) for all n in the domain of f does not increase (resp. decrease) $T(n)$.

Let $f(n) = O(n^c)$ for $c < \log_b a$. Then as a "bigger" function consider $f'(n) = r(n^c + 1)$ for r big enough. By Lemma 4 and the continuos Master theorem, $T(n) = O(n^{\log_b a})$.

As a "smaller" function, consider $f'(n) = 0$. By Lemma 4 and the continuos Master theorem, $T(n) = \Omega(n^{\log_b a})$.

Exercise. Both bigger and smaller functions satisfy the polynomial growth condition.

Case 2. Analogous.

Discrete Master theorem

Case 3.

$T(n) \geq f(n)$ and hence $T(n) = \Omega(f(n))$. It remains to show that $T(n) = O(f(n))$.

Regularity condition: $a_1 f(\lfloor n/b \rfloor) + a_2 f(\lceil n/b \rceil) \leq cf(n)$ for some $c < 1$ and all $n \geq p$.

For all $n < p$, there exists $s \geq 1$: $T(n) \leq sf(n)$. We show by induction that for all $n \in \mathbb{N}$, $T(n) \leq qf(n)$ for $q = s/(1-c)$.

- Base case: $n < p$ - by the choice of s
- Suppose that $n \geq p$ and the claim holds for all smaller n

$$T(n) = a_1 T(\lfloor n/b \rfloor) + a_2 T(\lceil n/b \rceil) + f(n) \leq a_1 q f(\lfloor n/b \rfloor) + a_2 q f(\lceil n/b \rceil) + f(n) \leq$$

$$\leq qcf(n) + f(n) = (\frac{sc}{1-c} + 1) f(n) = \frac{s - \overbrace{(1-c)s + 1 - c}^{\leq 0}}{1-c} f(n) \leq qf(n)$$

$\blacksquare$

Remarque 8.2.0.1 (Remarks on the Proof).

- *Lemmas 1 to 3 serve to show that the argument does not go too far when it is rounded up or down.*
- *Slide 36 Last Line :* $\frac{2}{\beta^i-1} < \frac{4}{\beta^i}$ *for* $i \geq i_0$. *Thus :* $\sum_{i=1}^{\infty} \frac{2}{\beta^i-1} < \sum_{i=1}^{\infty} \frac{4}{\beta^i} + \sum_{i=0}^{i_0} \frac{2}{\beta^i-1}$ *and that last sum is* $\mathcal{O}(1)$
- *Slide 37 Line 3 : The first inequalities comes from the Recursion-Tree, so that we can ensure the argument does not deviate to much, by the second inequalities.*

8.3 Use Cases

Using the Master Theorem we can show the complexity of many algorithms :

1. Merge Sort Complexity : $T(n) = T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \mathcal{O}(n) = \Theta(n \log n)$
2. Strassen's Algorithm for Matrix Multiplication : $T(n) = 7T(n/2) + \Theta(n^2) \Rightarrow T(n) = \mathcal{O}(n^{\log_2 7}) = \mathcal{O}(n^{2.8074})$

9 Fast Multiplication of Polynomials

The sum of two degree n polynomials can be done in $\mathcal{O}(n)$, Horner's rule for evaluation produces $\mathcal{O}(n)$ complexity. The naïve product can be done in $\mathcal{O}(n^2)$
Remembering Lagrange's Theorem on Polynomials (or Vandermonde's Determinant, or anything really), degree n polynomials are entirely represented by their point-value reprensentation over n distinct points (x_i, y_i). Then, by converting the coefficient reprensentation to point-value representation, then by point-wise multiplicating the polynomials, then by going back to the coefficient representation, we can have a better algorithm.

9.1 Point-Value Multiplication

It is easily done in $\mathcal{O}(n)$ if both polynomials are represented over the same axis.

9.2 Coefficient to Point-Value Conversion - Fast Fourier Transform

For $P = \sum_{i=0}^{n-1} a_i x^i$, we define :

$$\begin{aligned} P_{odd}(x) &= a_{n-1}x^{n/2-1} + a_{n-3}x^{n/2-3} + \ldots + a_1 x \\ P_{even}(x) &= a_{n-2}x^{n/2-2} + a_{n-4}x^{n/2-4} + \ldots + a_2 x^{2/2} + a_0 \end{aligned}$$

1. We have : $P = xP_{odd}(x^2) + P_{even}(x^2)$
2. We evaluate P_{odd}, P_{even} at $(\omega_n^i)^2$ recursively by the halving property.
3. We combine the result.

9.3 Point-Value to Coefficient Conversion - Inverse Fast Fourier Transform

Théorème 9.3.1. $V_n^{-1}[i,j] = \omega_n^{-ij}/n$

Part III
Lecture 3 - 12/1

Contents

Answering queries of appartenance on a set, maintaining a dictionary. Python dictionaries do not have upper bound guarantees, we shouldn't use them.
Suppose we have a set S over a universe U. We dnote by n the number of keys, m the size of the hash table

10 Naïve Array-based implementation

Here we assume that no two objects have equal keys. We store 1 in a bitvector of length $|U|$ at each position i such that i is in S.
This takes $\mathcal{O}(|U|)$ space, for $\mathcal{O}(1)$ search time, and $\mathcal{O}(1)$ modification time.

11 Chained Hash Tables

We give ourselves a function $h : U \to [1, m]$. We send the keys to their image by h in a table. However, since we have no guarantee that $|U| \leq m$, there might be collisions.
To deal with that, instead of storing in the table a boolean, we store a list of all the keys corresponding to $h(k)$.
Then we insert a key in constant time $\mathcal{O}(1)$, search a key in time $\mathcal{O}(|h[key]|)$ and delete a key in

time $\mathcal{O}(|h[key]|)$ since in the worst case the key is at the end of the list.

Further analysis leads to the following theorem.

Théorème 11.0.1 (Simple Uniform Hashing Assumption)**.** *Assuming SUHA : "h equally distributes the keys into the table slots", and assuming $h(x)$ can be computed in $\mathcal{O}(1)$, $E\left[T_{search}(n)\right] = \mathcal{O}\left(1 + \frac{n}{m}\right)$, and same for deletion time.*
Formally, SUHA is :

$$\forall y \in [1, |T|] \mathbb{P}\left(h(x) = y\right) = \frac{1}{|T|}$$

$$\forall y_1, y_2 \in [1, |T|]^2 \mathbb{P}\left(h(x_1) = y_1,\ h(x_2) = y_2\right) = \frac{1}{|t|^2}$$

Proof.

1. Unsuccessful Search : Suppose that $k_0, \ldots, k_{n-1}$ are keys in the dictionary and we perform an unsuccessful search for a key k. The number of comparisons is : $\sum_{i=0}^{n-1} \nVdash_{h(k)=h(k_i)}$. Then, the expected time is : $\mathbb{E}[T_{search}] = \frac{n}{|T|}$ by SUHA.
2. Successful Search : Suppose keys were introduced in order $k_0, \ldots, k_{n-1}$. k_i appears before any of $k_0, \ldots, k_{i-1}$ and after any of $k_{i+1}, \ldots, k_{n-1}$ that are in the same linked list. Then, to search for k_i, we need $\sum_{j=i+1}^{n-1} \nVdash_{h(k_j)=h(k_i)}$. Under SUHA, the expectation of each of these variables is $\frac{1}{|T|}$. Then, the average expected search time is : $\frac{1}{n}\sum_{i=0}^{n-1} = 1+\frac{1}{n}\sum_{i=0}^{n-1}\frac{(n-1-i)}{|T|} = \mathcal{O}(1+\frac{n}{|T|})$

This concludes the proof of the theorem. ■

Good hash functions are functions that distribute the keys evenly. Yet, we do not know what the keys are, and thus will need various heuristics to answer this question. At least, without loss of generality, we can assume that keys are integers.

11.1 Heuristic Hash Functions

- Division Method : $h(k) = k \mod m$. It is better to choose m to be a prime number, and avoid $m = 2^p$.
- Multiplication Method : $h(k) = \lfloor m\left\{k \cdot A\right\}\rfloor$. $A \in (0, 1)$ and $m = 2^p$.

Yet, fixing the function can allow anyone to construct a probability distribution for which the function will be "bad".

11.2 Universal Family of Hash Functions

$H = \{h : U \to [0, |T| - 1]\}$ is Universal if :

$$\forall k_1 \neq k_2 \in U,\ |\{h \in H \mid h(k_1) = h(k_2)\}| \leq \frac{|H|}{m}$$

Théorème 11.2.1. *If h is a hash function chosen uniformly at random from a universal family of hash functions. Suppose that $h(k)$ can be computed in constant time and there are at most n keys. Then the expected search time is $\mathcal{O}(1 + \frac{n}{|T|})$*

Proof. Same as the case when h satisfies SUHA. Observe that the probability comes this time from choosing the function. ■

Théorème 11.2.2. *Let $p \in \mathcal{P}$ such that $U \subseteq [0, p-1]$. Then $H = \big\{h_{a,b}(k) = ((ak + b) \mod p) \mod |T| \mid a \in \mathbb{Z}_p^*,\ b$ is a universal family.*

Proof. Let $k_1 \neq k_2 \in U$. Let $l_i = (ak_i + b) \mod p$. We have $l_1 \neq l_2$ and $a = ((l_1 - l_2)((k_1 - k_2)^{-1} \mod p) \mod p)$ and $b = (l_1 - ak_1) \mod p$. There is then one-to-one mapping between (a, b) and (l_1, l_2). The number of $h \in H$ such that $h(k_1) = h(k_2)$ is :

$$|\{(l_1, l_2) \mid l_1 \neq l_2 \in \mathbb{Z}_p,\ l_1 = l_2 \mod m\}| \leq \frac{p(p-1)}{|T|} \leq \frac{|H|}{|T|}$$

■

12 Open Addressing

Elements are stored in the table. To insert, we probe the hash table until we find x or an empty slot. If we find an empty slot, insert x here. To define which slots to probe, we use a hash function that depends on the key and the probe number. To search, we probe the hash table until we either find x (return YES) or an empty slot (return NO). In the analysis, we will assume h to be uniform.

Théorème 12.0.1 (Analysis)**.** *Given an open-address hash-table with load factor $\alpha = \frac{n}{|T|} < 1$, the expected number of probes in an unsuccessfulsearch is at most $\frac{1}{1-\alpha}$, assuming uniform hashing.*

Proof. An unsuccessful search on x means that every probed slot except the last one is occupied and does not contain x, and the last one is empty. We define A_i the event " The i-th probe occurs and is occupied.". By Bayes's Theorem, we must estimate :

$$\mathbb{P}[\# \text{ of probes} \geq i] = \prod_{k=1}^{i-1} \mathbb{P}[A_k \mid \bigcap_{j=1}^{k-1} A_j]$$

But we have : $\mathbb{P}[A_j \mid A_1 \cap \ldots \cap A_{j-1}] = \frac{n-j+2}{|T|-j+2}$ So we have : $\mathbb{P}[\# \text{ of probes} \geq i] \leq \frac{n}{|T|}^{i-1} = \alpha^{i-1}$
Then, the expected number of probes is :

$$\sum_{i=1}^{+\infty} \mathbb{P}[\# \text{ of probes} \geq i] \leq \sum_{i=1}^{+\infty} \alpha^{i-1} = \frac{1}{1-\alpha}$$

■

Corollaire 12.0.1.1. *The expected number of probes during insertion is at most $\frac{1}{1-\alpha}$.*

Proof. If we insert x we first ran an unsuccessful search for it ■

Théorème 12.0.2. *The expected number of probes during a Successful search is at most $\frac{1}{\alpha} \ln \frac{1}{1-\alpha}$.*

Proof. A successful search for x probes the same sequence of slots as insertion.
If x is the i-th element inserted into the table, insertion probes less than $\frac{1}{1-\frac{i}{|T|}}$ slots in expectation.
Therefore, the expected time of a successful search is at most :

$$\frac{1}{n} \sum_{i=0}^{n-1} \frac{|T|}{|T|-i} = \frac{|T|}{n} \sum_{i=0}^{n-1} \frac{1}{|T|-i} = \sum_{i=0}^{|T|} \frac{1}{i} - \sum_{i=0}^{|T|-n} \frac{1}{i} \leq \frac{|T|}{n} \ln \frac{|T|}{|T|-n} = \frac{1}{\alpha} \ln \frac{1}{1-\alpha}$$

■

This is hard to implement, so we will use heuristics.

12.1 Heuristic hash functions

Let $h^{'}, h^{''}$ be two auxiliary hash functions.

- Linear Probing : $h(k,i) = (h^{'}(k) + i) \mod |T|$ This is easy to implement but it suffers from clustering.
- Quadratic Probing : $h(k,i) = (h^{'}(k) + c_1 i + c_2 i^2) \mod |T|$ We must choose the constants c_1 and c_2 carefully, and this still suffers from clustering.
- Double Hashing : $h(k,i) = (h^{'}(k) + i h^{''}(k)) \mod |T|$ To use the whole table, $h^{''}(k)$ must be relatively prime to m, e.g. $h^{''}(k)$ is always odd, $m = 2^i$.

13 Cuckoo Hashing

Hashing scheme with search time constant in the worst case, as it maintains a hash function without collisions to achieve perfect hashing. This is possible if the set of keys is static. Assume that we have two hash functions h_1, h_2 that satisfy SUHA. We store x in either $T[h_1(x)]$ or $T[h_2(x)]$. Search for x is done in constant time.

```
function Insert(x)
    if x = T[h1(x)] or x = T[h2(x)] then return
    end if
    pos ← h1(x)
    for i ← 0 to n do
        if thenT[pos] = Null
            T[pos] = x; return
        end if
        x ⟷ T[pos]
        if pos = h1(x) then
            pos ← h2(x)
        else
            pos ← h1(x)
        end if
    end for
    Rehash
    Insert(x)
end function
```

Théorème 13.0.1. *This insertion is done in constant time.*

Lemme 13.0.2. *Suppose that $|T| \geq c \cdot n$ for some $c > 1$. For any i, j, the probability that there exists a path from i to j of length $l \geq 1$ which is a shortest path from i to j is at most $\frac{1}{c^l \cdot |T|}$*

Proof. By induction on l :

- Initialization : By SUHA, $\mathbb{P}[h_{1,2}(x) = y] = \frac{1}{|T|}$. Thus $\mathbb{P}[\text{there is an edge from } i \text{ to } j] = \frac{n}{|T|^2} \leq \frac{1}{c|T|}$
- Heredity : For $l \geq 1$ there must exist k such that there is a path of length $l-1$ from i to k and an edge from k to j.

■

Proof. We define the bucket of x as all the cells that can be reached either from $h_1(x)$ or $h_2(x)$. x, y are in the same bucket if and only if there is a path from $\{h_1(x), h_2(x)\}$ to $\{h_1(y), h_2(y)\}$. Then :

$$\mathbb{P}[x, y \text{ are in the same bucket}] \leq 4 \sum_{l=1}^{\infty} \frac{1}{c^l\,|T|} = \frac{4}{(c-1)\,|T|}$$

So :

$$\mathbb{E}[|\text{bucket of } x|] = \mathbb{E} \sum X_{x,y} = \sum \mathbb{E}[X_{x,y}] = \sum \mathbb{P}[x, y \text{ are in the same bucket}] \leq \frac{4}{c-1}$$

Hence, in the absence of rehash, expected insertion time in constant.

The probability that we need a rehash is at most probability that there is a cycle, i.e. a path from i to i : $\frac{4}{c-1} \leq \frac{1}{2}$. The probability that we will need d rehashes is then at most $\frac{1}{2^d}$. Thus, the expected time per insertion is :

$$\frac{1}{n} \cdot \mathcal{O}(n) \sum_{d=1}^{+\infty} \frac{1}{2^d} = \mathcal{O}(1)$$

■

13.1 Rolling Hash Functions

Définition 13.1.1. *The Karp-Rabin fingerprint of a string $S = s_1 \dots s_m$ is :*

$$\varphi(s_1 \dots s_m) = \sum_{i=1}^{m} s_i r^{m-1} \mod p$$

where p is a big enough prime and $r \in \mathbb{F}_p$.

Proposition 13.1.1.
- *If $S = T$, then $\varphi(S) = \varphi(T)$*
- *Else, $\varphi(S) \neq \varphi(T)$ with high probability.*

Proof. Let σ be the size of the alphabet, $p \geq max\{\sigma, n^c\}$ where $c > 1$ is a constant. We have:

$$\varphi(S) = \varphi(T) \Leftrightarrow \sum_{i=1}^{m} (s_i - t_i) \cdot r^{m-i} \mod p = 0$$

Hence, r is a root of $P(x) = \sum_{i=1}^{m} (s_i - t_i) \cdot x^{m-i}$ a polynomial over $\mathbb{F}_p$. The probability of such event is at most $\frac{m}{p} \leq \frac{1}{n^{c-1}}$. ∎

The Karp-Rabin algorithm is as follows :

- Compute the fingerprint of the pattern
- Compare it with the fingerprint of each m-length substring of the text. If the fingerprint is equal to the fingerprint of a substring, report it as an occurrence

Proposition 13.1.2. *We have :*

$$\varphi(s_1 \dots s_{j+1}) = \varphi(s_1 \dots s_j) \cdot r + s_{j+1} \mod p$$

Proof. Observe that :

$$\varphi(s_1 \dots s_{j+1}) = \sum_{i=1}^{j+1} s_i r^{j+1-i} \mod p$$

∎

14 TD 3 : Hashing - 13/10

14.1 Exercise 1

We can expect to have $\frac{n(n-1)}{m} - 1$ collisions, i.e. if we consider the sets of pairs of values with collisions, we can expect its cardinality to be $2 \times (\frac{n(n-1)}{m} - 1)$.

14.2 Exercise 2

By definition of h, the sets $\left(h^{-1}(i)\right)_{i \in [1,m]}$ form a partition of U. Thus, since $|U| > nm$, there exists i such that $\left|h^{-1}(i)\right| > n$, giving us the result.

14.3 Exercise 3

We get, by sum, if $k = k_n \dots k_0$, $h(k) = \sum_{i=0}^{n} k_i * (2^p)^i \mod 2^p - 1 = [\sum_{i=0}^{n} k_i] \mod 2^p - 1$. This formula shows the order of the caracters doesn't matter in the value by the hash function.

14.4 Exercise 4

It is sufficient to find the cardinality of the set $\{ih_2(k) \mid i \in [0, m-1]\}$. This is, by Euler's formula m if and only if $h_2(k) \wedge m = 1$

14.5 Exercise 5

14.6 Exercise 6

14.6.1 Question a.

We know that the number of probes during insertion is the number of probes seen during search. An unsuccessful search on x means that every probed slot except the last one is occupied and does not contain x, and the last one is empty. We define A_i the event " The i-th probe occurs and is occupied.". By Bayes's Theorem, we must estimate :

$$\mathbb{P}[\# \text{ of probes} \geq i] = \prod_{k=1}^{i-1} \mathbb{P}[A_k \mid \bigcap_{j=1}^{k-1} A_j]$$

But we have : $\mathbb{P}[A_j \mid A_1 \cap \ldots \cap A_{j-1}] = \frac{n-j+2}{|T|-j+2}$
So we have : $\mathbb{P}[\# \text{ of probes} \geq i] \leq \frac{n}{|T|}^{i-1} = \alpha^{i-1}$
Here, $\alpha \leq \frac{1}{2}$ thus we have the result.

Then, we get that the probability of the i-th insertion requiring strictly more than $2\log_2(n)$ probes is at most $\frac{1}{n^2}$

14.6.2 Question b.

We have : $\mathbb{P}[X > 2\log_2(n)] = \mathbb{P}\left[\bigcup_{i=1}^{n} X_i > 2\log_2(n)\right] < \sum_{i=1}^{n} \mathbb{P}\left[X_i > 2\log_2(n)\right] = \frac{1}{n}$. The last inequality is strict as these events are not independent.

14.6.3 Question c.

The expected length of the longest probe sequence is :

$$\mathbb{E}[X] \leq \mathbb{P}[X \leq 2\log_2(n)] \times 2\log_2(n) + \mathbb{P}[X > 2\log_2(n)] \times n = \mathcal{O}(\log_2(n))$$

14.7 Exercise 7

14.7.1 Question a.

If H is 2-independent, then for each pair of distinct keys x_1, x_2 and y_1, y_2 is $\mathbb{P}[h(x_1) = y_1, h(x_2) = y_2] = \frac{1}{m^2}$. If we fix $y_1 = y_2$, we have : $\mathbb{P}[h(x_1) = y_1 = h(x_2)] = \frac{1}{m^2}$. By summing over all values for y_1 : $\mathbb{P}[h(x_1) = h(x_2)] = \frac{1}{m}$.
However, as H is universal if and only if : $\mathbb{P}[h(x_1 = h(x_2)) \mid h \leftarrow^s H] \leq \frac{1}{m}$, we have the result.

14.7.2 Question b.

By the same argument as in the class, H is universal because we have a one-to-one mapping between U and H. Moreover, with $x = (0, \ldots, 0) \mod p$, we get the result.

14.7.3 Question c.

Again this family is universal. We have : $\mathbb{P}[h^{'}_{a,b}(x_1) = y_1 \wedge h^{'}_{a,b}(x_2) = y_2] = \frac{1}{m^2}$ since a and b are uniquely determined by $h^{'}(a, b)$.

14.7.4 Question d.

Given the fact that we H is 2-independent, we get the result since this comes from : $\mathbb{P}[h(m) = t \wedge h(m_0) = t_0] = \frac{1}{p}^2$. But, knowing the family, since this family is universal, the adversary might have better knowledge of which functions have same image over m, he can succeed to fool Bob with probability at most $\frac{1}{p}$.

14.8 Exercise 8.

14.8.1 Question a.

This is Langrange's Theorem.

14.8.2 Question b.

As there is an even number of elements in $\mathbb{F}$, there are as many elements in $\mathbb{F}$ ordered with an even and an odd number. Then, by question 1, since there is a one-to-one mapping between $\mathbb{F}^4$ and G, there are, for each tuple in $\{\pm 1\}^4$, $2^4 = m^4$ functions that suit. Thus, H is 4-independent.

Part IV
Lecture 4 : 26/10

Contents

15 Binary Search Trees

Définition 15.0.1. *A Binary Tree is a rooted tree with every node having at most two children. Binary search trees are binary trees verifying the following :*

If a node contains element l, the left subtree rooted at the left child contains only elements $\leq l$ and the right subtree elements $> l$.

Proposition 15.0.1. *To access a given element, the time needed is $\mathcal{O}(h)$ time where h is the height of the tree.*

h is not necessarily small, there are many implementations to bound h.

15.1 Red-Black Trees

Définition 15.1.1. *A red-black tree is a binary tree that satisfies the following red-black properties :*

1. *Every node has a color : red or black*
2. *The root is black*
3. *Every leaf (NIL) is black*
4. *If a node is red, both its children are black*

5. *For each node, all paths from the node to descendant leaves contain the same number of black nodes.*

Lemme 15.1.1. *The height of a red-black tree with n nodes is at most $2\log(n+1)$.*

Proof. We introduce $bh(x)$ the number of black nodes in a path from x to a leaf.
By induction on $bh(x)$, the subtree rooted at x contains at least $2^{bh(x)} - 1$:

- Initialization : $bh(x) = 0$ implies the tree has more than 0 nodes which is true.
- Heredity :
  - If x is black, let y_1, y_2 denote its children. We have $bh(y_1) = bh(x) - 1 = bh(y_2) - 1$. By summing, we get the result.
  - If x is red, both of its children are black by rule 4. Let us denote the grand children of x by $z_1, \ldots, z_4$. We have : $bh(z_i) = bh(x) - 1$. By sum, we get the result.

Then if h the black height of the tree, $n \geq 2^h - 1$ and $h \leq \log n + 1$. Since in any root-to-leaf path, the number of nodes is at most twice the number of black nodes per rule 4, the lemma follows. ■

Proposition 15.1.1 (Insertion)**.** *We insert a node into a n-node red-black tree by :*

- *Inserting the node into T as if it were an ordinary binary search tree*
- *We color it red*
- *We perform a number of rotations and node recolouring.*

This is done in $\mathcal{O}(\log n)$.

We define the left rotation on a node x whose right child is not null like this :

$$Tree(x, \alpha, Tree(y, \beta, \gamma)) \mapsto Tree(y, Tree(x, \alpha, \beta), \gamma)$$

We define right rotations to be the reverse operation.

Proposition 15.1.2. *Rotations do not break the BST properties.*

Only on right rotations. Elements in γ are greater than y, in β are greater than x and lower or equal than y, and elements in α are lower or equal than x. After the rotation, we still have those properties. ■

Proposition 15.1.3 (Restoring red-black properties)**.** *Rule* 1 *is never violated. Rules* 2 *and* 3 *can be fixed in* $\mathcal{O}(1)$ *time. Rule* 5 *is never violated as we colour the new node red.*
We will try and maintain the invariant that the properties are broken in at most one node. There are three cases in which Rule 4 *can be violated : Let z be the node that violates the red-black properties.*

1. *z's uncle y is red $\Rightarrow$ We colour $z.p$ black, colour y black and $z.p.p$ red. We moved the problematic node up by two layers.*

2. *z's uncle y is black and z is a right child $\Rightarrow$ We left-rotate around the edge $z.p$-z, we then have z's uncle y to be black and z to be a left child, i.e. case 3.*

3. *z's uncle y is black and z is a left child $\Rightarrow$ We color $z.p$ black, colour $z.p.p$ red and right-rotate.*

Proposition 15.1.4 (Complexity)**.** *Red black tree use $\mathcal{O}(n)$ space, $\mathcal{O}(\log n)$ in time for insertion and deletion in the worst case.*

15.2 Treaps

Définition 15.2.1. *Treaps are binary trees with every node having a key and a priority verifying :*

- *a binary search tree for the keys*
- *a min-heap for the priorities*

Proposition 15.2.1 (Search)**.** *The usual algorithm yields time in* $\mathcal{O}(depthofthenode)$ *for a successful search and* $\mathcal{O}(\max(\boldsymbol{depth}(v^-), \boldsymbol{depth}(v^+)))$ *where* $\boldsymbol{key}(v^-)$ *is the predecessor of the searched key.*

Proposition 15.2.2 (Insertion)**.** *The standard BST algorithm with added rotations to fix the heap properties (if* $p(z) \leq p(z.p)$*, rotate auround* $(z, z.p)$*) yields time complexity in* $\mathcal{O}(\max(\boldsymbol{depth}(v^-), \boldsymbol{depth}(v^+)))$.

Proposition 15.2.3 (Deletion)**.** *We run the insertion algorithm backwards : let* $\boldsymbol{light}(z)$ *be the child of* z *with smaller priority. The choice of this edge preserves the heap property everywhere except at* z*. As long as* z *is not a leaft, rotate around* $(z, \boldsymbol{ligth}(z))$*, then chop it off.*

Proposition 15.2.4 (Split)**.** *If we want to split the tree along a pivot* π*, we can insert a new node* z *with key* π *and priority* $-\infty$*. After the insertion, this new node is the root since it has the smallest priority, and all keys in the left subtree are smaller than the pivot and this subtree is a treap.*

Proposition 15.2.5 (Merge)**.** *Suppose that we want to merge two treaps* T_- *and* T_+*, where keys in the first tree are smaller than the keys in the second tree. We create a dummy root with priority* $-\infty$ *and then delete it.*

Proposition 15.2.6 (Complexity)**.**

- *Time for a successful search :* $\mathcal{O}(\boldsymbol{depth(v)})$ *where* $\boldsymbol{key}(v) = k$.
- *Time for an unsuccessful search :* $\mathcal{O}(\max(\boldsymbol{depth}(v^-), \boldsymbol{depth}(v^+)))$ *where* $\boldsymbol{key}(v^-)$ *is the predecessor of the searched key.*
- *Insertion Time :* $\mathcal{O}(\max(\boldsymbol{depth}(v^-), \boldsymbol{depth}(v^+)))$ *where* $\boldsymbol{key}(v^-)$ *is the predecessor of the searched key.*
- *Deletion Time :* $\mathcal{O}(tree\ depth)$
- *Split/Merge Time : same as insertion / deletion*

We will choose the priorities to be independently and uniformly distributed continuous random variables. We denote by $x_1, \ldots, x_n$ the nodes of the tree in the increasing order of the keys. We denote by $i \uparrow k$, the event x_i if a proper ancestor of x_k : $\mathtt{depth}(x_k) = \sum_{i=1}^{i=n} [i \uparrow k]$

Lemme 15.2.1. *Define*

$$X(i,k) = \begin{cases} x_i, \ldots, x_k & \text{if } i < k \\ x_k, \ldots, x_i & \text{otherwise.} \end{cases}$$

For all $i \neq k$ *we have* $i \uparrow k$ *iff* x_i *has the smallest priority in* $X(i,k)$.

Proof.

1. If x_i is the root, then x_i has the smallest priority
2. If x_k is the root, then x_i is not an ancestor of x_k and does not have the smallest priority
3. If x_j is the root with $i < j < k$, then x_i is not an ancestor of x_k and does not have the smallest priority.
4. If x_j is the root and either $i < k < j$ or $i < j < k$ then x_i and x_j are in the same subtree and the claim follow by induction.

■

Corollaire 15.2.1.1.

$$\mathbb{P}[i \uparrow k] = \begin{cases} \dfrac{1}{k-i+1} & \textit{if } i < k \\ 0 & \textit{if } i = k \\ \dfrac{1}{i-k+1} & \textit{if } i > k \end{cases}$$

Then :

$$\mathbb{E}[\texttt{depth}(x_k)] = \sum_{i=1}^{i=n} \mathbb{P}[i \uparrow k] = \sum_{i=1}^{k-1} \frac{1}{k-i+1} + \sum_{k+1}^{n} \frac{1}{i-k+1} = H_k - 1 + H_{n-k+1} - 1 < 2\ln n - 2$$

Thus, all treap operations take $\mathcal{O}(\log n)$ time in expectation

Remarque 15.2.1.1. *The fastest sorting algorithm is Quicksort, and it can be seen as inputing all the values in the list in a treap then taking the infix order of the treap.*

Remarque 15.2.1.2. *Tango trees are $\mathcal{O}(\log\log n)$-competitive and Splay trees are conjecture to be fastest in $\mathcal{O}(1)$-competitive, but we do not know if this is true.*

16 Lower Bound for Sorting

The question here is : can we sort in $\mathcal{O}(n \log n)$ or better ? If comparisons only are allowed then no : indeed, any comparison-based sorting program can be thought of as defining a decision tree of possible executions. Running the same program twice on the same permutation causes it to do the exact same thing bu running it on different permutations of the same data causes a different sequence of comparisons to be made on each.

16.1 Decision Tree

Définition 16.1.1. *A decision tree for a sorting algorithm is a binary tree that shows the possible executions of an algorithm on a set.*

Proposition 16.1.1. *The minimum height of a decision tree is the worst-case complexity of comparison-based sorting.*

Lemme 16.1.1. *The height of any decision tree is $\Omega(n \log n)$*

Proof. Since any two different permutations of n elements require a different sequence of steps to sort, there must be at least $n!$ different paths from the root to leaves in the decision tree. Thus there must be at least $n!$ different leaves in this binary tree. Since a binary tree of height h has at most 2^h leaves, we know that : $n! \leq 2^h$ i.e. $h \geq \log(n!)$. Finally, we have $\log n! = \Omega(n \log n)$. ∎

Théorème 16.1.2. *Any comparison-based sorting algorithm must use $\Omega(n \log n)$ time.*

17 Predecessor Problem

We want to maintain a set S of integers from a universe $U = [1, u]$ subject to insertions, deletions, predecessor and succesor queries. This problem is harder than dictionaries and hashing. BTSs give a solution in $\mathcal{O}(n)$ space and $\Theta(\log n)$ time.

17.1 van Emde Boas Trees

If the time per operation satisfies : $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$, by Substitution, $T(u) = \mathcal{O}(\log\log u)$. We will split U into $\sqrt{u}$ chunks of size $\sqrt{u}$ size, and the recurse.

Définition 17.1.1 (Recursive van Emde Boas Trees)**.**
- *`T.summary` is a vEB-tree of size $\sqrt{u}$ containing all i such that the i-th chunk is not empty.*

- *For each* $1 \leq i \leq \sqrt{u}$, `T.chunk[i]` *is a vEB-tree of size* $\sqrt{u}$ *containing* $x \texttt{mod} \sqrt{u}$ *for each* x *in the* i*-th tree.*
- `T.min` *is the smallest element in* T, *not stored recursively.*

We represent each integer $x = \langle c, i \rangle$ *where*

- c *is the chunk coordinates :* $c = x // \sqrt{u}$
- i *is the position of* x *in the chunk :* $i = x \texttt{mod} \sqrt{u}$

Proposition 17.1.1 (Successor Operation)**.** *With the following algorithm we get the wanted complexity :*

Algorithme 6 Successor Complexity Verifies : $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$

```
function (Successor)
    Input (T, x = ⟨c, i⟩)
    if x < T.min then
        return T.min
    end if
    if i < T.chunk[c].max then
        return ⟨c, Successor (T.chunk[c], i)⟩
    else
        c' = Successor(T.summary, c)
        return ⟨c', T.chunk[c'].min⟩
    end if
end function
```

Proposition 17.1.2 (Insertion)**.** *The algorithm below gets the correct complexity.*

Algorithme 7 Insertion Complexity Verifies : $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$

```
function (Insert)
    Input (T, x = ⟨c, i⟩)
    if T.min = None then
        T.min = T.max = x
        return
    end if
    if x < T.min then
        swap(x, T.min)
    end if
    if x > T.max then
        T.max = x
    end if
    if T.chunk[c].min = None then
        Insert(T.summary, c)
    end if
    Insert(T.chunk[c], i)
end function
```

Proposition 17.1.3 (Deletion)**.** *The algorithm below gets the correct complextiy.*

Proposition 17.1.4 (Complexity)**.**

- *All of these operations' complexities verify* $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$ *and thus have time complexity* $\mathcal{O}(\log \log u)$
- *Space complexity satifies :* $S(u) = (\sqrt{u} + 1)\dot{S}(\sqrt{u}) + \mathcal{O}(1)$ *and therefore* $S(u) = \mathcal{O}(u)$.

Algorithme 8 Deletion Complexity Verifies : $T(u) = T(\sqrt{u}) + \mathcal{O}(1)$

```
function (Delete)
    Input (T, x = ⟨c, i⟩)
    if x = T.min then
        c = T.summary.min
        if c = None then
            T.min = None. return
        end if
        x = T.min = ⟨c, i = T.chunk[c].min⟩
    end if
    Delete(T.chunk[c], i)
    if T.chunk[c].min = None then
        Delete(T.summary, c)
    end if
    if T.summary.min = None then
        T.max = T.min
    else
        c' = T.summary.max
        T.max = ⟨c', T.chunk[c'].max⟩
    end if
end function
```

Proposition 17.1.5 (Original van Emde Boad trees)**.** *For* `Successor`(x)*, as the path from the root to x is monotone, binary searching the path to find the lowest 1 gives us either the predecessor or the successor of x. By storing all nodes in an array of size $\mathcal{O}(u)$ to allow efficient binary search; a pointer from each node to the maximum and minimum of their subtree; all the elements as a doubly-linked list, we find the successor and the predecessor of x in $\mathcal{O}(\log\log u)$ time.*
Update is done in $\mathcal{O}(\log u)$ time, since we only need to update the element-to-root path, in $\Theta(u)$ space

17.2 Improvements

17.2.1 x-fast trees

Définition 17.2.1. *In an x-fast tree, we store every root-to-green node (nodes representing an element from the set) path, viewed in binary (left = 0, right = 1), via Cuckoo Hashing.*

Proposition 17.2.1. *Predecessor queries are done in $\mathcal{O}(\log\log u)$ time, updates in $\mathcal{O}(\log u)$ expected amortised time, but this tree only takes $\mathcal{O}(n \log u)$ space.*

Proof. To maintain successor and predecessor, we use the vEB-tree algorithm, giving us the same complexity, and same for updates. Yet, since we only store the root-to-element paths which are of $\log u$ length, we need $\mathcal{O}(n \log u)$ space. ■

17.2.2 y-fast trees

Définition 17.2.2. *We maintain elements in groups of size in $\left[\frac{\log u}{4}, 2\log u\right]$. For each group, we build a BST, and we store representatives of the group using an x-fast tree :*

- *If there are fewer than $\frac{\log u}{2}$ elements, we store them in a single BST.*
- *otherwise, suppose we add/delete an element. If a group becomes too large, we split it in two. If a group becomes too small, we merge it with its neighbour, then split if needed.*

Proposition 17.2.2. *Predecessor queries are in $\mathcal{O}(\log\log u)$ time, updates in $\mathcal{O}(\log\log u)$ expected amortised time, since insertion into the x-fast trie happens only once per $\Theta(\log u)$ new elements.*

Proof. This comes directly from the definition and the definition of the x-fast trees. ■

Part V
Lecture 5 : 9/11

Contents

18 General Definitions

Définition 18.0.1. *An alphabet Σ is a finite set. Elements of Σ are letters, characters or symbols, denoted by small letters. A string (or word), is a finite sequence of letters, denoted by a capital letter.*
The length of a string $S = s_1 s_2 \dots s_n$ where $s_i \in \Sigma$ is denoted by $|S|$ and defined to be equal to n. For $1 \le i \le j \le S$, we say that $S[i,j] = s_i \dots s_j$ is a substring of S. If $i = 1$, $S[i,j]$ is a prefix of s. If $j = |S|$, $S[i,j]$ is a suffix of S. $S[i,j]$ is an occurence of a string X in S if $S[i,j] = X$

19 Pattern Matching

Définition 19.0.1. *We define the pattern matching problem as such : Given a string P of length m (pattern), a string T of lenth $n \ge m$ (text), return all occurences of P in T specified by their starting positions.*

19.1 Naïve Algorithm

The naïve algorithm consists of looking, for each possible starting position in the text, check if the m following characters form a string equal to the pattern. This algorithm has time complexity $\mathcal{O}(nm)$ and space complexity $\mathcal{O}(n+m)$.

Remarque 19.1.0.1. *There are more than 80 algorithms for this problem. See S.Faro, T.Lecroq https://arxiv.org/pdf/1012.2547.pdf*

19.2 Knuth-Morris-Pratt Algorithms

Définition 19.2.1. *The border of a string P is a proper prefix of P that is equal to a suffix of P. The border array of length $|P|$ is such that $B[i]$ is the maximal length of a border of $P[1,i]$, 0 if undefined.*

Proposition 19.2.1. *B can be computed in $\mathcal{O}(|P|)$ time.*

Lemme 19.2.1. *First, note that :*

1. *$B[1] = 0$*
2. *If X is a border of $P[1,k]$ of length x, then $X^{'} = X[1, x-1]$ is a border of $P[1, k-1]$.*
3. *$B[k] - 1 = B[k-1]$ if and only if $P[k] = P[B[k-1] + 1]$.*
4. *If X is a border of X and Y is a border of X, Y is a border of S.*

Suppose that we have computed $B[1], \ldots, B[k-1]$. We will now compute $B[k]$.
By property 3, if $P[k] = P[B[k-1]+1]$, then $B[k] = B[k-1]+1$.
Else, if $P[k] \neq P[B[k-1]+1]$, consider $B^2[k-1] = B[B[k-1]]$. If $P[k] = P[B^2[k-1]+1]$, set $B[k] = B^2[k-1]+1$, else consider $B^3[k-1]$, and so on.

This algorithm is correct and in $\mathcal{O}(m)$.

Proof.
- Correctness : In fact, $B[1,k] \in \{B[k-1]+1, B^2[k-1]+1, \ldots, 0\}$.
  From property 2, any border of $P[1,k]$ can be obtained by appending $P[k]$ to some border of $P[1, k-1]$.
  From property 4, $B^j[k-1]$ is the j-th longest border of $P[1, k-1]$.
- Time : $\texttt{Time}(B[k]) \leq |B[k] - B[k-1]|$, indeed, if $B[k] = B^j[k-1]+1$, we use j steps. Observe that :

$$-1 \leq \sum_k (B[k] - B[k-1]) = \sum^{-} (B[k] - B[k-1]) + \sum^{+} (B[k] - B[k-1]) \leq m$$

  However, by property 2, the sum on the positive differences if $\mathcal{O}(m)$. Hence, the total time is in $\mathcal{O}\left(\sum |B[k] - B[k-1]|\right) = \mathcal{O}(m)$

■

Algorithme 9 Knuth-Morris-Pratt Algorithm : KMP

```
Input T, P, n, m
B ← BorderArray(P)
L ← []
i ← 1, j ← 1
while i ≤ n − m + j do
    (i, j) ← match(i, j, m)
    if j = m + 1 then
        L ← (i − m) :: L
    else
        if j = 1 then
            i ← i + 1
        else
            j ← B[j − 1] + 1
        end if
    end if
end whilereturn L
```

Proposition 19.2.2. *This algorithm is Correct*

Lemme 19.2.2. *No occurence of P can start in the shift interval.*

Proof. If $T[x, x+m-1] = P$, $P[1, j-1]$ has a border of length $> B[j-1]$, contradiction. ■

Lemme 19.2.3. *We have $P[1, B[j-1]] = T[i + (j - 1 - B[j-1]), i + j - 2]$.*
Hence, we do not need to compare the first $B[j-1]$ letters of the patter with the letters of the text after the shift.

Of the Correctness. This stands from 19.2.2 and 19.2.3. ■

Proposition 19.2.3. *This is done using $\mathcal{O}(m)$ extra space, in $\mathcal{O}(n+m)$ time.*

Proof.

- Space : $\mathcal{O}(m)$ to store B + constant
- Time : From **??**, $\mathcal{O}(m)$ to build B. To process T we need $\mathcal{O}(n+m)$ time, since $1 \leq i \leq n$ and $1 \leq j \leq n$ and their difference increases after any step.

■

Définition 19.2.2. *An algorithm is called online if can process its input item-by-item in a serial fashion, without having the entire input available from the start.*
An online algorithm is called real-time if it processes each data item in constant time.
KMP is an online algorithm, but it is not real-time.

Proposition 19.2.4. *We can construct a real-time version of KMP, by using $B^{'}[j, T[i]]$ to decide how to shift the pattern, where :*

$$B^{'}[j,a] = \max\{k \mid k < j,\ P[1,k] = P[j-k+1, j],\ P[k+1] = a\}$$

$B^{'}$ occupies $\mathcal{O}(m)$ space and is computed in $\mathcal{O}(m \cdot |\Sigma|)$ time. The modified KMP is real-time and correct.

20 Dictionaries, Multiple Pattern Matching

20.1 Tries

Définition 20.1.1. *A dictionary D is a set of string. A trie of D is a tree with every node being labeled by a letter such that :*

- *For every node, outgoing edges are labeled with different letters*
- *For every string $S \in D$, there is a root-to-node path that spells out S. The end of the path is labeled with the id of S.*
- *Every root-to-leaf path spells out a string in D.*

Proposition 20.1.1. *A trie for D uses $\mathcal{O}(\sum_{S \in D} |S|)$ space.*

Proof. By laying out each root-to-leaf path. ■

We will consider two applications :

1. Dictionary Look-Up : Given a string P, decide if it belongs to the dictionary.
2. Multiple Pattern Matching : Given a dictionary of patterns D, find all their occurences in a text T.

20.2 Dictionary Look-Up

Proposition 20.2.1. *For P a pattern, $P \in D \Leftrightarrow$ there is a root-to-node path that spells out P labeled by an id of a string from D. By starting at the root, if there is an from the root to its child u labeled by $P[1]$, the algorithm moves to u and continues recursively. This takes time in $\mathcal{O}(|P|)$.*

20.3 Aho-Corasick Algorithm

We will first assume that no pattern in D is a substring of another.

Définition 20.3.1. *The failure link from a node u labeled by a string S points to the deepest node v labeled by a proper suffix of S.*

Lemme 20.3.1. *The trie occupies $\mathcal{O}(m)$ space and can be constructed in $\mathcal{O}(m)$ time.*

Proof.

- Space : There are $\mathcal{O}(m)$ nodes, each node has exactly one failure link.
- Time : Failure links are built top-to-down. Links from nodes of depth 1 point to the root. Suppose that we have built links for all nodes to depth $\leq d-1$. The failure link from a node labeled with Sa must point to a node labeled with $S^{'}a$, where $S^{'}$ is a suffix of S. In other words, the parent of the end of the failure link from the node must belong to the failure link path from the node $p(u)$, and it must be the deepest node that has an outgoing edge labeled with a. The algorithm works as follows: we follow the failure link path from $p(u)$. The first node with an outgoing edge labeled with a is the end of the failure link for u. Consider the time needed to build the failure links for one root-to-leaf path. Let $root, u_1, \ldots, u_k$ be the nodes in this path, and denote by $f(u_i)$ the depth of the end of the failure link for u_i. The time to find the link for u_i is $\leq c \times (2 + f(u_{i-1} - f(u_i)))$. Thus the total time for the trie is $\mathcal{O}(m)$.

■

Algorithme 10 Aho-Corasick Algorithm

Input T, D, n
$curr\_node \leftarrow root, i \leftarrow 1, L \leftarrow []$
while $i \leq n$ **do**
  if $\exists e = (curr\_node, u)$ labeled with $T[i]$ **then**
    $curr\_node \leftarrow u$
    if $curr\_node$ corresponds to a pattern **then**
      $L \leftarrow i :: L$
    end if
    $i \leftarrow i + 1$
  else
    if $curr\_node = root$ **then**
      $i \leftarrow i + 1$
    else
      $curr_node = \texttt{failure\_link}(curr\_node)$
    end if
  end if
end while

Proposition 20.3.1. *This algorithm has space complexity $\mathcal{O}(m)$ and time complexity $\mathcal{O}(m+n)$.*

Proof.

- Space : m is the total length of the patterns
- Time : We need $\mathcal{O}(m)$ time to build the trie. Consider how the depth of *curr_node* changes during the algorithm.
  - Every time we do down an edge (happens $\leq n$ times), it increases by 1. Every time we follow a failure link, it decreases by ≥ 1.
  - Therefore, as the depth is always positive, we follow a failure link at most n times.

■

Théorème 20.3.2. *If no pattern is a substring of another, the multiple pattern matching problem can be solved in $\mathcal{O}(m)$ space and $\mathcal{O}(n+m)$ time.*

Définition 20.3.2. *When patterns are substrings of others, we add output links : an output link from a node u goes to the nearest node on the failure link path outgoing from u that corresponds to a pattern of the dictionary.*

Proposition 20.3.2. *Output links can be built in $\mathcal{O}(m)$ time by one top-down traversal of the failure link tree.*

We modify the algorithm : at each step it follows the path from *curr_node* and outputs the patterns in the output link path.

Théorème 20.3.3. *The multiple pattern matching problem can ve solved in $\mathcal{O}(m)$ space and $\mathcal{O}(n + m + occ)$ where occ is the total number of occurences of the patterns in the text.*

21 Suffix Trees and Applications

Définition 21.0.1. *A text index is a data structure that represents a text and supports pattern matching queries. The suffix tree is one of them.*

We define the compact trie for D as the trie for D where nodes with only one child have been replaced by an edge. The suffix tree of a string is the compact trie for the set of the suffixes of that string with \$ appended (\$ must not be in Σ).

21.1 Suffix Trees

Proposition 21.1.1. *A suffix tree for a string of length n has n leaves, and at most $2n - 1$ nodes and $2n - 2$ edges. Storing the labels on the edges can take $\Theta\left(|T|^2\right)$. To save space we represent each label as two numbers, the left and the right endpoints in T.*

Proof. By induction. ■

To implement algorithms on the suffix tree efficiently, we need to be able to identify, given a node u and a letter a, an edge u, v such that its label starts with a. Then, for each node u, we store an array A_u of size $|\Sigma|$, with $A_u[a]$ being a pointer to the child v of u such that the label of (u, v) starts with a.

Proposition 21.1.2. *This structure takes $\mathcal{O}(|Sigma| \cdot |T|)$ space.*

21.2 Pattern Matching

Remarque 21.2.0.1. *A suffix of T starts with P if and only if it corresponds to a leaf of the suffix tree that belongs to a subtree rooted at the end of the path labeled with P.*

To answer pattern matching queries, we start at the root. Then, we follow the path labeled by the letters of P, and use arrays A_u to find the next edge to follow. If there is no path labeled by P, there are no occurences of P in T. Otherwise, the starting positions of the occurences of P in T are the starting positions of the suffixes that are in the subtree rooted at the end of the path labeled by P. We retrieve the occurences by traversing the subtree depth-first.

Proposition 21.2.1. *This algorithm has total time $\mathcal{O}(m + occ)$.*

Proof. The arrays allow to decide which edge to follow next in $\mathcal{O}(1)$ time. In total, finding the path requires $\mathcal{O}(m)$ time. If there is a path labeled by P, the subtree rooted at its end has $\mathcal{O}(occ)$ leaves, where *occ* is the number of occurences of P in T. As the subtree does not have nodes of degree one (except possibly for the root), its size is $\mathcal{O}(occ)$. ■

21.3 Longest Common Substring

Given two strings, find the longest substring that occurs both in T_1 and T_2. We will denote $n = |T_1| + |T_2|$.

To find the longest common substring of T_1 and T_2, we need the suffix tree containing the suffixes of both string (sometimes called generalised suffix tree).

Proposition 21.3.1. *This suffix tree can be built in* $\mathcal{O}(|T_1| + |T_2|)$.

Remarque 21.3.0.1. *If a string X occurs at position i in T_1, and at position j in T_2, then the subtree rooted at the end of the path labeled by X contains the leaf corresponding to $T_1[i\ldots]$ and $T_2[j\ldots]$.*

By bottom-up traversal, we find the deepest node in the tree such that its subtree contains both leaves corresponding to suffixes of T_1 and suffixes of T_2. It is labeled by the longest common substring of T_1 and T_2.

Proposition 21.3.2. *This algorithm has time complexity* $\mathcal{O}(|T_1| + |T_2|)$.

We can generalise this to any m strings in time $\mathcal{O}(n)$ where $n = \sum_{k=1}^{m} |T_k|$. We will use the following fact : we can preprocess a tree of size $\mathcal{O}(n)$ in time $\mathcal{O}(n)$ to support lowest common ancestor (LCA) queries in constant time. (see) $LCA(u, v)$ must return the lowest node that is an ancestor of both u and v.

Open Problem 4. *We showed that the longest common substring problem for two string can be solved in $\mathcal{O}(n)$ space and $\mathcal{O}(n)$ time. It is also known that the problem can be solved in $\mathcal{O}(s)$ extra space and time*

$$\mathcal{O}\left(\frac{n^2 \log(n) \log^*(n)}{L \cdot s} + n \log n\right)$$

where L is answer. What is the optimal trade-off.

Part VI

Lecture 6 : 23/11

Contents

22 Disjoint-set Data Structure

Définition 22.0.1. *A disjoint-set data structure is a way to maintain a collection of disjoint sets with 3 operations :*

- `make_set(x)` *: create a new set containing one element, x*

- *`union_set`(x, y) : union the sets containing x and y*

- *`find_set`(x) : return a pointer to the representative of the set containing x*

We will parametrize by n the number of elements and m the number of operations.

It can be used to preprocess the connected components of a graph, so as to maintain the queries of the paths between two vertices.

It can also be used to represent the ways player play in the HEX game.

23 Linked-list Representation

Here, each set is a linked list, and the representative is the head of the list. For every element, we store a pointer to the node containing it. In that node, we store a pointer to the representative.

Proposition 23.0.1. *In this situation : `make_set` and `find_set` are done in $\mathcal{O}(1)$.*

For `union_set`, it requires more work.
Indeed, simply appending the set of y to the set of x takes constant time, but updating the pointers to the representatives is done in $\Theta\left(|\text{set of } y|\right)$. By doing the following sequence of operation :

$$\texttt{make\_set}(x_1), \texttt{make\_set}(x_2), \ldots, \texttt{make\_set}(x_n); \texttt{union\_set}(x_2, x_1), \texttt{union\_set}(x_3, x_2), \ldots, \texttt{union\_set}(x_n, x_{n-1})$$

we see that even the amortised time per operation can be large, as it takes a total time of $\Theta\left(n^2\right)$ and the amortised being $\Theta(n)$.
Then, we propose a weighted-union strategy : if the set of x is larger than the set of y, we append the latter to the former, else we append the set of x to the set of y.

Théorème 23.0.1. *Using linked-lists and the weighted-union strategy, doing m `make_set`, `union_set` and `find_set`, n of which are `union_set` takes $\mathcal{O}\left(m + n \log n\right)$*

Proof. At any moment, the largest size of a set is bounded by n. Therefore, an element can change its set representative at most $\log_2 n$, as every time it happends the size of the set at least doubles in size. Therefore, the total time for updating the representatives is $\mathcal{O}(n \log n)$. All other operations take $\mathcal{O}(m)$ time. ∎

24 Disjoint-set forests

Here, each set is a treee whose root is the representative. Each node stores a pointer to its parent. `make_set`(x) creates a tree containing only one node, `find_set`(x) follows the path from x to the root and `union_set`(x, y) makes the root of the tree of y become a child of the root of the tree of x.
We will use two strategies :

- Path Compression : During `find_set`(x), we make each node on the path to the root be a child of the root.

- Union by Rank : During `union_set`, the root of the tree with smaller rank becomes a child of the root of the tree with larger rank.

Définition 24.0.1. *We define ranks as upper bounds on the height of the node :*

- *When we add a new node x, we initialise the rank of x with 0.*

- *If we union two trees (i.e.,two sets) of different ranks, ranks do not change. If we union two trees of equal ranks, the rank of the root of the resulting tree increases by one.*

Algorithme 11 Union-Find

```
function (make_set)
    Input x
    x.p ← x
    x.rank ← 0
end function
function (union_set)
    Input x, y
    link(find_set(x), find_set(y))
end function
function (find_set)
    Input x
    if x ≠ x.p then
        x.p ← find_set(x.p)
    end ifreturn x.p
end function
function (link)
    Input x, y
    if x ≠ y then
        if x.rank > y.rank then
            y.p ← x
        else
            x.p ← y
            if x.rank = y.rank then
                y.rank ← y.rank + 1
            end if
        end if
    end if
end function
```

25 Time Analysis

We defined n as the number of elements, i.e. of `make_set` and m the total number of operations.

Théorème 25.0.1 (Tarjan)**.** *Using both Union by Rank and Path Compression, the time complexity for any sequence of Find and Union is $\mathcal{O}(m + n\alpha(n))$ where $\alpha(4)$ is the inverse of the Ackermann function. In any conceivable application $\alpha(n) \leq 4$. This time complexity is tight.*

25.1 A simpler Bound

We will first start with a simpler analysis showing $\mathcal{O}(m \log^\star n)$.

Proposition 25.1.1. *Suppose we convert a sequence of $m^{'}$ operations `make_set, union_set, find_set` into m operations `make_set, link, find_set`. If we can execute the latter in $\mathcal{O}(m \cdot \alpha(n))$, we can execute the former in $\mathcal{O}(m^{'} \cdot \alpha(n))$.*

Proof. $m^{'} \leq m \leq 3m^{'}$ ■

We will thus assume we have only `make_set, link, find_set` operations.

Proposition 25.1.2.

- *The rank of a node can only increase with time. Rank of non-root nodes are frozen forever.*
- *For every node x, $x.rank \leq x.p.rank$ and the inequality is strict unless x is a root node.*
- *The ranks of nodes in any path strictly increase*
- *The rank of any node is bounded by $n - 1$.*

Proposition 25.1.3 (Rank Lemma)**.** *At any moment, for every $r \in \mathbb{N}$, there are at most $\frac{n}{2^r}$.*

Proof. By induction, the number of nodes in a tree with a root of rank r is at least 2^r :

- Base Case : $r = 0$ is true
- Induction Step : After linking two trees with roots of rank r_1, r_2 :
  - If $r_1 < r_2$ the claim is true by induction for r_2.
  - If $r = r_1 = r_2$, then the root of the resulting tree has rank $r + 1$ and the tree has at least $2^r + 2^r = 2^{r+1}$ nodes.

Consider all nodes of rank r. None of them is not and was not an ancestor of another, so there are 2^r nodes corresponding to each of them in the forest, and these sets are independent. As we have n nodes in total, the bound follows. ■

25.2 Buckets

Définition 25.2.1. *We define buckets so they form a partition of $\mathbb{N}$ with : $B_0 = [0,0], B_1 = [1,1]$, $B_2 = \left[2, 2^2 - 1\right], B_3 = \left[2^2, 2^{(2^2)} - 1\right]$ and so on. They are not implemented, bu only used for the analysis. The total number of buckets is at most $\log^\star n$.*

Proposition 25.2.1. *Put a node with rank r into the bucket B containing r. The number of nodes in a bucket $B = \left[b, 2^b - 1\right]$ is at most :*

$$\sum_{r \in B} \frac{n}{2^r} \leq \frac{2n}{2^b}$$

Définition 25.2.2. *At a given time a node is good if :*

- *It is a root*

- *It is a child of a root*
- *It is in a smaller bucket than its parent*

It is bad otherwise.

A `find_set` visits $\mathcal{O}(\log^\star n)$ good nodes. Therefore, the total time complexity is $\mathcal{O}(m \log^\star n + |\{\text{bad nodes visited}\}|)$.

Proposition 25.2.2. *The total number of visits of bad nodes is* $\mathcal{O}(n \log^\star n)$

Proof. Bad nodes have frozen rank. When a bad node x is visited by a `find_set`, the new $x.p$ has rank strictly larger than the previous one. Hence, for each bad node x in a bucket $B = \left[b, 2^b - 1\right]$ the number of times x is visited while x is bad is at most $2^b - 1$, so by 25.2.1, it is at mode $\mathcal{O}\left(\frac{2n}{2^b}\right)$. So the number of times we visit a bad node in B is $\mathcal{O}(n)$. ■

Thus, the total complexity is $\mathcal{O}\left(m \log^\star n\right)$
In the following sections, we will try to improve this complexity.

25.3 Ackermann's function

Définition 25.3.1. *For $k \geq 0$, $j \geq 1$ two integers, the Ackermann function of k and j is defined as :*

$$A_k(j) = \begin{cases} j+1 & \text{if } k = 0 \\ A_{k-1}^{j+1}(j) & \text{if } k \geq 1 \end{cases}$$

Proposition 25.3.1 (Properties).

- *Lemma 1 :* $\forall j \geq 1, A_1(j) = 2j + 1$
- *Lemma 2 :* $\forall j \geq 1, A_2(j) = 2^{j+1}(j+1) - 1$

Proof. Both are done by induction. ■

Définition 25.3.2. *We define reverse Ackermann's function by :*

$$\alpha(n) = \min\left\{k \mid A_k(1) \geq n\right\}$$

We have :

$$\alpha(n) = \begin{cases} 0 & \text{if } 0 \leq n \leq 2 \\ 1 & n = 3 \\ 2 & 4 \leq n \leq 7 \\ 3 & 8 \leq n \leq 2047 \\ 4 & 2048 \leq n \leq A_4(1) \end{cases}$$

25.4 Potential Function

We define a potential function :

Définition 25.4.1.

- $\mathtt{level}(x) = \max\left\{k \mid x.p.rank \geq A_k(x.rank)\right\}$
- $\mathtt{iter}(x) = \max\left\{i \mid x.p.rank \geq A^{(i)}_{\mathtt{level}(x)}(x.rank)\right\}$

For a node x, we define :

$$\Phi_q(x) = \begin{cases} \alpha(n) \cdot x.rank & \text{if } x \text{ if root or } x.rank = 0 \\ (\alpha(n) - \mathtt{level}(x)) \cdot x.rank - \mathtt{iter}(x) & \text{otherwise} \end{cases}$$

For a forest, define :

$$\Phi_0(F) = 0 \text{ and } \Phi_q(F) = \sum_{x \in F} \Phi_q(x)$$

We will show that $\Phi_q(F) \geq 0$: Then, if we denote by t_q the time for the q-th operation :

$$\sum_q t_q \leq \sum_q t_q + \Phi_q(F) - \Phi_0(F) = \sum_q t_q + \sum_q (\Phi_{q+1}(F) - \Phi_q(F)) = \sum_q (t_q + \Phi_{q+1}(F) - \Phi_q(F))$$

Lemme 25.4.1. *If $x \neq x.p$ and $x.rank \geq 1$: $0 \leq \texttt{level}(x) \leq \alpha(n) - 1$*

Proof. We have $x.p.rank \geq x.rank + 1 = A_0(x.rank)$, hence $\texttt{level}(x) \geq 0$.
As $A_{\alpha(n)}(x.rank) \geq A_{\alpha(n)}(1) \geq n \geq x.p.rank$, we get the other inequality. ■

Lemme 25.4.2. *If $x.rank \geq 1$, $1 \leq \texttt{iter}(x) \leq x.rank$.*

Proof. We have $x.p.rank \geq A_{\texttt{level}(x)}(x.rank) = A^1_{\texttt{level}(x)}$ thus $\texttt{iter}(x) \geq 1$.
Since : $A^{x.rank+1}_{\texttt{level}(x)}(x.rank) = A_{\texttt{level}(x)+1}(x.rank) > x.p.rank$ and thus $\texttt{iter}(x) \leq x.rank$. ■

Lemme 25.4.3. $\forall x, q, 0 \leq \Phi_q(x) \leq \alpha(n)x.rank$

Proof. If x is the root or $x.rank = 0$, there is equality and we are done. Otherwise :

$$\Phi_q(x) = (\alpha(n) - \texttt{level}(x))\cdot x.rank - \texttt{iter}(x) \geq (\alpha(n) - (\alpha(n) - 1))\cdot x.rank - \texttt{iter}(x) \geq x.rank - x.rank = 0$$

Also :

$$\Phi_q(x) \leq \alpha(n) \cdot x.rank - 1 < \alpha(n)x.rank$$

■

Corollaire 25.4.3.1. *If x is not a root and $x.rank > 0$, then $\Phi_q(x) < \alpha(n)x.rank$*

Lemme 25.4.4. *If x is not a root, the q-th operation is **`link`** or **`find_set`** then $\Phi_q(x) \leq \Phi_{q-1}(x)$. Moreover, if $x.rank \geq 1$ and either **`level`**(x) or **`iter`**(x) changes, then $\Phi_q(x) \leq \Phi_{q-1}(x) - 1$.*

Proof. Since x is not a root, the rank of x is frozen.

1. If $x.rank = 0$ then $\Phi_q(x) = \Phi_{q-1}(x) = 0$
2. Assume $x.rank \geq 1$.
   - If $\texttt{level}(x)$ increases, $(\alpha(n) - \texttt{level}(x))\, x.rank$ drops by at least $x.rank$ but $\texttt{iter}(x)$ can drop by at most $x.rank - 1$ therefore, $\Phi_q(x) \leq \Phi_{q-1}(x) - 1$.
   - If $\texttt{level}(x)$ is unchanged, $\texttt{iter}(x)$ remains unchanged or increases. If it does not change $\Phi_q(x) = \Phi_{q-1}(x)$, otherwise, $\Phi_q(x) \leq \Phi_{q-1}(x) - 1$

■

25.5 Time Bound

More lemmas :

Lemme 25.5.1. *The amortised cost of each **`make_set`** is $\mathcal{O}(1)$*

Proof. `make_set` creates a node x with $\Phi_q(x) = 0$. Hence, $\Phi_q(F) = \Phi_{q-1}(F)$ and the amortised time is 1. ■

Lemme 25.5.2. *The amortised time of **`link`**(x, y) is $\mathcal{O}(\alpha(n))$.*

Proof. WLOG, y becomes the parent of x. By 25.4.4, the potentials of any node but x and y cannot increase. Before `link`, the node x is a root and $\Phi_{q-1}(x) = \alpha(n)x.rank$. If $x.rank = 0$, we have $\Phi_{q-1}(x) = \Phi_q(x) = 0$. Otherwise by 25.4.3.1, $\Phi_q(x) < \alpha(n)x.rank = \Phi_{q-1}(x)$.
For y, either $\Phi_{q-1}(y) = \Phi_q(y)$ or $\Phi_q(y) = \Phi_{q-1}(y) + \alpha(n)$.

Hence, $\Phi_q(F) - \Phi_{q-1}(F) \leq \alpha(n)$. The actual time of $\texttt{link}(x, y)$ is 1 and thus the lemma follows. ■

Lemme 25.5.3. *The amortised time of **`find_set`** is $\mathcal{O}(\alpha(n))$.*

Proof. Assume the find path contains s nodes :

1. There are no potential increases from 25.4.4.
2. The potential of at least $\max\{0, s-\alpha(n)-2\}$ decreases by at least one. Consider all nodes x on the find path with rank at least 1 and such that there exists y an ancestor of x with the same `level`. There are at least $s-(\alpha(n)+2)$ such nodes.

   Consider such a node x, let $k = \texttt{level}(x)$:

   Prior to path compression, $x.p.rank \geq A_k^{\texttt{iter}(x)}(x.rank)$ and $y.p.rank \geq A_k(y.rank)$. Then, by monotony of *rank* along a path, $y.rank \geq x.p.rank$. Let $i = \texttt{iter}(x)$. We have $y.p.rank \geq A_k\left(A_k^i(x.rank)\right)$

   After path compression, x and y have the same parent, the root so $x.p.rank = y.p.rank \geq A_k^{i+1}(x.rank)$. since $x.rank$ does not change. Therefore, either $\texttt{iter}(x)$ or $\texttt{level}(x)$ increases, and hence, $\Phi_q(x) \leq \Phi_{q-1}(x) - 1$ from 25.4.4.

As a corollary, we obtain that $\Phi_q(F) \leq \Phi_{q-1}(F) - \max\{0, s-(\alpha(n)+2)\}$. Since the actual time of `find_path` is s, the amortised time is $\mathcal{O}(\alpha(n)+2)$. ∎

Finally :

Théorème 25.5.4. *A sequence of m `make_set`, `union_set`, `find_set`, out of which n are `make_set` takes $\mathcal{O}(m\alpha(n))$ time. In other words, one operation takes $\mathcal{O}(\alpha(n))$ amortised time.*

Part VII
Lecture 7 : 16/11

Contents

26 Definitions

All along the course :

Définition 26.0.1.
- *n is the number of vertices*
- *m is the number of edges*
- *An algorithm in $\mathcal{O}(n+m)$ is said to be linear*
- *K_n is the complete graph on n vertices*

Table 1: Some Graph Parameters

$\delta(G)$	minimum degree
$\Delta(G)$	maximum degree
$\omega(G)$	clique number
$\alpha(G)$	size of a maximum independent set
$\chi(G)$	chromatic number
$\tau(G)$	vertex cover
$\kappa(G)$	vertex connectivity
$\lambda(G)$	edge connectivit
$\mu(G)$	size of a maximum matching
Girth	length of a shortest cycle

- *$G = (U, V, E)$ is a bipartite graph means U and V is the partition.*
- *$K_{a,b}$ denotes the complete bipartite graph with parts of size a and b.*
- *H is a subgraph of G if H can be obtained from G by deleting vertices and edges.*
- *H is a induced subgraph of G if H can be obtained from G by deleting vertices.*
- *H is a subdivision of G if H can be obtained from G by subdividing some edges.*
- *H is a minor of G if H can be obtained from G by deleting vertices, edges and contracting edges.*
- *A sink is a vertex of outdegree 0*

Définition 26.0.2 (Generic Graph Search)**.** *The problem is to visit the vertices following the edges of the graph. A search also outputs a search tree, and an ordering on the vertices.*

27 BFS, DFS and Graph Encoding

27.1 The Algorithms

Nan mais flemme à un moment.

27.2 Properties

Définition 27.2.1. *We will denote by $v.d$ the time at which a vertex is discovered, and $v.f$ the time at which a vertex is finished, i.e. all of its neighbour are discovered.*

Théorème 27.2.1 (Parenthesis Theorem)**.** *In any DFS of a graph, for any two vertices u and v either :*

- *$[u.d, u.f] \cap [v.d, v.f] = \varnothing$ and u is not a descendant nor an ancestor of v.*
- *$[u.d, u.f] \subset [v.d, v.f]$ and u is a descendant of v*
- *$[v.d, v.f] \subset [u.d, u.f]$ and v is a descendant of u*

Corollaire 27.2.1.1. *A vertex v is a descendant of u in the DFS forest if and only if $u.d < v.d < v.f < u.f$.*

Théorème 27.2.2 (White-path Theorem)**.** *In a DFS forest of a digraph, a vertex v is a descendant of a vertex u if and only if at time $u.d$, there is a (u, v)-path made of undiscovered vertices.*

Définition 27.2.2. *We define arc types in terms of the DFS forest G_π :*

- *Tree arcs are arcs of* G_π
- *Forward arcs are arcs* uv *such that* u *in an ancestor of* v
- *Back arcs are arcs* uv *such that* u *is a descendant of* v
- *Cross arc are arcs* uv *such that* u *is not an ancestor of* v *and* v *is not an ancestor of* u.

Proposition 27.2.1. *A digraph has a directed cycle if and only (any) DFS produces a back arc.*

27.3 Topological Sort

Définition 27.3.1. *A topological ordering of the vertices of a digraph is a labeling* f *such that* $uv \in Eonlyiff(u) < f(v)$.

Théorème 27.3.1. *G has a topological ordering if and only if it is a DAG.*

Théorème 27.3.2. *The problem of finding a topological is solved in linear time using a DFS.*

Lemme 27.3.3. *A digraph is acyclic if and only if a DFS of the graph yields no back edges.*

27.4 Strongly Connected Components

Définition 27.4.1. *A strong connected component (scc) of a directed graph G is a maximum set of vertices such that every pair of vertices as a directed path from the first to the second and from the second to the first.*
We define G^{SCC} *the quotient graph for* $u\mathcal{R}v \Leftrightarrow u \rightsquigarrow v \wedge v \rightsquigarrow u$.

Proposition 27.4.1. G^{SCC} *is a DAG.*

Lemme 27.4.1. *Let* $C, C^{'}$ *be two scc of a digraph. Let* $u, v \in C$ *and* $u^{'}, v^{'} \in C^{'}$. *If* $u \rightsquigarrow u^{'}$ *then there is no path from* $v^{'}$ *to* v.

Définition 27.4.2. *We extend the discovery and finishing times to sets of vertices :*

- $d(S) = \min\{u.d \mid u \in S\}$
- $f(S) = \max\{u.f \mid u \in S\}$

Lemme 27.4.2. *Let* C *and* $C^{'}$ *be two scc and let* $u \in C$, $v \in C^{'}$ *such that* $uv \in E$. *Then* $f(C) > f(C^{'})$.

Définition 27.4.3. *We define* $G^R = \left(V, E^R\right)$ *where* $E^R = \{uv \mid vu \in E\}$.

Proposition 27.4.2. G^R *has the same scc as* G

Algorithme 12 Kosaraju's Two Pass Algorithm

Call `DFS`(G)
Compute G^R
Call `DFS`(G^R) considering vertices in order of decreasing $u.f$ **return** Output the vertices of each tree in the DFS forest computed by `DFS`(G^R)

Théorème 27.4.3. *Kosaraju's two pass algorithm computes the scc in linear-time.*

28 Minimum Weighted Spanning Tree - MWST

28.1 Trees

Définition 28.1.1. *A tree is a connected acyclic graph. A forest is an acyclic graph. A leaf of a tree is a vertex of degree* 1.

Proposition 28.1.1.
- *Every tree contains at least two leaves*
- *Every tree T satifies :* $|E(T)| = |V(T)| - 1$.
- *Every subgraph of a tree is a forest.*

Proposition 28.1.2. *Let $G = (V, E)$ be a graph. The following are equivalent :*

- *G is a tree*
- *Any two vertices in G are linked by a unique path*
- *G is connected and $|E| = |V| - 1$*
- *G is acyclic and $|E| = |V| - 1$*
- *G is connected but for every edge uv, $G - \{uv\}$ has exactly two connected components, one containg u and the other v.*
- *G is acyclic but if any edge is added to G, the resulting graph contains a unique cycle going through this added edge.*

Définition 28.1.2. *$\omega : E \to \mathbb{R}$ is called a weight function. A spanning tree is a subgraph of G that is a tree and contains all vertices of G. The weight of a subgraph is the sum of the weights of its edges.*

Proposition 28.1.3 (Cut and Paste Technic)**.** *Let $G = (V, E)$ and let T be a spanning tree of G. Let $uv \in E(G) - T$ and let T_{uv} be the unique path linking u and v in T. Then for every edge xy of T_{uv} $T \setminus \{xy\} \cup \{uv\}$ is a spanning tree of T.*

Proof. Since T_{uv} is in the unique path of T linking u and v, removing any edge $xy \in T_{uv}$ from T breaks T into two connected components, one containing u and the other v. Adding uv reconnects the two parts and form a new spanning tree $T \setminus \{xy\} \cup \{uv\}$. ∎

Théorème 28.1.1. *Let $G = (V, E, \omega)$. A spanning tree T of G is a minimum spanning tree if and only if for every edge $e \in E \setminus T$:*

$$\omega(e) \geq \omega(f) \forall f \in \text{ the unique cycle of } T \cup \{e\}$$

Définition 28.1.3. *A set of edges A is said to be promising if it can be completed into a minimum spanning tree. At each step, we determine an edge uv that we can add to A without violating this invariant, that is $A \cup \{uv\}$ is still a promising set. Such an edge uv is said to be safe with respect to A.*

The algorithm works as follows : We start with an empty set, and while our set is not a spanning tree, we find a safe edge and add it to the set. How to find safe edges ?

Définition 28.1.4. *A cut $(S, V - S)$ is a partition of V. An edge of the cut or crossing edge is an edge with one end in S and the other in $V - S$. A cut respects a set of edges if no edge of the set is a crossing edge of the cut.*

Proposition 28.1.4. *Let A be a promising set of edges. Let $(S, V - S)$ be a cut respecting A and uv be a lightest crossing edge. Then uv is safe.*

Proof. Let T be a MST containing A. If T contains uv we are done, so assume it does not. Let T_{uv} be the unique path linking u and v in T. Since uv is a crossing edge of $(S, V - S)$, T_{uv} contains an edge xy of $(S, V - S)$. By the cut and paste technique, $T^{'} = (T \setminus \{xy\}) \cup \{uv\}$ is a spanning tree of G. By hypothesis, uv is a lightest edge of $(S, V - S)$, so $\omega(uv) \leq \omega(xy)$ and thus:

$$\omega(T^{'}) = \omega(T) + \omega(uv) - \omega(xy) \leq \omega(T)$$

So $T^{'}$ is a MST containing uv ∎

28.2 KRUSKAAAAAAAAL

Kruskal algorithm grows a promising forest A :

- Sort the edges by non-decreasing order of weight
- Start with the empty set
- Add a minimum weighted edge that does not create a cycle which is equivalent to add a minimum weighted edge that connects two connected components of the growing forest.

Algorithme 13 Kruskal

Input $G = (V, E, \omega)$
$A \leftarrow \varnothing$
for $u \in V$ **do**
  `Make-Set`(u)
end for
`Sort`(E) ▷ Sort by non-decreasing order of weight
for $uv \in E$ **do**
  if `Find`$(u) \neq$ `Find`(v) **then**
    $A \leftarrow A \cup \{uv\}$
    `Union`(u, v)
  end if
end forreturn A

Théorème 28.2.1. *The Kruskal algorithm answers the problem in $\mathcal{O}(m \log(n))$ time*

Proof.
- Complexity : from union-find, we do $\mathcal{O}(m)$ `Find-Set` and $\mathcal{O}(n)$ `Union` and `Make-Set` in $\mathcal{O}(m+n)\alpha(n)^4$. As we do one sort, the total complexity is $\mathcal{O}(m \log(n))$.
- Correctness : Comes from the safe property, by considering the loop invariant : Prior to each iteration, A is promising.

■

28.3 Prim

This algorithm works like Kruskal's, but instead of growing a forest, we grow a tree A. At each iteration, we add the lightest edge with exactly one extremity in $V(A)$.

Définition 28.3.1. *A Priority Queue is a data structure that maintains a set S of elements, each with an associated value called a key. TIt supports the following :*

- *`Insert`(S, x)*
- *`Minimum`(S)*
- *`Extract-Min`(S)*
- *`Decrease-Key`(S, x, k) does $x.key \leftarrow k$.*

Remarque 28.3.0.1. *This is for a min-queue, we can also do a max-queue.*

We implement priority queues using a heap.

Définition 28.3.2. *A max-heap is a complete binary tree where the key of a node is larger than the keys of its children. It can be implemented in an array : `Parent`$(i) = \lfloor \frac{i}{2} \rfloor$, `Left`$(i) = 2i + 1$, `Right`$(i) = 2i + 2$. It supports the same operations as a max-queue.*

Théorème 28.3.1. *Using a binary heap, insertion is done in $\log N$, max-retrieval is done in constant time and max-deletion is done in $\log N$.*

Algorithme 14 PRIM

```
Input G = (V, E, ω), r
for v ∈ V do
    v.π ← NIL and v.key ← +∞
end for
r.key ← 0
Q ← V
while Q ≠ ∅ do
    u ← Extract-Min(Q)
    for dov ∈ Adj[u]
        if thenv ∈ Q ∧ v.key > ω(uv)
            v.key ← ω(uv)
            v.π ← u
        end if
    end for
end while
```

Proof. See implementation. ■

Théorème 28.3.2. *This algorithm solves the problem in $\mathcal{O}(m + n \log n)$.*

Proof.
- Correctness comes from the safe property
- Complexity is in $\mathcal{O}(m \log n)$ with a min-heap and $\mathcal{O}(m + n \log n)$ with a Fibonacci Heap

■

29 Matroids and Greedy Algorithms

29.1 Definitions

Définition 29.1.1. *Let E be a set of elements and $\mathcal{I} \subseteq 2^E$. $(E, \mathcal{I})$ is a hereditary set system if it satisfies*

- *M1 : $\varnothing \in \mathcal{I}$*
- *M2 : If $X \subseteq Y \in \mathcal{I}$, $X \in \mathcal{I}$.*
- *E is called the ground set.*
- *Sets in $\mathcal{I}$ are called independent sets.*
- *Maximal independent sets are called bases.*
- *For $X \subseteq E$, the rank of X denoted by $rk(X)$ is the size of a maximum independent set included in X.*
- *Sets not in $\mathcal{I}$ are called dependent sets.*
- *Minimal dependent sets are called circuit.*

Given a positive weight function, we want to find a maximum independent set such that its weight is maximum. For example :

- MST : $\mathcal{I} = \{I \subseteq E \mid I \text{ is a forest }\}$
- TSP, SPP, Maximum Matching Problem, Maximum Stable Set, Knapsack are also of the sort.

Théorème 29.1.1 (Rado - Edmonds)**.** *The greedy algorithm for this problem is optimal for any weight function if and only if* $(E, \mathcal{I})$ *is a matroid.*

Définition 29.1.2. *A matroid is a hereditary set system* $E, \mathcal{I}$ *such that : if* $X, Y \in \mathcal{I}$ *and* $|X| > |Y|$*, there is* $x \in X \setminus Y$ *such that* $Y \cup \{x\} \in \mathcal{I}$