Homework 1

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1 Question 1

1.1 Question 1(a)

Initially, we have $d^* = 0$. Moreover, we always have $d^* \geq 0$ and an increase of d^* is caused by a relabeling. Thus, d^* can only increase $2n^2$ times (the maximum number of relabelings) and decrease as many times.
$\boxed{\text{There are thus at most } 4n^2 \text{ phases.}}$

1.2 Question 1(b)

- Relabeling v causes $\bar{d}(v)$ to increase but cannot cause $\bar{d}(w)$ to increase if $w \neq v$.
  $\boxed{\text{Thus, relabeling a node increases } \Phi \text{ by at most } \dfrac{n}{K}.}$
- A saturating push creates at most one new active node.
  $\boxed{\text{Thus, a saturating push increases } \Phi \text{ by at most } \dfrac{n}{K}.}$
- A nonsaturating push across the edge (u, v) deactivates node u and might activate node v. Then we have $\bar{d}(v) \leq \bar{d}(u)$, and hence a nonsaturating push does not increase Φ.
  During heavy phases, we execute $\rho > K$ nonsaturating pushes. Since d^* is constant during the phase, all ρ nonsaturating pushes must be from nodes at level d^*.
  Indeed, we choose nodes from the highest level, thus d^*.
  The phase terminates either when all nodes in level d^* are deactivated or when relabeling moves a node to level $d^* + 1$.
  Level d^* thus contains $\rho > K$ nodes (either active or inactive) throughout the phase.
  Hence, each nonsaturating push decreases Φ by at least one, since $\bar{d}(v) \leq \bar{d}(u) - 1$ for (u, v) with $|\{w \mid d(w) = d(u)\}| \geq K$.
  Finally, a heavy phase of non saturating push will decrease Φ by at least $\rho > K$.

For light phases, the bound is easier : the number of nonsaturating pushes is bounded K.

1.3 Question 1(c)

The total increase of Φ is bounded by $\frac{(2n^2+2nm)n}{K}$ and so the total decrease cannot be more than that (since $\Phi \geq 0$). Therefore, the number of nonsaturating push cannot be more than $\frac{2n^3+2n^2m}{K}$. The number of non saturating pushes in both phases, is then bounded by :

$$\frac{2n^3 + 2n^2m}{K} + 4n^2K$$

since $4n^2$ is the number of phases (and thus more than the number of light phases).

Finally, since $n = \mathcal{O}(m)$ (the graph being connex $m \geq n-1$ and $n \leq m+1$), taking $K = \sqrt{m}$ we get a complexity in $\mathcal{O}(n^2\sqrt{m})$.

2 Question 2

We will use Ford-Fulkerson's theorem on an appropriate graph to prove this property. Let us write $U = \{a_1, \dots, a_n\}$. We define a set of vertices V by :

$$V = \left\{s, \bar{S_1}, \dots, \bar{S_k}, \bar{a_1}, \dots, \bar{a_n}, \tilde{a_1}, \dots \tilde{a_n}, \bar{T_1}, \dots, \bar{T_n}, t\right\}$$

Then, we add arcs with a capacity function :

- An arc $(s, \bar{S_i})$ with capacity 1 for each $j \in [\![1, k]\!]$.
- An arc $(s, \tilde{a_i})$ with capacity 0 for each $i \in [\![1, n]\!]$.
- An arc $(\bar{S_j}, \bar{a_i})$ with infinite capacity for i, j such that $a_i \in S_j$.
- An arc $(\bar{a_i}, \tilde{a_i})$ with capacity 1 for each $i \in [\![1, n]\!]$.
- An arc $(\bar{a_i}, t)$ with capacity 0 for each $i \in [\![1, n]\!]$.
- An arc $(\tilde{a_i}, \bar{T_j})$ with infinite capacity for i, j such that $a_i \in T_j$.
- An arc $(\bar{T_j}, t)$ with capacity 1 for each $j \in [\![1, k]\!]$.